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REAL ANALYSIS
THIRD WEEK

Exercise 1.9.8

Give examples to show that if $r = 1$ in the statement of the Ratio Test, anything may happen.

First consider the harmonic series, $\sum_{n=1}^{\infty} \frac{1}{n}$. The $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = 1$

$$\lim_{n\to\infty} |\frac{\frac{1}{n+1}}{\frac{1}{n}}| = \lim_{n\to\infty} |\frac{n}{n+1}| = 1$$

However, we know that the series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Next consider $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$. Again $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = 1$

$$\begin{aligned}\lim_{n\to\infty} |\frac{\frac{(-1)^{n+2}}{n+1}}{\frac{(-1)^{n+1}}{n}}| &= \lim_{n\to\infty} |\frac{(-1)^{n+2}}{n+1}\frac{n}{(-1)^{n+1}}| \\ &= \lim_{n\to\infty} |\frac{-n}{n+1}| \\ &= 1\end{aligned}$$

This series, $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges since $|\sum_{k=m+1}^{n} \frac{(-1)^{k+1}}{k}| < \frac{1}{m}$ but not absolutely because $\sum_{n=1}^{\infty} |\frac{(-1)^n}{n}|$ goes to infinity.

Finally, consider the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ This too has $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = 1$.

$$\begin{aligned}\lim_{n\to\infty} |\frac{\frac{1}{n+1^2}}{\frac{1}{n^2}}| &= \lim_{n\to\infty} |\frac{1}{(n+1)^2}\frac{n^2}{1}| \\ &= \lim_{n\to\infty} \frac{n^2}{(n+1)^2} \\ &= 1\end{aligned}$$

The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is absolutely convergent because $\sum_{n=1}^{\infty} |\frac{1}{n^2}|$ converges.

Exercise 1.9.20

Give examples to show that if $r = 1$ in the statement of the Root Test, anything my happen.

Borrowing from the ideas in the last exercise, consider the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ and $\sum_{n=1}^{\infty} \frac{1}{n}$. Both have $r = 1$.

$$\limsup_{n\to\infty} |\frac{1}{n}|^{\frac{1}{n}} = 1$$

$$\limsup_{n\to\infty} |\frac{1}{n^2}|^{\frac{1}{n}} = 1$$

Anything may happen when $r = 1$

Exercise 1.9.26

Determine the radius of convergence of the following power series:

$$r = \limsup_{n\to\infty} |a_n|^{\frac{1}{n}}$$

Where the radius of convergence is $\frac{1}{r}$ if $r > 0$, ∞ if $r = 0$ and 0 if lim sup does not exist, $(r = \infty)$.

1. $\sum_{n=1}^{\infty} \frac{z^n}{n!}$

$$\sum_{n=1}^{\infty} \frac{z^n}{n!} = \sum_{n=1}^{\infty} \frac{1}{n!} z^n$$

$$\lim_{n\to\infty} |\frac{1}{n!}|^{\frac{1}{n}} = 0$$

Radius of convergence is ∞

2. $\sum_{n=2}^{\infty} \frac{z^n}{\ln(n)}$

$$\sum_{n=1}^{\infty} \frac{z^n}{\ln(n)} = \sum_{n=2}^{\infty} \frac{1}{\ln(n)} z^n$$

$$\lim_{n\to\infty} |\frac{1}{\ln(n)}|^{\frac{1}{n}} = 1$$

Radius of convergence is 1.

3. $\sum_{n=1}^{\infty} \frac{n^n}{n!} z^n$

$e^n = \sum_{n=1}^{\infty} \frac{n^n}{n!}$

$$\lim_{n\to\infty} |\frac{n^n}{n!}|^{\frac{1}{n}} = e$$

Radius of convergence is $\frac{1}{e}$

Exercise 2.5.8

Prove that the scalar product is a positive definite symmetric bilinear form on $\mathbb{E}^n$.

Proof. The scalar product of vectors $\mathbf{v} = (v_1, v_2, v_3, ..., v_n)$ and $\mathbf{w} = (w_1, w_2, w_3, ..., w_n)$ is $\langle \mathbf{v}, \mathbf{w} \rangle = v_1 w_1 + v_2 w_2 + v_3 w_3 + ... + v_n w_n$.

Let $V = \mathbb{E}^n$ be a vector space over $F = \mathbb{R}$. A bilinear form $\langle .,. \rangle$ on V is a map

$$\langle .,. \rangle : V \times V \to F$$

The satisfies linearity in both variables. That is, for all $\mathbf{v}, \mathbf{v}_1 \mathbf{v}_2, \mathbf{w}, \mathbf{w}_1, \mathbf{w}_2 \in V$ and all $\alpha \in F$

$$\begin{aligned}
\langle \mathbf{v}_1 + \mathbf{v}_2, \mathbf{w} \rangle &= (v_{1_1} + v_{2_1})w_1 + (v_{1_2} + v_{2_2})w_2 + (v_{1_3} + v_{2_3})w_3 + ... + (v_{1_n} + v_{2_n})w_n \\
&= v_{1_1}w_1 + v_{2_1}w1 + v_{1_2}w_2 + v_{2_2}w2 + v_{1_3}w_3 + v_{2_3}w3 + ... + v_{1_n}w_n + v_{2_n}w_n \\
&= (v_{1_1}w_1 + v_{1_2}w_2 + v_{1_3}w_3 + ... + v_{1_n}w_n) + (v_{2_1}w_1 + v_{2_2}w_2 + v_{2_3}w_3 + ... + v_{2_n}w_n) \\
&= \langle \mathbf{v_1}, \mathbf{w} \rangle + \langle \mathbf{v_2}, \mathbf{w} \rangle
\end{aligned}$$

$$\begin{aligned}
\langle \alpha\mathbf{v}, \mathbf{w} \rangle &= \alpha v_1 w_1 + \alpha v_2 w_2 + \alpha v_3 w_3 + ... + \alpha v_n w_n \\
&= \alpha(v_1 w_1) + \alpha(v_2 w_2) + \alpha(v_3 w_3) + ... + \alpha(v_n w_n) \\
&= \alpha(v_1 w_1 + v_2 w_2 + v_3 w_3 + ... + v_n w_n) \\
&= \alpha\langle \mathbf{v}, \mathbf{w} \rangle
\end{aligned}$$

$$\begin{aligned}
\langle \mathbf{v}, \mathbf{w}_1 + \mathbf{w}_2 \rangle &= v_1(w_{1_1} + w_{2_1}) + v_2(w_{1_2} + w_{2_2}) + v_3(w_{1_3} + w_{2_3}) + ... + v_n(w_{1_n} + w_{2_n}) \\
&= v_1 w_{1_1} + v_1 w_{2_1} + v_2 w_{1_2} + v_2 w_{2_2} + v_3 w_{1_3} + v_3 w_{2_3} + ... + v_n w_{1_n} + v_n w_{2_n} \\
&= (v_1 w_{1_1} + v_2 w_{1_2} + v_3 w_{1_3} + ... + v_n w_{1_n}) + (v_1 w_{2_1} + v_2 w_{2_2} + v_3 w_{2_3} + ... + v_n w_{2_n}) \\
&= \langle \mathbf{v}, \mathbf{w}_1 \rangle + \langle \mathbf{v}, \mathbf{w}_2 \rangle
\end{aligned}$$

$$\begin{aligned}
\langle \mathbf{v}, \alpha\mathbf{w} \rangle &= v_1 \alpha w_1 + v_2 \alpha w_2 + v_3 \alpha w_3 + ... + v_n \alpha w_n \\
&= \alpha(v_1 w_1) + \alpha(v_2 w_2) + \alpha(v_3 w_3) + ... + \alpha(v_n w_n) \\
&= \alpha(v_1 w_1 + v_2 w_2 + v_3 w_3 + ... + v_n w_n) \\
&= \alpha\langle \mathbf{v}, \mathbf{w} \rangle
\end{aligned}$$

Regarding being positive definite , consider $\langle \mathbf{v}, \mathbf{v} \rangle$

$$\begin{aligned}
\langle \mathbf{v}, \mathbf{v} \rangle &= v_1 v_1 + v_2 v_2 + v_3 v_3 + ... + v_n v_n \\
&= v_1^2 + v_2^2 + v_3^2 + ... + v_n^2 \\
&\geq 0
\end{aligned}$$

Notice also, that $\langle \mathbf{v}, \mathbf{v} \rangle = 0$ if and only if $\mathbf{v} = 0$

□

Exercise 2.5.11

Prove the following properties of the norm if $\mathbf{v}, \mathbf{w} \in \mathbb{E}^n$

1. $||\mathbf{v}|| \geq 0$

Proof. If $\mathbf{v} \in \mathbb{E}^n$ the norm is defined by

$$\begin{aligned}
||\mathbf{v}|| &= \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle} \\
&= \sqrt{(v_1 v_1 + v_2 v_2 + v_3 v_3 + ... + v_n v_n)} \\
&= \sqrt{(v_1^2 + v_2^2 + v_3^2 + ... + v_n^2)} \\
&= (v_1^2 + v_2^2 + v_3^2 + ... + v_n^2)^{\frac{1}{2}} \\
&\geq 0
\end{aligned}$$

$\square$

2. $||\mathbf{v}|| = 0$ if and only if $\mathbf{v} = 0$

From the definition of positive definite, $\langle \mathbf{v}, \mathbf{v} \rangle = 0$ if and only if $\mathbf{v} = 0$

Proof. "$\Rightarrow$" Let $\|\mathbf{v}\| = 0$. Then

$$\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle} = 0$$

$$\begin{aligned}
\sqrt{\langle \mathbf{v}, \mathbf{v} \rangle} &= 0 \\
\sqrt{(v_1 v_1 + v_2 v_2 + v_3 v_3 + ... + v_n v_n)} &= 0 \\
(v_1^2 + v_2^2 + v_3^2 + ... + v_n^2)^{\frac{1}{2}} &= 0 \\
(v_1^2 + v_2^2 + v_3^2 + ... + v_n^2) &= 0
\end{aligned}$$

So $\mathbf{v} = 0$ $\square$

Proof. "$\Leftarrow$"

Let $\mathbf{v} = 0$

So $\langle \mathbf{v}, \mathbf{v} \rangle = 0$.

$$\begin{aligned}
\|\mathbf{v}\| &= \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle} \\
&= \sqrt{0} \\
&= 0
\end{aligned}$$

$\square$

3. $\|\alpha \mathbf{v}\| = |\alpha| \|\mathbf{v}\|$, $\alpha \in$

Proof.

$$\begin{aligned}\|\alpha\mathbf{v}\| &= \sqrt{\langle \alpha\mathbf{v}, \alpha\mathbf{v}\rangle} \\ &= \sqrt{(\alpha^2 v_1^2 + \alpha^2 v_2^2 + \alpha^2 v_3^2 + ... + \alpha^2 v_n^2)} \\ &= \sqrt{\alpha^2 (v_1^2 + v_2^2 + v_3^2 + ... + v_n^2)} \\ &= |\alpha| \sqrt{(v_1^2 + v_2^2 + v_3^2 + ... + v_n^2)} \\ &= |\alpha| \sqrt{\langle \mathbf{v}, \mathbf{v}\rangle} \\ &= |\alpha| \|\mathbf{v}\|\end{aligned}$$

□

4. $\|\mathbf{v} + \mathbf{w}\| \leq \|\mathbf{v}\| + \|\mathbf{w}\|$

Proof. First recall the Pythagorean theorem, $a^2 + b^2 = c^2$ and Cauchy-Schwarz Inequality, let $\mathbf{v}, \mathbf{w} \in \mathbb{E}^n$, then $|\langle \mathbf{v}, \mathbf{w}\rangle| \leq \|\mathbf{v}\| \|\mathbf{w}\|$

$$\begin{aligned}\|\mathbf{v} + \mathbf{w}\|^2 &= \langle \mathbf{v} + \mathbf{w}, \mathbf{v} + \mathbf{w}\rangle \\ &= (v_1 + w_1)^2 + (v_2 + w_2)^2 + ... + (v_n + w_n)^2 \\ &= (v_1^2 + v_2^2 + ... + v_n^2) + (w_1^2 + w_2^2 + ... + w_n^2) + (v_1 w_1 + ... + v_n w_n) + (w_1 v_1 + ... + w_n v_n) \\ &= \langle \mathbf{v}, \mathbf{v}\rangle + \langle \mathbf{v}, \mathbf{w}\rangle + \langle \mathbf{w}, \mathbf{v}\rangle + \langle \mathbf{w}, \mathbf{w}\rangle \\ &\leq \|\mathbf{v}\|^2 + 2|\langle \mathbf{v}, \mathbf{w}\rangle| + \|\mathbf{w}\|^2 \\ &\leq \|\mathbf{v}\|^2 + 2\|\mathbf{v}\| \|\mathbf{w}\| + \|\mathbf{w}\|^2 \\ &= (\|\mathbf{v}\| + \|\mathbf{w}\|)^2\end{aligned}$$

□

5. $\|\mathbf{v} + \mathbf{w}\|^2 + \|\mathbf{v} - \mathbf{w}\|^2 = 2(\|\mathbf{v}\|^2 + \|\mathbf{w}\|^2)$

Proof.

$$\begin{aligned}\|\mathbf{v} + \mathbf{w}\|^2 + \|\mathbf{v} - \mathbf{w}\|^2 &= \langle \mathbf{v} + \mathbf{w}, \mathbf{v} + \mathbf{w}\rangle + \langle \mathbf{v} - \mathbf{w}, \mathbf{v} - \mathbf{w}\rangle \\ &= (v_1 + w_1)^2 + ... + (v_n + w_n)^2 + (v_1 - w_1)^2 + ... + (v_n - w_n)^2 \\ &= (v_1^2 + 2v_1 w_1 + w_1^2 + ... + v_n^2 + 2v_n w_n + w_n^2) + (v_1^2 - 2v_1 w_1 + w_1^2 + ... + v_n^2 - 2v_n w_n + w_n^2) \\ &= (2v_1^2 + ... + 2v_n^2) + (2w_1^2 + ... + 2w_n^2) \\ &= 2(v_1^2 + ... + v_n^2) + 2(w_1^2 + ... + w_n^2) \\ &= 2(\langle \mathbf{v}, \mathbf{v}\rangle + \langle \mathbf{w}, \mathbf{w}\rangle) \\ &= 2(\|\mathbf{v}\|^2 = \|\mathbf{w}\|^2)\end{aligned}$$

□