Chap13 — 3

含参变量的定积分

13.3.1 含参变量的定积分及其性质

定义 设$f(x, u)$在$[a, b]\times[\alpha, \beta]$上定义, 且$\forall u\in[\alpha, \beta]$, $f(x, u)$关于x 在$[a, b]$上可积, 则称

$$\varphi(u)=\int_a^b f(x,u)\mathrm{d}x$$

为含参变量的定积分, u称为参变量.

证：$\forall$ 固定 $u_0 \in [\alpha, \beta]$ 由 Cantor 定理 $\forall \varepsilon > 0, \exists \delta > 0$

$\downarrow$ $\forall (x', u')\ (x'', u'') \in I$ 且 $\sqrt{(x''-x')^2 + (u''-u')^2} < \delta$ 有

$$|f(x', u') - f(x'', u'')| < \frac{\varepsilon}{b-a}$$

从而 $\forall x \in [a, b]$, $|u - u_0| < \delta$ 有 $|f(x, u) - f(x, u_0)| < \frac{\varepsilon}{b-a}$

$$|\varphi(u) - \varphi(u_0)| = \left|\int_a^b f(x, u)dx - \int_a^b f(x, u_0)dx\right|$$

$$= \left|\int_a^b [f(x, u) - f(x, u_0)]\right| \leq \int_a^b |f(x, u) - f(x, u_0)|dx$$

$$< \frac{\varepsilon}{b-a}(b-a) = \varepsilon \ \#$$

定理(连续性) 设$f(x, u)$在$I = [a, b]\times[\alpha, \beta]$上连续, 则

$$\varphi(u) = \int_a^b f(x,u)\mathrm{d}x$$

在$[\alpha, \beta]$上连续.

➢ **积分号下取极限**

$$\boxed{\lim_{u\to u_0}\int_a^b f(x,u)\mathrm{d}x = \int_a^b f(x,u_0)\mathrm{d}x = \int_a^b \lim_{u\to u_0} f(x,u)\mathrm{d}x}$$

例1 计算极限 $\lim_{u\to 0^+}\int_0^1 \frac{\mathrm{d}x}{1+x^2\cos xu}$

$f(x,u) = \frac{dx}{1+x^2\cos xu} \in C[0,1]\times[0,1]$

$= \int_0^1 \frac{dx}{1+0} = 1$

定理(交换积分次序) 设$f(x, u) \in C[a, b]\times[\alpha, \beta]$, 则

$$\int_{\alpha}^{\beta}\left(\int_{a}^{b} f(x,u)\mathrm{d}x\right)\mathrm{d}u = \int_{a}^{b}\left(\int_{\alpha}^{\beta} f(x,u)\mathrm{d}u\right)\mathrm{d}x$$

或
$$\int_{\alpha}^{\beta}\mathrm{d}u\int_{a}^{b} f(x,u)\mathrm{d}x = \int_{a}^{b}\mathrm{d}x\int_{\alpha}^{\beta} f(x,u)\mathrm{d}u$$

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例2 计算积分

$$I = \int_0^1 \frac{x^b - x^a}{\ln x}\mathrm{d}x$$

$$\frac{x^b - x^a}{\ln x} = \frac{x^y}{\ln x}\Big|_{y=a}^{y=b} = \int_a^b x^y dy$$

其中$0 < a < b$. $\lim_{x\to 1^-} \cdots = \lim_{x\to 1^-}(bx^b - ax^a) = b - a$

$$I = \int_0^1 dx \int_a^b x^y dy = \int_a^b dy \int_0^1 x^y dx = \int_a^b \frac{dy}{y+1} = \ln(y+1)\Big|_a^b = \ln\frac{b+1}{a+1}$$

解 $\lim_{x\to 0^+} \frac{x^b - x^a}{\ln x} = 0$ 无瑕点.

➢ 常用积分变换式 $(0 < a < b)$

$$\frac{\sin bx - \sin ax}{x} = \int_a^b \cos xy \,\mathrm{d}y$$

$$\frac{\arctan bx - \arctan ax}{x} = \int_a^b \frac{1}{1+(xy)^2}\mathrm{d}y$$

$$\frac{\mathrm{e}^{bx} - \mathrm{e}^{ax}}{x} = \int_a^b \mathrm{e}^{xy}\mathrm{d}y$$

例3 计算 $\int_0^1 \left(\int_\pi^{2\pi} \frac{y\sin xy}{y-\sin y}\mathrm{d}y \right)\mathrm{d}x \overset{C}{=} \int_\pi^{2\pi} \mathrm{d}y \int_0^1 \frac{y\sin xy}{y-\sin y}\mathrm{d}x = \int_\pi^{2\pi} \frac{-\cos xy}{y-\sin y}\Big|_{x=0}^{x=1} \mathrm{d}y$

$= \int_\pi^{2\pi} \frac{1-\cos y}{y-\sin y}\mathrm{d}y = \ln(y-\sin y)\Big|_\pi^{2\pi} = \ln 2\pi - \ln\pi = \ln 2$

$\uparrow \frac{f'}{f}$

定理(可导性) 设$f(x,u)$, $f'_u(x,u)\in C[a,b]\times[\alpha,\beta]$, 则

$\varphi(u)\in C^{(1)}[\alpha,\beta]$, 且

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}u}\int_a^b f(x,u)\mathrm{d}x=\int_a^b\frac{\partial f}{\partial u}(x,u)\mathrm{d}x}$$

➢ **积分号下求导数**

例4 计算导数 $\dfrac{\mathrm{d}}{\mathrm{d}u}\displaystyle\int_0^1 \mathrm{e}^{-ux^2}\mathrm{d}x$

$$=\int_0^1\frac{\partial e^{-ux^2}}{\partial u}dx=-\int_0^1 x^2e^{-ux^2}dx$$

记 $g(u) = \int_a^b f'_u(x,u)\,dx$，下证 $\varphi'(u) = g(u)$

$u \in [\alpha,\beta]$

$$\int_\alpha^u g(t)\,dt = \int_\alpha^u dt \int_a^b \boxed{\overbrace{f_t(x,t)}^{\in C[a,b]\times[\alpha,u]}}\,dx$$

$$= \int_a^b dx \int_\alpha^u f_t(x,t)\,dt = \int_a^b f(x,t)\Big|_{t=\alpha}^{t=u} dx$$

$$= \int_a^b \big(f(x,u) - f(x,\alpha)\big)\,dx = \varphi(u) - \varphi(\alpha)$$

$$\varphi(u) - \varphi(\alpha) = \int_\alpha^u g(t)\,dt$$

$g(t)$ 连续 $\int_\alpha^u g(t)dt$ 可导 $\to \varphi(u)$ 可导

对 u 导

$$\Rightarrow \quad \varphi'(u) = g(u)$$

13.3.2 积分限含参变量的定积分

$$\psi(u)=\int_{a(u)}^{b(u)} f(x,u)\mathrm{d}x$$

定理(连续性) 设$f(x,u)$在$I=[a,b]\times[\alpha,\beta]$上连续, 又 $a(u)$, $b(u)$在$[\alpha,\beta]$上连续, 且 $a\le a(u),b(u)\le b$ ，则

$$\psi(u)=\int_{a(u)}^{b(u)} f(x,u)\mathrm{d}x$$

在$[\alpha,\beta]$上连续.

固定 $u_0\in[\alpha,\beta]$

$$\psi(u)=\left(\int_{a(u)}^{a(u_0)}+\int_{a(u_0)}^{b(u_0)}+\int_{b(u_0)}^{b(u)}\right)f(x,u)dx \overset{def}{=} I_1+I_2+I_3$$

$\lim_{u \to u_0} I_2 = \psi(u_0)$ $|I_1| < \int_{a(u)}^{a(u_0)} |f(x,u)| dx \le M |a(u_0) - a(u)| \xrightarrow{u \to u_0} 0$

同理 $\lim_{u \to u_0} I_3 = 0$ $f(x,u) \le M$

定理(可导性) 设 $f(x, u), f'_u(x, u) \in C[a, b] \times [\alpha, \beta]$, 又

$a(u), b(u)$在$[\alpha, \beta]$上可导, 且 $a \le a(u), b(u) \le b$, 则

$$\psi(u) = \int_{a(u)}^{b(u)} f(x,u) \mathrm{d}x$$

在$[\alpha, \beta]$上可导, 且

上学期: $\frac{d}{dx}\int_a^{x^2} f(x)\,dx = f(x^2) \cdot (x^2)' = f(x^2) \cdot 2x$

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}u}\int_{a(u)}^{b(u)} f(x,u) \mathrm{d}x = \int_{a(u)}^{b(u)} f'_u(x,u) \mathrm{d}x + f(b(u),u) b'(u) - f(a(u),u) a'(u)}$$

例5 计算导数 $\frac{\mathrm{d}}{\mathrm{d}u}\int_{u}^{u^2} \frac{\sin ux}{x} \mathrm{d}x$

令 $y=b(u),\ z=a(u)$

证：

令 $F(u,y,z)=\int_z^y f(x,u)\,dx$

$$\Psi(u)=F(u,b(u),a(u))$$

$$\Psi'(u)=F'_u+F'_y\,b'(u)+F'_z\,a'(u)$$

$$F'_u=F'_u(u,b(u),a(u))=\int_{a(u)}^{b(u)} f'_u(x,u)\,dx$$

$$F'_y=\frac{d}{dy}\int_z^y f(x,u)\,dx=f(y,u)=f(b(u),u)$$

$$F'_z=\frac{d}{dz}\int_z^y f(x,u)\,dx=-\frac{d}{dz}\int_y^z f(x,u)\,dx=-f(z,u)=-f(a(u),u)$$

eg5. $=\int_u^{u^2}\left(\frac{\sin ux}{x}\right)'_u dx+\frac{\sin u^3}{u^2}\cdot 2u-\frac{\sin u^2}{u}\cdot 1$

$$=\int_u^{u^2}\cos ux\,dx+\frac{2\sin u^3-\sin u^2}{u}$$

$$=\frac{\sin u^3-\sin u^2}{u}+\sim=\frac{3\sin u^3-2\sin u^2}{u}$$