3.6.25

i) Show that $\mathbb{Q}$ is dense in $\mathbb{R}$.

Consider $a, b \in \mathbb{R}$. WLOG, let $a < b$.

We want to show that $\exists\, q \in \mathbb{Q}$ s.t. $a < q < b$.

By the Archimedean principle, $\exists\, N \in \mathbb{N}$ s.t. $\frac{1}{N} < b - a$

Claim: $\exists\, k \in \mathbb{Z}$ s.t. $a < \frac{k}{N} < b$

For contradiction, suppose not.

Then $\exists\, k \in \mathbb{Z}$ s.t. $\frac{k}{N} \leqslant a$ and $\frac{k+1}{N} \geqslant b$

Then $b - a \leqslant \frac{k+1}{N} - \frac{k}{N} = \frac{1}{N}$ which contradicts the Archimedean property.

ii) Show that the Dyadic numbers are dense in $\mathbb{R}$.

Consider $a, b \in \mathbb{R}$. We want to show that $\exists\, d, N \in \mathbb{Z}$ s.t. $a < \frac{d}{2^N} < b$. Suppose not, for contradiction.

(Note that by the Archimedean principle, $\exists\, N \in \mathbb{N}$ s.t. $\frac{1}{N} < b - a$

$\exists\, d \in \mathbb{Z}$ s.t. $\frac{d}{2^N} \leqslant a$ and $\frac{d+1}{2^N} \geqslant b$.

Then $b - a \leqslant \frac{d+1}{2^N} - \frac{d}{2^N} = \frac{1}{2^N} < \frac{1}{N}$

$\therefore$ contradicting the Archimedean principle.

3.6.26 i) We want to show that $\forall x \in X$, $\exists x_0 \in X$, $\forall r > 0$, $\forall B_r(x)$, $x_0 \in B_r(x)$. But here simply pick $x_0 = x$.

ii) Suppose, for contradiction, $\exists A$ s.t. $A \neq X$ and is a dense subset of X.

That is, $\forall x \in X$, $\exists a \in A$ s.t. $\forall r > 0$, $a \in B_r(x)$.

Since $A \neq X$, $\exists x_0 \in X$ s.t. $x_0 \notin A$. For this x_0, we choose $r = \frac{1}{2}$. Then $B_{\frac{1}{2}}(x_0) = \{x_0\} \notin A$.

This contradicts the definition of A as a dense subset of X.

iii) Suppose X is the only dense subset of X.

Then $A = X - \{b\}$ is not dense.

Then $\{b\}$ is an open set.

We can generalize this to say that all singletons are open.

That implies that all subsets are open.

$\therefore$ satisfying the definition of a discrete space

3.6.30) X' is dense because the image of a dense set is dense.
Since X is countable, $F(X) = X'$ is also countable because the set of the image can never be larger than the set of domain
We have X' is dense and countable $\therefore$ X' is seperable.

3.6.31) $(R, \text{discrete})$ is not seperable.
Let us do a proof by contradiction. Suppose A is dense and countable.
let $A \subseteq R$. Let $x \in \mathbb{R}$ but $x \notin A$.
Note that if $A = \mathbb{R}$, then A is not countable.
Let us consider $B_r(x)$. If $r \leq 1$, $B_1(x) = \{x\}$ but $x \notin A$.
$\therefore$ Then A is not dense in $\mathbb{R}$. Contradiction.

Extra Q1) let $\varepsilon > 0$. let $|x-y| < \delta$. Choose $\delta = \frac{\varepsilon}{L}$.

$$
\begin{aligned}
|f_n(x) - f_n(y)| &\leq L\,|x-y| \quad \text{(as it is uniformly Lipschitz)} \\
&< L \cdot \delta \\
&= L \cdot \frac{\varepsilon}{L} = \varepsilon \quad \therefore f_n \text{ is equicontinuous}
\end{aligned}
$$

Since f_n is uniformly bounded and equicontinuous, f_n is sequentially compact.

Extra Q2) let $\varepsilon > 0$. let $|x-y| < \delta$. Choose $\delta = \left(\frac{\varepsilon}{C}\right)^{\frac{1}{\alpha}}$

$$
\begin{aligned}
|f_n(x) - f_n(y)| &< C\,|x-y|^{\alpha} \quad \text{(as it is uniformly } \alpha\text{- Holder)} \\
&< C\,\delta^{\alpha} \\
&= C\left[\left(\tfrac{\varepsilon}{C}\right)^{1/\alpha}\right]^{\alpha} \\
&= \varepsilon
\end{aligned}
$$

Since f_n is uniformly bounded and equicontinuous, f_n is sequentially compact

3.7.6)

i) $f(x) = x + 1$

ii) $f(x) = x^2$

iii) We will use the Intermediate Value Theorem.

Consider $g(x) = f(x) - x$

$g(0) = f(0) \geqslant 0$ and $g(1) = f(1) - 1 \leqslant 0$

Thus $\exists$ $g(x)$ s.t. $g(1) \leqslant g(x) \leqslant g(0)$ and $g(x) = 0$

That is $g(x) = f(x) - x = 0$ $\therefore f(x) = x$