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MATH CAMP 2016 EXERCISES

Exercise 1.1

Prove that

$$\|\mathbf{x}+\mathbf{y}\|^2 + \|\mathbf{x}-\mathbf{y}\|^2 = 2\|\mathbf{x}\|^2 + 2\|\mathbf{y}\|^2$$

if $\mathbf{x} \in R^k$ and $\mathbf{y} \in R^k$.

norm: $\|x\| = (\langle \mathbf{x}, \mathbf{x} \rangle)^{\frac{1}{2}}$

Proof.

$$\begin{aligned}
\|\mathbf{x}+\mathbf{y}\|^2 + \|\mathbf{x}-\mathbf{y}\|^2 &= \langle \mathbf{x}+\mathbf{y}, \mathbf{x}+\mathbf{y} \rangle + \langle \mathbf{x}-\mathbf{y}, \mathbf{x}-\mathbf{y} \rangle \\
&= \sum_1^k (x_i + y_i)^2 + \sum_1^k (x_i - y_i)^2 \\
&= \sum_1^k x_i^2 + \sum_1^k 2x_i y_i + \sum_1^k y_i^2 + \sum_1^k x_i^2 - \sum_1^k 2x_i y_i + \sum_1^k y_i^2 \\
&= 2\sum_1^k x_i^2 + 2\sum_1^k y_i^2 \\
&= 2\langle \mathbf{x}, \mathbf{x} \rangle + 2\langle \mathbf{y}, \mathbf{y} \rangle \\
&= 2\|\mathbf{x}\|^2 + 2\|\mathbf{y}\|^2
\end{aligned}$$

Consider parrallegram formed by $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}$. The sum of the length of the sides, squared will be equal to the sum of the diagonals squared.

□

Exercise 1.2

For $x \in \mathbf{R}^1$ and $y \in mathbfR^1$, determine whether each is a metric.

1. $d_1(x, y) = (x - y)^2$
   Fails subadditivity. Consider when $x = 4$, $y = -1$, $z = 2$.

$$d(4, -1) = 25 > 15 = d(4, 2) + d(2, -1)$$

2. $d_2(x, y) = \sqrt{|x - y|}$ This fails non-negativity. Suppose, $x = 12$ and $y = 3$.

$$d(12, 3) = \sqrt{|12 - 3|} = \pm 3$$

For subadditivity, first observe

$$
\begin{aligned}
|x-y| &= |x-y+z-z| && (1)\\
&= |x-z+z-x| && (2)\\
&\leq |x-z|+|z-x| && (3)\\
&\leq |x-z|+|z-x|+2\sqrt{|x-z|}\sqrt{|z-y|} && (4)\\
&= (\sqrt{|x-z|}+\sqrt{|z-y|})^2 && (5)
\end{aligned}
$$

Taking the square root of the resulting inequality

$$\sqrt{|x-y|} \leq \sqrt{|x-z|}+\sqrt{|z-y|}$$

Line 4, is not neccessarily true.

3. $d_3(x,y) = |x^2-y^2|$

   It is obvious this definition satisifies nonnegativity and symmetry.

   For subbadditivity, observe

$$
\begin{aligned}
|x^2-y^2| &= |x^2-z^2+z^2-y^2|\\
&\leq |x^2-z^2|+|z^2-y^2|
\end{aligned}
$$

   Hence $d(x,y) \leq d(x,z)+d(z,y)$

Exercise 3

Consider the following subset of $\mathbb{R}^2$ and discuss the closedness, openes, and boundedness of each.

1. $E = \mathbb{N}$ The set of all integers.

   Closedness: from (b), the set has no limit points. Hence, every limit point is contained in E. Therefore it it is closed.

   Openness: E has no interior points, hence E is not open. $\exists x \in E$ such that x is not an interior point.

   Boundedess: For $\forall x \in N$ there exists y such that $x < y$. (Archmedian principle). Unbounded.

2. $E = \{\frac{1}{n} \mid n \in \mathbb{N}^+\}$

   Closedness: 0 is an accumulation point of E, but $0 \notin E$ therefore E is not closed.

   Openness: $\forall x \in E$, $\exists r > 0$ such that $N_r(x) \not\subset E$ by the properaty for $\mathbb{Q}$ is dense in $\mathbb{R}$. Hence E has no interior points and is not open.

   boundedness: E is bounded. Choose $M = 2$, observe $\forall x \in E$, $d(x,(0,0)) < M$.

3. $E = \mathbb{R}^2$ This set is open, closed and unbounded. The first two easily follow from (d) and (f). For boundedness, observe that given any M such that $d(x,p) < M$ for $x \in E$ there exists a $p \in \mathbb{R}^2$ such that $d(x,p) > M$, using the standard metric. However, in the discrete metric this would be bounded.

Exercise 4

Determine whether each of the following sets is compact.

1. $[0,1]$
   Closed and bounded, hence compact

2. $[0,1)$ Observe that 1 is a limit point of the set. The set is not closed, so not compact. Also, let $B_n = (0, \frac{1}{n})$ for $n \in \mathbb{N}^+$ Observe that the the open cover of the set, $\{0\} \cup \bigcup_n^\infty B_n$ has no finite subcover.

3. $E = \{1,2,3\}$ Compact. The set is closed and bounded.

4. $E = \{\frac{1}{n} \mid n = 1,2,3...\}$ Observer the 0 is a limit point of E, but $0 \notin E$ so E is not closed. Therefore, not compact.

5. $E = \{\frac{1}{n} \mid n = 1,2,3...\} \cup \{0\}$ Closed and bounded, therefore compact.

Exercise 5

Calculate $\lim_{n\to\infty}(\sqrt{n^2+n} - n)$

First note

$$\begin{aligned}\sqrt{n^2+n} - n &= \sqrt{n^2+n} - n\frac{\sqrt{n^2+n}+n}{\sqrt{n^2+n}+n} \\ &= \frac{n}{\sqrt{n^2+n}+n} \\ &= \frac{1}{\sqrt{1+\frac{1}{n}}+1}\end{aligned}$$

Next, observe

$$\lim_{n\to\infty}\frac{1}{n} = 0$$

Therefore

$$\lim_{n\to\infty}(\sqrt{n^2+n} - n) = \frac{1}{2}$$

Exercise 6

Given
$$x_n = 1 + \frac{1}{2^2} + ... + \frac{1}{n^2}$$
prove the sequence $\{x_n\}$ converges.

Proof. Choose any $\epsilon > 0$. Then given sufficiently large $N \in \mathbb{N}$
$$d(x_n, x_{n+1}) = \frac{1}{(n+1)^2} < \epsilon$$
where $n \geq N$. Hence the sequence is Cauchy and, equivalently, convergent.

□

Exercise 7

If $s_1 = \sqrt{2}$ and $s_{n+1} = \sqrt{2+s_n}$ for $n = 1, 2, 3...$ prove that $\{s_n\}$ converges, and and that $s_n < 2$ for $n = 1, 2, 3....$

First, prove monotonicity using induction, i.e. $s_n < s_{n+1}$. Setting $n = 1$
$$s_1 = \sqrt{2} < \sqrt{2+\sqrt{2}} = s_2$$
as required.

Let n be an arbitrary natural number and suppose that $s_n < s_{n+1}$. Then
$$s_{n+1} = \sqrt{2+s_n} < \sqrt{2+s_{n+1}} = s_{n+2}$$

Since both the base case and the inductive have been performed, through mathematical induction, $s_n < s_{n+1}$ holds for all natural numbers.

Therefore, $\{s_n\}$ is monotonic.

Next show that $\{s_n\}$ is bounded. We are given that it is bounded below by $\sqrt{2}$. Using induction, show that $s_n < 2$ for $n = 1, 2, 3, ...$ Setting $n = 1$
$$s_1 = \sqrt{2} < 2$$

Let n be arbitrary and suppose $s_n < 2$. Then
$$\begin{aligned} s_{n+1} &= \sqrt{2+s_n} \\ &< \sqrt{2+2} \\ &= 2 \end{aligned}$$

Since both the base case and the inductive have been performed, through mathematical induction, $s_n < 2$ holds for all natural numbers.

Hence, $\{s_n\}$ is bounded above by 2 and converges.

Exercise 8

Find the upper and lower limits of the sequence $\{s_n\}$ defined by

$$s_n = \frac{(-1)^n}{1+\frac{1}{n}}$$

Observe that the odd numbered indexed sequences are negative and approach -1 while the even number indexed sequences are even and approach 1. Hence

$$\limsup s_n = 1 \text{ and } \liminf s_n = -1$$

Exercise 9

Calculate

$$\sum_{n=0}^{\infty}(n+1)x^n \text{ for } 0 \leq x < 1$$

We need to find a general formula of $S_n = \sum_{n=0}^{\infty}(n+1)x^n$. It will be usefull to recall the gemotric series

$$\sum_{i=0}^{n} r^i = \frac{1-r^{n+1}}{1-r} \text{ for } (-1 < r < 1)$$

and

$$\sum_{i=0}^{\infty} r^i = \frac{1}{1-r}$$

Calculating for S_n

$$\sum_{n=0}^{\infty}(n+1)x^n = 1 + 2x + 3x^2 + ... \tag{6}$$

$$= 1 + x + x^2 ... + x(1 + x + x^2 ...) + x^2(1 + x + x^3 ...) + ... \tag{7}$$

$$= (1 + x + x^2 + ...)^2 \tag{8}$$

$$= (\sum_{i=0}^{\infty} x^i)^2 \tag{9}$$

$$= \frac{1}{(1-x)^2} \tag{10}$$

It is important to note that step (8) to (9) works because we are given $0 \leq x < 1$.

Exercise 10

Let $x_n = \sum_{k=0}^{n} \frac{1}{n!}$. Prove that $\{x_n\}$. Converges.

Observe that

$$\frac{1}{n!} \leq \frac{1}{n(n-1)} \text{ for } n = 2, 3, 4, 5...$$

Which means

$$\begin{aligned}
\sum_{k=2}^{n} \frac{1}{n!} &= \frac{1}{2!} + ... + \frac{1}{n!} \\
&\leq \frac{1}{2} + ... + \frac{1}{n(n-1)} \\
&= \sum_{k=2}^{n} \frac{1}{k(k-1)} \\
&= 1 - \frac{1}{n} \\
&< 1
\end{aligned}$$

Since $\sum_{k=0}^{1} \frac{1}{n!} = 2$ we can conlude that

$$\sum_{k=0}^{n} \frac{1}{n!} < 3$$

Hence $\{x_n\}$ is bounded and monotonic sequence and therefore converges.

Exercise 11

Determine if the series $\sum a_n$ is convergent or divergent where

$$a_n = \frac{\sqrt{n+1} - \sqrt{n}}{n}$$

First observe

$$\begin{aligned}
a_n &= \frac{\sqrt{n+1} - \sqrt{n}}{n} \cdot \frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}} \\
&= \frac{1}{n(\sqrt{n+1} + \sqrt{n})}
\end{aligned}$$

It's helpful to know the sequence

For proof of this, see Week2 notes from Math 203

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

converge when $p > 1$.

By the comparison test

$$\frac{1}{n(\sqrt{n+1} + \sqrt{n})} =< \frac{1}{n^{3/2}}$$

The sequence converges.

Exercise 12

Determine whether the series $\sum a_n$ is convergent or divergent, where

$$a_n = (\sqrt[n]{n} - 1)^n.$$

Using the ratio test

$$\begin{aligned}\alpha &= \limsup_{n\to\infty} \sqrt[n]{|a_n|} \\ &= \limsup_{n\to\infty} \sqrt[n]{n} - 1\end{aligned}$$

A useful side note,

$$\begin{aligned}n^{\frac{1}{\log n}} &= x \\ \frac{1}{\log n}\log n &= \log x \\ 1 &= \log x \\ x &= e\end{aligned}$$

Which we can employ in calculating

$$\begin{aligned}\alpha &= \limsup_{n\to\infty} \sqrt[n]{n} - 1 \\ &= \limsup_{n\to\infty}(e^{\frac{\log n}{n}}) - 1 \\ &= \exp\big(\limsup_{n\to\infty} \frac{\log n}{n}\big) - 1 \\ &= 0\end{aligned}$$

Since $\alpha = 0 < 1$, the root test allows us to conlude that the series converges.

Exercise 13

Determine whether the series $\sum a_n$ converges or diverges, where

$$a_n = \frac{(-10)^n}{4^{2n+1}(n+1)}$$

Using the ratio test, $L = \lim_{n\to\infty} |\frac{a_{n+1}}{a_n}|$.
Compute

$$\frac{a_{n+1}}{a_n} = \frac{(-10)(n+1)}{16(n+2)}$$

$$\lim_{n\to\infty} |\frac{(-10)(n+1)}{16(n+2)}| = \frac{10}{16}$$

Since $L = \frac{10}{16} < 1$ the series converges.

Exercise 14

Show that the limit does not exist:

$$\lim_{x\to 0}\sin\left(\frac{1}{x}\right)$$

Consider the sequences $\{x_n\} = \frac{1}{\pi n}$ and $\{y_n\} = \frac{1}{\frac{\pi}{2}+2\pi n}$.

$$\lim_{n\to\infty} x_n = 0 \text{ and } \lim_{n\to\infty} y_n = 0$$

However,

$$\lim_{x\to 0}\sin\left(\frac{1}{x_n}\right) = 0 \text{ and } \lim_{x\to 0}\sin\left(\frac{1}{y_n}\right) = 1$$

therefore the limit does not exist.

Exercise 15

Compute

$$\lim_{x\to 0}\frac{\sqrt{1+x}-\sqrt{1-x}}{x}$$

Observe

$$\begin{aligned}\frac{\sqrt{1+x}-\sqrt{1-x}}{x} &= \frac{\sqrt{1+x}-\sqrt{1-x}}{x}\cdot\frac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\\ &= \frac{2x}{x(\sqrt{x+1}+\sqrt{1-x})}\\ &= \frac{2}{\sqrt{x+1}+\sqrt{1-x}}\end{aligned}$$

Hence,

$$\lim_{x\to 0}\frac{2}{\sqrt{x+1}+\sqrt{1-x}} = 1$$

Exercise 18

Suppose that $f(x)$ is continuous on $[a,b]$. Let

Exercises 16 and 17 don't appear in the notes I am following

$$\eta = \frac{1}{3}[f(x_1)+f(x_2)+f(x_3)]$$

where $x_1, x_2, x_3 \in [a,b]$. Prove that there exists $c \in [a,b]$ such that $f(c) = \eta$.

Notice that f is a continous mapping of over the closed interval $[a,b]$, then f attains it's min and maximum values. Let $f(d) = m$ and $f(c) = M$, the min and max values of the functions. Hence, $f(d) \le f(x) \le f(c)$ for all $x \in [a,b]$.

Let $x_1 \neq x_2 \neq x_3$, observe

$$f(c) = \frac{1}{3}[f(c) + f(c) + f(c)] < \frac{1}{3}[f(x_1) + f(x_2) + f(x_3)]$$

and

$$f(d) = \frac{1}{3}[f(d) + f(d) + f(d)] > \frac{1}{3}[f(x_1) + f(x_2) + f(x_3)].$$

Since $f(c) < \eta < f(d)$, by the Intermediate Value Theorem, there exists $c \in [a, b]$ such that $f(c) = \eta$.

Exercise 19

For $x, y \in \mathbb{R}$, define

$$d'(x, y) = \frac{|x - y|}{1 + |x - y|}.$$

Determine whether it is a metric or not.

Non-negativity and symmetry are obvious. For subadditivity, let $d(x, y) = |x - y|$, observe

$$\begin{aligned}
d'(x,z) + d'(z,y) &= \frac{d(x,z)}{1 + d(x,z)} + \frac{d(z,y)}{1 + d(z,y)} \\
&\geq \frac{d(x,z)}{1 + d(x,z) + d(z,y)} + \frac{d(z,y)}{1 + d(x,z) + d(z,y)} \\
&= \frac{d(x,z) + d(z,y)}{1 + d(x,z) + d(z,y)} \\
&= 1 - \frac{1}{1 + d(x,z) + d(z,y)} \\
&\geq 1 - \frac{1}{1 + d(x,y)} \\
&= \frac{d(x,y)}{1 + d(x,y)} \\
&= d'(x,y)
\end{aligned}$$

Therefore, $d'(x, y)$ is a metric.

Exercise 20

Let X be an infinite set. For $p, q \in X$, define

$$d(p,q) = \begin{cases} 1, & \text{if } p \neq q \\ 0, & \text{if } p = q \end{cases}$$

Prove that this is a metric. Which subsets of the resulting metric space are open? Which are closed? Which are compact?

Non-negativity and symmetry are obvious. For subadditivity prove by contradiction. Assume

$$d(x, y) > d(x, z) + d(y, z)$$

If $x = y$, there is an immediate contradiction. If $x \neq y$ then we must have $x = z$ and $y = z$, but then $x = y$ which is a contradiction.

Therefore

$$d(x, y) \leq d(x, z) + d(y, z)$$

All sets are open and closed. Let $A \subset X$ Since any ball of $e < 1$ around a point, $N_r(x) \subset A$, therefore every subset of X is the singleton $\{x\}$ and is open. Then $A^c = X \setminus A$ is open. So A is also closed.

Since X is infinite, there is no finite subcover of the open cover around each singleton point. Hence, X is not compact.

Exercise 21

Compute

$$\lim_{x\to\infty} \sin(\sqrt{x+1} - \sqrt{x}).$$

Observe

$$\begin{aligned} \sqrt{x+1} - \sqrt{x} &= \sqrt{x+1} - \sqrt{x} \cdot \frac{\sqrt{x+1} + \sqrt{x}}{\sqrt{x+1} + \sqrt{x}} \\ &= \frac{1}{\sqrt{x+1} + \sqrt{x}}. \end{aligned}$$

Then

$$\lim_{x\to\infty} \sin\left(\frac{1}{\sqrt{x+1} + \sqrt{x}}\right) = \sin\left(\lim_{x\to\infty} \frac{1}{\sqrt{x+1} + \sqrt{x}}\right) = 0$$

Exercise 22

Consider the sequence defined recursively by $a_1 = 1$ and $a_{n+1} = 3 + \frac{a_n}{2}$ for all $n \in \mathbb{N}$. Prove that $\{a_n\}$ converges. Find its limit.

Computing a few values of a_n by hand, the sequence might be bound above by 6. We can attempt to prove this by induction.

Base case, setting $n = 1$

$$a_1 = 1 < 6$$

Let n be arbitary and suppose $s_n < 6$.

$$\begin{aligned} a_{n+1} &= 3 + \frac{a_n}{2} \\ &< 3 + \frac{6}{2} \\ &= 6 \end{aligned}$$

Hence, a_{n+1} holds true and the sequence is bounded. Furthmore, it is trivially easy to see that this sequence is monotonic. Hence, the sequence converges.

Since the sequence converges,

$$\lim_{n\to\infty} 3 + \frac{a_n}{2} = L$$

must satisfy

$$L = 3 + \frac{L}{2}$$

Hence the limit is 6.

Exercise 23

Prove that there exists a number $x \in [0, \frac{\pi}{2}]$ such that $2x - 1 = \sin(x^2 + \frac{\pi}{4})$.

Let $f(x) = 2x - 1$, which is continuous function over $[0, \frac{\pi}{2}]$. Observe .

The inequality easily follows since the image of $\sin(x) = [-1, 1]$

$$f(0) = -1 \leq \sin(x^2 + \frac{\pi}{4}) < \pi - 1 = f(\frac{\pi}{2})$$

Therefore, by the Intermediate Value Theorem there exists $f(x) = \sin(x^2 + \frac{\pi}{4})$