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REAL ANALYSIS
FIRST WEEK

Section 1.5 Construction of the Real Numbers

Exercise 1.5.1

Show that for any $a, b \in \mathbb{Q}$, we have $||a| - |b|| \leq |a - b|$.

Proof. Since $a, b \in \mathbb{Q}$,

$$|a + b| \leq |a| + |b|$$

Absolute values on $\mathbb{Q}$ satisfy the Triangle Inequality

So

$$|a| = |a - b + b| \leq |a - b| + |b|$$

$$|b| = |a + b - a| \leq |b - a| + |a|$$

These can be rewritten as

$$|a| - |b| \leq |a - b|$$

$$|b| - |a| \leq |b - a|$$

Since $|a - b| = |b - a|$ and if $t \geq x$ and $t \geq -x$ then $t \geq |x|$, therefore

$$||a| - |b|| \leq |a - b|$$

$\square$

Exercise 1.5.5

If a sequence $(a_k)_{k \in \mathbb{N}}$ converges in $\mathbb{Q}$ show that $(a_k)_{k \in \mathbb{N}}$ is a Cauchy sequence in $\mathbb{Q}$.

Proof. By definition if $(a_k)_{k \in \mathbb{N}}$ converges in $\mathbb{Q}$ given any rational number $r > 0$ there exists an integer N such that if $n \geq N$ then $|a_n - a| < r$.

Suppose $(a_k)_{k \in \mathbb{N}}$ converges to $a, a \in \mathbb{Q}$. Let $r > 0$, since $(a_k)_{k \in \mathbb{N}}$ converges to a, $\exists N$ such that $\forall n \geq N$, $|a_n - a| < \frac{r}{2}$.

Then $\forall n, m > N$

$$|a_n - a_m| = |a_n - a + a - a_m| \leq |a_n - a| + |a - a_m|$$

Let $n, m > N$

$$|a_n - a| < \frac{r}{2}$$

and

$$|a - a_m| = |a_m - a| < \frac{r}{2}$$

therefore

$$|a_n - a_m| < \frac{r}{2} + \frac{r}{2} = r$$

$\square$

Exercise 1.5.6

Show that the limit of a convergent sequence is unique.

Proof. Suppose $(a_k)_{k\in\mathbb{Q}}$ converges in $\mathbb{Q}$ to L and M. Choose L and $M, L \neq M$ and let $r = \frac{|L-M|}{2}$. Then $\exists N_1 \in \mathbb{N}$ such that if $n \geq N_1$ then

$$|a_n - L| < r$$

and $\exists N_2 \in \mathbb{N}$ such that if $n \geq N_2$ then

$$|a_n - M| < r$$

Let $N = max(N_1, N_2)$. If $n \geq N$ then

$$|L-M| = |L-a_n+a_n-M| \leq |L-a_n|+|a_n-M| < 2(\frac{|L-M}{2}|) = |L-M|$$

Reducing the above, we have $|L-M| < |L-M|$ a contradiction, $\Rightarrow\Leftarrow$. Therefore, $L = M$.

□

Exercise 1.5.9

Show that the sum of two Cauchy sequences in $\mathbb{Q}$ is a Cauchy sequence in $\mathbb{Q}$.

Proof. Let $(a_k)_{k\in\mathbb{N}}$ and $(b_k)_{k\in\mathbb{N}}$ be Cauchy sequences $\mathbb{Q}$. Let $r > 0$, $\exists N_1 \in \mathbb{N}$ such that if $n, m \geq N_1$ then

$$|a_n - a_m| < \frac{r}{2}$$

and $\exists N_2 \in \mathbb{N}$ such that if $n, m \geq N_2$ then

$$|b_n - b_m| < \frac{r}{2}$$

Let $N = max(N_1, N_2)$ and choose $n, m \geq N$. This implies

$$|a_n - a_m| + |b_n - b_m| < \frac{r}{2} + \frac{r}{2} = r$$

$$|a_n - a_m + b_n - b_m| \leq |a_n - a_m| + |b_n - b_m|$$

Therefore

$$|(a_n + b_n) - (a_m + b_m)| < r$$

□

Exercise 1.5.13

Show that if a Cauchy sequence $(a_k)_{k\in\mathbb{N}}$ does not converge to 0, all the terms of the sequence eventually have the same sign.

Lem.: *1.5.12 Suppose $(a_k)_{k\in\mathbb{N}} \in \mathcal{C} \setminus \mathcal{I}$, then there exists a positive rational number r and an integer N such that $|a_n| \geq r$ for all $n \geq N$.*

Where $\mathcal{C}$ denotes the set of all Cauchy sequences of rational numbers and $\mathcal{I}$ denotes the set of all Cauchy sequences that converge to 0.

Proof. Suppose $(a_k)_{k\in\mathbb{N}}$ is a Cauchy sequence that does not converge to 0. Therefore given any $r > 0$ there exists an integer N such that if $n, m \geq N$, then $|a_n - a_m| < r$. From Lemma 1.5.2, we can choose $r > 0$ and N such that $|a_n| \geq r$ for all $n \geq N$.

Let $r > 0$ and $n, m \geq N$. Therefore

$$|a_n - a_m| < r \leq |a_n|$$

Suppose $a_n > 0$ and $a_m < 0$

In other words, they don't have the same sign.

$$|a_n - (-a_m)| = |a_n + a_m| < |a_n| \Rightarrow\Leftarrow$$

Likewise, suppose $a_n < 0$ and $a_m > 0$

$$|a_n - a_m| = |a_m - a_n|$$

$$|a_m - (-a_n)| = |a_m + a_n| < |a_n| \Rightarrow\Leftarrow$$

Therefore, all terms must eventually be the same sign.

□

Exercise 1.5.15

Show that $\sim$ defines an equivalence relation on $\mathcal{C}$ We need to show reflexivity, symmetry, and transitivity exist on Cauchy sequences that are equivalent.

$$(a_k)_{k\in\mathbb{N}} \sim (a_k)_{k\in\mathbb{N}}$$

For all $a_n \in (a_k)_{k\in\mathbb{N}}, |a_n - a_n| = 0$ Thus we can say that $(a_k - a_k)_{k\in\mathbb{N}}$ is in $\mathcal{I}$

$$(a_k)_{k\in\mathbb{N}} \sim (b_k)_{k\in\mathbb{N}}$$

Suppose that $(a_k)_{k\in\mathbb{N}}$ and $(b_k)_{k\in\mathbb{N}}$ are equivalent and $r > 0$. Then, there exist $N \in \mathbb{N}$, such that exists $|a_n - b_n| < r$ and $|b_n - a_n| < r$ for $n \geq N$ and $(b_k - a_k)_{k\in\mathbb{N}}$ in $\mathcal{I}$

$$(a_k)_{k\in\mathbb{N}} \sim (b_k)_{k\in\mathbb{N}}, (b_k)_{k\in\mathbb{N}} \sim (c_k)_{k\in\mathbb{N}} \Rightarrow (a_k)_{k\in\mathbb{N}} \sim (c_k)_{k\in\mathbb{N}}$$

Let $r > 0$, $\exists N_1 \in \mathbb{N}$ such that $|a_n - b_n| < \frac{r}{2}$ for all $n \geq N_1$ and $\exists N_2 \in \mathbb{N}$ such that $|b_n - c_n| < \frac{r}{2}$ for all $n \geq N_2$. This implies

$$|a_n - c_c| = |a_n - b_n + b_n - c_n| \leq |a_n - b_n| + |b_n - c_n| < \frac{r}{2} + \frac{r}{2} = r$$

for all $n = max(N_1, N_2)$. Therefore, $(a_k - c_k)_{k\in\mathbb{N}}$ is also in $\mathcal{I}$.

Exercise 1.5.17

Show that $\mathbf{R}$ is a commutative ring with 1, with $\mathcal{I}$ as the additive identity and $[a_k]$ such that $a_k = 1$ for all k as the multiplicative identity.

We know that if $(a_k), (b_k)$ are Cauchy sequences, $(a_n)_{n\in\mathbb{N}} + (a_n)_{n\in\mathbb{N}} = (a_n + b_n)_{n\in\mathbb{N}}$ and $(a_n)_{n\in\mathbb{N}}(b_n)_{n\in\mathbb{N}} = (a_n b_n)_{n\in\mathbb{N}}$ are well-defined.

Let $[a_k]$ be an equivalence class, addition and multiplication are defined as follows $[a_k] + [a_k] = [a_k + a_k]$ and $[a_k][a_k] = [a_k a_k]$

As examples, consider $(a_k)_{k\in\mathbb{N}}$ and $(a'_k)_{k\in\mathbb{N}}$ denoted as $\{a_k\}$ and $\{a'_k\}$, respectively. Likewise, $(b_k)_{k\in\mathbb{N}}$ and $(b'_k)_{k\in\mathbb{N}}$ denoted as $\{b_k\}$ and $\{b'_k\}$

For addition, let $\{a_k\} \sim \{a'_k\}$, $\{b_k\} \sim \{b'_k\}$ and $r > 0$. Then, $\exists N_1$ $\exists N_2$ in $\mathbb{N}$ such that

$$|a_n - a'_n| < \frac{r}{2} \text{ for } n \geq N_1$$

and

$$|b_n - b'_n| < \frac{r}{2} \text{ for } n \geq N_2$$

This implies

$$|(a_n + b_n) - (a'_n + b'_n)| = |a_n - a'_n + b_n - b'_n| \leq |a_n - a'n| - |b_n - b'n| < \frac{r}{2} + \frac{r}{2}$$

for $n \geq max(N_1, N_2)$. Therefore $[a_k + b_k]$ is in $\mathcal{I}$

It follows, if $[i_k]$ is $\mathcal{I}$ then $[a_k] + [i_k] = [a_k]$

Lem.: *1.5.8. Let $(a_k)_{k\in\mathbb{N}}$ be a Cauchy sequence of rational numbers. Then $(a_k)_{k\in\mathbb{N}}$ is a bounded sequence*

For multiplication, recall that $\{a_k\}$, $\{a'_k\}$, $\{b_k\}$ and $\{b'_k\}$ are bounded. $\exists M > 0$ such that $\{a_n\}$, $\{a'_n\}$, $\{b_n\}$,$\{b'_n\} \leq M$ for all $n \geq N \in \mathbb{N}$ such that

$$|a_n - a'_n| < \frac{r}{2M} \text{ for } n \geq N_1$$

and

$$|b_n - b'_n| < \frac{r}{2M} \text{ for } n \geq N_2$$

$$\begin{aligned}
2|a_n b_n - a'_n b'_n| &= |(a_n - a'_n)(b_n + b'_n) + (a_n + a'_n)(b_n - b'_n)| \\
&\leq |(a_n - a'_n)(b_n + b'_n)| + |(a_n + a'n)(b_n - b'_n)| \\
&= |a_n - a'n||b_n + b'_n| + |a_n + a'n||b_n - b'_n| \\
&\leq |a_n - a'n|(|b_n| + |b'_n|) + (|a_n| + |a'n|)|b_n - b'_n| \\
&< \frac{r}{2M}(2M) + \frac{r}{2M}(2M) \\
&= 2r
\end{aligned}$$

Therefore $|a_n b_n - a'_n b'_n| < r$ for all $n \geq max(N_1, N_2)$.

If $[i_k]$ with $i_k = 1$ for all k is the multiplicative identity it follows that $[a_k][i_k] = [a_k]$

Exercise 1.5.20

Show that order relation, defined below is well-defined and makes **R** and ordered field.

Def.: Let $a = [a_k]$ and $b = [b_k]$ be distinct elements of **R**. We define $a < b$ if $a_k < b_k$ eventually and $b < a$ if $b_k < a_k$ eventually.

Let $r > 0$, then there exists $n, m \geq N_1 \in \mathbb{N}$ such that

$$|c_n - c_m| = |(a_n - b_n) - (a_m - b_m)| < r$$

Since $[c_k]$ is not in $\mathcal{I}$ We can eventually find an $n > N_2 \in \mathbb{N}$ such that $|c_n| > r$ Therefore

$$|a_n - b_n| > r > 0$$

We also know, from exercise 1.5.13 that all terms at this point in the sequence need to have the same sign and it's easy to see that $a_n \neq b_n$. So it follows that either $a_k > b_k$ or $a_k < b_k$, eventually.

We can apply the above to the Order Axioms.

1. $(O1)$ **Trichotomy:** Since $[a] - [b]$ is not in $\mathcal{I}$, by definition either $a_k < b_k$ or $b_k > a_k$, eventually.
2. $(O2)$ **Transitivity:** For sake of argument, let $a_k < b_k$, eventually, and choose an additional arbitrary element of **R** $[c_k]$. Let $b_k < c_k$. Then $a_k < c_k$, eventually
3. (03) **Addition:** Let $a_k < b_k$ and choose $[c]$ to be in $\mathcal{I}$ it easily follows that $a_k + c_k < b_k + c_k$, eventually
4. (04) **Multiplication:** $a_k < b_k$ and let $[c_k]$ be the multiplicative identity $c_k = 1$ for all $k \in \mathbb{N}$, then $a_k c_k < b_k c_k$, eventually

Therefore order relation is well-defined and makes **R** and ordered field.

Exercise 1.6.11

Find a bounded sequence of real numbers that is not convergent.

Define $(a_k)_{k \in \mathcal{N}} = (-1)^k$, this sequence is bounded $[-1, 1]$. It is clear that $\{1, -1, 1, -1, ...\}$ does not converge.

Exercise 1.6.16

Prove Lemma 1.5.15

Lem.: *Lemma 1.5.15 Every bounded sequence in* $\mathbb{R}$ *has a convergent subsequence*

Let $(a_k)_{k\in\mathbb{N}}$ be a bounded sequence in $\mathbb{R}$.

Lem.: *Lemma 1.6.13: every bounded sequence in $\mathbb{R}$ has a monotonic subsequence.*

Lem.: *Lemma 1.6.14: Every bounded monotonic sequence in $\mathbb{R}$ converges to an element in $\mathbb{R}$.*

If $(a_k)_{k\in\mathbb{N}}$ does not have a monotonically increasing subsequence, $\exists n_1 \in \mathbb{N}$ such that $a_{n_1} > a_k$ for $k > n_1$. It follows that since $(a_k)_{k>n}$ is not monotonically increasing, there exists $a_{n_2} > a_k$ for $k > n_2$ and $a_{n_1} > a_{n_2}$ This process can be repeated over the set $(a_k)_{k\in\mathbb{N}}$ to create a strictly monotonic decreasing set $(a_{n_1}, a_{n_2}, ..., a_{n_k})$.

Alternatively, if $(a_k)_{k\in\mathbb{N}}$ does not have a strictly monotonic decreasing subsequence. We say $a_{n_1} < a_k$ for $k \geq n_1$ Repeating steps above to form a set $(a_{n_1}, a_{n_2}, ..., a_{n_k})$. Which is monotonic increasing.

Since (a_k) is bounded, (a_{k_j}) is bounded and we can apply Lemma 1.6.14 (a_{k_j}) converges to an element in R.

Lem.: *Every bounded monotonic sequence in $\mathbb{R}$ and conclude converges to an element in $\mathbb{R}$*

Exercise 1.6.20

Show that if $\limsup_{k\to\infty}(a_k) = \liminf_{k\to\infty}(a_k)$, then $(a_k)_{k\in\mathbb{N}}$ is convergent, and $\lim_{k\to\infty}(a_k) = \limsup_{k\to\infty}(a_k) = \liminf_{k\to\infty}(a_k)$.

It first helps to rewrite the definition of limit supremum and limit infimum.

$$\limsup_{k\to\infty}(a_k) = \lim_{n\to\infty}(b_n), \text{ where } b_n = sup\{a_k | k \geq n\}$$
$$\liminf_{k\to\infty}(a_k) = \lim_{n\to\infty}(c_n), \text{ where } c_n = inf\{a_k | k \geq n\}$$

These definitions combined with the information that the limit supremum of (a_k) being equal the the limit infenum of (a_k) imply that our sequence (a_n) converges to the same limit. The reason for this is that our infenum and supremem eventually are within epsilon of each other. Which implies that the whole sequence a_n converges to this same limit as the inf and sup.

Notice that c_n increases as $n \to \infty$ and b_n decreases as $n \to \infty$. Likewise, Since $c_n \leq a_n \leq b_n$ we can apply the Sandwich theorem and say $\lim_{n\to\infty} a_n = a$

Since c_k and b_k are items in $(a_k)_{k\in\mathbb{N}}$ and equal we can refer to the above as

$$\lim_{k\to\infty}(a_k)$$

Which implies $(a_k)_{k\in\mathbb{N}}$ converges and finally...

$$\lim_{k\to\infty}(a_k) = \limsup_{k\to\infty}(a_k) = \liminf_{k\to\infty}(a_k)$$