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REAL ANALYSIS

Section 1.5 Construction of the Real Numbers

Exercise 1.5.1

Show that for any $a, b \in \mathbb{Q}$, we have $||a| - |b|| \leq |a - b|$.

Proof. Since $a, b \in \mathbb{Q}$,

Absolute values on $\mathbb{Q}$ satisfy the Triangle Inequality

$$|a + b| \leq |a| + |b|$$

So

$$|a| = |a - b + b| \leq |a - b| + |b|$$

$$|b| = |a + b - a| \leq |b - a| + |a|$$

These can be rewritten as

$$|a| - |b| \leq |a - b|$$

$$|b| - |a| \leq |b - a|$$

Since $|a - b| = |b - a|$ and if $t \geq x$ and $t \geq -x$ then $t \geq |x|$, therefore

$$||a| - |b|| \leq |a - b|$$

□

Exercise 1.5.5

If a sequence $(a_k)_{k \in \mathbb{N}}$ converges in $\mathbb{Q}$ show that $(a_k)_{k \in \mathbb{N}}$ is a Cauchy sequence in $\mathbb{Q}$.

Proof. By definition if $(a_k)_{k \in \mathbb{N}}$ converges in $\mathbb{Q}$ given any rational number $r > 0$ there exists an integer N such that if $n \geq N$ then $|a_n - a| < r$.

Suppose $(a_k)_{k \in \mathbb{N}}$ converges to $a, a \in \mathbb{Q}$. Let $r > 0$, since $(a_k)_{k \in \mathbb{N}}$ converges to a, $\exists N$ such that $\forall n \geq N$, $|a_n - a| < \frac{r}{2}$.

Then $\forall n, m > N$

$$|a_n - a_m| = |a_n - a + a - a_m| \leq |a_n - a| + |a - a_m|$$

Since $n > N$ and $m > N$

$$|a_n - a| < \frac{r}{2}$$

and

$$|a - a_m| = |a_m - a| < \frac{r}{2}$$

therefore

$$|a_n - a_m| < \frac{r}{2} + \frac{r}{2} = r$$

□