3.5.9)

i) Counter example

Consider $A = (0,1)$ $X = \mathbb{R}$ with usual metric

$$f(x) = x$$

$f(A) = (0,1)$ $X' = \mathbb{C}$ with usual metric

ii) Counter example

$f : (\mathbb{R}, \text{discrete}) \to f : (\mathbb{R}, \text{usual})$

$f(x) = x$

$A = (0,1)$ $f(A) = (0,1)$

iii) Let B be a closed subset of X'.

$\therefore B^c$ is open

Since f is continuous, $f^{-1}[B^c]$ is open.

$\therefore (f^{-1}[B^c])^c = f^{-}[B]$ is closed

iv) Counter example

$f : (\mathbb{R}, \text{usual}) \to (\mathbb{R}, \text{usual})$

$A = (0,1)$ $f(x) = 1/x$, $f(A) = \{ 1/x \mid x \in A = (0,1) \}$

v) Counter example

$f^{-1} : (\mathbb{R}, \text{usual}) \to (\mathbb{R}, \text{usual})$

$B = (0,1)$ $f^{-1}(x) = 1/x$, $f^{-1}(B) = \{ 1/x \mid x \in B(0,1) \}$

3.5.9)

vi) Counterexample

$f : (\mathbb{R}, \text{usual}) \to (\mathbb{R}, \text{usual})$

$A = \{-1\} \cup [1, 5]$

$x_0 = \{-1\}$, $f(x) = |x|$, $f(A) = [1, 5]$

vii) Counterexample

$f : (\mathbb{R}, \text{usual}) \to (\mathbb{R}, \text{usual})$

$A = [0, 1]$, $f(x) = x$, $x_0 = 0$, $f(A) = \{0\}$, $f(x_0) = 0$

viii) Counterexample

$f : (\mathbb{R}, \text{usual}) \to (\mathbb{R}, \text{usual})$

$A = [0, 1]$, $f(x) = x$, $x_0 = 0$, $f(A) = \{0\}$, $f(x_0) = 0$

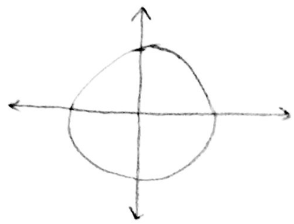
ix) Counterexample

$f : (\mathbb{R}, \text{discrete}) \to (\mathbb{R}, \text{usual})$

$A = [0, 1]$, $f(x) = x$, $f(A) = [0, 1]$, $f(x_0) = 0$, $x_0 = 0$

3.5.13)

$f(x) = e^{2\pi i x}$



From the graph, it is clear that f is continuous.
Now, we check for bijectivity.
Check for 1-1.
Suppose $f(x) = f(y)$

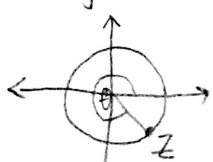
$\cos 2\pi x = \cos 2\pi y$ → real parts

$\sin 2\pi x = \sin 2\pi y$ → imaginary parts

$2\pi x = 2\pi y$

$x = y$

Check for onto. let $z \in \mathbb{C}$. $z = a + ib$



let $x = \frac{\theta}{2\pi}$. Then $f(x) = z$.

Now, let us show f^{-1} is not continuous at $1 \in \mathbb{T}$.
That is, we have to show $\exists \varepsilon > 0, \forall \delta > 0, \exists x \in \mathbb{T}, |x - 1| < \delta$ but $|f^{-1}(x) - f^{-1}(1)| \geq \varepsilon$
Choose $\varepsilon = \frac{1}{2}$, let $\delta > 0$. Choose $x \in \mathbb{T} \cap B_\delta(1)$ s.t. $x = a + ib$, $b < 0$.
Then $f^{-1}(1) = 0$ and $f^{-1}(x) \geq \frac{1}{2}$

$|f^{-1}(1) - f^{-1}(x)| \geq \frac{1}{2} = \varepsilon$ □

3.5.15)

$I(x) = x$ is continuous (This is trivial i.e $\forall \varepsilon > 0$ choose $\delta = \varepsilon$)

$I(a) = I(b) \Rightarrow a = b \therefore I$ is 1-1

Surjective property of I is also trivial, $f^{-1}(b) = a$.

To show, that I is not a homeomorphism, we show that f^{-1} is not continuous.

Consider $\underset{\text{discrete}}{I : \mathbb{R}} \rightarrow \underset{\text{usual}}{\mathbb{R}}$ $\quad I(x) = x$

$I^{-1} = J : \underset{\text{usual}}{\mathbb{R}} \rightarrow \underset{\text{discrete}}{\mathbb{R}}$ $\quad J(x) = x$

J is not continuous because $[0,1]$ is open in $(\mathbb{R}$, discrete$)$ but $J^{-1}([0,1]) = [0,1]$ is not open in $(\mathbb{R}$, usual$)$.

$\therefore I$ is not a homeomorphism

3.5.23)

i) $f(x) = x^3$.

First, we check if f is a bijection.

15 Check for 1-1.

Assume $f(a) = f(b)$

$$a^3 = b^3 \Rightarrow a = b \quad \therefore f \text{ is } 1\text{-}1$$

Check for onto.

$$y = x^3$$

$$x = \sqrt[3]{y}$$

$$f(\sqrt[3]{y}) = y = x^3 \quad \therefore f \text{ is onto}$$

$\therefore f$ is a bijection

- Check f is continuous.

Note that $g(x) = x$ is continuous.

Also note that the product of continuous functions are continuous

$\therefore f(x) = x^3$ is continuous

- Check $f^{-1}(x) = \sqrt[3]{x}$ is continuous.

If $x = 0$, then we choose $\delta = \varepsilon^3$.

Suppose $x > 0$. Let $\varepsilon > 0$. Choose $\delta = \min\{x, \varepsilon \cdot \sqrt[3]{x^2}\}$

Suppose $|x - y| < \delta$

$x - y \leq |x - y| < \delta \leq x \quad \therefore y > 0$

Then $|\sqrt[3]{x} - \sqrt[3]{y}| = \dfrac{x - y}{\sqrt[3]{x^2} + \sqrt[3]{xy} + \sqrt[3]{y^2}}$ $\qquad \sqrt[3]{x^2} + \sqrt[3]{xy} + \sqrt[3]{y^2} > \sqrt[3]{x^2} > 0$

$$< \frac{\delta}{\sqrt[3]{x^2}} \leq \varepsilon$$

To be continued

3.5.23

i) Part 2) Suppose $x < 0$.

Suppose $|y - x| < \delta$

$x - y \leqslant |y - x| < \delta < x \therefore y < 0$

Note that $\sqrt[3]{x^2} + \sqrt[3]{xy} + \sqrt[3]{y^2} > \sqrt[3]{x^2} > 0$

Then $$|\sqrt[3]{x} - \sqrt[3]{y}| = \frac{|x-y|}{\sqrt[3]{x^2} + \sqrt[3]{xy} + \sqrt[3]{y^2}}$$

$$< \frac{\delta}{\sqrt[3]{x^2}} \leqslant \varepsilon \qquad \square$$

Ex 3.5.23

ii) $f(x) = x + \sin x$

First, we check 1-1.

$f(x) = x + \sin x$

$f'(x) = 1 + \cos x \geq 0$

$\therefore$ f is increasing $\therefore$ f is 1-1

Note that f is a sum of continuous functions thus f is continuous.

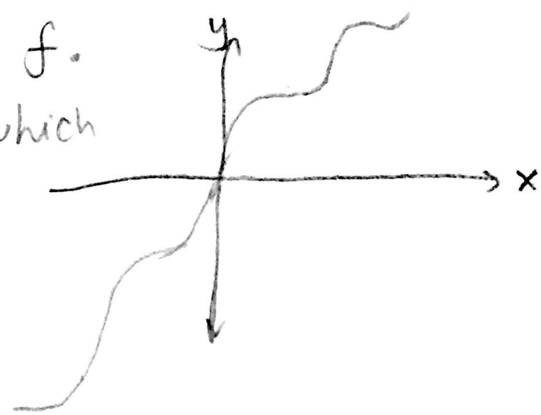
Now, we show that f is not bounded

Note that x is not bounded while $-1 \leq \sin x \leq 1$.

Since f is bounded and continuous thus it is onto.

Consider the graph of f.

We can see that f^{-1} which is the reflection of f at the line $y=x$ is continuous.



$\therefore$ f is an isomorphism

3.5.30) $f(x) = 0$, $\forall x \in (0,1)$. Let $\varepsilon > 0$.

We want N s.t. $\forall n > N$, $\forall x \in (0,1)$, $|f_n(x) - f(x)| < \varepsilon$

If $x \notin \mathbb{Q}$, then the sequence clearly converges.

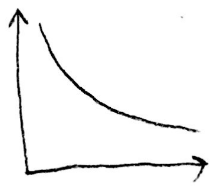
Choose N s.t. $\frac{1}{2^N} < \varepsilon$ which is possible since $2^N \to 0$.

Then for any $n \geq N$ and for any $x \in (0,1)$

$$|f_n(x) - f(x)| = \frac{1}{q^n} < \frac{1}{q^N} \leq \frac{1}{2^N} < \varepsilon$$

3.5.33)

i) $f(x) = 1/x$.



let $\varepsilon > 0$.

let $x = 1/n$, $y = 1/2n$

Suppose $|x - y| = |1/n - 1/2n| = \left|\frac{1}{2n}\right| < \delta$ where n is a natural number

$|f(x) - f(y)| = |n - 2n| = |n| > \varepsilon$ by Archimedean principle.

$\therefore f$ is not uniformly continuous.

3.5.33

ii) let $\varepsilon > 0$. We need $\delta > 0$ s.t. if $|x-y| < \delta \Rightarrow |\sqrt{x} - \sqrt{y}| < \varepsilon$

let $\varepsilon > 0$. Choose $\delta = \varepsilon^2$

Suppose $|x-y| < \delta$. Suppose WLOG that $x \geq y$

$\therefore y \leq x < y + \delta$

$$|\sqrt{x} - \sqrt{y}| = \sqrt{x} - \sqrt{y}$$
$$< \sqrt{y+\delta} - \sqrt{y}$$
$$= \frac{\delta}{\sqrt{y+\delta} + \sqrt{y}}$$
$$\leq \frac{\delta}{\sqrt{\delta}} = \sqrt{\delta} = \varepsilon$$

$\therefore f$ is uniformly continuous

iii) $f(x) = \ln(x)$.

Suppose, for contradiction, f is uniformly continuous.

i.e. $\forall \varepsilon > 0, \exists \delta > 0, \forall x, y \in (0,\infty)$, if $|x-y| < \delta$, then $|f(x) - f(y)| < \varepsilon$.

let $\varepsilon = \ln 1.5$

let $x = \delta$, $y = \delta/2$

$$|x-y| = |\delta - \delta/2| = |\delta/2| < \delta$$

$$|f(x) - f(y)| = |\ln \delta - \ln \delta/2| = \left|\ln \frac{\delta}{(\delta/2)}\right| = \ln 2 > \varepsilon$$

$\therefore$ There is a contradiction as needed

$\therefore f$ is not uniformly continuous

Ex 3.5.33

iv) $f(x) = x \ln x$

Suppose that f is uniformly continuous

Then $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $\forall x, y$,

$|x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$

let $\varepsilon = 1$. $\exists \delta > 0$ s.t. if $|x-y| < \delta \Rightarrow |f(x) - f(y)| < 1$

Choose x, y s.t. $x = y + \delta/2$

$$\begin{aligned} |f(x) - f(y)| &= |x \ln x - y \ln y| \\ &= |(y + \delta/2) \ln (y + \delta/2) - y \ln y| \\ &> (y + \delta/2) \ln y - y \ln y = \delta/2 \ln y > 1 \end{aligned}$$

for sufficiently large y.