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MATH CAMP 2016 EXERCISES

Exercise 1

Let
$$f(x) = \frac{x^2 + x + 1}{e^x}.$$
Calculate $f'(x)$

Observe
$$\begin{aligned} g(x) &= x^2 + x + 1 \\ g'(x) &= 2x + 1 \\ h(x) &= e^x \\ h'(x) &= e^x. \end{aligned}$$

Then rewrite
$$f(x) = (\frac{g}{h})(x).$$

From which it easily follows
$$\begin{aligned} f'(x) &= (\frac{g}{h})'(x) \\ &= \frac{g'(x)h(x) + g(x)h'(x)}{h^2(x)} \\ &= \frac{x - x^2}{e^x}. \end{aligned}$$

Exercise 2

Let $f(x) = e^{ax} \sin bx$. Calculate $f'(x)$.

Write
$$\begin{aligned} g(x) &= e^x \\ h(x) &= ax \\ j(x) &= \sin x \\ k(x) &= bx. \end{aligned}$$

Then
$$f(x) = g(h(x)) \cdot j(k(x)) = L(x)M(x) = (ML)(x).$$
and
$$f'(x) = L'(x)M(x) + L(x)M'(x)$$

Compute
$$\begin{aligned} L'(x) &= g'(h(x))h'(x) = e^{ax} \cdot a = ae^{ax} \\ M'(x) &= j'(k(x))k'(x) = \cos bx \cdot d = b \cos bx. \end{aligned}$$

Which leads to the result
$$f'(x) = e^{ax}(a \sin bx + b \cos bx).$$

Exercise 3

1. Let
$$f(x) = \begin{cases} x\sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

   Calculate $f'(x)$ for $x \neq 0$ and show that $f(x)$ is not differentiable at $x = 0$.

   Write
$$\begin{aligned} g(x) &= x \\ g'(x) &= 1 \\ h(x) &= \sin x \\ h'(x) &= \cos x \\ j(x) &= \frac{1}{x} \\ j'(x) &= -\frac{1}{x^2}. \end{aligned}$$

   Observe
$$f'(x) = g'(x)h(j(x)) = h'(j(x))j'(x)g(x) = \sin\frac{1}{x} - \frac{1}{x}\cos\frac{1}{x}.$$

   Finally, since $\frac{1}{x}$ is undefined when $x = 0$, $f(x)$ is not differentiable at 0.

2. Let
$$f(x) = \begin{cases} x^2\sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

   Write
$$\begin{aligned} g(x) &= x^2 \\ g'(x) &= 2x \\ h(x) &= \sin x \\ h'(x) &= \cos x \\ j(x) &= \frac{1}{x} \\ j'(x) &= -\frac{1}{x^2}. \end{aligned}$$

   Using the same method as in part (i)
$$f'(x) = 2x\sin\frac{1}{x} - \cos\frac{1}{x}.$$

Exercise 5

Prove that
$$\frac{b-a}{b} < \ln\frac{b}{a} < \frac{b-a}{a} \; (0 < a < b).$$

Proof. First observe $\ln\frac{b}{a} = \ln(b) - \ln(a)$. Rewriting the inequality
$$\frac{b-a}{b} < \ln(b) - \ln(a) < \frac{b-a}{a}$$
and diving all the terms by $(b-a)$
$$\frac{1}{b} < \frac{\ln(b)-\ln(a)}{b-a} < \frac{1}{a}.$$
From the mean value theorem
$$\frac{\ln(b)-\ln(a)}{b-a} = \ln'(x) = \frac{1}{x}$$
for some $x \in (a,b)$. The inequality is now
$$\frac{1}{b} < \frac{1}{x} < \frac{1}{a}.$$
Since $b > a \implies \frac{1}{b} < \frac{1}{a}$ the inequality holds. □

Exercise 6

Revisit Chap.1 Exercise 15. Compute
$$\lim_{x\to 0} \frac{\sqrt{1+x}-\sqrt{1-x}}{x}$$
using L'Hospital's rule.

Write
$$f(x) = \sqrt{1+x} - \sqrt{1-x} \text{ and } g(x) = x,$$
then
$$\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x) = 0$$
making this a problem a prime candidate for L'Hospital's rule, as suggested.

Observe
$$f'(x) = \frac{1}{2\sqrt{x+1}} - \frac{1}{2\sqrt{1-x}}$$
and
$$g'(x) = 1.$$
Therefore
$$\lim_{x\to 0} \frac{\sqrt{1+x}-\sqrt{1-x}}{x} = \lim_{x\to 0} \frac{f'(x)}{g'(x)} = 0$$

Exercise 7

Prove that
$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e.$$

Proof. First
$$\log\left((1+\frac{1}{x})^x\right) = x\log(1+\frac{1}{x}).$$

Write $f(x) = \log(1+\frac{1}{x})$ and $g(x) = \frac{1}{x}$. Hence
$$\log\left(\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x\right) = \lim_{x\to\infty}\frac{f(x)}{g(x)}.$$

Where $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = 0$, making this another PRIME candidate for L'Hospital's rule.

Find
$$\begin{aligned} f'(x) &= (\frac{1}{1+\frac{1}{x}})(-\frac{1}{x^2}) \\ g'(x) &= -\frac{1}{x^2}. \end{aligned}$$

Therefore
$$\lim_{x\to\infty}\frac{f'(x)}{g'(x)} = \lim_{x\to\infty}\frac{1}{1+\frac{1}{x}} = 1.$$

To recap we have found
$$\log\left(\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x\right) = 1$$

which means
$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e.$$

□

Exercise 8

Let $f(x) = e^x\cos x$. Prove that $f'' - 2f' + 2f = 0$.

First compute
$$\begin{aligned} f'(x) &= e^x(\cos x - \sin x) \\ f''(x) &= -2e^x\sin x. \end{aligned}$$

Then
$$\begin{aligned} f'' - 2f' + 2f &= -2e^x\sin x - 2e^x(\cos x - \sin x) + 2e^x\cos x \\ &= 0 \end{aligned}$$

Exercise 10

Find the Taylor polynomial of degree n for $f(x) = \frac{1}{1-x}$, centered at $x = 0$.

Observe

$$\begin{aligned} f'(x) &= \frac{1}{(x-1)^2} = \frac{1!}{(x-1)^2} \\ f''(x) &= \frac{2}{(1-x)^3} = \frac{2!}{(x-1)^3} \\ f'''(x) &= \frac{6}{(1-x)^4} = \frac{3!}{(x-1)^4} \\ f^n(x) &= \frac{n!}{(1-x)^{n+1}} \end{aligned}$$

Recall

$$f(\beta) = P(\beta) + \frac{f^n(x)}{n!}(\beta - \alpha)^n$$

and let $f(t) = \frac{1}{1-x}$, $\alpha = 0$, and $\beta = x$.

Hence

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + ... + x^n + \frac{1}{(1-s)^{n+1}}x^{n+1}$$

with $0 < s < x$.

Since

$$\lim_{x \to 0} \frac{1}{(1-s)^{n+1}}x^{n+1} = 0$$

this can be rewritten

I am not sure about the error term.

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + ... + x^n + o(x^n)\ ,\ (x \to 0).$$

Exercise 12

Let $f(x) = \cos^5 \sqrt{1+x^2}$. Calculate $f'(x)$.

Write

$$\begin{aligned} g(x) &= x^5 \\ h(x) &= \cos x \\ j(x) &= \sqrt{x} \\ k(x) &= 1 + x^2 \end{aligned}$$

then $f(x) = g(h(j(k(x))))$ and

$$f'(x) = g'(h(j(k(x))))h'(j(k(x)))j'(k(x))k'(x).$$

First

$$\begin{aligned} g'(x) &= 5x^4 \\ h(x) &= -\sin x \\ j(x) &= \frac{1}{2x} \\ k(x) &= 2x \end{aligned}$$

then put it all together

$$f'(x) = -\frac{5x\cos^4\sqrt{1+x^2}\cdot\sin\sqrt{1+x^2}}{\sqrt{1+x^2}}$$

Exercise 13

Use the mean value theorem to prove that

$$|\sin x - \sin y| \leq |x-y|$$

Proof. Let $f(x) = \sin x$ which is continous over some interval $[x, y]$ and differentiable over (x, y), then by the mean value theorem

$$f'(x) = \frac{f(x) - f(y)}{x - y}$$

for some $x \in (x, y)$. Taking the absolute value of both sides and noting that $|f'(x) \leq 1||$

$$|f(x) - f(y)| = |x-y||f'(x)| \leq |x-y|$$

$\square$

Exercise 14

Compute

$$\lim_{x\to 1}\left(\frac{1}{\ln x} - \frac{1}{x-1}\right).$$

Rewrite

$$\frac{1}{\ln x} - \frac{1}{x-1} = \frac{x-1-\ln x}{\ln x(x-1)}$$

where $g(x) = x - 1 - \ln x$ and $h(x) = \ln x(x-1)$.
Then

$$\lim_{x\to 1} f(x) = \lim_{x\to 1}\frac{g(x)}{h(x)} = \frac{0}{0}.$$

Observe

$$g'(x) = 1 - \frac{1}{x}$$
$$h'(x) = \frac{1}{x}(x-1) + \ln x$$

also tend to 0 as $x \to 1$. So we can take the second derivative

$$g''(x) = \frac{1}{x^2}$$
$$h''(x) = \frac{x-1}{x^2} + \frac{2}{x}.$$

Then observe, by L'Hospital's rule

$$\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{g'(x)}{h'(x)} = \lim_{x\to 1} \frac{g''(x)}{h''(x)} = \frac{1}{2}$$

Exercise 15

Find the Taylor polynomial of degree three for $f(x) = \sin x$, centered at $x = \frac{5\pi}{6}$.

Let $f(t) = \sin t$, $\alpha = \frac{5\pi}{6}$, and $\beta = x$.

$$f'(t) = \cos t$$
$$f''(t) = -\sin t$$
$$f^{(3)}(t) = -\cos t$$

Writing out the polynomial

$$\sin x = \frac{\sin\alpha}{0!}(x - \frac{5\pi}{6})^0 + \frac{\cos\alpha}{1!}(x - \frac{5\pi}{6})^1 + \frac{-\sin\alpha}{2!}(x - \frac{5\pi}{6})^2 + \frac{-\cos\alpha}{3!}(x - \frac{5\pi}{6})^3$$

Inserting the values as defined,

$$\sin x = \frac{1}{2} - \frac{\sqrt{3}}{2}(x - \frac{5\pi}{6}) - \frac{(x - \frac{5\pi}{6})^2}{4} + \frac{\sqrt{3}}{12}(x - \frac{5\pi}{6})^3.$$