JOE SEIDEL

REAL ANALYSIS
THIRD WEEK

Exercise 1.9.8

Give examples to show that if $r = 1$ in the statement of the Ratio Test, anything may happen.

First consider the harmonic series, $\sum_{n=1}^{\infty} \frac{1}{n}$. The $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = 1$

$$\lim_{n\to\infty} |\frac{\frac{1}{n+1}}{\frac{1}{n}}| = \lim_{n\to\infty} |\frac{n}{n+1}| = 1$$

However, we know that the series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Next consider $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$. Again $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = 1$

$$\begin{aligned}
\lim_{n\to\infty} |\frac{\frac{(-1)^{n+2}}{n+1}}{\frac{(-1)^{n+1}}{n}}| &= \lim_{n\to\infty} |\frac{(-1)^{n+2}}{n+1}\frac{n}{(-1)^{n+1}}| \\
&= \lim_{n\to\infty} |\frac{-n}{n+1}| \\
&= 1
\end{aligned}$$

This series, $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges since $|\sum_{k=m+1}^{n} \frac{(-1)^{k+1}}{k}| < \frac{1}{m}$ but not absolutely because $\sum_{n=1}^{\infty} |\frac{(-1)^n}{n}|$ goes to infinity.

Finally, consider the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ This too has $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = 1$.

$$\begin{aligned}
\lim_{n\to\infty} |\frac{\frac{1}{n+1^2}}{\frac{1}{n^2}}| &= \lim_{n\to\infty} |\frac{1}{(n+1)^2}\frac{n^2}{1}| \\
&= \lim_{n\to\infty} \frac{n^2}{(n+1)^2} \\
&= 1
\end{aligned}$$

The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is absolutely convergent because $\sum_{n=1}^{\infty} |\frac{1}{n^2}|$ converges.

Exercise 1.9.20

Give examples to show that if $r = 1$ in the statement of the Root Test, anything my happen.

Borrowing from the ideas in the last exercise, consider the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ and $\sum_{n=1}^{\infty} \frac{1}{n}$. Both have $r = 1$.

$$\begin{aligned}
\limsup_{n\to\infty} |\frac{1}{n}|^{\frac{1}{n}} &= \limsup_{n\to\infty} e^{\frac{1}{n}\ln\frac{1}{n}} \\
&= e^{(\limsup_{n\to\infty} -\ln n)(\limsup_{n\to\infty} \frac{1}{n})} \\
&= e^{(\limsup_{n\to\infty} -\ln n)(0)} \\
&= 1
\end{aligned}$$

$$\begin{aligned}\limsup_{n\to\infty} |\frac{1}{n^2}|^{\frac{1}{n}} &= \limsup_{n\to\infty} e^{\frac{1}{n}\ln\frac{1}{n^2}} \\ &= e^{(\limsup_{n\to\infty} -2\ln n)(\limsup_{n\to\infty}\frac{1}{n})} \\ &= e^{(\limsup_{n\to\infty} -2\ln n)(0)} \\ &= 1\end{aligned}$$

Anything may happen when $r = 1$

Exercise 1.9.26

Determine the radius of convergence of the following power series:

$$r = \limsup_{n\to\infty} |a_n|^{\frac{1}{n}}$$

Where the radius of convergence is $\frac{1}{r}$ if $r > 0$, ∞ if $r = 0$ and 0 if lim sup does not exist, $(r = \infty)$.

1. $\sum_{n=1}^{\infty} \frac{z^n}{n!}$

$$\sum_{n=1}^{\infty} \frac{z^n}{n!} = \sum_{n=1}^{\infty} \frac{1}{n!} z^n$$

$$\lim_{n\to\infty} |\frac{1}{n!}|^{\frac{1}{n}} = 0$$

Radius of convergence is ∞

2. $\sum_{n=2}^{\infty} \frac{z^n}{\ln(n)}$

$$\sum_{n=1}^{\infty} \frac{z^n}{\ln(n)} = \sum_{n=2}^{\infty} \frac{1}{\ln(n)} z^n$$

$$\lim_{n\to\infty} |\frac{1}{\ln(n)}|^{\frac{1}{n}} = 1$$

Radius of convergence is 1.

3. $\sum_{n=1}^{\infty} \frac{n^n}{n!} z^n$

$e^n = \sum_{n=1}^{\infty} \frac{n^n}{n!}$

$$\lim_{n\to\infty} |\frac{n^n}{n!}|^{\frac{1}{n}} = e$$

Radius of convergence is $\frac{1}{e}$

Exercise 2.5.8

Prove that the scalar product is a positive definite symmetric bilinear form on $\mathbb{E}^n$.

Proof. The scalar product of vectors $\mathbf{v} = (v_1, v_2, v_3, ..., v_n)$ and $\mathbf{w} = (w_1, w_2, w_3, ..., w_n)$ is $\langle \mathbf{v}, \mathbf{w} \rangle = v_1 w_1 + v_2 w_2 + v_3 w_3 + ... + v_n w_n$.

Let $V = \mathbb{E}^n$ be a vector space over $F = \mathbb{R}$. A bilinear form $\langle \cdot, \cdot \rangle$ on V is a map

$$\langle \cdot, \cdot \rangle : V \times V \to F$$

that satisfies linearity in both variables. That is, for all $\mathbf{v}, \mathbf{v}_1 \mathbf{v}_2, \mathbf{w}, \mathbf{w}_1, \mathbf{w}_2 \in V$ and all $\alpha \in F$

$$\begin{aligned}
\langle \mathbf{v}_1 + \mathbf{v}_2, \mathbf{w} \rangle &= (v_{1_1} + v_{2_1}) w_1 + (v_{1_2} + v_{2_2}) w_2 + (v_{1_3} + v_{2_3}) w_3 + ... + (v_{1_n} + v_{2_n}) w_n \\
&= v_{1_1} w_1 + v_{2_1} w1 + v_{1_2} w_2 + v_{2_2} w2 + v_{1_3} w_3 + v_{2_3} w3 + ... + v_{1_n} w_n + v_{2_n} w_n \\
&= (v_{1_1} w_1 + v_{1_2} w_2 + v_{1_3} w_3 + ... + v_{1_n} w_n) + (v_{2_1} w_1 + v_{2_2} w_2 + v_{2_3} w_3 + ... + v_{2_n} w_n) \\
&= \langle \mathbf{v_1}, \mathbf{w} \rangle + \langle \mathbf{v_2}, \mathbf{w} \rangle
\end{aligned}$$

$$\begin{aligned}
\langle \alpha \mathbf{v}, \mathbf{w} \rangle &= \alpha v_1 w_1 + \alpha v_2 w_2 + \alpha v_3 w_3 + ... + \alpha v_n w_n \\
&= \alpha (v_1 w_1) + \alpha (v_2 w_2) + \alpha (v_3 w_3) + ... + \alpha (v_n w_n) \\
&= \alpha (v_1 w_1 + v_2 w_2 + v_3 w_3 + ... + v_n w_n) \\
&= \alpha \langle \mathbf{v}, \mathbf{w} \rangle
\end{aligned}$$

$$\begin{aligned}
\langle \mathbf{v}, \mathbf{w}_1 + \mathbf{w}_2 \rangle &= v_1 (w_{1_1} + w_{2_1}) + v_2 (w_{1_2} + w_{2_2}) + v_3 (w_{1_3} + w_{2_3}) + ... + v_n (w_{1_n} + w_{2_n}) \\
&= v_1 w_{1_1} + v_1 w_{2_1} + v_2 w_{1_2} + v_2 w_{2_2} + v_3 w_{1_3} + v_3 w_{2_3} + ... + v_n w_{1_n} + v_n w_{2_n} \\
&= (v_1 w_{1_1} + v_2 w_{1_2} + v_3 w_{1_3} + ... + v_n w_{1_n}) + (v_1 w_{2_1} + v_2 w_{2_2} + v_3 w_{2_3} + ... + v_n w_{2_n}) \\
&= \langle \mathbf{v}, \mathbf{w}_1 \rangle + \langle \mathbf{v}, \mathbf{w}_2 \rangle
\end{aligned}$$

$$\begin{aligned}
\langle \mathbf{v}, \alpha \mathbf{w} \rangle &= v_1 \alpha w_1 + v_2 \alpha w_2 + v_3 \alpha w_3 + ... + v_n \alpha w_n \\
&= \alpha (v_1 w_1) + \alpha (v_2 w_2) + \alpha (v_3 w_3) + ... + \alpha (v_n w_n) \\
&= \alpha (v_1 w_1 + v_2 w_2 + v_3 w_3 + ... + v_n w_n) \\
&= \alpha \langle \mathbf{v}, \mathbf{w} \rangle
\end{aligned}$$

Regarding being positive definite , consider $\langle \mathbf{v}, \mathbf{v} \rangle$

$$\begin{aligned}
\langle \mathbf{v}, \mathbf{v} \rangle &= v_1 v_1 + v_2 v_2 + v_3 v_3 + ... + v_n v_n \\
&= v_1^2 + v_2^2 + v_3^2 + ... + v_n^2 \\
&\geq 0
\end{aligned}$$

Notice also, that $\langle \mathbf{v}, \mathbf{v} \rangle = 0$ if and only if $\mathbf{v} = 0$

$\square$

Exercise 2.5.11

Prove the following properties of the norm if $\mathbf{v}, \mathbf{w} \in \mathbb{E}^n$

1. $\|\mathbf{v}\| \geq 0$

   Proof. If $\mathbf{v} \in \mathbb{E}^n$ the norm is defined by

$$\begin{aligned} \|\mathbf{v}\| &= \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle} \\ &= \sqrt{(v_1 v_1 + v_2 v_2 + v_3 v_3 + ... + v_n v_n)} \\ &= \sqrt{(v_1^2 + v_2^2 + v_3^2 + ... + v_n^2)} \\ &= (v_1^2 + v_2^2 + v_3^2 + ... + v_n^2)^{\frac{1}{2}} \\ &\geq 0 \end{aligned}$$

$\square$

2. $||\mathbf{v}|| = 0$ if and only if $\mathbf{v} = 0$

   From the definition of positive definite, $\langle \mathbf{v}, \mathbf{v} \rangle = 0$ if and only if $\mathbf{v} = 0$

   Proof. "$\Rightarrow$" Let $\|\mathbf{v}\| = 0$. Then

$$\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle} = 0$$

$$\begin{aligned} \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle} &= 0 \\ \sqrt{(v_1 v_1 + v_2 v_2 + v_3 v_3 + ... + v_n v_n)} &= 0 \\ (v_1^2 + v_2^2 + v_3^2 + ... + v_n^2)^{\frac{1}{2}} &= 0 \\ (v_1^2 + v_2^2 + v_3^2 + ... + v_n^2) &= 0 \end{aligned}$$

   So $\mathbf{v} = 0$ $\square$

   Proof. "$\Leftarrow$"

   Let $\mathbf{v} = 0$

   So $\langle \mathbf{v}, \mathbf{v} \rangle = 0$.

$$\begin{aligned} \|\mathbf{v}\| &= \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle} \\ &= \sqrt{0} \\ &= 0 \end{aligned}$$

$\square$

3. $\|\alpha\mathbf{v}\| = |\alpha|\|\mathbf{v}\|$, $\alpha \in$

Proof.

$$\begin{aligned}
\|\alpha\mathbf{v}\| &= \sqrt{\langle \alpha\mathbf{v}, \alpha\mathbf{v}\rangle} \\
&= \sqrt{(\alpha^2 v_1^2 + \alpha^2 v_2^2 + \alpha^2 v_3^2 + ... + \alpha^2 v_n^2)} \\
&= \sqrt{\alpha^2 (v_1^2 + v_2^2 + v_3^2 + ... + v_n^2)} \\
&= |\alpha| \sqrt{(v_1^2 + v_2^2 + v_3^2 + ... + v_n^2)} \\
&= |\alpha| \sqrt{\langle \mathbf{v}, \mathbf{v}\rangle} \\
&= |\alpha|\|\mathbf{v}\|
\end{aligned}$$

□

4. $\|\mathbf{v} + \mathbf{w}\| \leq \|\mathbf{v}\| + \|\mathbf{w}\|$

Proof. First recall the Cauchy-Schwarz Inequality, let $\mathbf{v}, \mathbf{w} \in \mathbb{E}^n$, then $|\langle \mathbf{v}, \mathbf{w}\rangle| \leq \|\mathbf{v}\|\|\mathbf{w}\|$

$$\begin{aligned}
\|\mathbf{v} + \mathbf{w}\|^2 &= \langle \mathbf{v} + \mathbf{w}, \mathbf{v} + \mathbf{w}\rangle \\
&= (v_1 + w_1)^2 + (v_2 + w_2)^2 + ... + (v_n + w_n)^2 \\
&= (v_1^2 + v_2^2 + ... + v_n^2) + (w_1^2 + w_2^2 + ... + w_n^2) + (v_1 w_1 + ... + v_n w_n) + (w_1 v_1 + ... + w_n v_n) \\
&= \langle \mathbf{v}, \mathbf{v}\rangle + \langle \mathbf{v}, \mathbf{w}\rangle + \langle \mathbf{w}, \mathbf{v}\rangle + \langle \mathbf{w}, \mathbf{w}\rangle \\
&\leq \|\mathbf{v}\|^2 + 2|\langle \mathbf{v}, \mathbf{w}\rangle| + \|\mathbf{w}\|^2 \\
&\leq \|\mathbf{v}\|^2 + 2\|\mathbf{v}\|\|\mathbf{w}\| + \|\mathbf{w}\|^2 \\
&= (\|\mathbf{v}\| + \|\mathbf{w}\|)^2
\end{aligned}$$

□

5. $\|\mathbf{v} + \mathbf{w}\|^2 + \|\mathbf{v} - \mathbf{w}\|^2 = 2(\|\mathbf{v}\|^2 + \|\mathbf{w}\|^2)$

Proof.

$$\begin{aligned}
\|\mathbf{v} + \mathbf{w}\|^2 + \|\mathbf{v} - \mathbf{w}\|^2 &= \langle \mathbf{v} + \mathbf{w}, \mathbf{v} + \mathbf{w}\rangle + \langle \mathbf{v} - \mathbf{w}, \mathbf{v} - \mathbf{w}\rangle \\
&= (v_1 + w_1)^2 + ... + (v_n + w_n)^2 + (v_1 - w_1)^2 + ... + (v_n - w_n)^2 \\
&= (v_1^2 + 2v_1 w_1 + w_1^2 + ... + v_n^2 + 2v_n w_n + w_n^2) + (v_1^2 - 2v_1 w_1 + w_1^2 + ... + v_n^2 - 2v_n w_n + w_n^2) \\
&= (2v_1^2 + ... + 2v_n^2) + (2w_1^2 + ... + 2w_n^2) \\
&= 2(v_1^2 + ... + v_n^2) + 2(w_1^2 + ... + w_n^2) \\
&= 2(\langle \mathbf{v}, \mathbf{v}\rangle + \langle \mathbf{w}, \mathbf{w}\rangle) \\
&= 2(\|\mathbf{v}\|^2 = \|\mathbf{w}\|^2)
\end{aligned}$$

□

Exercise 2.5.20

1. Show that $\|\mathbf{v} \times \mathbf{w}\| = \|\mathbf{v}\|\|\mathbf{w}\| \sin\theta$, where θ is the angle between $\mathbf{v}$ and $\mathbf{w}$.

   Proof. First right out the cross product of $\mathbf{v} \times \mathbf{w}$

$$\mathbf{v} \times \mathbf{w} = (v_2w_3 - v_3w_2, v_3w_1 - v_1w_3, v_1w_2 - v_2w_1) \tag{1}$$

$$\begin{aligned}\|\mathbf{v} \times \mathbf{w}\|^2 &= (v_2w_3 - v_3w_2)^2 + (v_3w_1 - v_1w_3)^2 + (v_1w_2 - v_2w_1)^2 \\ &= v_2^2w_3^2 - 2v_2v_3w_2w_3 + v_3^2w_2^2 \\ &+ v_3^2w_1^2 - 2v_1v_3w_1w_3 + v_1^2w_3^2 \\ &+ v_1^2w_2^2 - 2v_1v_2w_1w_2 + v_2^2w_1^2\end{aligned}$$

So

$$\begin{aligned}\|\mathbf{v} \times \mathbf{w}\|^2 = v_1^2(w_2^2 + w_3^2) + v_2^2(w_1^2 + w_3^2) + v_3^2(w_1^2 + w_2^2) \\ - 2(v_2v_3w_2w_3 + v_1v_3w_1w_3 + v_1v_2w_1w_2)\end{aligned} \tag{2}$$

Recall that $\cos\theta = \frac{\langle \mathbf{v}, \mathbf{w} \rangle}{\|\mathbf{v}\|\|\mathbf{w}\|}$ implies $\langle \mathbf{v}, \mathbf{w} \rangle = \cos\theta \|\mathbf{v}\|\|\mathbf{w}\|$. Therefore

$$\cos^2\theta \|\mathbf{v}\|^2\|\mathbf{w}\|^2 = (\langle \mathbf{v}, \mathbf{w} \rangle)^2 \tag{3}$$

$$(\langle \mathbf{v}, \mathbf{w} \rangle)^2 = (v_1w_1 + v_2w_2 + v_3w_3)^2 \tag{4}$$

$$\begin{aligned}(v_1w_1 + v_2w_2 + v_3w_3)^2 &= (v_1w_1 + v_2w_2 + v_3w_3)(v_1w_1 + v_2w_2 + v_3w_3) \\ &= v_1^2w_1^2 + v_1v_2w_1w_2 + v_1v_3w_1w_3 \\ &+ v_1^2w_1^2 + v_1v_2w_1w_2 + v_2v_3w_2w_3 \\ &+ v_3^2w_3^2 + v_1v_3w_1w_3 + v_2v_3w_2w_3\end{aligned}$$

So applying (3) with our result from (4),

$$\begin{aligned}\cos^2\theta \|\mathbf{v}\|^2\|\mathbf{w}\|^2 = v_1^2w_1^2 + v_1^2w_1^2 + v_3^2w_3^2 + \\ 2(v_1v_2w_1w_2 + v_1v_3w_1w_3 + v_2v_3w_2w_3)\end{aligned} \tag{5}$$

Combining (4) with (2), $\|\mathbf{v}\times\mathbf{w}\|^2+\cos^2\theta\|\mathbf{v}\|^2\|\mathbf{w}\|^2$ we get

$$\begin{aligned}
&v_1^2w_1^2+v_1^2w_1^2+v_3^2w_3^2+2(v_1v_2w_1w_2+v_1v_3w_1w_3+v_2v_3w_2w_3)\\
&\quad+v_1^2(w_2^2+w_3^2)+v_2^2(w_1^2+w_3^2)+v_3^2(w_1^2+w_2^2)\\
&\quad-2(v_2v_3w_2w_3+v_1v_3w_1w_3+v_1v_2w_1w_2)\\
&\quad=v_1^2w_1^2+v_1^2w_1^2+v_3^2w_3^2\\
&\quad+v_1^2(w_2^2+w_3^2)+v_2^2(w_1^2+w_3^2)+v_3^2(w_1^2+w_2^2)\\
&\quad=v_1^2(w_1^2+w_2^2+w_3^2)+v_2^2(w_1^2+w_2^2+w_3^2)+v_3^2(w_1^2+w_2^2+w_3^2)\\
&\quad=(w_1^2+w_2^2+w_3^2)(v_1^2+v_2^2+v_3^2)
\end{aligned}$$

So from this we can conclude

$$\|\mathbf{v}\times\mathbf{w}\|^2+\cos^2\theta\|\mathbf{v}\|^2\|\mathbf{w}\|^2=\|\mathbf{w}\|^2\|\mathbf{v}\|^2 \tag{6}$$

The result of (6) implies that $\|\mathbf{v}\times\mathbf{w}\|^2=\|\mathbf{v}\|^2\|\mathbf{w}\|^2-\cos^2\theta\|\mathbf{v}\|^2\|\mathbf{w}\|^2$.

So our last steps

$$\begin{aligned}
\|\mathbf{v}\times\mathbf{w}\|^2&=\|\mathbf{v}\|^2\|\mathbf{w}\|^2-\cos^2\theta\|\mathbf{v}\|^2\|\mathbf{w}\|^2\\
&=\|\mathbf{v}\|^2\|\mathbf{w}\|^2(1-\cos^2\theta)\\
&=\|\mathbf{v}\|^2\|\mathbf{w}\|^2(\sin^2\theta)
\end{aligned}$$

We can take the square root to reach our conclusion.

$$\|\mathbf{v}\times\mathbf{w}\|=\|\mathbf{v}\|\|\mathbf{w}\|\sin\theta$$

□

2. Show that $\|\mathbf{v}\ \times\mathbf{w}\|$ is the area of a parallelogram spanned by $\mathbf{v}$ and $\mathbf{w}$.

   We just showed that

$$\|\mathbf{v}\times\mathbf{w}\|=\|\mathbf{v}\|\|\mathbf{w}\|\sin\theta$$

   The area of a parallelogram can be calculated using a simple base times height calculation. So area $=\|\mathbf{v}\|\|\mathbf{w}\|\sin\theta$ Where $\|\mathbf{v}\|$ is the length of the base and $\|\mathbf{w}\|\sin\theta$ is the height of the parallelogram.

Exercise 2.5.26

Let $\mathbf{v}_1,\mathbf{v}_2,...,\mathbf{v}_{n-1}$ be linearly independent vectors in $\mathbb{E}^n$. Let $\mathbf{p}_0$ be a point of $\mathbb{R}^n$. Let $\mathbf{H}$ be the hyperplane plane through $\mathbf{p}_0$ spanned by $\mathbf{v}_1,\mathbf{v}_2,...,\mathbf{v}_{n-1}$. If $\mathbf{p}$ is any point in $\mathbb{R}^n$, show that the distance from

$\mathbf{p}$ to $\mathbf{H}$, that is, $\inf\{\|\mathbf{p}-\mathbf{q}\| \mid \mathbf{q} \in \mathbf{H}\}$, is given by the length of the vector $\text{proj}_{\mathbf{v}}(\mathbf{p}-\mathbf{p_0})$ where $\mathbf{v}$ is the vector obtained in Theorem 2.5.21. Specialize this to obtain formulas for the distance from a point to a line in $\mathbb{R}^2$ and form a point to a plane in $\mathbb{R}^3$.

We are given that $\mathbf{v}$ is orthogonal to $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_{n-1}$ and that $\mathbf{H}$ is the hyperplane through $\mathbf{p}_0$ spanned by $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_{n-1}$.

Since $\mathbf{p}$ and $\mathbf{p}_0$ are points, we can think of $\text{proj}_{\mathbf{v}}(\mathbf{p}-\mathbf{p_0})$ as a projection of the vector $(\mathbf{p}-\mathbf{p}_0)$ onto $\mathbf{v}$ where $(\mathbf{p}-\mathbf{p}_0)$ is the hypotenuse of a right triangle formed by $\text{proj}_{\mathbf{v}}(\mathbf{p}-\mathbf{p_0})$. Which makes the $\text{proj}_{\mathbf{v}}(\mathbf{p}-\mathbf{p_0})$ the line adjacent to the angle formed by $\text{proj}_{\mathbf{v}}(\mathbf{p}-\mathbf{p_0})$ which is parallel with $\mathbf{v}$ which is orthogonal to $\mathbf{H}$. Since $\mathbf{q} \in \mathbf{H}$ This must be the shorted distance $\mathbf{p}-\mathbf{q}$ to $\mathbf{H}$, because if you took any other distance to the plane, it wouldn't form right angle with the plane and result in a greater distance.

Specializing this for $\mathbb{R}^2$ and $\mathbb{R}^3$. We consider the $\text{proj}_{\mathbf{v}}(\mathbf{p}-\mathbf{p_0})$. Let the vector $\mathbf{f} = (\mathbf{p}-\mathbf{p_0})$. We only care about the length of the line so we should take it's norm, which is given

$$\|\text{proj}_{\mathbf{v}}(\mathbf{f})\| = \frac{|\langle \mathbf{f}, \mathbf{v}\rangle|}{\|\mathbf{v}\|}$$

For $\mathbb{R}^2$, define vectors $\mathbf{v_1}$ and $\mathbf{v}$. Let $\mathbf{v}$ be non zero orthogonal to $\mathbf{v_1}$. This means the pair $\{\mathbf{v_1}, \mathbf{v}\}$ is a basis for $\mathbb{R}^2$. Let $\mathbf{H}$ be the plane through $\mathbf{p_0}$ be a point spanned by $\mathbf{v_1}$, and contain points (p_{0_1}, p_{0_2}). Let $\mathbf{p} = (p_1, p_2)$ to be any point in $\mathbb{R}^2$. by Let d be the distance we are trying to find.

$$\begin{aligned} d &= \frac{|\langle \mathbf{f}, \mathbf{v}\rangle|}{\|\mathbf{v}\|} \\ &= \frac{|(f_1 v_1 + f_2 v_2)|}{\|\mathbf{v}\|} \\ &= \frac{|(p_1 - p_{0_1})v_1 + (p_2 - p_{0_2})v_2|}{\|\mathbf{v}\|} \\ &= \frac{|(p_1 - p_{0_1})v_1 + (p_2 - p_{0_2})v_2|}{\sqrt{v_1^2 + v_2^2}} \end{aligned}$$

For $\mathbb{R}^3$, we have a similar result. define vectors $\mathbf{v_1}, \mathbf{v_2}$ to be linearly independent. Let $\mathbf{v}$ be the determinant of $\mathbf{v_1}, \mathbf{v_2}$. Resulting in $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v}\}$ as a basis $\mathbb{R}^3$ Let $\mathbf{H}$ be the plane through the point $\mathbf{p_0}$ spanned by $\mathbf{v_1}, \mathbf{v_2}$, and contain points $(p_{0_1}, p_{0_2}, p_{0_3})$. Let $\mathbf{p} = (p_1, p_2, p_3)$ to be any point in $\mathbb{R}^3$. by Let d be the distance we are trying to find.

$$
\begin{aligned}
d &= \frac{|\langle \mathbf{f}, \mathbf{v}\rangle|}{\|\mathbf{v}\|} \\
&= \frac{|(f_1 v_1 + f_2 v_2 + f_3 v_3)|}{\|\mathbf{v}\|} \\
&= \frac{|(p_1 - p_{0_1})v_1 + (p_2 - p_{0_2})v_2 + (p_3 - p_{0_3})v_3|}{\|\mathbf{v}\|} \\
&= \frac{|(p_1 - p_{0_1})v_1 + (p_2 - p_{0_2})v_2 + (p_3 - p_{0_3})v_3|}{\sqrt{v_1^2 + v_2^2 + v_3^2}}
\end{aligned}
$$

Exercise 2.5.30

Consider the vectors $\mathbf{v}_1 = (1,1,-1,0)$, $\mathbf{v}_2 = (1,0,0,-1)$ and $\mathbf{v}_3 = (0,1,1,1)$ in $\mathbb{E}^4$.

1. Use the Gram-Schmidt orthogonalization process on these three vectors to produce a set of three mutually orthogonal vectors that span the same subspace.

   Let $\mathbf{v}_1 = \tilde{\mathbf{v}}_1$

   For $\tilde{\mathbf{v}}_2 = \mathbf{v}_2 - \text{proj}_{\tilde{\mathbf{v}}_1}(\mathbf{v}_2)$

$$
\text{proj}_{\tilde{\mathbf{v}}_1}(\mathbf{v}_2) = \frac{\langle \mathbf{v}_2, \tilde{\mathbf{v}}_1\rangle}{\|\tilde{\mathbf{v}}_1\|}\frac{\tilde{\mathbf{v}}_1}{\|\tilde{\mathbf{v}}_1\|}
$$

$$
\begin{aligned}
\langle \mathbf{v}_2, \tilde{\mathbf{v}}_1\rangle &= (1)(1) + (0)(1) + (0)(-1) + (-1)(0) \\
&= 1 \\
\|\tilde{\mathbf{v}}_1\| &= \sqrt{1^2 + 1^2 + (-1)^2 + 0^2} \\
&= \sqrt{3}
\end{aligned}
$$

$$
\begin{aligned}
\text{proj}_{\tilde{\mathbf{v}}_1}(\mathbf{v}_2) &= \frac{1}{\sqrt{3}}\frac{\tilde{\mathbf{v}}_1}{\sqrt{3}} \\
&= \frac{1}{3}\tilde{\mathbf{v}}_\mathbf{1} \\
&= \frac{1}{3}\begin{bmatrix} 1 & 1 & -1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & 0 \end{bmatrix}
\end{aligned}
$$

$$\begin{aligned}\tilde{\mathbf{v}}_2 &= \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} - \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ -\frac{1}{3} \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & \frac{1}{3} & -1 \end{bmatrix}\end{aligned}$$

For $\tilde{\mathbf{v}}_3 = \mathbf{v}_3 - \text{proj}_{\tilde{\mathbf{v}}_1}(\mathbf{v}_3) - \text{proj}_{\tilde{\mathbf{v}}_2}(\mathbf{v}_3)$

$$\text{proj}_{\tilde{\mathbf{v}}_1}(\mathbf{v}_3) = \frac{\langle \mathbf{v}_3, \tilde{\mathbf{v}}_1 \rangle}{\|\tilde{\mathbf{v}}_1\|} \frac{\tilde{\mathbf{v}}_1}{\|\tilde{\mathbf{v}}_1\|}$$

$$\begin{aligned}\langle \mathbf{v}_3, \tilde{\mathbf{v}}_1 \rangle &= (0)(1) + (1)(1) + (1)(-1) + (1)(0) \\ &= 0 \\ \|\tilde{\mathbf{v}}_1\| &= \sqrt{3}\end{aligned}$$

$$\begin{aligned}\text{proj}_{\tilde{\mathbf{v}}_1}(\mathbf{v}_3) &= \frac{0}{\sqrt{3}} \frac{\tilde{\mathbf{v}}_1}{\sqrt{3}} \\ &= 0\tilde{\mathbf{v}}_1 \\ &= 0 \begin{bmatrix} 1 & 1 & -1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}\end{aligned}$$

$$\text{proj}_{\tilde{\mathbf{v}}_2}(\mathbf{v}_3) = \frac{\langle \mathbf{v}_3, \tilde{\mathbf{v}}_2 \rangle}{\|\tilde{\mathbf{v}}_2\|} \frac{\tilde{\mathbf{v}}_2}{\|\tilde{\mathbf{v}}_2\|}$$

$$\begin{aligned}\langle \mathbf{v}_3, \tilde{\mathbf{v}}_2 \rangle &= (0)(\frac{2}{3}) + (1)(-\frac{1}{3}) + (1)(\frac{1}{3}) + (1)(-1) \\ &= -1 \\ \|\tilde{\mathbf{v}}_2\| &= \sqrt{\frac{2}{3}^2 + -\frac{1}{3}^2 + \frac{1}{3}^2 + -1^2} \\ &= \sqrt{\frac{5}{3}}\end{aligned}$$

$$\begin{aligned}
\mathrm{proj}_{\tilde{\mathbf{v}}_2}(\mathbf{v}_3) &= -\frac{1}{\sqrt{\frac{5}{3}}}\frac{\tilde{\mathbf{v}}_2}{\sqrt{\frac{5}{3}}} \\
&= -\frac{3}{5}\tilde{\mathbf{v}}_2 \\
&= -\frac{3}{5}\begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & \frac{1}{3} & -1 \end{bmatrix} \\
&= \begin{bmatrix} \frac{6}{15} & -\frac{3}{15} & \frac{3}{15} & \frac{3}{5} \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
\tilde{\mathbf{v}}_3 &= \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} \frac{6}{15} \\ -\frac{3}{15} \\ \frac{3}{15} \\ \frac{3}{5} \end{bmatrix} \\
&= \begin{bmatrix} -\frac{6}{15} & \frac{18}{15} & \frac{12}{15} & \frac{2}{5} \end{bmatrix}
\end{aligned}$$

2. Extend the set of three vectors produced in part 1 to a mutually orthogonal basis for

Consider the matrix comprised of $\tilde{\mathbf{v}}_1, \tilde{\mathbf{v}}_2 \tilde{\mathbf{v}}_3$ and vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4 \in \mathbb{E}^4$ where 1 is in the the jth term and 0 otherwise.

$$\begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 & \mathbf{e}_4 \\ 1 & 1 & -1 & 0 \\ \frac{2}{3} & -\frac{1}{3} & \frac{1}{3} & -1 \\ -\frac{6}{15} & \frac{18}{15} & \frac{12}{15} & \frac{2}{5} \end{bmatrix} \tag{7}$$

The vector that will complete the basis can be created by taking the determinant of the matrix (7).

Step 1

$$\begin{aligned}
\mathbf{e}_1 \begin{bmatrix} 1 & -1 & 0 \\ -\frac{1}{3} & \frac{1}{3} & -1 \\ \frac{18}{15} & \frac{12}{15} & \frac{2}{5} \end{bmatrix} - \mathbf{e}_2 \begin{bmatrix} 1 & -1 & 0 \\ \frac{2}{3} & \frac{1}{3} & -1 \\ -\frac{6}{15} & \frac{12}{15} & \frac{2}{5} \end{bmatrix} \\
+ \mathbf{e}_3 \begin{bmatrix} 1 & 1 & 0 \\ \frac{2}{3} & -\frac{1}{3} & -1 \\ -\frac{6}{15} & \frac{18}{15} & \frac{2}{5} \end{bmatrix} - \mathbf{e}_4 \begin{bmatrix} 1 & 1 & -1 \\ \frac{2}{3} & -\frac{1}{3} & \frac{1}{3} \\ -\frac{6}{15} & \frac{18}{15} & \frac{12}{15} \end{bmatrix}
\end{aligned}$$

Then the next step...

$$
\begin{aligned}
&\mathbf{e}_1(1 \begin{bmatrix} \frac{1}{3} & -1 \\ \frac{12}{15} & \frac{2}{5} \end{bmatrix} - -1 \begin{bmatrix} -\frac{1}{3} & -1 \\ \frac{18}{15} & \frac{2}{5} \end{bmatrix} - \\
&\mathbf{e}_2(1 \begin{bmatrix} \frac{1}{3} & -1 \\ \frac{12}{15} & \frac{2}{5} \end{bmatrix} - -1 \begin{bmatrix} \frac{2}{3} & -1 \\ -\frac{6}{15} & \frac{2}{5} \end{bmatrix} + \\
&\mathbf{e}_3(1 \begin{bmatrix} -\frac{1}{3} & -1 \\ \frac{18}{15} & \frac{2}{5} \end{bmatrix} - 1 \begin{bmatrix} \frac{2}{3} & -1 \\ -\frac{6}{15} & \frac{2}{5} \end{bmatrix} - \\
&\mathbf{e}_4(1 \begin{bmatrix} -\frac{1}{3} & \frac{1}{3} \\ \frac{18}{15} & \frac{12}{15} \end{bmatrix} - 1 \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ -\frac{6}{15} & \frac{12}{15} \end{bmatrix} + -1 \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{6}{15} & \frac{18}{15} \end{bmatrix})
\end{aligned}
$$

...eventually arriving at...

$$\mathbf{e}_1(-\frac{2}{5}) - \mathbf{e}_2(\frac{4}{5}) + \mathbf{e}_3(\frac{6}{5}) - \mathbf{e}_4(-2)$$

We now compute our new vector to be $(-\frac{2}{5}, -\frac{4}{5}, \frac{6}{5}, 2)$ and define it as $\mathbf{v}$. Which gives us the basis $\{\tilde{\mathbf{v}}_\mathbf{1}, \tilde{\mathbf{v}}_\mathbf{2}, \tilde{\mathbf{v}}_\mathbf{3}, \mathbf{v}\}$ over $\mathbb{E}^4$.

In order to make this a mutually orthogonal basis for $\mathbb{E}^4$, apply the Gram-Schmidt orthogonalization process once again.

For $\tilde{\mathbf{v}} = \mathbf{v} - \text{proj}_{\tilde{\mathbf{v}}_1}(\mathbf{v}) - \text{proj}_{\tilde{\mathbf{v}}_2}(\mathbf{v}) - \text{proj}_{\tilde{\mathbf{v}}_3}(\mathbf{v})$

$$\langle \mathbf{v}, \tilde{\mathbf{v}}_1 \rangle = (-\frac{2}{5})(1) + (-\frac{4}{5})(1) + (\frac{6}{5})(-1) + (2)(0) = -\frac{12}{5}$$

$$
\begin{aligned}
\text{proj}_{\tilde{\mathbf{v}}_1}(\mathbf{v}) &= -\frac{-\frac{12}{5}}{\sqrt{3}} \frac{\tilde{\mathbf{v}}_1}{\sqrt{3}} \\
&= -\frac{4}{5}\tilde{\mathbf{v}}_1 \\
&= -\frac{4}{5}\begin{bmatrix} 1 & 1 & -1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} -\frac{4}{5} & -\frac{4}{5} & \frac{4}{5} & 0 \end{bmatrix}
\end{aligned}
$$

$$\langle \mathbf{v}, \tilde{\mathbf{v}}_2 \rangle = (-\frac{2}{5})(\frac{2}{3}) + (-\frac{4}{5})(-\frac{1}{3}) + (\frac{6}{5})(\frac{1}{3}) + (2)(-1) = -\frac{8}{5}$$

$$\|\tilde{\mathbf{v}}_2\| = \sqrt{\frac{5}{3}}$$

$$\begin{aligned}
\text{proj}_{\tilde{\mathbf{v}}_2}(\mathbf{v}) &= \frac{-\frac{8}{5}}{\sqrt{\frac{5}{3}}}\frac{\tilde{\mathbf{v}}_2}{\sqrt{\frac{5}{3}}} \\
&= -\frac{24}{25}\tilde{\mathbf{v}}_2 \\
&= -\frac{24}{25}\begin{bmatrix}\frac{2}{3} & -\frac{1}{3} & \frac{1}{3} & -1\end{bmatrix} \\
&= \begin{bmatrix}-\frac{16}{25} & \frac{8}{25} & -\frac{8}{25} & \frac{24}{25}\end{bmatrix}
\end{aligned}$$

$$\langle \mathbf{v}, \tilde{\mathbf{v}}_3 \rangle = (-\frac{2}{5})(-\frac{6}{15}) + (-\frac{4}{5})(\frac{18}{15}) + (\frac{6}{5})(\frac{12}{15}) + (2)(\frac{2}{5}) = -\frac{8}{5}$$

$$\|\tilde{\mathbf{v}}_3\| = \sqrt{\frac{12}{5}}$$

$$\begin{aligned}
\text{proj}_{\tilde{\mathbf{v}}_3}(\mathbf{v}) &= \frac{\frac{24}{25}}{\sqrt{\frac{5}{12}}}\frac{\tilde{\mathbf{v}}_3}{\sqrt{\frac{5}{12}}} \\
&= \frac{2}{5}\tilde{\mathbf{v}}_3 \\
&= \frac{2}{2}\begin{bmatrix}\frac{-12}{75} & -\frac{1}{3} & \frac{1}{3} & -1\end{bmatrix} \\
&= \begin{bmatrix}-\frac{4}{25} & \frac{12}{25} & \frac{6}{25} & \frac{4}{25}\end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
\tilde{\mathbf{v}} &= \begin{bmatrix}-\frac{2}{5} \\ -\frac{4}{5} \\ \frac{6}{5} \\ 2\end{bmatrix} - \begin{bmatrix}-\frac{4}{5} \\ -\frac{4}{5} \\ \frac{4}{5} \\ 0\end{bmatrix} - \begin{bmatrix}-\frac{16}{25} \\ \frac{8}{25} \\ -\frac{8}{25} \\ \frac{24}{25}\end{bmatrix} - \begin{bmatrix}\frac{-4}{25} \\ \frac{12}{25} \\ \frac{6}{25} \\ \frac{4}{25}\end{bmatrix} \\
&= \begin{bmatrix}\frac{6}{5} & \frac{-4}{5} & \frac{12}{25} & \frac{22}{25}\end{bmatrix}
\end{aligned}$$

We now have a mutually orthogonal basis for $\mathbb{E}^4$.

3. Normalize your basis so that it becomes and orthonormal basis for $\mathbb{E}^4$

$$\frac{\tilde{\mathbf{v}}_1}{\|\tilde{\mathbf{v}}_1\|} = \begin{bmatrix}\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} & 0\end{bmatrix}$$

$$\frac{\tilde{\mathbf{v}}_2}{\|\tilde{\mathbf{v}}_2\|} = \begin{bmatrix}\frac{\frac{2}{3}}{\sqrt{\frac{5}{3}}} & \frac{\frac{-1}{3}}{\sqrt{\frac{5}{3}}} & \frac{\frac{1}{3}}{\sqrt{\frac{5}{3}}} & \frac{-1}{\sqrt{\frac{5}{3}}}\end{bmatrix}$$

$$\frac{\tilde{\mathbf{v}}_3}{\|\tilde{\mathbf{v}}_3\|} = \begin{bmatrix}\frac{\frac{-2}{3}}{\sqrt{\frac{12}{5}}} & \frac{\frac{6}{5}}{\sqrt{\frac{12}{5}}} & \frac{\frac{4}{5}}{\sqrt{\frac{12}{5}}} & \frac{\frac{2}{5}}{\sqrt{\frac{12}{5}}}\end{bmatrix}$$

$$\frac{\tilde{\mathbf{v}}}{\|\tilde{\mathbf{v}}\|} = \begin{bmatrix}\frac{\frac{6}{5}}{\sqrt{\frac{1928}{625}}} & \frac{\frac{-4}{5}}{\sqrt{\frac{1928}{625}}} & \frac{\frac{12}{25}}{\sqrt{\frac{1928}{625}}} & \frac{\frac{22}{25}}{\sqrt{\frac{1928}{625}}}\end{bmatrix}$$