Randomized Experiments and Randomization Inference

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- Random assignment: the probability of assignment to treatment (and control) is equal for each subject.
- That means no subjects has a higher probability to be treated than another subject.

Random Assignment

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  1. Simple random assignment: Each subject is allocated to the treatment group with probability m/N, where m is the number of subjects assigned to treatment.
  2. Complete random assignment (exactly m units are assigned to treatment)

Recap: Potential outcomes

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- $Y_i(0) \mid d_i = 1$: untreated potential outcome for subjects that receive the treatment.
- $Y_i(1) \mid D_i = 0$: treated potential outcome for subjects that would not receive the treatment under a hypothetical random assignment.

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- We use D_i when talking about the statistical properties of treatments.

Example

- What's the effect of private tutoring on exam scores (ranging from 1 to 6)?

Full schedule of potential outcomes

	$Y_i(0)$	$Y_i(1)$
	Test score if	
subject i	not tutored	tutored
1	3	4.5
2	5	5
3	5	4.5
4	4.5	5
5	4	5.5
6	6	6

Definition of a subject-level treatment effect

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3	5	4.5	-0.5
4	4.5	5	0.5
5	4	5.5	1.5
6	6	6	0

Definition of Average Treatment Effect

$\mu(Y(1)) - \mu(Y(0))$,

where

$\mu(Y(1))$ is the average value of $Y_i(1)$ for all subjects, and

$\mu(Y(0))$ is the average value of $Y_i(0)$ for all subjects.

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4	4.5	5	0.5
5	4	5.5	1.5
6	6	6	0
Average	4.58	5.08	0.5

Potential and observed outcomes

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- The $Y_i(1)s$ are observed for subjects who are treated, and the $Y_i(0)s$ are observed for subjects who are not treated. For any given subject, we observe either $Y_i(1)$ or $Y_i(0)$, never both at the same time.

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- A subject's treatment effect is unobserved.

Potential and observed outcomes

- The connection between the observed outcome Y_i and the underlying potential outcomes is given by the "switching equation":

$$Y_i = d_i Y_i(1) + (1 - d_i) Y_i(0)$$

Independence assumption

Treatment status is statistically independent of potential outcomes and background attributes (X):

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If a subject is randomly assigned to treatment, knowing whether a subject is treated provides no information about the subject's potential outcomes, or background attributes.

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- The exclusion restriction breaks down if treatment assignment z_i sets in motion causes of Y_i other than the treatment d_i.

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- Each subject is unaffected by the treatments and assignments of other units.

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where $Pr[X = x]$ denotes the probability that X takes on the value x, and where the summation is taken over all possible values of x.

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- Under random assignment:

$$E[Y_i(1) \mid D_i = 1] = E[Y_i(1) \mid D_i = 0]$$

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$$= E[Y_i(1)] - E[Y_i(0)] = E[\tau_i] = ATE.$$

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- Our data set is just one of many possible data sets that could have been created via random assignment. If we would redo the exact same random assignment procedure, different units would be allocated to treatment and control groups!
- The average estimated ATE across all possible random assignments is equal to the true ATE.
- On average we recover the true ATE. Our *estimator* is unbiased.

Observed Outcomes

subject i	$Y_i(0)$ Test score if not tutored	$Y_i(1)$ Test score if tutored	τ_i Treatment effect
1	?	4.5	?
2	5	?	?
3	?	4.5	?
4	4.5	?	?
5	4	?	?
6	?	6	?

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4	4.5	?	?
5	4	?	?
6	?	6	?
Average	4.5	5	

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4	4.5	?	?
5	4	?	?
6	?	6	?
Average	4.5	5	0.5

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3	?	4.5	?
4	4.5	?	?
5	?	5.5	?
6	6	?	?

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1	?	4.5	?
2	5	?	?
3	?	4.5	?
4	4.5	?	?
5	?	5.5	?
6	6	?	?
Average	5.17	4.83	

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1	?	4.5	?
2	5	?	?
3	?	4.5	?
4	4.5	?	?
5	?	5.5	?
6	6	?	?
Average	5.17	4.83	-0.32

Sampling distribution of the ATE

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Sampling distribution of the ATE

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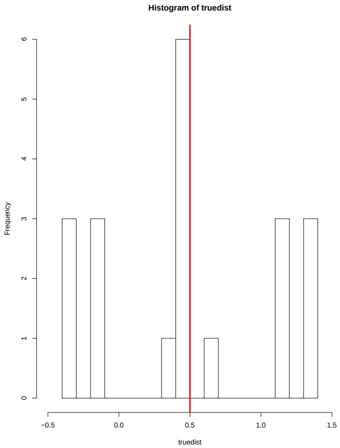
$$\frac{N!}{m!(N-m)!} = \frac{6!}{3!3!} = \frac{720}{36} = 20$$

ATEs

	Estimated ATEs	Frequency
1	-0.33	3
2	-0.17	3
3	0.33	1
4	0.50	6
5	0.67	1
6	1.17	3
7	1.33	3
	0.5	20

Sampling Distribution of estimated ATEs



Randomization inference

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- Sharp null hypothesis: The treatment effect is zero for each subject (Fisher's exact test).
- If the sharp null hypothesis is true, then $Y_i(0) = Y_i(1)$.
- Under the sharp null hypothesis, we can take the observed outcomes in our data set, and impute the counterfactual potential outcomes, re-assigning subjects to treatment and control group over and over again.

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Randomization inference

- Simulated randomizations provide the exact sampling distribution of the estimated ATE under the sharp null.
- Now we can calculate the number of times we obtain an estimated ATE at least as large as the one we obtained from our actual experiment if the treatment effect was zero for every subject.

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Randomization inference

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- Two-tailed test of sharp null: Count the *absolute* number of ATEs under the sharp null that are as large or larger than the actual ATE we obtain from our experiment

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- Two-tailed test of sharp null: Count the *absolute* number of ATEs under the sharp null that are as large or larger than the actual ATE we obtain from our experiment -> Two-tailed p-value.

Imputing the sharp null

subject i	$Y_i(0)$ Test score if not tutored	$Y_i(1)$ Test score if tutored	τ_i Treatment effect
1	?	4.5	?
2	5	?	?
3	?	4.5	?
4	4.5	?	?
5	4	?	?
6	?	6	?
Average	4.5	5	0.5

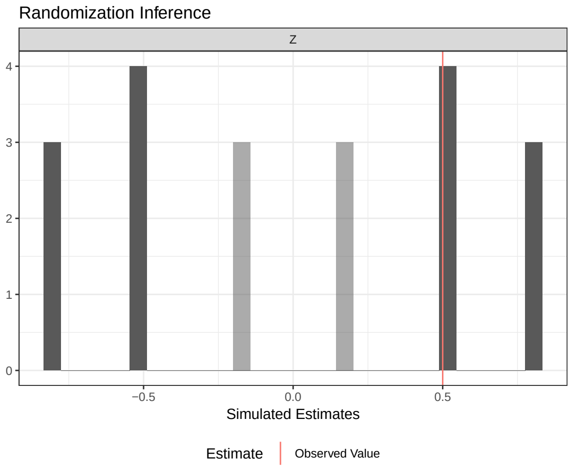
Imputing the sharp null

subject i	$Y_i(0)$ Test score if not tutored	$Y_i(1)$ Test score if tutored	τ_i Treatment effect
1	4.5	4.5	0
2	5	5	0
3	4.5	4.5	0
4	4.5	4.5	0
5	4	4	0
6	6	6	0
Average	4.75	4.75	0

$\widehat{ATE}$ under sharp null

subject i	$Y_i(0)$ Test score if not tutored	$Y_i(1)$ Test score if tutored	τ_i Treatment effect
1		4.5	
2	5		
3		4.5	
4	4.5		
5		4	
6	6		
Average	5.17	4.33	-0.84

Samling distribution of ATEs if sharp null is true



Let's do randomization inference using the ri2 package in R.