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# REAL ANALYSIS

## FIFTH WEEK

### Exercise 3.3.13

Let  $(X, d)$  be a metric space and let  $Y$  be an open set in  $X$ . Show that every open set in  $(Y, d')$ , where  $d'$  is the inherited metric, is also open in  $X$ .

Since  $Y$  is an open set,  $\forall y_0 \in Y$  and fix  $r > 0$  such that  $B_r(y_0) \subset Y$ . Note that  $B_r(y_0) = \{y \in Y \mid d'(y, y_0) < r\}$  where  $y_0, y \in Y (\subset X)$  and  $d'(y_0, y) = d(y_0, y)$ . Since  $y_0$  was arbitrary, any open set in  $Y$  is also open in  $X$ .

### Exercise 3.3.20

Show that  $\mathbb{Q}$  as a subset of  $\mathbb{R}$  with the usual metric is neither open or closed in  $\mathbb{R}$ . (Of course, if the metric space is simply  $\mathbb{Q}$  with the usual metric, then  $\mathbb{Q}$  is both open and closed in  $\mathbb{Q}$ .)

Rational numbers are dense in  $\mathbb{R}$ , which means between any  $p, t \in \mathbb{Q}, p \neq t$  there exists an irrational number,  $i$ . Without loss of generality say  $p < t$ , then  $p < i < t$ . Therefore, every open ball around  $q \in \mathbb{Q}$  contains points not in  $\mathbb{Q}$ , i.e. the open ball,  $\epsilon > 0$ ,  $B_\epsilon(q) \not\subset \mathbb{Q}$ . Therefore  $\mathbb{Q}$  is not open.

Similarly, irrational numbers lie between to rational numbers and none of  $\mathbb{Q}^c$  lie entirely in  $\mathbb{Q}$ . So,  $\mathbb{R} \setminus \mathbb{Q}$  is not open and  $\mathbb{Q}$  is not closed.

### Exercise 3.3.31

Suppose that  $A$  is a subset of a metric space  $X$ . Show that  $\overline{A} = A \cup \{\text{accumulation points of } A\}$

*Proof.* Let  $A' = \{\text{accumulation points of } A\}$ . Consider  $A \cup A'$ , then any  $a \in A \cup A'$  is either in  $A$  or  $A'$ . Consider  $a \in A$ , note that  $A \subset A'$ , so  $a \in \overline{A}$ . If  $a \in A'$ , since  $\overline{A}$  is closed it must contains all the accumulation points of  $A$  so  $a \in \overline{A}$ . Therefore  $A \cup A' \subset \overline{A}$

Now show  $\overline{A} \subset A \cup A'$ . Consider any  $a \in \overline{A}$ . Then  $a \in A$  or  $a \in \overline{A} \setminus A$ . The first case is trivial. If  $a \in \overline{A} \setminus A$ , then  $a \notin A$  but  $a \in \overline{A}$  so  $a$  must be an accumulation point of  $A$ . Therefore, for any  $a \in \overline{A}$ ,  $a \in A$  or  $a \in A'$  so conclude  $\overline{A} \subset A \cup A'$ .  $\square$

### Exercise 3.3.32

Suppose  $A$  is a subset of a metric space  $X$ . Prove or disprove  $\overline{A} = A \cup \partial A$

*Proof.* Want to show  $A \cup \partial A \subset \overline{A}$ . Consider any  $x \in A \cup \partial A$ . Then either  $x \in A$  or  $x \in \partial A$ . If  $x \in A$  then  $x \in \overline{A}$  since  $A \subset \overline{A}$ . If  $x \in \partial A$  then for any  $r > 0$ ,  $B_r(x) \cap A \neq \emptyset$  and  $B_r(x) \cap A^c \neq \emptyset$ . This means

$x$  is either an isolated point in  $A$ , so  $x \in A$ , or it is an accumulation point of  $A$ . Again since  $A \subset \bar{A}$  and  $\bar{A}$  contains all accumulation points,  $x \in \bar{A}$ . Since  $a$  was arbitrary,  $A \cup \partial A \subset \bar{A}$ .

Next, want to show  $\bar{A} \subset A \cup \partial A$ . Consider any  $x \in \bar{A}$ , then  $x \in A$  or  $x \notin A$ . If  $x \in A$  we are done. If  $x \notin A$ , then  $x \in X \setminus A$ . Suppose  $x$  is an exterior point of  $A$ , then  $x \in B_r(x)$ , there exists  $r > 0$  such that  $B_r(x) \cap A = \emptyset$ . However this contradicts  $x \in \bar{A}$  since  $\bar{A}$  is the intersection of every closed set containing  $A$ . Therefore if  $x \notin A$  then  $x \in \partial A$ . Then conclude,  $\bar{A} \subset A \cup \partial A$ .  $\square$

### Exercise 3.3.33

Suppose  $A$  is a subset of a metric space  $X$ . Prove that  $\partial A = \bar{A} \cap \overline{A^c}$ .

*Proof.* Choose any  $x \in \bar{A} \cap \overline{A^c}$ , then  $x \in \bar{A}$  and  $x \in \overline{A^c}$ . Suppose there exists  $r > 0$  such that  $B_r(x) \cap A = \emptyset$ , but then  $x \notin \bar{A}$ , which is a contradiction since  $\bar{A}$  is the intersection of all sets containing  $A$ . So  $x$  is either in  $\partial A$  or  $A$ . Now we should suppose  $x \in A$  and there exists  $r > 0$  such that  $B_r(x) \subset A$ . However, this contradicts that  $x \in \overline{A^c} (= X \setminus \bar{A})$ . Therefore  $x \in \partial A$ . This implies, since  $x$  was arbitrary,  $\bar{A} \cap \overline{A^c} \subset \partial A$ .

Now consider any  $x \in \partial A$ . This implies  $\forall r > 0$ ,  $B_r(x) \cap A \neq \emptyset$  and  $B_r(x) \cap A^c \neq \emptyset$ . Therefore  $x \in \bar{A}$ . Since  $A^c \subset \overline{A^c}$ ,  $x$  is also in  $\overline{A^c}$ . Again, since  $x$  was arbitrary  $\partial A \subset \bar{A} \cap \overline{A^c}$ .  $\square$

### Exercise 3.3.49

1. Describe the closed convex hull of the unit ball in  $\ell_n^p(\mathbb{R})$  for  $1 \leq p \leq \infty$ .

Let  $B_1(0)$  be the unit ball in  $\ell_n^p(\mathbb{R})$  and  $\mathcal{I} := \{p, q \in \overline{B_1(0)}\}$ . The closed convex hull of  $B_1(0)$  is  $\bigcap_{i \in \mathcal{I}} \{(1-t)p_i + t(q_i) | 0 \leq t \leq 1 \text{ with } t \in \mathbb{R}\}$

2. Suppose  $0 < p < 1$  For  $x \in \mathbb{R}^n$ , define,

$$\|x\|_p = \left( \sum_{k=1}^n |x_k|^p \right)^{\frac{1}{p}}$$

Define  $S_p = \{x \in \mathbb{R}^n | \|x\|_p \leq 1\}$ . Determine whether  $S_p$  is convex. If not, find the closed convex hull of  $S_p$ .

*Proof.* For  $S_p$  to be convex, the second derivative needs to be

greater than 0.

$$\begin{aligned} S'_p &= p|x_k|^{(p-1)} \\ S''_p &= p(p-1)|x_k|^{(p-2)} \end{aligned}$$

Since  $0 < p < 1$ ,  $p-1 < 0$  so  $S''_p < 0$  which means it is not convex. So now it is time to find the closed convex hull of  $S_p$ .

Consider the points at the corners of  $S_p$ , i.e.  $p, q \in S_p$  where  $p = (1, 0)$  and  $q = (0, 1)$  and take the norm of the line segment formed by these two points.

$$\begin{aligned} \|(1-t)p + tq\|_p &= \|1-t[0,1] + t[0,1]\|_p \text{ with } 0 \leq t \leq 1 \\ &= \|[1-t, 0] + [0, t]\|_p \\ &= \|(1-t, t)\|_p \\ &= (|(1-t)|^p + |t|^p)^{\frac{1}{p}} \\ &\geq ((1-t) + t)^{\frac{p}{p}} \\ &= 1 \end{aligned}$$

For  $0 < p < 1$ ,  $(|(1-t)|^p + |t|^p)^{\frac{1}{p}} \geq 1$ , so the line segment isn't contained in the  $S_p$ , confirming the above. However, for  $p = 1$   $(|(1-t)|^p + |t|^p)^{\frac{1}{p}} = 1$  so the closed convex hull of  $S_p$  is  $S_1 = \{x \in \mathbb{R}^n \mid \|x\|_1 \leq 1\}$

□

*Not in Book*

Work on the following problems.

1.  $(A^o)^c = \overline{A^c}$

*Proof.* Consider any  $x \in (A^o)^c$ , want so show  $(A^o)^c \subset \overline{A^c}$ . Then  $x \notin A^o$ . This means  $\nexists \epsilon > 0$  such that  $B_\epsilon(x) \subset A$ . Therefore  $\exists \epsilon > 0$  such that  $B_\epsilon(x) \cap A^c \neq \emptyset$ , so  $x \in \overline{A^c}$ . This implies  $(A^o)^c \subset \overline{A^c}$ .

Consider any  $x \in \overline{A^c}$ . This means  $\forall \epsilon > 0$ ,  $B_\epsilon(x) \cap A^c \neq \emptyset$ . Which means any open ball around  $x$  will intersect  $A^c$  so you cannot find an open ball where  $B_\epsilon(x) \subset A$ . This implies  $x \notin A^o$ . Therefore,  $\overline{A^c} \subset (A^o)^c$

□

2. An isolated point of  $A$  is an accumulation point of  $A^c$ .

*Proof.* Let  $x \in A$  be an isolated point of  $A$ . This means  $\exists \epsilon > 0$  such that  $B_\epsilon(x) \cap A = \{x\}$ . Since isolated points are boundary points, it also means, for any  $\epsilon > 0$   $B_\epsilon(x) \setminus \{x\} \cap A^c \neq \emptyset$ . Therefore, any isolated point of  $A$  is an accumulation point of  $A^c$ .

□

3. Construct an example of a set  $A$  such that  $\overline{A} = \emptyset$ .

The Cantor set in space  $\mathbb{R}$ . The interior of the Cantor set is empty, i.e.  $\mathcal{C}^\circ = \emptyset$ , since it contains no non-empty open intervals. The closure of  $\emptyset$  is also empty, i.e.  $\overline{\mathcal{C}^\circ} = \emptyset$ .

4. A set  $A$  such that  $\overline{A}^\circ = \emptyset$ .

Again, the Cantor set fulfils this requirement. The closure of the Cantor set is the Cantor set. The interior of the Cantor set is empty.