Chap 9 — 2

多变量函数的微分

9.2.1 多变量函数的偏微商

定义 设 $f(x,y)$在(x_0,y_0)某邻域有定义. 仅给x以增量Δx

相应有函数的增量(对x**偏增量**)

$$\Delta_x z = f(x_0+\Delta x, y_0) - f(x_0, y_0)$$

函数 f 在点(x_0,y_0)处对x的**偏微商(或偏导数)**

$$f'_x(x_0,y_0) \overset{\text{def}}{=} \lim_{\Delta x\to 0}\frac{\Delta_x z}{\Delta x} = \lim_{\Delta x\to 0}\frac{f(x_0+\Delta x, y_0) - f(x_0, y_0)}{\Delta x}$$

◆ 偏微商也可记为 $\dfrac{\partial f}{\partial x}(x_0, y_0)$

◆ 对变量y的偏微商类似

◆ **可偏导**: 两个偏导数都存在.

◆ **偏导(函)数**: $f_x'(x,y),\ f_y'(x,y)$ or $\dfrac{\partial f}{\partial x}(x,y), \dfrac{\partial f}{\partial y}(x,y)$

例 求函数$u = x^y\ (x > 0)$的偏导数.

■ 连续与可偏导

- 可偏导未必连续

例 考察 $f(x,y)=\begin{cases}\dfrac{xy}{x^2+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$

在(0,0)的情况.

- 连续未必可偏导

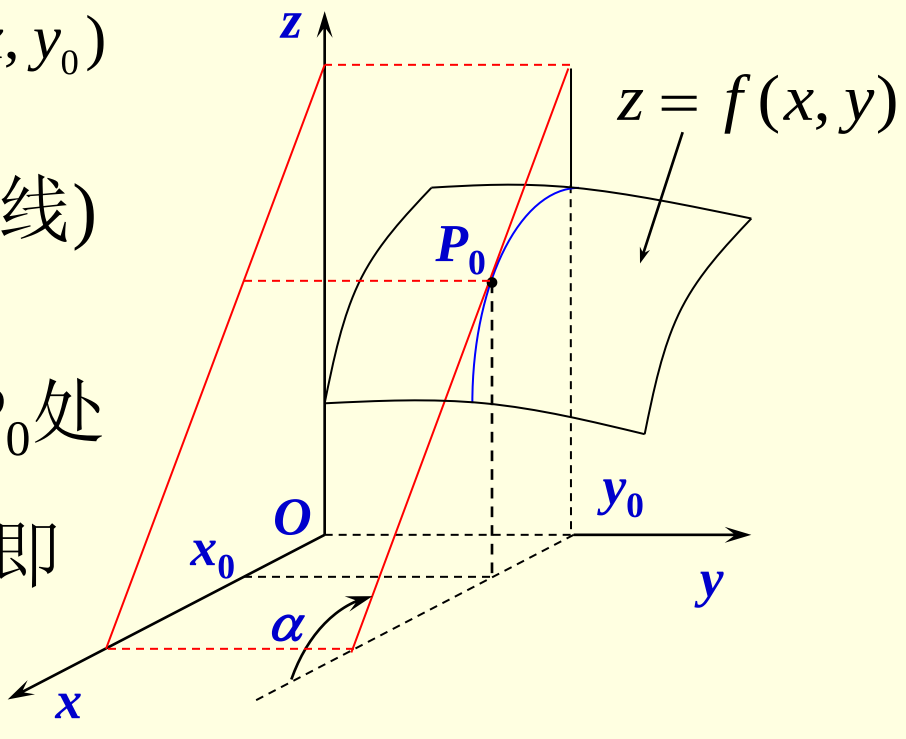
例 考察 $f(x,y) = |x| + |y|$ 在(0,0)的情况.

■ 偏导数的几何意义

曲面$z=f(x,y)$与平面$y=y_0$的交线

$$\begin{cases} z=f(x,y) \\ y=y_0 \end{cases} \Rightarrow \quad z=f(x,y_0)$$

(平面$y=y_0$上的曲线)



$f'_x(x_0,y_0)$是该曲线在P_0处

的切线关于x轴的**斜率**. 即

$$\boxed{f'_x(x_0,y_0)=\tan\alpha}$$

定义 $f(x,y)$在某邻域内的偏导数$f'_x(x,y)$, $f'_y(x,y)$的偏导数称为 f 的**二阶偏导数**. 记为

$$f''_{xx}=\frac{\partial^2 f}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right),\quad f''_{xy}=\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

$$f''_{yx}=\frac{\partial^2 f}{\partial y\partial x}=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right),\quad f''_{yy}=\frac{\partial^2 f}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

类似可定义三阶偏导数, 例如

$$f'''_{xxy}=\frac{\partial^3 f}{\partial x^2\partial y}=\frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial x^2}\right)$$

例 求函数 $z=\ln x+\mathrm{e}^y\sin x$ 的所有二阶偏导数.

问题: 混合偏导数是否总与求偏导次序无关?

例 设 $f(x,y)=\begin{cases} xy, & |x|\geq|y| \\ -xy, & |x|<|y| \end{cases}$, 求 $f''_{xy}(0,0), f''_{yx}(0,0)$.

分析

$$f''_{xy}(0,0)=\lim_{y\to 0}\frac{f'_x(0,y)-f'_x(0,0)}{y}$$

$$(y\neq 0)\qquad f'_x(0,y)=\lim_{x\to 0}\frac{f(x,y)-f(0,y)}{x}$$

$$f'_x(0,0)=\lim_{x\to 0}\frac{f(x,0)-f(0,0)}{x}$$

定理 若$f(x,y)$的二阶混合偏导数在(x,y)连续, 则

$$f''_{xy}(x,y)=f''_{yx}(x,y)$$

9.2.2 多变量函数的可微性

一元情形：若 $\Delta f = a\Delta x + o(\Delta x)$,则称 f 在x_0可微,

并把$a\Delta x$称为 f 在x_0处的微分, 记为 $\mathrm{d}f\big|_{x=x_0} = a\Delta x$

二元情形：对函数$z = f(x, y)$, 若**增量**

$$\Delta z = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)$$

$$= a\Delta x + b\Delta y + o(\rho)$$

其中a, b是常数, $\rho = \sqrt{(\Delta x)^2 + (\Delta y)^2}$, 则称 f 在(x_0,y_0)**可微**.

并把$a\Delta x + b\Delta y$称为 f 在(x_0,y_0)处的**微分**. 记为

$$\mathrm{d}f\big|_{(x_0, y_0)} = a\Delta x + b\Delta y$$

若 f 在区域 D 内处处可微, 则称 f 是 D 内的**可微函数**.

- **可微必连续**
- **可微必可偏导**, 且若

$$\left.\mathrm{d}f\right|_{(x_0,y_0)} = a\Delta x + b\Delta y$$
$$\Rightarrow \quad f_x'(x_0,y_0) = a,\ f_y'(x_0,y_0) = b$$

■ 微分公式

$$\boxed{\mathrm{d}f(x,y) = f_x'(x,y)\mathrm{d}x + f_y'(x,y)\mathrm{d}y}$$

例 求函数 $z = x^y$ 在点(1,1)处的微分.

例 求函数 $z = \arctan\dfrac{y}{x}$ 的微分.

定理 设$f(x, y)$在区域D内存在偏导数, 则

(1) 若$f'_x(x, y), f'_y(x, y)$在D内有界, 则f在D内连续;

(2) 若$f'_x(x, y), f'_y(x, y)$在D内连续, 则f在D内可微.

结论

$$\text{偏导数连续} \Rightarrow \text{可微} \Rightarrow \begin{cases} \text{连续} \\ \text{可偏导} \end{cases}$$

9.2.3 方向导数与梯度

定义 设 $\boldsymbol{e}=(\cos\alpha,\cos\beta)$，函数$z=f(x,y)$在$(x_0,y_0)$处沿 $\boldsymbol{e}$ 的**方向导数**定义为

$$\frac{\partial f}{\partial \boldsymbol{e}}(x_0,y_0)=\lim_{t\to 0}\frac{f(x_0+t\cos\alpha,y_0+t\cos\beta)-f(x_0,y_0)}{t}$$

- 偏导数是方向导数的特例,

$$\frac{\partial f}{\partial x}(x_0,y_0)=\frac{\partial f}{\partial \boldsymbol{e}}(x_0,y_0),\quad \boldsymbol{e}=(1,0)$$

定理 $z=f(x,y)$在$M_0(x_0,y_0)$可微， $\boldsymbol{e}=(\cos\alpha,\cos\beta)$

则f在M_0点存在方向导数, 且

$$\boxed{\frac{\partial f}{\partial \boldsymbol{e}}(x_0,y_0)=f_x'(x_0,y_0)\cos\alpha+f_y'(x_0,y_0)\cos\beta}$$

结论

● 可微 $\Rightarrow$ 方向导数存在 $\Rightarrow$ 可偏导

方向导数存在 $\not\Rightarrow$ 极限存在

方向导数存在 $\not\Rightarrow$ 连续

例 求 $r(x,y)=\sqrt{x^2+y^2}$ 沿$\boldsymbol{e}=(\cos\alpha,\cos\beta)$的方向导数

定义 函数 $f(x, y)$在点$M_0(x_0,y_0)$的**梯度**定义为

$$\operatorname{grad} f(x_0, y_0) \overset{\text{def}}{=} (f_x'(x_0, y_0), f_y'(x_0, y_0))$$

利用梯度符号, 得到

$$\frac{\partial f}{\partial \boldsymbol{e}}(M_0) = \operatorname{grad} f(M_0) \cdot \boldsymbol{e} = |\operatorname{grad} f(M_0)| \cos\theta$$

$\Rightarrow \theta = 0$时, 方向导数 $\dfrac{\partial f}{\partial \boldsymbol{e}}(M_0)$ 取最大值 $|\operatorname{grad} f(M_0)|$

结论 梯度的方向是方向导数取最大值时的方向, 其模就是方向导数的最大值.

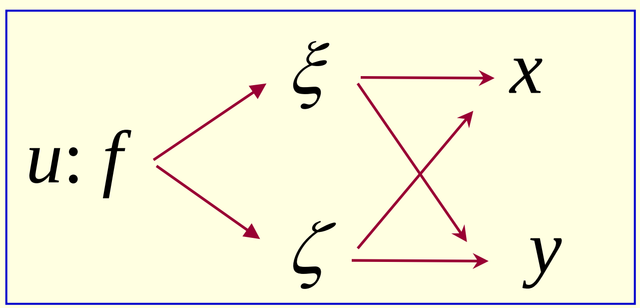
9.2.4 复合函数的微分

定理 设 $u=f(\xi,\zeta)$可微, $\xi=\xi(x,y)$, $\zeta=\zeta(x,y)$可微.

则复合函数 $u=f(\xi(x,y),\zeta(x,y))$ 也可微, 且

$$\frac{\partial u}{\partial x}=\frac{\partial f}{\partial \xi}\cdot\frac{\partial \xi}{\partial x}+\frac{\partial f}{\partial \zeta}\cdot\frac{\partial \zeta}{\partial x},\quad \frac{\partial u}{\partial y}=\frac{\partial f}{\partial \xi}\cdot\frac{\partial \xi}{\partial y}+\frac{\partial f}{\partial \zeta}\cdot\frac{\partial \zeta}{\partial y}$$

- **链法则**



想一想 m个中间变量n个自变量的链法则?

■ 一阶微分形式的不变性

函数 $u = f(\xi, \zeta)$的微分

$$\mathrm{d}u = \frac{\partial f}{\partial \xi}\mathrm{d}\xi + \frac{\partial f}{\partial \zeta}\mathrm{d}\zeta$$

若ξ, ζ又是x, y的可微函数$\xi = \xi(x, y), \zeta = \zeta(x, y)$, 则

复合函数 $u = f(\xi(x, y), \zeta(x, y))$ 的微分

$$\mathrm{d}u = \frac{\partial u}{\partial x}\mathrm{d}x + \frac{\partial u}{\partial y}\mathrm{d}y$$

$$= \left(\frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial f}{\partial \zeta}\frac{\partial \zeta}{\partial x}\right)\mathrm{d}x + \left(\frac{\partial f}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial f}{\partial \zeta}\frac{\partial \zeta}{\partial y}\right)\mathrm{d}y$$

$$= \frac{\partial f}{\partial \xi}\left(\frac{\partial \xi}{\partial x}\mathrm{d}x + \frac{\partial \xi}{\partial y}\mathrm{d}y\right) + \frac{\partial f}{\partial \zeta}\left(\frac{\partial \zeta}{\partial x}\mathrm{d}x + \frac{\partial \zeta}{\partial y}\mathrm{d}y\right)$$

注意到 $\mathrm{d}\xi = \frac{\partial \xi}{\partial x}\mathrm{d}x + \frac{\partial \xi}{\partial y}\mathrm{d}y, \quad \mathrm{d}\zeta = \frac{\partial \zeta}{\partial x}\mathrm{d}x + \frac{\partial \zeta}{\partial y}\mathrm{d}y$

从而
$$\mathrm{d}u = \frac{\partial f}{\partial \xi}\mathrm{d}\xi + \frac{\partial f}{\partial \zeta}\mathrm{d}\zeta$$

因此, 对于函数$u = f(\xi, \zeta)$, 无论ξ, ζ是自变量还是函数, 都有
$$\mathrm{d}u = \frac{\partial f}{\partial \xi}\mathrm{d}\xi + \frac{\partial f}{\partial \zeta}\mathrm{d}\zeta$$

想一想 n**元函数的一阶微分形式不变性?**

例 设函数$u = f(x, y, z)$可微, 而$z = z(x, y)$可偏导.

求复合函数$u = f(x, y, z(x, y))$对x的偏导数.

例 原点处电荷q产生电势 $u = q/r$, 其中r 是点

$\boldsymbol{r} = (x, y, z)$到原点的距离. 当$r \neq 0$时, 求$u$在$(x, y, z)$

处的梯度及沿方向$\boldsymbol{r}$的变化率, 并证明u满足方程

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \longleftarrow \textbf{Laplace方程}$$

9.2.5 向量值函数的微商和微分

设有单变量向量值函数(参数曲线)

$$t \mapsto \boldsymbol{r}(t), \quad t \in [\alpha, \beta]$$

它在t_0处的**微商**定义为

$$\boldsymbol{r}'(t_0) = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}(t_0) = \lim_{\Delta t \to 0} \frac{\boldsymbol{r}(t_0 + \Delta t) - \boldsymbol{r}(t_0)}{\Delta t}$$

- **几何意义** $\boldsymbol{r}'(t_0)$是曲线在参数为的t_0点处**切向量**,

  且指向参数增加方向.

➢ 坐标形式 若

$$\boldsymbol{r}(t)=x(t)\boldsymbol{i}+y(t)\boldsymbol{j}+z(t)\boldsymbol{k}$$

则有

$$\boldsymbol{r}'(t)=x'(t)\boldsymbol{i}+y'(t)\boldsymbol{j}+z'(t)\boldsymbol{k}$$

$$\frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2}(t)=\frac{\mathrm{d}^2x}{\mathrm{d}t^2}(t)\boldsymbol{i}+\frac{\mathrm{d}^2y}{\mathrm{d}t^2}(t)\boldsymbol{j}+\frac{\mathrm{d}^2z}{\mathrm{d}t^2}(t)\boldsymbol{k}$$

其**微分**定义为 $\mathrm{d}\boldsymbol{r}(t)=\dfrac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\mathrm{d}t$，即

$$\mathrm{d}\boldsymbol{r}=\mathrm{d}x\boldsymbol{i}+\mathrm{d}y\boldsymbol{j}+\mathrm{d}z\boldsymbol{k}$$

■ **运算性质** 设$\boldsymbol{a}(t)$, $\boldsymbol{b}(t)$是向量函数, $f(t)$是数量函数, 则

$$1^{\circ}\quad \frac{\mathrm{d}}{\mathrm{d}t}(f\boldsymbol{a}) = f\frac{\mathrm{d}\boldsymbol{a}}{\mathrm{d}t} + \frac{\mathrm{d}f}{\mathrm{d}t}\boldsymbol{a}$$

$$2^{\circ}\quad \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{a}\cdot\boldsymbol{b}) = \frac{\mathrm{d}\boldsymbol{a}}{\mathrm{d}t}\cdot\boldsymbol{b} + \boldsymbol{a}\cdot\frac{\mathrm{d}\boldsymbol{b}}{\mathrm{d}t}$$

$$3^{\circ}\quad \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{a}\times\boldsymbol{b}) = \frac{\mathrm{d}\boldsymbol{a}}{\mathrm{d}t}\times\boldsymbol{b} + \boldsymbol{a}\times\frac{\mathrm{d}\boldsymbol{b}}{\mathrm{d}t}$$

例 设$\boldsymbol{a}(t)$是向量函数, 且$|\boldsymbol{a}(t)| = c$(常数). 证明

$$\boldsymbol{a}\cdot\frac{\mathrm{d}\boldsymbol{a}}{\mathrm{d}t} = 0$$

设有两变量向量值函数(参数曲面)

$$\boldsymbol{r}(u,v)=x(u,v)\boldsymbol{i}+y(u,v)\boldsymbol{j}+z(u,v)\boldsymbol{k}\quad (u,v)\in D$$

它的**偏微商**定义为

$$\frac{\partial \boldsymbol{r}}{\partial u}=\frac{\partial x}{\partial u}\boldsymbol{i}+\frac{\partial y}{\partial u}\boldsymbol{j}+\frac{\partial z}{\partial u}\boldsymbol{k}$$

$$\frac{\partial \boldsymbol{r}}{\partial v}=\frac{\partial x}{\partial v}\boldsymbol{i}+\frac{\partial y}{\partial v}\boldsymbol{j}+\frac{\partial z}{\partial v}\boldsymbol{k}$$

其**微分**定义为

$$\mathrm{d}\boldsymbol{r}(u,v)=\mathrm{d}x(u,v)\boldsymbol{i}+\mathrm{d}y(u,v)\boldsymbol{j}+\mathrm{d}z(u,v)\boldsymbol{k}$$

$$=\frac{\partial \boldsymbol{r}}{\partial u}\mathrm{d}u+\frac{\partial \boldsymbol{r}}{\partial v}\mathrm{d}v$$

■ 向量值函数的微分

设有向量函数 $\boldsymbol{f}:\mathbf{R}^n \to \mathbf{R}^m$, 分量形式为

$$\boldsymbol{y}=(y_1,y_2,\cdots,y_m)=(f_1(\boldsymbol{x}),f_2(\boldsymbol{x}),\cdots,f_m(\boldsymbol{x}))$$

其中 $\boldsymbol{x}=(x_1, x_2, \ldots, x_n)$, $\boldsymbol{f}$ 的**微分**定义为

$$\mathrm{d}\boldsymbol{f}(\boldsymbol{x})=(\mathrm{d}f_1(\boldsymbol{x}),\mathrm{d}f_2(\boldsymbol{x}),\cdots,\mathrm{d}f_m(\boldsymbol{x}))$$

由于 $$\mathrm{d}f_j=\sum_{i=1}^{n}\frac{\partial f_j}{\partial x_i}\mathrm{d}x_i$$

导出

$$\begin{pmatrix}\mathrm{d}f_1\\ \vdots\\ \mathrm{d}f_m\end{pmatrix}=\begin{pmatrix}\dfrac{\partial f_1}{\partial x_1} & \cdots & \dfrac{\partial f_1}{\partial x_n}\\ \vdots & \ddots & \vdots\\ \dfrac{\partial f_m}{\partial x_1} & \cdots & \dfrac{\partial f_m}{\partial x_n}\end{pmatrix}\begin{pmatrix}\mathrm{d}x_1\\ \vdots\\ \mathrm{d}x_n\end{pmatrix}$$

或 $$\mathrm{d}\boldsymbol{f}(\boldsymbol{x})=\frac{\partial \boldsymbol{f}}{\partial x_1}\mathrm{d}x_1+\frac{\partial \boldsymbol{f}}{\partial x_2}\mathrm{d}x_2+\cdots+\frac{\partial \boldsymbol{f}}{\partial x_n}\mathrm{d}x_n$$

记 f 的**Jacobi矩阵**为

$$J_x(f)=\begin{pmatrix} \dfrac{\partial y_1}{\partial x_1} & \cdots & \dfrac{\partial y_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial y_m}{\partial x_1} & \cdots & \dfrac{\partial y_m}{\partial x_n} \end{pmatrix}$$

当$m=n$时, 该方阵的行列式

$$\det J_x(f)=\frac{\partial(y_1,y_2,\cdots,y_n)}{\partial(x_1,x_2,\cdots,x_n)}$$

称为 f 的**Jacobi行列式.**

➢ 向量值复合函数链法则

$$\boldsymbol{f}(\xi,\zeta)=\begin{pmatrix} f_1(\xi,\zeta) \\ f_2(\xi,\zeta) \end{pmatrix}, \qquad \boldsymbol{g}(x,y)=\begin{pmatrix} \xi(x,y) \\ \zeta(x,y) \end{pmatrix}$$

可微, 则 $\boldsymbol{f}\circ\boldsymbol{g}=\begin{pmatrix} u_1(x,y) \\ u_2(x,y) \end{pmatrix}$ 的**Jacobi矩阵**

$$\boldsymbol{J}(\boldsymbol{f}\circ\boldsymbol{g})=\begin{pmatrix} \frac{\partial u_1}{\partial x} & \frac{\partial u_1}{\partial y} \\ \frac{\partial u_2}{\partial x} & \frac{\partial u_2}{\partial y} \end{pmatrix}=\begin{pmatrix} \frac{\partial f_1}{\partial \xi} & \frac{\partial f_1}{\partial \zeta} \\ \frac{\partial f_2}{\partial \xi} & \frac{\partial f_2}{\partial \zeta} \end{pmatrix}\begin{pmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y} \\ \frac{\partial \zeta}{\partial x} & \frac{\partial \zeta}{\partial y} \end{pmatrix}=\boldsymbol{J}(\boldsymbol{f})\cdot\boldsymbol{J}(\boldsymbol{g})$$

➢ **想一想** $\boldsymbol{f}$ 为m维k元, $\boldsymbol{g}$为k维n元向量值函数的情形