Project 5: PDE simulation for bullet options

The students have to simulate a PDE of a bullet option pricing in local volatility models. They have also to use Crank-Nicolson scheme in order to simulate this family of options. To solve the tridiagonal systems involved, the students have first to start using the sequential Thomas algorithm then use PCR once given by students of Project 4.

As described in [6], Thomas algorithm allows to solve tridiagonal systems:

$$\begin{pmatrix} b_1 & c_1 & & & & \\ a_2 & b_2 & c_2 & & 0 & \\ & a_3 & b_3 & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & 0 & & \ddots & \ddots & c_{n-1} \\ & & & & a_n & b_n \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \tag{7}$$

using a forward phase

$$c'_1 = \frac{c_1}{b_1},\ y'_1 = \frac{y_1}{b_1},\ c'_i = \frac{c_i}{b_i - a_i c'_{i-1}},\ y'_i = \frac{y_i - a_i y'_{i-1}}{b_i - a_i c'_{i-1}} \text{ when } i = 2, ..., n \tag{8}$$

then a backward one

$$z_n = y'_n,\ z_i = y'_i - c'_i z_{i+1} \text{ when } i = n-1, ..., 1. \tag{9}$$

Using Thomas method, write a kernel that solves various tridiagonal systems at the same time.

Let $u(t,x,j) = e^{\int_t^T r_g(u)du} F(t, e^x, j)$ where F is the price of a bullet option $F(t,x,j) = e^{-\int_t^T r_g(u)du} E(X \big| S_t = x, I_t = j)$, $X = (S_T - K)_+ 1_{\{I_T \in [P_1, P_2]\}}$ with $I_t = \displaystyle\sum_{T_i \leq t} 1_{\{S_{T_i} < B\}}$ and

- K, T are respectively the contract's strike and maturity
- $T_0 = 0 < T_1 < ... < T_M = T = T_{M+1}$ is a predetermined schedule
- barrier B should be crossed I_T times $\in \{P_1, ..., P_2\} \subset \{0, ..., M\}$
- r_g is the risk-free rate, assumed piecewise constant
- $\sigma_{loc}(x,t)$ is a local volatility function: $\mathbb{R}_+^* \times \mathbb{R}_+ \to \mathbb{R}_+^*$ used in the model

$$dS_t = S_t r_g(t) dt + S_t \sigma_{loc}(S_t, t) dW_t, \quad S_0 = x_0.$$

$u(t,x,j)$ is then the solution of the PDE

$$\frac{1}{2}\sigma_{loc}^2(x,t)\frac{\partial^2 u}{\partial x^2}(t,x,j)+\mu(x,t)\frac{\partial u}{\partial x}(t,x,j)=-\frac{\partial u}{\partial t}(t,x,j)$$

with: $\mu(x,t)=r_g(t)-\dfrac{\sigma_{loc}^2(x,t)}{2}, (x,t,j)\in]0,\max(K_g)]\times[T_k,T_{k+1}[\times[0,P_2]$

. The final and boundary conditions are:

$$\begin{aligned}
u(T,x,j) &= \max(e^x-K,0) \text{ for any (x, j)}\\
u(t,\log[\min(Kg)],j) &= \texttt{pmin}=0\\
u(t,\log[\max(Kg)],j) &= \texttt{pmax}=\max(Kg)-K
\end{aligned}$$

Denoting $u_{k,i}=u(t_k,x_i,j)$, $\sigma=\sigma_{loc}(x_i,t_k)$, $\mu=\mu(x_i,t_k)$ and using a Crank Nicolson scheme we get

$$q_u u_{k,i+1}+q_m u_{k,i}+q_d u_{k,i-1}=p_u u_{k+1,i+1}+p_m u_{k+1,i}+p_d u_{k+1,i-1}$$

$$\begin{aligned}
q_u &= -\frac{\sigma^2\Delta t}{4\Delta x^2}-\frac{\mu\Delta t}{4\Delta x}, \quad q_m=1+\frac{\sigma^2\Delta t}{2\Delta x^2}, \quad q_d=-\frac{\sigma^2\Delta t}{4\Delta x^2}+\frac{\mu\Delta t}{4\Delta x}\\
p_u &= \frac{\sigma^2\Delta t}{4\Delta x^2}+\frac{\mu\Delta t}{4\Delta x}, \quad p_m=1-\frac{\sigma^2\Delta t}{2\Delta x^2}, \quad p_d=\frac{\sigma^2\Delta t}{4\Delta x^2}-\frac{\mu\Delta t}{4\Delta x}
\end{aligned}$$

The difficulty lies in the fact that I_T is not Markov. Indeed, for instance, we see that :

$$\sum_{i=1}^{M}\mathbb{1}_{\{S_{T_i}<B\}}=\begin{cases}\sum_{i=1}^{M-1}\mathbb{1}_{\{S_{T_i}<B\}} & \text{if } S_{T_M}\geq B\\ \\ \sum_{i=1}^{M-1}\mathbb{1}_{\{S_{T_i}<B\}}+1 & \text{if } S_{T_M}<B.\end{cases}$$

Therefore we obtain the following backward induction:

for any $t\in[T_M,T[$, $\quad u_t(x,j)=\mathbb{E}[(S_T-K)_+|S_t=x]$

for any $t\in[T_{M-1},T_M[$,

$$u_t(x,j)=\begin{cases}\mathbb{E}[(S_T-K)_+\mathbb{1}_{\{S_{T_M}\geq B\}}\mid S_t=x] & \text{if } j=P_2\\ \mathbb{E}[(S_T-K)_+\mid S_t=x] & \text{if } j\in[P_1,P_2-1]\\ \mathbb{E}[(S_T-K)_+\mathbb{1}_{\{S_{T_M}<B\}}\mid S_t=x] & \text{if } j=P_1-1\end{cases}$$

for any $t\in[T_{M-k-1},T_{M-k}[$, $k=M-1,...,1$,

$$u_t(x,j) = \begin{cases} \mathbb{E}[u_{T_{M-k}}(S_{T_{M-k}}, P_2)\mathbb{1}_{\{S_{T_{M-k}} \geq B\}} \mid S_t = x] & \text{if } j = P_2 \\ \mathbb{E}[u_{T_{M-k}}(S_{T_{M-k}}, P_k^1)\mathbb{1}_{\{S_{T_{M-k}} < B\}} \mid S_t = x] & \text{if } j = P_k^1 - 1 \\ \mathbb{E}\left[\begin{matrix} u_{T_{M-k}}(S_{T_{M-k}}, j)\mathbb{1}_{\{S_{T_{M-k}} \geq B\}} \\ +u_{T_{M-k}}(S_{T_{M-k}}, j+1)\mathbb{1}_{\{S_{T_{M-k}} < B\}} \end{matrix} \middle| S_t = x \right] & \text{if } j \in [P_k^1, P_2 - 1] \end{cases} \tag{10}$$

with $P_k^1 = max(P_1 - k, 0)$.

The figure below shows an example of how PDE's backward resolution algorithm (with M = 10, $P_1 = 3, P_2 = 8$) is deployed with time on the x-axis and the set of values of I_t in the ordinate.

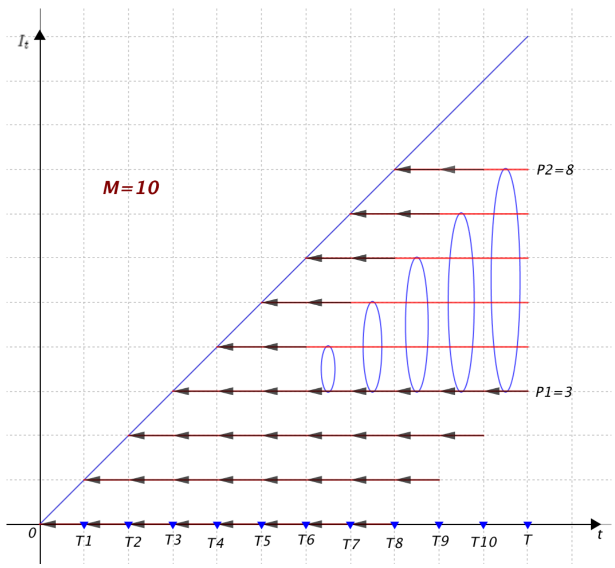

Figure 3: Backward induction scheme

Write the pricing code and compare the execution time when using either Thomas or PCR.

Project 6: Risk measures with the merge path sort and application on XVA simulation

Value at Risk (VaR) and Expected Shortfall (ES) are widespread used risk measures in financial literature. Their definitions involve capturing the distribution tail generally done by sorting the values. The students have to optimize a standard $O(n^2)$ sorting algorithm to capture the tails of a large number of arrays. They have also to explore the implementation of merging sort presented in [3]. Finally, they have to apply their solution to an X-Valuation Adjustment (XVA)