# Problem Set 1

## Q1.

a) 10!

If there are no restrictions on the seating arrangement, the first seat can be occupied by any one of the 10 people and the second seat can be occupied by any one of the remaining 9 people and so on till all the 10 seats are occupied. The choice of who sits in the first seat does not affect the choices available for the second seat and so on. So by the step rule of counting, the total number of seating arrangements is 10*9*8*... *1 = 10!

b) 18 * 8!

A and B have to sit together which leaves us with calculating permutations of the remaining 8 people and then checking the number of seats available for A and B.
The permutation for 8 people is 8! and the number of ways in which A and B can sit together is 18(9 times when A is sitting to the right of B and 9 times when B is sitting to the right of A)

c) 2 * 5! * 5!

There are 5 adults and 5 children. There are two possible ways to make them sit as either adults will sit on the odd seats and children will sit on the even seats or vice versa.
For both adults and children there are 5 choices for the first seat and 4 choices for the second seat and so on which means 5! For adults and 5! for children

d) 2^5 * 5!

If there are 5 married couples and each couple must sit together in a row, treat each couple as a single block, the 5 blocks can be arranged in 5! ways, within each couple a person can swap seats in 2 ways, and since there are 5 couples that give 2^5 arrangements.

# Q.2

## (a) 1120

If there are no restrictions on the selection, we must choose 3 bird species out of the 8 available and 3 reptile species out of the 6 available. The number of ways to choose 3 birds is 56, and the number of ways to choose 3 reptiles is 20. Since the choices of birds and reptiles are independent, we apply the step rule of counting and multiplying these values. So the total number of exhibits that can be formed is 56 * 20 = 1120.

## (b) 1000

If the restriction is that two specific bird species cannot both be chosen together, then we first calculate the number of forbidden cases. If both restricted birds are chosen, the third bird must be chosen from the remaining 6, which can be done in 6 ways. For the reptiles, there are still 20 ways to choose them. So the total number of forbidden exhibits is 6 * 20 = 120. The unrestricted total was 1120, so subtracting the forbidden cases gives 1120 - 120 = 1000. Therefore, the total number of valid exhibits when the two restricted birds cannot appear together is 1000.

## (c) 910

If the restriction is that a specific bird species B and a specific reptile species R cannot both appear together, then we must count the forbidden case. If B is included, the other 2 birds must be chosen from the remaining 7, which can be done in 21 ways. If R is also included, the other 2 reptiles must be chosen from the remaining 5, which can be done in 10 ways. Thus, the total number of forbidden exhibits is 21 * 10 = 210. Subtracting this from the unrestricted total of 1120 gives 1120 - 210 = 910. Therefore, the number of valid exhibits when B and R cannot appear together is 910.

# Q.3

## a) 286

If an investment must be made in each of the four companies and we must invest all \$20 million, first give each company its minimal investment of \$1, \$2, \$3 and \$4 million respectively; these sum to \$10 million, leaving \$10 million to distribute among the four companies. The remaining distribution is a stars-and-bars problem: the number of nonnegative integer solutions to $y_1+y_2+y_3+y_4=10$ is (10+4−1 choose 4−1) = (13 choose 3) = 286. So there are 286 possible investment strategies

## b) 680

If investments must be made in at least 3 of the 4 companies, split into two cases. Case 1 — all 4 companies get money: this is the same as above and gives (13 choose 3) = 286 ways. Case 2 — exactly 3 companies get money: choose which company is omitted (4 choices). If company n (with minimum n million) is omitted, the three chosen companies have their minima summing to 10−n, so after subtracting those minima there remain 10+n millions to distribute freely among 3 companies; the number of nonnegative distributions is ((10+n)+3−1 choose 3−1) = (12+n choose 2). Evaluating for omissions n=1,2,3,4 gives (13 choose 2)=78, (14 choose 2)=91, (15 choose 2)=105, (16 choose 2)=120 respectively, which sum to 394. Adding the all 4 case (286) gives total = 286 + 394 = 680 possible strategies when at least 3 companies must receive money.

## c) NOT DONE

# Q.4

## a) 0.48

There are 27 students in Java, 28 in C++, and 20 in Python. Since some students take more than one class, we apply the inclusion exclusion principle. Adding them directly gives 75, but then we subtract the overlaps: 12 students are in both Java and C++, 5 are in both Java and Python, and 8 are in both C++ and Python. This subtraction removes too much, so we add back the 2 students who are in all three classes. The total number of students in at least one programming class is 27 + 28 + 20 – 12 – 5 – 8 + 2 = 52. Since the university has 100 students in total, the number not enrolled in any programming class is 100 – 52 = 48. Therefore, the probability of randomly selecting a student not in any class is 48 out of 100, which is 0.48.

## (b) 0.31

To find this, we calculate how many students belong exclusively to Java, exclusively to C++, and exclusively to Python. For Java, we subtract students also in C++ or Python and then add back those triple-counted: 27 - 12 - 5 + 2 = 12. For C++, the count is 28 - 12 - 8 + 2 = 10. For Python, the count is 20 - 5 - 8 + 2 = 9. Adding these gives 12 + 10 + 9 = 31. Thus, 31 students are in exactly one programming class. The probability that a randomly chosen student falls in this group is 0.31.

## (c) 0.772

First, we calculate the probability that both chosen students are outside all programming classes. Since 48 students are not in any class, the probability the first student comes from this group is 48 out of 100. Once this happens, the second student must come from the remaining 47 out of the 99 students left. This gives a probability of (48/100) × (47/99) = 188/825 that both are outside. Therefore, the probability that at least one of the two chosen students is in a programming class is 1 – 188/825 = 637/825, which is approximately 0.772

# Q.5

From a deck of 52 cards, the number of distinct 5 card hand is (52 choose 5) which is 2598960

## a) 5148 / 2598960

A flush occurs when all five cards are from the same suit. To count such hands, we first choose a suit, which can be done in four ways. Within the chosen suit, there are thirteen cards, and we must pick any five of them. The number of ways to do this is 1,287. Multiplying by the four suits gives 5,148 possible flush hands. Since the total number of possible five-card hands from a 52-card deck is 2,598,960, the probability of being dealt a flush is 5,148 / 2,598,960.

## b) 123,552 / 2598960

Two pairs means the hand consists of two cards of one rank, two cards of a second rank, and a fifth card of a third rank. To count this, we first select which two ranks will form the pairs. There are 78 ways to choose these two ranks from the 13 available. For each chosen rank, we must select two suits out of the four available, which can be done in six ways each, giving 36 choices for the two pairs. Next, we select the rank of the remaining card, which must be different from the two pair ranks, leaving 11 options. Finally, we select one of the four suits for this single card. Multiplying these together, the total number of two-pair hands is 123,552. Dividing by the total number of possible hands, the probability of two pairs is 123552 / 2598960

## c) 624 / 2598960

A four-of-a-kind hand contains four cards of the same rank and one card of a different rank. To count these, we begin by choosing the rank of the four identical cards, which can be done in 13 ways. All four suits of that rank are automatically included, so there is only one way to complete that part. Then we choose the rank of the fifth card, which must be different from the first, leaving 12 possibilities. For the chosen rank, any of its four suits may be used, giving four options. Altogether, this yields 624 possible hands. When divided by the total number of 5-card hands, the probability of being dealt four of a kind is 624 / 2598960

# Q.6

There are 6 choices for each die roll and we roll six times, so every sequence of six rolls is one of 46656 equally likely outcomes.

## a) 300 / 46656

To get exactly two different numbers with each appearing three times, first choose which two face values appear; there are 15(6 choose 2) ways to pick that pair. Once the two values are chosen, we must place them in the six roll positions so that each appears exactly three times. The number of distinct placements of three of one kind and three of the other is 20(6! / 3! * 3!). Multiplying the 15 choices of values by the 20 placements gives 300 equally likely favorable sequences. Therefore the probability is 300 out of 46656.

## b) 7200 / 46656

Again there are 46,656 total equally likely sequences. To have exactly one value occur three times while the remaining three rolls are all different (and different from the tripled value), first choose which value is the tripled number (6 choices). Next choose which three of the six roll positions contain that value (20 choices). The remaining three positions must be filled with three distinct values chosen from the five remaining face values, and their order matters; that gives 60 ways to fill the three spots. Multiplying 6 by 20 by 60 gives 7,200 favorable sequences. Thus the probability is 7200 out of 46656

# Q.7

## Ans. 544597178125 / 6847565144260608

First choose which 4 users will be the “two-email” users (there are C(12,4) ways) and then choose which 3 of the remaining 8 users will be the “four-email” users (there are C(8,3) ways). Once the users with the specified counts are chosen, assign the 20 distinguishable emails to those users in a way that gives each chosen user their required count. The number of assignments that give specified counts (n1, n2, …, n12) equals the multinomial coefficient 20! divided by the product of the factorials of the counts; here that factor is 20! / [(2!)^4 (4!)^3 (0!)^5]. Multiplying together the choices of users and the multinomial count gives the total number of favorable assignments.

The total number of possible assignments of 20 distinguishable emails to 12 users (with each email equally likely to go to any user) is 12^20. Hence the exact probability is

C(12,4) × C(8,3) × [20! / ((2!)^4 (4!)^3)]  divided by 12^20.

Evaluating this gives an exact reduced fraction of

544597178125 / 6847565144260608

# Q.9

We are putting m strings randomly into N buckets. Each string is equally likely to go into any bucket. So, the chance that a string goes into the first bucket is $1/N$, and the chance it goes into some other bucket is $(N-1)/N$.

Now, we want exactly k strings to land in the first bucket. This is a binomial situation, because we have m independent trials (each string), and each trial has a “success” (goes to bucket 1) with probability $1/N$.

The formula for this probability is:

$P(\text{exactly } k \text{ in the first bucket}) = C(m, k) \times (1/N)^k \times ((N-1)/N)^{(m-k)}$.

Here $C(m, k)$ is the number of ways to choose which k strings go to the first bucket.

# Q.10

## a) **$2^{(n-1)} / n!$**

A completely degenerate BST is one where every node has at most one child, so the tree looks like a straight chain. This happens only if, during insertion, each next element is either the smallest or the largest among the remaining values.For the root, any element 1 through n can be chosen, no restriction.After choosing the root, to stay degenerate, the next inserted number must be either the smallest remaining or the largest remaining. That gives 2 valid choices out of the $(n-1)$ available.The next step again gives 2 valid choices out of the remaining $(n-2)$, and so on, until only one element is left.

So, the probability is the product of these fractions:

$(2 / (n-1)) * (2 / (n-2)) * \ldots * (2 / 2) * (2 / 1)$.

This simplifies to:

$P(\text{degenerate BST}) = 2^{(n-1)} / n!$

## b) 9

We want the smallest n where $2^{(n-1)} / n! < 0.001$.

- For $n = 6$: $2^5 / 6! = 32 / 720 \approx 0.0444$.
- For $n = 7$: $2^6 / 7! = 64 / 5040 \approx 0.0127$.
- For $n = 8$: $2^7 / 8! = 128 / 40320 \approx 0.00318$.
- For $n = 9$: $2^8 / 9! = 256 / 362880 \approx 0.000705$ (less than 0.001).

So, the smallest n is 9.