Chap 11 — 7

保守场与无源场

11.7.1 空间曲线积分与路径无关的条件

■ 空间区域的连通性

定义 设V为空间区域, 若V中的任意闭曲线都可在V中连续收缩为一点, 则称V为**一维(曲面)单连通.**

若V中的任意闭曲面可在V中连续收缩为一点, 则称V为**二维(空间)单连通.**

定理 设函数$P(x, y, z)$, $Q(x, y, z)$, $R(x, y, z)$在**一维单连通**区域V内有连续偏导数, 则下面四条等价:

(1) 在V内的任一条分段光滑闭曲线L上

$$\oint_L P\mathrm{d}x + Q\mathrm{d}y + R\mathrm{d}z = 0$$

(2) 在V内曲线积分 $\int_L P\mathrm{d}x + Q\mathrm{d}y + R\mathrm{d}z$ 与路径无关

(3) $P\mathrm{d}x + Q\mathrm{d}y + R\mathrm{d}z$是某三元函数$\varphi$的**全微分**, 即

$$\mathrm{d}\varphi = P\mathrm{d}x + Q\mathrm{d}y + R\mathrm{d}z$$

(4) 在V内恒成立 $\dfrac{\partial R}{\partial y} = \dfrac{\partial Q}{\partial z}$, $\dfrac{\partial P}{\partial z} = \dfrac{\partial R}{\partial x}$, $\dfrac{\partial Q}{\partial x} = \dfrac{\partial P}{\partial y}$ 无旋场

(3)⇒(4) 设 $\vec{v}=(P,Q,R)=\nabla\varphi$ 即

$$\frac{\partial\varphi}{\partial x}=P \quad \frac{\partial\varphi}{\partial y}=Q \quad \frac{\partial\varphi}{\partial z}=R$$

$$\operatorname{rot}\vec{v}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ P & Q & R\end{vmatrix}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ \frac{\partial\varphi}{\partial x} & \frac{\partial\varphi}{\partial y} & \frac{\partial\varphi}{\partial z}\end{vmatrix}=\left(\frac{\partial\varphi}{\partial y\partial z}-\frac{\partial\varphi}{\partial z\partial y}\right)\vec{i}+0\vec{j}+0\vec{k}=\vec{0}$$

二阶系数对应(4)中二阶等式

■ 保守场与势函数

定义 设 $\boldsymbol{v}=(P,Q,R)$是连通区域V内光滑向量场, 若

(1) 在V内曲线积分 $\int_L P\mathrm{d}x+Q\mathrm{d}y+R\mathrm{d}z$ 与路径无关

则称$\boldsymbol{v}$为V中的**保守场**.

(2) $\mathrm{d}\varphi = P\mathrm{d}x+Q\mathrm{d}y+R\mathrm{d}z$, 则称$\varphi$是$\boldsymbol{v}$的**势函数**, $\boldsymbol{v}$是φ的**梯度场**, 即$\boldsymbol{v}=\nabla\varphi$.

(3) 在V内恒有 $\mathrm{rot}\,\boldsymbol{v}=\boldsymbol{0}$, 则称$\boldsymbol{v}$为$V$中的**无旋场**.

定理 设$\boldsymbol{v}=(P,Q,R)$是一维单连通区域V内的光滑向量场, 则下面三条等价:

(1) $\boldsymbol{v}$为V中的保守场;

(2) $\boldsymbol{v}$为V中有势场(存在势函数); $\nabla\varphi$

(3) $\boldsymbol{v}$为V中无旋场. $\nabla\times\nabla\varphi=\vec{0}$

推论 设$\boldsymbol{v}$为V中的保守场, φ为$\boldsymbol{v}$的势函数, 则有

$$\int_A^B \boldsymbol{v}\cdot \mathrm{d}\boldsymbol{r}=\varphi\Big|_A^B=\varphi(B)-\varphi(A)$$

例1 证明向量场

$$\boldsymbol{v}=(x^2-yz)\boldsymbol{i}+(y^2-zx)\boldsymbol{j}+(z^2-xy)\boldsymbol{k}$$

为有势场, 并求出其一个势函数.

$$\mathrm{rot}\,\vec{v}=\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2-yz & y^2-zx & z^2-xy \end{vmatrix}=\vec{0}$$

$\vec{v}$ 为有势场

$$\varphi(x,y,z)=\int_0^{(x,y,z)}(x^2-yz)dx+(y^2-zx)dy+(z^2-xy)dz$$

$$=\left(\int_0^{(x,0,0)}+\int_{(x,0,0)}^{(x,y,0)}+\int_{(x,y,0)}^{(x,y,z)}\right)\cdots$$

$$=\int_0^x x^2dx+\int_0^y(y^2-0x)dy+\int_0^z(z^2-xy)dz$$

$= \frac{1}{3}x^2 + \frac{1}{3}y^3 + \frac{1}{3}z^3 - xyz$

11.7.2 向量势与无源场

定义 设 $\boldsymbol{v}$ 为光滑向量场, $V\subset \mathbf{R}^3$. 若存在向量场 $\boldsymbol{\alpha}$

使得 $\boldsymbol{v} = \text{rot}\ \boldsymbol{\alpha} = \nabla\times\boldsymbol{\alpha}$, 则称 $\boldsymbol{\alpha}$ 为 $\boldsymbol{v}$ 的向量势.

例2 设向量场 $\boldsymbol{\alpha} = x^2\boldsymbol{i} - zx\boldsymbol{j} + xy^2\boldsymbol{k}$, 令 $\boldsymbol{v} = \text{rot}\ \boldsymbol{\alpha}$,

求 $\boldsymbol{v}$ 及其散度div $\boldsymbol{v}$.

定义 设 $\boldsymbol{v}$ 为向量场, 若其散度div $\boldsymbol{v} = \nabla\cdot\boldsymbol{v} = 0$, 则称

$\boldsymbol{v}$ 为无源场.

定理 若$\boldsymbol{v}$为光滑向量场，则$\boldsymbol{v}$ 存在向量势的充要条件是$\boldsymbol{v}$为无源场.

命题 若$\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ 均为光滑向量场$\boldsymbol{v}$ 的向量势，则$\boldsymbol{\alpha}$ 与$\boldsymbol{\beta}$ 至多相差一个梯度场.

$\mathrm{rot}\,\vec{\alpha} = \vec{v} = \mathrm{rot}\,\vec{\beta}$

$\mathrm{rot}(\vec{\alpha} - \vec{\beta}) = \vec{0}$ （无旋场）

故 $\vec{\alpha} - \vec{\beta}$ 为有势场，

$\exists \varphi$, s.t. $\vec{\alpha} - \vec{\beta} = \nabla \varphi$

定理

证 ⇒ 设 $\vec{v} = \mathrm{rot}\, \vec{\alpha}$ $\vec{\alpha} = (A, B, C)$ 即

$$\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A & B & C \end{vmatrix} = \left(\frac{\partial C}{\partial y} - \frac{\partial B}{\partial z}\right)\vec{i} + \left(\frac{\partial A}{\partial z} - \frac{\partial C}{\partial x}\right)\vec{j} + \left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y}\right)\vec{k}$$

$$\mathrm{div}\, \vec{v} = \left(\frac{\partial^2 C}{\partial y \partial x} - \frac{\partial^2 B}{\partial z \partial x} + \frac{\partial^2 A}{\partial z \partial y} - \frac{\partial^2 C}{\partial x \partial y} + \frac{\partial^2 B}{\partial x \partial z} - \frac{\partial^2 A}{\partial y \partial z}\right) = 0$$

"⇐" 设 $\vec{v} = (P, Q, R)$ 且 $\mathrm{div}\, \vec{v} = 0$. 下证 $\exists \vec{\alpha}$ s.t. $\vec{v} = \mathrm{rot}\, \vec{\alpha}$

即 $\frac{\partial C}{\partial y} - \frac{\partial B}{\partial z} = P$

$\frac{\partial A}{\partial z} - \frac{\partial C}{\partial x} = Q$

$\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y} = R$

找 ABC.

不妨 $C = 0$. $\frac{\partial B}{\partial z} = -P$ $\frac{\partial A}{\partial z} = Q$

$$A = \int_{z_0}^{z} Q\, dz$$

$$B = \int_{z_0}^{z} -P\, dz + f(x, y)$$

$$\Rightarrow \quad \frac{\partial B}{\partial x} = -\int_{z_0}^{z} \frac{\partial P}{\partial x} dz + \frac{\partial f}{\partial x} \qquad \frac{\partial A}{\partial y} = \int_{z_0}^{z} \frac{\partial Q}{\partial y} dz$$

$$\frac{\partial \int_a^b E\,dx}{\partial y} \xlongequal[\text{积分内求导}]{13章} \int_a^b \frac{\partial E}{\partial y} dx$$

$$\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y} = -\int_{z_0}^{z} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) dz + \frac{\partial f}{\partial x}$$

$$\xlongequal[\operatorname{div}\vec{v}=0]{\text{无源场}} -\int_{z_0}^{z} \frac{\partial R}{\partial z} dz + \frac{\partial f}{\partial x}$$

$$= R(x,y,z) - R(x,y,z_0) + \frac{\partial f}{\partial x}$$

从而 $\frac{\partial f}{\partial x} = R(x,y,z_0)$ $\quad f(x,y) = \int_{x_0}^{x} R(x,y,z_0)\,dx$

$$A = \int_{z_0}^{z} Q\,dz \qquad B = -\int_{z_0}^{z} P\,dz + \int_{x_0}^{x} R(x,y,z_0)\,dx$$

$$C = 0$$