2. Differentiation

In this chapter we shall confine our attention to *real functions* defined on intervals or segments. This is not just a matter of convenience, since genuine differences appear when we pass from real functions to vector-valued ones. Differentiation of functions defined on R^k will be discussed in Chap.5: Multivariate Differentiation.

2.1. The Derivative of a Real Function

Definition 1. Let f be defined (and real-valued) on $[a,b]$. For any $x \in [a,b]$ form the quotient

$$\phi(t) = \frac{f(t) - f(x)}{t - x} \quad (a < t < b, t \neq x),$$

and define

$$f'(x) = \lim_{t \to x} \phi(t),$$

provided this limit exists.

We thus associate with the function f a function f' whose domain is the set of points x at which the limit exists; f' is called the *derivative* of f.

If f' is defined at a point x, we say that f is *differentiable* at x. If f' is defined at every point of a set $E \subset [a,b]$, we say that f is differentiable on E.

We also use the notation

$$\frac{df(x)}{dx} \quad \text{or} \quad \frac{d}{dx} f(x)$$

for $f'(x)$.

Remark. It is possible to consider right-hand and left-hand limits; this leads to the definition of right-hand and left-hand derivatives. In particular, at the endpoints a and b, the derivative, if it exists, is a right-hand or left-hand derivative, respectively.

Theorem 1. Let f be defined on $[a,b]$. If f is differentiable at a point $x \in [a,b]$, then f is continuous at x.

Remark. The converse of this theorem is not true. It is easy to construct continuous functions which fail to be differentiable at isolated points. Moreover, functions which are continuous on the whole line without being differentiable at any point can also be constructed. A very famous example - and by far the most important when it comes to the practical application (finance: option pricing) - is the *Wiener process*.

Theorem 2. Suppose f and g are defined on $[a,b]$ and are differentiable at a point $x \in [a,b]$. Then $f+g, fg$, and f/g are differentiable at x, and

(a) $(f+g)'(x) = f'(x) + g'(x)$;
(b) $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$;
(c) $\left(\frac{f}{g}\right)'(x) = \frac{g(x)f'(x) - g'(x)f(x)}{g^2(x)}$.
In (c), we assume of course that $g(x) \neq 0$.

Exercise 1. Let
$$f(x) = \frac{x^2+x+1}{e^x}.$$
Calculate $f'(x)$.

Theorem 3. (Chain Rule). Suppose f is continuous on $[a,b]$, $f'(x)$ exists at some point $x \in [a,b]$, g is defined on an interval I which contains the range of f, and g is differentiable at the point $f(x)$. If
$$h(x) = g(f(x)) \ \ (a \leq x \leq b),$$
then h is differentiable at x, and
$$h'(x) = g'(f(x))f'(x).$$

Exercise 2. Let $f(x) = e^{ax}\sin bx$. Calculate $f'(x)$.

Exercise 3. (i) Let
$$f(x) = \begin{cases} x\sin\frac{1}{x} & (x \neq 0) \\ 0 & (x = 0) \end{cases}$$
Calculate $f'(x)$ for $x \neq 0$, and show that $f(x)$ is not differentiable at $x = 0$

(ii) Let
$$f(x) = \begin{cases} x^2\sin\frac{1}{x} & (x \neq 0) \\ 0 & (x = 0) \end{cases}$$
Calculate $f'(x)$.

2.2. Mean Value Theorems

Theorem 4. Let f be defined on $[a,b]$; if f has a local maximum(minimum) at a point $x \in (a,b)$, and if $f'(x)$ exists, then $f'(x) = 0$.

Remark. The converse of this theorem is not true.

Theorem 5. (Generalized Mean Value Theorem). If f and g are continuous real functions on $[a,b]$ which are differentiable in (a,b), then there is a point $x \in (a,b)$ at which

$$[f(b) - f(a)]g'(x) = [g(b) - g(a)]f'(x).$$

Proof. Construct

$$h(t) = [f(b) - f(a)]g(t) - [g(b) - g(a)]f(t) \quad (a \leq t \leq b).$$

Then h is continuous on $[a,b]$ and differentiable in (a,b). Note that

$$h(a) = f(b)g(a) - f(a)g(b) = h(b).$$

To prove the theorem, we then need to show that $h'(x) = 0$ for some $x \in (a,b)$.(details left to readers.)

Theorem 6. (The Mean Value Theorem). If f is a real continuous function on $[a,b]$ which is differentiable in (a,b), then there is a point $x \in (a,b)$ at which

$$f(b) - f(a) = (b-a)f'(x).$$

Proof. Take $g(x) = x$ in Theorem 5.

Example 4. Prove that

$$\frac{x}{1+x} < \ln(1+x) < x, \quad (x > 0).$$

by using the mean value theorem.

Proof. Let $f(t) = \ln(1+t)$. Take $a = 0, b = x > 0$, then f is differentiable in (a,b). Apply the mean value theorem to f, there exists $c \in (a,b)$ such that

$$\ln(1+b) - \ln(1+a) = (b-a)\frac{1}{1+c}$$

i.e.

$$\ln(1+x) - \ln 1 = \ln(1+x) = x\frac{1}{1+c}.$$

Since $0 < c < x$, we have

$$\frac{x}{1+x} < \ln(1+x) < x.$$

Exercise 5. Prove that

$$\frac{b-a}{b} < ln\frac{b}{a} < \frac{b-a}{a} \quad (0 < a < b).$$

We have already seen [Exercise 3(b)] that a function f may have a derivative f' which exists at every point, but is discontinuous at some point. However, not every discontinous function can be a derivative. In particular, derivatives which exist at every point of an interval have one important property in common with functions which are continuous on an interval: Intermediate values are assumed (compare Chap.1 Theorem 31). We state the theorem and its proof as an important application of the mean value theorem.

Theorem 7. Suppose f is a real differentiable function on $[a,b]$ and suppose $f'(a) < \lambda < f'(b)$. Then there is a point $x \in (a,b)$ such that $f'(x) = \lambda$.

Proof. Construct $g(t) = f(t) - \lambda t$. Then $g'(a) < 0$, so that $g(t_1) < g(a)$ for some $t_1 \in (a,b)$, and $g'(b) > 0$, so that $g(t_2) < g(b)$ for some $t_2 \in (a,b)$. Hence g attains its minimum on $[a,b]$ (Chap.1 Theorem 29) at some point x such that $a < x < b$. By Theorem 4, $g'(x) = 0$. Hence $f'(x) = \lambda$.

2.3. L'Hospital's Rule

Theorem 8. Suppose f and g are real and differentiable in (a,b), and $g'(x) \neq 0$ for all $x \in (a,b)$, where $-\infty \leq a < b \leq +\infty$. Suppose

$$\lim_{x\to a}\frac{f'(x)}{g'(x)} = A.$$

If

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0 \text{ or } \pm\infty,$$

then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)} = A.$$

The analogous statement is of course also true if $x \to b$.

Exercise 6. Revisit Chap.1 Exercise 15. Compute

$$\lim_{x\to 0}\frac{\sqrt{1+x}-\sqrt{1-x}}{x}$$

using L'Hospital's rule.

Exercise 7. Prove that
$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e.$$

2.4. Taylor's Theorem

Taylor polynomials are used to approximate a function by its derivative. This theorem is critical because a lot of economics functions are highly nonlinear and can only be solved by firstly approximate them using Taylor polynormials.

Definition 2. If f has a derivative f' on an interval, and if f' is itself differentiable, we denote the derivative of f' by f'' and call f'' the second derivative of f. Continuing in this manner, we obtain functions
$$f, f', f'', f^{(3)}, \cdots, f^{(n)},$$
each of which is the derivative of the preceding one. $f^{(n)}$ is called the *nth derivative*, or *the derivative of order n*, of f.

Remark. In order for $f^{(n)}(x)$ to exist at a point x, $f^{(n-1)}(t)$ must exist in a neighborhood of x, and $f^{(n-1)}$ must be differentiable at x.

Exercise 8. Let $f(x) = e^x \cos x$. Prove that $f'' - 2f' + 2f = 0$.

Theorem 9. Suppose f is a real function on $[a,b]$, n is a positive integer, $f^{(n-1)}$ is continuous on $[a,b]$, $f^{(n)}$ exists for every $t \in (a,b)$. Let α, β be distinct points of $[a,b]$, and define
$$P(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(t-\alpha)^k.$$
Then there exists a point x between α and β such that
$$f(\beta) = P(\beta) + \frac{f^{(n)}(x)}{n!}(\beta-\alpha)^n.$$

Remark. For $n = 1$, this is just the mean value theorem. In general, the theorem shows that f can be approximated by a polynomial of degree $n-1$, and allows us to estimate the error, if we know bounds on $|f^{(n)}(x)|$.

Example 9. Let $f(t) = e^t, \alpha = 0, \beta = x$, we have
$$e^x = 1 + x + \frac{1}{2!}x^2 + \cdots + \frac{1}{n!}x^n + \frac{e^s}{(n+1)!}x^{n+1}$$

where s is some number between 0 and x.
Since

$$\lim_{x\to 0}\frac{\frac{e^s}{(n+1)!}x^{n+1}}{x^n}=0,$$

we also write

$$e^x=1+x+\frac{1}{2!}x^2+\cdots+\frac{1}{n!}x^n+o(x^n),\quad (x\to 0).$$

Remark. Let $x=1$ and $n\to\infty$, we show that $e=\sum_{n=0}^{\infty}\frac{1}{n!}$ (see Chap.1 Exercise 10.)

Exercise 10. Find the Taylor polynomial of degree n for $f(x)=\frac{1}{1-x}$, centered at $x=0$.

In very rare cases Taylor's theorem could be useless. See the following example.

Example 11. Let

$$f(x)=\begin{cases} e^{-1/x^2}, & x\neq 0,\\ 0, & x=0.\end{cases}$$

Since

$$f'(0)=\lim_{\Delta x\to 0}\frac{f(0+\Delta x)-f(0)}{\Delta x}=\lim_{\Delta x\to 0}\frac{e^{-1/(\Delta x)^2}}{\Delta x}=0,$$

we have

$$f'(x)=\begin{cases} \frac{2}{x^3}e^{-1/x^2}, & x\neq 0,\\ 0, & x=0.\end{cases}$$

Continue this procedure, we can prove that

$$f^{(n)}(0)=0,\quad n=1,2,\cdots.$$

Therefore, we learn nothing about this function from its Taylor expansion at $x=0$.

2.5. More Exercises

Exercise 12. Let $f(x)=\cos^5\sqrt{1+x^2}$. Calculate $f'(x)$.

Exercise 13. Use the mean value theorem to prove that

$$|\sin x-\sin y|\leq |x-y|.$$

Exercise 14. Compute

$$\lim_{x\to 1}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right).$$

Exercise 15. Find the Taylor polynomial of degree three for $f(x)=\sin x$, centered at $x=\frac{5\pi}{6}$.