Chap 9 — 5

多变量函数Taylor公式与极值

9.5.1 二元函数的微分中值定理

凸区域 若区域D中任意两点的连线都含于D.

微分中值定理 设f在凸区域D中可微, 则∃ $\theta\in(0,1)$:

$$\boxed{\begin{aligned}&f(x_0+h,y_0+k)-f(x_0,y_0)\\&=f_x'(x_0+\theta h,y_0+\theta k)h+f_y'(x_0+\theta h,y_0+\theta k)k\end{aligned}}$$

向量函数无微分中值定理

➢ 若$f(x,y)$在区域D可微, 且 $f_x'(x,y)=0,\ f_y'(x,y)=0$, 则

$$f(x,y)\equiv C$$

凸区域显然

9.5.2 二元函数的Taylor公式

凹

很多邻域需要曲线

证：$F(t) = f(x_0+th, y_0+tk)$ $F \in D[0,1]$

$F(1) - F(0) = F'[0+\theta(1-0)] = F'(\theta)$

$f(x_0+th, y_0+tk) = f_x'(x_0+\theta h, y_0+\theta k)h + f_y'(x_0+\theta h, y_0+\theta k)k$

$-f(x_0, y_0)$ $\quad \frac{\partial f}{\partial x}\frac{dx}{dt}$ $\quad \frac{\partial f}{\partial y}\frac{dy}{dt}$

Qed.

回忆 一元函数Taylor公式：

$$f(x_0+h)=\sum_{m=0}^{n}\frac{1}{m!}h^m\frac{\mathrm{d}^m f(x_0)}{\mathrm{d}x^m}+R_n \overset{\text{def}}{=} \sum_{m=0}^{n}\frac{1}{m!}\left(h\frac{\mathrm{d}}{\mathrm{d}x}\right)^m f(x_0)+R_n$$

定理(Taylor公式) 设函数 $f(x,y)$ 在 $B(P_0(x_0,y_0))$ 有 $n+1$ 阶连续偏导数，则 $\exists\ \theta\in(0,1)$:

$$f(x_0+h,y_0+k)=\sum_{m=0}^{n}\frac{1}{m!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^m f(x_0,y_0)+R_n$$

其中**Lagrange型余项**　Penao 余项 $o(\rho^n)$　$\rho=\sqrt{h^2+k^2}$

$$R_n=\frac{1}{(n+1)!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^{n+1} f(x_0+\theta h,y_0+\theta k)$$

对F use taylor 展开、 $\theta \in (0,1)$

$$F(t) = \sum_{m=0}^{n} \frac{1}{m!} F^{(m)}(0) t^m + \frac{F^{(n+1)}(\theta)}{(n+1)!}$$

$t=1$

$$F(1) = \sum_{m=0}^{n} \frac{1}{m!} F^{(m)}(0) + \frac{F^{(n+1)}(\theta)}{(n+1)!}$$

$$F'(t) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = \frac{\partial f}{\partial x} h + \frac{\partial f}{\partial y} k$$

$$F''(t) = \frac{\partial^2 f}{\partial x^2} h^2 + 2\frac{\partial^2 f}{\partial x \partial y} hk + \frac{\partial^2 f}{\partial y^2} k^2$$

Induction: assume $F^{(m)}(t) = \left(\frac{\partial}{\partial x} h + \frac{\partial}{\partial y} k\right)^m f$

$$F^{(m+1)}(t) = \frac{\partial \left(\frac{\partial}{\partial x} h + \frac{\partial}{\partial y} k\right)^m f}{\partial x} h + \frac{\partial \left(\frac{\partial}{\partial x} h + \frac{\partial}{\partial y} k\right)^m f}{\partial y} k$$

$$= \left(\frac{\partial}{\partial x} h + \frac{\partial}{\partial y} k\right) \cdot \left(\frac{\partial}{\partial x} h + \frac{\partial}{\partial y} k\right)^m f$$

$= \cdots$

代入即证

➢ m阶偏导数算子

$$\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^m=\sum_{i+j=m}\frac{m!}{i!\,j!}h^i k^j\frac{\partial^m}{\partial x^i\partial y^j}$$

➢ $n=0$时, 即**Lagrange中值定理**

➢ $n=1$时, **一阶Taylor公式**

$$\boxed{\begin{aligned}f(x_0+h,y_0+k)&=f(x_0,y_0)+\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)f(x_0,y_0)\\&\quad+\frac{1}{2}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^2 f(x_0+\theta h,y_0+\theta k)\end{aligned}}$$

例1 求 $f(x,y)=e^{x}\cos y$ 在(0, 0)处的二阶Taylor展式

想一想 三元及n元函数的Taylor公式?

9.5.3 二元函数的极值

一. 极值定义

若在点$P_0(x_0,y_0)$的某邻域内

$$f(x,y)\le f(x_0,y_0)\quad (\text{or } f(x,y)\ge f(x_0,y_0))$$

则称函数f在(x_0, y_0)处取极大值(or 极小值),

$P_0(x_0,y_0)$称为函数的极大值点(or极小值点).

$f'_x = e^x \cos y \quad f'_y = -e^x \sin y \quad f''_{xx} = e^x \cos y$

eg1: $e^x \cos y = 1 + f'_x(0,0)x + f'_y(0,0)y + \frac{1}{2!}\left[f''_{xx}(0,0)x^2 + 2f''_{xy}(0,0)xy + f''_{yy}(0,0)y^2\right] + o(\rho^2)$ where $\rho = \sqrt{x^2+y^2}$

$= 1 + x + 0y + \frac{1}{2}(x^2 + 0xy - y^2) + o(\rho^2)$

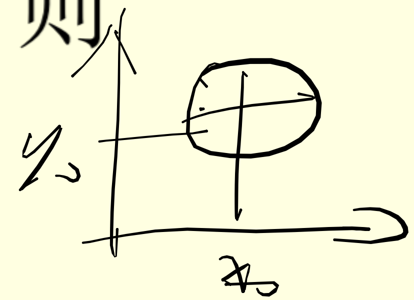
$= 1 + x + \frac{1}{2}(x^2 - y^2) + o(\rho^2)$

解二: $f(x,y) = \left(1 + x + \frac{x^2}{2!} + o(x^2)\right)\left(1 - \frac{y^2}{2!} + \cancel{\frac{y^4}{4!}} + o(y^2)\right)$

$= 1 + x + \frac{x^2}{2} - \frac{y^2}{2} + o(\rho^2)$

二. 极值的必要条件

若 $f(x,y)$ 可偏导，且在 $P_0(x_0,y_0)$ 取极值，则

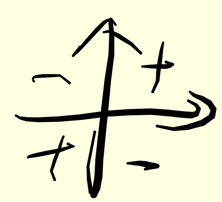


$$f'_x(x_0,y_0)=0=f'_y(x_0,y_0)$$

- 满足上式的点称为**驻点**(**驻点未必是极值点**)

例2 考察函数 $f(x,y)=xy$ 在(0,0)的情况.

不是极值点



- **极值点未必是驻点!**

例3 考察函数 $f(x,y)=(x^2+y^2)^{1/2}$ 在(0,0)的情况.

- **可偏导的极值点必是驻点!**

三. 极值的充分条件

定理 设 $f(x,y)$ 在$B(P_0(x_0,y_0))$的二阶偏导数连续, 且

$$f'_x(x_0,y_0)=0,\ f'_y(x_0,y_0)=0,$$

记**Hesse矩阵** $$H=\begin{pmatrix} f''_{xx} & f''_{xy} \\ f''_{yx} & f''_{yy} \end{pmatrix}_{P_0}$$

则有 **(1)** 若$\boldsymbol{H}$为正定矩阵, 则$f(x_0,y_0)$为严格极小值;

若$\boldsymbol{H}$为负定矩阵, 则$f(x_0,y_0)$为严格极大值

(2) 若$\boldsymbol{H}$为不定矩阵, 则$f(x_0,y_0)$非极值.

半定 (半正定 半负定) 不定

证明思路

$$\Delta f = f(x_0+h, y_0+k) - f(x_0, y_0)$$

$$= \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(x_0, y_0) + \frac{1}{2}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(x_0, y_0) + o(\rho^2)$$

$$= \frac{1}{2}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(x_0, y_0) + o(\rho^2)$$

$$= \frac{1}{2}[h^2 f''_{xx} + 2hk f''_{xy} + k^2 f''_{yy}]_{(x_0, y_0)} + o(\rho^2)$$ $\longrightarrow$ $h\,k$ 充分小 Δf 符号

$$= \frac{\rho^2}{2}\left\{[\xi^2 f''_{xx} + 2\xi\eta f''_{xy} + \eta^2 f''_{yy}]_{(x_0, y_0)} + o(1)\right\}$$

$$\Delta f = \frac{\rho^2}{2}\left[(\xi\ \eta) H \begin{pmatrix}\xi \\ \eta\end{pmatrix} + o(1)\right]$$

其中 $\rho = \sqrt{h^2 + k^2}, \xi = \frac{h}{\rho}, \eta = \frac{k}{\rho}$. 利用$\boldsymbol{H}$的型确定$\Delta f$的符号.

命题 设 $\boldsymbol{H}=\begin{pmatrix} A & B \\ B & C \end{pmatrix}$

二次型

$$Q(h,k)=(h \quad k)H\begin{pmatrix} h \\ k \end{pmatrix}=Ah^2+2Bhk+Ck^2$$

下

令 $S=\{(\xi,\eta)\mid\xi^2+\eta^2=1\}$

则有 (1) $Q(h,k)$是**定的** $\Leftrightarrow$ $|\boldsymbol{H}|>0$. 且 则 $(\xi\ \eta)H\begin{pmatrix}\xi\\ \eta\end{pmatrix}$在$S$上连续

$\det H$

当$A>0$时, Q为**正定**; 当$A<0$时, Q为**负定**. 从而有最值

(2) $Q(h,k)$是**不定的** $\Leftrightarrow$ $|\boldsymbol{H}|<0$.

(3) $Q(h,k)$是**半定的** $\Leftrightarrow$ $|\boldsymbol{H}|=0$.

若 正定，令 $S=\{(\xi,\eta) \mid \xi^2+\eta^2=1\}$ 则

$(\xi\ \eta) H \binom{\xi}{\eta}$ 在 S 上连续 S 有界闭，从而有最值

$\exists (\xi_0, \eta_0) \in S$ s.t. $m_0 \overset{def}{=} (\xi_0\ \eta_0) H \binom{\xi_0}{\eta_0} = \min$ 值 > 0

$\Delta f \geq \frac{\rho^2}{2}(m_0 + o(1)) > 0$

ρ 充分小时 $f(x_0, y_0)$ 极小值

若 不定 则 $\exists (h_1, k_1)$ 使 $(h_1\ k_1) H \binom{h_1}{k_1} > 0$ 记 $\rho_1 = \sqrt{h_1^2+k_1^2}$

$\xi_1 = \frac{h_1}{\rho_1}$ $\quad \eta = \frac{h_2}{\rho_2}$ $\quad (\xi_1\ \ \eta_1) H \binom{\xi_1}{\eta_1} = \frac{1}{\rho_1^2}(h_1\ k_1) H \binom{h_1}{k_1} > 0$

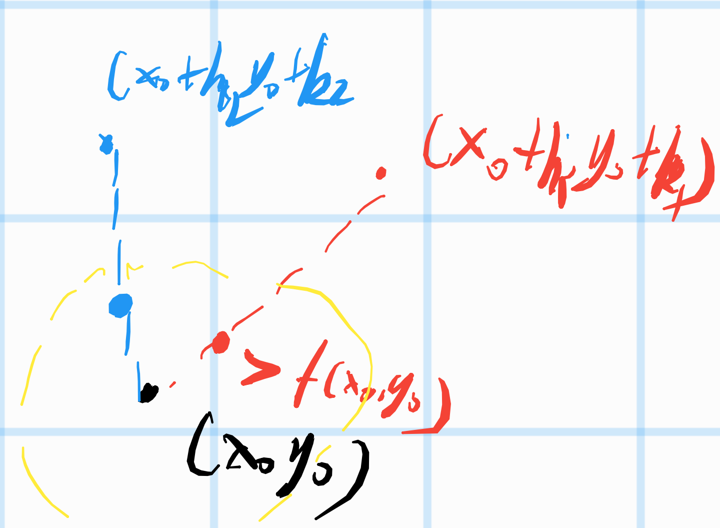
$\Delta f = f(x_0+\varepsilon h_1, y_0+\varepsilon k_1) - f(x_0\ y_0)$

$= \frac{(\varepsilon\rho_1)^2}{2}\left[(\xi_1\ \ \eta_1) H \binom{\xi_1}{\eta_1} + o(1)\right] > 0$

ε 充分小时才成立 (因为 $o(1)$)

That is to say. $\forall \mathring{U}(x_0, y_0)$ $\exists$ $x_0+\Delta x$ $y_0+\Delta y$ 在其中 $\Delta f = f(x_0+\Delta x, y_0-\Delta y)$
$- f(x_0, y_0) > 0$

$\therefore$ 不定，$\forall \mathring{U}(x_0, y_0)$ $\exists \Delta f < 0$.



推论 设 $f(x, y)$ 在$B(P_0(x_0,y_0))$的二阶偏导数连续, 且

$$f'_x(x_0, y_0) = 0,\ f'_y(x_0, y_0) = 0,$$

记 $A = f''_{xx}(x_0, y_0), B = f''_{xy}(x_0, y_0), C = f''_{yy}(x_0, y_0)$

(1) 若**Hesse行列式** $|\boldsymbol{H}| = AC - B^2 < 0$, 则 $f(x_0, y_0)$非极值

(2) 若 $|\boldsymbol{H}| > 0$, 则 $f(x_0, y_0)$为极值, 且当$A > 0$时, $f(x_0, y_0)$为严格极小值; 当$A < 0$时, $f(x_0, y_0)$为严格极大值.

想一想 当$|\boldsymbol{H}| = 0$时, 结论如何?

例4 求函数 $f(x, y) = x^3 + y^3 - 3xy$ 的极值.

994. $f_x'(x,y)=3x^2-3y$ $f_y'(x,y)=3y^2-3x$

$\begin{cases} f_x'(x,y)=0 \\ f_y'(x,y)=0 \end{cases}$, $\begin{cases} x=0 \\ y=0 \end{cases}$ 或 $\begin{cases} x=1 \\ y=1 \end{cases}$

$f''_{xx}=6x$ $f''_{yy}=6y$ $f''_{xy}=-3$

$H=\begin{pmatrix} 6x & -3 \\ -3 & 6y \end{pmatrix}$ $|H|=36xy-9$

$|H|\big|_{(0,0)}=-9$ 不定

$H\big|_{(1,1)}=27>0$.

$6x>0$, 正定,

$(1,1)\to$极小值,

9.5.4 条件极值

例5 求体积V_0固定, 表面积最小的圆柱体底圆半径与高之比.

在许多极值问题中, 函数自变量还要满足一些**约束条件**, 这类极值问题称为**条件极值**. 如求**目标函数**

$$u = f(x, y)$$

在约束条件 $\varphi(x, y) = 0$ 下的极值.

注 相对于条件极值, 对自变量无约束条件的极值问题称为**无条件极值**.

■ 直接法

如果可从约束条件 $\varphi(x, y) = 0$ 中解出一个变量, 比如 $y = y(x)$, 然后代入目标函数得

$$u = f(x, y(x))$$

再求此一元函数的无条件极值.

■ Lagrange乘数法

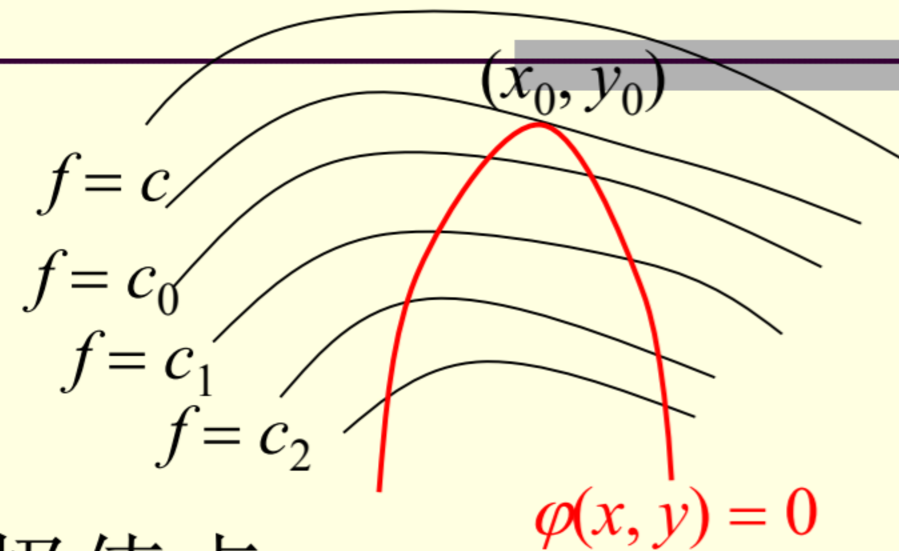
引进辅助函数(**Lagrange函数**)

$$L(x, y, \lambda) = f(x, y) + \lambda\varphi(x, y)$$

从其无条件极值的必要条件

几何意义

$$\begin{cases} L'_x = f'_x + \lambda\varphi'_x = 0 \\ L'_y = f'_y + \lambda\varphi'_y = 0 \\ L'_\lambda = \varphi(x,y) = 0 \end{cases}$$



求出的(x_0, y_0)是可能的条件极值点.

例6 在曲面 $z = x^2 + 2y^2$ 上求一点P, 使其到平面 $x - y + 2z + 6 = 0$ 的距离最短.

Ex. 在椭圆 $x^2 + 4y^2 = 4$ 上求一点P, 使其到直线 $2x + 3y - 6 = 0$ 的距离最短.

$$d=\frac{|x-y+2z+6|}{\sqrt{6}}$$

eg6. $f=(x-y+2z+6)^2$

$$g(x,y,z)=f(x,y,z)+\lambda(z-x^2-2y^2)$$

$$\frac{\partial g}{\partial x}=2(x-y+2z+6)-2\lambda x$$

$$\frac{\partial g}{\partial y}=-2(x-y+2z+6)-4\lambda y$$

$$\frac{\partial g}{\partial z}=4(x-y+2z+6)+\lambda$$

$$\frac{\partial g}{\partial \lambda}=z-x^2-2y^2$$

$$\lambda x=-2\lambda y=-\frac{\lambda}{4}$$

1° $\lambda=0$ $\begin{cases} z=x^2+2y^2 \\ x-y+2z+6=0 \end{cases}$ $\rightarrow$ 无解

2° $\lambda\neq0$, $x=-\frac{1}{4}$ $y=\frac{1}{8}$ $z=\frac{3}{32}$

$P(-\frac{1}{4},\frac{1}{8},\frac{3}{32})$

最小值显然存在，故 P 为答案

解2、$P(x,y,z)$、 $\vec{n}_p=(2x,4y,-1)$

$\vec{n}\parallel(1,-1,2)$ $\qquad \frac{2x}{1}=\frac{4y}{-1}=\frac{-1}{2}$

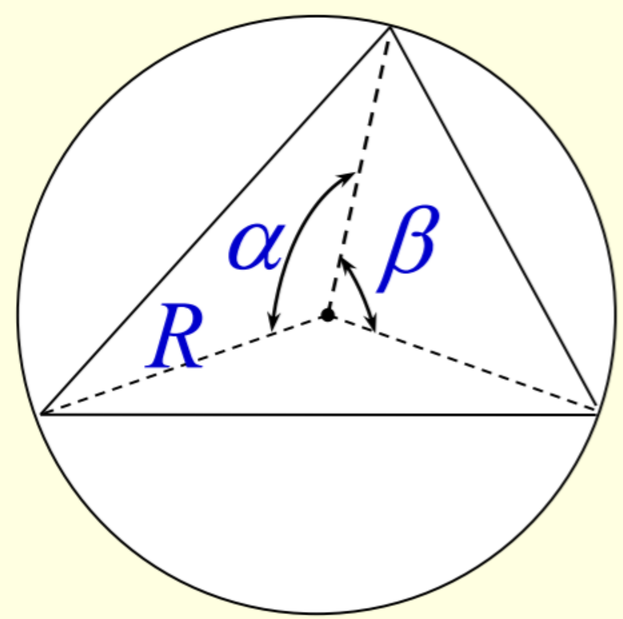
$P(-\frac{1}{4},\frac{1}{8},\frac{3}{32})$ 且平面与曲面不交

从而P为所求

■ 最值问题

原则 有界闭区域上的可微函数的最值在内部驻点或边界点取到. 实际问题中, 若最值必在区域内部取得又驻点唯一, 则此驻点就是最值点.

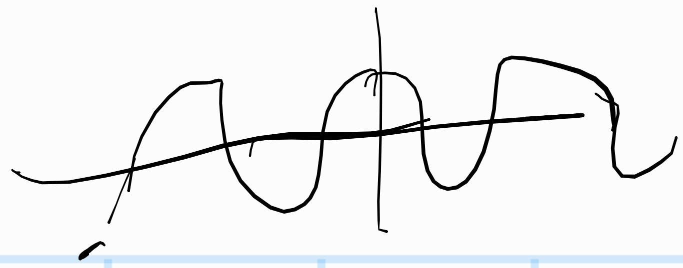
例7 在半径为R的圆中求面积最大的内接三角形的边长



解 $S=\dfrac{R^2}{2}\big(\sin\alpha+\sin\beta-\sin(\alpha+\beta)\big),$

其中 $\alpha,\beta\in(0,2\pi)$ 且 $\alpha+\beta<2\pi.$

$F = \frac{2}{R^2} S$



$$\frac{\partial F}{\partial \alpha} = \cos\alpha - \cos(\alpha+\beta)$$

$$\frac{\partial F}{\partial \beta} = \cos\beta - \cos(\alpha+\beta)$$

$\Rightarrow \cos\alpha = \cos\beta$

$\beta = 2k\pi + \alpha$

$\beta = 2k\pi - \alpha \quad \alpha+\beta < 2\pi.$

$\therefore \alpha = \beta$

$F = 2\sin\alpha + \sin 2\alpha$

$F' = 2\cos\alpha + 2\cos 2\alpha$

$\cos\alpha = -\cos 2\alpha$

$\alpha + 2\alpha = \pi$

例8 求函数 $f(x,y)=2x^2+6xy+y^2$ 在椭圆域

D° $x^2+2y^2\leq 3$ 上的最大值和最小值.

在 D° $f'_x=4x+6y=0$

$f'_y=6x+2y=0$ $(0,0)$; $f(0,0)=0$

在 ∂D $L=2x^2+6xy+y^2+\lambda(x^2+2y^2-3)$

$L'_x=4x+6y+2\lambda x=0$

$L'_y=6x+2y+4\lambda y=0$

$$\begin{cases}(\lambda+2)x+3y=0\\3x+(2\lambda+1)y=0\\x^2+2y^2=3\end{cases}$$

若 $\begin{vmatrix} \lambda+2 & 3 \\ 3 & 2\lambda+1 \end{vmatrix} \neq 0$、$\begin{cases} x=0=y \\ x^2+2y^2=3 \end{cases}$ 无解

$|\sim|=0$ $\lambda=1$ or $-\frac{7}{2}$

当 $\lambda=1$ 时 $\begin{cases} y=-x \\ x^2+2y^2=3 \end{cases} \Rightarrow (\pm1, \mp1)$ $f(\pm1,\mp1)=-3$

$\lambda=-\frac{7}{2}$ 时 $\begin{cases} x=2y \\ x^2+2y^2=3 \end{cases} \Rightarrow (\pm\sqrt{2}, \pm\frac{1}{\sqrt{2}})$ $f(\pm\sqrt{2}, \pm\frac{1}{\sqrt{2}})=10+\frac{1}{2}=\frac{21}{2}$