Chap 9 — 4

空间曲线与曲面

9.4.1 参数曲线

设参数曲线的方程为

$$\boldsymbol{r}=\boldsymbol{r}(t)=x(t)\boldsymbol{i}+y(t)\boldsymbol{j}+z(t)\boldsymbol{k},\quad t\in[\alpha,\beta]$$

对应参数t_0点$M_0(x_0,y_0,z_0)$的**切向量**(指向t增加方向)

$$\boldsymbol{r}'(t_0)=x'(t_0)\boldsymbol{i}+y'(t_0)\boldsymbol{j}+z'(t_0)\boldsymbol{k}$$

故M_0处**切线方程**

$$\boxed{\frac{x-x_0}{x'(t_0)}=\frac{y-y_0}{y'(t_0)}=\frac{z-z_0}{z'(t_0)}}$$

M_0处**法平面方程**(过切点且与切线垂直的平面)

$$\boxed{x'(t_0)(x-x_0)+y'(t_0)(y-y_0)+z'(t_0)(z-z_0)=0}$$

◆ **光滑曲线**: 切向量连续变化的曲线.

◆ **逐段光滑曲线**如何定义?

◆ **正则点**: $|\boldsymbol{r}'(t_0)| \neq 0$. **正则曲线**如何定义?

例1 证明螺旋线$(a, b > 0)$

$$x = a\cos t,\ y = a\sin t,\ z = bt, \quad t \in \mathbb{R}$$

在每点处的切线和z 轴成定角. 切向量 $\vec{\tau} = (-a\sin t, a\cos t, b)$

z轴向 $\vec{s} = (0, 0, 1)$

■ 弧长

光滑平面参数曲线

$$\boldsymbol{r}=\boldsymbol{r}(t)=x(t)\boldsymbol{i}+y(t)\boldsymbol{j},\quad t\in[\alpha,\beta]$$

的弧长

$$l=\int_{\alpha}^{\beta}|\boldsymbol{r}'(t)|\,\mathrm{d}t=\int_{\alpha}^{\beta}\sqrt{x'^{2}(t)+y'^{2}(t)}\,\mathrm{d}t$$

光滑空间参数曲线

$$\boldsymbol{r}=\boldsymbol{r}(t)=x(t)\boldsymbol{i}+y(t)\boldsymbol{j}+z(t)\boldsymbol{k},\quad t\in[\alpha,\beta]$$

的弧长

$$l=\int_{\alpha}^{\beta}|\boldsymbol{r}'(t)|\,\mathrm{d}t=\int_{\alpha}^{\beta}\sqrt{x'^{2}(t)+y'^{2}(t)+z'^{2}(t)}\,\mathrm{d}t$$

■ 弧长参数

正则曲线起点 A 到动点$M(t)$的弧长

$$s(t)=\int_{\alpha}^{t}|\boldsymbol{r}'(\tau)|\,\mathrm{d}\tau,\quad t\in[\alpha,\beta]$$

该变上限积分的导数

$$\frac{\mathrm{d}s}{\mathrm{d}t}=\sqrt{x'^2(t)+y'^2(t)+z'^2(t)}=|\boldsymbol{r}'(t)|>0$$

故$s(t)$严格增加, 存在反函数$t=t(s)$, 因此有**自然方程**

$$\boldsymbol{r}=\boldsymbol{r}(t)=\boldsymbol{r}(t(s))\overset{\text{def}}{=}\boldsymbol{r}(s)$$

取微分得 $\mathrm{d}\boldsymbol{r}=\boldsymbol{r}'(t)\mathrm{d}t=\boldsymbol{r}'(s)\mathrm{d}s$

由于

$$\frac{\mathrm{d}s}{\mathrm{d}t}=|\boldsymbol{r}'(t)|,\quad \frac{\mathrm{d}t}{\mathrm{d}s}=\frac{1}{|\boldsymbol{r}'(t)|}$$

故有

$$\boxed{\boldsymbol{r}'(s)=\frac{\boldsymbol{r}'(t)}{|\boldsymbol{r}'(t)|}}$$

进而有

$$1=|\boldsymbol{r}'(s)|^2=\left(\frac{\mathrm{d}x}{\mathrm{d}s}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}s}\right)^2+\left(\frac{\mathrm{d}z}{\mathrm{d}s}\right)^2$$

单位切向量$\boldsymbol{r}'(s)$的方向余弦为

$$\cos\alpha=\frac{\mathrm{d}x}{\mathrm{d}s},\quad \cos\beta=\frac{\mathrm{d}y}{\mathrm{d}s},\quad \cos\gamma=\frac{\mathrm{d}z}{\mathrm{d}s}$$

注意 对弧长参数s的微商记为

$$\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}s}=\dot{\boldsymbol{r}},\quad \frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}s^2}=\ddot{\boldsymbol{r}}$$

例2 求螺旋线

$$x=R\cos t,\ y=R\sin t,\ z=kt,\quad 0\le t\le 2\pi$$

的弧长.

$$r = \int_0^{2\pi} r'(t) = \int_0^{2\pi} \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}$$

$$= \int_0^{2\pi} \sqrt{(-R\sin t)^2 + (R\cos t)^2 + (b)^2}$$

$$= \sqrt{R^2 + b^2} \cdot 2\pi$$

■ 曲率

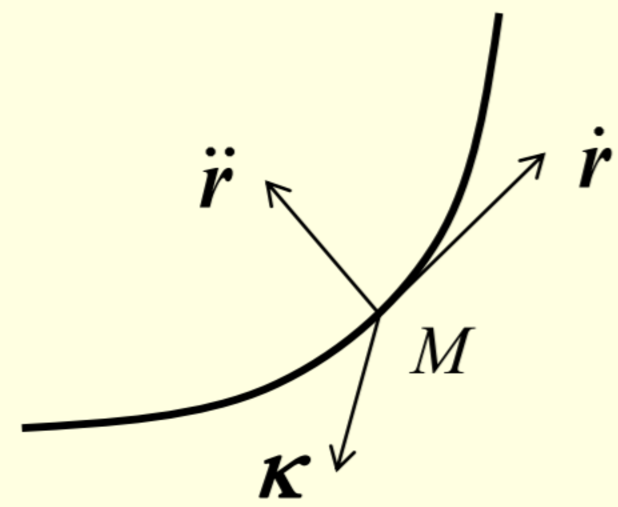
设正则曲线L: $\boldsymbol{r}=\boldsymbol{r}(t)$有二阶连续导数, 则

$$|\dot{\boldsymbol{r}}|=1 \quad \Rightarrow \quad \dot{\boldsymbol{r}}\cdot\ddot{\boldsymbol{r}}=0$$

结论 切向量 $\dot{\boldsymbol{r}}$ 与 $\ddot{\boldsymbol{r}}$ 总是正交. $\ddot{\boldsymbol{r}}$ 称为**主法向量**.

而 $\boldsymbol{\kappa}=\dot{\boldsymbol{r}}\times\ddot{\boldsymbol{r}}$ 称为**副法向量**, 且有

$$|\boldsymbol{\kappa}|=|\ddot{\boldsymbol{r}}|$$



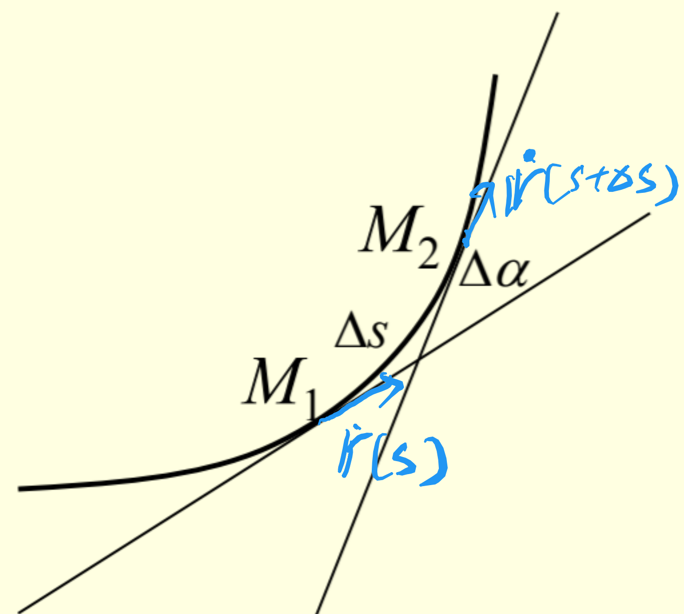
定义 设曲线上弧M_1M_2的长度为Δs, M_1处切线到M_2处切线的总转角为$\Delta\alpha$，弧M_1M_2的**平均曲率**定义为

$$\bar{\kappa} = \left|\frac{\Delta\alpha}{\Delta s}\right|$$

单位弧长切线转过角度

M_1点的**曲率**定义为

$$\kappa = \lim_{\Delta s\to 0}\left|\frac{\Delta\alpha}{\Delta s}\right|$$



结论

$$\kappa = |\ddot{\boldsymbol{r}}| = |\boldsymbol{\kappa}|$$

$\Delta\alpha$ = 弧长 $\approx |\dot{r}(s+\Delta s) - \dot{r}(s)|$

$= |\Delta\dot{r}(s)|$

$\lim\limits_{\Delta s\to 0}\frac{\Delta\varphi}{\Delta s}=\lim\limits_{\Delta s\to 0}\frac{|\Delta\dot{r}(s)|}{\Delta s}=|\ddot{r}(s)|$

命题 曲线用参数t表示时, 有

单位切向量 $\dot{\boldsymbol r}=\dfrac{\boldsymbol r'(t)}{|\boldsymbol r'(t)|}$

主法向量 $\ddot{\boldsymbol r}=\dfrac{1}{|\boldsymbol r'(t)|^2}\boldsymbol r''(t)+\dfrac{1}{|\boldsymbol r'(t)|}\left[\dfrac{\mathrm d}{\mathrm dt}\left(\dfrac{1}{|\boldsymbol r'(t)|}\right)\right]\boldsymbol r'(t)$ 太麻烦

副法向量 $\boldsymbol\kappa=\dot{\boldsymbol r}\times\ddot{\boldsymbol r}=\dfrac{\boldsymbol r'(t)\times\boldsymbol r''(t)}{|\boldsymbol r'(t)|^3}$

曲率 $\kappa=|\boldsymbol\kappa|=\dfrac{|\boldsymbol r'(t)\times\boldsymbol r''(t)|}{|\boldsymbol r'(t)|^3}$

$$\ddot{\mathbf{r}} = \frac{d\dot{\mathbf{r}}}{ds} = \frac{d\dot{\mathbf{r}}}{dt}\frac{dt}{ds} = \frac{d\dot{\mathbf{r}}}{dt}\frac{1}{|\mathbf{r}'(t)|} = \frac{d}{dt}\left(\frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}\right)\frac{1}{|\mathbf{r}'(t)|}$$

$$= \left[\frac{1}{|\mathbf{r}'(t)|}\mathbf{r}''(t) + \frac{d}{dt}\left(\frac{1}{|\mathbf{r}'(t)|}\right)\mathbf{r}'(t)\right]\frac{1}{|\mathbf{r}'(t)|}$$

$$\dot{\mathbf{r}} \times \ddot{\mathbf{r}} = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} \times (\cdots + \cdots)$$

$$= \frac{\mathbf{r}'(t) \times \mathbf{r}''(t)}{|\mathbf{r}'(t)|^3}$$

例3 求螺旋线$(a,b>0)$

$\kappa = \frac{a\sqrt{a^2+b^2}}{(\sqrt{a^2+b^2})^3} = \frac{a}{a^2+b^2}$

$$x = a\cos t,\ y = a\sin t,\ z = bt,\quad t\in\mathbb{R}$$

的曲率.

$r'(t) = (-a\sin t,\ a\cos t,\ b)$　$|r'(t)| = \sqrt{a^2+b^2}$

$r''(t) = (-a\cos t,\ -a\sin t,\ 0)$

$r'(t)\times r''(t) = (ab\sin t,\ -ab\cos t,\ a^2)$

$|\times| = a\sqrt{a^2+b^2}$

■ 平面曲线曲率

设xOy面内参数曲线

$$\boldsymbol{r} = \boldsymbol{r}(t) = x(t)\boldsymbol{i} + y(t)\boldsymbol{j},\quad t\in[\alpha,\beta]$$

副法向量

$$\boldsymbol{\kappa} = \frac{\boldsymbol{r}'(t)\times\boldsymbol{r}''(t)}{|\boldsymbol{r}'(t)|^3} = \frac{x'(t)y''(t)-x''(t)y'(t)}{[x'^2(t)+y'^2(t)]^{3/2}}\boldsymbol{k}$$

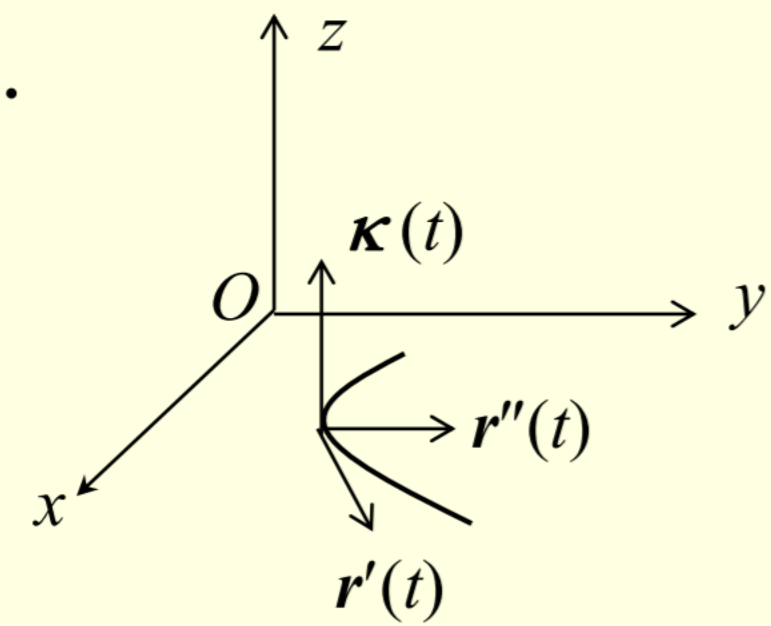
其**曲率**定义为

<0 上凸

$$\kappa = \frac{x'(t)y''(t) - x''(t)y'(t)}{[x'^2(t) + y'^2(t)]^{3/2}}$$

>0 下凸

结论 $\boldsymbol{r}''(t)$始终指向曲线的凹侧.



命题 曲线方程为$y = f(x)$时

$$\kappa = \frac{f''(x)}{[1 + f'^2(x)]^{3/2}} \boldsymbol{k}$$

$$\kappa = \frac{f''(x)}{[1 + f'^2(x)]^{3/2}}$$

例4 求摆线$(a>0)$

$$x=a(t-\sin t),\ y=a(1-\cos t),\quad t\in(0,2\pi)$$

的曲率.

$x'(t)=a-a\cos t$ $\quad x''(t)=a\sin t$

$y'(t)=a\sin t$ $\quad y''(t)=a\cos t$

$$K=\frac{(a-a\cos t)a\cos t-a^2\sin^2 t}{[(a-a\cos t)^2+(a\sin t)^2]^{\frac{3}{2}}}$$

$$=\frac{a^2\cos t-a^2}{(2a^2-2a\cos t)^{\frac{3}{2}}}=\frac{1}{a}\frac{\cos t-1}{(2-2\cos t)^{\frac{3}{2}}}$$

$$=-\frac{1}{2\sqrt{2}a}\frac{1}{\sqrt{1-\cos t}}=\frac{1}{4a\sin\frac{t}{2}}$$

9.4.2 参数曲面

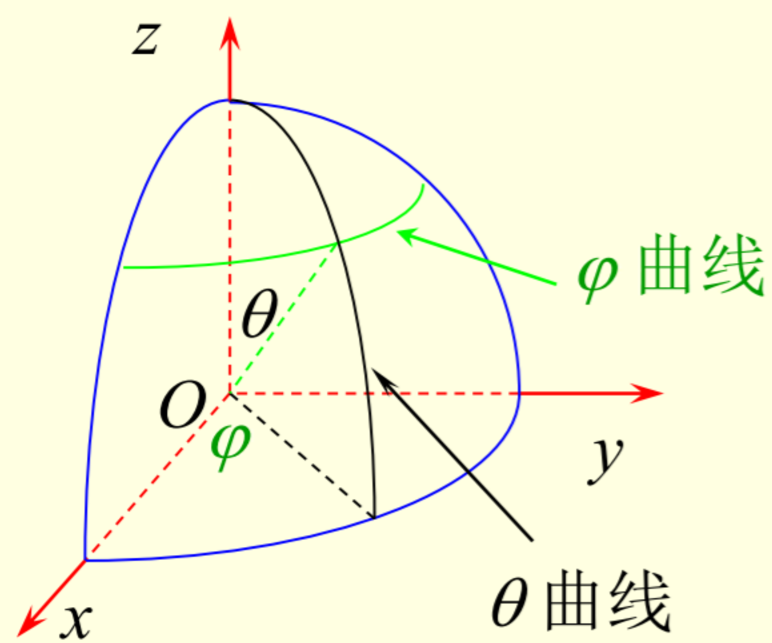
设参数曲面方程

$$\boldsymbol{r}(u,v)=x(u,v)\boldsymbol{i}+y(u,v)\boldsymbol{j}+z(u,v)\boldsymbol{k},\quad (u,v)\in D\subset\mathbb{R}^2$$

- 固定v可得u**曲线**, 类似定义v**曲线**

例5 球面方程

$$\begin{cases} x=R\sin\theta\cos\varphi, \\ y=R\sin\theta\sin\varphi, \\ z=R\cos\theta \end{cases}$$



其中$0\le\theta\le\pi,\ 0\le\varphi<2\pi.$

■ 切平面

设曲面S参数方程为

$$\boldsymbol{r}(u,v)=x(u,v)\boldsymbol{i}+y(u,v)\boldsymbol{j}+z(u,v)\boldsymbol{k},\quad (u,v)\in D$$

S上点$M_0\ (x_0, y_0, z_0)$参数(u_0, v_0), 过M_0的u, v曲线切向量

$$\boldsymbol{r}'_u(u_0,v_0),\quad \boldsymbol{r}'_v(u_0,v_0)$$

设$L\subset S$为过M_0的任一 光滑曲线, 它由D中曲线

$$u = u(t),\ v = v(t)$$

经$\boldsymbol{r}(u,v)$映射得到, 其中$u(t_0) = u_0,\ v(t_0) = v_0$, 即

$$L: \boldsymbol{r}(t)=\boldsymbol{r}(u(t),v(t))$$

L在M_0的切向量

$$\boldsymbol{r}'(t_0)=\boldsymbol{r}_u'(u(t_0),v(t_0))u'(t_0)+\boldsymbol{r}_v'(u(t_0),v(t_0))v'(t_0)$$

$$=\boldsymbol{r}_u'(u_0,v_0)u'(t_0)+\boldsymbol{r}_v'(u_0,v_0)v'(t_0)$$

它始终与 $\boldsymbol{r}_u'(u_0,v_0),\boldsymbol{r}_v'(u_0,v_0)$ 共面. 故S上过M_0的光滑曲线之切线共面, 该平面称为S在M_0的**切平面**. 其法向量

$$\boxed{\boldsymbol{n}_0=\boldsymbol{r}_u'(u_0,v_0)\times\boldsymbol{r}_v'(u_0,v_0)}$$

想一想 切平面的方程?

■ 法向量场

曲面S上参数为(u, v)的点处法向量

$$\boldsymbol{n}=\boldsymbol{r}_u'(u,v)\times\boldsymbol{r}_v'(u,v)=\frac{\partial(y,z)}{\partial(u,v)}\boldsymbol{i}+\frac{\partial(z,x)}{\partial(u,v)}\boldsymbol{j}+\frac{\partial(x,y)}{\partial(u,v)}\boldsymbol{k}$$

光滑曲面 法向量连续变化的曲面

记 $$E=|\boldsymbol{r}_u'|^2,\quad G=|\boldsymbol{r}_v'|^2,\quad F=\boldsymbol{r}_u'\cdot\boldsymbol{r}_v'$$

则 $$|\boldsymbol{r}_u'(u,v)\times\boldsymbol{r}_v'(u,v)|=\sqrt{|\boldsymbol{r}_u'|^2|\boldsymbol{r}_v'|^2-(\boldsymbol{r}_u'\cdot\boldsymbol{r}_v')^2}=\sqrt{EG-F^2}$$

而单位法向量 $$\boldsymbol{n}^0=\frac{\boldsymbol{r}_u'\times\boldsymbol{r}_v'}{|\boldsymbol{r}_u'\times\boldsymbol{r}_v'|}$$

9.4.3 隐式曲线和曲面

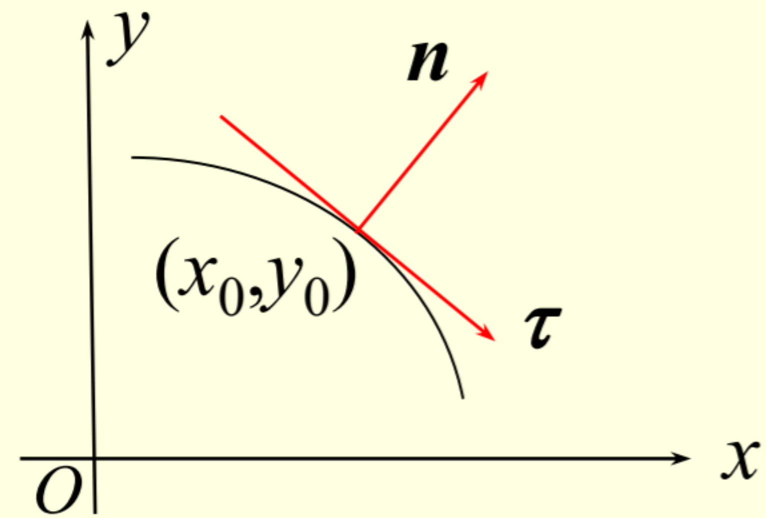
■ **平面隐式曲线** 设函数$F(x, y)$有连续偏导数, 则曲线$F(x, y) = 0$在(x_0,y_0)处的

切线方程 $$\boxed{F'_x(x_0, y_0)(x - x_0) + F'_y(x_0, y_0)(y - y_0) = 0}$$

切线~点法式

法向量 $\boldsymbol{n} = (F'_x, F'_y)\big|_{(x_0, y_0)}$

切向量 $\boldsymbol{\tau} = (F'_y, -F'_x)\big|_{(x_0, y_0)}$



例6 求椭圆 $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ 在点(x_0,y_0)处的切线方程.

$$F(x,y)=0 \xrightarrow{\text{确定}} y=y(x) \text{ 则 } \left.\frac{dy}{dx}\right|_{x=x_0} = -\frac{F(x_0,y_0)}{F_y'(x_0,y_0)}$$

切线 $y-y_0 = -\dfrac{F_x'(x_0,y_0)}{F_y'(x_0,y_0)}(x-x_0) \longrightarrow \sim$

eg6. $F = \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}-1$

$$\frac{\partial F}{\partial x} = \frac{2x}{a^2} \qquad \frac{\partial F}{\partial y} = \frac{2y}{b^2}$$

$$\frac{2x_0}{a^2}(x-x_0)+\frac{2y_0}{b^2}(y-y_0)=0$$

$$\frac{x_0}{a^2}(x-x_0)+\frac{y_0}{b^2}(y-y_0)=0$$

$$\frac{x_0x}{a^2}+\frac{y_0y}{b^2}=1$$

eg6' $F = Ax^2+Bxy+Cy^2+Dx+Ey+F$

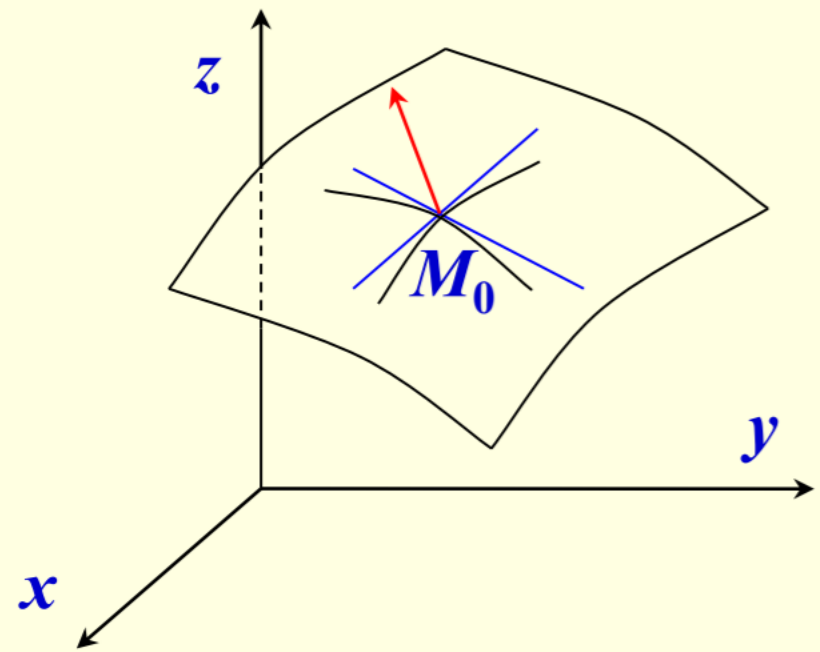
$$\frac{\partial F}{\partial x} = 2Ax+By+D \qquad \frac{\partial F}{\partial y} = Bx+2Cy+E$$

$$(2Ax_0+By_0+D)(x-x_0)+(Bx_0+2Cy_0+E)(y-y_0)=0$$

■ 空间隐式曲面

设曲面S的方程 $F(x, y, z) = 0$, $M_0(x_0, y_0, z_0) \in S$,

S上过M_0(对应参数t_0)的曲线为



$$\boldsymbol{r}(t) = x(t)\boldsymbol{i} + y(t)\boldsymbol{j} + z(t)\boldsymbol{k}$$

曲线在S上, 故 $F(x(t), y(t), z(t)) \equiv 0$

$$\Rightarrow (F'_x, F'_y, F'_z) \cdot (x'(t), y'(t), z'(t)) = 0 \xrightarrow{t=t_0} \nabla F(M_0) \cdot \boldsymbol{\tau}_0 = 0$$

故切向量 $\boldsymbol{\tau}_0 = (x'(t_0), y'(t_0), z'(t_0))$ 总与 $\nabla F(M_0)$ 正交.

曲面的法向量 $\boxed{\nabla F(M_0) = (F'_x, F'_y, F'_z)\big|_{M_0}}$

切平面方程

$$F'_x(M_0)(x-x_0)+F'_y(M_0)(y-y_0)+F'_z(M_0)(z-z_0)=0$$

法线方程 $$\frac{x-x_0}{F'_x(M_0)}=\frac{y-y_0}{F'_y(M_0)}=\frac{z-z_0}{F'_z(M_0)}$$

特别地, 若S: $z=f(x,y)$, 则法向量为

$$\boxed{\boldsymbol{n}=(-f'_x(x_0,y_0),-f'_y(x_0,y_0),1)}$$

切平面

$$f'_x(x_0,y_0)(x-x_0)+f'_y(x_0,y_0)(y-y_0)-(z-z_0)=0$$

■ 空间隐式曲线

设空间曲线L的方程为

$$\begin{cases} F(x,y,z)=0 \\ G(x,y,z)=0 \end{cases}$$

点$M_0 \in L$, 其切线为两曲面在M_0点切平面之交线, 故

切向量 $\tau = \nabla F(M_0) \times \nabla G(M_0)$

$$= \left.\frac{\partial(F,G)}{\partial(y,z)}\right|_{M_0} \boldsymbol{i} + \left.\frac{\partial(F,G)}{\partial(z,x)}\right|_{M_0} \boldsymbol{j} + \left.\frac{\partial(F,G)}{\partial(x,y)}\right|_{M_0} \boldsymbol{k}$$

想一想 切线的方程?

例7 求曲面 $z = x^2 + \dfrac{y^2}{4} - 1$ 上点$(-1,-2,1)$处的切平面和法线方程.

例8 求曲线 $\begin{cases} 2x^2 + 3y^2 + z^2 = 9 \\ z^2 = 3x^2 + y^2 \end{cases}$ 在点$(1,-1,2)$处的切线.

eg7. $\vec{n}=(2x,\frac{y}{2},-1)\big|_{(-1,-2,1)}=(-2,-1,-1)\parallel(2,1,1)$

切平面 $2(x+1)+(y+2)+(z-1)=0$

eg8. 令 $F=2x^2+\cdots-4$ $\quad\vec{n_1}=(4x,6y,2z)\big|_{(1,-1,2)}\parallel(2,-3,2)$

$G=\cdots$ $\quad\vec{n_2}=(6x,2y,-2z)\big|_{(1,-1,2)}\parallel(3,-1,-2)$

切线方向向量

$\vec{n_1}\times\vec{n_2}=(8,10,7)$

方程: $\frac{x-1}{8}=\frac{y+1}{10}=\frac{z-2}{7}$