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1. Basic Analysis/Topology

1.1. Euclidean Spaces

In graduate courses, the n-commodity economies will make heavy use of Euclidean spaces, which is the appropriate generalization of the two-dimensional models appeared in undergraduate economics courses.

Definition 1. For each positive integer k, let R^k be the set of all ordered k-tuples

$$\mathbf{x} = (x_1, x_2, \cdots, x_k),$$

where $x_1, \cdots, x_k$ are real numbers, called the *coordinates* of x. The elements of R^k are called points, or vectors, especially when $k > 1$. If $\mathbf{y} = (y_1, \cdots, y_k)$ and if α is a real number, put

$$\mathbf{x} + \mathbf{y} = (x_1 + y_1, \cdots, x_k + y_k),$$

$$\alpha\mathbf{x} = (\alpha x_1, \cdots, \alpha x_k)$$

so that $\mathbf{x} + \mathbf{y} \in R^k$ and $\alpha\mathbf{x} \in R^k$. These two operations, addition of vectors and multiplication of a vector by a real number, make R^k into a *vector space over the real field*[2]. The zero element of R^k (sometimes called the *origin* or the *null vector*) is the point 0, all of whose coordinates are 0.

We also define the so-called "inner product"[3] (or scalar product) of $\mathbf{x}$ and $\mathbf{y}$ by

$$\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^{k} x_i y_i$$

and the associated *norm* of $\mathbf{x}$ by

$$\|\mathbf{x}\| = (\mathbf{x} \cdot \mathbf{x})^{1/2} = \left(\sum_{1}^{k} x_i^2 \right)^{1/2}.$$

The structure now defined (the vector space R^k with the above inner product and norm) is called Euclidean k-space[4].

[2] The formal definition of vector space will be discussed in Chap.4: Linear Algebra.

[3] We have: $\mathbf{x} \cdot \mathbf{y} = \|\mathbf{x}\| \, \|\mathbf{y}\| \cos\theta$, where θ is the angle between the two vectors. Thus, inner product allows us to consider the geometric concept of "angle".

[4] For a generic vector space with a well-defined inner product, the structure is called a *inner product space*. Further, if the space is also a *complete metric space* with respect to the distance function induced by the inner product, then it's called a *Hilbert space*, which generalizes the notion of Euclidean space.

Theorem 1. Suppose $x,y,z \in R^k$, and α is real. Then
(a) $\|\mathbf{x}\| \geq 0$;
(b) $\|\mathbf{x}\| = 0$ iff $\mathbf{x} = \mathbf{0}$;
(c) $\|\alpha\mathbf{x}\| = |\alpha|\|\mathbf{x}\|$;
(d) $|\mathbf{x}\cdot\mathbf{y}| \leq \|\mathbf{x}\|\|\mathbf{y}\|$;
(e) $\|\mathbf{x}+\mathbf{y}\| \leq \|\mathbf{x}\|+\|\mathbf{y}\|$;
(f) $\|\mathbf{x}-\mathbf{z}\| \leq \|\mathbf{x}-\mathbf{y}\|+\|\mathbf{y}-\mathbf{z}\|$.

Remark. (i) Theorem 1(a), (b), and (f) will allow us to regard R^k as a *metric space.*
(ii) The proof of (d) follows from the Schwarz inequality: $\left(\sum_1^k a_i^2\right)\left(\sum_1^k b_i^2\right) \geq \left(\sum_1^k a_i b_i\right)^2$.

Exercise 1. Prove that

$$\|\mathbf{x}+\mathbf{y}\|^2 + \|\mathbf{x}-\mathbf{y}\|^2 = 2\|\mathbf{x}\|^2 + 2\|\mathbf{y}\|^2$$

if $\mathbf{x} \in R^k$ and $\mathbf{y} \in R^k$. Interpret this geometrically. This is known as the famous *parallelogram law*.

1.2. Metric Spaces and Some Basic Topology

An intuitive understanding of this section will aid in understanding a variety of topics in optimization as well as in consumer and producer theory.

Definition 2. A set X, whose elements we shall call *points*, is said to be a *metric space* if with any two points p and q of X there is associated a real number $d(p,q)$, called the *distance* from p to q, such that

(a)	$d(p,q) > 0$ if $p \neq q$; $d(p,p) = 0$;	(non-negativity)
(b)	$d(p,q) = d(q,p)$;	(symmetry)
(c)	$d(p,q) \leq d(p,r) + d(r,p)$, for any $r \in X$.	(subadditivity)

Any function with these three properties is called a *distance function*, or a *metric*.

Remark. (i) The most familiar metric space is 3-dimensional Euclidean space; the distance in R^k is defined by

$$d(\mathbf{x},\mathbf{y}) = \|\mathbf{x}-\mathbf{y}\| \ \ (\mathbf{x},\mathbf{y} \in R^k).$$

The conditions of Definition 2 are satisfied by Theorem 1(a),(b) and (f). In fact, a "metric" is the generalization of the "ordinary" distance between two points in Euclidean space.

(ii) It is also important to observe that every subset Y of a metric space X is a metric space in its own right, with the same distance function. For it is clear that if conditions (a) to (c) of Definition 2 hold for $p,q,r \in X$, they also hold if we restrict p,q,r to lie in Y. Thus every subset of a Euclidean space is a metric space.

Exercise 2. For $x \in R^1$ and $y \in R^1$, define

$$d_1(x,y) = (x-y)^2,$$
$$d_2(x,y) = \sqrt{|x-y|},$$
$$d_3(x,y) = |x^2 - y^2|.$$

Determine, for each of these, whether it is a metric or not.

Definition 3. Let X be a metric space. All points and sets mentioned below are understood to be elements and subsets of X.

(a) A *neighborhood* of p is a set $N_r(p)$ consisting of all q such that $d(p,q) < r$, for some $r > 0$. The number r is called the *radius* of $N_r(p)$.
(b) A point p is a *limit point* of the set E if *every* neighborhood of p contains a point $q \neq p$ such that $q \in E$.
(c) If $p \in E$ and p is not a limit point of E, then p is called an *isolated point* of E.
(d) E is *closed* if every limit point of E is a point of E.
(e) A point p is an *interior point* of E if there is a neighborhood N of p such that $N \subset E$.
(f) E is *open* if every point of E is an interior point of E.
(g) The *complement* of E (denoted by E^c) is the set of all points $p \in X$ such that $p \notin E$.
(h) E is *bounded* if there is a real number M and a point $q \in X$ such that $d(p,q) < M$ for all $p \in E$.

Remark. (i) It can be easily shown that every neighborhood is an open set, which is also called an *open ball*. Thus, a metric on a space naturally induces topological properties like open and closed balls. These open balls form the base for a topology on the space, making it a *topological space*[5].
(ii) A finite point set has no limit points.

[5] A *topological space* is an ordered pair (X,τ), where X is a set and τ is a collection of subsets of X, satisfying the following axioms:
(a) The empty set and X itself belong to τ.
(b) Any (finite or infinite) union of members of τ still belongs to τ.
(c) The intersection of any finite number of members of τ still belongs to τ.
The elements of τ are called *open sets* and the collection τ is called a *topology* on X.

Exercise 3. Let us consider the following subsets of R^2:

(a) The set of all integers.
(b) The set consisting of the numbers $1/n(n = 1,2,3,\cdots)$.
(c) The set of all points (that is, R^2).
(d) The segment (a,b).

Discuss the closedness, openness and boundedness for each of these.

Remark. Suppose $E \subset Y \subset X$, where X is a metric space. Example (d) shows that a set may be *open relative* to Y without being an open subset of X. Thus it is safer to mention the relative space when confusion may arise.

Theorem 2. A set E is open if and only if its complement is closed.

Definition 4. If X is a metric space, if $E \subset X$, and if E' denotes the set of all limit points of E in X, then the *closure* of E is the set $\overline{E} = E \cup E'$.

Theorem 3. If X is a metric space and $E \subset X$, then

(a) $\overline{E}$ is closed,
(b) $E = \overline{E}$ if and only if E is closed,
(c) $\overline{E} \subset F$ for every closed set $F \subset X$ such that $E \subset F$.
By (a) and (c), $\overline{E}$ is the *smallest* closed subset of X that contains E.

Definition 5. By an *open cover* of a set E in a metric space X we mean a collection $\{G_\alpha\}$ of open subsets of X such that $E \subset \bigcup_\alpha G_\alpha$.

Definition 6. A subset E of a metric space X is said to be *compact* if every *open cover* of E contains a *finite* subcover.
More explicitly, the requirement is that if $\{G_\alpha\}$ is an open cover of E, then there are finitely many indices $\alpha_1, \cdots, \alpha_n$ such that

$$E \subset G_{\alpha_1} \cup \cdots \cup G_{\alpha_n}.$$

The notion of compactness is of great importance in analysis, especially in connection with continuity, which we shall see later.

Theorem 4. Compact subsets of metric spaces are closed.

Theorem 5. Closed subsets of compact sets are compact.

Corollary. If F is closed and E is compact, then $F \cap E$ is compact.

Theorem 6. If a set is compact, then it is bounded.

Theorem 7. (Heine-Borel). Suppose E is a subset of Euclidean space R^k, the following two statements are equivalent:
(a) E is closed and bounded.
(b) E is compact.

Exercise 4. Determine whether each of the following sets is compact.
(a) [0,1];
(b) [0,1);
(c) $\{1,2,3\}$;
(d) The set of all positive integers;
(e) $\{1,1/2,1/3,\cdots\}$;
(f) $\{1,1/2,1/3,\cdots\}\cup\{0\}$.

Theorem 8. If E is an infinite subset of a compact set K, then E has a limit point in K.

Proof. If no point of K were a limit point of E, then each $q \in K$ would have a neighborhood V_q which contains at most one point of E (namely, q, if $q \in E$). It is clear that no finite subcollection of $\{V_q\}$ can cover E; and the same is true of K, since $E \subset K$. This contradicts the compactness of K.

Theorem 9. (Weierstrass) Every bounded infinite subset E of R^k has a limit point in R^k.

Proof. Boundedness of E allows us to construct a k-cell I consists of all points $\mathbf{x} = (x_1, \cdots, x_k)$ such that $a_j \leq x_j \leq b_j$ $(1 \leq j \leq k)$ and $E \subset I$. I is compact and so E has a limit point in I, by Theorem 8.

1.3. Numerical Sequences

A central concern in economic theory is the effect of a small change in one economic variable $\mathbf{x}$ on some other economic variable $\mathbf{y}$. Before we can make this effect precise, we need to have a working knowledge of the concepts of *small* change. What does a small change in prices mean? The following sections focus on these questions by studying notions of sequences, series and limits. The exposition follows a careful logical development as one principle is deduced from previous ones.

1.3.1. Convergent Sequences

Definition 7. A sequence $\{p_n\}$ in a metric space X is said to *converge* if there is a point $p \in X$ with the following property: For every $\varepsilon > 0$ there is an integer N such that $n \geq N$ implies that $d(p_n, p) < \varepsilon$.

In this case we also say that $\{p_n\}$ converges to p, or that p is the *limit* of $\{p_n\}$, and we write $p_n \to p$, or

$$\lim_{n\to\infty} p_n = p.$$

If $\{p_n\}$ does not converge, it is said to *diverge.*

We now summarize some important properties of convergent sequences in metric spaces.

Theorem 10. Let $\{p_n\}$ be a sequence in a metric space X.

(a) $\{p_n\}$ converges to $p \in X$ if and only if every neighborhood of p contains p_n for *all but finitely many n*.
(b) If $p \in X, p' \in X$, and if $\{p_n\}$ converges to p and to p', then $p' = p$.
(c) If $\{p_n\}$ converges, then $\{p_n\}$ is bounded.
(d) If $E \subset X$ and if p is a limit point of E, then there is a sequence $\{p_n\}$ in E such that $p = \lim\limits_{n\to\infty} p_n$.

Theorem 11. Suppose $\{s_n\}, \{t_n\}$ are real sequences, and $\lim\limits_{n\to\infty} s_n = s, \lim\limits_{n\to\infty} t_n = t$. Then

(a) $\lim\limits_{n\to\infty} (s_n + t_n) = s + t$;

(b) $\lim\limits_{n\to\infty} cs_n = cs, \lim\limits_{n\to\infty} (c + s_n) = c + s$, for any number c;

(c) $\lim\limits_{n\to\infty} s_n t_n = st$;

(d) $\lim\limits_{n\to\infty} \frac{1}{s_n} = \frac{1}{s}$, provided $s_n \neq 0 (n = 1, 2, 3, \cdots)$, and $s \neq 0$.

Exercise 5. Calculate $\lim\limits_{n\to\infty} (\sqrt{n^2 + n} - n)$.

Theorem 12. (a) Suppose $\mathbf{x}_n \in R^k$ $(n = 1, 2, 3, \cdots)$ and

$$\mathbf{x}_n = (\alpha_{1,n}, \cdots, \alpha_{k,n}).$$

Then $\{\mathbf{x}_n\}$ converges to $\mathbf{x} = (\alpha_1, \cdots, \alpha_k)$ if and only if

$$\lim_{n\to\infty} \alpha_{j,n} = \alpha_j \ \ (1 \leq j \leq k).$$

(b) Suppose $\{\mathbf{x}_n\}, \{\mathbf{y}_n\}$ are sequences in R^k, $\{\beta_n\}$ is a sequence of real numbers, and $\mathbf{x}_n \to \mathbf{x}, \mathbf{y}_n \to \mathbf{y}, \beta_n \to \beta$. Then

$$\lim_{n\to\infty} (\mathbf{x}_n + \mathbf{y}_n) = \mathbf{x} + \mathbf{y}, \ \ \lim_{n\to\infty} \mathbf{x}_n \cdot \mathbf{y}_n = \mathbf{x} \cdot \mathbf{y}, \ \ \lim_{n\to\infty} \beta_n \mathbf{x}_n = \beta \mathbf{x}.$$

Definition 8. Given a sequence $\{p_n\}$, consider a sequence $\{n_k\}$ of positive integers, such that $n_1 < n_2 < n_3 < \cdots$. Then the sequence $\{p_{n_i}\}$ is called a *subsequence* of $\{p_n\}$. If $\{p_{n_i}\}$ converges, its limit is called a *subsequential limit* of $\{p_n\}$.

It is clear that $\{p_{n_i}\}$ converges to p if and only if every subsequence of $\{p_{n_i}\}$ converges to p.

1.3.2. Cauchy Sequences

Definition 9. A sequence $\{p_n\}$ in a metric space X is said to be a *Cauchy sequence* if for every $\varepsilon > 0$ there is an integer N such that $d(p_n, p_m) < \varepsilon$ if $n \geq N$ and $m \geq N$.

Theorem 13. (a) In any metric space X, every convergent sequence is a Cauchy sequence.
(b) If X is a compact metric space and if $\{p_n\}$ is a Cauchy sequence in X, then $\{p_n\}$ converges to some point of X.
(c) In R^k, every Cauchy sequence converges.

Remark. (a) The difference between the definition of convergence and the definition of a Cauchy sequence is that the limit is explicitly involved in the former, but not in the latter. Thus Theorem 13(b) may enable us to decide whether or not a given sequence converges without knowledge of the limit to which it may converge.
The fact that a sequence converges in R^k if and only if it is a Cauchy sequence is usually called the *Cauchy criterion* for convergence.
(b) Counterexample: Consider the metric space of all rationals. The sequence $\{a_k\}$ of decimal expansions up to k decimal places of $\sqrt{2}$ is obviously a Cauchy sequence. However, it converges to $\sqrt{2} \notin \mathbb{Q}$.

Exercise 6. Let

$$x_n = 1 + \frac{1}{2^2} + \cdots + \frac{1}{n^2}.$$

Prove that the sequence $\{x_n\}$ converges.

Theorem 14. Suppose $\{s_n\}$ is *monotonic*. Then $\{s_n\}$ converges if and only if it is bounded.

Exercise 7. If $s_1 = \sqrt{2}$, and

$$s_{n+1} = \sqrt{2 + s_n} \ \ (n = 1, 2, 3, \cdots),$$

prove that $\{s_n\}$ converges, and that $s_n < 2$ for $n = 1, 2, 3, \cdots$.

1.3.3. Upper and Lower Limits

Definition 10. Let $\{s_n\}$ be a sequence of real numbers. Let E be the set of numbers x (in the extended real number system) such that $s_{n_k} \to x$ for some subsequences $\{s_{n_k}\}$. This set E contains all subsequential limits, plus possibly the numbers $+\infty, -\infty$.
We now put

$$s^* = sup\, E,$$

$$s_* = inf\, E.$$

The numbers s^*, s_* are called the *upper* and *lower limits* of $\{s_n\}$; we use the notation

$$\underset{n\to\infty}{lim\, sup}\, s_n = s^*, \quad \underset{n\to\infty}{lim\, inf}\, s_n = s_*$$

Remark. For a real-valued sequence $\{s_n\}$, $\lim_{n\to\infty} s_n = s$ if and only if

$$\underset{n\to\infty}{lim\, sup}\, s_n = \underset{n\to\infty}{lim\, inf}\, s_n = s.$$

Exercise 8. Find the upper and lower limits of the sequence $\{s_n\}$ defined by

$$s_n = \frac{(-1)^n}{1+\frac{1}{n}}.$$

1.4. Series

Definition 11. Given a sequence $\{a_n\}$, we define another sequence $\{s_n\}$, where

$$s_n = \sum_{k=1}^{n} a_k.$$

For $\{s_n\}$ we use the symbolic expression

$$\sum_{n=1}^{\infty} a_n.$$

The symbol we call an *infinite series*, or just a *series*. The numbers $\{s_n\}$ are called the *partial sums* of the series.If $\{s_n\}$ converges to s, we say that the series *converges*, and write

$$\sum_{n=1}^{\infty} a_n = s.$$

The number s is called the sum of the series. If $\{s_n\}$ diverges, the series is said to diverge.

Remark. It is clear that every theorem about sequences can be stated in terms of series (putting $a_1 = s_1$, and $a_n = s_n - s_{n-1}$ for $n > 1$), and vice versa. But it is nevertheless useful to consider both concepts.

Exercise 9. Calculate

$$\sum_{n=0}^{\infty} (n+1)x^n \quad (0 \leq x < 1).$$

The Cauchy criterion (Theorem 13) can be restated in the following form:

Theorem 15. (Cauchy criterion) $\sum a_n$ converges if and only if for every $\varepsilon > 0$ there is an integer N such that

$$\left| \sum_{k=n}^{m} a_k \right| \leq \varepsilon$$

if $m \geq n \geq N$. In particular, by taking $m = n$, it becomes

$$|a_n| \leq \varepsilon \ \ (n \geq N).$$

In other words:

Theorem 16. If $\sum a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

Remark. The condition $a_n \to 0$ is not, however, sufficient to ensure convergence of $\sum a_n$. For instance, the series $\sum \frac{1}{n}$ diverges.

Theorem 14, concerning monotonic sequences, also has an immediate counterpart for series.

Theorem 17. A series of nonnegative terms converges if and only if its partial sums form a bounded sequence.

Exercise 10. Let
$$x_n = 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{n!}$$
where $n! = 1 \cdot 2 \cdots n, 0! = 1$. Prove that the sequence $\{x_n\}$ converges.

Remark. The limit is defined as the number $e = \sum_{n=0}^{\infty} \frac{1}{n!}$

Theorem 18. (Comparison Test). (a) If $|a_n| \leq c_n$ for $n \geq N_0$, where N_0 is some fixed integer, and if $\sum c_n$ converges, then $\sum a_n$ converges.
(b) If $a_n \geq d_n > 0$ for $n \geq N_0$, and if $\sum d_n$ diverges, then $\sum a_n$ diverges.

Exercise 11. Determine whether the series $\sum a_n$ is convergent or divergent, where
$$a_n = \frac{\sqrt{n+1} - \sqrt{n}}{n}.$$

Theorem 19. (Root Test). Given $\sum a_n$, put $\alpha = \underset{n\to\infty}{lim\ sup} \sqrt[n]{|a_n|}$.
Then
(a) if $\alpha < 1$, $\sum a_n$ converges;
(b) if $\alpha > 1$, $\sum a_n$ diverges;
(c) if $\alpha = 1$, the test gives no information.

Exercise 12. Determine whether the series $\sum a_n$ is convergent or divergent, where
$$a_n = (\sqrt[n]{n} - 1)^n.$$

Theorem 20. (Ratio Test) Given $\sum a_n$. Define $L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$
Then
(a) if $L < 1$ the series is convergent.
(b) if $L > 1$ the series is divergent.
(c) if $L = 1$ the test gives no information.

Exercise 13. Determine whether the series $\sum a_n$ is convergent or divergent, where

$$a_n = \frac{(-10)^n}{4^{2n+1}(n+1)}.$$

Remark. The ratio test is frequently easier to apply than the root test, since it is usually easier to compute ratios than nth roots. However, the root test has wider scope. More precisely: Whenever the ratio test shows convergence, the root test does too; whenever the root test is inconclusive, the ratio test is too.

1.5. Continuity

1.5.1. Limits of Functions

We begin with the $\varepsilon - \delta$ definition of the limit of a function.

Definition 12. Let X and Y be metric spaces; suppose $E \subset X$, f maps E into Y, and p is a limit point of E. We write $f(x) \to q$ as $x \to p$, or

$$\lim_{x \to p} f(x) = q$$

if there is a point $q \in Y$ with the following property: For every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$d_Y(f(x), q) < \varepsilon$$

for all points $x \in E$ for which

$$0 < d_X(x, p) < \delta.$$

The symbols d_X and d_Y refer to the metrics in X and Y, respectively.

Remark. It should be noted that $p \in X$, but that p need not be a point of E in the above definition. Moreover, even if $p \in E$, we may very well have $f(p) \neq \lim_{x \to p} f(x)$.

We can recast the definition in terms of limits of sequences:

Theorem 21. Let X, Y, E, f, and p be as in Definition 12. Then

$$\lim_{x \to p} f(x) = q$$

if and only if

$$\lim_{x\to p} f(p_n) = q$$

for every sequence $\{p_n\}$ in E such that

$$p_n \neq p, \ \lim_{n\to\infty} p_n = p.$$

Corollary. If f has a limit at p, this limit is unique.

Remark. A non-existence proof for a limit directly from Definition 12 is often awkward. Theorem 21 gives a convenient way to show that a limit of a function does not exist.

Exercise 14. Show that the limit does not exist:

$$\lim_{x\to 0} sin\left(\frac{1}{x}\right)$$

Theorem 22. Suppose $E \subset X$, a metric space, p is a limit point of E, f and g are real functions on E, and

$$\lim_{x\to p} f(x) = A, \ \lim_{x\to p} g(x) = B.$$

Then
(a) $\lim_{x\to p}(f+g)(x) = A+B$;
(b) $\lim_{x\to p}(fg)(x) = AB$;
(c) $\lim_{x\to p}\frac{f}{g}(x) = \frac{A}{B}$, if $B \neq 0$.

Remark. In view of Theorem 21, these assertions follow immediately from the analogous properties of sequences (Theorem 11).

Exercise 15. Compute

$$\lim_{x\to 0}\frac{\sqrt{1+x}-\sqrt{1-x}}{x}.$$

Definition 13. Let f be defined on (a,b). Consider any point x such that $a \leq x < b$. We define *right limit* as

$$f(x+) = q$$

if $f(t_n) \to q$ as $n \to \infty$, for all sequences $\{t_n\}$ in (x,b) such that $t_n \to x$. To obtain the definition of *left limit* $f(x-)$, for $a < x \leq b$, we restrict ourselves to sequences $\{t_n\}$ in (a,x).

It is clear that for any point x of (a,b), $\lim_{t\to x} f(t)$ exists if and only if

$$f(x+) = f(x-) = \lim_{t\to x} f(t)$$

1.5.2. Continuous Functions

Definition 14. Suppose X and Y are metric spaces, $E \subset X, p \in E$ and f maps E into Y. Then f is said to be *continuous at* p if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$d_Y(f(x), f(p)) < \varepsilon$$

for all points $x \in E$ for which $d_X(x, p) < \delta$.

If f is continuous at every point of E, then f is said to be *continuous on* E.

Remark. (i) It should be noted that f has to be defined at the point p in order to be continuous at p. (Compare this with the remark following Definition 12.)
(ii) If p is an isolated point of E, then our definition always holds and thus f is continuous at p. It p is a limit point, then we have the following theorem.

Theorem 23. In the situation given in Definition 14, assume also that p is a limit point of E. Then f is continuous at p if and only if $\lim_{x \to p} f(x) = f(p)$.

Theorem 24. Suppose X, Y, Z are metric spaces, $E \subset X$, f maps E into Y, g maps the range of f, $f(E)$ into Z, and h is the mapping of E into Z defined by

$$h(x) = g(f(x)) \ \ (x \in E).$$

If f is continuous at a point $p \in E$ and if g is continuous at the point $f(p)$, then h is continuous at p.
This function h is called the *composition* or the *composite* of f and g, frequently noted as $h = g \circ f$.

Remark. A brief statement of the above theorem is that a continuous function of a continuous function is continuous.

Theorem 25. Let f and g be real continuous functions on a metric space X. Then $f+g, fg$, and f/g $(g(x) \neq 0,$ for all $x \in X)$ are continuous on X.

Theorem 26. A mapping f of a metric space X into a metric space Y is continuous on X if and only if $f^{-1}(V)$ is open in X for every open set V in Y.

Example 16. Define $f : R \to R$ by $f(x) = x^2$, and consider the open interval $I = (-1, 1)$. Then

$$f(I) = [0, 1), \ \ f^{-1}(I) = (-1, 1),$$

so the inverse image of the open interval I is open, but the image is not.

Remark. A continuous function needn't map open sets to open sets. However, the inverse image of an open set under a continuous function is always open. This property is the topological definition of a continuous function.

1.5.3. Continuity and Compactness

Definition 15. A mapping $\mathbf{f}$ of a set E into R^k is said to be *bounded* if there is a real number M such that $\|\mathbf{f}(x)\| \leq M$ for all $x \in E$.

Theorem 27. Suppose f is a continuous mapping of a compact metric space X into a metric space Y. Then $f(X)$ is compact.

Remark. Compactness is preserved by continuous maps. However, not all topological properties are preserved by continuous maps, e.g. openness and closedness as we saw in Example 16.

We shall now deduce some consequences of Theorem 27.

Theorem 28. If f is a continuous mapping of a compact metric space X into R^k, then $f(X)$ is closed and bounded. Thus, f is bounded.

Theorem 29. Suppose f is a continuous real function on a compact metric space X, and

$$M = \underset{p \in X}{sup}\, f(p), \;\; m = \underset{p \in X}{inf}\, f(p).$$

Then there exist points $p, q \in X$ such that $f(p) = M$ and $f(q) = m$.

Remark. The conclusion may also be simply stated as follows: f attains its maximum (at p) and its minimum (at q).

Theorem 30. Suppose f is a continuous $1-1$ mapping of a compact metric space X onto a metric space Y. Then the inverse mapping f^{-1} defined on Y by

$$f^{-1}(f(x)) = x \;\; (x \in X)$$

is a continuous mapping of Y onto X.

Remark. The compactness is essential in this theorem.

Example 17. Let X be the half-open interval $[0, 2\pi)$ on the real line, and let $\mathbf{f}$ be the mapping of X onto the unit circle Y consisting of all points whose distance from the origin is 1, given by

$$\mathbf{f}(t) = (cos\, t, sin\, t) \ \ (0 \leq t < 2\pi).$$

$\mathbf{f}$ is a continuous $1-1$ mapping of X onto Y. However, the inverse mapping fails to be continuous at the point $(1,0) = \mathbf{f}(0)$. Of course, X is not compact in this example.

1.6. The Intermediate Value Theorem

Theorem 31. Suppose that $f : [a,b] \to R$ is continuous. Suppose $f(a) > f(b)$ for some $a, b \in E$. Then for any η such that $f(b) < \eta < f(a)$, there exists $c \in [a,b]$ such that $f(c) = \eta$.

Exercise 18. Suppose that $f(x)$ is continuous on $[a,b]$. Let

$$\eta = \frac{1}{3}[f(x_1) + f(x_2) + f(x_3)],$$

where $x_1, x_2, x_3 \in [a,b]$. Prove that there exists $c \in [a,b]$ such that $f(c) = \eta$.

1.7. More Exercises

Exercise 19. For $x \in R$ and $y \in R$, define

$$d(x,y) = \frac{|x-y|}{1+|x-y|}$$

Determine whether it is a metric or not.

Exercise 20. Let X be an infinite set. For $p \in X$ and $q \in X$, define

$$d(p,q) = \begin{cases} 1 & (\text{if } p \neq q) \\ 0 & (\text{if } p = q) \end{cases}$$

Prove that this is a metric. Which subsets of the resulting metric space are open? Which are closed? Which are compact?

Exercise 21. Compute

$$\lim_{x \to \infty} \sin(\sqrt{x+1} - \sqrt{x}).$$

Exercise 22. Consider the sequence defined recursively by $a_1 = 1$ and $a_{n+1} = 3 + \frac{a_n}{2}$ for all $n \in \mathbb{N}$. Prove that $\{a_n\}$ converges. Find its limit.

Exercise 23. Prove that there exists a number $x \in [0, \frac{\pi}{2}]$ such that $2x - 1 = \sin(x^2 + \frac{\pi}{4})$.