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Ex 3.3.13) let Y be an open subset of X.
We have to show that if $A \subseteq Y$ is open in Y, it is also open in X.
Since A is open in Y, $\exists\, r > 0$, $B_r(a) \subseteq A$

We have $B_r^Y(a) = \{y \in Y \mid d(y,a) < r\} \subseteq A$

We have to show that $B_r^X(a) = \{x \in X \mid d(x,a) < r\} \subseteq A$

$B_r^y(a) = B_r^X(a) \cap Y$. Note that Y is open.

The intersection of open sets is open so $B_r^y(a)$ open in X.

$\exists\, r_2$, $B_{r_2}^X(a) \subseteq B_r^Y(a) \subseteq A$

Ex 3.3.20) Show that $\mathbb{Q} \subseteq \mathbb{R}$ with usual metric is neither open nor closed.

First, we show that $\mathbb{Q}$ is not open.
That is, $\exists\, x \in \mathbb{Q}$, $\forall B_r(x) \not\subseteq \mathbb{Q}$.
Consider $0 \in \mathbb{Q}$. Since $\mathbb{Q}^c$ is dense in $\mathbb{R}$,
then $\forall r > 0$, $\exists\, b \in \mathbb{Q}^c$ s.t. $-r < b < r$
So the open ball $B_r(0) \nsubseteq \mathbb{Q}$ $\therefore \mathbb{Q}$ is not open.

Second, we show that $\mathbb{Q}$ is not closed.
That is, $\mathbb{Q}^c$ is not open.
Consider $\sqrt{2} \in \mathbb{Q}^c$, Since $\mathbb{Q}$ is dense in $\mathbb{R}$,
$\forall r > 0$, $\exists\, b \in \mathbb{Q}$ s.t. $\sqrt{2} - r < b < \sqrt{2} + r$
So the open ball $B_r(\sqrt{2}) \nsubseteq \mathbb{Q}^c$ $\therefore \mathbb{Q}^c$ is not open
$\therefore \mathbb{Q}$ is not closed.

Ex 3.3.31) $\bar{A} = A \cup \{\text{acc pts of } A\}$

First, we show that $A \cup \{\text{acc pts of } A\}$ is closed.
That is, we have to show $(A \cup \{\text{acc pts of } A\})^c$ is open.
let $x \in (A \cup \{\text{acc pts of } A\})^c = A^c \cap \{\text{acc pts}\}^c$
by De Morgan's Law.

Since x is not an accumulation point of A,
we have $(B_r(x) - \{x\}) \cap A = \emptyset$ but since $x \in A^c$
we have $B_r(x) \cap A = \emptyset$.

Also, we have to show $B_r(x) \cap \{\text{acc pt of } A\} = \emptyset$.
Suppose, for contradiction, that $\exists\, y \in B_r(x) \cap \{\text{acc pt of } A\}$.
We have $d(x,y) < r$ and $B_r(y) - \{y\} \cap A \neq \emptyset$

Now choose $r_0 = r - d(x,y)$.
$\exists\, a \in A$ s.t. $a \in B_{r-d(x,y)}(y)$
We now show that $B_{r-d(x,y)}(y) \subseteq B_r(x)$

let $z \in B_{r-d(x,y)}$, $d(y,z) < r - d(x,y)$
By triangle inequality, $d(x,z) \leq d(x,y) + d(y,z) < r$
$\therefore$ $z \in B_r(x)$

So we have $B_{r-d(x,y)}(y) \subseteq B_r$ but $B_r(x) \not\cap A$

$\therefore$ we have our contradiction.

Since $B_r(x) \cap A = \emptyset$ and $B_r(x) \cap \{\text{acc pt of } A\} = \emptyset$
we have $B_r(x) \subseteq (A \cup \{\text{acc pt}\})^c$.

So $A \cup \{\text{acc pt}\}$ is closed.
Proof continued on next page.

Ex 3.3.31) (Continued)

Now, let F be a closed set containing A.
We must show that $A \cup \{acc\ pt\} \subseteq F$.

let $x \in A \cup \{acc\ pt\}$

F is closed, so F contains all its acc pts.
We also know that F contains A.

$\{acc\ pt\ of\ A\} \subseteq \{acc\ pt\ of\ F\} \subseteq A$

Then $A \cup \{acc\ pt\} \subseteq F$

Also, obviously, $A \cup \{acc\ pts\}$ contains A.

To recap, we have shown that
- $A \cup \{acc\ pt\}$ is closed
- $A \cup \{acc\ pt\} \supseteq A$
- If F is closed and contains A, $A \cup \{acc\ pts\} \subseteq F$.

$\therefore A \cup \{acc\ pt\ of\ A\} = \bar{A}$

Ex 3.3.32) $\bar{A} = A \cup \partial A$. $A \subseteq X$.

First, we try to show $\bar{A} \subseteq A \cup \partial A$.
let $x \in \bar{A}$. Note that $\bar{A} = A \cup \{\text{acc pts of } A\}$.
If $x \in A$ it is trivial. Now we have to show $\{\text{acc pts of } A\} \subseteq A \cup \partial A$.
let $x \in \{\text{acc pts of } A\}$. Then $(B_r(x) - \{x\}) \cap A \neq \emptyset$.
Suppose $x \notin A$. Then $x \in A^c$ and also $(B_r(x) - \{x\}) \cap A \neq \emptyset$
$\therefore$ $x \in \partial A$ $\therefore$ $\bar{A} \subseteq A \cup \partial A$

Second, we try to show $A \cup \partial A \subseteq \bar{A} = A \cup \{\text{acc pts of } A\}$.
let $x \in A \cup \partial A$. Again, if $x \in A$ it is trivial.
Now we have to show $\partial A \subseteq A \cup \{\text{acc pts of } A\}$
Suppose $x \notin A$. We have $B_r(x) \cap A \neq \emptyset$ and also
$B_r(x) \cap A^c \neq \emptyset$ by definition of ∂A.
But since $x \notin A$ then $B_r(x) - \{x\} \cap A \neq \emptyset$ $\therefore$
$x \in \{\text{acc pts of } A\}$ $\therefore$ $A \cup \partial A \subseteq \bar{A}$.

$\square$

E: Ex 3.3.3? $\partial A = \bar{A} \cap \bar{A^c}$

First, we show $\partial A \subseteq \bar{A} \cap \bar{A^c}$.
let $x \in \partial A$. $\forall r > 0$, $B_r(x) \cap A \neq \emptyset$ and $B_r(x) \cap A^c \neq \emptyset$
Suppose that $x \notin A$.
We have $B_r(x) \cap A \neq 0$
But $x \notin A$ so $(B_r(x) - \{x\}) \cap A \neq \emptyset$
$\therefore$ x is an accumulation point of A

Second, we show $\bar{A} \cap \bar{A^c} \subseteq \partial A$. let $x \in \bar{A} \cap \bar{A^c}$.
Then $x \in \bar{A}$ and $x \in \bar{A^c}$.
let $r > 0$. $B_r(x) \cap A \neq \emptyset$
$B_r(x) \cap A^c \neq \emptyset$
Then $x \in \partial A$ by definition.

$\square$

3.3.49)

i)

If A is convex, closed convex hull $A = \bar{A}$

So the closed convex hull is equal to the closure of the unit ball for each p i.e. $\bar{B}_1^p = \{x \in \mathbb{R}^n \mid d_p(x,0) \leq 1\}$

ii) $S_p = \{x \in \mathbb{R}^n \mid \|x\|_p \leq 1\}$ for $0 < p < 1$.

Consider $(0,1)$ and $(1,0)$. The line connecting the two points is not in S_p thus it is not convex.

Consider $\left(\frac{1}{2}, \frac{1}{2}\right)$.

$$\left(\frac{1}{2}\right)^p + \left(\frac{1}{2}\right)^p = 2 \cdot \left(\frac{1}{2}\right)^p$$

$$= 2^{1-p} > 1 \quad \because 1-p > 0$$

$\therefore \left(\frac{1}{2}, \frac{1}{2}\right) \notin S_p$

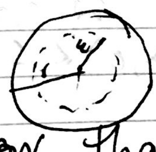
The closed convex hull of S_p is $B_1^1(0) = \{x \in \mathbb{R}^n \mid d_1(x,0) \leq$

Extra Q1) $\text{Int } A = \bigcup_{\substack{G \text{ open} \\ G \subseteq A}} G$

$G \subseteq A$
$G^c \supseteq A^c$

$(\text{Int } A)^c = \Big(\bigcup_{\substack{G \text{ open} \\ G \subseteq A}} G \Big)^c$

$= \Big(\bigcap_{\substack{G \text{ open} \\ G \subseteq A}} G^c \Big) = \bigcap_{\substack{G^c \text{ closed} \\ G^c \supseteq A^c}} G^c = \overline{A^c}$

Q2)  $(\mathbb{R}^n, \text{ standard metric})$

Show that:
An isolated point of A is the acc pt of A^c.

let x be an isolated point of A.
Then $\exists\, r > 0$ s.t. x is the only point of A in $B_r(x)$
We know that $x = (x_1, x_2, x_3, \dots, x_n) \in \mathbb{R}^n$
We want to show $\forall r > 0$, $\exists y \neq x$, $y \in A^c$ and $y \in B_\varepsilon(x)$

Let $\varepsilon > 0$. let $m = \min\{r, \varepsilon\}$
Choose $y = (x_1 + \frac{m}{2}, x_2, \dots, x_n)$. Then $d(x,y) = \frac{m}{2} < \varepsilon$.
$d(x,y) < r \Rightarrow y \in B_r(x)$ but $y \neq x$ thus $y \in A^c$.

Extra Q 3)

i) $A = \{ x \in \mathbb{R}^n \mid d_2(x, a) = r, \text{ where } a, r \in \mathbb{R} \}$

ii) Same set as i