Ex 3.4.8) $Y \subseteq X$. X is complete.
We have to show Y is complete $\Leftrightarrow$ Y is closed

First, we show that Y is complete $\Rightarrow$ Y is closed.
Consider $(x_n) \in Y$ and $\lim_{n \to \infty} (x_n) = a \in \overline{Y}$.

Since (x_n) converges, (x_n) is Cauchy in Y.
But since Y is complete, (x_n) must converge to $b \in Y$.
But the limit is unique so $a = b$.
We have $a \in Y \therefore Y$ is closed.

Second, we show that Y is closed $\Rightarrow$ Y is complete.
Consider a Cauchy sequence $(y_n) \in Y \subseteq X$.
By completeness of X, $\lim_{n \to \infty} y_n = a \in X$.

But since Y is closed, it must contain all of its limit point
so $a \in Y \therefore Y$ is complete

3.4.9) First, we show that $(x_n^{(1)}, x_n^{(2)}, \dots)$, a Cauchy sequence in $(\mathbb{R}^n, d_p)$ implies $(x_n^{(1)})$ is Cauchy in $(\mathbb{R}, d_p)$

Since $(x_n^{(1)}, x_n^{(2)}, \dots)$ is Cauchy, $\forall \varepsilon > 0, \exists N$ s.t $\forall n, m \geq N$

$$\left[|x_n^{(1)} - x_m^{(1)}|^p + |x_n^{(2)} - x_m^{(2)}|^p + \dots + |x_n^{(n)} - x_m^{(n)}|^p \right]^{\frac{1}{p}} < \varepsilon$$

$$|x_n^{(1)} - x_m^{(1)}|^p + |x_n^{(2)} - x_m^{(2)}|^p + \dots + |x_n^{(n)} - x_m^{(n)}|^p < \varepsilon^p$$

Note that $|x_n^{(k)} - x_m^{(k)}|^p \geq 0$

So $0 < |x_n^{(1)} - x_m^{(1)}|^p < \varepsilon^p$

We have that $\forall n, m \geq N$, $\left(|x_n^{(1)} - x_m^{(1)}|^p \right)^{1/p} = |x_n^{(1)} - x_m^{(1)}| < \varepsilon$

$\therefore$ $(x_n^{(1)})$ is Cauchy and so $\lim (x_n^{(1)})$ exists.
Since $\mathbb{R}$ is complete, $\lim x^{(n)}$ exists.

Claim: A Cauchy sequence $(x_n^{(1)}, x_n^{(2)}, \dots, x_n^{(n)}) \to (\lim_{n \to \infty} x^{(1)}, \dots, \lim_{n\to\infty} x^{(n)})$

Denote $\lim_{n \to \infty} x_n^{(1)} = x^{(1)'}$.

let $\varepsilon > 0$. $\exists N_1$ s.t. $\forall n \geq N_1$, $|x_n^{(1)} - x^{(1)'}| < \frac{\varepsilon}{\sqrt[p]{n}}$

$\vdots$

$\exists N_n$ s.t. $\forall n \geq N_n$, $|x_n^{(n)} - x_n^{(n)'}| < \frac{\varepsilon}{\sqrt[p]{n}}$

Choose $N = \max\{N_1, N_2, \dots, N_n\}$.

$\forall n \geq N$, $\sqrt[p]{(x_n^{(1)} - x^{(1)'})^p + \dots + (x_n^{(n)} - x^{(n)'})} < \sqrt[p]{\left(\frac{\varepsilon}{\sqrt[p]{n}}\right)^p + \dots + \left(\frac{\varepsilon}{\sqrt[p]{n}}\right.}$

$= \varepsilon$

$\therefore \ell_n^p(\mathbb{R})$ is complete. Using the same proof method, $\ell_n^p(\mathbb{C}$ is complete.

3.4.18)

i) $f_n(x) = \sin\left(\frac{x}{n}\right)$

$f_n(x)$ converges $\forall x \in [0, 2\pi]$.

Point wise limit function $f(x) = 0$

ii) $f_n(x) = \sin(nx)$

$f_n(x)$ converges for $x = 0, \pi, 2\pi$.

$f_n(x)$ does not have a pointwise limit function.

iii) $f_n(x) = \sin^n(x)$

$f_n(x)$ converges $\forall x \in [0, 2\pi] \setminus \{\frac{3\pi}{2}\}$

Hence, there is no point wise limit function.

$\exists\ \varepsilon > 0$

3.4.22)

i) Consider $f_n(x) = x^n$
Since $|x| < 1$, then $x^n \to 0$, $\forall x \in (-1,1)$.

ii) let $\varepsilon > 0$.

If $\varepsilon \geq 1$. Choose $N = 1$. Then $\forall x \in [-\frac{1}{2}, \frac{1}{2}]$ and $\forall n \geq N$,
$|x^n| = |x|^n \leq \frac{1}{2}^n < 1 < \varepsilon$.

If $\varepsilon < 1$. Choose $N = \left\lceil \frac{\ln \varepsilon}{\ln \frac{1}{2}} \right\rceil$ ← least greatest integer symbol.

Then $\forall x \in [-\frac{1}{2}, \frac{1}{2}]$ and $\forall n \geq N$,

$$n \ln |x| \leq N \ln |x| \leq \frac{\ln \varepsilon \ \ln |x|}{\ln \frac{1}{2}} \leq \ln \varepsilon$$

so $\ln |x|^n \leq \varepsilon$

$\therefore\ |x|^n < \varepsilon$

iii) Suppose $(f_n)_{n\in\mathbb{N}}$ is uniformly convergent to $f(x) = 0$.

Then $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$, $\forall n \geq N$, $\forall x \in (-1,1)$

$$|x|^n < \varepsilon$$
$$|x|^N < \varepsilon, \quad \forall x \in (-1,1)$$
$$N \ln |x| < \ln \varepsilon$$
$$N > \frac{\ln \varepsilon}{\ln |x|}$$
$$N > \frac{\ln \frac{1}{2}}{\ln |x|}$$

This is a contradiction because as $x \to 1$, $\ln(x) \to 0$
then $\frac{\ln \frac{1}{2}}{\ln |x|} \to \infty$.

3.5.2) f is continuous at $x_0 \iff (x_n) \to x_0 \Rightarrow f(x_n) \to f(x_0)$

let $\varepsilon > 0$. Since f is continuous at x_0, $\exists\, \delta$ s.t.
$d(x_n, x_0) < \delta \Rightarrow d'(f(x_n), f(x_0)) < \varepsilon$.
Since (x_n) converges to x_0, $\exists\, N \in \mathbb{N}$ s.t. $\forall n \geq N$, $d(x_n, x_0) < \delta$.
Thus $d(f(x_n), f(x_0)) < \varepsilon \quad \forall n \geq N$.
$\therefore$ $f(x_n)$ converges to $f(x_0)$

Now for the backward direction,
Let $\varepsilon > 0$. Assume that $\forall (x_n) \to x_0 \Rightarrow f(x_n) \to f(x_0)$.
Assume that f is not continuous at x.
So we want to show that $\exists$ a sequence (x_n) s.t
$x_n \to x_0$ but $f(x_n) \not\to f(x_0)$
f not continuous at x_0 $\therefore$ $\exists\, \varepsilon > 0$ s.t $\forall\, \delta > 0$,
$\exists\, x$ s.t. $d(x, x_0) < \delta \wedge d'(f(x), f(x_0)) \geq \varepsilon$
Then $\forall n \in \mathbb{N}$, $\exists\, x_n$ s.t.
$d(x_n, x_0) < \frac{1}{n}$ and $d'(f(x_n), f(x_0)) \geq \varepsilon$

Now choose x_n $\forall n \in \mathbb{N}$.
So we have a sequence (x_n) s.t. $(x_n) \to x_0$ and $f(x_n) \to f(x_0)$

Ex 3.5.3) f is cont at $x_0 \in \mathbb{R}$

$f(x) = a_m x^m + a_{m-1} x^{m-1} + \dots + a_1 x + a_0$

Assume $(x_n) \to (x_0)$ i.e. $\lim_{n\to\infty} x_n = 0$

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} \left(a_m x_n^m + a_{m-1} x_n^{m-1} + \dots + a_0\right)$$

$$= \lim_{n\to\infty} a_m x_n^m + \dots + \lim a_0$$

$$= a_m \lim_{n\to\infty} x_n^m + \dots + a_1 \lim x_n + \lim a_0$$

$$= a_m \left(\lim_{n\to\infty} x_n\right)^m + \dots + a_1 \lim x_n + \lim a_0$$

$$= a_m (x_0)^m + \dots + a_1 x_0 + a_0 = f(x_0)$$

3.5.4 → Show that f is continuous at 0.

let $\varepsilon > 0$. Choose $\delta = \varepsilon$.
Suppose $|x| < \delta$

$$|f(x)| = \begin{cases} \frac{1}{q} & \text{if } x = \frac{p}{q} \to |f(x)| = \frac{1}{q} \leq \left|\frac{p}{q}\right| = |x| < \delta = \varepsilon \\ 0 & \text{if } x \notin \mathbb{Q} \to |f(x)| < \varepsilon \end{cases}$$

$\square$

→ Show that f is continuous at any irrational point

$\forall n \in \mathbb{N}, \quad \exists k \in \mathbb{Z}$ s.t. $\frac{k}{n} < \alpha < \frac{k+1}{n}$

Let $s_n = \min\left\{\alpha - \frac{k}{n}, \frac{k+1}{n} - \alpha\right\}$

i.e. the shortest distance from α to a rational number with denominator n.

let $\varepsilon > 0$. $\exists N \in \mathbb{N}$ s.t. $\frac{1}{N} < \varepsilon$.

let $\delta = \min\{s_1, s_2, \ldots, s_N\}$

Suppose that $|x - \alpha| < \delta$.

Then $s_q \leq \left|\frac{p}{q} - \alpha\right| = |x - \alpha| < \delta \leq s_i$

$\forall\ 1 \leq i \leq N$.

$\therefore q > N \therefore \frac{1}{q} < \frac{1}{N} < \varepsilon$

$$|f(x)| = \begin{cases} 0 & \text{if } x \notin \mathbb{Q}. \quad \text{(Trivial Case)} \\ \frac{1}{q} < \varepsilon & \text{if } x = \frac{p}{q} \end{cases}$$

$\square$