Chap 9— 3

隐函数和逆映射定理

9.3.1 隐函数的存在性和微商

①

定理 设函数 F 在$M_0(x_0,y_0)$邻域内有连续偏导数, 且

3+3

$$F(x_0,y_0)=0,\quad F'_y(x_0,y_0)\neq 0$$

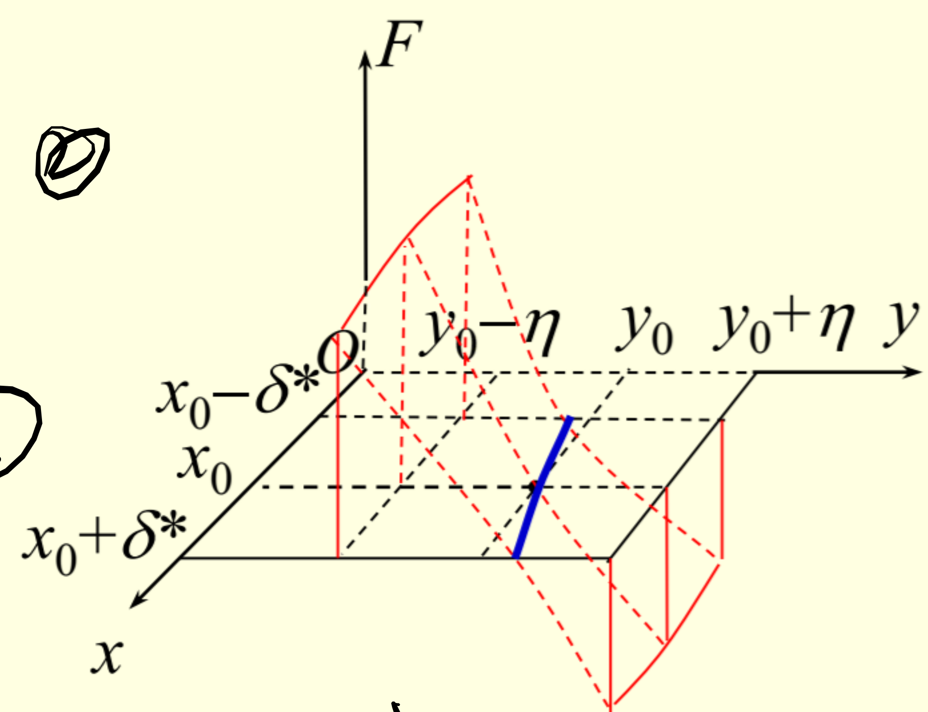
③ ②

则方程$F(x,y)=0$在M_0某邻域内存在**唯一连续可导**

①

隐函数$y=f(x)$, 满足

$$F(x,f(x))\equiv 0,\ y_0=f(x_0)$$ ②



和

$$\boxed{\frac{\mathrm{d}y}{\mathrm{d}x}=f'(x)=-\frac{F'_x(x,y)}{F'_y(x,y)}}$$ ③

充分不必要

证：先证（存在，唯一性）

由 $F_y'(x_0,y_0)\neq 0$ 不妨 <0. 且 F_y' 连续

据保号性 $\exists B(M_0)$ 邻域 使 $F_y'(x,y)<0$. 故 $F(x,y)$ 关于 y 严格 $\downarrow$

又 $F(x_0,y_0)=0$ 故 $\exists \eta>0$ s.t. $F(x_0,y_0-\eta)>0$

由条件，$F(x,y)$ 连续

$\exists \delta_1,\delta_2>0$ s.t. $\forall x\in U(x_0,\delta_1)$: $F(x,y_0-\eta)>0$

$\forall x\in U(x_0,\delta_2)$ $F(x,y_0+\eta)<0$

let $\delta^*=\min\{\delta_1,\delta_2\}$ $\forall x\in U(x_0,\delta^*)$ 有 $F(x,y_0-\eta)>0$, $F(x,y_0+\eta)<0$

零值 Th. $\Longrightarrow$ $\forall x\in U(x_0,\delta^*)$ $\exists$ unique $y=f(x)\in(y_0-\eta,y_0+\eta)$

s.t. $F(x,f(x))=0$

再证 $y=f(x)$ 在 $U(x_0,\delta^*)$ 连续

$\forall$ 固定 $\overline{x}\in U(x_0,\delta)$ 由 $F(\overline{x},f(\overline{x}))=0$ 故

$\forall\varepsilon>0$ 　　则 $F(\overline{x},f(\overline{x})-\varepsilon)>0$.

$F(\overline{x},f(\overline{x})+\varepsilon)<0$.

从而 $\exists\delta>0$, $\forall x\in U(\overline{x},\delta)$ 有 $F(x,f(\overline{x})-\varepsilon)>0$; $F(x,f(\overline{x})+\varepsilon)<0$

零值 $\Rightarrow \exists y \in (f(\bar{x})-\varepsilon, f(\bar{x}+\varepsilon))$ s.t. $F(x,y)=0$.

据唯一性 $y=f(x)$ 从而

$$|f(x)-f(\bar{x})|<\varepsilon$$

最后 可导 $x \in U(x_0, \delta^*)$ 给增量 设 $\Delta y = f(x+\Delta x)-f(x)$

则 $F(x,y)=0 \quad F(x+\Delta x, y+\Delta y)=0$

$$0 = F(x+\Delta x, y+\Delta y) - F(x,y)$$

$$= [F(x+\Delta x, y+\Delta y) - F(x, y+\Delta y)] + [F(x, y+\Delta y) - F(x,y)]$$

$$\overset{\exists \theta_1\theta_2 \in (0,1)}{=} F_x'(x+\theta_1\Delta x, y+\Delta y)\Delta x + F_y'(x, y+\theta_2\Delta y)\Delta y$$

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = -\lim_{\Delta x \to 0} \frac{F_x'(x+\theta_1\Delta x, y+\Delta y)}{F_y'(x, y+\theta_2\Delta y)} \underset{\uparrow}{=} -\frac{F_x'(x,y)}{F_y'(x,y)}$$

$\Delta x \to 0 \Rightarrow \Delta y \to 0$ (连续)

即 $$\frac{dy}{dx} = -\frac{F_x'(x,y)}{F_y'(x,y)}$$

let $f(x,y) =$ $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(1 - \frac{1}{\cos(x+y)+1}\right) = \frac{-\sin(x+y)}{[\cos(x+y)+1]^2}(1+y') = \frac{\sin(x+y)}{(\cos(x+y)+1)^3}$

例1 设方程 $\sin(x+y)+2x+y=0$ 在(0, 0)附近确定

隐函数 $y=y(x)$, 求 $\frac{dy}{dx}$ 和 $\frac{d^2y}{dx^2}$

$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} = -\frac{\cos(x+y)+2}{\cos(x+y)+1}$

例2 讨论方程 $F(x, y)=y^3-x=0$ 在(0, 0)附近确定

隐函数的情况.

$F_x'(x,y) \neq 0$，有唯一 $x=x(y)$ 隐函数

注意 $x=y^3$

讨论 $x=x(y)$ 的导数，可判断是

$\frac{\partial F}{\partial x} = -1$ $\frac{\partial F}{\partial y} = 3y^2$

例3 讨论方程 $F(x, y)=x^2+y^2-1=0$ 在(0, 1)和(1, 0) 否存

反函数

即是否有唯一 $y=y(x)$

附近确定隐函数的情况.

$y'(0)=0$

$\frac{\partial F}{\partial x} = 2x$ $\frac{\partial F}{\partial y} = 2y$ 在(0,1)有 $y=y(x)$ $y'(x) = -\frac{x}{y}$

(1,0)有 $x=x(y)$

例4 设有方程 $e^{xy}-\cos x-\sin y=0.$

(1) 证明: 在(0,0)点的某邻域内, 上述方程可确定唯一的隐函数$y=y(x)$满足$y(0)=0$.

(2) 上述方程能否在(0,0)点的某邻域内, 确定隐函数$x=x(y)$? 为什么.

提示: $$y'(x)=\frac{\sin x+ye^{xy}}{\cos y-xe^{xy}},\quad y'(0)=0$$

$$y''(0)=\lim_{x\to0}\frac{y'(x)-y'(0)}{x}=\lim_{x\to0}\frac{\overbrace{\frac{\sin x}{x}}^{\to 1}+\overbrace{\frac{y}{x}}^{\to y'(0)}e^{xy}}{\underbrace{\cos y-xe^{xy}}_{\to 1}}=1$$

$\neq 0$ ∴无唯一 $x(y)$

$y(0)=0$是极小值

Q4 (1) $\frac{\partial f}{\partial x} = ye^{xy} + \sin x$ $\frac{\partial f}{\partial y} = xe^{xy} - \cos y$

$y'(x) = \frac{\sin x + ye^{xy}}{\cos y - xe^{xy}}$ $y'(0) = \frac{0 + y(0)}{\cos y(0)} \to 0$

$\Rightarrow y(0) = 0$

(2) $y''(0) =$

Ex. 设有方程 $x^2+y+\sin xy=0.$

(1) 证明: 在(0,0)点的某邻域内, 上述方程可确定唯一的隐函数$y=y(x)$满足$y(0)=0$.

(2) 上述方程能否在(0,0)点的某邻域内, 确定隐函数$x=x(y)$?为什么.

提示: $y'(x)=-\dfrac{2x+y\cos xy}{1+x\cos xy}$, $y'(0)=0$

$$y''(0)=\lim_{x\to 0}\frac{y'(x)-y'(0)}{x}=\lim_{x\to 0}-\frac{2+\dfrac{y}{x}\cos xy}{1+x\cos xy}=-2$$

■ 多元隐函数

定理 设函数F在$M_0(x_0,y_0,z_0)$的邻域内有连续偏导数, 且

$$F(x_0,y_0,z_0)=0,\quad F'_z(x_0,y_0,z_0)\neq 0$$

则方程$F(x,y,z)=0$在M_0某邻域内存在唯一具有连续偏导数的**隐函数**$z=f(x,y)$, 满足

$$F(x,y,f(x,y))\equiv 0,\quad z_0=f(x_0,y_0)$$

且有
$$\boxed{\frac{\partial z}{\partial x}=-\frac{F'_x}{F'_z},\ \frac{\partial z}{\partial y}=-\frac{F'_y}{F'_z}}$$

例5 设方程 $\mathrm{e}^z-xyz=0$ 确定隐函数$z=z(x,y)$, 求z'_x.

$$\frac{\partial z}{\partial x} = -\frac{F_x'}{F_z'} = -\frac{-yz}{e^z - xy}$$

■ 隐映射存在定理

若函数 $F(x,y,u,v), G(x,y,u,v)$ 在点 $P_0(x_0,y_0,u_0,v_0)$

某一邻域内有连续的偏导数, 且 $F(x_0,y_0,u_0,v_0)=0$,

$G(x_0,y_0,u_0,v_0)=0$, **Jacobi行列式**

$$J_0 = \left.\frac{\partial(F,G)}{\partial(u,v)}\right|_{P_0} = \begin{vmatrix} \frac{\partial F}{\partial u} & \frac{\partial F}{\partial v} \\ \frac{\partial G}{\partial u} & \frac{\partial G}{\partial v} \end{vmatrix}_{P_0} \neq 0,$$

因变量　自变量

则 $\begin{cases} F(x,y,u,v)=0 \\ G(x,y,u,v)=0 \end{cases}$ 可唯一确定**隐映射** $\begin{cases} u=u(x,y) \\ v=v(x,y) \end{cases}$

满足此方程组及 $\begin{cases} u_0=u(x_0,y_0) \\ v_0=v(x_0,y_0) \end{cases}$ 且有连续偏导数

$$\frac{\partial u}{\partial x}=-\frac{1}{J}\frac{\partial(F,G)}{\partial(x,v)}=-\frac{\begin{vmatrix}F_x & F_v\\ G_x & G_v\end{vmatrix}}{\begin{vmatrix}F_u & F_v\\ G_u & G_v\end{vmatrix}}\qquad \frac{\partial u}{\partial y}=-\frac{1}{J}\frac{\partial(F,G)}{\partial(y,v)}=-\frac{\begin{vmatrix}F_y & F_v\\ G_y & G_v\end{vmatrix}}{\begin{vmatrix}F_u & F_v\\ G_u & G_v\end{vmatrix}}$$

$$\frac{\partial v}{\partial x}=-\frac{1}{J}\frac{\partial(F,G)}{\partial(u,x)}=-\frac{\begin{vmatrix}F_u & F_x\\ G_u & G_x\end{vmatrix}}{\begin{vmatrix}F_u & F_v\\ G_u & G_v\end{vmatrix}}\qquad \frac{\partial v}{\partial y}=-\frac{1}{J}\frac{\partial(F,G)}{\partial(u,y)}=-\frac{\begin{vmatrix}F_u & F_y\\ G_u & G_y\end{vmatrix}}{\begin{vmatrix}F_u & F_v\\ G_u & G_v\end{vmatrix}}$$

例6 设函数$\begin{cases}u=u(x,y)\\ v=v(x,y)\end{cases}$由方程组$\begin{cases}x^2+y^2-uv=0 & \hat{=}F\\ xy-u^2+v^2=0 & \hat{=}G\end{cases}$确定,

求 $\frac{\partial u}{\partial x},\frac{\partial v}{\partial y}$.

$$\frac{\partial u}{\partial x}=\frac{4xv+yu}{2(u^2+v^2)},\qquad \frac{\partial v}{\partial y}=\frac{4yu-xv}{2(u^2+v^2)}$$

微分理解：$F_x' dx + F_y' dy + F_u' du + F_v' dv = 0$

$G_x' dx + G_y' dy + G_u' du + G_v' dv = 0$

$$\begin{cases} F_u' du + F_v' dv = -(F_x' dx + F_y' dy) \\ G_u' du + G_v' dv = -(G_x' dx + G_y' dy) \end{cases}$$

克莱姆法则

$$du = \frac{-1}{\begin{vmatrix} F_u' & F_v' \\ G_u' & G_v' \end{vmatrix}} \begin{vmatrix} F_x' dx + F_y' dy & F_v' \\ G_x' dx + G_y' dy & G_v' \end{vmatrix}$$

$$= -\frac{1}{J} \begin{vmatrix} F_x' & F_v' \\ G_x' & G_v' \end{vmatrix} dx - \frac{1}{J} \begin{vmatrix} F_y' & F_v' \\ G_y' & G_v' \end{vmatrix} dy$$

同理 $dv = -\frac{1}{J} \begin{vmatrix} F_u' & F_x' \\ G_u' & G_x' \end{vmatrix} dx - \frac{1}{J} \begin{vmatrix} F_u' & F_y' \\ G_u' & G_y' \end{vmatrix} dy$

eg6.

$J = \frac{\partial(F,G)}{\partial(u,v)} = \begin{vmatrix} -v & -u \\ -2u & +2v \end{vmatrix} = -2(u^2+v^2)$

$$\frac{\partial u}{\partial x} = -\frac{1}{J} \frac{\partial(F,G)}{\partial(x,v)} = \frac{\begin{vmatrix} 2x & -u \\ y & +2v \end{vmatrix}}{2(u^2+v^2)} = \frac{4xv + yu}{2(u^2+v^2)}$$

9.3.2 从微分看隐函数定理

由 $F(x,y(x))\equiv 0$, 两端取微分得

$$\mathrm{d}F=F_x'\mathrm{d}x+F_y'\mathrm{d}y=0 \quad \to y=y(x)$$

$0=F_x'(x,y(x))+F_y'(x,y(x))$

导出

$$\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{F_x'(x,y(x))}{F_y'(x,y(x))}$$

想一想 两个三元方程的方程组 $\begin{cases}F(x,y,z)=0\\G(x,y,z)=0\end{cases}$ 情形?

及n个$m+n$元方程的方程组 $\boldsymbol{F}(\boldsymbol{x},\boldsymbol{y})=\boldsymbol{0}$ 情形? 其中

$$\boldsymbol{F}:D\subset\mathbb{R}^{m+n}\to\mathbb{R}^n,\boldsymbol{x}=(x_1,x_2,\cdots,x_m),\boldsymbol{y}=(y_1,y_2,\cdots,y_n)$$

$$J = \frac{\partial F}{\partial y} = \frac{\partial(F_1 F_2 \dots F_n)}{\partial(y_1 y_2 \dots y_n)}$$

$$\frac{\partial y_i}{\partial x_j} = \frac{\frac{\partial F}{\partial y(y_i \leftarrow x_j)}}{J}$$

eg7. $dz = dx + dy - d(xe^z)$

$$= dx + dy - (e^z dx + xe^z dz)$$

$$dz = \frac{1-e^z}{1+xe^z dz} dx + \frac{1}{1+xe^z dz} dy$$

eg8.

$$dz = dx + d\varphi(x+y)$$

let $u = x+y$

$$d\varphi(x+y) = \frac{d\varphi(u)}{du} \cdot d(x+y)$$

$$= dx + \varphi' \cdot (dx + dy)$$

$$F_x' dx + F_y' dy + F_z' dz = 0$$

解得 $dy =$ balala dx $dz =$ balabla dx

- 求隐函数所有偏导数时，**微分法**比较简便

例7 设$z=f(x,y)$由方程 $z=x+y-x\mathrm{e}^{z}$ 确定，求z'_x，z'_y.

例8 函数 $y=y(x)$，$z=z(x)$由方程组

$$\begin{cases} z=x+\varphi(x+y) \\ F(x,y,z)=0 \end{cases}$$

确定，其中φ，F均可微，$F'_y+\varphi' F'_z\neq 0$，求 $\dfrac{\mathrm{d}y}{\mathrm{d}x}$，$\dfrac{\mathrm{d}z}{\mathrm{d}x}$.

$$\boxed{\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{F'_x+(1+\varphi')F'_z}{F'_y+\varphi' F'_z},\qquad \frac{\mathrm{d}z}{\mathrm{d}x}=-\frac{\varphi' F'_x-(1+\varphi')F'_y}{F'_y+\varphi' F'_z}}$$

一元情形

(1) $y=f(x)$ 有连续导数

(2) $y_0=f(x_0)$ (3) $f'(x_0)\neq 0$

则在 $U(y_0)$ 内 $y=f(x)$ 存在反函数

邻域 $x=f^{-1}(y)$ 满足 $x_0=f^{-1}(y_0)$

且 $\frac{dx}{dy}\frac{dy}{dx}=1$

9.3.3 逆映射存在定理

设函数 $u=u(x,y), v=v(x,y)$ 在 $B(P_0(x_0,y_0))$ 有连续偏导数, 且 $u_0=u(x_0,y_0)$, $v_0=v(x_0,y_0)$, Jacobi行列式

$$J_0=\left.\frac{\partial(u,v)}{\partial(x,y)}\right|_{P_0}\neq 0,$$

则 $\begin{cases}u=u(x,y)\\ v=v(x,y)\end{cases}$ 在$B(u_0,v_0)$存在逆映射 $\begin{cases}x=x(u,v)\\ y=y(u,v)\end{cases}$

满足 $\begin{cases}x_0=x(u_0,v_0)\\ y_0=y(u_0,v_0)\end{cases}$ 且有连续偏导数

记

令：$\begin{cases} F = u(x,y) - u = 0 & \text{四元函数缺 } v \\ G = v(x,y) - v = 0 & \text{四元函数缺 } u \end{cases}$

（上方标注：x, y　$, u$）

偏导连续：$F(x_0,y_0,u_0,v_0) = 0 = G(x_0,y_0,u_0,v_0)$

$$\left.\frac{\partial(F,G)}{\partial(x,y)}\right|_{P_0} = \left.\frac{\partial(u,v)}{\partial(x,y)}\right|_{P_0} \neq 0$$

隐映射定理：

$$\begin{cases} x = X(u,v) \\ y = y(u,v) \end{cases} \qquad J = \frac{\partial(F,G)}{\partial(x,y)}$$

$$\frac{\partial x}{\partial u} = -\frac{1}{J}\frac{\partial(F,G)}{\partial(u,y)} = -\frac{1}{J}\begin{vmatrix} -1 & \frac{\partial u}{\partial y} \\ 0 & \frac{\partial v}{\partial y} \end{vmatrix} = \frac{1}{J}\frac{\partial v}{\partial y}$$

$$\frac{\partial x}{\partial u} = \frac{1}{J}\cdot\frac{\partial v}{\partial y}, \qquad \frac{\partial x}{\partial v} = -\frac{1}{J}\cdot\frac{\partial u}{\partial y}$$

$$\frac{\partial y}{\partial u} = -\frac{1}{J}\cdot\frac{\partial v}{\partial x}, \qquad \frac{\partial y}{\partial v} = \frac{1}{J}\frac{\partial u}{\partial x}$$

推论 同前定理条件, 则有

$$\boxed{\frac{\partial(x,y)}{\partial(u,v)}\cdot\frac{\partial(u,v)}{\partial(x,y)} = 1}$$

证 记

$$\boldsymbol{f} = \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix}, \text{逆映射 } \boldsymbol{f}^{-1} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x(u,v) \\ y(u,v) \end{pmatrix}$$

则复合映射

$$f^{-1}\circ f:\begin{pmatrix}x\\y\end{pmatrix}\to\begin{pmatrix}x\\y\end{pmatrix}$$ 即恒等映射 $$I:\begin{cases}x=x\\y=y\end{cases}$$

单位矩阵

由 $f^{-1}\circ f=I$，两端求Jacobi矩阵, 得 $J\left(f^{-1}\right)\cdot J(f)=E$

取行列式得 $\left|J\left(f^{-1}\right)\cdot J(f)\right|=\left|J\left(f^{-1}\right)\right|\cdot\left|J(f)\right|=\left|E\right|=1$

即

$$\boxed{\frac{\partial(x,y)}{\partial(u,v)}\cdot\frac{\partial(u,v)}{\partial(x,y)}=1}$$

例9 求极坐标变换 $x=r\cos\theta, y=r\sin\theta$ 的逆变换偏微商

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

由 $x = r\cos\theta$, $\xrightarrow{x偏导}$ $\frac{\partial r}{\partial x}\cos\theta + r\frac{\partial \cos\theta}{\partial \theta}\frac{\partial \theta}{\partial x} = 1$

$$\Rightarrow \frac{\partial r}{\partial x}\cos\theta - \frac{\partial \theta}{\partial x} r\sin\theta = 1$$

由 $y = r\sin\theta$ $\xrightarrow{y}$ $\frac{\partial r}{\partial x}\sin\theta + \frac{\partial \theta}{\partial x} r\cos\theta = 0$

解得 $\frac{\partial r}{\partial x} = \cos\theta = \frac{x}{r} = \frac{x}{\sqrt{x^2+y^2}}$

$$\frac{\partial \theta}{\partial y} = \frac{-\sin\theta}{r} = \frac{-y}{\sqrt{x^2+y^2}}$$