JOE SEIDEL

REAL ANALYSIS SIXTH WEEK

3.4.8

Prove that a subset Y of a complete metric space X is also complete metric space with the inherited metric if and only if Y is closed as a subset of X.

Proof. "$\Rightarrow$"

Suppose Y is closed. Let (y_n) be a a Cauchy sequence in Y. Since $Y \subset X$, (y_n) is a Cauchy sequence in X. Since X is complete, (y_n) converges to y for some $y \in X$. Since Y is closed, $y \in Y$, hence Y is complete.

"$\Leftarrow$" Let Y be a complete metric space and suppose Y is open. Then a Cauchy sequence $(y_n) \in Y$ converges to $y_n \notin Y$, but this contradicts that Y is complete, so Y is closed.

$\square$

3.4.9

Show that, for $1 \leq p \leq \infty$, the space $\ell_n^p(\mathbb{R})$ and $\ell_n^p(\mathbb{C})$ are complete metric spaces.

Proof. Define $\|x\|_p = (\sum_{k=1}^n |x_k|^p)^{\frac{1}{p}}$ and let V be a vector space in $\mathbb{R}^n or \mathbb{C}^n$.

Let the set $\{e_i\}_{i=1}^n$ be a base for V. Recall that that norms for $1 \leq p \leq \infty$ are equivalent on finite dimensional spaces, therefore we can choose $p = 1$ and completeness is preserved on these equivalent norms.

We can choose $L, M > 0 \in \mathbb{R}$ or $\mathbb{C}$ Such that $L\|w\| \leq \|w\| \leq M\|w\|$ for all $w \in V$. This implies, $\forall \epsilon > 0$ there exists $N \in \mathbb{N}$ such that if $n, m > N$

$$\begin{aligned} L|v_{n_i} - v_{k_i}| &\leq L \sum_{i=1}^n |v_{n_i} - v_{k_i}| \\ &= L\|v_n - v_m\| \leq \|v_n - v_m\| < \epsilon \end{aligned}$$

for all $1 \leq i \leq n$. Hence, (v_{k_i}) is a Cauchy sequence in $\mathbb{R}$ or $\mathbb{C}$ for each i. Since $\mathbb{R}$ and $\mathbb{C}$ are complete, there exists $u_i \in \mathbb{R}$ or $\mathbb{C}$ such that $u_i = \lim_{k\to\infty} v_{k_i}$ for each i. Let $u = (u_1, ..., u_n) = \sum_{i=1}^n u_i e_i$ which means that $u \in V$. Finally, to show completeness, need to show $\lim_{k\to\infty} \|v_k - u\| = 0$.

$$\begin{aligned}\lim_{k\to\infty}\|v_k-u\| &\leq M\lim_{k\to\infty}\|v_k-u\| \\ &= M\lim_{k\to\infty}\sum_{i=1}^{n}|v_{k_i}-u_i| \\ &= M\sum_{i=1}^{n}\lim_{k\to\infty}|v_{k_i}-u_i| \\ &= 0\end{aligned}$$

$\square$

3.4.18

For the following sequences $(f_n)_{n\in\mathbb{N}}$ of functions, where $f_n:[0,2\pi]\to\mathbb{R}$ for all $n\in\mathbb{N}$, find all values of $x\in[0,2\pi]$ such that the sequence $(f_n(x))_{n\in\mathbb{N}}$ converges and find the pointwise limit function $f:[0,2\pi]\to\mathbb{R}$ if it exists.

1. $f_n(x)=\sin(\frac{x}{n})$
   Since $1\leq n$ this function is always defined. For all values $x\in[0,2\pi]$ the sequence $(f_n(x))_{n\in\mathbb{N}}$ converges to 0 so $f:[0,2\pi]\to\mathbb{R}$ is given by $f(x)=0$.

2. $f_n(x)=sin(nx)$.
   Since the sin function oscillates between -1 and 1. Consider $f_n(x)=1$ when $n<\frac{\pi}{2x}$ and again when $2\pi(x)<n<\frac{5\pi}{2x}$ and so forth. Consider when $f_n(x)=0$, when $\frac{\pi}{2x}<n<\frac{\pi}{x}$ and again when $\frac{3\pi}{2x}<n<\frac{2\pi}{x}$ and so forth. Next, when $(f_n(x)=-1$ whenever $\frac{\pi}{x}<n<\frac{3\pi}{2x}$ and again $3\pi<n<\frac{7\pi}{2}$ and so forth.

   Hence, there are no values in the domain $[0,2\pi]$ such that $(f_n(x))_{n\in\mathbb{N}}$ converges. Hence, the pointwise limit function does not exist.

3. $f_n(x)=\sin^n(x)$.

$$f_n(x)=\begin{cases}0, & \text{if } x\neq\frac{3\pi}{2} \text{ and } x\neq\frac{\pi}{2}\\ 1, & \text{if } x=\frac{\pi}{2}\\ -1 \text{ or } 1, & \text{if } x=\frac{3\pi}{2}\end{cases}$$

   Since the sequence does not converge when $x=\frac{3\pi}{2}$ we cannot define $f:[0,2\pi]\to\mathbb{R}$ as the point wise limit function of $f_n(x)$.

3.4.22

Let $f_n(x)=x^n$ for $n\in\mathbb{N}$.

1. Show that the sequence $(f_n)_{n\in\mathbb{N}}$ converges pointwise to the function $f(x) = 0$ on the interval $(-1, 0)$.

   When $0 < x < 1$, this implies $x = \frac{1}{a}$ where $a > 1$ which implies $\lim_{n\to\infty} x^n = \lim_{n\to\infty} \frac{1}{a^n} = 0$. When $-1 < x < 0$, it implies $x = (-1)\frac{1}{a}$ where $a > 1$ which means $(-1)\lim_{n\to\infty} \frac{1}{a^n} = 0$. When $x = 0$, $\lim_{n\to\infty} x^n = 0$.

   Therefore all values in the domain $(-1, 1)$, $\lim_{n\to\infty} f_n(x) = f(x)$.

2. Show that if we restrict to the domain $[-\frac{1}{2}, \frac{1}{2}]$, the sequence $f(n)_{n\in\mathbb{N}}$ converges uniformly to the function $f(x) = 0$.

   Proof. A sequence converges uniformly to a function if given $\epsilon > 0$, $\exists N_\epsilon \in \mathbb{B}$ such that $\sup_{x\in X} |f_n(x) - f(x)| < \epsilon$ for $n \geq N_\epsilon$.

   Since $f(x) = 0$, $|f_n(x) - f(x)| = |x^n| < \epsilon$ if $x < \epsilon^{\frac{1}{n}}$. Since $\epsilon^{\frac{1}{n}} < 1$ for all n the sequence converges uniformly for the domain $[-\frac{1}{2}, \frac{1}{2}]$. □

3. Show that the sequence $(f_n)_{n\in\mathbb{N}}$ does not converge uniformly on the domain $(-1, 1)$.

   Proof. Consider again, the expression from above, $|f_n(x) - f(x)| = |x^n| < \epsilon$. The inequality $x < \epsilon^{\frac{1}{n}}$ fails when x gets within ϵ of 1. To see this, choose $x \in (-1, 1)$ such that $1 - \epsilon = x$. Notice that $1 - \epsilon < \epsilon^{\frac{1}{n}}$ is clearly false. Therefore, $(f_n)_{n\in\mathbb{N}}$ does not converge uniformly. □

3.5.2

3.5.3

3.5.4