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MATH CAMP 2016 EXERCISES

Exercise 1

Is the set $X = \{(x, y) \in R^2 \mid xy < 1\}$ convex?

Paul Sally Jr's definition of convex sets eliminated some ambiguity over choice of λ in the definition provided in the math camp nots.

Def. (Convex Sets): Let A be a non-empty subset of $\mathbb{R}^n$. We say A is *convex* if, given any two points $\mathbf{p}, \mathbf{q} \in A$, the "line segment" with endpoints $\mathbf{p}$ and $\mathbf{q}$, that is, the set

$$\{(1-t)\mathbf{p} + t\mathbf{q} \mid t \in \mathbb{R}, 0 \leq t \leq 1\}$$

is a subset of A.

Hence, choose $\mathbf{x} = (a, 0)$ and $\mathbf{y} = (0, a)$, where $a > 0$ and large enough. Then $\mathbf{x}, \mathbf{y} \in X$, but $z = (\lambda a, (1-\lambda)a) \notin X$ for some $\lambda \in [0, 1]$. Therefore, X is not convex.

E.g.: $(2, 0)$ and $(0, 2)$. The point $(1, 1) \in \{z\}$ but $(1, 1) \notin X$.

Exercise 3

Prove Theorem 5.

Let f be concave, $\lambda \in [0, 1]$ and suppose $f(x) > f(y)$.

$$\begin{aligned}
f(\lambda x + (1-\lambda)y) &\geq \lambda f(x) + (1-\lambda)f(y) && (1)\\
&\geq \lambda f(y) + (1-\lambda)f(y) && (2)\\
&= f(y) && (3)\\
&= \min\{f(x), f(y)\} && (4)
\end{aligned}$$

Therefore f is quasi-concave. By the the definition of concave, the inequality becomes strict when $\lambda \in (0, 1)$, hence also stricltly quasi-concave.

A similar proof follows for convex implies quasi-convex. Flip the inequality in (2) and (3) and choose $f(y) > f(x)$, e.g $f(y) = \max\{f(x), f(y)\}$.