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# REAL ANALYSIS SECOND WEEK

## Exercise 1.6.35

1. Show the  $\emptyset$  and  $\mathbb{R}$  are the only subsets of  $\mathbb{R}$  that are both open and closed.

*Proof.* Let  $S$  be a non-empty open and closed set of  $\mathbb{R}$ . Fix  $x_0 \in S$  and  $S \neq \mathbb{R}$ . Then,  $\exists y \in \mathbb{R} \setminus S$ . Without loss of generality we may assume  $y > x_0$ .

Therefore we can form a set

$$I = \{x \in \mathbb{R} | x > x_0, x \notin S\}$$

By construction this set is bounded below by  $x_0$  and not empty because  $y > x_0$ , therefore  $y \in I$ . Therefore we can let  $i = \inf I$

greatest lower bound

Suppose  $i \in S$ . Since  $S$  is open it contains the open interval  $(i - \epsilon, i + \epsilon)$  for  $\epsilon > 0$ . However, this interval contradicts that  $i = \inf I$  because it implies a sequence  $i_n > i, |i - i_n| < \frac{1}{n}$ , where  $i_n \in I$ , which means  $i_n \notin S$ . Which is not possible because  $[i, i + \epsilon) \subset S$

Now suppose  $i \notin S$ . Since  $S$  is closed,  $S^c$  which means that it contains an open interval  $(i - \epsilon, i + \epsilon)$ , but this contradicts the definition  $i = \inf I$  because then we can find  $i - \frac{\epsilon}{2}$  that is in  $S^c$ .

Therefore  $S = \mathbb{R}$ . □

2. Show that every non-empty open set in  $\mathbb{R}$  can be written as a countable union of pairwise disjoint open intervals.

*Proof.* Let  $U \subset \mathbb{R}$ . Let  $\mathcal{O}$  be the set of open intervals that are a subsection of  $U$ . For  $I, J \in \mathcal{O}$  define  $I \sim J$  iff there are

$$I_0 = I, I_1, I_2, \dots, I_n = J \in \mathcal{O}$$

Such that  $I_k \cap I_{k+1} \neq \emptyset$  for  $k = 0, \dots, n - 1$ . Then  $\sim$  defines an equivalence relation on  $\mathcal{O}$ . For  $I \in \mathcal{O}$  let  $[I]$  be the  $\sim$  of  $I$ . Then  $\{[I] \text{ for } I \in \mathcal{O}\}$  is decomposition of  $U$  into pairwise disjoint intervals. By construction, these intervals are countable. □

3. Show that an arbitrary union of open sets in  $\mathbb{R}$  is open in  $\mathbb{R}$ .

*Proof.* Suppose  $\{A_i \subset \mathbb{R} | i \in I\}$  is an arbitrary collection of open sets.

If  $x \in \bigcup A_i$  then  $x \in A_i$  for some  $i \in I$ . Since  $A_i$  is open, there  $(x - \epsilon, x + \epsilon) \subset A_i$  for  $\epsilon > 0$ . Therefore,

$$(x - \epsilon, x + \epsilon) \subset \bigcup_{i \in I} A_i$$

since  $A_i \subset \mathbb{R}$  we arrive at our conclusion: The arbitrary union of open sets,  $\bigcup_{i \in I} A_i$  is open in  $\mathbb{R}$ .  $\square$

4. Show that a finite intersection of open sets in  $\mathbb{R}$  is open in  $\mathbb{R}$

*Proof.* Suppose  $\{A_i \subset \mathbb{R} | i = 1, 2, \dots, n\}$  is a finite collection of open sets. If  $x \in \bigcap_{i=1}^n A_i$  then  $x \in A_i$  for every  $1 \leq i \leq n$ . Since  $A_i$  is open, there are  $\epsilon_i > 0$  such that  $(x - \epsilon_i, x + \epsilon_i) \subset A_i$ .

Let  $\epsilon = \min(\epsilon_1, \dots, \epsilon_n) > 0$ . This shows  $(x - \epsilon, x + \epsilon) \subset \bigcap_{i=1}^n A_i$ .  $\square$

5. Show, by example, that an infinite intersection of open sets is not necessarily open. The open interval  $(-\frac{1}{n}, \frac{1}{n})$  is open  $\forall n \in \mathbb{N}$ . However, its intersection

$$\bigcap_{n=1}^{\infty} (-\frac{1}{n}, \frac{1}{n}) = \{0\} = [0, 0]$$

which is closed.  $\square$

6. Show that an arbitrary intersection of closed sets in  $\mathbb{R}$  is a closed set in  $\mathbb{R}$ .

*Proof.* Let  $\{A_i \subset \mathbb{R}, i \in I\}$  be an arbitrary collection of closed sets. Let  $\bigcap_{i \in I} A_i$  be the intersection of the closed sets. If this intersection is  $\emptyset$  we are done. Supposing it is not, we'll continue.

By definition  $\mathbb{R} \setminus \bigcap_{i \in I} A_i = \bigcup_{i \in I} (\mathbb{R} \setminus A_i)$ .

DeMorgan's Law

Since  $\{A_i\} \subset \mathbb{R}$  are closed,  $(\mathbb{R} \setminus A_i) \subset \mathbb{R}$  is made up of open sets. So we have an arbitrary union of open sets in  $\mathbb{R}$  which we have already shown to be open. This means  $\mathbb{R} \setminus \bigcap_{i \in I} A_i$  is also open.

Therefore its complement  $\bigcap_{i \in I} A_i$  is closed.  $\square$

7. Show that a finite union of closed sets in  $\mathbb{R}$  is a closed set in  $\mathbb{R}$

*Proof.* Let  $\{A_i \subset \mathbb{R}, i = 1, 2, \dots, n\}$  be a finite collection of some  $n \in \mathbb{N}$  closed sets in  $\mathbb{R}$ . Let  $\bigcup_{i=1}^n A_i$  be the union of the finite closed subsets.

Summoning DeMorgan's Law, once more

$$\mathbb{R} \setminus \bigcup_{i=1}^n A_i = \bigcap_{i=1}^n (\mathbb{R} \setminus A_i)$$

Since  $\{A_i\} \subset \mathbb{R}$  are closed,  $(\mathbb{R} \setminus A_i) \subset \mathbb{R}$  is made up of open sets. This means we have a finite intersection of open sets, which we have already shown to be open. Therefore  $\mathbb{R} \setminus \bigcup_{i=1}^n A_i$  is also open. Which allows us to conclude, the complement,  $\bigcup_{i=1}^n A_i$  must be closed.

□

8. Show, by example, that an infinite union of closed sets in  $\mathbb{R}$  is not necessarily closed in  $\mathbb{R}$

One example is  $\bigcup_{n=1}^{\infty} [\frac{1}{n}, n]$ . The union of this is open because you will never find a point of the union that lives at the boundary of the union.

Also, consider that  $\mathbb{R}$  can be constructed of an infinite union of closed sets. For example, each one point set of one point in  $\mathbb{R}$ , however we know that  $\mathbb{R}$  is not closed.

### Exercise 1.6.36

Show that a subset of  $\mathbb{R}$  is closed iff it contains all its accumulation points.

*Proof.* " $\Rightarrow$ " (Given in class)

Suppose  $S$  is closed and  $x$  is an accumulation point. We prove by contradiction. Suppose  $x \notin S$  then  $x \in S^c$ . This means,  $\exists \epsilon$  such that

$$(x - \epsilon, x + \epsilon) \subset S^c$$

That is to say  $(x - \epsilon, x + \epsilon) \cap S = \emptyset$ . Which contradicts that  $x$  is an accumulation point. □

*Proof.* " $\Leftarrow$ " Suppose  $S$  contains all its limit points and let  $S$  be open, prove by contradiction. Therefore,  $\exists x \in S^c$  such that  $(x - \epsilon, x + \epsilon)$  contains at least one element of  $S$ . Symbolically written as

$$(x - \epsilon, x + \epsilon) \cap S \neq \emptyset, \forall \epsilon > 0$$

For all  $n \in \mathbb{N}$  let  $x_n \in (x, \frac{1}{n}) \cap S$

Notice,  $x_n$  is a sequence in  $S$  that converges to  $x \notin S$ , meaning that  $x$  is an accumulation point of  $S$  that is not contained in the open  $S$ , so  $S$  must be closed. □