6. Prerequisites for Convex Analysis

6.1. Concave Functions, Convex Functions and Definiteness

Definition 1. A *convex set* is a set $C \subset X$, for some vector space X, such that for any $\mathbf{x}, \mathbf{y} \in C$ and $\lambda \in [0,1]$ then

$$\mathbf{z} = \lambda \mathbf{x} + (1-\lambda)\mathbf{y} \in C.$$

And $\mathbf{z}$ is called a *convex combination* of $\mathbf{x}$ and $\mathbf{y}$.

Exercise 1. Is the set $\{(x,y) \in R^2 | xy < 1\}$ convex?

Definition 2. Let X be a convex subset of R^k. A function $f : X \to R$ is said to be *concave* if and only if for any $\mathbf{x}, \mathbf{y} \in X$ and $\lambda \in [0,1]$ then

$$f(\lambda \mathbf{x} + (1-\lambda)\mathbf{y}) \geq \lambda f(\mathbf{x}) + (1-\lambda) f(\mathbf{y}).$$

The function f is *convex* if and only if

$$f(\lambda \mathbf{x} + (1-\lambda)\mathbf{y}) \leq \lambda f(\mathbf{x}) + (1-\lambda) f(\mathbf{y}).$$

If the above condition holds with strict inequality for all $\lambda \in (0,1)$, we say that f is *strictly concave* or *strictly convex*, respectively.

Theorem 1. Given functions $g : R^k \to R$ and $f : R^n \to R^k$, their composition $h = g(f) : R^n \to R$ is convex if:
(a) g is convex non-decreasing, and f is convex; or
(b) g is convex non-increasing, and f is concave;
and the composite function h is concave if:
(c) g is concave non-decreasing, and f is concave; or
(d) g is concave non-increasing, and f is convex.

Theorem 2. Let $f_i : X \to R$ be concave (convex) functions. If $\alpha_i \geq 0$ for all $i = 1, \cdots, n$, then $f = \sum_{i=1}^{n} \alpha_i f_i$ is concave (convex). If, in addition, at least one f_j is strictly concave (convex) and corresponding α_j is strictly greater than 0, then f is strictly concave (convex).

Theorem 3. Let f be a $\mathscr{C}^2$ function on an open convex set X of R^k. Then f is concave (convex) on X if and only if the Hessian $H(f)$ is negative (positive) semi-definite for all $\mathbf{x} \in X$. f is strictly concave (convex) on X if $H(f)$ is negative (positive) definite.

Example 2. The Hessian of the function $f(x,y) = x^4 + x^2y^2 + y^4 - 3x - 8y$ is

$$H(f) = \begin{pmatrix} 12x^2 + 2y^2 & 4xy \\ 4xy & 2x^2 + 12y^2 \end{pmatrix}.$$

The principal minors, $B_1^{(1)} = 12x^2 + 2y^2, B_1^{(2)} = 2x^2 + 12y^2$ and $B_2 = 24x^4 + 132x^2y^2 + 24y^2$ are all weakly positive for all values of (x,y) on R^2. Therefore, f is a convex function on R^2.

6.2. Quasi-Concave and Quasi-Convex Functions

Definition 3. Let X be a convex subset of R^k and f be a real-valued function on X. We say that f is *quasi-concave* if and only if for all $\mathbf{x}, \mathbf{y} \in X$ and $\lambda \in [0,1]$ then

$$f(\lambda \mathbf{x} + (1-\lambda)\mathbf{y}) \geq \min\{f(\mathbf{x}), f(\mathbf{y})\}.$$

The function is *quasi-convex* if and only if

$$f(\lambda \mathbf{x} + (1-\lambda)\mathbf{y}) \leq \max\{f(\mathbf{x}), f(\mathbf{y})\}.$$

If the above condition holds with strict inequality for all $\lambda \in (0,1)$, we say that f is *strictly quasi-concave* or *strictly quasi-convex*, respectively.

Theorem 4. Let f be a real-valued function defined on a convex subset X of R^k. Then f is quasi-concave (quasi-convex) if and only if the upper contour sets (lower contour sets) of f are all convex. That is, if for any $a \in R$ the set

$$U_a = \{\mathbf{x} \in X : f(\mathbf{x}) \geq a\} \quad (L_a = \{\mathbf{x} \in X : f(\mathbf{x}) \leq a\})$$

is convex.

Theorem 5. If a function f is concave (convex), then it is quasi-concave (quasi-convex). Moreover, if f is strictly concave (convex), then f is strictly quasi-concave (quasi-convex).

Exercise 3. Prove Theorem 5.

The critical feature of quasi-concavity is that it is preserved by monotone transformation.

Theorem 6. If a function $f : X \to R$ is quasi-concave and $\phi : f(X) \to R$ is strictly increasing, then $g = \phi(f)$ is also quasi-concave.