JOE SEIDEL

REAL ANALYSIS SIXTH WEEK

Exercise 3.4.8

Prove that a subset Y of a complete metric space X is also complete metric space with the inherited metric if and only if Y is closed as a subset of X.

Proof. "$\Rightarrow$"

Suppose Y is closed. Let (y_n) be a a Cauchy sequence in Y. Since $Y \subset X$, (y_n) is a Cauchy sequence in X. Since X is complete, (y_n) converges to y for some $y \in X$. Since Y is closed, $y \in Y$, hence Y is complete.

"$\Leftarrow$" Let Y be a complete metric space and suppose Y is open. Then a Cauchy sequence $(y_n) \in Y$ converges to $y_n \notin Y$, but this contradicts that Y is complete, so Y is closed. □

Exercise 3.4.9

Show that, for $1 \leq p \leq \infty$, the space $\ell_n^p(\mathbb{R})$ and $\ell_n^p(\mathbb{C})$ are complete metric spaces.

Proof. Suppose that $(x_k)_{k=1}^{\infty}$ is a sequence of points where $x_k = (x_{1_k}, x_{2_k}, ..., x_{n_k})$ in $\mathbb{R}$ that is Cauchy with respect to $\|\cdot\|$, defined as $\|x\| = (\sum_{k=1}^{n} |x_k|^p)^{\frac{1}{p}}$. Since for $1 \leq p < \infty$ for every $x \in \mathbb{R}$ $\|x\|_\infty \leq \|x\|_p \leq n^{\frac{1}{p}} \|x\|_\infty$, implies $|x_{i_j} - x_{i_k}| \leq \|x_j - x_k\|_p$.

Therefore, each coordinate sequence $(x_{i_k})_{k=1}^{\infty}$ is Cauchy in $\mathbb{R}$. Since $\mathbb{R}$ is complete, $\lim_{k\to\infty} x_{i_k} = x_i$ for some $x_i \in \mathbb{R}$.

Let $x = (x_1, x_2, ..., x_n)$, then

$$\|x_k - x\|_p \leq Cmax\{|x_{i_k} - x_i| : i = 1, 2, ..., n\}$$

where $C = n^{\frac{1}{p}}$ if $1 \leq p < \infty$ or $C = 1$ if $p = \infty$. Given $\epsilon > 0$ chose $N_i \in \mathbb{N}$ such that $|x_{i_k} - x_i| < \frac{\epsilon}{C}$ for all $k > N_i$. Let $N = maxN_1, N_2, ...N_n$, then $k > N$ implies that $\|x_k - x\| < \epsilon$ which proves that $\lim_{k\to\infty} x_k = x$ in $ell_p^n(\mathbb{R})$ making this space complete.

The same proof works for $\mathbb{C}$. □

scratch work

Grader, please disregard

Define $\|x\|_p = (\sum_{k=1}^{n} |x_k|^p)^{\frac{1}{p}}$ and let V be a vector space in $\mathbb{R}^n or \mathbb{C}^n$.

Let the set $\{e_i\}_{i=1}^{n}$ be a base for V. Recall that that norms for $1 \leq p \leq \infty$ are equivalent on finite dimensional spaces, therefore we can choose $p = 1$ and completeness is preserved on these equivalent norms.

We can choose $L, M > 0 \in \mathbb{R}$ or $\mathbb{C}$ Such that $L\|w\| \leq \|w\| \leq M\|w\|$ for all $w \in V$. This implies, $\forall \epsilon > 0$ there exists $N \in \mathbb{N}$ such that if

$n, m > N$

$$\begin{aligned} L|v_{n_i} - v_{k_i}| &\leq L \sum_{i=1}^{n} |v_{n_i} - v_{k_i}| \\ &= L\|v_n - v_m\| \leq \|v_n - v_m\| < \epsilon \end{aligned}$$

for all $1 \leq i \leq n$. Hence, (v_{k_i}) is a Cauchy sequence in $\mathbb{R}$ or $\mathbb{C}$ for each i. Since $\mathbb{R}$ and $\mathbb{C}$ are complete, there exists $u_i \in \mathbb{R}$ or $\mathbb{C}$ such that $u_i = \lim_{k\to\infty} v_{k_i}$ for each i. Let $u = (u_1, ..., u_n) = \sum_{i=1}^{n} u_i e_i$ which means that $u \in V$. Finally, to show completeness, need to show $\lim_{k\to\infty} \|v_k - u\| = 0$.

$$\begin{aligned} \lim_{k\to\infty} \|v_k - u\| &\leq M \lim_{k\to\infty} \|v_k - u\| \\ &= M \lim_{k\to\infty} \sum_{i=1}^{n} |v_{k_i} - u_i| \\ &= M \sum_{i=1}^{n} \lim_{k\to\infty} |v_{k_i} - u_i| \\ &= 0 \end{aligned}$$

Exercise 3.4.18

For the following sequences $(f_n)_{n\in\mathbb{N}}$ of functions, where $f_n : [0, 2\pi] \to \mathbb{R}$ for all $n \in \mathbb{N}$, find all values of $x \in [0, 2\pi]$ such that the sequence $(f_n(x))_{n\in\mathbb{N}}$ converges and find the pointwise limit function $f : [0, 2\pi] \to \mathbb{R}$ if it exists.

1. $f_n(x) = \sin(\frac{x}{n})$
   Since $1 \leq n$ this function is always defined. For all values $x \in [0, 2\pi]$ the sequence $(f_n(x))_{n\in\mathbb{N}}$ converges to 0 so $f : [0, 2\pi] \to \mathbb{R}$ is given by $f(x) = 0$.

2. $f_n(x) = \sin(nx)$.
   Since the sin function oscillates between -1 and 1. Consider $f_n(x) = 1$ when $n < \frac{\pi}{2x}$ and again when $2\pi(x) < n < \frac{5\pi}{2x}$ and so forth. Consider when $f_n(x) = 0$, when $\frac{\pi}{2x} < n < \frac{\pi}{x}$ and again when $\frac{3\pi}{2x} < n < \frac{2\pi}{x}$ and so forth. Next, when $(f_n(x) = -1$ whenever $\frac{\pi}{x} < n < \frac{3\pi}{2x}$ and again $3\pi < n < \frac{7\pi}{2}$ and so forth.

   There are no values in the domain $[0, 2\pi]$ such that $(f_n(x))_{n\in\mathbb{N}}$ converges. Hence, the pointwise limit function does not exist.

3. $f_n(x) = \sin^n(x)$.

$$f_n(x) = \begin{cases} 0, & \text{if } x \neq \frac{3\pi}{2} \text{ and } x \neq \frac{\pi}{2} \\ 1, & \text{if } x = \frac{\pi}{2} \\ -1 \text{ or } 1, & \text{if } x = \frac{3\pi}{2} \end{cases}$$

Since the sequence does not converge when $x = \frac{3\pi}{2}$ we cannot define $f : [0, 2\pi] \to \mathbb{R}$ as the point wise limit function of $f_n(x)$.

Exercise 3.4.22

Let $f_n(x) = x^n$ for $n \in \mathbb{N}$.

1. Show that the sequence $(f_n)_{n\in\mathbb{N}}$ converges pointwise to the function $f(x) = 0$ on the interval $(-1, 1)$.

   When $0 < x < 1$, this implies $x = \frac{1}{a}$ where $a > 1$ which implies $\lim_{n\to\infty} x^n = \lim_{n\to\infty} \frac{1}{a^n} = 0$. When $-1 < x < 0$, it implies $x = (-1)\frac{1}{a}$ where $a > 1$ which means $(-1)\lim_{n\to\infty} \frac{1}{a^n} = 0$. When $x = 0$, $\lim_{n\to\infty} x^n = 0$.

   Therefore all values in the domain $(-1, 1)$, $\lim_{n\to\infty} f_n(x) = f(x)$.

2. Show that if we restrict to the domain $[-\frac{1}{2}, \frac{1}{2}]$, the sequence $f(n)_{n\in\mathbb{N}}$ converges uniformly to the function $f(x) = 0$.

   Proof. A sequence converges uniformly to a function if given $\epsilon > 0$, $\exists N_\epsilon \in \mathbb{N}$ such that $\sup_{x\in X} |f_n(x) - f(x)| < \epsilon$ for $n \geq N_\epsilon$.

   Since $f(x) = 0$, $|f_n(x) - f(x)| = |x^n| < \epsilon$ if $x < \epsilon^{\frac{1}{n}}$. Since $\epsilon^{\frac{1}{n}} < 1$ for all n the sequence converges uniformly for the domain $[-\frac{1}{2}, \frac{1}{2}]$. □

3. Show that the sequence $(f_n)_{n\in\mathbb{N}}$ does not converge uniformly on the domain $(-1, 1)$.

   Proof. Consider again, the expression from above, $|f_n(x) - f(x)| = |x^n| < \epsilon$. The inequality $x < \epsilon^{\frac{1}{n}}$ fails when x gets within ϵ of 1. To see this, we can choose $x \in (-1, 1)$ such that $1 - \epsilon = x$. Notice that $1 - \epsilon < \epsilon^{\frac{1}{n}}$ is clearly false. Therefore, $(f_n)_{n\in\mathbb{N}}$ does not converge uniformly. □

Exercise 3.5.2

Suppose that X and X' are metric spaces as above and the $x_0 \in X$. Show that f is continuous at x_0 iff for every sequence $(x_n)_{n\in\mathbb{N}}$ in X which converges to x_0 in X we have

$$\lim_{n\to\infty} f(x_n) = f(x_0)$$

in X'.

Proof. "⇒"

Suppose $f : X \to X'$ is continuous at $x_0 \in X$. Let $\epsilon > 0$ and supposed $\nexists \delta > 0$ such that $d(x, x_0) < \delta$ implies $d'(f(x), f(x_0)) < \epsilon$ and prove by contradiction.

Let $d = (\frac{1}{n})$ for any $n \in \mathbb{N}$, then there is an $x_n \in B_{\frac{1}{n}}(x_0)$ for which $f(x_n) \notin B_\epsilon(f(x_0))$. Therefore there is a sequence $\{x_n\}$ that converges to x_0 but the sequence $\{f(x_n)\}$ does not converge to $f(x_0)$, contradicting the assumption that f is continuous.

"$\Leftarrow$" Assume that $\forall \epsilon > 0, \exists \delta > 0$ such that $x \in B_\delta(x_0)$ implies $f(x) \in B_\epsilon(f(x_0))$. Let $\{x_n\} \in X$ be a sequence that converges to x_0. In order to show that $\{f(x_n)\}$ converges to $f(x_0)$, let $\epsilon > 0$. Therefore $\exists \delta > 0$ for which $x \in B_\delta(x_0)$ implies $f(x) \in B_\epsilon(f(x_0))$.

Since $\{x_n\}$ converges to x_0 we can choose $n_0 \in \mathbb{N}$ such that for $n > n_0$, $x_n \in B_\delta(x_0)$, but then $n > n_0$ implies $f(x_n) \in B_\epsilon(f(x_0))$ i.e. $\{f(x_n)\}$ converges to $f(x_0)$. Therefore f is continuous at x_0. $\square$

Exercise 3.5.3

Let $f : \mathbb{R} \to \mathbb{R}$ be a polynomial function, where $\mathbb{R}$ is the usual metric. Show that f is continuous.

Proof. First, note that for a linear function $f(x) = \alpha x + \beta$, we can choose α and β such that the $\lim_{x \to x_0} f(x) = x_0$ is continuous. Second, through repeated application of the produce rule for limits of functions, $\forall n \in \mathbb{N}$, $\lim_{x \to x_0} x^n = x_0^n$.

Now observe $f(x) = A_n x^n + A_{n-1}x^{n-1} + ... + A_0$. Using algebraic limit laws and the two facts above, notice

$$\begin{aligned}
\lim_{x \to x_0} f(x) &= \lim_{x \to x_0} A_n x^n + A_{n-1}x^{n-1} + ... + A_0 \\
&= \lim_{x \to x_0} A_n x^n + \lim_{x \to x_0} A_{n-1}x^{n-1} + ... + \lim_{x \to x_0} A_0 \\
&= \lim_{x \to x_0} A_n (\lim_{x \to x_0} x^n) + \lim_{x \to x_0} A_{n-1} (\lim_{x \to x_0} x^{n-1}) + ... + \lim_{x \to x_0} A_0 \\
&= A_n x^n + A_{n-1}x^{n-1} + ... + A_0 \\
&= f(x_0)
\end{aligned}$$

Hence for a polynomial function $f : \mathbb{R} \to \mathbb{R}$, we have $\lim_{x \to x_0} f(x) = f(x_0)$ which implies continuity. $\square$

Exercise 3.5.4

Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} \frac{1}{q}, & \text{if } x = \frac{p}{q} \text{ (reduced to lowest terms, } x \neq 0) \\ 0, & \text{if } x = 0 \text{ or } x \notin \mathbb{Q} \end{cases}$$

Show that f is continuous at 0 and any irrational point. Show that f is not continuous at any nonzero rational point.

To see why f is continuous at 0 and any irrational point, let X be $\{x \in \mathbb{R} | x = 0 \text{ or } x \in R \setminus \mathbb{Q}\}$. Choose any $x_0 \in X$, want to show that $\forall \epsilon > 0$, $\exists \delta > 0$ such that $d(x, x_0) < \delta$ implies $d(f(x), f(x_0)) < \epsilon$. Since we are in in $\mathbb{R}$, define distance to be $|x - y|$ for $x, y \in \mathbb{R}$.

For any range of length 1, in particular, consider $[x_0 - \frac{1}{q}, x_0 + \frac{1}{q})$, $f(x)$ takes the values of $\frac{1}{q}$, $q - 1$ times. We can remove these points up to $\frac{1}{\epsilon}$. and have removed only finitely many points. What's left is points $f(x) < \frac{1}{\frac{1}{\epsilon}} \leq \epsilon$. Since x_0 is irrational or 0, it was not a point removed. So we can form an open ball $B_\delta(x_0)$ that does not contain any rational points, with $\delta > 0$ small enough. This δ implies continuity.

To show discontinuity at any nonzero rational point, consider $0 < \epsilon < \frac{1}{q}$. and any $\delta > 0$. Since $\mathbb{R}$ is dense in $\mathbb{Q}$,there are irrational numbers y such that $|x - y| < \delta$, but then $|f(x) - f(y)| = \frac{1}{q} > \epsilon$. So f is discontinuous $\forall x \in \mathbb{Q}$, $x \neq 0$.