JOE SEIDEL

REAL ANALYSIS
FIFTH WEEK

Exercise 3.3.13

Let (X, d) be a metric space and let Y be an open set in X. Show that every open set in (Y, d'), where d' is the inherited metric, is also open in X.

Since Y is an open set, $\forall y_0 \in Y$ and fix $r > 0$ such that $B_r(y_0) \subset Y$. Note that $B_r(y_0) = \{y \in Y | d'(y, y_0) < r\}$ where $y_0, y \in Y(\subset X)$ and $d'(y_0, y) = d(y_0, y)$. Since y_0 was arbitrary, any open set in Y is also open in X.

Exercise 3.3.20

Show that $\mathbb{Q}$ as a subset of $\mathbb{R}$ with the usual metric is neither open or closed in $\mathbb{R}$. (Of course, if the metric space is simply $\mathbb{Q}$ with the usual metric, then $\mathbb{Q}$ is both open and closed in $\mathbb{Q}$.)

Rational numbers are dense in $\mathbb{R}$, which means between any $p, t \in \mathbb{Q}$, $p \neq t$ there exists an irrational number, i. Without loss of generality say $p < t$, then $p < i < t$. Therefore, every open ball around $q \in \mathbb{Q}$ contains points not in $\mathbb{Q}$, i.e. the open ball, $\epsilon > 0$, $B_\epsilon(q) \not\subset \mathbb{Q}$. Therefore $\mathbb{Q}$ is not open.

Similarly, irrational numbers lie between any two rational numbers and none of $\mathbb{Q}^c$ lie entirely in $\mathbb{Q}$. So, $\mathbb{R} \setminus \mathbb{Q}$ is not open and $\mathbb{Q}$ is not closed.

Exercise 3.3.31

Suppose that A is a subset of a metric space X. Show that $\overline{A} = A \cup \{\text{accummulation points of } A\}$

Proof. Let $A' = \{\text{accummulation points of } A\}$. Consider $A \cup A'$, then any $a \in A \cup A'$ is either in A or A'. Consider $a \in A$, note that $A \subset \overline{A}$, so $a \in \overline{A}$. If $a \in A'$, since $\overline{A}$ is closed it must contains all the accumulation points of A so $a \in \overline{A}$. Therefore $A \cup A' \subset \overline{A}$

Now show $\overline{A} \subset A \cup A'$. Consider any $a \in \overline{A}$. Then $a \in A$ or $a \in \overline{A} \setminus A$. The first case is trivial. If $a \in \overline{A} \setminus A$, then $a \notin A$ but $a \in \overline{A}$ so a must be an accumulation point of A. Therefore, for any $a \in \overline{A}$, $a \in A$ or $a \in A'$ so conclude $\overline{A} \subset A \cup A'$. $\square$

Exercise 3.3.32

Suppose A is a subset of a metric space X. Prove or disprove $\overline{A} = A \cup \partial A$

Proof. Want to show $A \cup \partial A \subset \overline{A}$. Consider any $x \in A \cup \partial A$. Then either $x \in A$ or $x \in \partial A$. If $x \in A$ then $x \in \overline{A}$ since $A \subset \overline{A}$. If $x \in \partial A$ then for any $r > 0$, $B_r(x) \cap A \neq \emptyset$ and $B_r(x) \cap A^c \neq \emptyset$. This means

x is either an isolated point in A, so $x \in A$, or it is an accumulation point of A. Again since $A \subset \overline{A}$ and $\overline{A}$ contains all accumulation points, $x \in \overline{A}$. Since a was arbitrary, $A \cup \partial A \subset \overline{A}$

Next, want to show $\overline{A} \subset A \cup \partial A$. Consider any $x \in \overline{A}$, then $x \in A$ or $x \notin A$. If $x \in A$ we are done. If $x \notin A$, then $x \in X \setminus A$. Suppose x is an exterior point of A, then $x \in B_r(x)$, there exists $r > 0$ such that $B_r(x) \cap A = \emptyset$. However this contradicts $x \in \overline{A}$ since $\overline{A}$ is the intersection of every closed set containing A. Therefore if $x \notin A$ then $x \in \partial A$. Then conclude, $\overline{A} \subset A \cup \partial A$. □

Exercise 3.3.33

Suppose A is a subset of a metric space X. Prove that $\partial A = \overline{A} \cap \overline{A^c}$.

Proof. Choose any $x \in \overline{A} \cap \overline{A^c}$, then $x \in \overline{A}$ and $x \in \overline{A^c}$. Suppose there exists $r > 0$ such that $B_r(x) \cap A = \emptyset$, but then $x \notin \overline{A}$, which is a contradiction since $\overline{A}$ is the intersection of all sets containing A. So x is either in ∂A or A. Now we should suppose $x \in A$ and there exists $r > 0$ such that $B_r(x) \subset A$. However, this contradicts that $x \in \overline{A^c}(= X \setminus \overline{A})$. Therefore $x \in \partial A$. This implies, since x was arbitrary, $\overline{A} \cap \overline{A^c} \subset \partial A$.

Now consider any $x \in \partial A$. This implies $\forall r > 0$, $B_r(x) \cap A \neq \emptyset$ and $B_r(x) \cap A^c \neq \emptyset$. Therefore $x \in \overline{A}$. Since $A^c \subset \overline{A^c}$, x is also in $\overline{A^c}$. Again, since x was arbitrary $\partial A \subset \overline{A} \cap \overline{A^c}$.

□

Exercise 3.3.49

1. Describe the closed convex hull of the unit ball in $\ell_n^p(\mathbb{R})$ for $1 \leq p \leq \infty$.

   Let $B_1(0)$ be the unit ball in $\ell_n^p(\mathbb{R})$ and $\mathcal{I} := \{p, q \in \overline{B_1(0)}\}$. The closed convex hull of $B_1(0)$ is $\bigcap_{i \in \mathcal{I}} \{(1-t)p_i + t(q_i) | 0 \leq t \leq 1 \text{ with } t \in \mathbb{R}\}$

2. Suppose $0 < p < 1$ For $x \in \mathbb{R}^n$, define,

$$\|x\|_p = \left(\sum_{k=1}^{n} |x_k|^p \right)^{\frac{1}{p}}$$

   Define $S_p = \{x \in \mathbb{R}^n | \|x\|_p \leq 1\}$. Determine whether S_p is convex. If not, find the closed convex hull of S_p.

   Proof. For S_p to be convex, the second derivative needs to be greater than 0.

$$S_p' = p|x_k|^{(p-1)}$$
$$S_p'' = p(p-1)''|x_k|^{(p-2)}$$

Since $0 < p < 1$, $p - 1 < 0$ so $S_p'' < 0$ which means it is not convex. So now it is time to find the closed convex hull of S_p.

Consider the points at the corners of S_p, i.e. $p, q \in S_p$ where $p = (1,0)$ and $q = (0,1)$ and take the norm of the line segment formed by these two points.

$$\begin{aligned}
\|(1-t)p + tq\|_p &= \|1 - t[0,1] + t[0,1]\|_p \text{ with } 0 \leq t \leq 1 \\
&= \|[1-t, 0] + [0, t]\|_p \\
&= \|(1-t, t)\|_p \\
&= (|(1-t)|^p + |t|^p)^{\frac{1}{p}} \\
&\geq ((1-t) + t)^{\frac{p}{p}} \\
&= 1
\end{aligned}$$

For $0 < p < 1$, $(|(1-t)|^p + |t|^p)^{\frac{1}{p}} \geq 1$, so the line segment isn't contained in the S_p, confirming the above. However, for $p = 1$ $(|(1-t)|^p + |t|^p)^{\frac{1}{p}} = 1$ so the closed convex hull of S_p is $S_1 = \{x \in \mathbb{R}^n \mid \|x\|_1 \leq 1\}$

□

Not in Book

Work on the following problems.

1. $(A^o)^c = \overline{A^c}$

Proof. Consider any $x \in (A^o)^c$, want so show $(A^o)^c \subset \overline{A^c}$. Then $x \notin A^o$. This means $\nexists \epsilon > 0$ such that $B_\epsilon(x) \subset A$. Therefore $\exists \epsilon > 0$ such that $B_\epsilon(x) \cap A^c \neq 0$, so $x \in \overline{A^c}$. This implies $(A^o)^c \subset \overline{A^c}$.

Consider any $x \in \overline{A^c}$. This means $\forall \epsilon > 0$, $B_\epsilon(x) \cap A^c \neq \emptyset$. Which means any open ball around x will intersect A^c so you cannot find an open ball where $B_\epsilon(x) \subset A$. This implies $x \notin A^o$. Therefore, $\overline{A^c} \subset (A^o)^c$

□

2. An isolated point of A is an accumulation point of A^c.

Proof. Let $x \in A$ be an isolated point of A. This means $\exists \epsilon > 0$ such that $B_\epsilon(x) \cap A = \{x\}$. Since isolated points are boundary points, it also means, for any $\epsilon > 0$ $B_\epsilon(x) \setminus \{x\} \cap A^c \neq \emptyset$. Therefore, any isolated point of A is an accumulation point of A^c.

□

3. Construct an example of a set A such that $\overline{A} = \emptyset$.

   The Cantor set in space $\mathbb{R}$. The interior of the Cantor set is empty, i.e. $\mathcal{C}^o = \emptyset$, since it contains no non-empty open intervals. The closure of $\emptyset$ is also empty, i.e. $\overline{\mathcal{C}^o} = \emptyset$.

4. A set A such that $\overline{A}^o = \emptyset$.

   Again, the Cantor set fulfils this requirement. The closure of the Cantor set is the Cantor set. The interior of the Cantor set is empty.