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# REAL ANALYSIS THIRD WEEK

*Exercise 1.9.8*

Give examples to show that if  $r = 1$  in the statement of the Ratio Test, anything may happen.

First consider the harmonic series,  $\sum_{n=1}^{\infty} \frac{1}{n}$ . The  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$

$$\lim_{n \rightarrow \infty} \left| \frac{\frac{1}{n+1}}{\frac{1}{n}} \right| = \lim_{n \rightarrow \infty} \left| \frac{n}{n+1} \right| = 1$$

However, we know that the series  $\sum_{n=1}^{\infty} \frac{1}{n}$  diverges.

Next consider  $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ . Again  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{\frac{(-1)^{n+2}}{n+1}}{\frac{(-1)^{n+1}}{n}} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+2}}{n+1} \cdot \frac{n}{(-1)^{n+1}} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{-n}{n+1} \right| \\ &= 1 \end{aligned}$$

This series,  $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$  converges since  $\left| \sum_{k=m+1}^n \frac{(-1)^{k+1}}{k} \right| < \frac{1}{m}$  but not absolutely because  $\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n} \right|$  goes to infinity.

Finally, consider the series  $\sum_{n=1}^{\infty} \frac{1}{n^2}$ . This too has  $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$ .

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} \right| &= \lim_{n \rightarrow \infty} \left| \frac{1}{(n+1)^2} \cdot \frac{n^2}{1} \right| \\ &= \lim_{n \rightarrow \infty} \frac{n^2}{(n+1)^2} \\ &= 1 \end{aligned}$$

The series  $\sum_{n=1}^{\infty} \frac{1}{n^2}$  is absolutely convergent because  $\sum_{n=1}^{\infty} \left| \frac{1}{n^2} \right|$  converges.

*Exercise 1.9.20*

Give examples to show that if  $r = 1$  in the statement of the Root Test, anything may happen.

Borrowing from the ideas in the last exercise, consider the series  $\sum_{n=1}^{\infty} \frac{1}{n^2}$  and  $\sum_{n=1}^{\infty} \frac{1}{n}$ . Both have  $r = 1$ .

$$\begin{aligned} \limsup_{n \rightarrow \infty} \left| \frac{1}{n} \right|^{\frac{1}{n}} &= \limsup_{n \rightarrow \infty} e^{\frac{1}{n} \ln \frac{1}{n}} \\ &= e^{(\limsup_{n \rightarrow \infty} -\ln n)(\limsup_{n \rightarrow \infty} \frac{1}{n})} \\ &= e^{(\limsup_{n \rightarrow \infty} -\ln n)(0)} \\ &= 1 \end{aligned}$$

$$\begin{aligned}
\limsup_{n \rightarrow \infty} \left| \frac{1}{n^2} \right|^{\frac{1}{n}} &= \limsup_{n \rightarrow \infty} e^{\frac{1}{n} \ln \frac{1}{n^2}} \\
&= e^{(\limsup_{n \rightarrow \infty} -2 \ln n)(\limsup_{n \rightarrow \infty} \frac{1}{n})} \\
&= e^{(\limsup_{n \rightarrow \infty} -2 \ln n)(0)} \\
&= 1
\end{aligned}$$

Anything may happen when  $r = 1$

### Exercise 1.9.26

Determine the radius of convergence of the following power series:

$$r = \limsup_{n \rightarrow \infty} |a_n|^{\frac{1}{n}}$$

Where the radius of convergence is  $\frac{1}{r}$  if  $r > 0$ ,  $\infty$  if  $r = 0$  and 0 if  $\limsup$  does not exist, ( $r = \infty$ ).

1.  $\sum_{n=1}^{\infty} \frac{z^n}{n!}$

$$\sum_{n=1}^{\infty} \frac{z^n}{n!} = \sum_{n=1}^{\infty} \frac{1}{n!} z^n$$

$$\lim_{n \rightarrow \infty} \left| \frac{1}{n!} \right|^{\frac{1}{n}} = 0$$

Radius of convergence is  $\infty$

2.  $\sum_{n=2}^{\infty} \frac{z^n}{\ln(n)}$

$$\sum_{n=1}^{\infty} \frac{z^n}{\ln(n)} = \sum_{n=2}^{\infty} \frac{1}{\ln(n)} z^n$$

$$\lim_{n \rightarrow \infty} \left| \frac{1}{\ln(n)} \right|^{\frac{1}{n}} = 1$$

Radius of convergence is 1.

3.  $\sum_{n=1}^{\infty} \frac{n^n}{n!} z^n$

$$e^n = \sum_{n=1}^{\infty} \frac{n^n}{n!}$$

$$\lim_{n \rightarrow \infty} \left| \frac{n^n}{n!} \right|^{\frac{1}{n}} = e$$

Radius of convergence is  $\frac{1}{e}$

## Exercise 2.5.8

Prove that the scalar product is a positive definite symmetric bilinear form on  $\mathbb{E}^n$ .

*Proof.* The scalar product of vectors  $\mathbf{v} = (v_1, v_2, v_3, \dots, v_n)$  and  $\mathbf{w} = (w_1, w_2, w_3, \dots, w_n)$  is  $\langle \mathbf{v}, \mathbf{w} \rangle = v_1 w_1 + v_2 w_2 + v_3 w_3 + \dots + v_n w_n$ .

Let  $V = \mathbb{E}^n$  be a vector space over  $F = \mathbb{R}$ . A bilinear form  $\langle \cdot, \cdot \rangle$  on  $V$  is a map

$$\langle \cdot, \cdot \rangle : V \times V \rightarrow F$$

that satisfies linearity in both variables. That is, for all  $\mathbf{v}, \mathbf{v}_1, \mathbf{v}_2, \mathbf{w}, \mathbf{w}_1, \mathbf{w}_2 \in V$  and all  $\alpha \in F$

$$\begin{aligned} \langle \mathbf{v}_1 + \mathbf{v}_2, \mathbf{w} \rangle &= (v_{1_1} + v_{2_1})w_1 + (v_{1_2} + v_{2_2})w_2 + (v_{1_3} + v_{2_3})w_3 + \dots + (v_{1_n} + v_{2_n})w_n \\ &= v_{1_1}w_1 + v_{2_1}w_1 + v_{1_2}w_2 + v_{2_2}w_2 + v_{1_3}w_3 + v_{2_3}w_3 + \dots + v_{1_n}w_n + v_{2_n}w_n \\ &= (v_{1_1}w_1 + v_{1_2}w_2 + v_{1_3}w_3 + \dots + v_{1_n}w_n) + (v_{2_1}w_1 + v_{2_2}w_2 + v_{2_3}w_3 + \dots + v_{2_n}w_n) \\ &= \langle \mathbf{v}_1, \mathbf{w} \rangle + \langle \mathbf{v}_2, \mathbf{w} \rangle \end{aligned}$$

$$\begin{aligned} \langle \alpha \mathbf{v}, \mathbf{w} \rangle &= \alpha v_1 w_1 + \alpha v_2 w_2 + \alpha v_3 w_3 + \dots + \alpha v_n w_n \\ &= \alpha(v_1 w_1) + \alpha(v_2 w_2) + \alpha(v_3 w_3) + \dots + \alpha(v_n w_n) \\ &= \alpha(v_1 w_1 + v_2 w_2 + v_3 w_3 + \dots + v_n w_n) \\ &= \alpha \langle \mathbf{v}, \mathbf{w} \rangle \end{aligned}$$

$$\begin{aligned} \langle \mathbf{v}, \mathbf{w}_1 + \mathbf{w}_2 \rangle &= v_1(w_{1_1} + w_{2_1}) + v_2(w_{1_2} + w_{2_2}) + v_3(w_{1_3} + w_{2_3}) + \dots + v_n(w_{1_n} + w_{2_n}) \\ &= v_1 w_{1_1} + v_1 w_{2_1} + v_2 w_{1_2} + v_2 w_{2_2} + v_3 w_{1_3} + v_3 w_{2_3} + \dots + v_n w_{1_n} + v_n w_{2_n} \\ &= (v_1 w_{1_1} + v_2 w_{1_2} + v_3 w_{1_3} + \dots + v_n w_{1_n}) + (v_1 w_{2_1} + v_2 w_{2_2} + v_3 w_{2_3} + \dots + v_n w_{2_n}) \\ &= \langle \mathbf{v}, \mathbf{w}_1 \rangle + \langle \mathbf{v}, \mathbf{w}_2 \rangle \end{aligned}$$

$$\begin{aligned} \langle \mathbf{v}, \alpha \mathbf{w} \rangle &= v_1 \alpha w_1 + v_2 \alpha w_2 + v_3 \alpha w_3 + \dots + v_n \alpha w_n \\ &= \alpha(v_1 w_1) + \alpha(v_2 w_2) + \alpha(v_3 w_3) + \dots + \alpha(v_n w_n) \\ &= \alpha(v_1 w_1 + v_2 w_2 + v_3 w_3 + \dots + v_n w_n) \\ &= \alpha \langle \mathbf{v}, \mathbf{w} \rangle \end{aligned}$$

Regarding being positive definite, consider  $\langle \mathbf{v}, \mathbf{v} \rangle$

$$\begin{aligned} \langle \mathbf{v}, \mathbf{v} \rangle &= v_1 v_1 + v_2 v_2 + v_3 v_3 + \dots + v_n v_n \\ &= v_1^2 + v_2^2 + v_3^2 + \dots + v_n^2 \\ &\geq 0 \end{aligned}$$

Notice also, that  $\langle \mathbf{v}, \mathbf{v} \rangle = 0$  if and only if  $\mathbf{v} = 0$

□

*Exercise 2.5.11*

Prove the following properties of the norm if  $\mathbf{v}, \mathbf{w} \in \mathbb{E}^n$

1.  $\|\mathbf{v}\| \geq 0$

*Proof.* If  $\mathbf{v} \in \mathbb{E}^n$  the norm is defined by

$$\begin{aligned}\|\mathbf{v}\| &= \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle} \\ &= \sqrt{(v_1v_1 + v_2v_2 + v_3v_3 + \dots + v_nv_n)} \\ &= \sqrt{(v_1^2 + v_2^2 + v_3^2 + \dots + v_n^2)} \\ &= (v_1^2 + v_2^2 + v_3^2 + \dots + v_n^2)^{\frac{1}{2}} \\ &\geq 0\end{aligned}$$

□

2.  $\|\mathbf{v}\| = 0$  if and only if  $\mathbf{v} = 0$

From the definition of positive definite,  $\langle \mathbf{v}, \mathbf{v} \rangle = 0$  if and only if  $\mathbf{v} = 0$

*Proof.* " $\Rightarrow$ " Let  $\|\mathbf{v}\| = 0$ . Then

$$\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle} = 0$$

$$\begin{aligned}\sqrt{\langle \mathbf{v}, \mathbf{v} \rangle} &= 0 \\ \sqrt{(v_1v_1 + v_2v_2 + v_3v_3 + \dots + v_nv_n)} &= 0 \\ (v_1^2 + v_2^2 + v_3^2 + \dots + v_n^2)^{\frac{1}{2}} &= 0 \\ (v_1^2 + v_2^2 + v_3^2 + \dots + v_n^2) &= 0\end{aligned}$$

So  $\mathbf{v} = 0$

□

*Proof.* " $\Leftarrow$ "

Let  $\mathbf{v} = 0$

So  $\langle \mathbf{v}, \mathbf{v} \rangle = 0$ .

$$\begin{aligned}\|\mathbf{v}\| &= \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle} \\ &= \sqrt{0} \\ &= 0\end{aligned}$$

□

3.  $\|\alpha \mathbf{v}\| = |\alpha| \|\mathbf{v}\|, \alpha \in \mathbb{R}$

*Proof.*

$$\begin{aligned}
 \|\alpha \mathbf{v}\| &= \sqrt{\langle \alpha \mathbf{v}, \alpha \mathbf{v} \rangle} \\
 &= \sqrt{(\alpha^2 v_1^2 + \alpha^2 v_2^2 + \alpha^2 v_3^2 + \dots + \alpha^2 v_n^2)} \\
 &= \sqrt{\alpha^2 (v_1^2 + v_2^2 + v_3^2 + \dots + v_n^2)} \\
 &= |\alpha| \sqrt{(v_1^2 + v_2^2 + v_3^2 + \dots + v_n^2)} \\
 &= |\alpha| \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle} \\
 &= |\alpha| \|\mathbf{v}\|
 \end{aligned}$$

□

4.  $\|\mathbf{v} + \mathbf{w}\| \leq \|\mathbf{v}\| + \|\mathbf{w}\|$

*Proof.* First recall the Cauchy-Schwarz Inequality, let  $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ , then  $|\langle \mathbf{v}, \mathbf{w} \rangle| \leq \|\mathbf{v}\| \|\mathbf{w}\|$

$$\begin{aligned}
 \|\mathbf{v} + \mathbf{w}\|^2 &= \langle \mathbf{v} + \mathbf{w}, \mathbf{v} + \mathbf{w} \rangle \\
 &= (v_1 + w_1)^2 + (v_2 + w_2)^2 + \dots + (v_n + w_n)^2 \\
 &= (v_1^2 + v_2^2 + \dots + v_n^2) + (w_1^2 + w_2^2 + \dots + w_n^2) + (v_1 w_1 + \dots + v_n w_n) + (w_1 v_1 + \dots + w_n v_n) \\
 &= \langle \mathbf{v}, \mathbf{v} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle + \langle \mathbf{w}, \mathbf{v} \rangle + \langle \mathbf{w}, \mathbf{w} \rangle \\
 &\leq \|\mathbf{v}\|^2 + 2|\langle \mathbf{v}, \mathbf{w} \rangle| + \|\mathbf{w}\|^2 \\
 &\leq \|\mathbf{v}\|^2 + 2\|\mathbf{v}\| \|\mathbf{w}\| + \|\mathbf{w}\|^2 \\
 &= (\|\mathbf{v}\| + \|\mathbf{w}\|)^2
 \end{aligned}$$

□

5.  $\|\mathbf{v} + \mathbf{w}\|^2 + \|\mathbf{v} - \mathbf{w}\|^2 = 2(\|\mathbf{v}\|^2 + \|\mathbf{w}\|^2)$

*Proof.*

$$\begin{aligned}
 \|\mathbf{v} + \mathbf{w}\|^2 + \|\mathbf{v} - \mathbf{w}\|^2 &= \langle \mathbf{v} + \mathbf{w}, \mathbf{v} + \mathbf{w} \rangle + \langle \mathbf{v} - \mathbf{w}, \mathbf{v} - \mathbf{w} \rangle \\
 &= (v_1 + w_1)^2 + \dots + (v_n + w_n)^2 + (v_1 - w_1)^2 + \dots + (v_n - w_n)^2 \\
 &= (v_1^2 + 2v_1 w_1 + w_1^2 + \dots + v_n^2 + 2v_n w_n + w_n^2) + (v_1^2 - 2v_1 w_1 + w_1^2 + \dots + v_n^2 - 2v_n w_n + w_n^2) \\
 &= (2v_1^2 + \dots + 2v_n^2) + (2w_1^2 + \dots + 2w_n^2) \\
 &= 2(v_1^2 + \dots + v_n^2) + 2(w_1^2 + \dots + w_n^2) \\
 &= 2(\langle \mathbf{v}, \mathbf{v} \rangle + \langle \mathbf{w}, \mathbf{w} \rangle) \\
 &= 2(\|\mathbf{v}\|^2 + \|\mathbf{w}\|^2)
 \end{aligned}$$

□

## Exercise 2.5.20

1. Show that  $\|\mathbf{v} \times \mathbf{w}\| = \|\mathbf{v}\|\|\mathbf{w}\| \sin \theta$ , where  $\theta$  is the angle between  $\mathbf{v}$  and  $\mathbf{w}$ .

*Proof.* First right out the cross product of  $\mathbf{v} \times \mathbf{w}$

$$\mathbf{v} \times \mathbf{w} = (v_2w_3 - v_3w_2, v_3w_1 - v_1w_3, v_1w_2 - v_2w_1) \quad (1)$$

$$\begin{aligned} \|\mathbf{v} \times \mathbf{w}\|^2 &= (v_2w_3 - v_3w_2)^2 + (v_3w_1 - v_1w_3)^2 + (v_1w_2 - v_2w_1)^2 \\ &= v_2^2w_3^2 - 2v_2v_3w_2w_3 + v_3^2w_2^2 \\ &\quad + v_3^2w_1^2 - 2v_1v_3w_1w_3 + v_1^2w_3^2 \\ &\quad + v_1^2w_2^2 - 2v_1v_2w_1w_2 + v_2^2w_1^2 \end{aligned}$$

So

$$\begin{aligned} \|\mathbf{v} \times \mathbf{w}\|^2 &= v_1^2(w_2^2 + w_3^2) + v_2^2(w_1^2 + w_3^2) + v_3^2(w_1^2 + w_2^2) \\ &\quad - 2(v_2v_3w_2w_3 + v_1v_3w_1w_3 + v_1v_2w_1w_2) \end{aligned} \quad (2)$$

Recall that  $\cos \theta = \frac{\langle \mathbf{v}, \mathbf{w} \rangle}{\|\mathbf{v}\|\|\mathbf{w}\|}$  implies  $\langle \mathbf{v}, \mathbf{w} \rangle = \cos \theta \|\mathbf{v}\|\|\mathbf{w}\|$ . Therefore

$$\cos^2 \theta \|\mathbf{v}\|^2 \|\mathbf{w}\|^2 = (\langle \mathbf{v}, \mathbf{w} \rangle)^2 \quad (3)$$

$$(\langle \mathbf{v}, \mathbf{w} \rangle)^2 = (v_1w_1 + v_2w_2 + v_3w_3)^2 \quad (4)$$

$$\begin{aligned} (v_1w_1 + v_2w_2 + v_3w_3)^2 &= (v_1w_1 + v_2w_2 + v_3w_3)(v_1w_1 + v_2w_2 + v_3w_3) \\ &= v_1^2w_1^2 + v_1v_2w_1w_2 + v_1v_3w_1w_3 \\ &\quad + v_1^2w_1^2 + v_1v_2w_1w_2 + v_2v_3w_2w_3 \\ &\quad + v_3^2w_3^2 + v_1v_3w_1w_3 + v_2v_3w_2w_3 \end{aligned}$$

So applying (3) with our result from (4),

$$\begin{aligned} \cos^2 \theta \|\mathbf{v}\|^2 \|\mathbf{w}\|^2 &= v_1^2w_1^2 + v_1^2w_1^2 + v_3^2w_3^2 + \\ &\quad 2(v_1v_2w_1w_2 + v_1v_3w_1w_3 + v_2v_3w_2w_3) \end{aligned} \quad (5)$$

Combining (4) with (2),  $\|\mathbf{v} \times \mathbf{w}\|^2 + \cos^2 \theta \|\mathbf{v}\|^2 \|\mathbf{w}\|^2$  we get

$$\begin{aligned}
 & v_1^2 w_1^2 + v_1^2 w_2^2 + v_3^2 w_3^2 + 2(v_1 v_2 w_1 w_2 + v_1 v_3 w_1 w_3 + v_2 v_3 w_2 w_3) \\
 & + v_1^2 (w_2^2 + w_3^2) + v_2^2 (w_1^2 + w_3^2) + v_3^2 (w_1^2 + w_2^2) \\
 & - 2(v_2 v_3 w_2 w_3 + v_1 v_3 w_1 w_3 + v_1 v_2 w_1 w_2) \\
 & = v_1^2 w_1^2 + v_1^2 w_2^2 + v_3^2 w_3^2 \\
 & + v_1^2 (w_2^2 + w_3^2) + v_2^2 (w_1^2 + w_3^2) + v_3^2 (w_1^2 + w_2^2) \\
 & = v_1^2 (w_1^2 + w_2^2 + w_3^2) + v_2^2 (w_1^2 + w_2^2 + w_3^2) + v_3^2 (w_1^2 + w_2^2 + w_3^2) \\
 & = (w_1^2 + w_2^2 + w_3^2)(v_1^2 + v_2^2 + v_3^2)
 \end{aligned}$$

So from this we can conclude

$$\|\mathbf{v} \times \mathbf{w}\|^2 + \cos^2 \theta \|\mathbf{v}\|^2 \|\mathbf{w}\|^2 = \|\mathbf{w}\|^2 \|\mathbf{v}\|^2 \quad (6)$$

The result of (6) implies that  $\|\mathbf{v} \times \mathbf{w}\|^2 = \|\mathbf{v}\|^2 \|\mathbf{w}\|^2 - \cos^2 \theta \|\mathbf{v}\|^2 \|\mathbf{w}\|^2$ .

So our last steps

$$\begin{aligned}
 \|\mathbf{v} \times \mathbf{w}\|^2 &= \|\mathbf{v}\|^2 \|\mathbf{w}\|^2 - \cos^2 \theta \|\mathbf{v}\|^2 \|\mathbf{w}\|^2 \\
 &= \|\mathbf{v}\|^2 \|\mathbf{w}\|^2 (1 - \cos^2 \theta) \\
 &= \|\mathbf{v}\|^2 \|\mathbf{w}\|^2 (\sin^2 \theta)
 \end{aligned}$$

We can take the square root to reach our conclusion.

$$\|\mathbf{v} \times \mathbf{w}\| = \|\mathbf{v}\| \|\mathbf{w}\| \sin \theta$$

□

### Exercise 2.5.26

Let  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{n-1}$  be linearly independent vectors in  $\mathbb{E}^n$ . Let  $\mathbf{p}_0$  be a point of  $\mathbb{R}^n$ . Let  $\mathbf{H}$  be the hyperplane plane through  $\mathbf{p}_0$  spanned by  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{n-1}$ . If  $\mathbf{p}$  is any point in  $\mathbb{R}^n$ , show that the distance from  $\mathbf{p}$  to  $\mathbf{H}$ , that is,  $\inf\{\|\mathbf{p} - \mathbf{q}\| \mid \mathbf{q} \in \mathbf{H}\}$ , is given by the length of the vector  $\text{proj}_{\mathbf{v}}(\mathbf{p} - \mathbf{p}_0)$  where  $\mathbf{v}$  is the vector obtained in Theorem 2.5.21. Specialize this to obtain formulas for the distance from a point to a line in  $\mathbb{R}^2$  and from a point to a plane in  $\mathbb{R}^3$ .

We are given that  $\mathbf{v}$  is orthogonal to  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{n-1}$  and that  $\mathbf{H}$  is the hyperplane through  $\mathbf{p}_0$  spanned by  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{n-1}$ .

Since  $\mathbf{p}$  and  $\mathbf{p}_0$  are points, we can think of  $\text{proj}_{\mathbf{v}}(\mathbf{p} - \mathbf{p}_0)$  as a projection of the vector  $(\mathbf{p} - \mathbf{p}_0)$  onto  $\mathbf{v}$  where  $(\mathbf{p} - \mathbf{p}_0)$  is the hypotenuse of a right triangle formed by  $\text{proj}_{\mathbf{v}}(\mathbf{p} - \mathbf{p}_0)$ . Which makes the  $\text{proj}_{\mathbf{v}}(\mathbf{p} - \mathbf{p}_0)$  the line adjacent to the angle formed by  $\text{proj}_{\mathbf{v}}(\mathbf{p} - \mathbf{p}_0)$  which is parallel with  $\mathbf{v}$  which is orthogonal to  $\mathbf{H}$ . Since  $\mathbf{q} \in \mathbf{H}$



This must be the shorted distance  $\mathbf{p} - \mathbf{q}$  to  $\mathbf{H}$ , because if you took any other distance to the plane, it wouldn't form right angle with the plane and result in a greater distance.

Specializing this for  $\mathbb{R}^2$  and  $\mathbb{R}^3$ . We consider the  $\text{proj}_{\mathbf{v}}(\mathbf{p} - \mathbf{p}_0)$ . Let the vector  $\mathbf{f} = (\mathbf{p} - \mathbf{p}_0)$ . We only care about the length of the line so we should take it's norm, which is given

$$\|\text{proj}_{\mathbf{v}}(\mathbf{f})\| = \frac{|\langle \mathbf{f}, \mathbf{v} \rangle|}{\|\mathbf{v}\|}$$

For  $\mathbb{R}^2$ , define vectors  $\mathbf{v}_1$  and  $\mathbf{v}$ . Let  $\mathbf{v}$  be non zero orthogonal to  $\mathbf{v}_1$ . This means the pair  $\{\mathbf{v}_1, \mathbf{v}\}$  is a basis for  $\mathbb{R}^2$ . Let  $\mathbf{H}$  be the plane through  $\mathbf{p}_0$  be a point spanned by  $\mathbf{v}_1$ , and contain points  $(p_{01}, p_{02})$ . Let  $\mathbf{p} = (p_1, p_2)$  to be any point in  $\mathbb{R}^2$ . by Let  $d$  be the distance we are trying to find.

$$\begin{aligned} d &= \frac{|\langle \mathbf{f}, \mathbf{v} \rangle|}{\|\mathbf{v}\|} \\ &= \frac{|(f_1 v_1 + f_2 v_2)|}{\|\mathbf{v}\|} \\ &= \frac{|(p_1 - p_{01})v_1 + (p_2 - p_{02})v_2|}{\|\mathbf{v}\|} \\ &= \frac{|(p_1 - p_{01})v_1 + (p_2 - p_{02})v_2|}{\sqrt{v_1^2 + v_2^2}} \end{aligned}$$

For  $\mathbb{R}^3$ , we have a similar result. define vectors  $\mathbf{v}_1, \mathbf{v}_2$  to be linearly independent. Let  $\mathbf{v}$  be the determinant of  $\mathbf{v}_1, \mathbf{v}_2$ . Resulting in  $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}\}$  as a basis  $\mathbb{R}^3$  Let  $\mathbf{H}$  be the plane through the point  $\mathbf{p}_0$  spanned by  $\mathbf{v}_1, \mathbf{v}_2$ , and contain points  $(p_{01}, p_{02}, p_{03})$ . Let  $\mathbf{p} = (p_1, p_2, p_3)$  to be any point in  $\mathbb{R}^3$ . by Let  $d$  be the distance we are trying to find.

$$\begin{aligned} d &= \frac{|\langle \mathbf{f}, \mathbf{v} \rangle|}{\|\mathbf{v}\|} \\ &= \frac{|(f_1 v_1 + f_2 v_2 + f_3 v_3)|}{\|\mathbf{v}\|} \\ &= \frac{|(p_1 - p_{01})v_1 + (p_2 - p_{02})v_2 + (p_3 - p_{03})v_3|}{\|\mathbf{v}\|} \\ &= \frac{|(p_1 - p_{01})v_1 + (p_2 - p_{02})v_2 + (p_3 - p_{03})v_3|}{\sqrt{v_1^2 + v_2^2 + v_3^2}} \end{aligned}$$

### Exercise 2.5.30

Consider the vectors  $\mathbf{v}_1 = (1, 1, -1, 0)$ ,  $\mathbf{v}_2 = (1, 0, 0, -1)$  and  $\mathbf{v}_3 = (0, 1, 1, 1)$  in  $\mathbb{E}^4$ .

1. Use the Gram-Schmidt orthogonalization process on these three vectors to produce a set of three mutually orthogonal vectors that span the same subspace.

Let  $\mathbf{v}_1 = \tilde{\mathbf{v}}_1$

For  $\tilde{\mathbf{v}}_2 = \mathbf{v}_2 - \text{proj}_{\tilde{\mathbf{v}}_1}(\mathbf{v}_2)$

$$\text{proj}_{\tilde{\mathbf{v}}_1}(\mathbf{v}_2) = \frac{\langle \mathbf{v}_2, \tilde{\mathbf{v}}_1 \rangle}{\|\tilde{\mathbf{v}}_1\|} \frac{\tilde{\mathbf{v}}_1}{\|\tilde{\mathbf{v}}_1\|}$$

$$\begin{aligned}\langle \mathbf{v}_2, \tilde{\mathbf{v}}_1 \rangle &= (1)(1) + (0)(1) + (0)(-1) + (-1)(0) \\ &= 1 \\ \|\tilde{\mathbf{v}}_1\| &= \sqrt{1^2 + 1^2 + (-1)^2 + 0^2} \\ &= \sqrt{3}\end{aligned}$$

$$\begin{aligned}\text{proj}_{\tilde{\mathbf{v}}_1}(\mathbf{v}_2) &= \frac{1}{\sqrt{3}} \frac{\tilde{\mathbf{v}}_1}{\sqrt{3}} \\ &= \frac{1}{3} \tilde{\mathbf{v}}_1 \\ &= \frac{1}{3} \begin{bmatrix} 1 & 1 & -1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & 0 \end{bmatrix}\end{aligned}$$

$$\begin{aligned}\tilde{\mathbf{v}}_2 &= \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} - \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ -\frac{1}{3} \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & \frac{1}{3} & -1 \end{bmatrix}\end{aligned}$$

For  $\tilde{\mathbf{v}}_3 = \mathbf{v}_3 - \text{proj}_{\tilde{\mathbf{v}}_1}(\mathbf{v}_3) - \text{proj}_{\tilde{\mathbf{v}}_2}(\mathbf{v}_3)$

$$\text{proj}_{\tilde{\mathbf{v}}_1}(\mathbf{v}_3) = \frac{\langle \mathbf{v}_3, \tilde{\mathbf{v}}_1 \rangle}{\|\tilde{\mathbf{v}}_1\|} \frac{\tilde{\mathbf{v}}_1}{\|\tilde{\mathbf{v}}_1\|}$$

$$\begin{aligned}\langle \mathbf{v}_3, \tilde{\mathbf{v}}_1 \rangle &= (0)(1) + (1)(1) + (1)(-1) + (1)(0) \\ &= 0 \\ \|\tilde{\mathbf{v}}_1\| &= \sqrt{3}\end{aligned}$$

$$\begin{aligned}
\text{proj}_{\tilde{\mathbf{v}}_1}(\mathbf{v}_3) &= \frac{0}{\sqrt{3}} \frac{\tilde{\mathbf{v}}_1}{\sqrt{3}} \\
&= 0\tilde{\mathbf{v}}_1 \\
&= 0 \begin{bmatrix} 1 & 1 & -1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}
\end{aligned}$$

$$\text{proj}_{\tilde{\mathbf{v}}_2}(\mathbf{v}_3) = \frac{\langle \mathbf{v}_3, \tilde{\mathbf{v}}_2 \rangle}{\|\tilde{\mathbf{v}}_2\|} \frac{\tilde{\mathbf{v}}_2}{\|\tilde{\mathbf{v}}_2\|}$$

$$\begin{aligned}
\langle \mathbf{v}_3, \tilde{\mathbf{v}}_2 \rangle &= (0)\left(\frac{2}{3}\right) + (1)\left(-\frac{1}{3}\right) + (1)\left(\frac{1}{3}\right) + (1)(-1) \\
&= -1 \\
\|\tilde{\mathbf{v}}_2\| &= \sqrt{\frac{2^2}{3} + -\frac{1^2}{3} + \frac{1^2}{3} + -1^2} \\
&= \sqrt{\frac{5}{3}}
\end{aligned}$$

$$\begin{aligned}
\text{proj}_{\tilde{\mathbf{v}}_2}(\mathbf{v}_3) &= -\frac{1}{\sqrt{\frac{5}{3}}} \frac{\tilde{\mathbf{v}}_2}{\sqrt{\frac{5}{3}}} \\
&= -\frac{3}{5} \tilde{\mathbf{v}}_2 \\
&= -\frac{3}{5} \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & \frac{1}{3} & -1 \end{bmatrix} \\
&= \begin{bmatrix} \frac{6}{15} & -\frac{3}{15} & \frac{3}{15} & \frac{3}{5} \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
\tilde{\mathbf{v}}_3 &= \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} \frac{6}{15} \\ -\frac{3}{15} \\ \frac{3}{15} \\ \frac{3}{5} \end{bmatrix} \\
&= \begin{bmatrix} -\frac{6}{15} & \frac{18}{15} & \frac{12}{15} & \frac{2}{5} \end{bmatrix}
\end{aligned}$$

2. Extend the set of three vectors produced in part 1 to a mutually orthogonal basis for

Consider the matrix comprised of  $\tilde{\mathbf{v}}_1, \tilde{\mathbf{v}}_2, \tilde{\mathbf{v}}_3$  and vectors  $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4 \in \mathbb{E}^4$  where 1 is in the  $j$ th term and 0 otherwise.

$$\begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 & \mathbf{e}_4 \\ 1 & 1 & -1 & 0 \\ \frac{2}{3} & -\frac{1}{3} & \frac{1}{3} & -1 \\ -\frac{6}{15} & \frac{18}{15} & \frac{12}{15} & \frac{2}{5} \end{bmatrix} \quad (7)$$

The vector that will complete the basis can be created by taking the determinant of the matrix (7).

Step 1

$$\begin{aligned} \mathbf{e}_1 \begin{bmatrix} 1 & -1 & 0 \\ -\frac{1}{3} & \frac{1}{3} & -1 \\ \frac{18}{15} & \frac{12}{15} & \frac{2}{5} \end{bmatrix} - \mathbf{e}_2 \begin{bmatrix} 1 & -1 & 0 \\ \frac{2}{3} & \frac{1}{3} & -1 \\ -\frac{6}{15} & \frac{12}{15} & \frac{2}{5} \end{bmatrix} \\ + \mathbf{e}_3 \begin{bmatrix} 1 & 1 & 0 \\ \frac{2}{3} & -\frac{1}{3} & -1 \\ -\frac{6}{15} & \frac{18}{15} & \frac{2}{5} \end{bmatrix} - \mathbf{e}_4 \begin{bmatrix} 1 & 1 & -1 \\ \frac{2}{3} & -\frac{1}{3} & \frac{1}{3} \\ -\frac{6}{15} & \frac{18}{15} & \frac{12}{15} \end{bmatrix} \end{aligned}$$

Then the next step...

$$\begin{aligned} \mathbf{e}_1 \left( 1 \begin{bmatrix} \frac{1}{3} & -1 \\ \frac{12}{15} & \frac{2}{5} \end{bmatrix} - -1 \begin{bmatrix} -\frac{1}{3} & -1 \\ \frac{18}{15} & \frac{2}{5} \end{bmatrix} - \right. \\ \mathbf{e}_2 \left( 1 \begin{bmatrix} \frac{1}{3} & -1 \\ \frac{12}{15} & \frac{2}{5} \end{bmatrix} - -1 \begin{bmatrix} \frac{2}{3} & -1 \\ -\frac{6}{15} & \frac{2}{5} \end{bmatrix} + \right. \\ \mathbf{e}_3 \left( 1 \begin{bmatrix} -\frac{1}{3} & -1 \\ \frac{18}{15} & \frac{2}{5} \end{bmatrix} - 1 \begin{bmatrix} \frac{2}{3} & -1 \\ -\frac{6}{15} & \frac{2}{5} \end{bmatrix} - \right. \\ \left. \left. \mathbf{e}_4 \left( 1 \begin{bmatrix} -\frac{1}{3} & \frac{1}{3} \\ \frac{18}{15} & \frac{12}{15} \end{bmatrix} - 1 \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ -\frac{6}{15} & \frac{12}{15} \end{bmatrix} + -1 \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{6}{15} & \frac{18}{15} \end{bmatrix} \right) \right) \end{aligned}$$

...eventually arriving at...

$$\mathbf{e}_1 \left( -\frac{2}{5} \right) - \mathbf{e}_2 \left( \frac{4}{5} \right) + \mathbf{e}_3 \left( \frac{6}{5} \right) - \mathbf{e}_4 (-2)$$

We now compute our new vector to be  $(-\frac{2}{5}, -\frac{4}{5}, \frac{6}{5}, 2)$  and define it as  $\mathbf{v}$ . Which gives us the basis  $\{\tilde{\mathbf{v}}_1, \tilde{\mathbf{v}}_2, \tilde{\mathbf{v}}_3, \mathbf{v}\}$  over  $\mathbb{E}^4$ .

In order to make this a mutually orthogonal basis for  $\mathbb{E}^4$ , apply the Gram-Schmidt orthogonalization process once again.

For  $\tilde{\mathbf{v}} = \mathbf{v} - \text{proj}_{\tilde{\mathbf{v}}_1}(\mathbf{v}) - \text{proj}_{\tilde{\mathbf{v}}_2}(\mathbf{v}) - \text{proj}_{\tilde{\mathbf{v}}_3}(\mathbf{v})$

$$\langle \mathbf{v}, \tilde{\mathbf{v}}_1 \rangle = \left( -\frac{2}{5} \right) (1) + \left( -\frac{4}{5} \right) (1) + \left( \frac{6}{5} \right) (-1) + (2)(0) = -\frac{12}{5}$$

$$\begin{aligned}
\text{proj}_{\tilde{\mathbf{v}}_1}(\mathbf{v}) &= -\frac{-\frac{12}{5}}{\sqrt{3}} \frac{\tilde{\mathbf{v}}_1}{\sqrt{3}} \\
&= -\frac{4}{5} \tilde{\mathbf{v}}_1 \\
&= -\frac{4}{5} \begin{bmatrix} 1 & 1 & -1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} -\frac{4}{5} & -\frac{4}{5} & \frac{4}{5} & 0 \end{bmatrix}
\end{aligned}$$

$$\langle \mathbf{v}, \tilde{\mathbf{v}}_2 \rangle = \left(-\frac{2}{5}\right)\left(\frac{2}{3}\right) + \left(-\frac{4}{5}\right)\left(-\frac{1}{3}\right) + \left(\frac{6}{5}\right)\left(\frac{1}{3}\right) + (2)(-1) = -\frac{8}{5}$$

$$\|\tilde{\mathbf{v}}_2\| = \sqrt{\frac{5}{3}}$$

$$\begin{aligned}
\text{proj}_{\tilde{\mathbf{v}}_2}(\mathbf{v}) &= \frac{-\frac{8}{5}}{\sqrt{\frac{5}{3}}} \frac{\tilde{\mathbf{v}}_2}{\sqrt{\frac{5}{3}}} \\
&= -\frac{24}{25} \tilde{\mathbf{v}}_2 \\
&= -\frac{24}{25} \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & \frac{1}{3} & -1 \end{bmatrix} \\
&= \begin{bmatrix} -\frac{16}{25} & \frac{8}{25} & -\frac{8}{25} & \frac{24}{25} \end{bmatrix}
\end{aligned}$$

$$\langle \mathbf{v}, \tilde{\mathbf{v}}_3 \rangle = \left(-\frac{2}{5}\right)\left(-\frac{6}{15}\right) + \left(-\frac{4}{5}\right)\left(\frac{18}{15}\right) + \left(\frac{6}{5}\right)\left(\frac{12}{15}\right) + (2)\left(\frac{2}{5}\right) = -\frac{8}{5}$$

$$\|\tilde{\mathbf{v}}_3\| = \sqrt{\frac{12}{5}}$$

$$\begin{aligned}
\text{proj}_{\tilde{\mathbf{v}}_3}(\mathbf{v}) &= \frac{\frac{24}{25}}{\sqrt{\frac{5}{12}}} \frac{\tilde{\mathbf{v}}_3}{\sqrt{\frac{5}{12}}} \\
&= \frac{2}{5} \tilde{\mathbf{v}}_3 \\
&= \frac{2}{5} \begin{bmatrix} -\frac{12}{75} & -\frac{1}{3} & \frac{1}{3} & -1 \end{bmatrix} \\
&= \begin{bmatrix} -\frac{4}{25} & \frac{12}{25} & \frac{6}{25} & \frac{4}{25} \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
\tilde{\mathbf{v}} &= \begin{bmatrix} -\frac{2}{5} \\ -\frac{4}{5} \\ \frac{6}{5} \\ 2 \end{bmatrix} - \begin{bmatrix} -\frac{4}{5} \\ -\frac{4}{5} \\ \frac{4}{5} \\ 0 \end{bmatrix} - \begin{bmatrix} -\frac{16}{25} \\ \frac{8}{25} \\ -\frac{8}{25} \\ \frac{24}{25} \end{bmatrix} - \begin{bmatrix} -\frac{4}{25} \\ \frac{12}{25} \\ \frac{6}{25} \\ \frac{4}{25} \end{bmatrix} \\
&= \begin{bmatrix} \frac{6}{5} & -\frac{4}{5} & \frac{12}{25} & \frac{22}{25} \end{bmatrix}
\end{aligned}$$

We now have a mutually orthogonal basis for  $\mathbb{E}^4$ .

3. Normalize your basis so that it becomes an orthonormal basis for  $\mathbb{E}^4$

$$\frac{\tilde{\mathbf{v}}_1}{\|\tilde{\mathbf{v}}_1\|} = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} & 0 \end{bmatrix}$$

$$\frac{\tilde{\mathbf{v}}_2}{\|\tilde{\mathbf{v}}_2\|} = \begin{bmatrix} \frac{\frac{2}{3}}{\sqrt{\frac{5}{3}}} & \frac{\frac{-1}{3}}{\sqrt{\frac{5}{3}}} & \frac{\frac{1}{3}}{\sqrt{\frac{5}{3}}} & \frac{\frac{-1}{3}}{\sqrt{\frac{5}{3}}} \end{bmatrix}$$

$$\frac{\tilde{\mathbf{v}}_3}{\|\tilde{\mathbf{v}}_3\|} = \begin{bmatrix} \frac{\frac{-2}{3}}{\sqrt{\frac{12}{5}}} & \frac{\frac{6}{5}}{\sqrt{\frac{12}{5}}} & \frac{\frac{4}{5}}{\sqrt{\frac{12}{5}}} & \frac{\frac{2}{5}}{\sqrt{\frac{12}{5}}} \end{bmatrix}$$

$$\frac{\tilde{\mathbf{v}}}{\|\tilde{\mathbf{v}}\|} = \begin{bmatrix} \frac{\frac{6}{5}}{\sqrt{\frac{1928}{625}}} & \frac{\frac{-4}{5}}{\sqrt{\frac{1928}{625}}} & \frac{\frac{12}{25}}{\sqrt{\frac{1928}{625}}} & \frac{\frac{22}{25}}{\sqrt{\frac{1928}{625}}} \end{bmatrix}$$