

Chap 9 — 6

向量场的微商

9.6.1 向量场

场 空间区域中每一点都对应一个量. 若对应的量是向量(数量), 则称之为**向量场(数量场)**.

如空间向量场

$$\boldsymbol{v}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$$

$(x, y, z) \in D \subset \mathbf{R}^3$. 平面向量场

$$\boldsymbol{v}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}, \quad (x, y) \in D \subset \mathbb{R}^2$$

位置向量

$$\boldsymbol{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

定义 设有向量场 $\nu(x, y, z)$, 它对 x 的偏微商定义为

$$\frac{\partial \nu}{\partial x} = \lim_{\Delta x \rightarrow 0} \frac{\nu(x + \Delta x, y, z) - \nu(x, y, z)}{\Delta x}$$

想一想 对 y, z 的偏微商? 显见

$$\frac{\partial \nu}{\partial x} = \frac{\partial P(x, y, z)}{\partial x} \mathbf{i} + \frac{\partial Q(x, y, z)}{\partial x} \mathbf{j} + \frac{\partial R(x, y, z)}{\partial x} \mathbf{k}$$

光滑向量场 P, Q, R 具有连续偏导数.

例 1 设 $\nu(x, y, z) = \sin xyz \mathbf{i} + x^2 \mathbf{j} + e^{x+2y+z} \mathbf{k}$, 求 $\frac{\partial \nu}{\partial x}, \frac{\partial \nu}{\partial y}, \frac{\partial \nu}{\partial z}$

9.6.2 梯度、散度与旋度

设有数量场 $\varphi(x, y, z)$, 根据微分及点乘公式, 有

$$\begin{aligned} d\varphi &= \frac{\partial \varphi}{\partial x} dx + \frac{\partial \varphi}{\partial y} dy + \frac{\partial \varphi}{\partial z} dz \\ &= \left(\frac{\partial \varphi}{\partial x} \mathbf{i} + \frac{\partial \varphi}{\partial y} \mathbf{j} + \frac{\partial \varphi}{\partial z} \mathbf{k} \right) \cdot (dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k}) \\ &= \mathbf{grad} \varphi \cdot d\mathbf{r} \end{aligned}$$

- 函数微分是梯度与位置向量微分之点乘.

➤ Hamilton算子

$$\nabla = \frac{\partial}{\partial x} \mathbf{i} + \frac{\partial}{\partial y} \mathbf{j} + \frac{\partial}{\partial z} \mathbf{k}$$

➤ ∇ 算子兼具**偏微商**和**向量**属性, 它‘数乘’函数得

$$\nabla \varphi = \mathbf{grad} \varphi = \frac{\partial \varphi}{\partial x} \mathbf{i} + \frac{\partial \varphi}{\partial y} \mathbf{j} + \frac{\partial \varphi}{\partial z} \mathbf{k}$$

➤ 数量场的**梯度是向量场**

定义 设有向量场 $\boldsymbol{v}(x, y, z) = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$.

$$1^\circ \ \boldsymbol{v} \text{ 的散度} \quad \operatorname{div} \boldsymbol{v} \stackrel{\text{def}}{=} \nabla \cdot \boldsymbol{v} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

➤ 向量场的散度是数量场

2° ν 的旋度

$$\text{rot } \nu \stackrel{\text{def}}{=} \nabla \times \nu = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathbf{k}$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

➤ 向量场的旋度是向量场

命题(运算性质) 设 φ, ψ 是数量场, \mathbf{a}, \mathbf{b} 是向量场, 则有

$$\nabla f(\varphi) = f'(\varphi) \nabla \varphi$$

$$\nabla(\varphi + \psi) = \nabla \varphi + \nabla \psi$$

$$\nabla \cdot (\mathbf{a} + \mathbf{b}) = \nabla \cdot \mathbf{a} + \nabla \cdot \mathbf{b}$$

$$\nabla \times (\mathbf{a} + \mathbf{b}) = \nabla \times \mathbf{a} + \nabla \times \mathbf{b}$$

$$\nabla(\varphi\psi) = \psi \nabla \varphi + \varphi \nabla \psi$$

$$\nabla \cdot (\varphi \mathbf{a}) = \nabla \varphi \cdot \mathbf{a} + \varphi \nabla \cdot \mathbf{a}$$

$$\nabla \times (\varphi \mathbf{a}) = \nabla \varphi \times \mathbf{a} + \varphi \nabla \times \mathbf{a}$$

$$\nabla \cdot (\mathbf{a} \times \mathbf{b}) = \nabla \times \mathbf{a} \cdot \mathbf{b} - \nabla \times \mathbf{b} \cdot \mathbf{a}$$

此外, 还有

$$\text{rot grad } \varphi = \nabla \times \nabla \varphi = \mathbf{0}$$

$$\text{div rot } \mathbf{a} = \nabla \cdot \nabla \times \mathbf{a} = 0$$

➤ Laplace算子

$$\Delta = \nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

它作用函数 φ 得到

$$\Delta \varphi = \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2}$$

例2 设 $f(x, y, z) = g(r)$, $r = \sqrt{x^2 + y^2 + z^2}$

- 1) 求 Δf ;
- 2) 若 $\Delta f = 0$, 求 f .