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REAL ANALYSIS
FOURTH WEEK

Exercise 3.2.5

Let X be any non-empty set and, for $x_1, x_2 \in X$, define

$$d(x_1, x_2) = \begin{cases} 0, & \text{if } x_1 = x_2 \\ 1, & \text{if } x_1 \neq x_2 \end{cases}$$

Show that d is a metric on X. This is called the *discrete metric*, the pair (X, d) is referred to as a *discrete metric space*.

In order to be a metric, need to show positive definiteness, symmetry and triangle inequality.

1. For $x_1, x_2 \in X$, $d(x_1, x_2) \geq 0$, and $d(x_1, x_2) = 0$ if and only if $x_1 = x_2$.
   This is trivial considering the given case. If $x_1 \neq x_2$ $d(x_1, x_2) = 1 \geq 0$. If $x_1 = x_2$ then $d(x_1, x_2) = 0$. Likewise, if $d(x_1, x_2) = 0$ then $x_1 = x_2$.

2. For any $x_1, x_2 \in X$, we have $d(x_1, x_2) = d(x_2, x_1)$.
   Consider $x_1 = x_2$, $d(x_1, x_2) = 0$ and $d(x_2, x_1) = 0$ therefore $d(x_1, x_2) = d(x_2, x_1)$. Consider $x_1 \neq x_2$, $d(x_1, x_2) = 1$ and $d(x_2, x_1) = 1$ therefore $d(x_1, x_2) = d(x_2, x_1)$.

3. For any $x_1, x_2, x_3 \in X$, we have

   $$d(x_1, x_2) \leq d(x_1, x_3) + d(x_3, x_2)$$

   Since $d(x_i, x_j)$ is equal to either 1 or 0 for $i \neq j$.
   Case 1, choose $x_1, x_2, x_3 \in X$. let $x_1 = x_2$ and let x_3 be arbitrary. Then $d(x_1, x_2) = 0$ and $d(x_1, x_3) + d(x_3, x_2) \geq 0$. Satisfying the triangle inequality.
   Case 2, choose $x_1, x_2, x_3 \in X$, let $x_1 \neq x_2$. Then $d(x_1, x_2) = 1$ Choose x_3 such that $d(x_1, x_3) = 0$ and $d(x_2, x_3) = 0$ This implies $x_1 = x_3$ and $x_2 = x_3$ which means $x_1 = x_2$ which is a contradiction. So atleast one of $d(x_1, x_3) = 1$ or $d(x_3, x_2) = 1$. Satisfying the triangle inequality.

Exercise 3.2.6

(NOT ASSIGNED)

Let (X, d) be a metric space, and let Y be a proper subset of X. Show that (Y, d') is a metric space, where we define $d'(y_1, y_2) = d(y_1, y_2)$. We call d' the *inherited metric* on Y.

Keeping in mind that Y is a proper subset of X.

1. Consider $d'(y_1, y_2)$, since (X, d) is a metric space $d(y_1, y_2) \geq 0$ and equal to zero if and only if $y_1 = y_2$, then by definition of $d'(y_1, y_2) = d(y_1, y_2)$ the same positive definiteness holds for (Y, d').

2. Since since (X, d) is a metric space $d(y_1, y_2) = d(y_2, y_1)$ which implies $d(y_2, y_1) = d'(y_2, y_1)$ Therefore $d'(y_1, y_2) = d'(y_2, y_1)$.

3. Consider $d'(y_1, y_2) = d(y_1, y_2)$, $d'(y_1, y_3) = d(y_1, y_3)$ and $d'(y_3, y_2) = d(y_3, y_2)$. Then $d(y_1, y_2) \leq d(y_1, y_3) + d(y_3, y_2)$ implies $d'(y_1, y_2) \leq d'(y_1, y_3) + d'(y_3, y_2)$ satisfying the triangle inequality.

Exercise 3.2.8

(NOT ASSIGNED)

Prove that d_p is a metric on $\mathbb{R}^n$ for $p > 1$. *Hint:* The triangle inequality of the only hard part. The proof depends on on Hölder's Inequality. To begin, observe that

$$\|x+y\|_p^p - \sum_i |x_i + y_i|^p \leq \sum_i |x_i + y_i|^{p-1}|x_i| + \sum_i |x_i + y_i|^{p-1}|y_i|$$

Proof. To prove the triangle inequality, apply Hölder's Inequality using $q = \frac{p}{p-1}$.

$$\begin{aligned}
\|x+y\|_p^p &\leq (\sum_i |x_i + y_i|^{p-1})^{\frac{p}{p-1}})^{\frac{p-1}{p}} (\sum_i |x_i|^p)^{\frac{1}{p}} \\
&+ (\sum_i |x_i + y_i|^{p-1})^{\frac{p}{p-1}})^{\frac{p-1}{p}} (\sum_i |y_i|^p)^{\frac{1}{p}} \\
&= \|x\|_p (\sum_i |x_i + y_i|^p)^{\frac{p-1}{p}} + \|y\|_p (\sum_i |x_i + y_i|^p)^{\frac{p-1}{p}} \\
&= \|x\|_p \|x+y\|_p^{p-1} + \|y\|_p \|x+y\|_p^{p-1} \\
&= (\|x\|_p + \|y\|_p)\|x+y\|_p^{p-1}
\end{aligned}$$

We can divide both sides by $\|x+y\|_p^{p-1}$ to get

$$\|x+y\|_p \leq \|x\|_p + \|y\|_p$$

□

Exercise 3.2.9

Note that Hölder's Inequality only works for $p, q > 1$. Prove the triangle inequality for the d_1 metric.

Proof. In the d_1 metric, $d(x, y) = \|x - y\|_p$

$$\begin{aligned}
\|x+y\|_1 = \sum_i |x_i + y_1|^1 &\leq \sum_i |x_i|^1 + \sum |y_i|^1 \\
&= \|x\|_1 + \|x\|_1
\end{aligned}$$

□

Exercise 3.2.10

Prove that d_∞ defines a metric on $\mathbb{R}^n$

Proof. Need to show positive definiteness, symmetry, and triangle inequality.

1. Positive definiteness. $d_\infty(x,y) = max_{1\leq j\leq n}|x_i - y_i| \geq 0$ and 0 if and only if $x = y$.

2. $d_\infty(x,y) = max_{1\leq j\leq n}|x_i - y_i| = max_{1\leq j\leq n}|y_i - x_i| = d_\infty(y,x)$

3. Triangle inequality. Consider $x, y \in X$.

$$\begin{aligned}\|x+y\|_\infty &= max_{1\leq j\leq n}|x_i + y_i| \\ &\leq max_{1\leq j\leq n}|x_i| + max_{1\leq j\leq n}|y_i| = \|x\|_\infty + \|y\|_\infty\end{aligned}$$

□

Exercise 3.3.5

If $1 \leq p < q$, show that the unit ball in $\ell_n^p(\mathbb{R})$ is contained in the unit ball in $\ell_n^q(\mathbb{R})$.

if $0 < a \leq 1$ then $(\sum_i |x_i|)^a \leq \sum_i |x_i|^a$

Proof. Suppose $1 \leq p < q$

$$\begin{aligned}\|x\|_q = (\sum_i |x_i|^q)^{\frac{1}{q}} &= (\sum_i |x_i|^q)^{\frac{p}{qp}} \\ &\leq (\sum_i |x_i|^{q\frac{p}{q}})^{\frac{1}{p}} = (\sum_i |x_i|^p)^{\frac{1}{p}} = \|x\|_p\end{aligned}$$

Finally, supposed $\|x\|_p < 1$. This implies $\|x\|_q \leq \|x\|_p < 1$ which means any point of $\|x\|_p$ is contained in $\|x\|_q$, which implies $\ell_n^p \subset \ell_n^q$.

□

Exercise 3.3.6

Choose p with $1 \leq p \leq \infty$, and let $\epsilon > 0$. Show that $B_\epsilon(0) = \{\epsilon \cdot x | x \in B_1(0)\}$.

Let (X, d) be a metric space. Define $B_1(0) = \{x \in X | d(x,0) < 1\}$ and $B_\epsilon(0) = \{x \in X | d(x,0) < \epsilon\}$. If we take any point from $B_1(0)$ we have the inequality $d(x,0) < 1$. Therefore, we can multiply both sides of the inequality by ϵ with the result $\epsilon \cdot d(x,0) < \epsilon$ for all $x \in B_1(0)$. Which defines $B_\epsilon(0)$.

Exercise 3.3.7

Consider a point $x \in \mathbb{R}^2$ that lies outside the unit ball in $\ell_2^1(\mathbb{R})$ and inside the unit ball in ℓ_2^∞. Is there a p between 1 and ∞ such that $\|x\|_p = 1$? Do the same problem for $\mathbb{R}^n$.

For $\mathbb{R}^2$.

Proof. Choose $x = (x_1, x_2)$ $x \in \mathbb{R}^2$ such that $|x_1| \leq 1$ and $|x_2| \leq 1$ and $|x_1| + |x_2| > 1$ for $p = 1$. Consider $f(p) = |x_1|^p + |x_2|^p$. As p approaches infinity $f(p)$ gets approaches 0. Since $f(p)$ is a continuous function, there is some p between one and ∞ where $f(p) = 1$ making $\|x\|_p = 1$. □

For $\mathbb{R}^n$.

Proof. Chose $x = (x_1, x_2, ..., x_n)$, $x \in \mathbb{R}^n$ such that $|x_i| \leq 1$ and $|x_1| + |x_2| + ... + |x_n| > 1$ for $p = 1$. Again consider $f(p) = |x_1|^p + |x_2|^p + ... + |x_n|^p$. As p approaches infinity $f(p)$ approaches 0. Again, since $f(p)$ is a continuous function, there is some p between 1 and ∞ such that $f(p) = 1$ making $\|x\|_p = 1$. □

Exercise 3.3.10

Prove that the following are open sets.

1. The "first quadrant", that is, $\{(x, y) \in \mathbb{R}^2 | x > 0 \text{ and } y > 0\}$, in the usual metric.

   Proof. Choose $x, y \in \mathbb{R}^2$ such that $x > 0$ and $y > 0$ and otherwise let them be arbitrary. Since we know $\mathbb{R}$ is open, For x choose $\epsilon_x > 0$ such that $(x - \epsilon_x, x + \epsilon_x) \subset \mathbb{R}$ and choose $\epsilon_y > 0$ such that $(y - \epsilon_y, y + \epsilon_y) \subset \mathbb{R}$ Choose $\epsilon = min(\epsilon_x, \epsilon_y)$, then we have $B_\epsilon(x, y) \subset \mathbb{R}^2$. Since x, y only need to be positive, we can draw an open ball around any point (x, y) in $\mathbb{R}_+^2$ making the first quadrant open. □

2. any subset of a discrete metric.

   Proof. Let (X, d) be a discrete metric. Recall that in a discrete metric $d(x, x) = 0$ and $d(x, y) = 1$, $x \neq y$. You can take $0 < r < 1$ such that the only point the ball contains is the point that it is centered on. If $r > 1$ then it contains all the points. In either case $B_r(x) \subset X$. □

Exercise 3.3.12

Let $X = [-1, 1]$ with the metric inherited above (*mathbbR*). Describe the open balls of $B_r(1)$ for various values of r.

1. Consider $0 < r \leq 2$. $B_r(1) = (-r, 1]$
2. Consider $r > 2$. $B_r(1) = [-1, 1]$