REAL ANALYSIS
FIRST MIDTERM

Question 1

Determine if the following series converge is *absolutely convergent* or not.

$$\sum_{n=1}^{\infty} \frac{(-1)^n n^3}{2^n}, \sum_{n=1}^{\infty} \frac{3+2^{-n}}{n^{\frac{1}{2}}}, \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n^2+n+1}}$$

Answer

To show absolute converge of a series $\sum_{k=0}^{\infty} a_k$, it is enough to show convergence of $\sum_{k=0}^{\infty} |a_k|$.

For $\sum_{n=1}^{\infty} \frac{(-1)^n n^3}{2^n}$, use root test.

$$\lim_{n\to\infty} |\frac{(-1)^n n^3}{2^n}|^{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^{\frac{3}{n}}}{2} = \frac{1}{2} < 1$$

This series is convergent.

For $\sum_{n=1}^{\infty} \frac{3+2^{-n}}{n^{\frac{1}{2}}}$ consider $\frac{3+2^{-n}}{n^{\frac{1}{2}}} > \frac{3}{n^{\frac{1}{2}}}$ It is known that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges for $p \leq 1$ so the series in question diverges.

For $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n^2+n+1}}$

$$|\frac{(-1)^n}{\sqrt{n^2+n+1}}| = \frac{1}{\sqrt{n^2+n+1}} > \frac{1}{\sqrt{n^2+2n+1}} = \frac{1}{n+1}$$

Therefore, this series is not absolutely convergent.

Question 2

Find the radius of convergence of the following series.

$$\sum_{n=1}^{\infty} \frac{(-1)^n 2^n z^n}{n(n+1)}$$

$$|\frac{(-1)^n 2^n z^n}{n(n+1)}|^{\frac{1}{n}} = \frac{2}{(n(n+1))^{\frac{1}{n}}}$$

$$lim_{n\to\infty} = \frac{2}{(n(n+1))^{\frac{1}{n}}} = 2$$

The radius of convergence is $\frac{1}{2}$

Question 3

It is known that arbitrary union of open sets is open and arbitrary intersection of closed sets is closed. Construct an example satisfying: Countably many closed sets whose union is open but not closed.

$$\bigcup_{n=1}^{\infty} [-1+\frac{1}{n}, 1-\frac{1}{n}] = (-1,1)$$

Question 4

1. Prove that closed subsets of a compact set are compact

   Proof. Suppose S is compact and $A(\subset S)$ is closed. Denote X by the full set. Consider any open cover $\mathcal{U}$ of A. We add to the open cover another open set $X \setminus A$, then we get an open cover of S. Since S is compact, there must be a finite subcover. Removing $X \setminus A$ from the inite sets, the resulting finitely many open sets form an open cover of A. Therefore any open cover of A has a finite subcover, hence A is compact. □

2. Show that the interval $(0, 1)$ on the real line is not open as a subset of $\mathbb{C}$.

   Proof. To be an open set in $\mathbb{C}$, as set S has to satisfy the following: $\forall x \in S$ there exists ϵ such that $B_\epsilon(x) \subset S$ where

   $$B_\epsilon(x) = \{y \in \mathbb{C} : |y - x| < \epsilon\}$$

   Consider any point $x \in (0, 1) \subset \mathbb{R}$. For any ϵ the point $x + \frac{i\epsilon}{2} \in B_\epsilon(x)$. However, $x + \frac{i\epsilon}{2} \notin (0, 1)$ Therefore, the ball $B_\epsilon(x)$ is not a subset of $(0, 1)$, hence $(0, 1)$ is not open in $\mathbb{C}$. □

Question 5

We introduce the bilinear form $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^2$ by

$$\langle v, w \rangle = av_1w_1 + b(v_1w_2 + v_2w_1) + cv_2w_2$$

where $v = (v_1, v_2)$, $w = (w_1, w_2)$. Find necessary and sufficient conditions on a, b, c such that this bilinear form is an inner product.

Answer

To be an inner product, the bilinear form has to be symmetric and positive definite. The symmetricity is easy to verify. It remains to guarantee positive definiteness, ie. $\langle v, v \rangle = av_1^2 + 2bv_1v_2 + cv_2^2 \geq 0$ and 0 if and only if $v = 0$. Now suppose $v \neq 0$ then either $v_1 \neq 0$ or $v_2 \neq 0$. We need $\langle v, v \rangle > 0$. Suppose $v_1 \neq 0$. Then denote $x = \frac{v_2}{v_1}$. This implies

$$\langle v, v \rangle = v_1^2(a + 2bx + cx^2) > 0$$

For all x. We have to require $c > 0$ and $(2b)^2 - 4ac < 0$ to guarantee the parabola never touches the x-axis. Similarly, when $v_2 \neq 0$. we get $a > 0$ and $(2b)^2 - 4ac < 0$.

To summarize we need $a > 0$, $c > 0$ and $b^2 < ac$.

Question 6

Consider number $\alpha \in [0,1]$. For fixed $c > 0$ and $\sigma > 0$, we say a number $\alpha \in DC(c,\sigma)$, if the following inequality is satisfied for all rational numbers $\frac{p}{q} \in [0,1]$,

$$\left|\alpha - \frac{p}{q}\right| \geq \frac{c}{q^{2+\sigma}}$$

Such numbers are called Diophantine numbers. Prove the following

1. Show that for c small enough, the set $DC(c,\sigma)$ is not empty. Hint: Consider the set $DC(c,\sigma)$ as the resulting set by removing the interval $(\frac{p}{q} - \frac{c}{q^{2+\sigma}}, \frac{p}{q} + \frac{c}{q^{2+\sigma}})$ of length $2\frac{c}{q^{2+\sigma}}$ center at each rational number $\frac{p}{q}$. For each q, there are at most q rational numbers with denominator q, i.e. $\frac{1}{q}, \frac{2}{q}, ..., \frac{q}{q}$. So for each q, the total length of removed intervals is at most

$$q \times \frac{2c}{q^{2+\sigma}} = \frac{2c}{q^{1+\sigma}}$$

Proof. Keeping the hint in mind, we sum over $q \in \mathbb{N}$. The total length of removed intervals is less than or equal to $\sum_{q=1}^{\infty} \frac{2c}{q^{1+\sigma}}$ since $1 + \sigma > 1$ the series converges. The sum can be made arbitrarily small by choosing c small. The removed set has has total length as small as we wish. Therefore, the remaining set $DC(c,\sigma)$ has total length as close to 1 as we wish. Therefore it cannot be empty. □

2. Prove that for each c, σ, the set $DC(c,\sigma)$ is closed and nowhere dense. (Hint, this set is very similar to the Cantor set. Notice the set does not contain rational numbers).

Proof. Since we always remove open sets of the form $(\frac{p}{q} - \frac{c}{q^{2+\sigma}}, \frac{p}{q} + \frac{c}{q^{2+\sigma}})$ the union of open sets is open. So the resulting $DC(c,\sigma)$ is closed. Suppose $DC(c,\sigma)$ is not nowhere dense. Since $DC(c,\sigma)$ is closed it must contain interval. Since rational points are dense, there are always rational points in any interval. This is a contradiction since $DC(c,\sigma)$ does not contain any rational points. □