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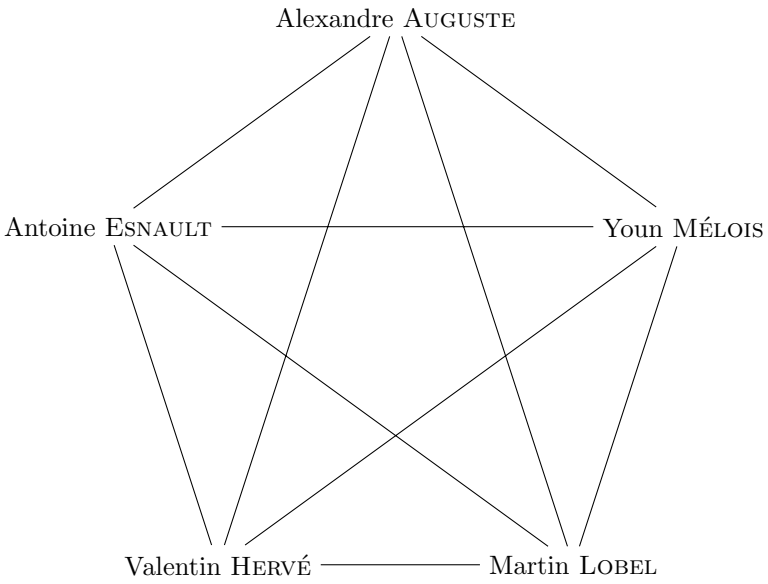
Final Project
The Maximum Edge Weight Clique Problem



Abstract
The Maximum Edge Weight Clique (MEWC) problem is an optimization problem in graph theory that asks for the clique (a subset of vertices, all adjacent to one another) with the maximum total weight in an edge-weighted undirected graph. In the MEWC problem, each edge has a weight, and the weight of a clique is the sum of the weights of its edges. The goal is to find a clique with the maximum possible weight.

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1 Introduction

1.1 Presentation

Graph theory is a branch of mathematics that deals with the study of graphs, which are mathematical structures used to model pairwise relationships between objects. Graphs consist of vertices (also called nodes) that are connected by edges. The edges can be either directed (one-way) or undirected (two-way) and can also have a weight.

The Maximum Edge Weight Clique (MEWC) problem is an optimization problem in graph theory that asks for the clique (a subset of vertices, all adjacent to one another) with the maximum total weight in an edge-weighted undirected graph. In the MEWC problem, each edge has a weight, and the weight of a clique is the sum of the weights of its edges. The goal is to find a clique with the maximum possible weight.

Now, the MEWC problem is **NP-hard**, which means that it is not possible to find an efficient algorithm to solve it in polynomial time or that this problem is at least as hard as the hardest problems in NP. It is also the generalization of the Maximum Clique Problem (MCP), which is the special case where all edges have the same weight.

For example, the following graph $G = (V, E)$ has for its set of vertices $V = \{1, 2, 3, 4, 5, 6\}$ and for its set of edges $E = \{(1, 2), (1, 5), (2, 3), (2, 5), (3, 4), (4, 5), (4, 6)\}$. As we can see, the red edges form a clique of size 3 and the other colored edges are each cliques of size 1. We can also easily deduct that the maximum clique of G is the red clique of size 3.

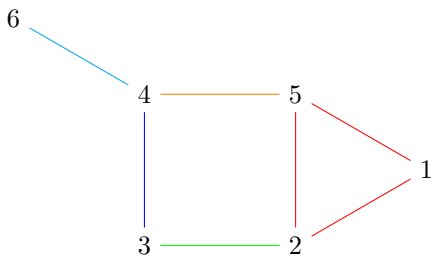


Figure 1: Basic graph example

The MEWC problem can be used to model various types of real-world situations where the goal is to find a subset of objects with the maximum total weight, and the objects are connected by weighted edges. Here are a few examples of such situations:

- **Network design**: In a communication network, the MEWC problem can be used to find the optimal subset of devices (vertices) to include in the network, such that the total cost of communication between the devices (edges) is maximum.
- **Protein interaction**: In biology, the MEWC problem can be used to find the optimal subset of proteins (vertices) in a protein-protein interaction network, such that the total interaction strength (edges) between the proteins is maximum.
- **Social network analysis**: In a social network, the MEWC problem can be used to find the optimal subset of individuals (vertices) with the maximum total relationship strength (edges) between them.

1.2 Configuration

- Ajouter les configurations que l'on a utilise pour les tests du style la config

For this project we used **C++** to develop and implement our algorithms. It is a general-purpose programming language that has been used for a long time and is still widely used today. It is a compiled language, which means that it is translated into machine code before being executed. This allows it to be very fast and efficient. It is also a statically typed language, which means that the type of each variable must be declared before it is used. This allows the compiler to check the type of each variable and to detect errors at compile time.

Though C++ is a very powerful language, it is also very complex and has a lot of features. This can make it difficult to use. That's why we had a debate about which language to use for this project. Our choices were C++ and Python. We eventually chose C++ because it is a language that we are all familiar with and that we have all used before. We also chose it because it is a very powerful language that allows us to implement very efficient algorithms.

To be able to work on the project efficiently and to be able to share the code between us, we used **Github**[1]. It is a web-based platform for version control, collaboration, and sharing of code, as well as a community of developers who contribute to open source projects and share their knowledge. It was a tool that was difficult for some to get used to quickly, especially on the configuration of the project at home, but which brought us a significant gain in efficiency once we had understood how to use it. And to share information and communicate between us, we used **Discord**.

1.3 Example of real-life situations

As we said before, the MEWC has many real-world applications in various fields such as social networks, chemistry, bioinformatics. Now, we will give an concrete example of a real-life situation that can be modelled as a MEWC problem.

The team formation process for a project, or for the search for a particular social group, is a situation that can be viewed as a MEWC problem. Let's imagine that the Student Office of ISEN Nantes is looking to reinforce the video games club of its school. Indeed, the latter has no succession for the following year and is thus led to die if no member presents himself. The future members of the office will have to be in contact with each other during a whole year and it is thus important to find people with common interests so that no tension is formed during their studies. The office has access to the Steam profiles of the students within ISEN (Steam is a video game digital distribution service that gives information about the games played by each one) as well as a record made by the gaming club of the different games played by their members. The fact of playing games in common could bring some people closer, and this makes it a good criterion to create a group that could take over the club because it would share common interests.

To model this problem, we can represent each student as a vertex and each the games played in common as an edge. The weight of each edge will be the number of games played in com-

[1] `https://github.com/sehnryr/Final-Graph-Project-ISEN-CIR3`

mon. The goal of the office is to find a group of students that will take over the club. To do this, we will use the MEWC algorithm to find a group of students that has the highest affinity.

In the following example, we will restrict ourselves to 9 students of the CIR3 class of ISEN Nantes since it would be too complicated to represent every student of the school. We will also assume that the students have played the game listed in the table below.

Students	**Game played**
Youn	Minecraft, Civilisations, Lost ARK, Among US
Martin	Minecraft
Valentin	Genshin, Minecraft, Civilisations
Bastien	Genshin, Minecraft, Lost ARK, Among US
Guillaume	CSGO, Genshin, Overwatch, Stardew Valley
Dorian	CSGO, Paladins, Overwatch
Antoine	League of Legends, Stardew Valley
Thomas	League of Legends, The Last of US
Alexandre	League of Legends, Dofus

Which would give us this graph :

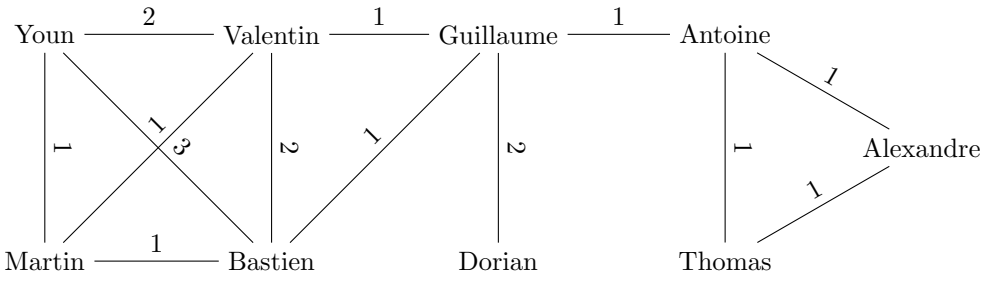


Figure 2: Graph representing the games played between the students

The goal of the Maximum Edge Weight Clique problem in this context would be to find a complete subset of individuals such that the sum of the weights of the edges between the individuals is maximized. In this example, the maximum weight clique would be the clique consisting of nodes Youn, Valentin, Martin, Maxence with a total weight of $10(2+1+1+1+3+2)$.

2 Exact Algorithm

2.1 Presentation

An exact algorithm for solving the MEWC problem works by exploring all potential cliques in the graph and selecting the clique with the maximum weight. To do this, the algorithm uses a recursive function to explore all possible subsets of V. For each subset, the algorithm computes the weight of the clique formed by the vertices in the subset. If the weight is greater than the current maximum weight, the clique is selected as the current maximum. This process is repeated until all possible cliques have been explored, at which point the algorithm returns the clique the the maximum weight.

The time complexity of this approach is $\mathcal{O}(n^2 \times 2^n)$, where n is the number of vertices in the graph. This is due to the fact that there are 2^n possible subsets of V, and the algorithm must compute the weight of the clique, which by itself has a complexity of $\mathcal{O}(n^2)$ because there are at most $\frac{n(n-1)}{2}$ edges in a complete graph, and compare it to the current maximum weight. Therefore, the algorithm is only feasible for small-scale graphs.

However, there is a better algorithm called the **Bron-Kerbosch** algorithm[1] that can find all maximal cliques in a graph with a time complexity of $\mathcal{O}(3^{\frac{n}{3}})$, where n is the number of vertices in the graph. This algorithm is optimal, as it has been proven by J. W. Moon & L. Moser in 1965[2] that there are at most $3^{\frac{n}{3}}$ maximal cliques in any n-vertex graph.

We can use the Bron-Kerbosch algorithm to find all maximal cliques in the graph, and then compute the weight of each clique. This approach has a time complexity of $\mathcal{O}(n^2 \times 3^{\frac{n}{3}})$, which is much better than the previous approach. However, this algorithm is still not feasible for large-scale graphs.

2.2 How it works

As said before, our algorithm first uses the Bron-Kerbosch algorithm to obtain all the maximal cliques of the input graph. Then, it iterate through those maximal cliques to looks for which cliques have the highest weight.

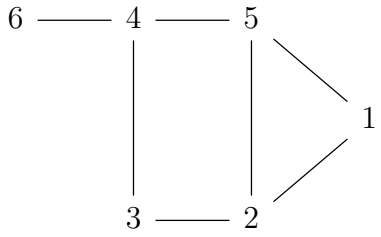
The Bron-Kerbosch pivot algorithm that we use is a more efficient variant of the initial algorithm. The basic form of the algorithm is inefficient in the case of graphs with many non-maximal cliques as it makes a recursive call for every clique, maximal or not. To save time and allow the algorithm to backtrack more quickly in branches of the search that contain no maximal cliques, Bron and Kerbosch introduced a variant of the algorithm involving a "pivot vertex".

At each step, the algorithm keeps track of three groups of vertices: R which is which is the subset of vertices that must be in a clique, P which is the candidates that could be included in the clique and X which is the excluded vertices that already have been searched. In each call of the algorithm, P and X are disjoint sets whose union consists of vertices that form cliques when added to R. When P and X are both empty there are no further elements that can be added to R, so R is a maximal clique and the algorithm outputs R.

The recursion is initiated by setting R and X to be the empty set and P to be the vertex set of the graph. Within each recursive call, the algorithm considers the vertices P in turn; if there a no such vertices, it either reports R as a maximal clique if X is also empty, or continue. Then, a pivot vertex u is chosen from $P \cup X$ since any maximal clique must include either u or one of its non-neighbors, for otherwise the clique could be augmented by adding u to it. Only u and its non-neighbors needs to be tested as the choices for the vertex v that is added to R in each recusive call to the algorithm. For each vertex v chosen from $P \backslash N(u)$, with $N(u)$ being the neighbor set of u, it makes a recursive call in which v is added to R and in which P and X are restriced to $N(v)$, which finds and reports all cliques extensions of R that contains v. Then, it moves v from P to X to exclude it from consideration in future cliques and continues with the next vertex in $P \backslash N(u)$.

To illustrate the Bron-Kerbosch algorithm, let's use the example in Figure 1 on page 3:

Step 0 :

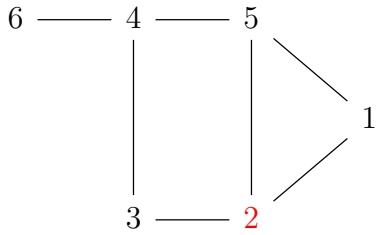


We have here our graph as an example. As said before, the Bron-Kerbosh pivot algorithm will take 3 parameters. R, the set of vertices already selected. P, the set of candidate vertices. X, the set of vertices that have been considered but not selected.

Here, we have :

$$R = \{\} \quad P = \{1, 2, 3, 4, 5, 6\} \quad X = \{\}$$

Step 1 :

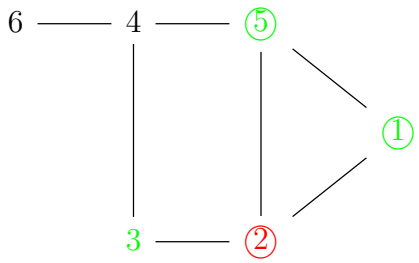


Now, to begin the algorithm, we will need to chose a pivot u. The pivot u should be chosen as one of the degree-three vertices, to minimize the number of recursive calls. Now, we suppose that u is chosen to be vertex 2. We see, that there is 2 vertices that are not adjacent to 2 which are 4 and 5.

Then we know that we will work by starting with these 3 configurations (we know the first, but we don't know yet for the vertex 4 and 6 because some vertex could be added to X or removed from P when we process on 2 or 4) :

$$R = \{2\} \quad P = \{1, 3, 5\} \quad X = \{\}$$
$$R = \{4\}$$
$$R = \{6\}$$

Step 2 :

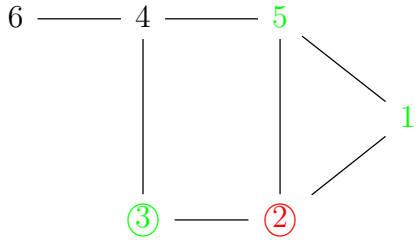


The algorithm begins by looking at vertex 2 in the graph. It makes a recursive call with using vertex 2 as a starting point. In this recursive call, the algorithm looks at the first group of vertices it was given (vertices 1 and 5) and selects one of them as the pivot vertex. Let's say it selects vertex 1. It then makes a first second-level recursive call for vertex 5. That will eventually find the clique $(1, 2, 5)$.

The recursive call process like this :

$R = \{2\}$	$P = \{1, 3, 5\}$	$X = \{\}$
$R = \{2, 5\}$	$P = \{1\}$	$X = \{\}$
$R = \{2, 5, 1\}$	$P = \{\}$	$X = \{\}$

$\rightarrow P = \emptyset$ and $X = \emptyset$ then $(2, 5, 1)$ is a clique.



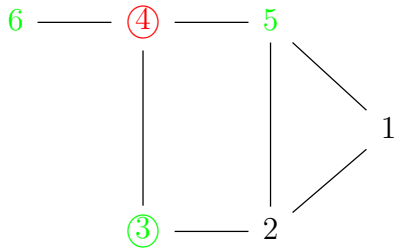
The algorithm then makes a second second-level recursive calls for vertex 3. That will eventually find the clique $(2, 3)$. After these two second level recursive calls have completed, vertex 2 is added to X and removed from P.

The recursive call process like this :

$R = \{2\}$	$P = \{1, 3, 5\}$	$X = \{\}$
$R = \{2, 3\}$	$P = \{\}$	$X = \{\}$
$R = \{\}$	$P = \{1, 3, 4, 5, 6\}$	$X = \{2\}$

$\rightarrow P = \emptyset$ and $X = \emptyset$ then $(2, 3)$ is a clique.

Step 3 :

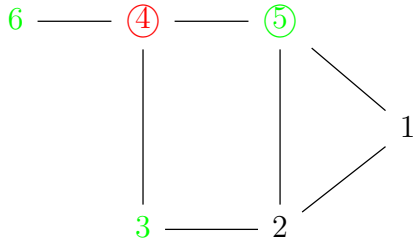


it will now do an iteration taking 4 as vertices. We note that the vertex 2 belongs to the set X in the outer call to the algorithm, but it is not a neighbor of the vertex 4 and is excluded from the subset of X passed to the recursive call. The algorithm makes a first second-level recursive call for vertex 3. That will eventually find the clique $(3, 4)$.

The recursive call process like this :

$R = \{4\}$	$P = \{3, 5, 6\}$	$X = \{\}$
$R = \{4, 3\}$	$P = \{\}$	$X = \{\}$

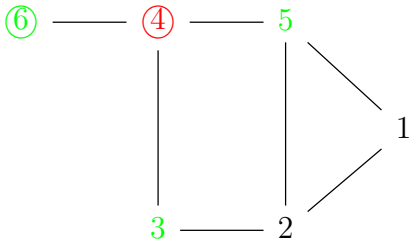
$\rightarrow P = \emptyset$ and $X = \emptyset$ then $(4, 3)$ is a clique.



The algorithm makes a second second-level recursive call for vertex 5. That will eventually find the clique $(4, 5)$.

The recursive call process like this :

$R = \{4\}$	$P = \{3, 5, 6\}$	$X = \{\}$	$\rightarrow P = \emptyset$ and $X = \emptyset$ then $(4, 5)$ is a clique.
$R = \{4, 5\}$	$P = \{\}$	$X = \{\}$	

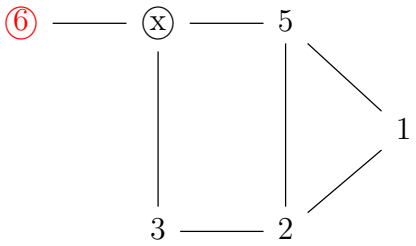


The algorithm makes a third second-level recursive call for vertex 6. That will eventually find the clique $(4, 6)$. After these three second level recursive calls have completed, vertex 4 is added to X and removed from P.

The recursive call process like this :

$R = \{4\}$	$P = \{3, 5, 6\}$	$X = \{\}$	
$R = \{4, 6\}$	$P = \{\}$	$X = \{\}$	$\rightarrow P = \emptyset$ and $X = \emptyset$ then $(4, 6)$ is a clique.
$R = \{\}$	$P = \{1, 3, 5, 6\}$	$X = \{2, 4\}$	

Step 4 :



It will now do a third and final iteration taking 6 as vertex. It makes a recursive call but because P is empty and X is nom empty (the vertex 4 was added to X and removed from P in step 3). Because of that, the algorithm immediately stops searching for cliques and backtracks, because there can be no maximal clique that includes the vertex 6 and excludes the vertex 4.

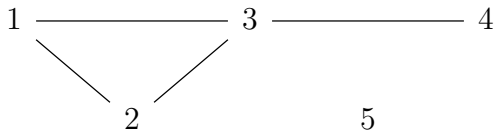
The recursive call process like this :

$R = \{6\}$	$P = \{\}$	$X = \{4\}$	$\rightarrow$ No clique was found

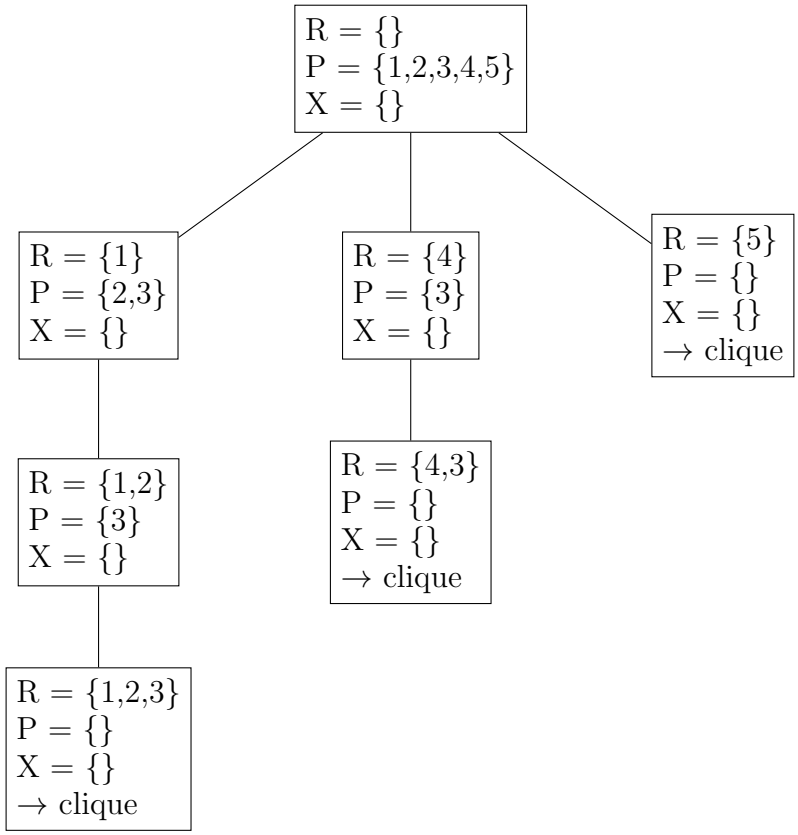
The algorithm is therefore **finished**, and we have obtained the cliques :

$$\{(2, 3), (1, 2, 5), (3, 4), (4, 5), (5, 6)\}$$

To summarize things and try to make things even clearer, we will take another example and make the call tree of it (the precedent example is too big) :



His call tree :



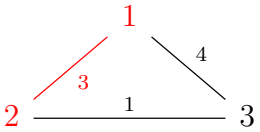
After getting every clique, the exact algorithm will iterate all maximal clique and apply a function that will return the total weight of it.

This function work by iterating every possible pairs of vertices in the clique and the weight of it if there is an edge between them in a variable that start at 0. The variable will be returned as the total weight of the clique.

Some quick example of it :

Step 1 :

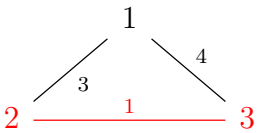
Total Weight = 0 (start)



The function will take the vertex 1 and 2, the edge between them have a weight of 3. It will add 3 in the variable "Total Weight".

Total Weight = 0 + 3 = 3

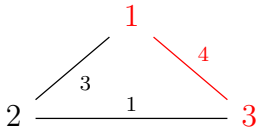
Step 2 :



The function will take the vertex 2 and 3, the edge between them have a weight of 1. It will add 1 in the variable "Total Weight".

Total Weight = 3 + 1 = 4

Step 3 :



The function will take the vertex 1 and 3, the edge between them have a weight of 4. It will add 4 in the variable "Total Weight".

$$\text{Total Weight} = 4 + 4 = 8$$

The total weight of this clique is 8.

The algorithm then takes the clique with the greatest weight. And **the MEWC is solved**.

2.3 Pseudo code

2.4 Complexity

2.5 Bad Instance

The exact algorithm have no bad instance, he will always find the optimal solution each time. However, it will take a fairly long time to do so.

2.6 Experiments

2.7 Analysis

3 Constructive Algorithm

3.1 Presentation

A heuristic algorithm is a type of algorithm that uses rules of thumb to try to find an approximate solution to a problem, rather than an exact one. Heuristic algorithms are often used when it is not possible to find an exact solution to a problem, or when an exact solution would be too time-consuming to compute. They are also used when an approximate solution is good enough, or when finding an exact solution is not the primary goal. Heuristic algorithms are commonly used in artificial intelligence, computer science, and other fields. They are often used to solve optimization problems, search problems, and other types of problems where an exact solution is not necessary or practical.

A constructive heuristic algorithm is a type of heuristic algorithm that is used to find a solution to a problem by building it incrementally. Constructive heuristics start with a partial solution and gradually add to it until a complete solution is found. They are commonly used to solve optimization problems, where the goal is to find the optimal solution, or the solution that is the best among all possible solutions.

Constructive heuristics can be contrasted with other types of heuristics, such as local search heuristics, which try to find a solution by making small changes to an existing solution, or random heuristics, which generate solutions randomly and then choose the best one. Constructive heuristics are often used when it is important to find a solution that is complete and comprehensive, rather than just a local improvement.

Examples of some famous problems that are solved using constructive heuristics are the flow shop scheduling, the vehicle routing problem and the open shop problem.

3.2 How it works

3.3 Pseudo code

3.4 Complexity

3.5 Instance

3.6 Experiments

3.7 Analysis

4 Local Search Algorithm

4.1 Presentation

4.2 How it works

4.3 Pseudo code

4.4 Complexity

4.5 Instance

4.6 Experiments

4.7 Analysis

5 Grasp Algorithm

5.1 Presentation

5.2 How it works

5.3 Pseudo code

5.4 Complexity

5.5 Instance

5.6 Experiments

5.7 Analysis

6 Conclusion

References

[1] C. Bron and J. Kerbosch. Algorithm 457: finding all cliques of an undirected graph. *Communications of the ACM*, September 1973. https://doi.org/10.1145/362342.362367.

[2] J.W. Moon and L. Moser. On cliques in graphs. *Israel J. Math.*, Match 1965. https://doi.org/10.1007/BF02760024.