Basic of Large Scale Structure Formation

Sehighs

June 2025

Contents

1 Spherical Collapse

1.1 Friedmann Equation

We begin with the Friedmann equations:

$$H^2(t) = \frac{8\pi G}{3}\rho_m(t) + \frac{\Lambda}{3} - \frac{k}{R^2(t)}\,, \tag{1}$$

here $H^2(t)$ is Hubble parameter, k and $R^2(t)$ is the curvature and radius of the universe. We have relation $a(t) = \frac{R(t)}{R_0}$, so we get:

$$\begin{aligned} H^2(t) &= \frac{8\pi G}{3}\rho_m(t) + \frac{\Lambda}{3} - \frac{k}{a^2(t)R_0^2} \\ \dot{a}^2(t) &= \frac{8\pi G}{3}\rho_m(t)a^2(t) + \frac{\Lambda}{3}a^2(t) - \frac{k}{R_0^2}\,. \end{aligned} \tag{2}$$

We also have the relation:

$$
\begin{aligned}
H_0^2 &= \frac{8\pi G}{3}\rho_{m,0} + \frac{\Lambda}{3} - \frac{k}{R_0^2} \\
1 &= \frac{8\pi G}{3H_0^2}\rho_{m,0} + \frac{\Lambda}{3H_0^2} - \frac{k}{H_0^2 R_0^2} \\
1 &= \frac{\rho_{m,0}}{\rho_{c,0}} + \frac{\Lambda}{3H_0^2} - \frac{k}{H_0^2 R_0^2} \\
1 &= \Omega_{m,0} + \Omega_{\Lambda,0} + \Omega_{k,0}\,.
\end{aligned}
\tag{3}
$$

Now combine the relation and the our equation:

$$
\begin{aligned}
\dot{a}^2(t) &= H_0^2\big(\frac{8\pi G}{3H_0^2}\rho_m(t)a^2(t) + \frac{\Lambda}{3H_0^2}a^2(t) - \frac{k}{H_0^2 R_0^2}\big) \\
\dot{a}^2(t) &= H_0^2\big(\frac{8\pi G}{3H_0^2}\frac{\rho_{m,0}}{a^3(t)}a^2(t) + a^2(t)\,\Omega_{\Lambda,0} + \Omega_{k,0}\big) \\
\dot{a}^2(t) &= H_0^2\big(\frac{\Omega_{m,0}}{a(t)} + a^2(t)\,\Omega_{\Lambda,0} + \Omega_{k,0}\big) \\
\dot{a}^2(t) &= H_0^2\big(\frac{\Omega_{m,0}}{a(t)} + a^2(t)\,\Omega_{\Lambda,0} + (1 - \Omega_{m,0} - \Omega_{\Lambda,0})\big) \\
\dot{a}^2(t) &= H_0^2\big[\Omega_{m,0}(\frac{1}{a(t)} - 1) + \Omega_{\Lambda,0}(a^2(t) - 1) + 1\big]\,.
\end{aligned}
\tag{4}
$$

Finally we get the Friedmann equation we will use, and ignore the $\Omega_{\Lambda,0}$ in Matter domain universe, consequently the matter erengy density fraction $\Omega_{m,0}$ is not our universe's, it should be 1 we will use, but it is true because it is just a helpful quantity:

$$
\begin{aligned}
\dot{a}(t) &= H_0\big[\Omega_{m,0}(\frac{1}{a(t)} - 1) + \Omega_{\Lambda,0}(a^2(t) - 1) + 1\big]^{\frac{1}{2}} \\
\dot{a}(t) &= H_0\big[\Omega_{m,0}(\frac{1}{a(t)} - 1) + 1\big]^{\frac{1}{2}}\,.
\end{aligned}
\tag{5}
$$

1.2 Collapse

The equation (5) has parametric solutions as below:

$$
\begin{aligned}
a(\theta) &= A(1 - \cos\theta) \\
t(\theta) &= B(\theta - \sin\theta)\,.
\end{aligned}
\tag{6}
$$

The derivative of a with respect to t is:

$$
\frac{da}{dt} = \frac{da}{d\theta}\frac{d\theta}{dt} = \frac{A\sin\theta}{B(1 - \cos\theta)} = H_0\big[\Omega_{m,0}(\frac{1}{a(t)} - 1) + 1\big]^{\frac{1}{2}}\,. \tag{7}
$$

Let $\theta = \pi$, therefore $a(\pi) = A(1+1) = 2A$ we have:

$$\begin{aligned} \Omega_{m,0}(\frac{1}{a(\theta)} - 1) + 1 = 0 \\ \Omega_{m,0}(\frac{1}{2A} - 1) + 1 = 0 \\ A = \frac{\Omega_{m,0}}{2(\Omega_{m,0} - 1)}\,; \end{aligned} \tag{8}$$

Let $\theta = \pi/2$, therefore $a(\pi/2) = A$ we have:

$$\begin{aligned} \frac{A^2}{B^2} = \left[\Omega_{m,0}(\frac{1}{A} - 1) + 1\right]^2 \\ B = \frac{\Omega_{m,0}}{2H_0(\Omega_{m,0} - 1)^{\frac{3}{2}}}\,. \end{aligned} \tag{9}$$

We get:

$$\begin{aligned} a(\theta) = \frac{\Omega_{m,0}}{2(\Omega_{m,0} - 1)}(1 - \cos\theta) \\ t(\theta) = \frac{\Omega_{m,0}}{2H_0(\Omega_{m,0} - 1)^{\frac{3}{2}}}(\theta - \sin\theta)\,. \end{aligned} \tag{10}$$

$$\frac{da}{dt} = \frac{da}{d\theta}\frac{d\theta}{dt} = \frac{\frac{\Omega_{m,0}}{2(\Omega_{m,0}-1)}\sin\theta}{\frac{\Omega_{m,0}}{2H_0(\Omega_{m,0}-1)^{\frac{3}{2}}}(1-\cos\theta)} = H_0(\Omega_{m,0} - 1)^{\frac{1}{2}}\frac{\sin\theta}{(1-\cos\theta)}\,. \tag{11}$$

We can see the scale factor will grows to its maximum at $\theta = \pi$, the scale factor and the time is $\frac{\Omega_{m,0}}{(\Omega_{m,0}-1)}$ and $\pi B = \frac{\pi\Omega_{m,0}}{2H_0(\Omega_{m,0}-1)^{\frac{3}{2}}}$, we note the symbol $a_{\max}$, t_{a} corresponding to the "turn around". And the a down to zero when $\theta = 2\pi$ we call it collapse, the symbols are $t_{\mathrm{col}} = 2\pi B = 2t_{\mathrm{a}}$. Note a is impossible down to zero, we should confirm its value using Virial equilibrium $-W_{\mathrm{vir}} = 2K_{\mathrm{vir}}$, because the total energy $E = W_{\mathrm{vir}} + K_{\mathrm{vir}}$, we get:

$$E = -W_{\mathrm{vir}}/2 = -\frac{GM}{2R_{\mathrm{vir}}} = -\frac{GM}{R_{\max}}\,, \tag{12}$$

therefore $a_{\mathrm{col}} = 2a_{\max}$. Mark the density in the "independent small universe" at "turn around time" is ρ_{a}, we have the relation $\rho_{\mathrm{col}} = 8\rho_{\mathrm{a}}$.

To solve the spherical collapse model, we must set the initial conditions that the density is $\rho_i(t_i = 0)$, here the start time t_i is zero, Define the density contrast $1 + \delta = \dfrac{\rho}{\bar{\rho}}$, here the $\bar{\rho}$ is the average density, or we usually call it background density.

For an Einstein de-Sitter model, it is Matter Domain(MD), the background density have:

$$\begin{aligned} \dot{a}(t) \propto t^{\frac{2}{3}} \qquad \frac{\dot{a}(t)}{a(t)} = \frac{2}{3}t = H(t)^{\frac{1}{2}} = \sqrt{\frac{8\pi G}{3}\bar{\rho}(t)} \\ \bar{\rho}(t) = \frac{1}{6\pi G t^2}\,. \end{aligned} \tag{13}$$

Now we write the background density and density in the close small universe during the time t:

$$\begin{aligned}
\rho(t) &= \rho_i(t_i)\frac{a_i^3}{a^3(t)} = \rho_i(t_i)\frac{A^3(1-\cos\theta_i)^3}{A^3(1-\cos\theta)^3} \\
\bar{\rho}(t) &= \bar{\rho}(t_i)\frac{6\pi G t^2}{6\pi G t_i^2} = \bar{\rho}(t_i)\frac{6\pi G B^2(\theta-\sin\theta)^2}{6\pi G B^2(\theta_i-\sin\theta_i)^2}\,.
\end{aligned} \tag{14}$$

Note at the initial time $\bar{\rho}(t_i) = \rho_i(t_i)$, and expansion first for $\theta_i \to 0$, we have $(1-\cos\theta_i) \sim \frac{\theta^2}{2}$ and $(\theta - \sin\theta) \sim \frac{\theta^3}{6}$, the density contrast become:

$$1+\delta(t) = \frac{\rho(t)}{\bar{\rho}(t)} = \frac{(\theta-\sin\theta)^2}{(1-\cos\theta)^3}\lim\frac{(\theta_i-\sin\theta_i)^2}{(1-\cos\theta_i)^3} = \frac{9}{2}\frac{(\theta-\sin\theta)^2}{(1-\cos\theta)^3}\,. \tag{15}$$

At the turn-around point $\theta = \pi$ and collapse point $\theta = 2\pi$, we find:

$$\begin{aligned}
1+\delta_a &= \frac{9\pi}{16} \\
1+\delta_{\mathrm{col}} &= 18\pi^2 \simeq 178\,.
\end{aligned} \tag{16}$$

Wait! It seems the result not depend on the form of A and B? Now we introduce the linear-regime, at early times, the δ is very small so we can expand the Eq .15, note this is the second expanding:

$$\begin{aligned}
\frac{(\theta-\sin\theta)^2}{(1-\cos\theta)^3} &= \frac{2}{9} + \frac{\theta^2}{30} + \dots \\
t = \frac{\theta^3}{6}B &\qquad B = \frac{6t_i}{\theta^3} \\
\delta = \frac{3}{20}\theta^2 &= \frac{3}{20}\left(\frac{6t}{B}\right)^{2/3} \\
\delta = \frac{3}{20}\left(\frac{6t}{6t_i}\right)^{2/3}\theta_i^2 &= \frac{3}{5}\delta_i\left(\frac{t}{5_i}\right)^{2/3}\,.
\end{aligned} \tag{17}$$

In the linear-regime At the turn-around point $\theta = \pi$ and collapse point $\theta = 2\pi$, we find:

$$\begin{aligned}
\delta_a &= \frac{3}{5}\left(\frac{3\pi}{4}\right)^{2/3} \simeq 1.06 \\
\delta_{\mathrm{col}} &= \frac{3}{5}\left(\frac{3\pi}{2}\right)^{2/3} \simeq 1.686\,.
\end{aligned} \tag{18}$$

2 Power Spectrum

The Press-Schechter (PS) formalism describes the comoving number density of dark matter halos as a function of mass and redshift. Let us review some key definitions and hypothesis of LSS formation.

- Density contrast field:

$$\delta(\mathbf{x}) = \frac{\rho(\mathbf{x}) - \bar{\rho}}{\bar{\rho}} \tag{19}$$

- Gaussian random field:

$$P(\delta) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\delta^2/(2\sigma^2)} \tag{20}$$

(mean is 0, and variance is σ^2)

- Linear power spectrum: The two-point correlation in Fourier space:

$$P(k, z) = \langle |\delta(k, z)|^2 \rangle \tag{21}$$

(Note the $P(k)$ is not exactly the variance of the overdensity field in Fourier space, but rather its spectral density that describes how the variance is distributed at different scales (wavenumbers k).

$$\sigma^2(z) = \int \frac{d^3k}{(2\pi)^3} \langle |\delta_{\vec{k}}|^2 \rangle = \int \frac{d^3k}{(2\pi)^3} P(k) \tag{22}$$

$$\begin{aligned} \sigma^2(z) &= \int_0^\infty \frac{dk}{2\pi^2} k^2 P(k) \\ &= \int_0^\infty \Delta^2(k) \frac{dk}{k} \end{aligned} \tag{23}$$

where the **dimensionless power spectrum** $\Delta^2(k)$ is defined as:

$$\Delta^2(k, z) = \frac{k^3 P(k, z)}{2\pi^2} \tag{24}$$

  - $P(k)$ is the *spectral density* of the variance per unit k-space volume.
  - $\Delta^2(k)$ represents the *contribution to the total variance per logarithmic interval in* k:

$$\Delta^2(k) \approx \frac{d\sigma^2}{d \ln k} \tag{25}$$

3 Halo Mass Function

3.1 Variance relation

To study structure formation at mass scale M, we smooth the density field with a window function W:

$$\delta_M(\mathbf{x}) = \int \delta(\mathbf{x}') W(|\mathbf{x}' - \mathbf{x}|; R) d^3\mathbf{x}' \tag{26}$$

The smoothing scale R relates to mass M as below (just a ball):

$$M = \frac{4\pi}{3}\bar{\rho}R^3 \quad \text{(Top-hat filter)} \tag{27}$$

Review the critical threshold for collapse comes from the spherical collapse model:

$$\delta_c = \frac{3(12\pi)^{2/3}}{20} \approx 1.686 \tag{28}$$

For the top-hat filter in real space, the Fourier transform is:

$$W(kR) = \frac{3}{(kR)^3}(\sin kR - kR\cos kR) \tag{29}$$

The variance of the smoothed density field (**sigma_z0**2 (Note the return sqrt(result)) as below in code**) characterizes the amplitude of fluctuations at scale M:

$$\sigma^2(M,z) = \langle \delta^2(M,z) \rangle = \frac{1}{2\pi^2}\int_0^\infty k^2 P(k) W^2(kR) dk \tag{30}$$

```
double dsigma_dk(double k, void *params){
    double p, w, T, gamma, q, aa, bb, cc, kR;
    double Radius;

    Radius = *(double *)params;
    kR = k*Radius;

    if ( (global_params.FILTER == 0) || (sigma_norm < 0) ){ // top hat
        if ( (kR) < 1.0e-4 ){ w = 1.0;} // w converges to 1 as (kR) -> 0
        else { w = 3.0 * (sin(kR)/pow(kR, 3) - cos(kR)/pow(kR, 2));}
    }
    else if (global_params.FILTER == 1){ // gaussian of width 1/R
        w = pow(E, -kR*kR/2.0);
    }
    return k*k*p*w*w;
}
```

```
double sigma_z0(double M){
    double result, error, lower_limit, upper_limit;
    double kstart, kend;
    double Radius;

//    R = MtoR(M);
    Radius = MtoR(M);
    // now lets do the integral for sigma and scale it with sigma_norm

```

```
      kstart = 1.0e-99/Radius;
      kend = 350.0/Radius;

    lower_limit = kstart;//log(kstart);
    upper_limit = kend;//log(kend);

    F.function = &dsigma_dk;
    F.params = &Radius;

    int status;

    status = gsl_integration_qag (&F, lower_limit, upper_limit,
     0, rel_tol,1000, GSL_INTEG_GAUSS61, w, &result, &error);

    return sigma_norm * sqrt(result);
}
```

The derivative of the variance with respect to mass (**dsigmasqdm_z0 as below in code**) is:

$$\frac{d\sigma^2(M,z)}{dM} = \frac{1}{2\pi^2}\int_0^\infty k^2 P(k)\cdot 2W(kR)\frac{dW}{dR}\frac{dR}{dM}dk \tag{31}$$

For the top-hat filter:

$$\frac{dR}{dM} = \frac{1}{4\pi\bar{\rho}R^2} \tag{32}$$

The window function derivative:

$$\frac{dW}{dR} = \frac{9k\cos(kR)}{(kR)^3} + \frac{3k\sin(kR)}{kR}\left(1 - \frac{3}{(kR)^2}\right) \tag{33}$$

```
double dsigmasq_dm(double k, void *params){
    double p, w, T, gamma, q, aa, bb, cc, dwdr, drdm, kR;
    double Radius;
    Radius = *(double *)params;

    // now get the value of the window function
    kR = k * Radius;
    if (global_params.FILTER == 0){ // top hat
        if ( (kR) < 1.0e-4 ){ w = 1.0; }// w converges to 1 as
   (kR) -> 0
        else { w = 3.0 * (sin(kR)/pow(kR, 3) - cos(kR)/pow(kR,
   2));}

        // now do d(w^2)/dm = 2 w dw/dr dr/dm
        if ( (kR) < 1.0e-10 ){  dwdr = 0;}
        else{
        dwdr = 9*cos(kR)*k/pow(kR,3) + 3*sin(kR)*(1 - 3/(kR*kR)
   )/(kR*Radius);}
        //3*k*( 3*cos(kR)/pow(kR,3) + sin(kR)*(-3*pow(kR, -4) +
    1/(kR*kR)) );}
        //      dwdr = -1e8 * k / (R*1e3);
```

```
        drdm = 1.0 / (4.0*PI * cosmo_params_ps->OMm*RHOcrit *
    Radius*Radius);
    }
    else if (global_params.FILTER == 1){ // gaussian of width
    1/R
        w = pow(E, -kR*kR/2.0);
        dwdr = - k*kR * w;
        drdm = 1.0 / (pow(2*PI, 1.5) * cosmo_params_ps->OMm*
    RHOcrit * 3*Radius*Radius);
    }
//    return k*k*p*2*w*dwdr*drdm * d2fact;
    return k*k*p*2*w*dwdr*drdm;
}
```

```
double dsigmasqdm_z0(double M){
    double result, error, lower_limit, upper_limit;
    double kstart, kend;
    double Radius;

//    R = MtoR(M);
    Radius = MtoR(M);
      kstart = 1.0e-99/Radius;
      kend = 350.0/Radius;

    lower_limit = kstart;//log(kstart);
    upper_limit = kend;//log(kend);

    F.function = &dsigmasq_dm;
    F.params = &Radius;

    int status;

    gsl_set_error_handler_off();

    status = gsl_integration_qag (&F, lower_limit, upper_limit,
     0, rel_tol,1000, GSL_INTEG_GAUSS61, w, &result, &error);

    gsl_integration_workspace_free (w);

//    return sigma_norm * sigma_norm * result /d2fact;
    return sigma_norm * sigma_norm * result;
}
```

3.2 Dicke!

The linear growth factor $D(z)$ scales fluctuations with redshift:

$$\begin{aligned} \sigma(M,z) &= \sigma(M,0)\cdot D(z), \\ \sigma^2(M,z) &= \sigma^2(M,0)\cdot D^2(z) \end{aligned} \tag{34}$$

and:

$$\begin{aligned}\frac{d\sigma(M,z)}{dM} &= D(z)\cdot\frac{d\sigma(M,0)}{dM}\\ \frac{d\sigma(M,0)}{dM} &= \frac{1}{2\sigma(M,0)}\frac{d\sigma^2(M,0)}{dM}.\end{aligned} \tag{35}$$

We get:

$$\begin{aligned}\frac{d\sigma(M,z)}{dM} &= D(z)\cdot\left[\frac{1}{2\sigma(M,0)}\frac{d\sigma^2(M,0)}{dM}\right]\\ &= \frac{D^2(z)}{2\sigma(M,0)}\cdot\frac{d\sigma^2(M,0)}{dM}\end{aligned} \tag{36}$$

```
double dNdM_st(double growthf, double M) {
    double sigma = sigma_z0(M);        //
    double dsigmadm = dsigmasqdm_z0(M); //
    // Apply redshift evolution
    sigma *= growthf;  //
    dsigmadm *= (growthf*growthf/(2.*sigma));  //
    // ......
}
```

The factor `(growthf*growthf/(2.*sigma))` combines the growth factor evolution and the conversion from $\frac{d\sigma^2(M,z)}{dM}(0)$ to $\frac{d\sigma_M}{dM}(z)$.

3.3 Mass Function Derivation

The fundamental quantity is the fraction of mass contained in halos above mass M:

$$F(>M) = \int_{\delta_c}^{\infty} P(\delta_M)d\delta_M \tag{37}$$

For the Gaussian field (overdensity field):

$$F(>M)(z) = \frac{1}{2}\,\text{erfc}\left(\frac{\delta_c}{\sqrt{2}\sigma(M,z)}\right) \tag{38}$$

The factor of $1/2$ arises because only overdense regions collapse, but Press & Schechter recognized that underdense regions would be incorporated into larger structures, leading to the "cloud-in-cloud" problem. Their solution was to multiply by a factor 2:

$$F(>M)(z) = \text{erfc}\left(\frac{\nu(z)}{\sqrt{2}}\right), \quad \nu(z) \equiv \frac{\delta_c}{\sigma(M,z)}. \tag{39}$$

Note the complete explain is come from random walk theory.

The comoving number density of halos in the mass range $[M, M+dM]$ is:

$$n(M,z)dM = \frac{\bar{\rho}}{M}\left|\frac{dF(z)}{dM}\right|dM \tag{40}$$

Differentiating F with respect to M:

$$\frac{dF(z)}{dM} = \frac{dF}{d\nu(z)}\frac{d\nu(z)}{dM} \tag{41}$$

First term:

$$\frac{dF}{d\nu(z)} = -\sqrt{\frac{2}{\pi}}e^{-\nu(z)^2/2} \tag{42}$$

Second term:

$$\frac{d\nu(z)}{dM} = -\frac{\delta_c}{\sigma^2(M,z)}\frac{d\sigma(M,z)}{dM} \tag{43}$$

We get:

$$\frac{dF(z)}{dM} = \sqrt{\frac{2}{\pi}}e^{-\nu(z)^2/2} \cdot \frac{\delta_c}{\sigma^2(M,z)}\left|\frac{d\sigma(M,z)}{dM}\right| \tag{44}$$

So the mass function is:

$$n(M,z) = \sqrt{\frac{2}{\pi}}\frac{\bar{\rho}}{M}\frac{\delta_c}{\sigma(M,z)}\left|\frac{1}{\sigma(M,z)}\frac{d\sigma(M,z)}{dM}\right| e^{-\delta_c^2/(2\sigma^2(M,z))} \tag{45}$$

Because of the $\frac{d\ln\sigma_M}{d\ln M} = \frac{M}{\sigma_M}\frac{d\sigma_M}{dM}$, we get:

$$\begin{aligned}
\frac{dn}{dM}(M,z) &= -\frac{\bar{\rho}}{M}f(\nu(z))\frac{d\ln\sigma(M,z)}{dM} \\
n(M,z)dM &= -\frac{\bar{\rho}}{M^2}f(\nu(z))\left|\frac{d\ln\sigma(M,z)}{d\ln M}\right|dM \\
f(\nu(z)) =& \sqrt{\frac{2}{\pi}}\frac{\delta_c}{\sigma(M,z)}\exp(-\frac{\delta_c^2}{2\sigma^2(M,z)}) = \sqrt{\frac{2}{\pi}}\nu\exp(-\frac{\nu^2}{2}).
\end{aligned} \tag{46}$$

Confirm Cooray & Sheth 2002: They gave this HMF form:

$$\begin{aligned}
\frac{M^2 n_c(M,z)}{\bar{\rho}}\frac{dM}{M} &= \nu_c f(\nu_c)\frac{d\nu_c}{\nu_c} \\
\nu_c f(\nu_c) = \sqrt{\frac{\nu_c}{2\pi}}\exp(-\frac{\nu_c}{2}) &\qquad \nu_c = \frac{\delta_c^2}{\sigma^2(M,z)}.
\end{aligned} \tag{47}$$

Note that $n_c(M,z)$ in Cooray is just $\frac{dn}{dm}$, ν_c is our ν^2 , now we transform to our symbol and simplify the formula:

$$\begin{aligned}
\frac{M dn(M,z)}{\bar{\rho}\, dM} &= f(\nu)\frac{d\nu^2}{dM} \\
f(\nu) =& \sqrt{\frac{1}{2\pi}}\frac{1}{\nu}\exp(-\frac{\nu^2}{2}) \\
d\nu^2 = 2\nu d\nu \qquad \frac{dn}{dM}(M,z) &= \frac{\bar{\rho}}{M}\sqrt{\frac{2}{\pi}}\exp(-\frac{\nu^2}{2})\frac{d\nu}{dM}.
\end{aligned} \tag{48}$$

Therefore:

$$\begin{aligned}
&\frac{d\nu}{dM} = \frac{d\nu}{d\sigma}\frac{d\sigma}{dM} \qquad \frac{d\nu}{d\sigma} = (\frac{\delta_c}{\sigma(M,z)})' = -\frac{\delta_c}{\sigma^2(M,z)} \\
&\frac{d\nu}{dM} = -\frac{\delta_c}{\sigma(M,z)}\frac{1}{\sigma(M,z)}\frac{d\sigma}{dM} = -\nu\frac{d\ln\sigma}{dM} \\
&\frac{dn}{dM}(M,z) = -\frac{\bar{\rho}}{M}\sqrt{\frac{2}{\pi}}\nu\exp(-\frac{\nu^2}{2})\frac{d\ln\sigma}{dM},
\end{aligned} \tag{49}$$

we can see it is indeed our HMF, summary the HMF form as below:

$$\begin{aligned}
&\frac{dn}{dM}(M,z) = -\frac{\bar{\rho}}{M}\sqrt{\frac{2}{\pi}}\nu\exp(-\frac{\nu^2}{2})\frac{d\ln\sigma}{dM} \\
&\frac{dn}{dM}(M,z) = \frac{\bar{\rho}}{M}\sqrt{\frac{2}{\pi}}\exp(-\frac{\nu^2}{2})\frac{d\nu}{dM} \\
&\frac{dn}{dM}(M,z) = \frac{\bar{\rho}}{M}\sqrt{\frac{1}{2\pi}}\frac{1}{\nu}\exp(-\frac{\nu^2}{2})\frac{d\nu^2}{dM}
\end{aligned} \tag{50}$$

A better fit to the number density of halos in simulations of gravitational clustering in the CDM family of models is given by Sheth and Tormen:

$$f_{\mathrm{ST}}(\nu) = A\sqrt{\frac{2}{\pi}}\,[(1+(a\nu^2)^{-p})\,\nu\exp(-\frac{a\nu^2}{2})], \quad \nu = \nu(z) \tag{51}$$

```c
double dNdM_st(double growthf, double M){

    double sigma, dsigmadm, nuhat;

    float MassBinLow;
    int MassBin;


        sigma = sigma_z0_CDM(M);
        dsigmadm = dsigmasqdm_z0_CDM(M);

    sigma = sigma * growthf;
    dsigmadm = dsigmadm * (growthf*growthf/(2.*sigma));

    nuhat = sqrt(SHETH_a) * Deltac / sigma;

    return (-(cosmo_params_ps->OMm)*RHOcrit/M) * (dsigmadm/
     sigma) * sqrt(2./PI)*SHETH_A * (1+ pow(nuhat, -2*SHETH_p))
     * nuhat * pow(E, -nuhat*nuhat/2.0);
}
```

3.3.1 Conditional HMF

Consider the environmental effect, for example, the given denser cells at redshift z with volume V and corresponding mass M_V may be thought of as regions in which the critical density for collapse is easier to reach, detail see Cooray & Sheth 2002, define the new halo multiplicity:

$$\nu_{10}^2 \equiv \frac{[\delta_c - \delta(z)]^2}{\sigma^2(M,z) - \sigma^2(M_V,z)}. \tag{52}$$

This is just conditional HMF, and we calculate the PS conditional HMF:

$$\begin{aligned}
\frac{d\nu_{10}^2}{dM} &= \frac{d\nu_{10}^2}{d\sigma}\frac{d\sigma}{dM} = d\left(\frac{[\delta_c - \delta(z)]^2}{\sigma^2(M,z) - \sigma^2(M_V,z)}\right)\frac{d\sigma}{dM} \\
&= \frac{2\sigma(M,z)[\delta_c - \delta(z)]^2}{[\sigma^2(M,z) - \sigma^2(M_V,z)]^2}\frac{d\sigma}{dM} \\
\frac{dn}{dM}(M,z) &= \frac{\bar{\rho}}{M}\sqrt{\frac{1}{2\pi}}\frac{1}{\nu_{10}}\exp\left(-\frac{\nu_{10}^2}{2}\right)\frac{d\nu_{10}^2}{dM} \\
&= \frac{\bar{\rho}}{M}\sqrt{\frac{1}{2\pi}}\frac{\sqrt{\sigma^2(M,z) - \sigma^2(M_V,z)}}{\delta_c - \delta(z)}\frac{2\sigma(M,z)[\delta_c - \delta(z)]^2}{[\sigma^2(M,z) - \sigma^2(M_V,z)]^2}\exp\left(-\frac{\nu_{10}^2}{2}\right)\frac{d\sigma}{dM} \\
&= \frac{\bar{\rho}}{M}\sqrt{\frac{1}{2\pi}}[\delta_c - \delta(z)]\exp\left(-\frac{\nu_{10}^2}{2}\right)\frac{2\sigma(M,z)}{[\sigma^2(M,z) - \sigma^2(M_V,z)]^{1.5}}\frac{d\sigma}{dM},
\end{aligned} \tag{53}$$

consistent with the code as below:

```
float dNdM_conditional(float growthf, float M1, float M2, float
    delta1, float delta2, float sigma2){

    float sigma1, dsigmadm,dsigma_val;

    sigma1 = sigma_z0(exp(M1));
    dsigmadm = dsigmasqdm_z0(exp(M1));


    M1 = exp(M1);
    M2 = exp(M2);

    sigma1 = sigma1*sigma1;
    sigma2 = sigma2*sigma2;

    dsigmadm = dsigmadm/(2.0*sigma1); // This is actually
   sigma1^{2} as calculated above, however, it should just be
   sigma1. It cancels with the same factor below. Why I have
   decided to write it like that I don't know!

    if((sigma1 > sigma2)) {

        return -(( delta1 - delta2 )/growthf)*( 2.*sigma1*
   dsigmadm )*( exp( - ( delta1 - delta2 )*( delta1 - delta2 )
   /( 2.*growthf*growthf*( sigma1 - sigma2 ) ) ) )/(pow(
   sigma1 - sigma2, 1.5));
    }
    else if(sigma1==sigma2) {

```

```
        return -(( delta1 - delta2 )/growthf)*( 2.*sigma1*dsigmadm )*( exp( - ( delta1 - delta2 )*( delta1 - delta2 )/( 2.*growthf*growthf*( 1.e-6 ) ) ) )/(pow( 1.e-6, 1.5));

    }
    else {
        return 0.;
    }
}
```