3. Integration

Integration plays an important role in probability theory. We shall only review some primary concepts in this chapter.

3.1. Definition and Existence of the Riemann Integral

Definition 1. Let $[a,b]$ be a given interval. By a *partition* P of $[a,b]$ we mean a finite set of points $x_0, x_1, \cdots, x_n$, where

$$a = x_0 \leq x_1 \leq \cdots \leq x_{n-1} \leq x_n = b.$$

We write

$$\Delta x_i = x_i - x_{i-1} \quad (i = 1, \cdots, n).$$

Now suppose f is a bounded real function defined on $[a,b]$. Corresponding to each partition P of $[a,b]$ we put

$$M_i = \sup f(x) \quad (x_{i-1} \leq x \leq x_i),$$

$$m_i = \inf f(x) \quad (x_{i-1} \leq x \leq x_i),$$

$$U(P,f) = \sum_{i=1}^{n} M_i \Delta x_i,$$

$$L(P,f) = \sum_{i=1}^{n} m_i \Delta x_i,$$

and finally

$$\overline{\int_a^b} f dx = \inf U(P,f),$$

$$\underline{\int_a^b} f dx = \sup L(P,f),$$

where the inf and the sup are taken over all partitions P of $[a,b]$. The left members above are called the *upper* and *lower Riemann integrals* of f over $[a,b]$, respectively.

If the upper and lower integrals are equal, we say that f is *Riemann integrable* on $[a,b]$, we write $f \in \mathscr{R}$ (that is, $\mathscr{R}$ denotes the set of Riemann integrable functions), and we denote the common value by

$$\int_a^b f dx,$$

or by

$$\int_a^b f(x) dx.$$

This is the *Riemann integral* of f over $[a,b]$.

Remark. Since f is bounded, there exist two numbers, m and M, such that

$$m \le f(x) \le M \quad (a \le x \le b).$$

Hence, for every P,

$$m(b-a) \le L(P,f) \le U(P,f) \le M(b-a),$$

so that the numbers $L(P,f)$ and $U(P,f)$ form a bounded set. This shows that *the upper and lower integrals are defined* for *every* bounded function f. The question of their equality, and hence the question of the integrability of f, is a more delicate one. We'll show some brief results as follows.

Theorem 1. $\underline{\int_a^b} f\,dx \le \overline{\int_a^b} f\,dx$.

Theorem 2. (Cauchy Criterion for Riemann Integrability) $f \in \mathscr{R}$ on $[a,b]$ if and only if for every $\varepsilon > 0$ there exists a partition P such that

$$U(P,f) - L(P,f) < \varepsilon.$$

Remark. Theorem 2 furnishes a convenient criterion for integrability. It leads to the following fundamental results.

Theorem 3. If f is continuous on a compact interval $[a,b]$ then $f \in \mathscr{R}$ on $[a,b]$.

Example 1. Define $f : [0,1] \to R$ by

$$f(x) = \begin{cases} 1/x & 0 < x \le 1, \\ 0 & x = 0. \end{cases}$$

Then

$$\int_0^1 \frac{1}{x} dx$$

isn't defined as a Riemann integral because f is unbounded.

Remark. The compactness is essential in this theorem. A continuous function on a compact interval is bounded.

Theorem 4. If f is monotonic on $[a,b]$ then $f \in \mathscr{R}$ on $[a,b]$.

Theorem 5. Suppose f is bounded on $[a,b]$, f has only finitely many points of discontinuity on $[a,b]$, then $f \in \mathscr{R}$.

Example 2. The Dirichlet function $f : [0,1] \to R$ is defined by

$$f(x) = \begin{cases} 1 & x \text{ is rational,} \\ 0 & x \text{ is irrational.} \end{cases}$$

This function is not Riemann integrable. If $P = \{x_1, x_2, \cdots, x_n\}$ is a partition of $[0,1]$, then

$$M_i = \sup_{x_i} f = 1, \quad m_i = \inf_{x_i} f = 0,$$

since every interval of non-zero length contains both rational and irrational numbers. It follows that

$$U(P,f) = 1, \quad L(P,f) = 0$$

for every partition P of $[0,1]$, so $\underline{\int}_a^b f dx$ and $\overline{\int}_a^b f dx$ are not equal.

Remark. (i) This is an example that a bounded function on a compact interval whose Riemann integral doesn't exist. In fact, the Dirichlet function is discontinuous at every point of $[0,1]$, and the moral of this example is that the Riemann integral of a highly discontinuous function need not exist.

(ii) The Riemann integral is the simplest integral to define, and it allows one to integrate every continuous function as well as some not-too-badly discontinuous functions. Another important integral is the *Lebesgue integral*. The Lebesgue integral allows one to integrate unbounded or highly discontinuous functions whose Riemann integrals do not exist, and it has better mathematical properties than the Riemann integral. The definition of the Lebesgue integral requires the use of measure theory, which we will not describe here.

Theorem 6. Suppose $f \in \mathscr{R}$ on $[a,b]$, $m \leq f \leq M$, ϕ is continuous on $[m,M]$, and $h(x) = \phi(f(x))$ on $[a,b]$. Then $h \in \mathscr{R}$ on $[a,b]$.

3.2. Properties of the Riemann Integral

Theorem 7. (a) If $f_1 \in \mathscr{R}$ and $f_2 \in \mathscr{R}$, then $f_1 + f_2 \in \mathscr{R}$, $cf \in \mathscr{R}$ for every constant c, and

$$\int_a^b (f_1 + f_2) dx = \int_a^b f_1 dx + \int_a^b f_2 dx,$$

$$\int_a^b cf dx = c \int_a^b f dx.$$

(b) If $f_1(x) \leq f_2(x)$ on $[a,b]$, then

$$\int_a^b f_1 dx \leq \int_a^b f_2 dx.$$

(c) If $f \in \mathscr{R}$ on $[a,b]$ and if $a < c < b$, then $f \in \mathscr{R}$ on $[a,c]$ and on $[c,b]$, and

$$\int_a^c f dx + \int_c^b f dx = \int_a^b f dx.$$

Theorem 8. If $f \in \mathscr{R}$ and $g \in \mathscr{R}$ on $[a,b]$, then $fg \in \mathscr{R}$. If, in addition, $g \neq 0$ and $1/g$ is bounded, then $f/g \in \mathscr{R}$.

Theorem 9. If $f \in \mathscr{R}$, then $|f| \in \mathscr{R}$ and

$$\left| \int_a^b f dx \right| \leq \int_a^b |f| dx.$$

Theorem 10. If $f : [a,b] \to R$ is continuous, then there exists $x \in [a,b]$ such that

$$f(x) = \frac{1}{b-a} \int_a^b f dx.$$

Proof. Let $M = \sup\limits_{[a,b]} f, m = \inf\limits_{[a,b]} f$, then $m \leq f \leq M$ on $[a,b]$. Theorem 7(b) implies that

$$m(b-a) = \int_a^b m dx \leq \int_a^b f dx \leq \int_a^b M dx = M(b-a),$$

i.e.

$$m \leq \frac{1}{b-a} \int_a^b f dx \leq M.$$

By the intermediate value theorem (Chap.1 Theorem 31), f takes on every value between m and M, and the result follows.

3.3. The Fundamental Theorem of Calculus

Theorem 11. Let $f \in \mathscr{R}$ on $[a,b]$. For $a \leq x \leq b$, put

$$F(x) = \int_a^x f(t) dt.$$

Then F is continuous on $[a,b]$; furthermore, if f is continuous at a point x_0 of $[a,b]$, then F is differentiable at x_0, and

$$F'(x_0) = f(x_0).$$

Remark. This result can be thought of as a continuous analog of the corresponding identity for differences of sums,

$$\sum_{j=1}^{k} a_j - \sum_{j=1}^{k-1} a_j = a_k.$$

Example 3. If

$$f(x) = \begin{cases} 1 & x \geq 0, \\ 0 & x < 0. \end{cases}$$

then

$$F(x) = \int_0^x f(t)dt = \begin{cases} x & x \geq 0, \\ 0 & x < 0. \end{cases}$$

The function F is continuous but not differentiable at $x = 0$, where f is discontinuous, since the left and right derivatives of F at 0, given by $F'(0-) = 0$ and $F'(0+) = 1$, are different.

Theorem 12. (The Fundamental Theorem of Calculus). If $f \in \mathscr{R}$ on $[a,b]$ and if there is a differentiable function F on $[a,b]$ such that $F' = f$, then

$$\int_a^b f(x)dx = F(b) - F(a).$$

Remark. This is a continuous analog of the corresponding identity for sums of differences (compare with the remark of Chap.1 Definition 11),

$$\sum_{k=1}^{n} (s_k - s_{k-1}) = s_n - s_0.$$

Exercise 4. Compute

$$\int_0^1 (e^x + x)dx.$$

Exercise 5. Revisit Chap.2 Exercise 3(ii). Let

$$f(x) = \begin{cases} -\cos\frac{1}{x} + 2x\sin\frac{1}{x}, & 0 < x \leq 1 \\ 0, & x = 0 \end{cases}$$

Compute $\int_0^1 f(x)dx$.

Remark. (i) The fundamental theorem of calculus is the basic computational tool in integration. It allows us to compute the integral of a function f if we can find an antiderivative; that is, a function F such that $F' = f$. There is no systematic procedure for finding antiderivatives.
(ii) Two important general consequences of the fundamental theorem of calculus are *integration by parts* and *change of variable*, which we will not describe here.