

JOE SEIDEL

REAL ANALYSIS SECOND WEEK

Exercise 1.6.35

1. Show the \emptyset and \mathbb{R} are the only subsets of \mathbb{R} that are both open and closed.

Proof. Let S be a non-empty open and closed set of \mathbb{R} . Fix $x_0 \in S$ and $S \neq \mathbb{R}$. Then, $\exists y \in \mathbb{R} \setminus S$. Without loss of generality we may assume $y > x_0$.

Therefore we can form a set

$$I = \{x \in \mathbb{R} | x > x_0, x \notin S\}$$

By construction this set is bounded below by x_0 and not empty because $y > x_0$, therefore $y \in I$. Therefore we can let $i = \inf I$

greatest lower bound

Suppose $i \in S$. Since S is open it contains the open interval $(i - \epsilon, i + \epsilon)$ for $\epsilon > 0$. However, this interval contradicts that $i = \inf I$ because it implies a sequence $i_n > i, |i - i_n| < \frac{1}{n}$, where $i_n \in I$, which means $i_n \notin S$. Which is not possible because $[i, i + \epsilon) \subset S$

Now suppose $i \notin S$. Since S is closed, S^c which means that it contains an open interval $(i - \epsilon, i + \epsilon)$, but this contradicts the definition $i = \inf I$ because then we can find $i - \frac{\epsilon}{2}$ that is in S^c .

Therefore $S = \mathbb{R}$. □

2. Show that every non-empty open set in \mathbb{R} can be written as a countable union of pairwise disjoint open intervals.

Proof. Let $U \subset \mathbb{R}$. Let \mathcal{O} be the set of open intervals that are a subsection of U . For $I, J \in \mathcal{O}$ define $I \sim J$ iff there are

$$I_0 = I, I_1, I_2, \dots, I_n = J \in \mathcal{O}$$

Such that $I_k \cap I_{k+1} \neq \emptyset$ for $k = 0, \dots, n - 1$. Then \sim defines an equivalence relation on \mathcal{O} . For $I \in \mathcal{O}$ let $[I]$ be the \sim of I . Then $\{[I] \text{ for } I \in \mathcal{O}\}$ is decomposition of U into pairwise disjoint intervals. By construction, these intervals are countable. □

3. Show that an arbitrary union of open sets in \mathbb{R} is open in \mathbb{R} .

Proof. Suppose $\{A_i \subset \mathbb{R} | i \in I\}$ is an arbitrary collection of open sets.

If $x \in \cup A_i$ then $x \in A_i$ for some $i \in I$. Since A_i is open, there $(x - \epsilon, x + \epsilon) \subset A_i$ for $\epsilon > 0$. Therefore,

$$(x - \epsilon, x + \epsilon) \subset \bigcup_{i \in I} A_i$$

since $A_i \subset \mathbb{R}$ we arrive at our conclusion: The arbitrary union of open sets, $\bigcup_{i \in I} A_i$ is open in \mathbb{R} . \square

4. Show that a finite intersection of open sets in \mathbb{R} is open in \mathbb{R}

Proof. Suppose $\{A_i \subset \mathbb{R} | i = 1, 2, \dots, n\}$ is a finite collection of open sets. If $x \in \bigcap_{i=1}^n A_i$ then $x \in A_i$ for every $1 \leq i \leq n$. Since A_i is open, there are $\epsilon_i > 0$ such that $(x - \epsilon_i, x + \epsilon_i) \subset A_i$

Let $\epsilon = \min(\epsilon_1, \dots, \epsilon_n) > 0$. This shows $(x - \epsilon, x + \epsilon) \subset \bigcap_{i=1}^n A_i$ \square

5. Show, by example, that an infinite intersection of open sets is not necessarily open. The open interval $(-\frac{1}{n}, \frac{1}{n})$ is open $\forall n \in \mathbb{N}$. However, it's intersection

$$\bigcap_{n=1}^{\infty} (-\frac{1}{n}, \frac{1}{n}) = 0 = [0, 0]$$

which is closed. \square

6. Show that an arbitrary intersection of closed sets in \mathbb{R} is a closed set in \mathbb{R} .

Proof. Let $\{A_i \subset \mathbb{R}, i \in I\}$ be an arbitrary collection of closed sets. Let $\bigcap_{i \in I} A_i$ be the intersection of the closed sets. If this intersection is \emptyset we are done. Supposing it is not, we'll continue.

By definition $\mathbb{R} \setminus \bigcap_{i \in I} A_i = \bigcup_{i \in I} (\mathbb{R} \setminus A_i)$.

DeMorgan's Law

Since $\{A_i\} \subset \mathbb{R}$ are closed, $(\mathbb{R} \setminus A_i) \subset \mathbb{R}$ is made up of open sets. So we have an arbitrary union of open sets in \mathbb{R} which we have already shown to be open. This means $\mathbb{R} \setminus \bigcap_{i \in I} A_i$ is also open.

Therefore it's complement $\bigcap_{i \in I} A_i$ is closed. \square

7. Show that a finite union of closed sets in \mathbb{R} is a closed set in \mathbb{R}

Proof. Let $\{A_i \subset \mathbb{R}, i = 1, 2, \dots, n\}$ be a finite collection of some $n \in \mathbb{N}$ closed sets in \mathbb{R} . Let $\bigcup_{i=1}^n A_i$ be the union of the finite closed subsets.

Summoning DeMorgan's Law, once more

$$\mathbb{R} \setminus \bigcup_{i=1}^n A_i = \bigcap_{i=1}^n (\mathbb{R} \setminus A_i)$$

Since $\{A_i\} \subset \mathbb{R}$ are closed, $(\mathbb{R} \setminus A_i) \subset \mathbb{R}$ is made up of open sets. This means we have a finite intersection of open sets, which we have already shown to be open. Therefore $\mathbb{R} \setminus \bigcup_{i=1}^n A_i$ is also open. Which allows us to conclude, the complement, $\bigcup_{i=1}^n A_i$ must be closed.

□

8. Show, by example, that an infinite union of closed sets in \mathbb{R} is not necessarily closed in \mathbb{R}

One example is $\bigcup_{n=1}^{\infty} [\frac{1}{n}, n]$. The union of this is open because you will never find a point of the union that lives at the boundary of the union.

Also, consider that \mathbb{R} can be constructed of an infinite union of closed sets. For example, each one point set of one point in \mathbb{R} , however we know that \mathbb{R} is not closed.

Exercise 1.6.36

Show that a subset of \mathbb{R} is closed iff it contains all its accumulation points.

Proof. " \Rightarrow " (Given in class)

Suppose S is closed and x is an accumulation point. We prove by contradiction. Suppose $x \notin S$ then $x \in S^c$. This means, $\exists \epsilon$ such that

$$(x - \epsilon, x + \epsilon) \subset S^c$$

That is to say $(x - \epsilon, x + \epsilon) \cap S = \emptyset$. Which contradicts that x is an accumulation point. □

Proof. " \Leftarrow " Suppose S contains all its limit points and let S be open, prove by contradiction. Therefore, $\exists x \in S^c$ such that $(x - \epsilon, x + \epsilon)$ contains at least one element of S . Symbolically written as

$$(x - \epsilon, x + \epsilon) \cap S \neq \emptyset, \forall \epsilon > 0$$

For all $n \in \mathbb{N}$ let $x_n \in (x, \frac{1}{n}) \cap S$

Notice, x_n is a sequence in S that converges to $x \notin S$, meaning that x is an accumulation point of S that is not contained in the open S , so S must be closed. \square

Exercise 1.6.37

1. The Cantor set is closed.

Let $\mathcal{C} = \bigcap_{n \in \mathbb{N}} C_n$ Where C_n is finite union of closed sets. As, shown above the arbitrary intersection of closed sets, this is closed.

2. The Cantor set consist of all numbers in the closed $[0, 1]$ whose ternary expansion consists of only 0s and 2s and may end infinitely in 2s.

In every stage of constructing the Cantor set, the middle third of the of the set is removed, i.e. the numbers whose ternary decimal expansion notation has a 1 in the n th decimal place.

For example in the first stage, all numbers with 1 in the first decimal place are removed, in the second all numbers 1 in the second decimal place and so on.

3. The Cantor set is uncountable

Proof. Suppose \mathcal{C} is countable.

$$\mathcal{C} = \{x_1^1, x_1^2, x_1^3, \dots, x_1^n\}$$

for each x write

$$x^i = 0.d_1^i, 0.d_2^i, 0.d_3^i, 0.d_4^i, \dots, 0.d_m^i$$

Let $(d_1, d_2, d_3, d_4, \dots, d_m)$ be the sequence that differs from the diagonal sequence of $d^i = d_1^i, d_2^i, d_3^i, d_4^i, \dots, d_m^i$ such that

$$d_m = \begin{cases} 2 & \text{if } d_m^m = 0 \\ 0 & \text{if } d_m^m = 2 \end{cases}$$

The ternary expansion d_m , does not appear in the list x^i since $d_m \neq d_m^m$. So, $x = 0.d_1d_2d_3d_4\dots d_m$ is in \mathcal{C} but no element in \mathcal{C} has two different ternary expansions using only 0s and 2s. So x doesn't appear in the list above, which is a contradiction, so \mathcal{C} is uncountable. \square

4. Every point in the Cantor set is an accumulation point of the Cantor set.

Every point of the Cantor set can be captured in an arbitrary union of closed sets. So for any $x \in \mathcal{C}$, you can write x in ternary as a limit of other numbers that have 0s or 2s in their ternary expansion.

Proof. Let $C_n = \bigcup S_m$ where S_m is some number m of closed sets. For any $x \in S_m$ let \bar{x} be the right end point of S_m , unless $x = \bar{x}$ then let \bar{x} be the left endpoint. By construction, each subinterval of S_m has length $\frac{1}{3^m}$. Which means $|x - \bar{x}| < \frac{1}{3^m}$.

This implies that the sequence of x_m converges to x and such is an accumulation point. Since we chose x to be arbitrary, this applies to all $x \in \mathcal{C}$. \square

5. The complement of the Cantor Set in $[0, 1]$ is a dense subset of $[0, 1]$

In layman's terms, we delete all the middle intervals at every stage of constructing the Cantor set. Which would mean the complement is dense everywhere.

Proof. Let, $\mathcal{C} = \bigcap_{n \in \mathbb{N}} C_n$ where $C_n = \bigcup S_n$

The length of intervals in S_n is $\frac{1}{3^n} = 3^{-n}$. Let $0 \leq a < b \leq 1$. Then interval $(a, b) \subset [0, 1]$. Fix $n \in \mathbb{N}$ such that $3^{-n} < b - a$ so that the length of every interval in S_n is $3^{-n} < b - a$.

Therefore the intervals of S_n do not overlap any interval of length $(b - a)$. Therefore no interval of length $b - a$ is contained in C_n .

Likewise, since $C_n \subset \mathcal{C}$, no interval of length $b - a$ is contained in \mathcal{C} .

Since lengths of a, b were arbitrary, there are no open intervals in \mathcal{C} , which means \mathcal{C} is nowhere dense and \mathcal{C}^c is dense. \square

Exercise 1.6.42

Show that a compact subset of \mathbb{R} is both closed and compact.

Proof. Suppose A is a compact subset of \mathbb{R} . $A \subset \bigcup_{k=1}^{\infty} U_k$, where U_k are open sets.

By compactness of A , $\exists n \in \mathbb{N}$ such that $A \subset \bigcup_{k=1}^n U_k$. Thus we can say A is bounded.

Consider A^c and define it as $X = \mathbb{R} \setminus A$ and take any $x \in X$. For every $a \in A$ there are open sets $U_a = (a - \epsilon, a + \epsilon)$ and $V_a = (x - \epsilon, x + \epsilon)$, for some $\epsilon > 0$ such that $U_a \cap V_a = \emptyset$.

The sets $\{U_a | a \in A\}$ form an open cover over A and since A is compact, there are finitely many points, $m \in \mathbb{N}$ such that $A \subset \bigcup_{j=1}^m U_{a_j}$,

denote this as U_A and $V_A = \bigcap_{j=0}^m V_{a_j}$. Then U_A and V_A are open and $U_A \cap V_A = \emptyset$.

Notice that $V_A \subset X = A^c$ and since $x \in V_A$ and we chose x to be arbitrary, A^c is open, making A closed.

□

Exercise 1.6.46

A subset of \mathbb{R} is compact if and only if it is sequentially compact.

Proof. " \Rightarrow " (Given in class) Suppose S is compact. Consider any infinite subsequence $(X_n) \subset S$. We want to show that (X_n) has a convergent subsequence whose limit is in S .

We cover any point of S by $(x - 1, x + 1)$. This gives us an open cover. Since S is compact, there is a finite cover. Therefore the sequence (X_n) is covered by finitely many intervals of length 2, hence there is an interval with infinitely many points in (x_n) . Next, apply the theorem that every bounded sequence has a convergent subsequence.

We still need to show that the subsequence converges to a point x^* in S . Since x^* is an accumulation point and S is bounded and closed, S is closed and therefore $x \in S$

□

Proof. " \Leftarrow " Suppose S is sequentially compact. To show that S is also compact, let's assume S is not.

Since S is not compact, there is an infinite subsequence that is not bounded in S . Since this subsequence is not bounded, it does not converge to a point in S which contradicts that S is sequentially compact and therefore S is compact.

□

Exercise 1.9.6

1. If $N \in \mathbb{N}$ and $z \neq 1$, show that $S_N = \sum_{n=0}^N z^n = \frac{1-z^{N+1}}{1-z}$

Proof. By induction. Base case. $n = 0$.

$$z^0 = \frac{1 - z^{0+1}}{1 - z}$$

$$1 = 1$$

Induction step.

$$\begin{aligned}
 \sum_{n=0}^{n+1} z^n &= z^0 + z^1 + \dots + z^n + z^{n+1} \\
 &= \frac{1 - z^{n+1}}{1 - z} + z^{n+1} \\
 &= \frac{1 - z^{n+1} + (1 - z)z^{n+1}}{1 - z} \\
 &= \frac{1 - z^{n+1} + z^{n+1} + z^{n+2}}{1 - z} \\
 &= \frac{1 - z^{n+2}}{1 - z}
 \end{aligned}$$

□

2. If $|z| < 1$, show that $\lim_{n \rightarrow \infty} z^n = 0$

Let the sequence $(z_n)_{n \in \mathbb{N}}$ be defined as the sequence of numbers where $z_n = z^n$. If $|z| < 1$ it's easy to see that for any $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for any $n > N$ $|z_n| < \epsilon$. Which is to say the $\lim_{n \rightarrow \infty} z^n = 0$

3. If $|z| > 1$, show that $\lim_{n \rightarrow \infty} z^n$ does not exist.

Defining $(z_n)_{n \in \mathbb{N}}$ as above, with the caveat that $|z| > 1$, this sequence is ever increasing, such that as n gets larger the distance between n and $n + 1$ gets larger. Therefore, the limit does not exist.

Exercise 1.9.16

1. If $p \in \mathbb{R}$ and $p < 1$, show that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges.

A key idea here is the contrapositive of the Theorem $\sum_{n=1}^{\infty} a_n$ converges then $\lim_{n \rightarrow \infty} a_n = 0$. The contrapositive can be worded to say, if the terms of the series a_n do not go to zero, then the series must diverge.

Then given any $\epsilon > 0$ there must be an item in the sequence that is makes the statement $|a_n| < \epsilon$ true.

If $0 < p < 1$, case we have $n^{-p} < \epsilon$. This not possible, because $n \in \mathbb{N}$.

Similarly, if $p < 0$ then we have to satisfy $n^p < \epsilon$ which is also not possible.

Therefore, the series must diverge $\sum_{n=1}^{\infty} \frac{1}{n^p}$.