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REAL ANALYSIS MIDTERM 2

Question 1

Prove that the ℓ^p norm on $\mathbb{R}^2$ is equivalent to the ℓ^∞ norm for all $p \geq 1$.

Proof. The ℓ^p norm of x is $\|x\|_p = (|x_1|^p + |x_2|^p)^{\frac{1}{p}}$ $\forall x = (x_1, x_2) \in \mathbb{R}^2$. The ℓ^∞ norm of x is $\|x\|_\infty = max\{|x_1|, |x_2|\}$. Suppose, without loss of generality, $\|x\|_\infty = |x_1|$, i.e. $|x_1| \geq |x_2|$.

First

$$\|x\|_\infty = |x_1| = (|x_1|^p)^{\frac{1}{p}} \leq (|x_1|^p + |x_2|^p)^{\frac{1}{p}} = \|x\|_p$$

which implies $\|x\|_\infty \leq \|x\|_p$. Next

$$\|x\|_p = (|x_1|^p + |x_2|^p)^{\frac{1}{p}} \leq (|x_1|^p + |x_1|^p)^{\frac{1}{p}} = 2^{\frac{1}{p}}|x_1|^{\frac{1}{p}} = 2^{\frac{1}{p}}\|x\|_\infty$$

So

$$\|x\|_\infty \leq \|x\|_p \leq 2^{\frac{1}{p}}\|x\|_\infty$$

$\square$

Question 2

Suppose $f : X \to X'$ is a bijection (one-to-one and onto) and continuous where $X(\subset \mathbb{R}$ is compact and $X' \subset \mathbb{R}$. Prove that f is in fact a homeomorphism.

Proof. It remains to show f^{-1} is continuous. We need to show for any open set $U \subset X$, $(f^{-1})^{-1}(U) = f(U)$ is open in X'. Equivently for any closed set $V \subset X$, $(f^{-1})^{-1}(V) = f(V)$ is closed in X'.

Since X is compact, we have any closed subset $V(\subset X)$ is compact (midterm 1). Next f is continuous implies that f maps compact sets to compact sets. So $f(V)$ is compact. Since X' is bounded in $\mathbb{R}$, $f(V)$ as a compact set in $\mathbb{R}$ is bounded and closed. This shows that for any closed set $V \subset X$, its image $f(V)$ is closed. This shows f^{-1} is continuous, so f is a homeomorphism. $\square$

Question 3

Show the sequence

$$\{\cos^n x \,|\, x \in [-\frac{\pi}{2}, \frac{\pi}{2}]\}$$

does not converge uniformly.

Proof. A simple proof is by the Dini Theorem. If $\cos^n x$ for $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ converges uniformly, by Dini theorem, the limiting function should be continuous. However, the pointwise limit is

$$\cos^n x \to f(x) = \begin{cases} 1, & \text{if } x = 0 \\ 0, & \text{if } x \neq 0 \end{cases}$$

discontinous. ∎

Proof. Or we can show $\exists \epsilon$ such that for all n, there exists x_n such that $|\cos^n x_n - f(x_n)| > \epsilon$. We choose $\epsilon = \frac{1}{2}$ Since $f(x) = 0$ for $x \neq 0$ it is enough to find x_n satisfying $\cos x_n > (\frac{1}{2})^{\frac{1}{n}}$. Since $0 < (\frac{1}{2})^{\frac{1}{n}} < 1$, such x_n always exists.

∎

Question 4

Find the closure, interior and boundary of the following sets.

1. The interval $(0, 1)$ as a subset of $\mathbb{C}$.
   Closure $[0, 1] \subset \mathbb{C}$. Interior $\emptyset$. Boundary $[0, 1] \subset \mathbb{C}$
2. The set of rational numbers $\mathbb{Q}$ as a subset of $\mathbb{R}$
   Closure $\mathbb{R}$. Interior $\emptyset$. Boundary $\mathbb{R}$.
3. The Cantor set as a subset of $\mathbb{R}$
   Closure Cantor set. Interor $\emptyset$. Boundary Cantor set.

Question 5

Consider two vectors $v_1 = (1, 1, 0)$ and $v_2 = (3, 0, 4)$ in $\mathbb{R}^3$ endowed with the standard inner product. The two vectors span a plane

$$P = \{sv_1 + tv_2 \mid s, t \in \mathbb{R}\}.$$

Use Gram-Schmidt to produce an orthonormal basis for the plane P.
$\tilde{v}_1 = v_1$ and $\tilde{v}_2 = v_2 - \text{proj}_{v_1}(v_1)$ Then normalize v_1 and $\tilde{v}_2$.

Question 6

Consider a metric space (X, d). Suppose both two sets $S_1, S_2 \subset X$ are open and dense in X. Prove that $S_1 \cap S_2$ is open and dense in X.

Proof. The intersection of finite open sets is open, so $S_1 \cap S_2$ is open. To show $S_1 \cap S_2$ is dense, we consider any nonempty open set $U \subset X$. Since S_1 is dense, we have $S_1 \cap U \neq \emptyset$. Pick $x \in S_1 \cap U$, since $S_1 \cap U$ is open, we have that $\exists \epsilon$ such that $B_\epsilon(x) \subset S_1 \cap U$. Since S_2 is dense, we get $B_\epsilon(x) \cap S_2 \neq \emptyset$. This implies $B_\epsilon(x) \cap S_2 \subset S_1 \cap S_2 \cap U$. Hence $S_1 \cap S_2$ is dense. ∎

Question 7

We introduce the metric $d(x, y) = \frac{|x-y|}{1+|x-y|}$ on $\mathbb{R}$. Show that $\mathbb{R}$ is complete under this metric. You do not need to prove that d is a metric.

Proof. It is enough to show any Cauchy sequence has a limit in $\mathbb{R}$. Suppose $\{x_n\}$ is a Cauchy sequence in the new metric, i.e. $\forall \epsilon \; \exists N$ such that when $m, n > N$ we have

$$d(x_m, x_n) = \frac{|x_m - x_n|}{1 + |x_m - x_n|} < \epsilon$$

.

Then we have $|x_m - x_n| < \epsilon + |x_m - x_n|\epsilon$ for $\epsilon < \frac{1}{2}$, we have $|x_m - x_n| < \frac{\epsilon}{1-\epsilon} < 2\epsilon$. This implies $\forall \epsilon < \frac{1}{2} \; \exists N$ such that when $m, n > N$ we have $|x_m - x_n| < 2\epsilon$. Hence $\{x_n\}$ is a Cauchy sequence in $\mathbb{R}$ in the usual metric. $\mathbb{R}$ is complete in this metric, so $\exists x \in \mathbb{R}$ such that $\lim_{n \to \infty} x_n = x$, in the usual metric.

Furthermore, $d(x_n, x) = \frac{|x_n - x|}{1+|x_n+x|} < |x_n - x|$. Therefore $\{x_n\}$ converges to x also in the new metric.

$\square$