## 1

چون $Y=y$ را ثابت در نظر گرفته‌ایم، randomness ای ندارد و کافی است نسبت به $X$ انتگرال بگیریم:

$$p(X<x, Y=y) \xlongequal{\text{در بازه‌ی } x,y>0} \int_0^x \frac{e^{-\frac{x}{y}} e^{-y}}{y}\, dx = \frac{e^{-y}}{y}\left(-y e^{-\frac{x}{y}}\right)\Big|_0^x = e^{-y}\left(1-e^{-\frac{x}{y}}\right)$$

$$\longrightarrow f(Y=y) \xlongequal{y>0} \int_0^\infty \frac{e^{-\frac{x}{y}} e^{-y}}{y}\, dx = e^{-y}$$

$$\xrightarrow{\div} p(X<x \mid Y=y) = \begin{cases} 1-e^{-\frac{x}{y}} & x>0,\ y>0 \\ 0 & \text{otherwise} \end{cases}$$

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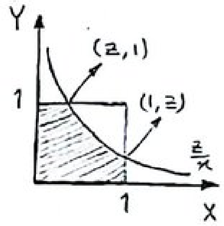
## 2.a

$$Z = X+Y \longrightarrow f_Z(z) = \int_0^1 \underbrace{f_{XY}(x, z-x)}_{x+z-x}\, dx = zx\Big|_0^1 = z$$

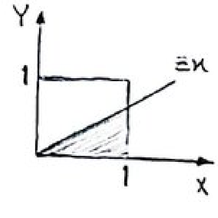
$$\longrightarrow f_Z(z) = \begin{cases} z & 0 \leqslant z \leqslant 2 \\ 0 & \text{otherwise} \end{cases}$$




## 2.b

$$Z = XY \longrightarrow F_Z(z) = 1 - \int_z^1 \int_{\frac{z}{x}}^1 f_{XY}(x,y)\, dy\, dx = 1 - \int_z^1 xy + \frac{1}{2}y^2 \Big|_{\frac{z}{x}}^1 dx$$

$$= 2z - z^2 \longrightarrow \boxed{f_Z(z) = 2 - 2z \qquad 0 \leqslant z \leqslant 1}$$

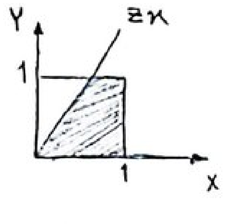



## 2.c

$$Z = \frac{Y}{X} \xrightarrow{0 \leqslant z \leqslant 1} F_Z(z) = \int_0^1 \int_0^{zx} f_{XY}(x,y)\, dy\, dx = \int_0^1 \underbrace{xy + \frac{1}{2}y^2 \Big|_0^{zx}}_{zx^2 + \frac{1}{2}z^2x^2} dx$$

$$= \int_0^1 \left(z + \frac{1}{2}z^2\right) x^2\, dx = \frac{z+\frac{1}{2}z^2}{3} x^3 \Big|_0^1$$

$$= \frac{1}{3}z + \frac{1}{6}z^2 \longrightarrow f_Z(z) = \frac{dF_Z(z)}{dz} = \boxed{\frac{1}{3} + \frac{1}{3}z \qquad 0 \leqslant z \leqslant 1}$$




$$\xrightarrow{z \geqslant 1} F_Z(z) = 1 - \int_0^{\frac{1}{z}} \int_{zx}^1 f_{XY}(x,y)\, dy\, dx = 1 - \int_0^{\frac{1}{z}} x + \frac{1}{2} - zx^2 - \frac{1}{2}z^2x^2\, dx$$

$$= 1 - \frac{2z+1}{6z^2} \longrightarrow \boxed{f_Z(z) = \frac{z+1}{3z^3} \qquad z \geqslant 1}$$

$\boxed{3.a}$ $f_Y(y \mid X=x) = \begin{cases} \frac{1}{2x} & -x \leq y \leq x \quad |y| \leq x \\ 0 & \text{otherwise} \end{cases}$

$\longrightarrow f_{XY}(x,y) = \begin{cases} \overset{1}{\cancel{\frac{1}{2x}2x}} & |y| \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$

$\boxed{3.b}$ $f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\, dx = \int_{|y|}^{1} 1\, dx = x \Big|_{|y|}^{1} = 1 - |y| \qquad |y| \leq 1$

$\boxed{3.c}$ $\mathbb{P}\left(|Y| < X^3\right) = \int_0^1 \underbrace{\mathbb{P}\left(|Y| < x^3 \mid X=x\right)}_{\frac{2x^3}{2x}\text{ زیرا Uniform است}} \underbrace{f_X(x)}_{2x}\, dx = \int_0^1 2x^3\, dx = \frac{1}{2}x^4 \Big|_0^1 = \frac{1}{2}$

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$\boxed{4.a}$ طول عمر زنبور را $X$ می‌نامیم . می‌دانیم که در توزیع نمایی $\mu_x = \frac{1}{\lambda}$ پس داریم $\lambda = \frac{1}{45}$

$X \sim \exp\left(\frac{1}{45}\right)$

$\mathrm{Var}[X] = \frac{1}{\lambda^2} = 45^2 \longrightarrow \boxed{\sigma_x = 45}$

$\boxed{4.b}$ $\mathbb{P}(X > 47) = 1 - \mathbb{P}(X \leq 47) = 1 - F_X(47) = 1 - \left(1 - e^{\frac{-47}{45}}\right) = e^{\frac{-47}{45}} \simeq 0.35$

$\boxed{4.c}$ $X_1 \cdots X_{1000} \overset{iid}{\sim} \exp\left(\frac{1}{45}\right) \begin{cases} \nearrow \mu_x = 45 \\ \searrow \sigma_x^2 = 45^2 \end{cases}$

$Y = \frac{X_1 + \cdots + X_{1000}}{1000} \longrightarrow$ تقریباً می‌توان گفت $Y \sim \mathcal{N}\left(\mu_x, \frac{\sigma_x^2}{1000}\right)$

$\mathbb{P}(Y > 47) = 1 - F_Y(47) = 1 - \Phi\left(\frac{47-45}{\frac{45}{10\sqrt{10}}}\right) = 1 - \underbrace{\Phi(1.4)}_{0.92} = 0.08$

## 4.d

سوال آن است که به ازای چه $a$ ای ، داریم : $F_Y(a) = \Phi\left(\frac{a-45}{\frac{45}{10\sqrt{10}}}\right) = 0.1$

طبق جدول می دانیم که $\Phi(-1.28) \simeq 0.1$ $\longrightarrow \frac{(a-45)\,10\sqrt{10}}{45} = -1.28 \longrightarrow a \simeq 43.18$

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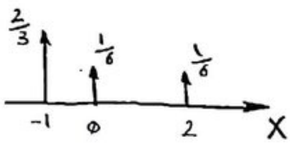
## 5.a




$$E[X] = \frac{-2}{3} + \frac{2}{6} = \boxed{\frac{-1}{3}} = \mu_x$$

$$E[X^2] = \frac{2}{3} + \frac{4}{6} = \frac{4}{3} \longrightarrow Var[X] = \frac{4}{3} - \frac{1}{9} = \frac{11}{9}$$

$$\longrightarrow \sigma_x = \boxed{\frac{\sqrt{11}}{3}}$$

## 5.b

$Y = X_1 + \dots + X_{100} \longrightarrow$ تقریبا می توان گفت $Y \sim \mathcal{N}(100\mu_x \,,\, 100\sigma_x^2)$

$$E[Y] = 100\mu_x = \frac{-100}{3}$$

$$SE[Y] = \frac{10\sigma_x}{10} = \frac{\sqrt{11}}{3}$$

## 5.c

$Z = \frac{X_1 + \dots + X_{100}}{100} \longrightarrow$ تقریبا می توان گفت $Z \sim \mathcal{N}\left(\mu_x \,,\, \frac{\sigma_x^2}{100}\right)$

$$P(Z>0) = 1 - F_Z(0) = 1 - \Phi\left(\frac{0+\frac{1}{3}}{\frac{\sqrt{11}}{30}}\right) = 1 - \underbrace{\Phi\left(\frac{10}{\sqrt{11}}\right)}_{0.998} = 0.002$$

## 5.d

سوال آن است که به ازای چه $a$ای داریم : $F_Z(a) = \Phi\left(\frac{a+\frac{1}{3}}{\frac{\sqrt{11}}{30}}\right) = 0.9$

طبق جدول می دانیم که $\Phi(1.29) \simeq 0.9$ $\longrightarrow \frac{(a+\frac{1}{3})\,30}{\sqrt{11}} = 1.29 \longrightarrow a \simeq -0.19$

## $\boxed{6.a}$

$$\mathbb{E}[X] = \mathbb{E}\left[3Y + W + 3\right] = 3(2) + 0 + 3 = \boxed{9}$$

$$\text{Var}[X] = \mathbb{E}\left[(X-9)^2\right] = \mathbb{E}\left[(3Y + W - 6)^2\right] = \mathbb{E}\left[\underbrace{9Y^2} + 6YW - \underbrace{36Y} + W^2 - 12W \underbrace{+36}\right]$$

$$= 9\underbrace{\mathbb{E}\left[Y^2 - 4Y + 4\right]}_{\text{Var}[Y]=16} + \underbrace{\mathbb{E}\left[W^2\right]}_{\text{Var}[W]=4} - 12\underbrace{\mathbb{E}[W]}_{0} + 6\underbrace{\mathbb{E}[YW]}_{\underbrace{\text{cov}(Y,W)}_{\underbrace{r\sigma_y\sigma_w}_{\frac{1}{4}\times 4\times 2 = 2}} + \underbrace{\mathbb{E}[Y]\,\mathbb{E}[W]}_{0}}$$

$$= 144 + 4 + 12 = \boxed{160}$$

## $\boxed{6.b}$

می دانیم که اگر $Y, W \sim J.G$ آنگاه هر ترکیب خطی آنها نیز گاوسی است $\longrightarrow$ $X$ گاوسی است

$$X \sim \mathcal{N}(9, 160)$$

$$\mathbb{P}(X \geq 20) = 1 - F_X(20) = 1 - \Phi\Bigg(\overbrace{\frac{20-9}{\sqrt{160}}}^{0.869}\Bigg) \simeq 1 - 0.8 = \boxed{0.2}$$

## $\boxed{6.c}$

می دانیم که اگر $Y, W \sim J.G$ آنگاه 2 ترکیب خطی دلخواه از آن‌ها نیز $J.G$ است $\longrightarrow$ $Y, X \sim J.G$

می دانیم که اگر $Y, X \sim J.G$ آنگاه $f_{Y|X=x}$ گاوسی است و داریم:

$$f_{Y|X=x}(y) \sim \mathcal{N}\left(\mu_y + \underbrace{\frac{r\sigma_y}{\sigma_x}}_{\frac{\text{cov}(X,Y)}{\text{Var}(X)}}(x - \mu_y)\,,\,(1-r^2)\,\sigma_y^2\right)$$

$$\longrightarrow \boxed{\mathbb{E}[Y|X] \overset{*}{=} 2 + \frac{50}{160}(X - 2)}$$

$$\text{cov}(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\,\mathbb{E}[Y] = 68 - 9\times 2 = 50 \qquad (*)$$

$$\mathbb{E}[XY] = \mathbb{E}\left[3Y^2 + WY + 3Y\right] = 3\underbrace{\mathbb{E}\left[Y^2\right]}_{\underbrace{\text{Var}[Y] + \mathbb{E}[Y]^2}_{20}} + \mathbb{E}[WY] + 3\,\mathbb{E}[Y] = 60 + 2 + 6 = 68$$

$\boxed{6.d}$ $$\mathbb{E}\left[\left(Y-2-\frac{5}{16}(X-2)\right)^2\right] = \mathbb{E}\left[\left(Y-\frac{5}{16}X-\frac{11}{8}\right)^2\right]$$

$$= \mathbb{E}\left[Y^2 - \frac{5}{8}XY - \frac{11}{4}Y + \frac{25}{256}X^2 + \frac{55}{64}X + \frac{121}{64}\right]$$

$$= 20 - \frac{5}{8}(68) - \frac{11}{4}(2) + \frac{25}{256}(241) + \frac{55}{64}(9) + \frac{121}{64} \simeq 5.16$$

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$\boxed{7.a}$

متغیرهای تصادفی $T_1, T_2-T_1, T_3-T_2, \dots$ را $Z_1, Z_2, \dots$ می‌نامیم.
همچنین متغیر تصادفی $N$ را تعریف می‌کنیم که نشان دهنده آن است که اولین ایمیل سالم، $N$اُمین ایمیل بوده است. بنابراین داریم:
واضح است که

$$Z_i \overset{iid}{\sim} \exp(\lambda) \qquad X = Z_1 + \dots + Z_N \qquad N \sim \text{geom}(p)$$

حال به شرط دانستن $N$ می‌توانیم موارد مربوط به $X$ را حساب کنیم:

$$\mathbb{E}[X] = \mathbb{E}\left[\mathbb{E}[X|N]\right] = \mathbb{E}\left[\mathbb{E}[Z_1+\dots+Z_N]\,|\,N\right] = \mathbb{E}\left[N \times \frac{1}{\lambda}\right] = \frac{1}{\lambda} \times \frac{1}{p} = \boxed{\frac{1}{\lambda p}}$$

$$\text{Var}[X] = \mathbb{E}\left[\text{Var}[X|N]\right] + \text{Var}\left[\mathbb{E}[X|N]\right] = \mathbb{E}\left[N \underbrace{\text{Var}[Z_1]}_{\frac{1}{\lambda^2}}\right] + \text{Var}\left[N \times \frac{1}{\lambda}\right]$$

$$= \frac{1}{p\lambda^2} + \frac{1}{\lambda^2} \times \frac{1-p}{p^2} = \frac{p+1-p}{p^2\lambda^2} = \boxed{\frac{1}{p^2\lambda^2}}$$

$\boxed{7.b}$ $$M_X(t) = \mathbb{E}\left[e^{tX}\right] = \mathbb{E}\left[\mathbb{E}[e^{tX}|N]\right] = \mathbb{E}\left[\mathbb{E}[e^{tZ_1}\dots e^{tZ_N}|N]\right]$$

$$= \mathbb{E}\left[\underbrace{\mathbb{E}[e^{tZ_1}|N]}_{M_Z(t)} \cdots \underbrace{\mathbb{E}[e^{tZ_N}|N]}_{M_Z(t)}\right] = \mathbb{E}\left[M_Z(t)^N\right]$$

$*$ برای توزیع نمایی، MGF را می‌دانیم و داریم: $M_Z(t) = \dfrac{\lambda}{\lambda - t}$

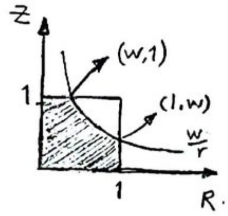
$$\longrightarrow M_X(t) \overset{*}{=} \mathbb{E}\left[\left(\frac{\lambda}{\lambda-t}\right)^N\right] = \sum_{k=1}^{\infty} \left(\frac{\lambda}{\lambda-t}\right)^k (1-p)^{k-1} p = \frac{p}{1-p} \sum_{k=1}^{\infty} \left(\frac{\lambda}{\lambda-t}(1-p)\right)^k$$

$$\overset{\text{سری هندسی}}{=} \frac{p}{1-p} \times \frac{\frac{\lambda(1-p)}{\lambda-t}}{1-\frac{\lambda(1-p)}{\lambda-t}} = \boxed{\frac{p\lambda}{\lambda p - t}} \xrightarrow{*} \text{MGF برای } \exp(\lambda p) \rightarrow X \sim \exp(\lambda p)$$

## $\boxed{8.a}$

$$Z = I^2 \longrightarrow F_Z(z) = \int_0^{\sqrt{z}} f_I(x)\,dx = \int_0^{\sqrt{z}} 6x - 6x^2\,dx$$

$$= 3x^2 - 2x^3 \Big|_0^{\sqrt{z}} = 3z - 2z^{\frac{3}{2}}$$




$$\longrightarrow f_Z(z) = 3 - 3\sqrt{z} \qquad 0 \leqslant z \leqslant 1$$

$$F_W(w) = 1 - \int_w^1 \int_{\frac{w}{r}}^1 \overbrace{(3 - 3\sqrt{z})(2r)}^{f_{ZR}(z,r)}\,dz\,dr = 1 - \int_w^1 2r\left(1 - 3\left(\frac{w}{r}\right) + 2\left(\frac{w}{r}\right)^{\frac{3}{2}}\right)dr$$

$$= 1 - \int_w^1 2r - 6w + 4w^{\frac{3}{2}}r^{-\frac{1}{2}}\,dr = 3w^2 - 8w^{\frac{3}{2}} + 6w$$

$$\longrightarrow \boxed{f_W(w) = 6w - 12\sqrt{w} + 6} \qquad 0 \leqslant w \leqslant 1$$

## $\boxed{8.b}$

$$\mathbb{E}[W] = \int_{-\infty}^{\infty} w\, f_W(w)\,dw = \int_0^1 6w^2 - 12w^{\frac{3}{2}} + 6w\,dw$$

$$= 2w^3 - \frac{24}{5}w^{\frac{5}{2}} + 3w^2 \Big|_0^1 = 2 - 4.8 + 3 = \boxed{0.2}$$

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