PDA - probabilistic data association filter

#MTT

References: @Bar-shalom2009, @Brekke2020

The notation is adopted from Bar-Shalom, just the time step goes to the index for easier reading.

Model

We assume the standard linear state-space model

$$\begin{aligned} x_k &= F_{k-1}x_{k-1} + v_k, & v_k &\sim \mathcal{N}(0, Q), \\ z_k &= H_k x_k + w_k, & w_k &\sim \mathcal{N}(0, R). \end{aligned}$$

Furthermore:

- λ - clutter rate (expected number of clutter measurements per volume)
- P_D - probability of detection (approximately: air radar 0.9, naval applications 0.5)
- P_G - gate probability. Simply assume 1.

PDA algorithm

A) KF prediction (independent of measurements)
Predict the state x of the target and its covariance P at time k based on the knowledge of the past state at $k-1$:

$$\begin{aligned} \hat{x}_{k|k-1} &= F_{k-1}\hat{x}_{k-1|k-1}, \\ P_{k|k-1} &= F_{k-1}P_{k-1|k-1}F_{k-1}^{\intercal} + Q_{k-1}. \end{aligned}$$

B) Measurement prediction (independent of measurements)
Predict the measurement and its covariance S at time k based on the predicted state:

$$\begin{aligned} \hat{z}_{k|k-1} &= H_k \hat{x}_{k|k-1}, \\ S_k &= H_k P_{k|k-1} H_k^{\intercal} + R_k. \end{aligned}$$

C) Measurement validation - Ellipsoidal Gating
Select measurements covered by the validation region of size γ:

$$\mathcal{V}_k(\gamma) = \{z : (z - \hat{z}_{k|k-1}) S_k^{-1} (z - \hat{z}_{k|k-1})^{\intercal} \leq \gamma\}.$$

Only the measurements in $\mathcal{V}_k(\gamma)$ will be used in subsequent steps. *They will be denoted by $z_{1,k}, \dots, z_{m_k,k}$ where m_k is their number!*

D) Data association probabilities
Calculate the probabilities β_i of measurements z_i in $\mathcal{V}_k(\gamma)$. Let the likelihood and the Bar-Shalom's scaled likelihood be

$$l_{i,k} = \mathcal{N}(z_{i,k}; \hat{z}_{k|k-1}, S_k) \qquad \text{and} \qquad \mathcal{L}_{i,k} = \frac{P_D \cdot l_{i,k}}{\lambda},$$

respectively. Then

$$\beta_{i,k} = \begin{cases} \frac{\mathcal{L}_{i,k}}{1 - P_D P_G + \sum_{j=1}^{m_k} \mathcal{L}_{j,k}} \propto P_D \cdot l_{i,k} & \text{for measurements } z_1, \dots, z_{m_k}, \\ \frac{1 - P_D P_G}{1 - P_D P_G + \sum_{j=1}^{m_k} \mathcal{L}_{j,k}} \propto \lambda(1 - P_D P_G) & \text{if no meas. is correct.} \end{cases}$$

- first case: one of $z_1, \dots, z_{m_k}$ is a true measurement generated by the target. The probabilities are proportional to the likelihoods.
- second case: all $z_1, \dots, z_{m_k}$ are clutter measurements. The probability is proportional to λ scaled by $\Pr(\neg\text{detected} \cap \neg\text{in gate})$. The original formulas in @Bar-shalom2009 are unnecesarilly complicated, the proportionalities elucidate this better. The denominators are identical, and the separation of λ from $\mathcal{L}$ simplifies understanding. *Don't forget to normalize the probabilities!*

E) KF measurement update steps

1. *Kalman gain W_k (independent of measurements)*
   The Kalman gain W_k (usually denoted by K_k) is independent of measurements. There are multiple equivalent formulas for it, Bar-Shalom's paper exploits

$$W_k = P_{k|k-1} H_k^{\intercal} S_k^{-1}.$$

2. *Innovations $\nu_{i,k}$*

   Innovations - prediction errors - are the differences between the measured and predicted value of the measurement,

$$\begin{aligned}\nu_{i,k} &= z_{i,k} - \hat{z}_{k|k-1} \\ &= z_{i,k} - H_k \hat{x}_{k|k-1}, \qquad i = 1, \ldots, m_k.\end{aligned}$$

   The *combined innovation* in the PDA filter is hence a weighted (convex) combination

$$\nu_k = \sum_{i=1}^{m_k} \beta_{i,k} \nu_{i,k}.$$

   Since the probabilities $\beta_{i,k}$ sum to one (cf. **Data association probabilities** above), the combined innovation corresponds to a single "pseudomeasurement".

3. *Kalman update of the state*

   The Kalman state update incorporates the measurements into the knowledge of the state. The formula reads

$$\begin{aligned}\hat{x}_{k|k} &= \hat{x}_{k|k-1} + W_k(z_k - H_k \hat{x}_{k|k-1}) \qquad &&\text{(in standard KF!)} \\ &= \hat{x}_{k|k-1} + \nu_k,\end{aligned}$$

   where the first formula is from the standard KF. The bracket - innovation - works only for a single measurement. However, the second line uses the innovation ν_k and is hence compatible with the combined innovation calculated above.

4. *Kalman update of the state covariance*

   There are several equivalent formulas for the Kalman covariance update. Two of them are

$$\begin{aligned}P_{k|k} &= (I - W_k H_k) P_{k|k-1} \\ &= P_{k|k-1} - W_k S_k K_k.^{\intercal}\end{aligned}$$

   The first one is omnipresent, the second one is used by Bar-Shalom. We denote the result by $P_{k|k}^C$. This case occurs with probability $1 - \sum_{i=1}^{m_k} \beta_{i,k} = 1 - \beta_{0,k}$. However, in PDA, we need to reflect the possibility that no measurement is generated by the target and the predicted covariance remains untouched. That is, with probability $\beta_{0,k}$,

$$P_{k|k} = P_{k|k-1}.$$

   Multiple measurements correspond to multiple posterior distributions weighted by probabilities $\beta_{i,k}$. The mean of the mixture corresponds to the convex combination, hence the KF update of the state (above) is correct. We also need to reflect the spread of the distributions in the covariance. This corresponds to

$$\tilde{P}_k = W_k \left[\sum_{i=1}^{m_k} \beta_{i,k} \nu_{i,k} \nu_{i,k}^{\intercal} - \nu_k \nu_k^{\intercal} \right] W_k^{\intercal}.$$

   As a result, the PDA covariance update reads

$$P_{k|k} = \beta_{0,k} P_{k|k-1} + (1 - \beta_{0,k}) P_{k|k}^C + \tilde{P}_k.$$