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REAL ANALYSIS
THIRD WEEK

Exercise 1.9.8

Give examples to show that if $r = 1$ in the statement of the Ratio Test, anything may happen.

First consider the harmonic series, $\sum_{n=1}^{\infty} \frac{1}{n}$. The $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = 1$

$$\lim_{n\to\infty} |\frac{\frac{1}{n+1}}{\frac{1}{n}}| = \lim_{n\to\infty} |\frac{n}{n+1}| = 1$$

However, we know that the series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Next consider $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$. Again $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = 1$

$$\lim_{n\to\infty} |\frac{\frac{(-1)^{n+2}}{n+1}}{\frac{(-1)^{n+1}}{n}}| = \lim_{n\to\infty} |\frac{(-1)^{n+2}}{n+1} \frac{n}{(-1)^{n+1}}|$$

$$= \lim_{n\to\infty} |\frac{-n}{n+1}|$$

$$= 1$$

This series, $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges since $|\sum_{k=m+1}^{n} \frac{(-1)^{k+1}}{k}| < \frac{1}{m}$ but not absolutely because $\sum_{n=1}^{\infty} |\frac{(-1)^n}{n}|$ goes to infinity.

Finally, consider the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ This too has $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = 1$.

$$\lim_{n\to\infty} |\frac{\frac{1}{n+1^2}}{\frac{1}{n^2}}| = \lim_{n\to\infty} |\frac{1}{(n+1)^2} \frac{n^2}{1}|$$

$$= \lim_{n\to\infty} \frac{n^2}{(n+1)^2}$$

$$= 1$$

The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is absolutely convergent because $\sum_{n=1}^{\infty} |\frac{1}{n^2}|$ converges.

Exercise 1.9.20

Give examples to show that if $r = 1$ in the statement of the Root Test, anything my happen.

Borrowing from the ideas in the last exercise, consider the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ and $\sum_{n=1}^{\infty} \frac{1}{n}$. Both have $r = 1$.

$$\limsup_{n\to\infty} |\frac{1}{n}|^{\frac{1}{n}} = \limsup_{n\to\infty} e^{\frac{1}{n} \ln \frac{1}{n}}$$

$$= e^{(\limsup_{n\to\infty} - \ln n)(\limsup_{n\to\infty} \frac{1}{n})}$$

$$= e^{(\limsup_{n\to\infty} - \ln n)(0)}$$

$$= 1$$

$$\limsup_{n\to\infty} \left|\frac{1}{n^2}\right|^{\frac{1}{n}} = \limsup_{n\to\infty} e^{\frac{1}{n}\ln\frac{1}{n^2}}$$

$$= e^{(\limsup_{n\to\infty} -2\ln n)(\limsup_{n\to\infty}\frac{1}{n})}$$

$$= e^{(\limsup_{n\to\infty} -2\ln n)(0)}$$

$$= 1$$

Anything may happen when $r = 1$

Exercise 1.9.26

Determine the radius of convergence of the following power series:

$$r = \limsup_{n\to\infty} |a_n|^{\frac{1}{n}}$$

Where the radius of convergence is $\frac{1}{r}$ if $r > 0$, ∞ if $r = 0$ and 0 if lim sup does not exist, $(r = \infty)$.

1. $\sum_{n=1}^{\infty} \frac{z^n}{n!}$

$$\sum_{n=1}^{\infty} \frac{z^n}{n!} = \sum_{n=1}^{\infty} \frac{1}{n!} z^n$$

$$\lim_{n\to\infty} \left|\frac{1}{n!}\right|^{\frac{1}{n}} = 0$$

Radius of convergence is ∞

2. $\sum_{n=2}^{\infty} \frac{z^n}{\ln(n)}$

$$\sum_{n=1}^{\infty} \frac{z^n}{\ln(n)} = \sum_{n=2}^{\infty} \frac{1}{\ln(n)} z^n$$

$$\lim_{n\to\infty} \left|\frac{1}{\ln(n)}\right|^{\frac{1}{n}} = 1$$

Radius of convergence is 1.

3. $\sum_{n=1}^{\infty} \frac{n^n}{n!} z^n$ $\qquad\qquad e^n = \sum_{n=1}^{\infty} \frac{n^n}{n!}$

$$\lim_{n\to\infty} \left|\frac{n^n}{n!}\right|^{\frac{1}{n}} = e$$

Radius of convergence is $\frac{1}{e}$

Exercise 2.5.8

Prove that the scalar product is a positive definite symmetric bilinear form on $\mathbb{E}^n$.

Proof. The scalar product of vectors $\mathbf{v} = (v_1, v_2, v_3, ..., v_n)$ and $\mathbf{w} = (w_1, w_2, w_3, ..., w_n)$ is $\langle \mathbf{v}, \mathbf{w} \rangle = v_1 w_1 + v_2 w_2 + v_3 w_3 + ... + v_n w_n$.

Let $V = \mathbb{E}^n$ be a vector space over $F = \mathbb{R}$. A bilinear form $\langle ., . \rangle$ on V is a map

$$\langle ., . \rangle : V \times V \to F$$

The satisfies linearity in both variables. That is, for all $\mathbf{v}, \mathbf{v_1} \mathbf{v_2}, \mathbf{w}, \mathbf{w_1}, \mathbf{w_2} \in V$ and all $\alpha \in F$

$$\begin{aligned}
\langle \mathbf{v_1} + \mathbf{v_2}, \mathbf{w} \rangle &= (v_{1_1} + v_{2_1})w_1 + (v_{1_2} + v_{2_2})w_2 + (v_{1_3} + v_{2_3})w_3 + ... + (v_{1_n} + v_{2_n})w_n \\
&= v_{1_1} w_1 + v_{2_1} w1 + v_{1_2} w_2 + v_{2_2} w2 + v_{1_3} w_3 + v_{2_3} w3 + ... + v_{1_n} w_n + v_{2_n} w_n \\
&= (v_{1_1} w_1 + v_{1_2} w_2 + v_{1_3} w_3 + ... + v_{1_n} w_n) + (v_{2_1} w_1 + v_{2_2} w_2 + v_{2_3} w_3 + ... + v_{2_n} w_n) \\
&= \langle \mathbf{v_1}, \mathbf{w} \rangle + \langle \mathbf{v_2}, \mathbf{w} \rangle
\end{aligned}$$

$$\begin{aligned}
\langle \alpha \mathbf{v}, \mathbf{w} \rangle &= \alpha v_1 w_1 + \alpha v_2 w_2 + \alpha v_3 w_3 + ... + \alpha v_n w_n \\
&= \alpha(v_1 w_1) + \alpha(v_2 w_2) + \alpha(v_3 w_3) + ... + \alpha(v_n w_n) \\
&= \alpha(v_1 w_1 + v_2 w_2 + v_3 w_3 + ... + v_n w_n) \\
&= \alpha \langle \mathbf{v}, \mathbf{w} \rangle
\end{aligned}$$

$$\begin{aligned}
\langle \mathbf{v}, \mathbf{w_1} + \mathbf{w_2} \rangle &= v_1(w_{1_1} + w_{2_1}) + v_2(w_{1_2} + w_{2_2}) + v_3(w_{1_3} + w_{2_3}) + ... + v_n(w_{1_n} + w_{2_n}) \\
&= v_1 w_{1_1} + v_1 w_{2_1} + v_2 w_{1_2} + v_2 w_{2_2} + v_3 w_{1_3} + v_3 w_{2_3} + ... + v_n w_{1_n} + v_n w_{2_n} \\
&= (v_1 w_{1_1} + v_2 w_{1_2} + v_3 w_{1_3} + ... + v_n w_{1_n}) + (v_1 w_{2_1} + v_2 w_{2_2} + v_3 w_{2_3} + ... + v_n w_{2_n}) \\
&= \langle \mathbf{v}, \mathbf{w_1} \rangle + \langle \mathbf{v}, \mathbf{w_2} \rangle
\end{aligned}$$

$$\begin{aligned}
\langle \mathbf{v}, \alpha \mathbf{w} \rangle &= v_1 \alpha w_1 + v_2 \alpha w_2 + v_3 \alpha w_3 + ... + v_n \alpha w_n \\
&= \alpha(v_1 w_1) + \alpha(v_2 w_2) + \alpha(v_3 w_3) + ... + \alpha(v_n w_n) \\
&= \alpha(v_1 w_1 + v_2 w_2 + v_3 w_3 + ... + v_n w_n) \\
&= \alpha \langle \mathbf{v}, \mathbf{w} \rangle
\end{aligned}$$

Regarding being positive definite , consider $\langle \mathbf{v}, \mathbf{v} \rangle$

$$\begin{aligned}
\langle \mathbf{v}, \mathbf{v} \rangle &= v_1 v_1 + v_2 v_2 + v_3 v_3 + ... + v_n v_n \\
&= v_1^2 + v_2^2 + v_3^2 + ... + v_n^2 \\
&\geq 0
\end{aligned}$$

Notice also, that $\langle \mathbf{v}, \mathbf{v} \rangle = 0$ if and only if $\mathbf{v} = 0$

$\square$

Exercise 2.5.11

Prove the following properties of the norm if $\mathbf{v}, \mathbf{w} \in \mathbb{E}^n$

1. $||\mathbf{v}|| \geq 0$

   Proof. If $\mathbf{v} \in \mathbb{E}^n$ the norm is defined by

   $$
   \begin{aligned}
   ||\mathbf{v}|| &= \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle} \\
   &= \sqrt{(v_1 v_1 + v_2 v_2 + v_3 v_3 + \ldots + v_n v_n)} \\
   &= \sqrt{(v_1^2 + v_2^2 + v_3^2 + \ldots + v_n^2)} \\
   &= (v_1^2 + v_2^2 + v_3^2 + \ldots + v_n^2)^{\frac{1}{2}} \\
   &\geq 0
   \end{aligned}
   $$

   $\square$

2. $||\mathbf{v}|| = 0$ if and only if $\mathbf{v} = 0$

   From the definition of positive definite, $\langle \mathbf{v}, \mathbf{v} \rangle = 0$ if and only if $\mathbf{v} = 0$

   Proof. "$\Rightarrow$" Let $||\mathbf{v}|| = 0$. Then

   $$
   ||\mathbf{v}|| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle} = 0
   $$

   $$
   \begin{aligned}
   \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle} &= 0 \\
   \sqrt{(v_1 v_1 + v_2 v_2 + v_3 v_3 + \ldots + v_n v_n)} &= 0 \\
   (v_1^2 + v_2^2 + v_3^2 + \ldots + v_n^2)^{\frac{1}{2}} &= 0 \\
   (v_1^2 + v_2^2 + v_3^2 + \ldots + v_n^2) &= 0
   \end{aligned}
   $$

   So $\mathbf{v} = 0$ $\square$

   Proof. "$\Leftarrow$"

   Let $\mathbf{v} = 0$

   So $\langle \mathbf{v}, \mathbf{v} \rangle = 0$.

   $$
   \begin{aligned}
   ||\mathbf{v}|| &= \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle} \\
   &= \sqrt{0} \\
   &= 0
   \end{aligned}
   $$

   $\square$

3. $\|\alpha \mathbf{v}\| = |\alpha| \|\mathbf{v}\|, \alpha \in$

   Proof.

$$\|\alpha \mathbf{v}\| = \sqrt{\langle \alpha \mathbf{v}, \alpha \mathbf{v} \rangle}$$
$$= \sqrt{(\alpha^2 v_1^2 + \alpha^2 v_2^2 + \alpha^2 v_3^2 + ... + \alpha^2 v_n^2)}$$
$$= \sqrt{\alpha^2(v_1^2 + v_2^2 + v_3^2 + ... + v_n^2)}$$
$$= |\alpha| \sqrt{(v_1^2 + v_2^2 + v_3^2 + ... + v_n^2)}$$
$$= |\alpha| \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$$
$$= |\alpha| \|\mathbf{v}\|$$

   $\square$

4. $\|\mathbf{v} + \mathbf{w}\| \leq \|\mathbf{v}\| + \|\mathbf{w}\|$

   Proof. First recall the Pythagorean theorem, $a^2 + b^2 = c^2$ and
   Cauchy-Schwarz Inequality, let $\mathbf{v}, \mathbf{w} \in \mathbb{E}^n$, then $|\langle \mathbf{v}, \mathbf{w} \rangle| \leq \|\mathbf{v}\| \|\mathbf{w}\|$

$$\|\mathbf{v} + \mathbf{w}\|^2 = \langle \mathbf{v} + \mathbf{w}, \mathbf{v} + \mathbf{w} \rangle$$
$$= (v_1 + w_1)^2 + (v_2 + w_2)^2 + ... + (v_n + w_n)^2$$
$$= (v_1^2 + v_2^2 + ... + v_n^2) + (w_1^2 + w_2^2 + ... + w_n^2) + (v_1 w_1 + ... + v_n w_n) + (w_1 v_1 + ... + w_n v_n)$$
$$= \langle \mathbf{v}, \mathbf{v} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle + \langle \mathbf{w}, \mathbf{v} \rangle + \langle \mathbf{w}, \mathbf{w} \rangle$$
$$\leq \|\mathbf{v}\|^2 + 2|\langle \mathbf{v}, \mathbf{w} \rangle| + \|\mathbf{w}\|^2$$
$$\leq \|\mathbf{v}\|^2 + 2\|\mathbf{v}\| \|\mathbf{w}\| + \|\mathbf{w}\|^2$$
$$= (\|\mathbf{v}\| + \|\mathbf{w}\|)^2$$

   $\square$

5. $\|\mathbf{v} + \mathbf{w}\|^2 + \|\mathbf{v} - \mathbf{w}\|^2 = 2(\|\mathbf{v}\|^2 + \|\mathbf{w}\|^2)$

   Proof.

$$\|\mathbf{v} + \mathbf{w}\|^2 + \|\mathbf{v} - \mathbf{w}\|^2 = \langle \mathbf{v} + \mathbf{w}, \mathbf{v} + \mathbf{w} \rangle + \langle \mathbf{v} - \mathbf{w}, \mathbf{v} - \mathbf{w} \rangle$$
$$= (v_1 + w_1)^2 + ... + (v_n + w_n)^2 + (v_1 - w_1)^2 + ... + (v_n - w_n)^2$$
$$= (v_1^2 + 2v_1 w_1 + w_1^2 + ... + v_n^2 + 2v_n w_n + w_n^2) + (v_1^2 - 2v_1 w_1 + w_1^2 + ... + v_n^2 - 2v_n w_n + w_n^2)$$
$$= (2v_1^2 + ... + 2v_n^2) + (2w_1^2 + ... + 2w_n^2)$$
$$= 2(v_1^2 + ... + v_n^2) + 2(w_1^2 + ... + w_n^2)$$
$$= 2(\langle \mathbf{v}, \mathbf{v} \rangle + \langle \mathbf{w}, \mathbf{w} \rangle)$$
$$= 2(\|\mathbf{v}\|^2 = \|\mathbf{w}\|^2)$$

   $\square$

2.5.20

1. Show that $\|\mathbf{v} \times \mathbf{w}\| = \|\mathbf{v}\|\|\mathbf{w}\| \sin\theta$, where θ is the angle between $\mathbf{v}$ and $\mathbf{w}$.

Proof. First right out the cross product of $\mathbf{v} \times \mathbf{w}$

$$\mathbf{v} \times \mathbf{w} = (v_2 w_3 - v_3 w_2, v_3 w_1 - v_1 w_3, v_1 w_2 - v_2 w_1) \qquad (1)$$

$$\begin{aligned}
\|\mathbf{v} \times \mathbf{w}\|^2 &= (v_2 w_3 - v_3 w_2)^2 + (v_3 w_1 - v_1 w_3)^2 + (v_1 w_2 - v_2 w_1)^2 \\
&= v_2^2 w_3^2 - 2 v_2 v_3 w_2 w_3 + v_3^2 w_2^2 \\
&\quad + v_3^2 w_1^2 - 2 v_1 v_3 w_1 w_3 + v_1^2 w_3^2 \\
&\quad + v_1^2 w_2^2 - 2 v_1 v_2 w_1 w_2 + v_2^2 w_1^2
\end{aligned}$$

So

$$\begin{aligned}
\|\mathbf{v} \times \mathbf{w}\|^2 &= v_1^2 (w_2^2 + w_3^2) + v_2^2 (w_1^2 + w_3^2) + v_3^2 (w_1^2 + w_2^2) \\
&\quad - 2(v_2 v_3 w_2 w_3 + v_1 v_3 w_1 w_3 + v_1 v_2 w_1 w_2)
\end{aligned}$$

$$\qquad (2)$$

Recall that $\cos\theta = \frac{\langle \mathbf{v}, \mathbf{w} \rangle}{\|\mathbf{v}\|\|\mathbf{w}\|}$ implies $\langle \mathbf{v}, \mathbf{w} \rangle = \cos\theta \|\mathbf{v}\|\|\mathbf{w}\|$. Therefore

$$\cos^2\theta \|\mathbf{v}\|^2 \|\mathbf{w}\|^2 = (\langle \mathbf{v}, \mathbf{w} \rangle)^2 \qquad (3)$$

$$(\langle \mathbf{v}, \mathbf{w} \rangle)^2 = (v_1 w_1 + v_2 w_2 + v_3 w_3)^2 \qquad (4)$$

$$\begin{aligned}
(v_1 w_1 + v_2 w_2 + v_3 w_3)^2 &= (v_1 w_1 + v_2 w_2 + v_3 w_3)(v_1 w_1 + v_2 w_2 + v_3 w_3) \\
&= v_1^2 w_1^2 + v_1 v_2 w_1 w_2 + v_1 v_3 w_1 w_3 \\
&\quad + v_1^2 w_1^2 + v_1 v_2 w_1 w_2 + v_2 v_3 w_2 w_3 \\
&\quad + v_3^2 w_3^2 + v_1 v_3 w_1 w_3 + v_2 v_3 w_2 w_3
\end{aligned}$$

So applying (3) with our result from (4),

$$\begin{aligned}
\cos^2\theta \|\mathbf{v}\|^2 \|\mathbf{w}\|^2 &= v_1^2 w_1^2 + v_1^2 w_1^2 + v_3^2 w_3^2 + \\
&\quad 2(v_1 v_2 w_1 w_2 + v_1 v_3 w_1 w_3 + v_2 v_3 w_2 w_3) \quad (5)
\end{aligned}$$

Combining (4) with (2), $\|\mathbf{v} \times \mathbf{w}\|^2 + \cos^2\theta \|\mathbf{v}\|^2 \|\mathbf{w}\|^2$ we get

$$
\begin{aligned}
v_1^2 w_1^2 &+ v_1^2 w_1^2 + v_3^2 w_3^2 + 2(v_1 v_2 w_1 w_2 + v_1 v_3 w_1 w_3 + v_2 v_3 w_2 w_3) \\
&+ v_1^2(w_2^2 + w_3^2) + v_2^2(w_1^2 + w_3^2) + v_3^2(w_1^2 + w_2^2) \\
&- 2(v_2 v_3 w_2 w_3 + v_1 v_3 w_1 w_3 + v_1 v_2 w_1 w_2) \\
&= v_1^2 w_1^2 + v_1^2 w_1^2 + v_3^2 w_3^2 \\
&+ v_1^2(w_2^2 + w_3^2) + v_2^2(w_1^2 + w_3^2) + v_3^2(w_1^2 + w_2^2) \\
&= v_1^2(w_1^2 + w_2^2 + w_3^2) + v_2^2(w_1^2 + w_2^2 + w_3^2) + v_3^2(w_1^2 + w_2^2 + w_3^2) \\
&= (w_1^2 + w_2^2 + w_3^2)(v_1^2 + v_2^2 + v_3^2)
\end{aligned}
$$

So from this we can conclude

$$
\|\mathbf{v} \times \mathbf{w}\|^2 + \cos^2\theta \|\mathbf{v}\|^2 \|\mathbf{w}\|^2 = \|\mathbf{w}\|^2 \|\mathbf{v}\|^2 \qquad (6)
$$

The result of (6) implies that $\|\mathbf{v} \times \mathbf{w}\|^2 = \|\mathbf{v}\|^2 \|\mathbf{w}\|^2 - \cos^2\theta \|\mathbf{v}\|^2 \|\mathbf{w}\|^2$.

So our last steps

$$
\begin{aligned}
\|\mathbf{v} \times \mathbf{w}\|^2 &= \|\mathbf{v}\|^2 \|\mathbf{w}\|^2 - \cos^2\theta \|\mathbf{v}\|^2 \|\mathbf{w}\|^2 \\
&= \|\mathbf{v}\|^2 \|\mathbf{w}\|^2 (1 - \cos^2\theta) \\
&= \|\mathbf{v}\|^2 \|\mathbf{w}\|^2 (\sin^2\theta)
\end{aligned}
$$

We can take the square root to reach our conclusion.

$$
\|\mathbf{v} \times \mathbf{w}\| = \|\mathbf{v}\| \|\mathbf{w}\| \sin\theta
$$

$\square$

Exercise 2.5.26

Let $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_{n-1}$ be linearly independent vectors in $\mathbb{E}^n$. Let $\mathbf{p}_0$ be a point of $\mathbb{R}^n$. Let $\mathbf{H}$ be the hyperplane plane through $\mathbf{p}_0$ spanned by $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_{n-1}$. If $\mathbf{p}$ is any point in $\mathbb{R}^n$, show that the distance from $\mathbf{p}$ to $\mathbf{H}$, that is, $\inf\{\|\mathbf{p} - \mathbf{q}\| \mid \mathbf{q} \in \mathbf{H}\}$, is given by the length of the vector $\text{proj}_{\mathbf{v}}(\mathbf{p} - \mathbf{p_0})$ where $\mathbf{v}$ is the vector obtained in Theorem 2.5.21. Specialize this to obtain formulas for the distance from a point to a line in $\mathbb{R}^2$ and form a point to a plane in $\mathbb{R}^3$.

We are given that $\mathbf{v}$ is orthogonal to $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_{n-1}$ and that $\mathbf{H}$ is the hyperplane through $\mathbf{p}_0$ spanned by $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_{n-1}$.

Since $\mathbf{p}$ and $\mathbf{p}_0$ are points, we can think of $\text{proj}_{\mathbf{v}}(\mathbf{p} - \mathbf{p_0})$ as a projection of the vector $(\mathbf{p} - \mathbf{p_0})$ onto $\mathbf{v}$ where $(\mathbf{p} - \mathbf{p_0})$ is the hypotenuse of a right angle formed by $\text{proj}_{\mathbf{v}}(\mathbf{p} - \mathbf{p_0})$. Which makes the $\text{proj}_{\mathbf{v}}(\mathbf{p} - \mathbf{p_0})$ the line adjacent to the angle formed by $\text{proj}_{\mathbf{v}}(\mathbf{p} - \mathbf{p_0})$ which is

parallel with $\mathbf{v}$ which is orthogonal to $\mathbf{H}$. By the Pythagorean theorem, this must be the shorted distance from $\mathbf{p}$ to $\mathbf{H}$, because if you took any other distance to the plane, it wouldn't form right angle with the plane and result in a greater distance.

Specializing this for $\mathbb{R}^2$ and $\mathbb{R}^3$. We consider the $\text{proj}_{\mathbf{v}}(\mathbf{p} - \mathbf{p_0})$. Let the vector $\mathbf{f} = (\mathbf{p} - \mathbf{p_0})$. We only care about the length of the line so we should take it's norm, which is given

$$\|\text{proj}_{\mathbf{v}}(\mathbf{f})\| = \frac{|\langle \mathbf{f}, \mathbf{v} \rangle|}{\|\mathbf{w}\|}$$

For $\mathbb{R}^2$, define vectors $\mathbf{v_1}$ and $\mathbf{v}$. Let $\mathbf{v}$ be non zero orthogonal to $\mathbf{v_1}$. This means the pair $\{\mathbf{v_1}, \mathbf{v}\}$ is a basis for $\mathbb{R}^2$. Let $\mathbf{H}$ be the plane through $\mathbf{p_0}$ be a point spanned by $\mathbf{v_1}$, and contain points (p_{0_1}, p_{0_2}). Let $\mathbf{p} = (p_1, p_2)$ to be any point in $\mathbb{R}^2$. by Let d be the distance we are trying to find.

$$\begin{aligned} d &= \frac{|\langle \mathbf{f}, \mathbf{v} \rangle|}{\|\mathbf{v}\|} \\ &= \frac{|(f_1 v_1 + f_2 v_2)|}{\|\mathbf{v}\|} \\ &= \frac{|(p_1 - p_{0_1})v_1 + (p_2 - p_{0_2})v_2|}{\|\mathbf{v}\|} \\ &= \frac{|(p_1 - p_{0_1})v_1 + (p_2 - p_{0_2})v_2|}{\sqrt{v_1^2 + v_2^2}} \end{aligned}$$

For $\mathbb{R}^3$, we have a similar result. define vectors $\mathbf{v_1}, \mathbf{v_2}$ to be linearly independent. Let $\mathbf{v}$ be the determinant of $\mathbf{v_1}, \mathbf{v_2}$. Resulting in $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v}\}$ as a basis $\mathbb{R}^3$ Let $\mathbf{H}$ be the plane through the point $\mathbf{p_0}$ spanned by $\mathbf{v_1}, \mathbf{v_2}$, and contain points $(p_{0_1}, p_{0_2}, p_{0_3})$. Let $\mathbf{p} = (p_1, p_2, p_3)$ to be any point in $\mathbb{R}^3$. by Let d be the distance we are trying to find.

$$\begin{aligned} d &= \frac{|\langle \mathbf{f}, \mathbf{v} \rangle|}{\|\mathbf{v}\|} \\ &= \frac{|(f_1 v_1 + f_2 v_2 + f_3 v_3)|}{\|\mathbf{v}\|} \\ &= \frac{|(p_1 - p_{0_1})v_1 + (p_2 - p_{0_2})v_2 + (p_3 - p_{0_3})v_3|}{\|\mathbf{v}\|} \\ &= \frac{|(p_1 - p_{0_1})v_1 + (p_2 - p_{0_2})v_2 + (p_3 - p_{0_3})v_3|}{\sqrt{v_1^2 + v_2^2 + v_3^2}} \end{aligned}$$

Exercise 2.5.30

Consider the vectors $\mathbf{v}_1 = (1,1,-1,0)$, $\mathbf{v}_2 = (1,0,0,-1)$ and $\mathbf{v}_3 = (0,1,1,1)$ in $\mathbb{E}^4$.

1. Use the Gram-Schmidt orthogonalization process on these three vectors to produce a set of three mutually orthogonal vectors that span the same subspace.

    Let $\mathbf{v}_1 = \tilde{\mathbf{v}}_1$

    For $\tilde{\mathbf{v}}_2 = \mathbf{v}_2 - \text{proj}_{\tilde{\mathbf{v}}_1}(\mathbf{v}_2)$

    $$\text{proj}_{\tilde{\mathbf{v}}_1}(\mathbf{v}_2) = \frac{\langle \mathbf{v}_2, \tilde{\mathbf{v}}_1 \rangle}{\|\tilde{\mathbf{v}}_1\|} \frac{\tilde{\mathbf{v}}_1}{\|\tilde{\mathbf{v}}_1\|}$$

    $$\langle \mathbf{v}_2, \tilde{\mathbf{v}}_1 \rangle = (1)(1) + (0)(1) + (0)(-1) + (-1)(0)$$
    $$= 1$$
    $$\|\tilde{\mathbf{v}}_1\| = \sqrt{1^2 + 1^2 + (-1)^2 + 0^2}$$
    $$= \sqrt{3}$$

    $$\text{proj}_{\tilde{\mathbf{v}}_1}(\mathbf{v}_2) = \frac{1}{\sqrt{3}} \frac{\tilde{\mathbf{v}}_1}{\sqrt{3}}$$
    $$= \frac{1}{3}\tilde{\mathbf{v}}_1$$
    $$= \frac{1}{3}\begin{bmatrix} 1 & 1 & -1 & 0 \end{bmatrix}$$
    $$= \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & 0 \end{bmatrix}$$

    $$\tilde{\mathbf{v}}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} - \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ -\frac{1}{3} \\ 0 \end{bmatrix}$$
    $$= \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & \frac{1}{3} & -1 \end{bmatrix}$$

    For $\tilde{\mathbf{v}}_3 = \mathbf{v}_3 - \text{proj}_{\tilde{\mathbf{v}}_1}(\mathbf{v}_3) - \text{proj}_{\tilde{\mathbf{v}}_2}(\mathbf{v}_3)$

    $$\text{proj}_{\tilde{\mathbf{v}}_1}(\mathbf{v}_3) = \frac{\langle \mathbf{v}_3, \tilde{\mathbf{v}}_1 \rangle}{\|\tilde{\mathbf{v}}_1\|} \frac{\tilde{\mathbf{v}}_1}{\|\tilde{\mathbf{v}}_1\|}$$

$$\langle \mathbf{v}_3, \tilde{\mathbf{v}}_1 \rangle = (0)(1) + (1)(1) + (1)(-1) + (1)(0)$$
$$= 0$$
$$\|\tilde{\mathbf{v}}_1\| = \sqrt{3}$$

$$\text{proj}_{\tilde{\mathbf{v}}_1}(\mathbf{v}_3) = \frac{0}{\sqrt{3}} \frac{\tilde{\mathbf{v}}_1}{\sqrt{3}}$$
$$= 0\tilde{\mathbf{v}}_1$$
$$= 0 \begin{bmatrix} 1 & 1 & -1 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{proj}_{\tilde{\mathbf{v}}_2}(\mathbf{v}_3) = \frac{\langle \mathbf{v}_3, \tilde{\mathbf{v}}_2 \rangle}{\|\tilde{\mathbf{v}}_2\|} \frac{\tilde{\mathbf{v}}_2}{\|\tilde{\mathbf{v}}_2\|}$$

$$\langle \mathbf{v}_3, \tilde{\mathbf{v}}_2 \rangle = (0)\left(\frac{2}{3}\right) + (1)\left(-\frac{1}{3}\right) + (1)\left(\frac{1}{3}\right) + (1)(-1)$$
$$= -1$$
$$\|\tilde{\mathbf{v}}_2\| = \sqrt{\frac{2}{3}^2 + -\frac{1}{3}^2 + \frac{1}{3}^2 + -1^2}$$
$$= \sqrt{\frac{5}{3}}$$

$$\text{proj}_{\tilde{\mathbf{v}}_2}(\mathbf{v}_3) = -\frac{1}{\sqrt{\frac{5}{3}}} \frac{\tilde{\mathbf{v}}_2}{\sqrt{\frac{5}{3}}}$$
$$= -\frac{3}{5}\tilde{\mathbf{v}}_2$$
$$= -\frac{3}{5} \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & \frac{1}{3} & -1 \end{bmatrix}$$
$$= \begin{bmatrix} \frac{6}{15} & -\frac{3}{15} & \frac{3}{15} & \frac{3}{5} \end{bmatrix}$$

$$\tilde{\mathbf{v}}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} \frac{6}{15} \\ -\frac{3}{15} \\ \frac{3}{15} \\ \frac{3}{5} \end{bmatrix}$$
$$= \begin{bmatrix} -\frac{6}{15} & \frac{18}{15} & \frac{12}{15} & \frac{2}{5} \end{bmatrix}$$

2. Extend the set of three vectors produced in part 1 to a mutually orthogonal basis for

Consider the matrix comprised of $\tilde{\mathbf{v}}_1, \tilde{\mathbf{v}}_2 \tilde{\mathbf{v}}_3$ and vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4 \in \mathbb{E}^4$ where 1 is in the the jth term and 0 otherwise.

$$
\begin{bmatrix}
\mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 & \mathbf{e}_4 \\
1 & 1 & -1 & 0 \\
\frac{2}{3} & -\frac{1}{3} & \frac{1}{3} & -1 \\
-\frac{6}{15} & \frac{18}{15} & \frac{12}{15} & \frac{2}{5}
\end{bmatrix}
\tag{7}
$$

The vector that will complete the basis can be created by taking the determinant of the matrix (7).

Step 1

$$
\mathbf{e}_1 \begin{bmatrix} 1 & -1 & 0 \\ -\frac{1}{3} & \frac{1}{3} & -1 \\ \frac{18}{15} & \frac{12}{15} & \frac{2}{5} \end{bmatrix} - \mathbf{e}_2 \begin{bmatrix} 1 & -1 & 0 \\ \frac{2}{3} & \frac{1}{3} & -1 \\ -\frac{6}{15} & \frac{12}{15} & \frac{2}{5} \end{bmatrix}
$$

$$
+ \mathbf{e}_3 \begin{bmatrix} 1 & 1 & 0 \\ \frac{2}{3} & -\frac{1}{3} & -1 \\ -\frac{6}{15} & \frac{18}{15} & \frac{2}{5} \end{bmatrix} - \mathbf{e}_4 \begin{bmatrix} 1 & 1 & -1 \\ \frac{2}{3} & -\frac{1}{3} & \frac{1}{3} \\ -\frac{6}{15} & \frac{18}{15} & \frac{12}{15} \end{bmatrix}
$$

Then the next step...

$$
\mathbf{e}_1 (1 \begin{bmatrix} \frac{1}{3} & -1 \\ \frac{12}{15} & \frac{2}{5} \end{bmatrix} - -1 \begin{bmatrix} -\frac{1}{3} & -1 \\ \frac{18}{15} & \frac{2}{5} \end{bmatrix} -
$$

$$
\mathbf{e}_2 (1 \begin{bmatrix} \frac{1}{3} & -1 \\ \frac{12}{15} & \frac{2}{5} \end{bmatrix} - -1 \begin{bmatrix} \frac{2}{3} & -1 \\ -\frac{6}{15} & \frac{2}{5} \end{bmatrix} +
$$

$$
\mathbf{e}_3 (1 \begin{bmatrix} -\frac{1}{3} & -1 \\ \frac{18}{15} & \frac{2}{5} \end{bmatrix} - 1 \begin{bmatrix} \frac{2}{3} & -1 \\ -\frac{6}{15} & \frac{2}{5} \end{bmatrix} -
$$

$$
\mathbf{e}_4 (1 \begin{bmatrix} -\frac{1}{3} & \frac{1}{3} \\ \frac{18}{15} & \frac{12}{15} \end{bmatrix} - 1 \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ -\frac{6}{15} & \frac{12}{15} \end{bmatrix} + -1 \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{6}{15} & \frac{18}{15} \end{bmatrix})
$$

...eventually arriving at...

$$
\mathbf{e}_1(-\frac{2}{5}) - \mathbf{e}_2(\frac{4}{5}) + \mathbf{e}_3(\frac{6}{5}) - \mathbf{e}_4(-2)
$$

We now compute our new vector to be $(-\frac{2}{5}, -\frac{4}{5}, \frac{6}{5}, 2)$ and define it as $\mathbf{v}$. Which gives us the basis $\{\tilde{\mathbf{v}}_1, \tilde{\mathbf{v}}_2, \tilde{\mathbf{v}}_3, \mathbf{v}\}$ over $\mathbb{E}^4$.

In order to make this a mutually orthogonal basis for $\mathbb{E}^4$, apply the Gram-Schmidt orthogonalization process once again.

Let $\bar{\mathbf{v}} = \mathbf{v}$

For $\bar{\mathbf{v}}_1 = \tilde{\mathbf{v}}_1 - \text{proj}_{\bar{\mathbf{v}}}(\tilde{\mathbf{v}}_1)$

$$\text{proj}_{\bar{\mathbf{v}}}(\tilde{\mathbf{v}}_1) = \frac{\langle \mathbf{v}_1, \bar{\mathbf{v}} \rangle}{\|\bar{\mathbf{v}}\|} \frac{\bar{\mathbf{v}}}{\|\bar{\mathbf{v}}\|}$$

$$\langle \tilde{\mathbf{v}}_1, \bar{\mathbf{v}} \rangle = (1)(-\frac{2}{5}) + (1)(-\frac{4}{5}) + (-1)(-\frac{6}{5}) + (0)(2)$$
$$= -\frac{12}{5}$$

$$\|\bar{\mathbf{v}}\| = \sqrt{-\frac{2}{5}^2 + -\frac{4}{5}^2 + \frac{6}{5}^2 + 2^2} = \sqrt{\frac{156}{25}}$$

$$\text{proj}_{\bar{\mathbf{v}}}(\tilde{\mathbf{v}}_1) = -\frac{-\frac{12}{5}}{\sqrt{\frac{156}{25}}} \frac{\bar{\mathbf{v}}}{\sqrt{\frac{156}{25}}}$$
$$= -\frac{5}{13}\bar{\mathbf{v}}$$
$$= -\frac{5}{13}\begin{bmatrix} -\frac{2}{5} & -\frac{4}{5} & \frac{6}{5} & 2 \end{bmatrix}$$
$$= \begin{bmatrix} \frac{2}{13} & \frac{4}{13} & -\frac{6}{13} & -\frac{10}{13} \end{bmatrix}$$

$$\bar{\mathbf{v}}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix} - \begin{bmatrix} \frac{2}{13} \\ \frac{4}{13} \\ -\frac{6}{13} \\ -\frac{10}{13} \end{bmatrix} = \begin{bmatrix} \frac{11}{13} & \frac{9}{13} & -\frac{7}{13} & \frac{10}{13} \end{bmatrix}$$

For $\bar{\mathbf{v}}_2 = \tilde{\mathbf{v}}_2 - \text{proj}_{\bar{\mathbf{v}}}(\tilde{\mathbf{v}}_2) - \text{proj}_{\bar{\mathbf{v}}_1}(\tilde{\mathbf{v}}_2)$. $\|\bar{\mathbf{v}}_1\| = \sqrt{\frac{27}{13}}$

$$\text{proj}_{\bar{\mathbf{v}}}(\tilde{\mathbf{v}}_2) = \frac{-\frac{8}{5}}{\sqrt{\frac{156}{25}}} \frac{\bar{\mathbf{v}}}{\sqrt{\frac{156}{25}}} = -\frac{10}{39}\bar{\mathbf{v}} = \begin{bmatrix} \frac{4}{39} & \frac{8}{39} & -\frac{12}{39} & -\frac{20}{39} \end{bmatrix}$$

$$\text{proj}_{\bar{\mathbf{v}}_1}(\tilde{\mathbf{v}}_2) = -\frac{\frac{12}{13}}{\sqrt{\frac{27}{13}}} \frac{\bar{\mathbf{v}}_1}{\sqrt{\frac{27}{13}}} = -\frac{4}{9}\bar{\mathbf{v}}_1 = \begin{bmatrix} -\frac{44}{117} & -\frac{4}{13} & \frac{28}{117} & -\frac{40}{117} \end{bmatrix}$$

$$\bar{\mathbf{v}}_2 = \begin{bmatrix} \frac{2}{3} \\ -\frac{1}{3} \\ \frac{1}{3} \\ -1 \end{bmatrix} - \begin{bmatrix} \frac{4}{39} \\ \frac{8}{39} \\ -\frac{12}{39} \\ -\frac{20}{39} \end{bmatrix} - \begin{bmatrix} -\frac{44}{117} \\ -\frac{4}{13} \\ \frac{28}{117} \\ -\frac{40}{117} \end{bmatrix} = \begin{bmatrix} \frac{110}{117} & -\frac{17}{39} & \frac{47}{117} & -\frac{17}{117} \end{bmatrix}$$

For $\bar{\mathbf{v}}_3 = \tilde{\mathbf{v}}_3 - \text{proj}_{\bar{\mathbf{v}}}(\tilde{\mathbf{v}}_3) - \text{proj}_{\bar{\mathbf{v}}_1}(\tilde{\mathbf{v}}_3) - \text{proj}_{\bar{\mathbf{v}}_2}(\tilde{\mathbf{v}}_3)$. Note, $\|\bar{\mathbf{v}}_2\| = \sqrt{\frac{49}{39}}$

$$\text{proj}_{\bar{\mathbf{v}}}(\tilde{\mathbf{v}}_3) = \frac{\frac{72}{75}}{\sqrt{\frac{156}{25}}\sqrt{\frac{156}{25}}}\bar{\mathbf{v}} = \frac{2}{13}\bar{\mathbf{v}} = \begin{bmatrix} -\frac{4}{65} & -\frac{8}{65} & \frac{12}{65} & \frac{4}{13} \end{bmatrix}$$

$$\text{proj}_{\bar{\mathbf{v}}_1}(\tilde{\mathbf{v}}_3) = \frac{8}{45}\bar{\mathbf{v}}_1 = \begin{bmatrix} \frac{88}{585} & \frac{72}{585} & -\frac{56}{585} & \frac{80}{585} \end{bmatrix}$$

$$\text{proj}_{\bar{\mathbf{v}}_2}(\tilde{\mathbf{v}}_3) = \frac{124}{245}\bar{\mathbf{v}}_2 = \begin{bmatrix} \frac{2728}{5733} & -\frac{2108}{9555} & \frac{5828}{28665} & -\frac{2108}{28665} \end{bmatrix}$$

$$\bar{\mathbf{v}}_3 = \begin{bmatrix} -\frac{9806}{9555} & \frac{12398}{9555} & \frac{19848}{28665} & \frac{9654}{28665} \end{bmatrix}$$

We now have a mutually orthogonal basis for $\mathbb{E}^4$.