# Note for 21cmFAST

Sehighs

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## 1 Reionzation

Now we talk about the emission rate of ionizing photons into IGM per baryon can be written as:

$$\dot{n}_{ion} = \frac{d}{dt}[\rho_b^{-1} \int_{M_{min}}^{\infty} dM_h \frac{dn}{dlnM_h} f_b f_* f_{esc} N_b] \tag{1}$$

The term $\frac{dn}{dlnM_h}$ is halo mass function(HMF), and in 21cmfast, the HMF is obtained by Sheth-Tormen HMF but the parameters are using Jenkins. HMF can written as bellow in detail:

$$\frac{dn}{dM} = -\frac{\rho_m}{M} f(\nu) \frac{dln\sigma}{dM} \tag{2}$$

$$\rho_m = \frac{8\pi G}{3H_0^2} \tag{3}$$

and the term $\frac{dln\sigma}{dM}$ also can equality written as $\frac{dln\sigma}{dlnM}\frac{1}{M}$ or $\frac{d\sigma}{dM}\frac{1}{\sigma}$ and 21cmfast using the latter.
$\rho_b^{-1}$ is the baryon mass density. And $f_*$ is the fraction of baryon which forming star. $f_b$ is the mass fraction of baryon in halo mass. $f_*$ is the fraction of halo baryons ending up in stars. $N_\gamma$ is number of ionizing photon generated by per star baryon. The last one $f_{esc}$ is the fraction of ionizing photon can escape into IGM.The unit of HMF is number per halo mass. The unit of term $f_b f_* f_{esc} N_b$ is the ionizing photon into IGM number per baryon in star from halo, in fact finally the unit is number. So the unit of term integral is a mass density, but it is from stars total mass, aiming to get the photon number, we should set the $\rho_b$ transfer the mass density to number.
Note the lower limit of integration $M_{min}$ is the halo which have enough mass to format and host stellar, so we define the costant $\zeta$ as bellow:

$$\begin{aligned} &\zeta = f_b f_* f_{esc} N_b, \\ &\zeta = 0, when\ \ M < M_{min} \end{aligned} \tag{4}$$

Now we define the mass dependence model. In simple case we take $\zeta$ depending mass entirely, show as:

$$\zeta = \zeta_0 (\frac{M_h}{M_{min}})^{\alpha} \tag{5}$$

So the emission rate can be written as:

$$\dot{n}_{ion} = \frac{d}{dt}[\frac{\zeta_0}{\rho_b^{-1}} \int_{M_{min}}^{\infty} dM_h \frac{dn}{dlnM_h} (\frac{M_h}{M_{min}})^{\alpha}] \tag{6}$$

More detail, we will take $f_{esc}$ and $f_*$ distinctly. We have:

$$\begin{aligned} f_{esc} &= f_{esc,10}(\frac{M_h}{10^{10}M_\odot})^{\alpha_{esc}} \\ f_* &= f_{*,10}(\frac{M_h}{10^{10}M_\odot})^{\alpha_*} \end{aligned} \tag{7}$$

Now we can write emission rate as:

$$\dot{n}_{ion} = \frac{d}{dt}[\rho_b^{-1} \int_{M_{min}}^{\infty} dM_h \frac{dn}{dlnM_h} (\frac{M_h}{10^{10}M_\odot})^{\alpha_*+\alpha_{esc}} f_b f_{*,10} f_{esc,10} N_b] \tag{8}$$

Time to talk about what is $f_b$, in former analyze, we set $f_b$ is the baryon mass fraction of halo, so the term $f_b f_* M_h$ will be the total star mass in a given mass halo. But connection to the cosmic mean value of fraction $\Omega_b/\Omega_m$, we hope set a coefficient $f_b'$ to unit $f_b$ by $\Omega_b/\Omega_m$, so the emission can written as:

$$\dot{n}_{ion} = \frac{d}{dt}[\rho_m^{-1} \int_{M_{min}}^{\infty} dM_h \frac{dn}{dlnM_h} (\frac{M_h}{10^{10}M_\odot})^{\alpha_*+\alpha_{esc}} f_b' f_{*,10} f_{esc,10} N_b] \tag{9}$$

This is the form of mesinger's note, the $f_b'$ is them note's $f_b$:the baryon fraction of a halo, in units of the cosmic mean value, $\Omega_b/\Omega_m$. And the term $\rho_b^{-1}$ absorb the term $\Omega_b/\Omega_m$ become $\rho_m^{-1}$ because of $\rho_b^{-1} = \rho_m^{-1}/\frac{\Omega_b}{\Omega_m}$. We can also explain the from another sight: $f_b$ transfer the halo mass to baryon mass, so we need the term $\rho_b$, but $f_b'$ not, the mass is still relative to halo, so we keep $\rho_m$.

NOTE:$f_b'$ will SET to 1 BELLOW !!!!(because park and 21cmFAST code, and I don't want to set.)

Combine $f_*$ and $\frac{\Omega_b}{\Omega_m}$, we have:

$$M_* = f_*(\frac{\Omega_b}{\Omega_m})M_h \tag{10}$$

So emission rate become:

$$\dot{n}_{ion} = \frac{d}{dt}[\rho_b^{-1} \int_{M_{min}}^{\infty} dM_h \frac{dn}{dM_h} (\frac{M_h}{10^{10}M_\odot})^{\alpha_{esc}} M_* f_{esc,10} N_b] \tag{11}$$

But the story not finished, note not all haloes can generate and host stars and galaxies, so we should define a fraction of how much mass of a halo can host star, written as:

$$f_{duty} = exp(-\frac{M_{turn}}{M_h}) \tag{12}$$

So the emission rate become:

$$\dot{n}_{ion} = \frac{d}{dt}[\rho_b^{-1}\int_{M_{min}}^{\infty} dM_h \frac{dn}{dM_h}(\frac{M_h}{10^{10}M_\odot})^{\alpha_{esc}} f_{duty} M_* f_{esc,10} N_b] \tag{13}$$

For convenience, we written mass related and costant independently:

$$\begin{aligned} &\zeta = f_{esc,10} f_{*,10} N_b \\ &\dot{n}_{ion} = \frac{d}{dt}[\frac{\zeta}{\rho_m}\int_{M_{min}}^{\infty} dM_h \frac{dn}{dM_h}(\frac{M_h}{10^{10}M_\odot})^{\alpha_{esc}+\alpha_*} exp(-\frac{M_{turn}}{M_h})M_h] \end{aligned} \tag{14}$$

Note the transfer of $\rho_b$ to $\rho_m$ because the $\Omega_b/\Omega_m$ have been absorbed. Function 13 and 14 is our finally form and $f_b'$ have set to 1 aiming to compare with Park and Code.

okay, let us comparing!!! So we written the equation of the code!

$$\begin{aligned} &\zeta = f_{esc,10} f_{*,10} N_b \\ &\dot{n}_{ion} = \frac{d}{dt}[\frac{\zeta}{\rho_m}\int_{M_{min}}^{\infty} dlnM_h \frac{dn}{dM_h}(\frac{M_h}{10^{10}M_\odot})^{\alpha_{esc}+\alpha_*} exp(-\frac{M_{turn}}{M_h})M_h^2] \end{aligned} \tag{15}$$

The term $M_h^2$ will become $M_h$ because of the integral variance $lnM_h$, so it is consistent to Park18.
And written Park2018!

$$\dot{n}_{ion} = \frac{d}{dt}[\rho_b^{-1}\int_0^{\infty} dM_h \frac{dn}{dM_h} f_{duty} M_* f_{esc} N_b] \tag{16}$$

or

$$\begin{aligned} &\zeta = f_{esc,10} f_{*,10} N_b \\ &\dot{n}_{ion} = \frac{d}{dt}[\frac{\zeta}{\rho_m}\int_0^{\infty} dM_h \frac{dn}{dM_h}(\frac{M_h}{10^{10}M_\odot})^{\alpha_{esc}+\alpha_*} exp(-\frac{M_{turn}}{M_h})M_h] \end{aligned} \tag{17}$$

We can see the consistent of our form and Park and code!

We should talk about the lower limit of integration. In prime Note, the lower limit is $M_{min}$ which is the minimum mass of the halo can host star. But in Park2018, we find the new label $M_{turn}$ is displacing $M_{min}$.
But in Park2018 we notice the lower limit is zero, and it is not physical(I think), and in the code, the lower limit also is $M_{min}$, and it is defined by $M_{turn}/50$, I also think it is not physical but I can understand: I guess their model can not guarantee the $M_{turn}$ is turely physical lower limit, so they roughly defined the $M_{min}$ is $M_{turn}/50$.

## 2 X-ray Effect

Now we talk about the X-ray effect. In general, X-ray will heating the IGM and 21cm spin temperature have coupled with kinetic temperature of gas because of the WF effect, so the curve of $\delta T_b$ will rise as the redshift decreases.

## 2.1 Radiation Transfer Function

We review the transfer function of astronomy, and analysis unit of ever physical quantity.
First physical quantity is specific intensity $I_\nu$, written as:

$$I_\nu = \frac{dE}{d\nu dt cos\theta d\sigma d\Omega} \tag{18}$$

So the specific intensity is the energy pass the per unit area per unit frequency and time from unit solid angle of source, the $\theta$ is the angle formed by the plane normal between the directions of the light. The term $cos\theta d\sigma$ ensure the surface of $d\Omega$ paralleling to the receiving surface. It's unit is $erg\ cm^{-2}s^{-1}sr^{-1}Hz^{-1}$. Maybe we can say the specific intensity is just a area density of power but limited by source solid angle and plane of receiving area.
And the mean specific intensity can describe by the integral of the solid angle:

$$J_\nu = \frac{1}{4\pi}\int I_\nu d\Omega \tag{19}$$

The unit is same as $I_\nu$.
And the specific flux is the fact energy passing the defined receiving area, it can be written as:

$$\pi F = \frac{dE}{dt d\sigma d\nu} \tag{20}$$

So

$$\pi F = \int_{4\pi} I_\nu cos\theta d\Omega \tag{21}$$

The unit of specific flux is $erg\ cm^{-2}s^{-1}Hz^{-1}$.
The radiation energy density is the energy per unit volume(per frequency):

$$u_\nu = \frac{dE}{dV d\nu d\Omega} \tag{22}$$

And the unit volume can describe the unit passing area times velocity times unit time.

$$dV = dAcdt \tag{23}$$

So

$$u_\nu = \frac{dE}{cdtdAd\Omega d\nu} \tag{24}$$

By compare with the former quantities, we can write the energy density as:

$$u_\nu = \frac{I_\nu}{c} = \frac{4\pi}{c}J_\nu \tag{25}$$

The unit of radiation energy density is $erg\ cm^{-3}Hz^{-1}sr^{-1}$.
Finally we should talk about the radiation transfer(RT) function. We defined specific emissivity is the energy radiation from the unit volume medium per unit

time per frequency per unit solid angle along the certain direction. It can be written as:

$$j_\nu = \frac{dE}{dtdVd\nu d\Omega} \tag{26}$$

If we write $dV$ as $dAds$, we will find the relation between emissivity and former quantities:

$$\begin{aligned} dI_\nu &= j_\nu ds \\ du_\nu &= j_\nu dt \end{aligned} \tag{27}$$

So we can say the specific emissivity is energy volume density raising rate and define a given solid angle and receiving area, the specific intensity will raise along the path. We can image if we defined a solid angle and receiving area, that means a given volume, and the energy will raise when radiation passing the path, the increment is up to the length of path or equal to the passing time because light velocity is a constant. The unit of specific emissivity is $erg\ \ s^{-1}cm^{-3}sr^{-1}Hz^{-1}$.
Now we define absorption $\kappa$ and write the RT function:

$$\begin{aligned} dI_\nu &= -\kappa_\nu I_\nu ds \\ \tau &= \int \kappa_\nu ds \end{aligned} \tag{28}$$

RT function is

$$\begin{aligned} \frac{dI_\nu}{ds} &= -\kappa_\nu I_\nu + j_\nu \\ S_\nu &= j_\nu/\kappa_\nu \\ I_\nu(B) &= I_\nu(A)e^\tau + \int_0^\tau S_\nu d\tau \end{aligned} \tag{29}$$

Or if no initial specific intensity in the start position, we just write:

$$J_\nu = \frac{1}{4\pi}\int_{path} j_\nu ds \tag{30}$$

Now we summarize all of quantities as below:

Table 1: **Radiation Quantities**

| quantities | name | unit | relation |
|---|---|---|---|
| $I_\nu$ | specific intensity | $erg\ \ cm^{-2}s^{-1}sr^{-1}Hz^{-1}$ | nan |
| $J_\nu$ | mean intensity | $erg\ \ cm^{-2}s^{-1}sr^{-1}Hz^{-1}$ | $\frac{1}{4\pi}I_\nu$ |
| $\pi F$ | specific flux | $erg\ \ cm^{-2}s^{-1}Hz^{-1}$ | $\frac{dE}{dtd\sigma d\nu}$ |
| $u_\nu$ | energy density | $erg\ \ cm^{-3}sr^{-1}Hz^{-1}$ | $\frac{I_\nu}{c}$ and $\int j_\nu dt$ |
| $j_\nu$ | specific emissivity | $erg\ \ cm^{-3}s^{-1}sr^{-1}Hz^{-1}$ | $\frac{du_\nu}{dt}$ and $\frac{dI_\nu}{ds}$ |
| $\kappa$ | absorption | $cm^{-1}$ | nan |
| $S_\nu$ | source function | $erg\ \ cm^{-2}s^{-1}sr^{-1}Hz^{-1}$ equal to $I_{nu}$ | $I_\nu = \int S_\nu d\tau$ |
| $\tau$ | optical depth | nan | $\int_{path} \kappa ds$ |

## 2.2 X-ray Effect

The X-ray emissivity is $\epsilon_X$, and now we will transfer the 'per frequency' to 'per photon energy', that means all quantities will become energy distribution.
Write down the RT function of X-ray:

$$J(x, E, z) = \frac{(1+z)^3}{4\pi} \int_z^\infty dz' \frac{cdt}{dz'} \epsilon_X e^{-\tau} \tag{31}$$

The term $\frac{cdt}{dz'}$ is used to transfer the integral of path to redshift. $(1+z)^3$ is related to emissivity evolution with redshift, and the energy evolution have disappeared because of the the term $erg\ \ keV^{-1}$ included by unit $erg\ \ cm^{-3}s^{-1}sr^{-1}keV^{-1}$
And the emissivity written by Park is:

$$\epsilon(x, E, z') = \frac{L_X}{SFR}[(1+\delta_{nl}) \int_0^\infty dM_h \frac{dn}{dM_h} f_{duty} \dot{M}_*] \tag{32}$$

Now we analyze the function. The term $\frac{dn}{dM_h}$ is HMF. The unit of HMF is number density per sun mass, it is $V^{-1}M_\odot^{-1}$. The integrand function HMF times $f_{duty}$ means the number density of the halo which can host stars and galaxies.
Now we talk about what's meaning of $\frac{dn}{dM_h} f_{duty} M_*$, it is the number density of enough mass halos times the how much stellar mass can formation per unit halo mass. So the unit is $M_* V_h^{-1} M_h^{-1}$, we can image in a given region $V$ where are many halos, $n_M$ is the number of halos which mass range is $M$ to $M + dM$, and these halos will generate stellar, ever halo will host the stellar which mass is $M_*$, the total stellar mass in the given region in the given halo mass range is $n_M M_*$, so the mass density is $n_M M_*/V$, is $nM_*$.And the true total mass density should integral with halo mass $M$, it is just $\int_0^\infty dM_h \frac{dn}{dM_h} f_{duty} M_*$. The unit of the rate of the integral is density $M/V\ \ T^{-1}$.
The term $(1+\delta_{nl})$ can write as $\rho_{nl}/\rho$, the useful of this fraction is UNKNOW. $SFR$ is the mean $SFR$, the unit is $MT^{-1}$. The last or we should say the first term $L_X$ is luminosity per energy of X-ray, it's unit is $erg\ \ s^{-1}keV^{-1}$. In 21cmFAST,

$$L_X = E^{\alpha_X}, \alpha_X\ \ set\ \ to\ \ 1 \tag{33}$$

So the unit of the X-ray emissivity is $erg\ \ cm^{-3}s^{-1}keV^{-1}$, WHERE IS THE $sr^{-1}$???
Our X-ray parameter $L_X/SFR$ in fact is

$$L_{X<2keV}/SFR = \int_{E_0}^{2keV} dE L_X/SFR \tag{34}$$

where $E_0$ is the X-ray energy threshold below which photons are absorbed inside the host galaxies. So if we vary this parameter is correponding to vary $\alpha_X$ because the determined upper and lower integral limit, and finally it is equal to vary the X-ray specific emissivity and mean specific intensity.

# 3 Gnedin & Abel 2001

## 3.1 FT equation

We write the RT equation along an affine parameter $s$:

$$\frac{dI_\nu}{ds} = -\,\kappa_\nu I_\nu + j_\nu. \tag{35}$$

Photon phase space: $(t, \vec{x}, \nu, \vec{n})$. Apply the chain rule:

$$\frac{dI_\nu}{ds} = \frac{dt}{ds}\,\partial_t I_\nu + \frac{d\vec{x}}{ds}\cdot\nabla I_\nu + \frac{d\nu}{ds}\,\partial_\nu I_\nu + \frac{d\vec{n}}{ds}\cdot\partial_{\vec{n}} I_\nu. \tag{36}$$

$$\frac{dt}{ds} = \frac{1}{c}, \qquad \frac{d\vec{x}}{ds} = \vec{n}, \qquad \frac{d\nu}{ds} = -\,\frac{H\nu}{c} \quad \text{(redshift)}, \qquad \frac{d\vec{n}}{ds} = 0. \tag{37}$$

Insert (37) into (36):

$$\frac{1}{c}\partial_t I_\nu + \vec{n}\cdot\nabla I_\nu - \frac{H\nu}{c}\,\partial_\nu I_\nu = j_\nu - \kappa_\nu I_\nu. \tag{38}$$

$$\partial_t I_\nu = \frac{dI_\nu}{dt} - \partial_\nu I_\nu \frac{d\nu}{dt}\,. \tag{39}$$

Now handle the term $\vec{n}\cdot\nabla I_\nu$:

$$\begin{aligned}
\frac{d\vec{x}}{dt} &= c\cdot\vec{n} \\
\nabla\cdot\dot{\vec{x}}\,I_\nu &= \dot{\vec{x}}\cdot\nabla I_\nu + I_\nu\,\nabla\cdot\dot{\vec{x}} \\
&= c\,\vec{n}\cdot\nabla I_\nu \;+\; I_\nu\,\nabla\cdot\dot{\vec{x}} \qquad \nabla\cdot\dot{\vec{x}} = 3H \\
\Rightarrow\quad \vec{n}\cdot\nabla I_\nu &= \frac{1}{c}(-3HI_\nu + \nabla\cdot\dot{\vec{x}}\,I_\nu)\,.
\end{aligned} \tag{40}$$

We get:

$$\begin{aligned}
\frac{1}{c}\partial_t I_\nu + \vec{n}\cdot\nabla I_\nu - \frac{H\nu}{c}\,\partial_\nu I_\nu &= j_\nu - \kappa_\nu I_\nu \\
\frac{1}{c}\partial_t I_\nu + \frac{1}{c}(-3HI_\nu + \nabla\cdot\dot{\vec{x}}\,I_\nu) - \frac{H\nu}{c}\,\partial_\nu I_\nu &= j_\nu - \kappa_\nu I_\nu \\
\partial_t I_\nu + \nabla\cdot\dot{\vec{x}}\,I_\nu - 3HI_\nu - H\nu\,\partial_\nu I_\nu &= c(j_\nu - \kappa_\nu I_\nu) \\
\partial_t I_\nu + \nabla\cdot\dot{\vec{x}}\,I_\nu - H(3I_\nu + \nu\,\partial_\nu I_\nu) &= c(j_\nu - \kappa_\nu I_\nu)
\end{aligned} \tag{41}$$

The $-H(\nu\partial_\nu I_\nu + 3I_\nu)$ term captures cosmological redshift.

## 3.2 Background

Define an angular/spatial average:

$$\overline{I}_\nu \equiv \langle I_\nu(t, \vec{x}, \vec{n})\rangle. \tag{42}$$

Background (homogeneous) equation ($\nabla \overline{I}_\nu = 0$):

$$\partial_t \overline{I}_\nu - H\big(\nu\,\partial_\nu \overline{I}_\nu - 3\overline{I}_\nu\big) = \overline{j}_\nu - \overline{\kappa}_\nu\,\overline{I}_\nu, \qquad \overline{\kappa}_\nu \equiv \frac{\langle \kappa_\nu I_\nu\rangle}{\overline{I}_\nu}\ \ (I_\nu\text{-weighted}). \tag{43}$$

Factorize fluctuations as

$$I_\nu \equiv f_\nu\,\overline{I}_\nu, \qquad f_\nu \equiv \frac{I_\nu}{\overline{I}_\nu}, \qquad \langle f_\nu\rangle = 1. \tag{44}$$

Insert $I_\nu = f_\nu \overline{I}_\nu$ into (41). Start with the full (inhomogeneous) RT equation (dividing (41) by $c$ if desired is equivalent):

$$\partial_t I_\nu + c\,\vec{n}\cdot\nabla I_\nu - H(\nu\,\partial_\nu I_\nu + 3I_\nu) = c\,(j_\nu - \kappa_\nu I_\nu). \tag{45}$$

Using $I_\nu = f_\nu \overline{I}_\nu$:

$$\partial_t(f_\nu\overline{I}_\nu)\ +\ c\,\vec{n}\cdot\nabla(f_\nu\overline{I}_\nu)\ -\ H\big[\nu\,\partial_\nu(f_\nu\overline{I}_\nu) - 3f_\nu\overline{I}_\nu\big] = c\,\big(j_\nu - \kappa_\nu f_\nu\overline{I}_\nu\big)\,. \tag{46}$$

Expand each term step by step:

$$\partial_t(f_\nu\overline{I}_\nu) = f_\nu\,\partial_t\overline{I}_\nu + \overline{I}_\nu\,\partial_t f_\nu, \tag{47}$$

$$\vec{n}\cdot\nabla(f_\nu\overline{I}_\nu) = f_\nu\,\vec{n}\cdot\nabla\overline{I}_\nu + \overline{I}_\nu\,\vec{n}\cdot\nabla f_\nu, \tag{48}$$

$$\nu\,\partial_\nu(f_\nu\overline{I}_\nu) = \nu\big[(\partial_\nu f_\nu)\,\overline{I}_\nu + f_\nu\,\partial_\nu\overline{I}_\nu\big]. \tag{49}$$

For the homogeneous background, $\nabla\overline{I}_\nu = 0$, hence

$$f_\nu\,\partial_t\overline{I}_\nu + \overline{I}_\nu\,\partial_t f_\nu + c\,\overline{I}_\nu\,\vec{n}\cdot\nabla f_\nu - H\left[\nu\,(\partial_\nu f_\nu)\,\overline{I}_\nu + \nu f_\nu\,\partial_\nu\overline{I}_\nu - 3f_\nu\overline{I}_\nu\right] = c\,\big(j_\nu - \kappa_\nu f_\nu\overline{I}_\nu\big)\,. \tag{50}$$

Group terms proportional to $f_\nu$ and to $\overline{I}_\nu$:

$$f_\nu\Big[\partial_t\overline{I}_\nu - H\big(\nu\,\partial_\nu\overline{I}_\nu - 3\overline{I}_\nu\big)\Big] + \overline{I}_\nu\Big[\partial_t f_\nu + c\,\vec{n}\cdot\nabla f_\nu - H\,\nu\,\partial_\nu f_\nu\Big] = c\,\big(j_\nu - \kappa_\nu f_\nu\overline{I}_\nu\big)\,. \tag{51}$$

Use the background equation (43) to replace the square bracket:

$$f_\nu\,\big(\overline{j}_\nu - \overline{\kappa}_\nu\,\overline{I}_\nu\big) + \overline{I}_\nu\Big[\partial_t f_\nu + c\,\vec{n}\cdot\nabla f_\nu - H\,\nu\,\partial_\nu f_\nu\Big] = c\,\big(j_\nu - \kappa_\nu f_\nu\overline{I}_\nu\big)\,. \tag{52}$$

Move the $f_\nu(\cdots)$ term to the RHS and divide by $\overline{I}_\nu$:

$$\partial_t f_\nu + c\,\vec{n}\cdot\nabla f_\nu - H\,\nu\,\partial_\nu f_\nu = \frac{c\,j_\nu}{\overline{I}_\nu} - f_\nu\,\frac{c\,\overline{j}_\nu}{\overline{I}_\nu} - c\,\kappa_\nu f_\nu + c\,\overline{\kappa}_\nu f_\nu. \tag{53}$$

Define

$$\hat{\kappa}_\nu \equiv \kappa_\nu - \overline{\kappa}_\nu + \frac{\overline{j}_\nu}{\overline{I}_\nu}, \qquad \psi_\nu \equiv \frac{j_\nu}{\overline{I}_\nu}. \tag{54}$$

Then (53) becomes

$$\partial_t f_\nu + c\,\vec{n}\cdot\nabla f_\nu - H\,\nu\,\partial_\nu f_\nu = -\,c\,\hat{\kappa}_\nu\, f_\nu + c\,\psi_\nu. \tag{55}$$

Equivalently, multiplying by $a/c$ (Gnedin & Abel convention):

$$\boxed{\frac{a}{c}\,\partial_t f_\nu + \vec{n}\cdot\nabla f_\nu - \frac{aH}{c}\,\nu\,\partial_\nu f_\nu = -\,\hat{\kappa}_\nu^{(a)}\, f_\nu + \psi_\nu^{(a)}} \tag{56}$$

with the rescaled

$$\hat{\kappa}_\nu^{(a)} \equiv \frac{a}{c}\,\hat{\kappa}_\nu, \qquad \psi_\nu^{(a)} \equiv \frac{a}{c}\,\psi_\nu, \tag{57}$$

which matches the compact form often quoted as "Eq.(6)" in Gnedin & Abel (2001) when the redshift term is neglected (see below).

**Small-band / small-scale approximation.** If the $\partial_\nu f_\nu$ term is neglected, (56) reduces to

$$\boxed{\frac{a}{c}\,\partial_t f_\nu + \vec{n}\cdot\nabla f_\nu = -\,\hat{\kappa}_\nu^{(a)}\, f_\nu + \psi_\nu^{(a)}} \tag{58}$$

## 3.3 Moment

Define the angular moments

$$J_\nu(t,\vec{x}) \equiv \frac{1}{4\pi}\int d\Omega\, f_\nu(t,\vec{x},\vec{n}), \tag{59}$$

$$F_\nu^i(t,\vec{x}) \equiv \frac{1}{4\pi}\int d\Omega\, n^i f_\nu(t,\vec{x},\vec{n}), \tag{60}$$

$$\frac{1}{4\pi}\int d\Omega\, n^i n^j f_\nu(t,\vec{x},\vec{n}) \equiv h_\nu^{ij}(t,\vec{x})\, J_\nu(t,\vec{x}) \qquad \text{(Eddington tensor).} \tag{61}$$

Useful angular identities:

$$\int d\Omega\, n^i = 0, \qquad \int d\Omega\, n^i n^j = \frac{4\pi}{3}\,\delta^{ij}. \tag{62}$$

Integrate (58) over solid angle:

$$\frac{a}{c}\,\partial_t\left[\frac{1}{4\pi}\int d\Omega\, f_\nu\right] + \frac{1}{4\pi}\int d\Omega\,\vec{n}\cdot\nabla f_\nu = -\,\hat{\kappa}_\nu^{(a)}\left[\frac{1}{4\pi}\int d\Omega\, f_\nu\right] + \frac{1}{4\pi}\int d\Omega\,\psi_\nu^{(a)}. \tag{63}$$

Using $\frac{1}{4\pi}\int d\Omega\, f_\nu = J_\nu$ and $\frac{1}{4\pi}\int d\Omega\,\vec{n}\cdot\nabla f_\nu = \nabla\cdot\mathbf{F}_\nu$,

$$\boxed{\frac{a}{c}\,\partial_t J_\nu + \nabla\cdot\mathbf{F}_\nu = -\,\hat{\kappa}_\nu^{(a)} J_\nu + \Psi_\nu, \qquad \Psi_\nu \equiv \frac{1}{4\pi}\int d\Omega\,\psi_\nu^{(a)}} \tag{64}$$

Multiply (58) by $n^i$ and integrate:

$$\frac{a}{c}\,\partial_t\left[\frac{1}{4\pi}\int d\Omega\, n^i f_\nu\right] + \partial_{x_j}\left[\frac{1}{4\pi}\int d\Omega\, n^i n^j f_\nu\right] = -\,\hat{\kappa}_\nu^{(a)}\left[\frac{1}{4\pi}\int d\Omega\, n^i f_\nu\right] + \frac{1}{4\pi}\int d\Omega\, n^i \psi_\nu^{(a)}. \tag{65}$$

With the definitions above and assuming an isotropic source so that $\int n^i \psi_\nu^{(a)} d\Omega = 0$,

$$\boxed{\frac{a}{c}\,\partial_t F_\nu^i + \partial_{x_j}\big(J_\nu\, h_\nu^{ij}\big) = -\,\hat{\kappa}_\nu^{(a)}\, F_\nu^i} \tag{66}$$

By definition,

$$\frac{1}{4\pi}\int d\Omega\, n^i n^j f_\nu = h_\nu^{ij}\, J_\nu, \tag{67}$$

so a closure (e.g. Eddington $h_\nu^{ij} = \delta^{ij}/3$, M1, OTVET, or RT-derived $h_\nu^{ij}$) is required to close the system (64)–(66).

*Remark.* One may also start from

$$\frac{a}{c}\,\partial_t f_\nu + \vec{n}\cdot\nabla f_\nu - \frac{aH}{c}\,\nu\,\partial_\nu f_\nu = -\,\hat{\kappa}_\nu^{(a)} f_\nu + \psi_\nu^{(a)}, \tag{68}$$

and repeat the moment-taking steps including the redshift term; the angular integrals then add $-(aH/c)\,\nu\,\partial_\nu J_\nu$ to (64) and $-(aH/c)\,\nu\,\partial_\nu F_\nu^i$ to (66).

# 4 xH

Table 2: The parameters constrained by HERA-like and SKA-Low in FDM universe with $m_{22} = 10$ which ionization efficiency is mass dependent. The constraint ability we shown in this table is $1\sigma$ range.

| redshift | xH | author | class |
|---|---|---|---|
| 5.6 | $< 0.04 + 0.05$ | McGreer | dark pixel |
| 5.9 | $< 0.06 + 0.05$ | McGreer | dark pixel |
| 6.3 | $< 0.79 \pm 0.04$ | Jin | dark pixel |
| 6.5 | $< 0.87 \pm 0.03$ | Jin | dark pixel |
| 6.7 | $0.94^{+0.06}_{-0.09}$ | Jin | dark pixel |
| 7 | $0.64^{+0.19}_{-0.23}$ | Greig 2022 | DW |
| 7.51 | $0.27^{+0.21}_{-0.17}$ | Greig 2022 | DW |
| 7.09 | $0.44^{+0.23}_{-0.24}$ | Greig 2022 | DW |
| 7.54 | $0.31^{+0.18}_{-0.19}$ | Greig 2022 | DW |
| 7.29 | $0.49 \pm 0.11$ combine | Greig 2022 | DW |
| 6.15 | $0.2^{+0.14}_{-0.12}$ | Greig 2024 | DW |
| 6.35 | $0.29^{+0.14}_{-0.13}$ | Greig 2024 | DW |
| 5.8, 5.95, 6.05, 6.55 | $< 0.21, 0.20, 0.21, 0.18$ | Greig 2024 | DW |
| $7.6 \pm 0.6$ | $0.88^{+0.05}_{-0.10}$ | Hoag 2019 | equivalent-width |
| 7 | $0.59^{+0.11}_{-0.15}$ | Mason 2018 | equivalent-width |
| 8 | $> 0.76$ | Mason 2019 | equivalent-width |
| 7.3 | $0.83^{+0.06}_{-0.07}$ | Morales 2021 | LF |
| 6.6 | $0.08^{+0.08}_{-0.05}$ | Morales 2021 | LF |
| 7 | $0.28 \pm 0.05$ | Morales 2021 | LF |
| 5.6 | $0.19 \pm 0.07$ | Spina 2024 | DW |
| 5.9 | $< 0.44$ | Spina 2024 | DW |
| 7 | $< 0.5$ | Sobacchi 2015 | equivalent-width |