Problem Set 3

Q2.

a) 8.21

$E[X] = 2 - 4e^{(-2)} \approx 1.4587$.
$E[\text{Profit}] = 7(1 + E[X]) - 9 \approx 8.21$.

b) 0.1813

Rate = 2/300 per sec. Mean over 30s = 0.2.
$P(\geq 1) = 1 - e^{(-0.2)} \approx 0.1813$.

Q3.

Ans 426.22

Pool positive probability = $1 - 0.95^5 \approx 0.2262$.
Expected positive pools = $200 \times 0.2262 = 45.24$.
Phase 2 tests = $5 \times 45.24 \approx 226.22$.
Total = $200 + 226.22 = 426.22$.

Q4.

Ans 0.7982

Prior: 0.8 effective, 0.2 not.
Likelihood($4 \mid \lambda=2$) = 0.09022, Likelihood($4 \mid \lambda=7$) = 0.09123.
Posterior = $(0.8\times0.09022)/(0.8\times0.09022 + 0.2\times0.09123) \approx 0.7982$.

Q5.

a) $c = 3/8$.

b) CDF:

$F(x) = 0$ for $x \le -1$.

$F(x) = (3/4)x - (1/4)x^3 + 1/2$ for $-1 < x < 1$.

$F(x) = 1$ for $x \ge 1$.

c) $E[X] = 0$.

Q7.

$E[\text{nonempty buckets}] = \Sigma\, [1 - (1 - p_i)^n]$.

Q8

a) 0.0149

Estimate event rate $\hat{\lambda} = 2 / 400 = 0.005$ per hour.

Probability of at least one event in 3 hours $= 1 - \exp(-0.005 \cdot 3) \approx 1 - \exp(-0.015) \approx 0.0149$.

b) 0.029

Treat number of affected users as Binomial(10000,p) with $p \approx 0.0157$ (or use 0.0149);

Using $p \approx 0.0157$ gives mean $\mu = 157$ and sd ≈ 12.43. Continuity-corrected normal approximation: $Z = (180.5 - 157)/12.43 \approx 1.89$, so $P \approx P(Z > 1.89) \approx 0.029$.

Q9.

a) 0.362

P(single > 3.0e6) ≈ 0.0475. Binomial(100, 0.0475). Normal approx: mean=4.75, sd=2.13.
$P(Y \geq 6) \approx 0.362$.

b) 0.717

P(single < 1.9e6) ≈ 0.672. Binomial(100, 0.672). Normal approx: mean=67.2, sd=4.70.
$P(W \geq 65) \approx 0.717$.

Q.11

Discrete PMF: S = 0,1,2,3,4,5,6,7 with probabilities 0.00, 0.11, 0.26, 0.22, 0.16, 0.09, 0.06, 0.04 (these sum to 0.94).

Continuous tail for $7.5 \le x < 30$ with total mass 0.06:

$f_1(x) = K_1 / x$ (fat tail)

$f_2(x) = K_2 / x^3$ (thin tail)

Assume $P(S \ge 30) = 0$.

a) 0.06

Reason: discrete mass below 7.5 sums to 0.94, so remaining tail probability = 1 − 0.94 = 0.06.

b) 7.20

$K_1 = 0.06 / \int_{7.5}^{30} (1/x)\, dx = 0.06 / \ln(30/7.5) = 0.06 / \ln(4) \approx 0.04328$.

$K_2 = 0.06 / \int_{7.5}^{30} x^{-3}\, dx = 0.06 / [\, (1/2)(7.5^{-2} - 30^{-2})\,] = 7.20$.

c) 3.20

Under f_1: $P = K_1 \int_{10}^{30} (1/x)\, dx = K_1 \ln(30/10) = K_1 \ln(3) \approx 0.04328 \times 1.09861 \approx 0.0476$ (≈ 4.76%).

Under f_2: $P = K_2 \int_{10}^{30} x^{-3}\, dx = K_2 \cdot (1/2)(10^{-2} - 30^{-2}) = 7.20 \times 0.00444444 = 0.0320$ (≈ 3.20%).

d) 3.99 and 3.74

Discrete contribution = $\Sigma\, s \cdot P(s)$ for s=1..7 = 3.02 (exact sum).

Continuous contribution (f_1): $\int_{7.5}^{30} x \cdot (K_1/x)\, dx = K_1 \cdot (30 - 7.5) = K_1 \cdot 22.5 \approx 0.04328 \times 22.5 \approx 0.97$.

E[S] (f_1) ≈ 3.02 + 0.97 = **3.99**.

Continuous contribution (f_2): $\int_{7.5}^{30} x \cdot (K_2\, x^{-3})\, dx =$

$K_2 \int x^{-2}\, dx = K_2 (1/7.5 - 1/30) = K_2 \times 0.1 = 7.20 \times 0.1 = 0.72$.

$E[S]$ (f_2) = 3.02 + 0.72 = **3.74**.

e) 30.32 and 22.05

Discrete contribution = $\Sigma\, s^2 \cdot P(s)$ for s=1..7 = 12.06 (exact sum).

Continuous contribution (f_1):

$\int_{7.5}^{30} x^2 \cdot (K_1/x)\, dx = K_1 \int x\, dx = K_1 \cdot (1/2)(30^2 - 7.5^2) = K_1 \cdot 421.875 \approx 0.04328 \times 421.875 \approx 18.26$.

$E[R]$ (f_1) ≈ 12.06 + 18.26 = **30.32**.

Continuous contribution (f_2):

$\int_{7.5}^{30} x^2 \cdot (K_2 x^{-3})\, dx = K_2 \int x^{-1}\, dx = K_2 \ln(30/7.5) = K_2 \ln(4) = 7.20 \times 1.386294 \approx 9.99$.

$E[R]$ (f_2) ≈ 12.06 + 9.99 = **22.05**.