JOE SEIDEL

REAL ANALYSIS
EIGHTH WEEK

Exercise 3.6.12-DROPPED

Suppose that A and B are nonempty subsets of a metric space X. The *distance* between A and B is defined by

$$d(A,B) = \inf\{d(a,b) \mid a \in A, b \in B\}$$

We say that $d(A,B)$ is *assumed* if there exists $a_0 \in A$ and $b_0 \in B$ such that $d(A,B) = d(a_0,b_0)$. Determine whether or not the distance between A and B is necessarily assumed in $(i) - (iii)$.

1. A is closed and B is closed.
   Suppose $A = \{x, \frac{-1}{x} |, x < 0\}$ and $B = \{x, \frac{1}{x} |, x > 0\}$ in $\mathbb{R}^2$. Then $d(A,B) = 0$ but, $\forall a \in A$ and $\forall b \in B$ distance is $\sqrt{(-x-x)^2 + (\frac{1}{x} - \frac{1}{x})^2} > 0$, i.e. $d(a,b) > 0$. Thefore, distance is not neccessarily assumed when A and B are closed.

2. A is compact and B is closed. Considering the same example of above, but with A now being compact. This means that any sequence in A converges in A, so it achieves it's minimum and maximum values. However,B is still closed and there is a sequence in $(b_n) \in B$ such that $\lim_{n\to\infty} b_n = 0$, however $0 \notin B$ which is required in order to make $d(A,B) = d(a_0,b_0) = 0$ for some $a_0 \in A$ and $b_0 \in B$. So distance in this case is not neccessrily assumed.

3. A is compact and B is compact.
   There exists a sequence $(a_n) \in A$ and $(b_n) \in B$ such that

   $$\lim_{n\to\infty} d(a_n, b_n) = d(A,B)$$

   Being compact in a metric is equivelance to being sequentially compact, so A and B are sequentially compact. This means

   $$\lim_{k\to\infty} d(a_{n_k}) = a \text{ with } a \in A$$

   Additionally, note

   $$\lim_{k\to\infty} d(a_{n_k}, b_{n_k}) = d(A,B)$$

   Since B is sequentially compact,

   $$\lim_{j\to\infty} b_{n_{k_j}} = b \text{ with } b \in B$$

   Note once more,

   $$\lim_{j\to\infty} d(a_{n_{k_j}}, b_{n_{k_j}}) = d(A,B)$$

   As long as d is a continuous function, the above implies $d(a,b) = d(A,B)$ with $a \in A$ and $b \in B$. Hence, distance can be assumed under these conditions.

4. What happens to the above cases if we assume X is complete?
   We proved in exercise 3.4.8 (hw 6) that a closed subset of a complete space is also complete. Therefore, all cauchy sequences in X converge in X. Hence, in the above cases distance can be assumed.

Exercise 3.6.25

1. In the usual metric, $\mathbb{Q}$ is dense in $\mathbb{R}$.
   Consider $a, b \in \mathbb{R}$, without loss of generality let $a < b$. We want to show that $\exists q \in \mathbb{Q}$ such that $a < q < b$. By Archmedean ordering principle $\exists N \in \mathbb{N}$ such that $\frac{1}{N} < b - a$.

   Claim: $\exists k \in mathbbZ$ such that $a < \frac{k}{N} < b$. Suppose this is not true and prove by contridiction. Then $\exists k \in \mathbb{Z}$ such that $\frac{k}{n} \leq a$ and $\frac{k+1}{n} \geq b$. Then $b - a \leq \frac{k+1}{N} - \frac{k}{N} = \frac{1}{n}$ which contradicts Archmedean ordering principle.

2. The "dyadic numbers," that is, the set $D = \{\frac{a}{2^n} \in \mathbb{Q} \,|\, a, n \in \mathbb{Z}\}$, are dense in $\mathbb{R}$ in the usual mertic.
   Consider $a < b \in \mathbb{R}$. By Archimedian property, $\exists n \in \mathbb{N}$ such that $0 < \frac{1}{n} < b - a$ which implies $0 < \frac{1}{2^n} < \frac{1}{n} < b - a$.

   Therefore, $1 < (2^n * b) - (2^n * a)$, and note that, $(2^n * b) > 1$ and $(2^n * a) > 1$ so there exists an integer m such that $2^n * a < m < 2^n * b \Rightarrow a < \frac{m}{2^n} < b$ where $2^n \neq 0$, Hence between any two rational numbers there exists $d \in D$ and between any two dyadic numbers, there exists a real number. So similar to $\mathbb{Q} \subset \mathbb{R}$, $\overline{D} = \mathbb{R} \setminus (D^c)^o \mathbb{R} \setminus \emptyset = \mathbb{R}$

Exercise 3.6.26

1. Show that in any metric space X, X is dense in X.
   $\overline{X} = X \setminus (X^c)^o = X \setminus (X \setminus X)^o = X \setminus (\emptyset)^o = X$.

2. Show that in any discrete metric space X, the only dense subset of X is X itself.
   Any proper subset $S \subset X$ contains a single point $\{x\}$. Hence $\overline{S} = S$. Therefore, there does not exists $S \subset X$ such that $\overline{S} = X$ unless $S = X$, as seen in item one of this exercise.

3. Show that if the only dense subset of a metric X is X itself, then X is discrete.
   Suppose X is a metric space and the only dense subset of X is X. This means that no subset $X \setminus \{x\}, \forall x \in X$ is a dense subset of X. Since $\overline{X \setminus \{x\}} \neq X$, x is an isolated point in X. Since x was arbitrary, $\forall x \in X$ are isolated and X is a discrete metric space.

Exercise 3.6.30

Suppose X and X' are metric spaces with X separable. Let $f : X \to X'$ be continuous surjection. Show that X' is separable.

Proof. Pick any nonempty open set $U \subset X'$, want to show that $U \cap f(S) \neq \emptyset$, i.e. $f(S)$ is dense in X'.

Since f is continuous, we have $f^{-1}(U)$ is open and not empty. Next, since X is separable we can find a countable set $S \subset X$ that is dense in X. Therefore $f^{-1}(U) \cap S \neq \emptyset$. Pick any $x \in f^{-1}(U) \cap S$, we get $f(x) \in f(S) \cap U$. Therefore $f(S)$ is dense in X'. Additionally, since S is countable, and f is surjective then for any $y \in X'$ there exists an $x \in X$ such that $f(x) = y$, so $f(S)$ is countable, since S is countable. To conclude we have a subset of X' that is countable and dense in X' which means X' is separable. □

Exercise 3.6.31

Find a metric d on $\mathbb{R}$ such that $(\mathbb{R}, d)$ is not separable.

The discrete metric. Suppose $X = (\mathbb{R}, d)$ where d is the discrete metric. Choose $A \subset \mathbb{R}$ such that $\overline{A} = \mathbb{R}$. However, since X is discrete, $\overline{A} = \mathbb{R}$ implies $A = \mathbb{R}$, but A is uncountable. As seen is exercise 3.6.26, in discrete space, the only dense subset of X is X. Therefore, there exists no countable subset of X that is dense in X. So X is not separable.

Exercise 3.7.6

1. Find a continuous function $f : \mathbb{R} \to \mathbb{R}$ that does not have a fixed point.
   $f(x) = x + 1$.

2. Find a continuous function $f : (0,1) \to (0,1)$ that does not have a fixed point.
   $f(x) = x^2$

3. Let $f : [0,1] \to [0,1]$ be continuous. Show that f has a fixed point.
   Since f is continuous, then it could have a fixed point $f(0) = 0$ or $f(1) = 1$. If it does not then $f(0) > 0$ and $f(1) - 1 < 0$. Consider the function $g(x) = f(x) - x$. Since $f(x)$ is continuous, $g(x)$ is continuous. Note that $g(x)$ is positive at $x = 0$ and negative at $x = 1$. By the intermediate value theorem, there is some point x_0 such that $g(x_0) = 0$. Which is to say $f(x_0) - x_0 = 0$ hence x_0 is a fixed point.

Exercise Supplement 1

Consider a family of functions $f_n(x) : [0,1] \to \mathbb{R}$ that is uniformly bounded and uniformly Lipschitz, where uniformly Lipschitz means that there exists a constant $L > 0$ such that for all n and all pair x, y, we have

$$|f_n(x) - f_n(y)| \leq L|x - y|.$$

Prove that this family is sequentially compact.

Proof. We are already given that f_n is uniformly bounded. We only need to show that f_n is equicontinuous and then apply Arzela-Ascoli. Let $\epsilon > 0$ and choose $\delta = \frac{\epsilon}{L}$. Then, $|x - y| < \frac{\epsilon}{L}$ but then since the family of functions is uniformly Lipschitz,

$$|f_n(x) - f_n(y)| \leq L|x - y| < L\frac{\epsilon}{L} = \epsilon$$

For any n. Therefore, f is a equicontinuous and uniformly bounded meaning that it is sequentially compact. $\square$

Exercise Supplement 2

Consider a family of functions $f_n(x) : [0,1] \to \mathbb{R}$ that is uniformly bounded and uniformly α-Hölder, where uniformly α-Hölder means that there exists a constant $C > 0$ and $0 < \alpha < 1$ such that for all n and pair x, y we have

$$|f_n(x) - f_n(y)| \leq C|x - y|^\alpha.$$

Prove that this family is sequentially compact.

Proof. We are given that $f_n(x) : [0,1] \to \mathbb{R}$ is uniformly bounded, it remains to be shown that this family of functions is equicontinuous. Since f_n is uniformly α-Hölder there $\exists C > 0$ and $0 < \alpha < 1$ such that

$$|f_n(x) - f_n(y)| \leq C|x - y|^\alpha$$

For $\forall n$ and $\forall x, y \in [0,1]$. Since $0 < \alpha < 1$, $|x - y|^\alpha \leq |x - y|$. Choose $\delta = \frac{\epsilon}{C}$ which implies

$$|f_n(x) - f_n(y)| \leq C|x - y|^\alpha \leq C|x - y| < C\frac{\epsilon}{C} = \epsilon.$$

Hence, f_n is equicontinuous. Applying Arzela-Ascoli, this family of functions is sequentially compact. $\square$