Chap 9 — 2

多变量函数的微分

9.2.1 多变量函数的偏微商

空心邻域

定义 设 $f(x,y)$ 在 (x_0,y_0) 某邻域有定义. 仅给 x 以增量 Δx

相应有函数的增量(对 x **偏增量**)

$$\Delta_x z = f(x_0+\Delta x, y_0) - f(x_0, y_0)$$

函数 f 在点 (x_0,y_0) 处对 x 的**偏微商(或偏导数)**

$$f_x'(x_0,y_0) \overset{\text{def}}{=} \lim_{\Delta x\to 0}\frac{\Delta_x z}{\Delta x} = \lim_{\Delta x\to 0}\frac{f(x_0+\Delta x, y_0) - f(x_0, y_0)}{\Delta x}$$

◆ 偏微商也可记为 $\dfrac{\partial f}{\partial x}(x_0, y_0)$

◆ 对变量y的偏微商类似

◆ **可偏导**: 两个偏导数都存在.

◆ **偏导(函)数**: $f_x'(x,y), f_y'(x,y)$ or $\dfrac{\partial f}{\partial x}(x,y), \dfrac{\partial f}{\partial y}(x,y)$

例 求函数$u = x^y\ (x > 0)$的偏导数.

■ 连续与可偏导

➢ 可偏导未必连续

例 考察 $f(x,y)=\begin{cases}\dfrac{xy}{x^2+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$

在(0,0)的情况.

➢ 连续未必可偏导

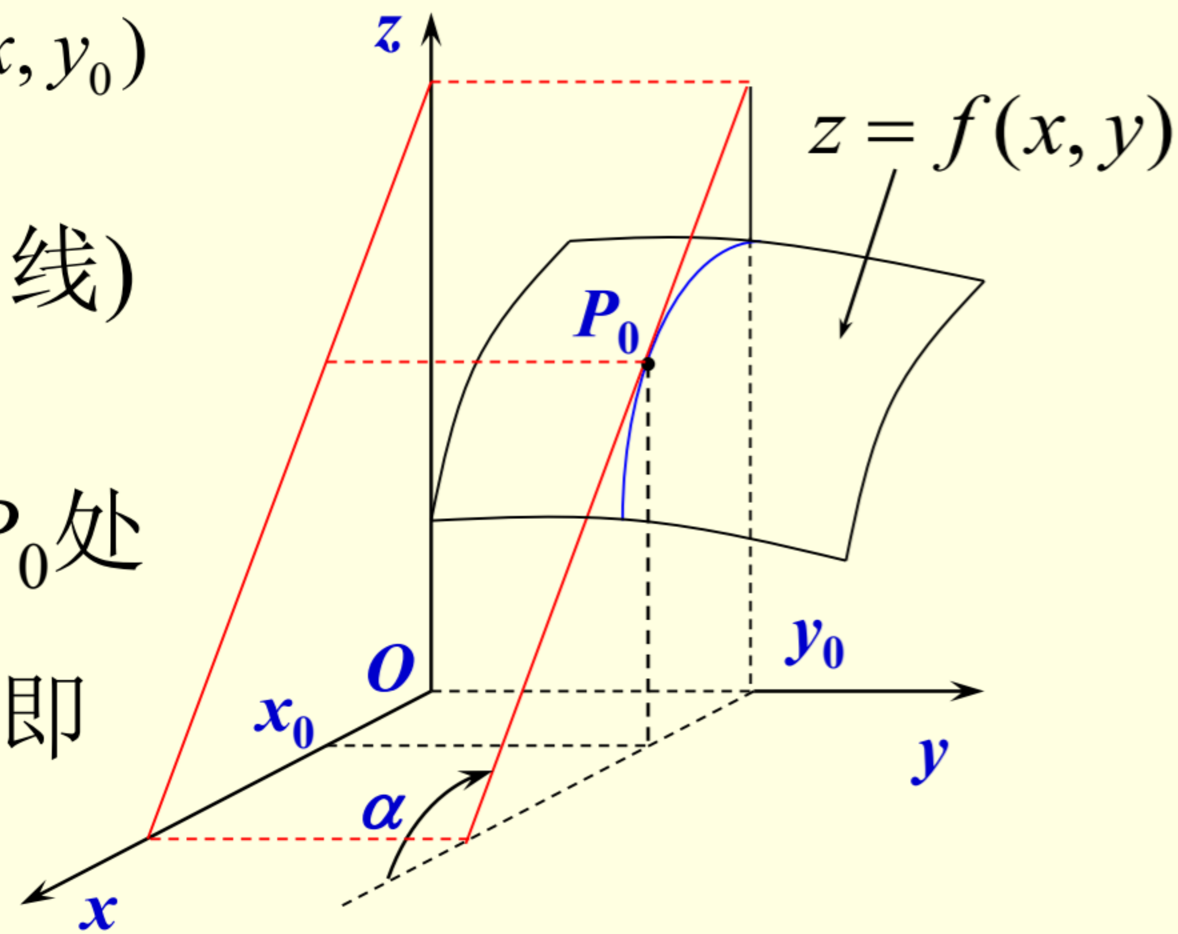
例 考察 $f(x,y)=|x|+|y|$ 在(0,0)的情况.

■ 偏导数的几何意义

曲面$z=f(x,y)$与平面$y=y_0$的交线

$$\begin{cases} z=f(x,y) \\ y=y_0 \end{cases} \Rightarrow \quad z=f(x,y_0)$$

(平面$y=y_0$上的曲线)



$f'_x(x_0,y_0)$是该曲线在P_0处

的切线关于x轴的**斜率**. 即

$$\boxed{f'_x(x_0,y_0)=\tan\alpha}$$

定义 $f(x,y)$在某邻域内的偏导数$f'_x(x,y), f'_y(x,y)$的偏导数称为f的**二阶偏导数**. 记为

warning!

$$f''_{xx}=\frac{\partial^2 f}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right),\quad f''_{xy}=\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

$$\frac{\partial^2 f}{\partial y\partial x}$$

先对x再对y

$$f''_{yx}=\frac{\partial^2 f}{\partial y\partial x}=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right),\quad f''_{yy}=\frac{\partial^2 f}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

类似可定义三阶偏导数, 例如

$$f'''_{xxy}=\frac{\partial^3 f}{\partial x^2\partial y}=\frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial x^2}\right)$$

例 求函数 $z=\ln x+e^y\sin x$ 的所有二阶偏导数.

问题: 混合偏导数是否总与求偏导次序无关? 不一定

例 设 $f(x,y)=\begin{cases} xy, & |x|\ge|y| \\ -xy, & |x|<|y| \end{cases}$, 求 $f''_{xy}(0,0), f''_{yx}(0,0)$.

分析

$$f''_{xy}(0,0)=\lim_{y\to 0}\frac{f'_x(0,y)-f'_x(0,0)}{y} = -1$$

$(y\neq 0)$ $$f'_x(0,y)=\lim_{x\to 0}\frac{f(x,y)-f(0,y)}{x} = \lim \frac{-xy-0}{x} = -y$$

$$f'_x(0,0)=\lim_{x\to 0}\frac{f(x,0)-f(0,0)}{x} = 0$$

混合偏导次序无关定理

定理 若 $f(x,y)$ 的二阶混合偏导数在 (x,y) 连续, 则

$$f''_{xy}(x,y)=f''_{yx}(x,y)$$

$f''_{yx}(x,y)=1$ ~~证明~~ 略去

9.2.2 多变量函数的可微性

线性主部

一元情形：若 $\Delta f = a\Delta x + o(\Delta x)$,则称 f 在 x_0 可微,

并把 $a\Delta x$ 称为 f 在 x_0 处的微分, 记为 $\mathrm{d}f\big|_{x=x_0} = a\Delta x$

二元情形：对函数 $z = f(x, y)$, 若**增量** (全增量)

$$\Delta z = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)$$

$$= a\Delta x + b\Delta y + o(\rho)$$

其中 a, b 是常数, $\rho = \sqrt{(\Delta x)^2 + (\Delta y)^2}$, 则称 f 在 (x_0,y_0) **可微**.

并把 $a\Delta x + b\Delta y$ 称为 f 在 (x_0,y_0) 处的**微分**. 记为

又称全微分

$$\mathrm{d}f\big|_{(x_0,y_0)} = a\Delta x + b\Delta y$$

若 f 在区域 D 内处处可微, 则称 f 是 D 内的**可微函数**.

- 可微必连续
- 可微必可偏导, 且若

$$\boxed{\begin{aligned} &\mathrm{d}f\Big|_{(x_0,y_0)} = a\Delta x + b\Delta y \\ \Rightarrow\quad & f_x'(x_0,y_0) = a,\ f_y'(x_0,y_0) = b \end{aligned}}$$

证: 令 $\Delta y = 0$

$\Delta_x f = a\Delta x + o(|\Delta x|)$

$f_x'(x_0,y_0) = \lim\limits_{\Delta x\to 0}\dfrac{\Delta_x f}{\Delta x}$

$= \lim\limits_{\Delta x\to 0}\left(a + \dfrac{o(|\Delta x|)}{\Delta x}\right)$

$= a$

微分公式

$$\boxed{\mathrm{d}f(x,y) = f_x'(x,y)\mathrm{d}x + f_y'(x,y)\mathrm{d}y}$$

例 求函数 $z = x^y$ 在点 $(1,1)$ 处的微分.

$\dfrac{\partial z}{\partial x}\Big|_{(1,1)} = yx^{y-1}\Big|_{(1,1)} = 1$

$\dfrac{\partial z}{\partial y}\Big|_{(1,1)} = x^y\ln x\Big|_{(1,1)} = 0$

$dz|_{(1,1)} = 1\cdot dx + 0\cdot dy = dx$

例 求函数 $z = \arctan\dfrac{y}{x}$ 的微分.

定理 设$f(x, y)$在区域D内存在偏导数, 则

(1) 若$f'_x(x, y), f'_y(x, y)$在D内有界, 则f在D内连续;

(2) 若$f'_x(x, y), f'_y(x, y)$在D内连续, 则f在D内可微.

结论

$$\text{偏导数连续} \overset{\not\Leftarrow}{\Rightarrow} \text{可微} \Rightarrow \begin{cases} \text{连续} \\ \text{可偏导} \end{cases}$$

eg $\frac{\partial z}{\partial x}=\frac{-\frac{y}{x^2}}{1+\frac{y^2}{x^2}}=\frac{-y}{x^2+y^2}$ $\frac{\partial z}{\partial y}=\frac{\frac{1}{x}}{1+\frac{y^2}{x^2}}=\frac{x}{x^2+y^2}$

$dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy=\cdots\cdots$

定理(1) $\Delta f=f(x+\Delta x,y+\Delta y)-f(x,y)$

$=[f(x+\Delta x,y+\Delta y)-f(x,y+\Delta y)]+[f(x,y+\Delta y)-f(x,y)]$

$\underset{\forall\theta_2\in[0,1]}{\overset{\forall\theta_1\in[0,1]}{=}} f_x'(x+\theta_1\Delta x,y+\Delta y)\Delta x+f_y'(x+\Delta x,y+\theta_2\Delta y)\Delta y$

$\because x\ y\in D$

$\Delta f<M_1\Delta x+M_2\Delta y$ (有界)

$\Delta x\to 0$
$\Delta y\to 0$ $\Delta f\to 0$.

可微的 a,b不能依赖Δx Δy

(2) 由(1) $\Delta f=[f_x'(x,y)+\alpha]\Delta x+[f_y'(x,y)+\beta]\Delta y$

其中α β为无穷小

$\Delta f=f_x'(x,y)\Delta x+f_y'(x,y)\Delta y+\alpha\Delta x+\beta\Delta y$

$\frac{|\alpha\Delta x+\beta\Delta y|}{\rho}<|\alpha|+|\beta|\to 0.$ $(\Delta x,\Delta y\to 0,0)$

9.2.3 方向导数与梯度

定义 设 $\boldsymbol{e}=(\cos\alpha,\cos\beta)$, 函数$z=f(x,y)$在$(x_0, y_0)$处沿 $\boldsymbol{e}$ 的**方向导数**定义为

$$\frac{\partial f}{\partial \boldsymbol{e}}(x_0,y_0)=\lim_{t\to 0}\frac{f(x_0+t\cos\alpha, y_0+t\cos\beta)-f(x_0,y_0)}{t}$$

- 偏导数是方向导数的特例,

$$\frac{\partial f}{\partial x}(x_0,y_0)=\frac{\partial f}{\partial \boldsymbol{e}}(x_0,y_0),\quad \boldsymbol{e}=(1,0)$$

方向角：$\vec{e}$与x轴 y轴正向夹角

定理 $z = f(x, y)$在$M_0(x_0, y_0)$可微， $\boldsymbol{e} = (\cos\alpha, \cos\beta)$

则f在M_0点存在方向导数，且

$\Delta f = f(x_0 + t\cos\alpha, y_0 + t\cos\beta) - f(x_0, y_0)$

$= f_x'(x_0, y_0)t\cos\alpha + f_y'(x_0, y_0)t\cos\beta + o(|t|)$

$$\boxed{\frac{\partial f}{\partial \boldsymbol{e}}(x_0, y_0) = f_x'(x_0, y_0)\cos\alpha + f_y'(x_0, y_0)\cos\beta}$$

结论

● 可微 $\Rightarrow$ 方向导数存在 $\not\Rightarrow$ 极限存在

方向导数存在 $\Rightarrow$ 可偏导

方向导数存在 $\not\Rightarrow$ 连续

例 求 $r(x, y) = \sqrt{x^2 + y^2}$ 沿$\boldsymbol{e} = (\cos\alpha, \cos\beta)$的方向导数

$(x, y) \neq (0, 0)$

(0,0)不可微

$$\frac{\partial}{\partial e}\sqrt{x^2+y^2} = \frac{\partial\sqrt{x^2+y^2}}{\partial x}\cos\alpha + \frac{\partial\sqrt{x^2+y^2}}{\partial y}\cos\beta$$

$$= \frac{x}{\sqrt{x^2+y^2}}\cos\alpha + \frac{y}{\sqrt{x^2+y^2}}\cos\beta$$

定义 函数 $f(x, y)$在点$M_0(x_0,y_0)$的**梯度**定义为

$$\operatorname{grad} f(x_0, y_0) \overset{\text{def}}{=} (f_x'(x_0, y_0), f_y'(x_0, y_0))$$

利用梯度符号, 得到 ∇f ∇(nabla) 也指梯度

$$\frac{\partial f}{\partial \boldsymbol{e}}(M_0) = \operatorname{grad} f(M_0) \cdot \boldsymbol{e} = |\operatorname{grad} f(M_0)| \cos\theta$$

$\Rightarrow \theta = 0$时, 方向导数 $\dfrac{\partial f}{\partial \boldsymbol{e}}(M_0)$ 取最大值 $|\operatorname{grad} f(M_0)|$

结论 梯度的方向是方向导数取最大值时的方向, 其模就是方向导数的最大值.

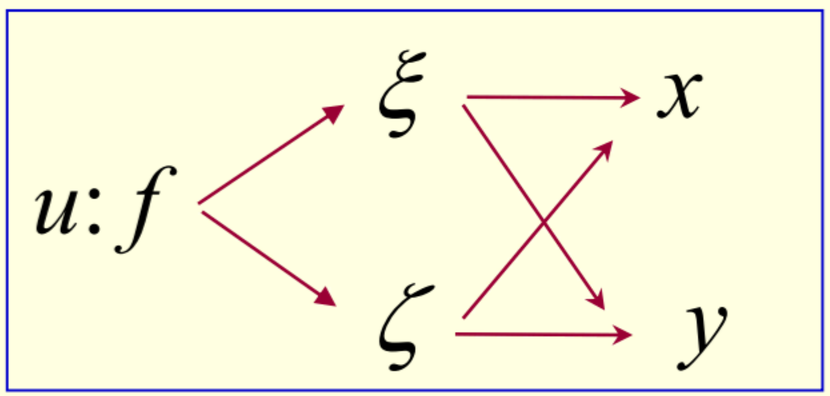
9.2.4 复合函数的微分

定理 设 $u=f(\xi,\zeta)$可微, $\xi=\xi(x,y)$, $\zeta=\zeta(x,y)$可微.

则复合函数 $u=f(\xi(x,y),\zeta(x,y))$ 也可微, 且

$$\frac{\partial u}{\partial x}=\frac{\partial f}{\partial \xi}\cdot\frac{\partial \xi}{\partial x}+\frac{\partial f}{\partial \zeta}\cdot\frac{\partial \zeta}{\partial x},\quad \frac{\partial u}{\partial y}=\frac{\partial f}{\partial \xi}\cdot\frac{\partial \xi}{\partial y}+\frac{\partial f}{\partial \zeta}\cdot\frac{\partial \zeta}{\partial y}$$

- **链法则**



$u=f(\xi_1, \xi_2 \cdots \xi_m)$

$\xi_i=\xi_i(x_1, x_2 \cdots x_n)$

$\frac{\partial u}{\partial x_j}=\sum_{i=1}^{m}\frac{\partial f}{\partial \xi_i}\frac{\partial x_i}{\partial \xi_j}$

想一想 m个中间变量n个自变量的链法则?

给增量Δx Δy

$$\Delta\xi = \frac{\partial\xi}{\partial x}\Delta x + \frac{\partial\xi}{\partial y}\Delta y + o(\rho) \quad \text{where } \rho = \sqrt{\Delta x^2 + \Delta y^2}$$

$$\stackrel{def}{=} a_1\Delta x + a_2\Delta y + o(\rho)$$

$$\Delta\zeta = \frac{\partial\zeta}{\partial x}\Delta x + \frac{\partial\zeta}{\partial y}\Delta y + o(\rho) \stackrel{def}{=} b_1\Delta x + b_2\Delta y + o(\rho)$$

$$\Delta u = \frac{\partial f}{\partial\xi}\Delta\xi + \frac{\partial f}{\partial\zeta}\Delta\zeta + o(\tilde{\rho}) \quad \text{where } \tilde{\rho} = \sqrt{\Delta\xi^2 + \Delta\zeta^2}$$

$$= \frac{\partial f}{\partial\xi}\left(\frac{\partial\xi}{\partial x}\Delta x + \frac{\partial\xi}{\partial y}\Delta y + o(\rho)\right) + \frac{\partial f}{\partial\zeta}\left(\frac{\partial\zeta}{\partial x}\Delta x + \frac{\partial\zeta}{\partial y}\Delta y + o(\rho)\right) + o(\tilde{\rho})$$

$$= \left(\frac{\partial f}{\partial\xi}\frac{\partial\xi}{\partial x} + \frac{\partial f}{\partial\zeta}\frac{\partial\zeta}{\partial x}\right)\Delta x + \left(\frac{\partial f}{\partial\xi}\frac{\partial\xi}{\partial y} + \frac{\partial f}{\partial\zeta}\frac{\partial\zeta}{\partial y}\right)\Delta y + o(\rho) + o(\tilde{\rho})$$

$$\frac{o(\tilde{\rho})}{\rho} = \frac{o(\tilde{\rho})}{\tilde{\rho}}\frac{\tilde{\rho}}{\rho}$$ 只需说明$\frac{\tilde{\rho}}{\rho}$有界量

$$\left(\frac{\tilde{\rho}}{\rho}\right)^2 = \frac{\Delta\xi^2 + \Delta\zeta^2}{\Delta x^2 + \Delta y^2} = \frac{1}{\rho^2}\left[(a_1\Delta x + a_2\Delta y + o(\rho))^2 + (b_1\Delta x + b_2\Delta y + o(\rho))^2\right]$$

$$= \left(\frac{a_1\Delta x}{\rho} + \frac{a_2}{\rho}\Delta y + \frac{o(\rho)}{\rho}\right)^2 + (\cdots)^2$$

$$\leq (a_1 + a_2 + M_1)^2 + (b_1 + b_2 + M_2)^2$$

有界

$$\therefore \Delta u = (\cdots)\Delta x + (\cdots)\Delta y + o(\rho)$$

■ 一阶微分形式的不变性

函数 $u = f(\xi, \zeta)$的微分

$$\mathrm{d}u = \frac{\partial f}{\partial \xi}\mathrm{d}\xi + \frac{\partial f}{\partial \zeta}\mathrm{d}\zeta$$

若ξ, ζ又是x, y的可微函数$\xi = \xi(x, y), \zeta = \zeta(x, y)$, 则

复合函数 $u = f(\xi(x,y),\zeta(x,y))$ 的微分

$$\mathrm{d}u = \frac{\partial u}{\partial x}\mathrm{d}x + \frac{\partial u}{\partial y}\mathrm{d}y$$

$$= \left(\frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial f}{\partial \zeta}\frac{\partial \zeta}{\partial x}\right)\mathrm{d}x + \left(\frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial f}{\partial \zeta}\frac{\partial \zeta}{\partial y}\right)\mathrm{d}y$$

$$= \frac{\partial f}{\partial \xi}\left(\frac{\partial \xi}{\partial x}\mathrm{d}x + \frac{\partial \xi}{\partial y}\mathrm{d}y\right) + \frac{\partial f}{\partial \zeta}\left(\frac{\partial \zeta}{\partial x}\mathrm{d}x + \frac{\partial \zeta}{\partial y}\mathrm{d}y\right)$$

注意到 $\mathrm{d}\xi = \frac{\partial \xi}{\partial x}\mathrm{d}x + \frac{\partial \xi}{\partial y}\mathrm{d}y, \quad \mathrm{d}\zeta = \frac{\partial \zeta}{\partial x}\mathrm{d}x + \frac{\partial \zeta}{\partial y}\mathrm{d}y$

从而 $$\mathrm{d}u = \frac{\partial f}{\partial \xi}\mathrm{d}\xi + \frac{\partial f}{\partial \zeta}\mathrm{d}\zeta$$

因此, 对于函数$u = f(\xi, \zeta)$, 无论ξ, ζ是自变量还是函数, 都有

$$\mathrm{d}u = \frac{\partial f}{\partial \xi}\mathrm{d}\xi + \frac{\partial f}{\partial \zeta}\mathrm{d}\zeta$$

想一想 n元函数的一阶微分形式不变性?

$$\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial \xi} & \frac{\partial f}{\partial \zeta} \end{pmatrix} \begin{pmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y} \\ \frac{\partial \zeta}{\partial x} & \frac{\partial \zeta}{\partial y} \end{pmatrix}$$

例 设函数$u=f(x,y,z)$可微, 而$z=z(x,y)$可偏导.

求复合函数$u=f(x,y,z(x,y))$对x的偏导数.

例 原点处电荷q产生电势 $u=q/r$, 其中r 是点

$\boldsymbol{r}=(x,y,z)$到原点的距离. 当$r\neq 0$时, 求u在(x,y,z)

处的梯度及沿方向$\boldsymbol{r}$的变化率, 并证明u满足方程

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0 \longleftarrow \textbf{Laplace方程}$$

① $\frac{\partial u}{\partial x} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial x}$

$$\begin{tikzcd} & & x \\ u \arrow[urr] \arrow[r] \arrow[drr] & z \arrow[ur] \arrow[dr] & \\ & & y \end{tikzcd}$$

② $u = \frac{q}{\sqrt{x^2+y^2+z^2}}$

$\frac{\partial u}{\partial x} = -q(x^2+y^2+z^2)^{-\frac{3}{2}} x$

$\frac{\partial u}{\partial y} = -q(x^2+y^2+z^2)^{-\frac{3}{2}} 2y$

$\frac{\partial u}{\partial z} = -q(x^2+y^2+z^2)^{-\frac{3}{2}} 2z$

梯度 $-q(x^2+y^2+z^2)^{-\frac{3}{2}}(x, y, z)$

$\frac{\partial u}{\partial r}$ =负梯度的~~模~~

$\frac{\partial u}{\partial x} = \left(\frac{q}{r}\right)' \frac{\partial r}{\partial x}$

$= -\frac{q}{r^2}\frac{x}{r}$

$= -\frac{qx}{r^3}$

$\nabla u =$

$-\frac{q}{r^3}(x, y, z)$

$= -\frac{q}{r^3}\vec{r}$

$\frac{\partial u}{\partial \vec{r}} = \nabla u \cdot \frac{\vec{r}}{r}$ 单位化

$= -\frac{q}{r^3}\vec{r} \cdot \frac{\vec{r}}{r}$

$= -\frac{q}{r^2}$

$\frac{\partial^2 u}{\partial x^2} = -\frac{q}{r^3} + \frac{3q}{r^4}\frac{x}{r}x$

$= -\frac{q}{r^3} + \frac{3qx^2}{r^5}$

$\frac{\partial u^2}{\partial x^2} + \frac{\partial u^2}{\partial y^2} + \frac{\partial u^2}{\partial z^2} = -\frac{3q}{r^3} + \frac{3q^2}{r^5}(x^2+y^2+z^2)$

$= -\frac{3q}{r^3} + \frac{3q^2}{r^3} = 0$

9.2.5 向量值函数的微商和微分

设有单变量向量值函数(参数曲线)

$$t \mapsto \boldsymbol{r}(t), \quad t \in [\alpha, \beta]$$

它在t_0处的**微商**定义为

$$\boldsymbol{r}'(t_0) = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}(t_0) = \lim_{\Delta t \to 0} \frac{\boldsymbol{r}(t_0 + \Delta t) - \boldsymbol{r}(t_0)}{\Delta t}$$

- **几何意义** $\boldsymbol{r}'(t_0)$是曲线在参数为的t_0点处**切向量**,

  且指向参数增加方向.

➢ 坐标形式 若

$$\boldsymbol{r}(t)=x(t)\boldsymbol{i}+y(t)\boldsymbol{j}+z(t)\boldsymbol{k}$$

则有

$$\boldsymbol{r}'(t)=x'(t)\boldsymbol{i}+y'(t)\boldsymbol{j}+z'(t)\boldsymbol{k}$$

$$\frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2}(t)=\frac{\mathrm{d}^2x}{\mathrm{d}t^2}(t)\boldsymbol{i}+\frac{\mathrm{d}^2y}{\mathrm{d}t^2}(t)\boldsymbol{j}+\frac{\mathrm{d}^2z}{\mathrm{d}t^2}(t)\boldsymbol{k}$$

其微分定义为 $\mathrm{d}\boldsymbol{r}(t)=\dfrac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\mathrm{d}t$，即

$$\mathrm{d}\boldsymbol{r}=\mathrm{d}x\boldsymbol{i}+\mathrm{d}y\boldsymbol{j}+\mathrm{d}z\boldsymbol{k}$$

■ 运算性质 设$\boldsymbol{a}(t)$, $\boldsymbol{b}(t)$是向量函数, $f(t)$是数量函数, 则

$$1^\circ \quad \frac{\mathrm{d}}{\mathrm{d}t}(f\boldsymbol{a}) = f\frac{\mathrm{d}\boldsymbol{a}}{\mathrm{d}t} + \frac{\mathrm{d}f}{\mathrm{d}t}\boldsymbol{a}$$

乘积求导

$$2^\circ \quad \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{a}\cdot\boldsymbol{b}) = \frac{\mathrm{d}\boldsymbol{a}}{\mathrm{d}t}\cdot\boldsymbol{b} + \boldsymbol{a}\cdot\frac{\mathrm{d}\boldsymbol{b}}{\mathrm{d}t}$$

$$3^\circ \quad \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{a}\times\boldsymbol{b}) = \frac{\mathrm{d}\boldsymbol{a}}{\mathrm{d}t}\times\boldsymbol{b} + \boldsymbol{a}\times\frac{\mathrm{d}\boldsymbol{b}}{\mathrm{d}t}$$

例 设$\boldsymbol{a}(t)$是向量函数, 且$|\boldsymbol{a}(t)| = c$(常数). 证明

$$\boldsymbol{a}\cdot\frac{\mathrm{d}\boldsymbol{a}}{\mathrm{d}t} = 0$$

$$\frac{d(\vec{a}\times\vec{b})}{dt} = \frac{d}{dt}\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1' & a_2' & a_3' \\ b_1 & b_2 & b_3 \end{vmatrix} + \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1' & b_2' & b_3' \end{vmatrix}$$

$$= \frac{d\vec{a}}{dt}\times\vec{b} + \vec{a}\times\frac{d\vec{b}}{dt}$$

例 $\vec{a}\cdot\vec{a} = C^2$

求導 $\frac{d(\vec{a}\cdot\vec{a})}{dt} = 2\frac{d\vec{a}}{dt}\cdot\vec{a}$

$0 =$

设有两变量向量值函数(参数曲面)

$$\boldsymbol{r}(u,v)=x(u,v)\boldsymbol{i}+y(u,v)\boldsymbol{j}+z(u,v)\boldsymbol{k}\quad (u,v)\in D$$

它的**偏微商**定义为

$$\frac{\partial \boldsymbol{r}}{\partial u}=\frac{\partial x}{\partial u}\boldsymbol{i}+\frac{\partial y}{\partial u}\boldsymbol{j}+\frac{\partial z}{\partial u}\boldsymbol{k}$$

$$\frac{\partial \boldsymbol{r}}{\partial v}=\frac{\partial x}{\partial v}\boldsymbol{i}+\frac{\partial y}{\partial v}\boldsymbol{j}+\frac{\partial z}{\partial v}\boldsymbol{k}$$

其**微分**定义为

$$dx(u,v)\vec{i} = \frac{\partial x}{\partial u}du\vec{i} + \frac{\partial x}{\partial v}dv\vec{i}$$

$$\mathrm{d}\boldsymbol{r}(u,v) = \mathrm{d}x(u,v)\boldsymbol{i} + \mathrm{d}y(u,v)\boldsymbol{j} + \mathrm{d}z(u,v)\boldsymbol{k}$$

$$= \frac{\partial \boldsymbol{r}}{\partial u}\mathrm{d}u + \frac{\partial \boldsymbol{r}}{\partial v}\mathrm{d}v$$

■ 向量值函数的微分

m维n元

设有向量函数 $\boldsymbol{f}: \mathbf{R}^n \to \mathbf{R}^m$, 分量形式为

$$\boldsymbol{y} = (y_1, y_2, \cdots, y_m) = (f_1(\boldsymbol{x}), f_2(\boldsymbol{x}), \cdots, f_m(\boldsymbol{x}))$$

其中 $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$, $\boldsymbol{f}$ 的**微分**定义为

$$\mathrm{d}\boldsymbol{f}(\boldsymbol{x}) = (\mathrm{d}f_1(\boldsymbol{x}), \mathrm{d}f_2(\boldsymbol{x}), \cdots, \mathrm{d}f_m(\boldsymbol{x}))$$

$$df_j = \left(\frac{\partial f_j}{\partial x_1} \quad \frac{\partial f_j}{\partial x_2} \cdots \right) \begin{pmatrix} dx_1 \\ dx_2 \\ \vdots \\ dx_n \end{pmatrix}$$

由于 $$\mathrm{d}f_j = \sum_{i=1}^{n} \frac{\partial f_j}{\partial x_i} \mathrm{d}x_i$$

导出

$$\begin{pmatrix} \mathrm{d}f_1 \\ \vdots \\ \mathrm{d}f_m \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix} \begin{pmatrix} \mathrm{d}x_1 \\ \vdots \\ \mathrm{d}x_n \end{pmatrix}$$

或 $$\mathrm{d}\boldsymbol{f}(\boldsymbol{x}) = \frac{\partial \boldsymbol{f}}{\partial x_1} \mathrm{d}x_1 + \frac{\partial \boldsymbol{f}}{\partial x_2} \mathrm{d}x_2 + \cdots + \frac{\partial \boldsymbol{f}}{\partial x_n} \mathrm{d}x_n$$

记 $\boldsymbol{f}$ 的**Jacobi矩阵**为

$$\boldsymbol{J}_{\boldsymbol{x}}(\boldsymbol{f})=\begin{pmatrix} \dfrac{\partial y_1}{\partial x_1} & \cdots & \dfrac{\partial y_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial y_m}{\partial x_1} & \cdots & \dfrac{\partial y_m}{\partial x_n} \end{pmatrix}$$

当$m=n$时, 该方阵的行列式

$$\det \boldsymbol{J}_{\boldsymbol{x}}(\boldsymbol{f})=\frac{\partial(y_1,y_2,\cdots,y_n)}{\partial(x_1,x_2,\cdots,x_n)}$$

称为 $\boldsymbol{f}$ 的**Jacobi行列式.**

例 $\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} \sin t + \cos s \\ st \\ e^s t \end{pmatrix}$ $y = f(s,t)$

$$J(f) = \begin{pmatrix} -\sin s, & \cos t \\ t, & s \\ e^s t, & e^s \end{pmatrix}$$

➢ 向量值复合函数链法则

$$\boldsymbol{f}(\xi,\zeta)=\begin{pmatrix} f_1(\xi,\zeta) \\ f_2(\xi,\zeta) \end{pmatrix}, \qquad \boldsymbol{g}(x,y)=\begin{pmatrix} \xi(x,y) \\ \zeta(x,y) \end{pmatrix}$$

可微, 则 $\boldsymbol{f}\circ\boldsymbol{g}=\begin{pmatrix} u_1(x,y) \\ u_2(x,y) \end{pmatrix}$ 的**Jacobi矩阵**

$$\boldsymbol{J}(\boldsymbol{f}\circ\boldsymbol{g})=\begin{pmatrix} \dfrac{\partial u_1}{\partial x} & \dfrac{\partial u_1}{\partial y} \\ \dfrac{\partial u_2}{\partial x} & \dfrac{\partial u_2}{\partial y} \end{pmatrix}=\begin{pmatrix} \dfrac{\partial f_1}{\partial \xi} & \dfrac{\partial f_1}{\partial \zeta} \\ \dfrac{\partial f_2}{\partial \xi} & \dfrac{\partial f_2}{\partial \zeta} \end{pmatrix}\begin{pmatrix} \dfrac{\partial \xi}{\partial x} & \dfrac{\partial \xi}{\partial y} \\ \dfrac{\partial \zeta}{\partial x} & \dfrac{\partial \zeta}{\partial y} \end{pmatrix}=\boldsymbol{J}(\boldsymbol{f})\cdot\boldsymbol{J}(\boldsymbol{g})$$

△

➢ **想一想** $\boldsymbol{f}$ 为m维k元, $\boldsymbol{g}$为k维n元向量值函数的情形