CPHD filter

Source: @voAnalytic2007

Proposition 1: CPHD Prediction

Suppose at time $k-1$ that the posterior intensity v_{k-1} and posterior cardinality distribution p_{k-1} are given. Then, the predicted cardinality distribution $p_{k|k-1}$ and predicted intensity $v_{k|k-1}$ are given by

$$p_{k|k-1}(n) = \sum_{j=0}^{n} \underbrace{p_{\Gamma,k}(n-j)}_{\text{births}} \underbrace{\Pi_{k|k-1}\left[v_{k-1}, p_{k-1}\right](j)}_{\text{survived}} \qquad \text{(convolution)}$$

$$v_{k|k-1}(x) = \int p_{S,k}(\zeta) f_{k|k-1}(x|\zeta) v_{k-1}(\zeta) d\zeta + \gamma_k(x) \qquad \text{(same as PHD)}$$

where

$$\begin{aligned}
\Pi_{k|k-1}[v,p](j) &= \sum_{\ell=j}^{\infty} C_j^{\ell} \frac{\langle p_{S,k}, v\rangle^j \langle 1-p_{S,k}, v\rangle^{\ell-j}}{\langle 1, v\rangle^{\ell}} p(\ell) \\
&= \sum_{\ell=j}^{\infty} \binom{\ell}{j} \frac{\left[\int p_{S,k} \cdot v_{k-1}(x) dx\right]^j \cdot \left[\int (1-p_{S,k}(x)) \cdot v_{k-1}(x) dx\right]^{\ell-j}}{\left[\int v_{k-1}(x) dx\right]^j \cdot \left[\int v_{k-1}(x) dx\right]^{\ell-j}} p(\ell) \\
&= \underbrace{\sum_{\ell=j}^{\infty}}_{\text{for } j \text{ or more targets}} \underbrace{\binom{\ell}{j} \left(\frac{\#\text{ survived}}{\#\text{ targets at } k-1}\right)^j \cdot \left(\frac{\#\text{ died}}{\#\text{ targets at } k-1}\right)^{\ell-j}}_{\text{Probability of } j \text{ survivals in binomial distribution}} \cdot p(\ell)
\end{aligned}$$

- $f_{k|k-1}(\cdot|\zeta)$ = single-target transition density at time k given previous state ζ
- $p_{S,k}(\zeta)$ = probability of target existence at time k given previous state ζ
- $\gamma_k(\cdot)$ = intensity of spontaneous births at time k
- $p_{\Gamma,k}(\cdot)$ = cardinality distribution of births at time k

> **Explanation of $\Pi_{k|k-1}[v,p](j)$:**
> Whatever the number ℓ of targets there is:
> - select probability of ℓ from the cardinality distribution $p(\cdot)$
> - multiply it by the probability that there are j survivals and $\ell - j$ deaths given by the binomial distribution
>   (cf. eq. 23 - there it's clear as the survival prob. is constant! :)

Proposition 2: CPHD update

Suppose at time k that the predicted intensity $v_{k|k-1}$ and predicted cardinality distribution $p_{k|k-1}$ are given. Then, the updated cardinality distribution p_k and updated intensity v_k are given by

$$p_k(n) = \frac{\Upsilon_k^0\left[v_{k|k-1}, Z_k\right](n) p_{k|k-1}(n)}{\left\langle \Upsilon_k^0\left[v_{k|k-1}, Z_k\right], p_{k|k-1}\right\rangle} = \frac{\text{likelihood of the number of } Z_k \times \text{predicted cardinality}}{\text{normalizing constant}}$$

$$v_k(x) = \frac{\left\langle \Upsilon_k^1\left[v_{k|k-1}, Z_k\right], p_{k|k-1}\right\rangle}{\left\langle \Upsilon_k^0\left[v_{k|k-1}, Z_k\right], p_{k|k-1}\right\rangle}\left[1-p_{D,k}(x)\right] v_{k|k-1}(x) + \sum_{z \in Z_k} \frac{\left\langle \Upsilon_k^1\left[v_{k|k-1}, Z_k \backslash \{z\}\right], p_{k|k-1}\right\rangle}{\left\langle \Upsilon_k^0\left[v_{k|k-1}, Z_k\right], p_{k|k-1}\right\rangle} \cdot \frac{\langle 1, \kappa_k\rangle}{\kappa_k(z)} p_{D,k}(x) \cdot g_k(z|x) v_{k|k-1}(x)$$

$$\text{cf. ordinary PHD:} \qquad v_k(x) = \left[1-p_{D,k}(x)\right] v_{k|k-1}(x) + \sum_{z \subset Z_k} \frac{p_{D,k}(x) g_k(z|x) v_{k|k-1}(x)}{k_k(z) + \int p_{D,k}(\xi) g_k(z|\xi) v_{k|k-1}(\xi) d\xi}$$

where

$$\Upsilon_k^u[v, Z](n) = \sum_{j=0}^{\min(|Z|,n)} (|Z|-j) p_{K,k}(|Z|-j) P_{j+u}^n \frac{\langle 1 - p_{D,k}, v\rangle^{n-(j+u)}}{\langle 1, v\rangle^n} e_j\left(\Xi_k(v, Z)\right)$$

$$= \underbrace{\sum_{j=0}^{\min(|Z|,n)}}_{\substack{\text{possible \# of} \\ \text{assocs } n\leftrightarrow Z}} \underbrace{(|Z|-j)\cdot p_{K,k}(|Z|-j)}_{\substack{\text{no. of unassociated} \\ \text{and its probability}}} \times \underbrace{P_{j+u}^n}_{\substack{\text{\# possible assocs.} \\ n\leftrightarrow Z}} \underbrace{(1-p_D)^{n-(j+u)}}_{\text{prob. misdetections}} \times \underbrace{\sum_{S\subseteq Z, |S|=j}}_{\forall j\text{-tuples from } Z} \underbrace{\prod_{z\in S} p_D \int \frac{g_k(z|x) v_k(x) dx}{"\lambda"}}_{\text{predictive likelihoods}}$$

(the orange eq is simplified and employs the ordinary λ for Poisson clutter rate)
and

- $\psi_{k,z}(x) = \frac{\langle 1,\kappa_k\rangle}{\kappa_k(z)} g_k(z|x) p_{D,k}(x)$
- $\Xi_k(v, Z) = \{\langle v, \psi_{k,z}\rangle : z \in Z\}$
- Z_k = measurement set at time k
- $g_k(\cdot|x)$ = single-target measurement likelihood at time k given current state x
- $p_{D,k}(x)$ = probability of target detection at time k given current state x
- $\kappa_k(\cdot)$ = intensity of clutter measurements at time k
- $p_{K,k}(\cdot)$ = cardinality distribution of clutter at time k .