Chap 11 — 5

Gauss定理和Stokes定理

11.5.1 Gauss定理

定理(Gauss公式) 设 $\boldsymbol{v} = (P, Q, R)$为空间有界闭域 V上的光滑向量场, ∂V 是分片光滑闭曲面, 则有

$$\boxed{\oiint_{\partial V^{+}} P\mathrm{d}y\mathrm{d}z + Q\mathrm{d}z\mathrm{d}x + R\mathrm{d}x\mathrm{d}y = \iiint_{V}\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right)\mathrm{d}V}$$

注 三重积分与其边界上第二型曲面积分的关系

分析 先证 $\oiint_{\partial V^{+}} R\mathrm{d}x\mathrm{d}y = \iiint_{V} \frac{\partial R}{\partial z}\mathrm{d}V$

再证关于P, Q的等式, 三式相加即证.

底: z_1 顶 z_2

LHS:

证: $$\oiint R\,dxdy = \oiint_{S_{顶}\cup S_{侧}\cup S_{底}} R\,dxdy = +\iint_D R(x,y,z_2(x,y))\,dxdy - \iint_D R(x,y,z_1(x,y))\,dxdy$$

$$= \iint_D \left[R(x,y,z_2(x,y)) - R(x,y,z_1(x,y))\right] dxdy$$

RHS: $$\iiint_V \frac{\partial R}{\partial z}\,dV = \iint_D dxdy \int_{z_1(x,y)}^{z_2(x,y)} \frac{\partial R}{\partial z}\,dz$$

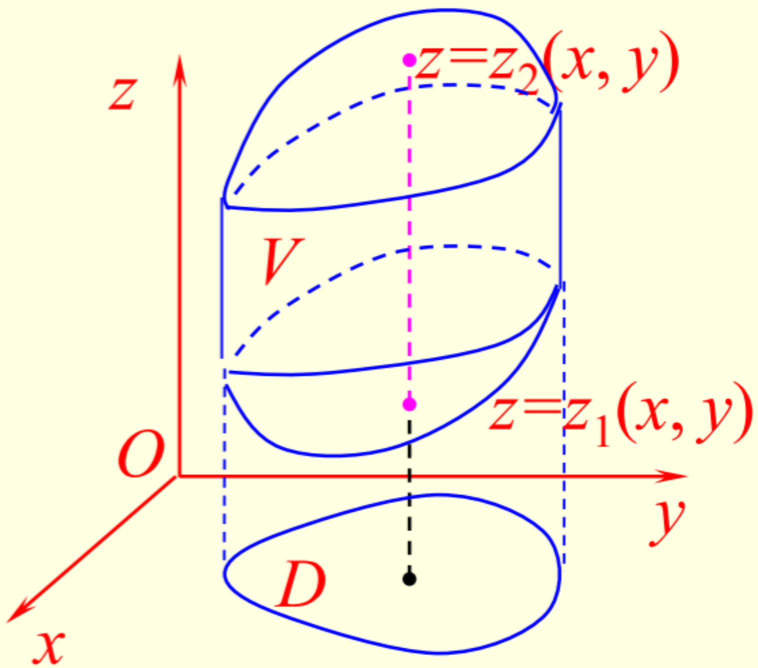
xy型区域

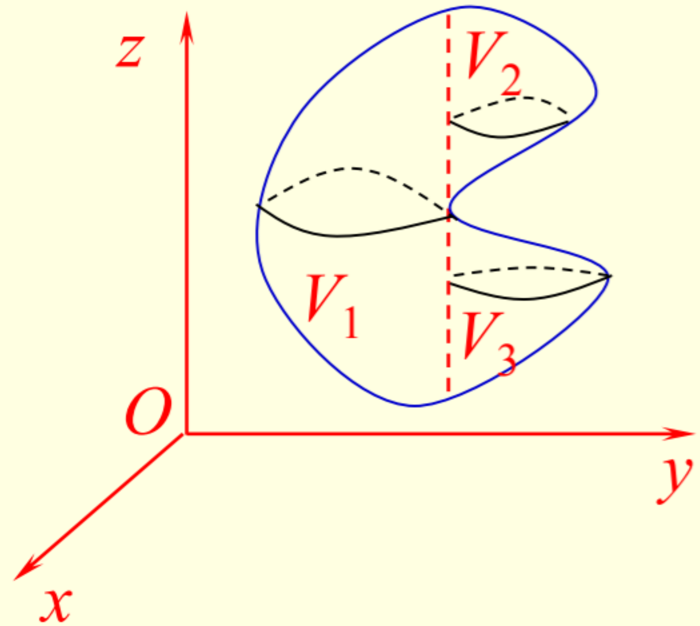
$$\iint_D R(x,y,z)\Big|_{z=z_1(x,y)}^{z=z_2(x,y)} dxdy$$ 故得证,

先证 $\displaystyle\oiint_{\partial V^+} R\,\mathrm{d}x\mathrm{d}y = \iiint_V \frac{\partial R}{\partial z}\mathrm{d}V$

证 1) 当V是xy型区域, 即

$$V=\{(x,y,z)\,|\,z_1(x,y)\le z\le z_2(x,y),\ (x,y)\in D\}$$





2) 当V是一般区域, 用母线平行z轴柱面分成若干

可分割证明

xy型区域并运用1)的结论.

推论 设空间有界闭域V的边界分片光滑，则其体积

$$\boxed{V=\frac{1}{3}\oiint_{\partial V^+} x\mathrm{d}y\mathrm{d}z+y\mathrm{d}z\mathrm{d}x+z\mathrm{d}x\mathrm{d}y}$$

例1 计算积分

$$I=\oiint_S x\mathrm{d}y\mathrm{d}z+y\mathrm{d}z\mathrm{d}x+z\mathrm{d}x\mathrm{d}y$$

其中S为椭球面 $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ 的外侧. $\boxed{4\pi abc}$

$I=3\iiint_V(x^2+y^2+z^2)dV$

例2 计算积分

$$I = \iint_S x\mathrm{d}y\mathrm{d}z + 2xy\mathrm{d}z\mathrm{d}x - 2z\mathrm{d}x\mathrm{d}y$$

其中S为锥面 $z = \sqrt{x^2 + y^2}, (0 \le z \le h)$ 的下侧.

$$\boxed{\frac{5\pi h^3}{3}}$$

例3 计算积分

$$I = \iint_S \frac{xz^2\mathrm{d}y\mathrm{d}z + (x^2 y - z^3)\mathrm{d}z\mathrm{d}x + (2xy + y^2 z)\mathrm{d}x\mathrm{d}y}{\sqrt{x^2 + y^2 + z^2}}$$

其中S为球面 $z = \sqrt{R^2 - x^2 - y^2}, (R > 0)$ 的上侧.

$$\boxed{\frac{2\pi R^4}{5}}$$

2 解 添加 S_0 $z=h$，上侧

$$I=\oiint_{S\cup S_0}\sim-\iint_{S_0}\sim$$

$$=\iiint_{\Omega}(1+2x-2)\,dxdydz-\iint_{D}-2z\,dxdy$$

$$=2\iiint_{\Omega}x\,dxdydz-\iiint_{P}dxdydz+2h\cdot\pi h^2$$

$$=0-\frac{1}{3}\pi h^3+2\pi h^3$$

$$=\frac{5}{3}\pi h^3$$

eg4. $I=\frac{1}{R}\iint_{S}xz^2dydz+(x^2y-z^3)dzdx+(2xy+y^2z)dxdy$

$$=\frac{1}{R}\iint_{S+S_0}\sim+\frac{1}{R}\iint_{S_0}(2xy+y^2z)\,dxdy$$

下侧

$$=\frac{1}{R}\iiint_{\Omega}(z^2+x^2+y^2)\,dxdydz+\frac{1}{R}\int_0^{2\pi}d\theta\int_0^R 2r^2\cos\theta\sin\theta\, r\,dr$$

$$=\frac{1}{R}\int_0^{\frac{\pi}{2}}d\varphi\int_0^{2\pi}d\theta\int_0^R\rho^2\cdot\rho^2\sin\varphi\,d\rho+\frac{2}{R}\int_0^{2\pi}\cos\theta\sin\theta\,d\theta\int_0^R r^3\,dr$$

$$=\frac{1}{R}\,2\pi\,\frac{1}{5}\rho^5\Big|_0^R+0$$

$$=\frac{2}{5}R^4+0=\frac{2}{5}R^4$$

例4 计算积分

$\frac{\boldsymbol{r}\cdot\boldsymbol{n}}{|\boldsymbol{r}|}$ 不妨 $\boldsymbol{n}$ 单位

$$I=\iint_S \frac{\cos(\boldsymbol{r},\boldsymbol{n})}{|\boldsymbol{r}|^2}\,\mathrm{d}S$$

第一型

其中S是包围原点的封闭光滑闭曲面，$\boldsymbol{n}$是S上点(x, y, z)处的外法向量，$\boldsymbol{r}=(x, y, z)$.

有奇点(0,0)，挖 原点为球心，球体

$I + I_{挖} = \iiint\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\right)$ $\boxed{4\pi}$

$=$

$\oiint_{S^+}\frac{\boldsymbol{r}}{|\boldsymbol{r}|^3}\cdot d\boldsymbol{S}$

$=\oiint_{S^+}\frac{1}{|\boldsymbol{r}|^3}(x\,dydz+y\,dzdx+z\,dxdy)$

$=\oiint_{S^+}\frac{x\,dydz+y\,dzdx+z\,dxdy}{(x^2+y^2+z^2)^{3/2}}$

$x\,(x^2+y^2+z^2)^{-\frac{3}{2}}$

$(x^2+y^2+z^2)^{-\frac{3}{2}}+x\left(-\frac{5}{2}\right)$

■ Gauss公式的向量形式

由于 $$\oiint_{\partial V^+} P\,\mathrm{d}y\mathrm{d}z + Q\,\mathrm{d}z\mathrm{d}x + R\,\mathrm{d}x\mathrm{d}y = \oiint_{\partial V^+} \boldsymbol{v}\cdot\mathrm{d}\boldsymbol{S}$$

$$\iiint_V \left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\right)\mathrm{d}V = \iiint_V \nabla\cdot\boldsymbol{v}\,\mathrm{d}V$$

where $v=(P,Q,R)$

故有

$$\boxed{\oiint_{\partial V^+} \boldsymbol{v}\cdot\mathrm{d}\boldsymbol{S} = \iiint_V \nabla\cdot\boldsymbol{v}\,\mathrm{d}V = \iiint_V \mathrm{div}\,\boldsymbol{v}\,\mathrm{d}V}$$

体积

单位体积上散发出去的流量

散度物理意义 $$\mathrm{div}\,\boldsymbol{v}(M) = \lim_{V\to M}\frac{1}{\mathrm{Vol}(V)}\oiint_{\partial V^+}\boldsymbol{v}\cdot\mathrm{d}\boldsymbol{S}$$

取极限

通量 $N = \iiint \nabla\cdot\vec{v}\,dv = \mathrm{div}\,\vec{v}(M^*)\cdot V \quad (M^*\in V) \Rightarrow \mathrm{div}\,\vec{v} = \frac{N}{V}$

—— 积分中值定理 ——

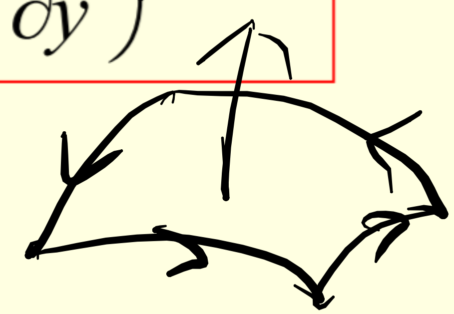
$\mathrm{div}\,\vec{v}(M)$ $\begin{cases} >0 & \text{源（点）} \\ =0 & \text{无源场} \\ <0 & \text{汇（点）} \end{cases}$

11.5.2 Stokes定理

定理(Stokes公式) 设 $\boldsymbol{v}=(P,Q,R)$为空间光滑曲面 S上的光滑向量场, ∂S 是分段光滑闭曲线, 则有

$$\boxed{\begin{aligned}&\oint_{\partial S} P\mathrm{d}x+Q\mathrm{d}y+R\mathrm{d}z\\ &=\iint_S\left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right)\mathrm{d}y\mathrm{d}z+\left(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}\right)\mathrm{d}z\mathrm{d}x+\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)\mathrm{d}x\mathrm{d}y\end{aligned}}$$

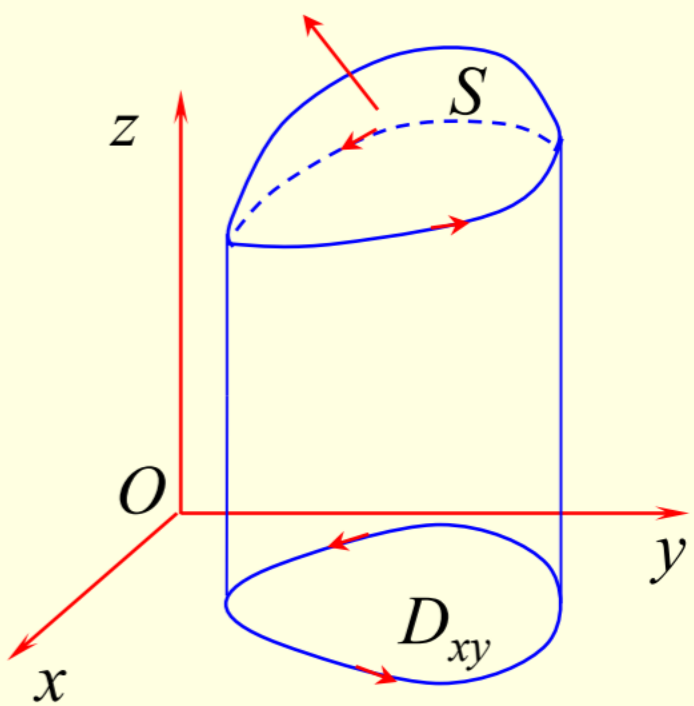
其中∂S的方向与S的侧向按右手法则联系.

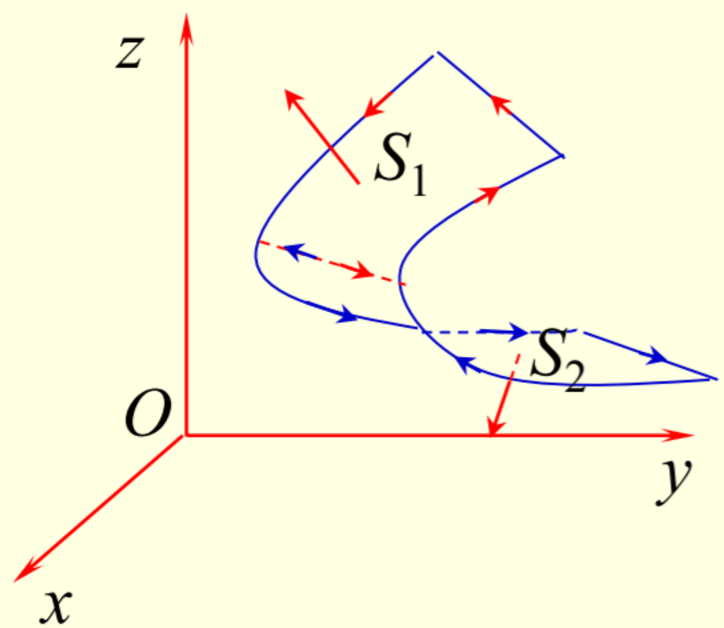


注 第二型**曲面积分**与其边界上第二型**曲线积分**关系

分析 先证 $\oint_{\partial S} P\mathrm{d}x = \iint_S \frac{\partial P}{\partial z}\mathrm{d}z\mathrm{d}x - \frac{\partial P}{\partial y}\mathrm{d}x\mathrm{d}y$

证 1) 当S是z型曲面, 即 $S:\ z = z(x,y),\ (x,y)\in D_{xy}$ 有显式方程





2) 一般曲面S可分成若干z型曲面并运用1)的结论

再证 关于Q, R的等式, 三式相加即证.

证：$\oint_{\partial S} P\,dx = \int_{\alpha}^{\beta} P(x(t), y(t), z(x(t),y(t)))\, x'(t)\,dt$

$\partial S: \mathbf{r}=\mathbf{r}(t)= x(t)\mathbf{i}+y(t)\mathbf{j}+z(x(t),y(t))\mathbf{k}$ $t: \alpha \to \beta$

↓ 投影

∂D_{xy} $\quad x(t)\mathbf{i}+y(t)\mathbf{j}$

$= \oint_{\partial D(x,y)} P(x,y,z(x,y))\,dx = -\iint_{D_{xy}} \left(\frac{\partial P}{\partial y}+\frac{\partial P}{\partial z}\cdot z'_y\right) dx\,dy$

Green公式

$\iint_S \frac{\partial P}{\partial z}dz\,dx - \frac{\partial P}{\partial y}dx\,dy = +\iint_{D_{xy}} \left(-\frac{\partial P}{\partial z}\cdot z'_y + -\frac{\partial P}{\partial y}\right) dx\,dy$

合一投影

■ 借助行列式, Stokes公式可记为

$$\oint_{\partial S} P\mathrm{d}x + Q\mathrm{d}y + R\mathrm{d}z = \iint_S \begin{vmatrix} \mathrm{d}y\mathrm{d}z & \mathrm{d}z\mathrm{d}x & \mathrm{d}x\mathrm{d}y \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

$$= \iint_S \begin{vmatrix} \cos\alpha & \cos\beta & \cos\gamma \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} \mathrm{d}S$$

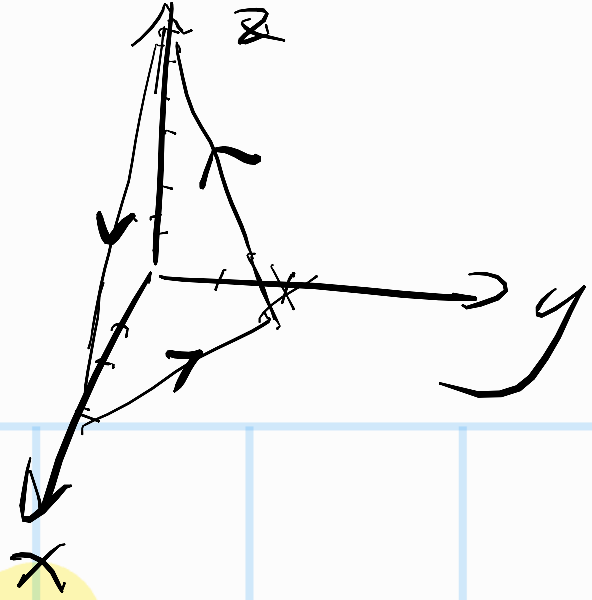
其中∂S定向与$(\cos\alpha, \cos\beta, \cos\gamma)$按右手法则联系

例5 计算积分

$$I=\oint_C z\mathrm{d}x+x\mathrm{d}y+y\mathrm{d}z$$

其中C是平面$2x+3y+z=6$被三个坐标平面所截的三角形S的边界，其方向与S上侧满足右手法则. $\boxed{18}$

eg5. $\frac{x}{3}+\frac{y}{2}+\frac{z}{6}=1$



$\vec{n}_0=\left(\frac{2}{\sqrt{14}},\frac{3}{\sqrt{14}},\frac{1}{\sqrt{14}}\right)$

第Ⅰ型

$$I=\iint_S \frac{1}{\sqrt{14}}\begin{vmatrix} 2 & 3 & 1 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & y \end{vmatrix} dS$$

注意第Ⅰ型

$$=\iint_S \frac{1}{\sqrt{14}}\left(2\left(\frac{\partial y}{\partial y}-\overset{0}{\frac{\partial x}{\partial z}}\right)+3\left(\frac{\partial z}{\partial z}-\overset{0}{\frac{\partial y}{\partial x}}\right)+\left(\frac{\partial x}{\partial x}-\overset{0}{\frac{\partial z}{\partial y}}\right)\right)dS$$

$$=\frac{6}{\sqrt{14}}\iint_S dS=\frac{6}{\sqrt{14}}S=\frac{6}{\sqrt{14}}\cdot 3\sqrt{14}=18$$

$$S=\frac{1}{2}\left|(-3,0,6)\times(-3,2,0)\right|$$

$$=3\sqrt{14}$$

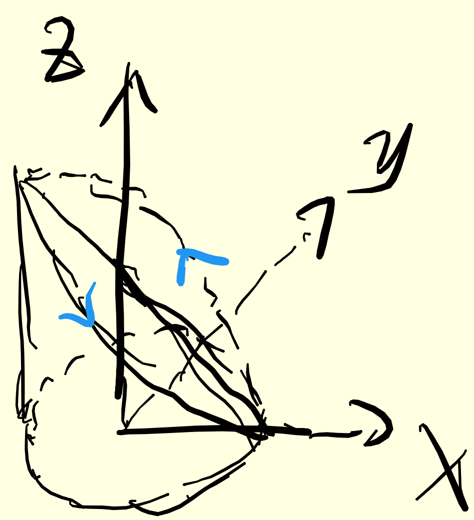
第Ⅱ型的也可

例6 计算积分

$$I=\oint_C (y-z)\mathrm{d}x+(z-x)\mathrm{d}y+(x-y)\mathrm{d}z$$

其中曲线C为 $\begin{cases} x^2+y^2=R^2 \\ x+z=R \end{cases}$，积分方向从$Ox$轴正向看沿逆时针.

$\boxed{-4\pi R^2}$



设以L为边界椭圆为S $\qquad \vec{n}_0 = \left(\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}\right)$

$$I = \iint_S \begin{vmatrix} \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y-z & z-x & x-y \end{vmatrix} dS = \iint_S \left(\frac{\sqrt{2}}{2}(-2) + 0 + \frac{\sqrt{2}}{2}(-2)\right) dS$$

$$= -2\sqrt{2}\iint_S dS = -2\sqrt{2}\cdot\pi\cdot\sqrt{2}R\cdot R$$

$$= -4\pi R^2$$

$$I = \iint_S \begin{vmatrix} dydz & dzdx & dxdy \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y-z & z-x & x-y \end{vmatrix} = \iint_S -2dydz - 2dzdx - 2dxdy$$

$$= -2\iint_D (-z'_x - z'_y + 1)\,dxdy$$

$$= -2\iint_D (1 + 0 + 1)\,dxdy = -4\iint_D dxdy = -4\cdot\pi R^2$$

■ Stokes公式的向量形式

记$\boldsymbol{r}=(x, y, z)$, 则$\mathrm{d}\boldsymbol{r}=(\mathrm{d}x, \mathrm{d}y, \mathrm{d}z)$, Stokes公式可写成

$$\boxed{\begin{aligned}\oint_{\partial S}\boldsymbol{v}\cdot\mathrm{d}\boldsymbol{r}&=\iint_S \operatorname{rot}\boldsymbol{v}\cdot\mathrm{d}\boldsymbol{S}\\&=\iint_S \operatorname{rot}\boldsymbol{v}\cdot\boldsymbol{n}^\circ\,\mathrm{d}S\end{aligned}}$$

$d\boldsymbol{S}=(dydz, dzdx, dxdy)$

其中∂S的定向与S的定侧($\boldsymbol{n}^\circ$)满足右手法则

旋度物理意义 $\operatorname{rot}\boldsymbol{v}\cdot\boldsymbol{n}^\circ\Big|_M=\lim\limits_{S\to M}\dfrac{1}{\operatorname{Area}(S)}\oint_{\partial S}\boldsymbol{v}\cdot\mathrm{d}\boldsymbol{r}$

$\Phi=\left[\operatorname{rot}\vec{v}\cdot\vec{n}^\circ\right]_{M^*}\cdot A_S$

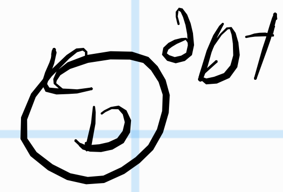
n^0和 rot v 共线时转速最快 |rot v| 方向是螺旋桨轴的方向

调整螺旋桨 使其最快

review:

N-L $f(x)\Big|_a^b = \int_a^b f'(x)\,dx$

Green. $\oint_{\partial D^+} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) d\sigma$



Gross $\oiint_{\partial V^+} P\,dy\,dz + Q\,dz\,dx + R\,dx\,dy = \iiint_V \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dV$

Stokes $\oint_{\partial S} P\,dx + Q\,dy + R\,dz = \iint_S \mathrm{rot}\,\vec{v} \cdot d\vec{S}$

微分形式 ω. 外微分 d 外乘 ∧ 流形上的微积分

$\int_{\partial S} \omega = \int d\omega$

$\omega_0 = f(x)$: $f(x)\Big|_a^b = \int df(x)\Big|_a^b$

$\omega_1 = P\,dx + Q\,dy + R\,dz$

$d\omega_1 = \frac{\partial P}{\partial x} dx \wedge dx + \frac{\partial P}{\partial y} dx \wedge dy + \frac{\partial P}{\partial z} dx \wedge dz$

$$\frac{\partial Q}{\partial x}dy\wedge dx+\frac{\partial Q}{\partial y}dy\wedge dy+\frac{\partial Q}{\partial z}dy\wedge dz$$

$$\frac{\partial R}{\partial x}dz\wedge dx+\frac{\partial R}{\partial y}dz\wedge dy+\frac{\partial R}{\partial z}dz\wedge dz$$

$$=-\left(\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right)dy\wedge dz+\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)dz\wedge dx+\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)dx\wedge dy\right)$$

反了!

$$w_2=P\,dy\wedge dz+Q\,dz\wedge dx+R\,dx\wedge dy$$

$$dw_2=\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\right)dx\wedge dy\wedge dz$$

$$dw=(dx+dy+dz)\wedge w$$

其中 $du\wedge P\,dv=\frac{\partial P}{\partial u}du\wedge dv$

二维

$$w=P\,dx+Q\,dy$$

$$dw=\frac{\partial P}{\partial x}dx\wedge dx+\frac{\partial P}{\partial y}dy\wedge dx+\frac{\partial Q}{\partial x}dx\wedge dy+\frac{\partial Q}{\partial y}dy\wedge dy$$

$$=\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dx\wedge dy \quad (\text{Green})$$

习题课

Green: $\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\wedge dy = \oint_{L=\partial D} Pdx + Qdy$

$D \qquad \partial D$

$dx\wedge dy \Longleftrightarrow$ L逆时针

$w^1 = Pdx + Qdy \qquad \iint_D dw_1 = \oint_{\partial D} w_1$

外微分 $dw_1 = d(Pdx) + d(Qdy)$

$= dP\wedge dx + dQ\wedge dy$

$= \left(\frac{\partial P}{\partial x}dx + \frac{\partial P}{\partial y}dy\right)\wedge dx + \left(\frac{\partial Q}{\partial x}dx + \frac{\partial Q}{\partial y}dy\right)\wedge dy$

$= \frac{\partial P}{\partial y}dy\wedge dx + \frac{\partial Q}{\partial x}dx\wedge dy$

$= \cdots$

Gass. $w^2 = Pdy\wedge dz + Qdz\wedge dx + Pdx\wedge dy$

$\iiint_V dw^2 = \oiint_{V=\partial D} w^2$

Stokes $\iint_D dw^1 = \oint_{L=\partial D} w^1$

Stokes Th.

$$\int_V d\omega = \int_{\partial V} \omega$$