Chap 9 — 6

向量场的微商

9.6.1 向量场

场 空间区域中每一点都对应一个量. 若对应的量是向量(数量), 则称之为**向量场(数量场)**.

如空间向量场

$$\boldsymbol{v}(x,y,z)=P(x,y,z)\boldsymbol{i}+Q(x,y,z)\boldsymbol{j}+R(x,y,z)\boldsymbol{k}$$

$(x, y, z) \in D \subset \mathbf{R}^3$. 平面向量场

$$\boldsymbol{v}(x,y)=P(x,y)\boldsymbol{i}+Q(x,y)\boldsymbol{j},\quad (x,y)\in D\subset\mathbb{R}^2$$

位置向量 $\boldsymbol{r}=x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k}.$

定义 设有向量场$\boldsymbol{v}(x, y, z)$, 它对x的**偏微商**定义为

$$\frac{\partial \boldsymbol{v}}{\partial x}=\lim_{\Delta x\to 0}\frac{\boldsymbol{v}(x+\Delta x, y, z)-\boldsymbol{v}(x, y, z)}{\Delta x}$$

想一想 对y, z的偏微商？显见

$$\frac{\partial \boldsymbol{v}}{\partial x}=\frac{\partial P(x, y, z)}{\partial x}\boldsymbol{i}+\frac{\partial Q(x, y, z)}{\partial x}\boldsymbol{j}+\frac{\partial R(x, y, z)}{\partial x}\boldsymbol{k}$$

光滑向量场 P, Q, R具有连续偏导数.

例 1 设$\boldsymbol{v}(x, y, z)=\sin xyz\boldsymbol{i}+x^2\boldsymbol{j}+\mathrm{e}^{x+2y+z}\boldsymbol{k}$, 求 $\frac{\partial \boldsymbol{v}}{\partial x}, \frac{\partial \boldsymbol{v}}{\partial y}, \frac{\partial \boldsymbol{v}}{\partial z}$

$$d\vec{v} = J(\vec{v})\begin{bmatrix} dx \\ dy \\ dz \end{bmatrix}$$

$$\frac{\partial v}{\partial x} = yz\cos xyz\,\vec{i} + 2x\vec{j} + e^{2x+2y+z}\vec{k}$$

9.6.2 梯度、散度与旋度

设有数量场$\varphi(x, y, z)$, 根据微分及点乘公式, 有

$$\mathrm{d}\varphi = \frac{\partial\varphi}{\partial x}\mathrm{d}x + \frac{\partial\varphi}{\partial y}\mathrm{d}y + \frac{\partial\varphi}{\partial z}\mathrm{d}z$$

$$= \left(\frac{\partial\varphi}{\partial x}\boldsymbol{i} + \frac{\partial\varphi}{\partial y}\boldsymbol{j} + \frac{\partial\varphi}{\partial z}\boldsymbol{k}\right)\cdot(\mathrm{d}x\boldsymbol{i} + \mathrm{d}y\boldsymbol{j} + \mathrm{d}z\boldsymbol{k})$$

$$= \mathbf{grad}\,\varphi\cdot\mathrm{d}\boldsymbol{r}$$

- 函数微分是梯度与位置向量微分之点乘.

➢ Hamilton算子

$$\nabla = \frac{\partial}{\partial x}\boldsymbol{i} + \frac{\partial}{\partial y}\boldsymbol{j} + \frac{\partial}{\partial z}\boldsymbol{k}$$

➢ ∇算子兼具**偏微商**和**向量**属性, 它‘数乘’函数得

$$\nabla\varphi = \mathbf{grad}\,\varphi = \frac{\partial\varphi}{\partial x}\boldsymbol{i} + \frac{\partial\varphi}{\partial y}\boldsymbol{j} + \frac{\partial\varphi}{\partial z}\boldsymbol{k}$$

➢ 数量场的**梯度是向量场**

定义 设有向量场$\boldsymbol{v}(x, y, z) = P\boldsymbol{i} + Q\boldsymbol{j} + R\boldsymbol{k}.$

$(\frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k)\cdot(Pi+Qj+Rk)$

1° $\boldsymbol{v}$ 的**散度** $\operatorname{div}\boldsymbol{v} \overset{\text{def}}{=} \nabla\cdot\boldsymbol{v} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$

➢ 向量场的**散度是数量场**

2° $\boldsymbol{v}$ 的旋度

$$\mathbf{rot}\,\boldsymbol{v} \overset{\text{def}}{=} \nabla\times\boldsymbol{v} = \left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right)\boldsymbol{i}+\left(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}\right)\boldsymbol{j}+\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)\boldsymbol{k}$$

$$=\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

➢ 向量场的**旋度是向量场**

命题(运算性质) 设φ, ψ是数量场, $\boldsymbol{a}$, $\boldsymbol{b}$是向量场, 则有

1 $\nabla f(\varphi) = f'(\varphi)\nabla\varphi$

2 $\nabla(\varphi+\psi) = \nabla\varphi + \nabla\psi$

3 $\nabla\cdot(\boldsymbol{a}+\boldsymbol{b}) = \nabla\cdot\boldsymbol{a} + \nabla\cdot\boldsymbol{b}$

4 $\nabla\times(\boldsymbol{a}+\boldsymbol{b}) = \nabla\times\boldsymbol{a} + \nabla\times\boldsymbol{b}$

5 $\nabla(\varphi\psi) = \psi\nabla\varphi + \varphi\nabla\psi$

6 $\nabla\cdot(\varphi\boldsymbol{a}) = \nabla\varphi\cdot\boldsymbol{a} + \varphi\nabla\cdot\boldsymbol{a}$

7 $\nabla\times(\varphi\boldsymbol{a}) = \nabla\varphi\times\boldsymbol{a} + \varphi\nabla\times\boldsymbol{a}$

8 $\nabla\cdot(\boldsymbol{a}\times\boldsymbol{b}) = \nabla\times\boldsymbol{a}\cdot\boldsymbol{b} - \nabla\times\boldsymbol{b}\cdot\boldsymbol{a}$

此外, 还有

无旋场 $\Longleftrightarrow$ 梯度场

$$\boxed{\begin{array}{c}\textbf{rot grad}\,\varphi = \nabla\times\nabla\varphi = \mathbf{0} \\[2ex] \text{div}\,\textbf{rot}\,\boldsymbol{a} = \nabla\cdot\nabla\times\boldsymbol{a} = 0\end{array}}$$

梯度的旋度为0

旋度的散度为0

1. $\nabla f(\varphi) = \frac{\partial}{\partial x} f(\varphi)\vec{i} + \frac{\partial}{\partial y} f(\varphi)\vec{j} + \frac{\partial}{\partial z} f(\varphi)\vec{k}$

$= f'(\varphi)\left[\frac{\partial \varphi}{\partial x}\vec{i} + \sim\right]$

$= f'(\varphi)\nabla\varphi$

4. let $\vec{a} = (a_1, a_2, a_3)$ $\vec{b} = (b_1, b_2, b_3)$

$$\nabla \times (\vec{a}+\vec{b}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_1+b_1 & a_2+b_2 & a_3+b_3 \end{vmatrix} = \begin{vmatrix} \sim \\ a_1 \; a_2 \; a_3 \end{vmatrix} + \begin{vmatrix} \sim \\ b_1 \; b_2 \; b_3 \end{vmatrix}$$

8. $$\nabla \cdot (\vec{a}\times\vec{b}) = \frac{\partial}{\partial x}\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \frac{\partial}{\partial y}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \frac{\partial}{\partial z}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

$$\begin{vmatrix} \frac{\partial a_2}{\partial x} & \frac{\partial a_3}{\partial x} \\ b_2 & b_3 \end{vmatrix}$$

$+$

$$\begin{vmatrix} a_2 & a_3 \\ \frac{\partial b_2}{\partial x} & \frac{\partial b_3}{\partial x} \end{vmatrix}$$

$$\mathrm{rot}(\mathrm{grad}\,\varphi) = \nabla\times(\nabla\varphi) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\varphi}{\partial x} & \frac{\partial\varphi}{\partial y} & \frac{\partial\varphi}{\partial z} \end{vmatrix} = \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\varphi}{\partial y} & \frac{\partial\varphi}{\partial z} \end{vmatrix}\hat{i} + \cdots$$

$= 0\hat{i} + 0\hat{j} + 0\hat{k} = \vec{0}$

➢ Laplace算子

$$\Delta = \nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

它作用函数φ得到

$$\Delta \varphi = \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2}$$

例2 设 $f(x,y,z) = g(r), r = \sqrt{x^2 + y^2 + z^2}$

1) 求Δf； **2)** 若$\Delta f = 0$, 求f.

$\cancel{g}$ 2.
$$\frac{\partial f}{\partial x} = g'(r)\frac{\partial r}{\partial x} = g'(r)\frac{x}{r}$$

$$\frac{\partial^2 f}{\partial x} = g''(r)\frac{x^2}{r^2} + g'(r)\left(\frac{1}{r} + x\left(-\frac{1}{r^2}\right)\frac{\partial r}{\partial x}\right)$$

$$= g''(r)\frac{x^2}{r^2} + g(r)\left(\frac{1}{r} - \frac{x^2}{r^3}\right)$$

$$\Delta f = g''(r) + g'(r)\frac{2}{r}$$

$$\frac{\partial f}{\partial x \partial r}\frac{\partial r}{\partial x} \qquad \frac{f'}{\partial r}$$