1.9.18) Give examples to show that if $r = 1$ in the ratio test, then anything can happen

Consider $\sum_{n=0}^{\infty} \frac{n+2}{2n+7}$

Applying the ratio test $\lim_{n\to\infty} \left| \frac{n+3}{2n+9} \cdot \frac{2n+7}{n+2} \right| = 1$

But $\lim_{n\to\infty} \frac{n+2}{2n+7} = \frac{1}{2} \neq 0$ $\therefore$ the series diverges

Now consider $\sum_{n=0}^{\infty} \frac{(-1)^n}{n^2+1}$

Applying the ratio test $\lim_{n\to\infty} \left| \frac{(-1)^{n+1}}{(n+1)^2+1} \cdot \frac{(n^2+1)}{(-1)^n} \right|$

$= \lim_{n\to\infty} \left| \frac{n^2+1}{(n+1)^2+1} \right| = 1$

Since this is an alternating series test, we check two conditions for convergence

1) $\lim_{n\to\infty} a_n = 0$

2) $|a_{n+1}| < |a_n|$

For 1) $\lim_{n\to\infty} \frac{(-1)^n}{n^2+1} = 0$

2) $\frac{1}{(n+1)^2+1} < \frac{1}{n^2+1}$

$\therefore$ the series converges

$\therefore$ when $r = 1$ in the ratio test we can have either convergence or divergence.

1.9.20 Give examples to show that if $r=1$ in the root test then anything can happen

Consider $\sum_{n=1}^{\infty} 1^n$

$\limsup_{n\to\infty} (1^n)^{1/n} = 1$

But $\lim_{n\to\infty} 1^n = 1 \neq 0$ $\therefore$ the series diverges

Consider $\sum_{n=1}^{\infty} 1/n^2$

$\limsup_{n\to\infty} \left(1/n^2\right)^{1/n} = \limsup_{n\to\infty} 1/n^{2/n}$

To find this limit consider $\lim_{n\to\infty} n^{2/n}$.

$\ln(n^{2/n}) = \frac{2}{n} \ln n$

$\lim_{n\to\infty} \frac{2 \ln n}{n} = \lim_{n\to\infty} \frac{2 \cdot 1/n}{1} = 0$

$\lim_{n\to\infty} e^{\ln(n^{2/n})} = e^0$

$\lim_{n\to\infty} n^{2/n} = 1$ $\therefore$ $\limsup_{n\to\infty} 1/n^{2/n} = 1$

But using the comparison test, we see that it is a p-series where $p=2$ so it converges.

$\therefore$ if $r=1$ in the root test we can have either convergence or divergence

.9.26) Determine the radius of convergence for the following series.

i.) $\sum_{n=1}^{\infty} \frac{z^n}{n!}$

Using ratio test $\lim_{n\to\infty} \left| \frac{z^{n+1}}{(n+1)!} \cdot \frac{n!}{z^n} \right| = \lim_{n\to\infty} \left(\frac{z}{n+1} \right) = 0$

$\therefore \ r = \infty$

ii) $\sum_{n=2}^{\infty} \frac{z^n}{\ln n}$

Using ratio test $\lim_{n\to\infty} \left| \frac{z^{n+1}}{\ln(n+1)} \cdot \frac{\ln(n)}{z^n} \right|$

$= \lim_{n\to\infty} \left| z \cdot \frac{\ln n}{\ln(n+1)} \right|$

$= \lim_{n\to\infty} \left| z \cdot \frac{1/n}{1/n+1} \right| = \lim_{n\to\infty} \left| z \cdot \frac{n+1}{n} \right| = |z$

$\therefore \ |r| < 1$ by the ratio test

iii) $\sum_{n=1}^{\infty} \frac{n^n}{n!} z^n$

Using ratio test, $\lim_{n\to\infty} \left| \frac{(n+1)^{n+1} \ z^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n \cdot z^n} \right|$

$= \lim_{n\to\infty} \left| z \cdot \left(\frac{n+1}{n} \right)^n \right|$

$= \lim_{n\to\infty} \left| z \cdot \left(1 + \frac{1}{n} \right)^n \right| = |ez| < 1$

$\therefore |r| < \frac{1}{|e|}$

2.5.8) Prove that the scalar product is a positive definite symmetric bilinear form.

First, we show that the scalar product is a bilinear form.

$$\langle v_1 + v_2, w \rangle = (v_{11}+v_{21})w_1 + (v_{12}+v_{22})w_2 + \dots + (v_{1n}+v_{2n})w_n$$
$$= v_{11}w_1 + v_{21}w_1 + \dots + v_{1n}w_n + v_{2n}w_n$$
$$= (v_{11}w_1 + v_{12}w_2 + \dots + v_{1n}w_n) + (v_{21}w_1 + \dots + v_{2n}w_n)$$
$$= \langle v_1, w\rangle + \langle v_2, w\rangle$$

$$\langle \alpha v, w\rangle = \alpha v_1 w_1 + \dots + \alpha v_n w_n$$
$$= \alpha(v_1w_1+\dots+v_nw_n)$$
$$= \alpha\langle v,w\rangle$$

$$\langle v, w_1+w_2\rangle = v_1(w_{11}+w_{21}) + \dots + v_n(w_{1n}+w_{2n})$$
$$= v_1w_{11} + v_1 w_{21} + \dots + v_n w_{1n} + v_n w_{2n}$$
$$= (v_1 w_{11} + \dots + v_n w_{1n}) + (v_1 w_{21}+\dots+v_n w_{2n})$$
$$= \langle v, w_1\rangle + \langle v, w_2\rangle$$

$$\langle v, \alpha w\rangle = v_1(\alpha w_1) + \dots + v_n(\alpha w_n)$$
$$= \alpha(v_1w_1+\dots+v_nw_n)$$
$$= \alpha\langle v,w\rangle$$

Next, we show that the scalar product is symmetric

$$\langle v,w\rangle = v_1w_1+\dots+v_nw_n$$
$$= w_1v_1+\dots+w_nv_n$$
$$= \langle w,v\rangle$$

Now, we show that the scalar product is positive definite.

$$\langle v,v\rangle = v_1v_1+\dots+v_nv_n = v_1^2+\dots+v_n^2$$
$$\therefore \langle v,v\rangle \geq 0 \quad \forall v\in V$$

Suppose $\langle v,v\rangle = 0$ and $v\neq 0$.

We have $v_1^2+\dots+v_{n-1}^2 = -v_n^2$ but the left side is positive while the right is negative $\therefore \langle v,v\rangle = 0 \iff v = 0$

2.5.11) Prove the following properties of norms. let $v, w \in \mathbb{C}^n$.

i) $\|v\| \geq 0$

$\|v\| = \sqrt{\langle v, v\rangle} = \sqrt{v_1^2 + v_2^2 + \ldots + v_n^2}$

Since $\langle v, v\rangle$ is positive definite $\langle v, v\rangle \geq 0$ then

$\sqrt{\langle v, v\rangle} \geq 0 \quad \therefore \quad \|v\| \geq 0$

ii) $\|v\| = 0$ iff $\vec{v} = 0$

$\|v\| = \sqrt{\langle v, v\rangle} = 0 \quad \therefore \langle v, v\rangle = 0$ and $\vec{v} = 0$ by positive definite property. If $\vec{v} = 0$, $\|v\| = \sqrt{v \cdot v} = \sqrt{0} = 0$.

iii) $\|\alpha v\| = |\alpha| \|v\|$.

$\|\alpha v\| = \sqrt{\langle \alpha v, \alpha v\rangle} = \sqrt{\alpha^2 v_1^2 + \ldots + \alpha^2 v_n^2}$

$= \alpha \sqrt{v_1^2 + \ldots + v_n^2}$

$= \alpha \|v\|$

iv) $\|v + w\| \leq \|v\| + \|w\|$

$\|v + w\|^2 = \langle v + w, v + w\rangle$

$= \langle v, v + w\rangle + \langle w, v + w\rangle$

$= \langle v, v\rangle + \langle v, w\rangle + \langle w, v\rangle + \langle w, w\rangle$ — ①

$(\|v\| + \|w\|)^2 = \|v\|^2 + 2\|v\|\|w\| + \|w\|^2$

$= \langle v, v\rangle + 2\|v\|\|w\| + \langle w, w\rangle$ — ②

Note from ① that $\langle v, w\rangle + \langle w, v\rangle \geq 0$ because $\|v + w\|^2 \geq 0$, $\langle v, v\rangle \geq 0$, and $\langle w, w\rangle$

Then from Cauchy-Schwarz Inequality we have

$|\langle v, w\rangle| \leq \|v\| \|w\|$

$\therefore \langle v, v\rangle + 2\langle v, w\rangle + \langle w, w\rangle \leq \langle v, v\rangle + 2\|v\|\|w\| + \langle w$

$\therefore \|v + w\|^2 \leq (\|v\| + \|w\|)^2$

$\therefore \|v + w\| \leq \|v\| + \|w\|$

2.5.11) v) $\|v+w\|^2 + \|v-w\|^2 = 2\left(\|v\|^2 + \|w\|^2\right)$

$$\|v+w\|^2 + \|v-w\|^2 = \langle v+w, v+w\rangle + \langle v-w, v-w\rangle$$
$$= \langle v, v+w\rangle + \langle w, v+w\rangle + \langle v, v-w\rangle + \langle -w, v-w\rangle$$
$$= \langle v, v\rangle + \langle v, w\rangle + \langle w, v\rangle + \langle w, w\rangle + \langle v, v\rangle$$
$$+ \langle v, -w\rangle + \langle -w, v\rangle + \langle -w, -w\rangle$$
$$= 2\langle v, v\rangle + 2\langle w, w\rangle + \langle v, 0\rangle + \langle 0, v\rangle$$
$$= 2\langle v, v\rangle + 2\langle w, w\rangle$$
$$= 2\left(\|v\|^2 + \|w\|^2\right)$$

2.5.20)

i) Show that $\|v \times w\| = \|v\|\,\|w\| \sin\theta$ where θ is the angle between v and w.

lemma $\|v \times w\|^2 + \langle v, w\rangle^2 = \|v\|^2 \|w\|^2$

proof. $\langle v\times w, v\times w\rangle + \langle v, w\rangle$

$$= (v_2 w_3 - v_3 w_2)^2 + (v_3 w_1 - v_1 w_3)^2 + (v_1 w_2 - v_2 w_1)^2$$
$$+ (v_1 w_1 + v_2 w_2 + v_3 w_3)^2$$
$$= (v_1^2 + v_2^2 + v_3^2)(w_1^2 + w_2^2 + w_3^2)$$
$$= \|v\|^2 \|w\|^2 \qquad \square$$

$$\|v \times w\|^2 = \|v\|^2 \|w\|^2 - \langle v, w\rangle^2$$
$$= \|v\|^2 \|w\|^2 - \|v\|^2 \|w\|^2 \cos^2\theta$$
$$= \|v\|^2 \|w\|^2 (1 - \cos^2\theta)$$
$$= \|v\|^2 \|w\|^2 \sin^2\theta$$

ii) Show that $\|v \times w\|$ is the area of a parallelogram

Area of triangle $= \frac{1}{2}\|v\|\,\|w\| \sin\theta = \frac{1}{2}\|v + w\|$

A parallelogram is made up of 2 triangles so the area is twice as large.

2.5.26 We want to show that

$$\| \text{proj}_v (p - p_0) \| = \inf \{ \|p - q\| \mid q \in H \}$$

Let $q \in H$. Then $p - p_0 = p - q + q - p_0$

Then $\langle p - p_0, v \rangle = \langle p - q, v \rangle + \langle q - p_0, v \rangle$
$= \langle p - q, v \rangle$

(Since $q - p_0 \in \text{span}\{v_1, \dots, v_{n-1}\}$ and v is orthogonal to $v_1, \dots, v_{n-1}$)

So $\| \text{proj}_v (p - p_0) \| = \dfrac{|\langle p - p_0, v \rangle|}{\|v\|}$

$$= \frac{|\langle p - q, v \rangle}{\|v\|}$$

$$= \frac{\|p - q\| \, \|v\|}{\|v\|}$$

$$= \|p - q\|$$

$\therefore$ $\| \text{proj}_v (p - p_0) \|$ is a lower bound of $\{ \|p - q\| \mid q \in H$

Note that $(p - p_0) - \text{proj}_v (p - p_0)$ is in the $\text{span}\{v_1, \dots, v$

so $p - \text{proj}_v (p - p_0) \in H$

and $\| p - (p - \text{proj}_v (p - p_0)) \| = \| \text{proj}_v (p - p_0) \|$

$$\therefore \| \text{proj}_v (p - p_0) \| = \inf \{ \|p - q\| \mid q \in H \}$$

2.5.30

i) $\tilde{v}_1 = (1, 1, -1, 0)$

$\tilde{v}_2 = v_2 - \text{proj}_{\tilde{v}_1} v_2 = \left(\frac{2}{3}, -\frac{1}{3}, \frac{1}{3}, -1\right)$

$\tilde{v}_3 = v_3 - \text{proj}_{\tilde{v}_1}(v_3) - \text{proj}_{\tilde{v}_2}(v_3) = \left(\frac{2}{5}, \frac{4}{5}, \frac{6}{5}, \frac{2}{5}\right)$

ii)
$$\begin{pmatrix} e_1 & e_2 & e_3 & e_4 \\ 1 & 1 & -1 & 0 \\ \frac{2}{3} & -\frac{1}{3} & \frac{1}{3} & -1 \\ \frac{2}{5} & \frac{4}{5} & \frac{6}{5} & \frac{2}{5} \end{pmatrix}$$

$$= \det\begin{vmatrix} 1 & 1 & -1 \\ \frac{2}{3} & -\frac{1}{3} & \frac{1}{3} \\ \frac{2}{5} & \frac{4}{5} & \frac{6}{5} \end{vmatrix} i - \det\begin{vmatrix} 1 & -1 & 0 \\ \frac{2}{3} & \frac{1}{3} & -1 \\ \frac{2}{5} & \frac{6}{5} & \frac{2}{5} \end{vmatrix} j$$

$$+ \det\begin{vmatrix} 1 & 1 & 0 \\ \frac{2}{3} & -\frac{1}{3} & -1 \\ \frac{2}{5} & \frac{4}{5} & \frac{6}{5} \end{vmatrix} k + \det.\begin{vmatrix} 1 & 1 & -1 \\ \frac{2}{3} & -\frac{1}{3} & \frac{1}{3} \\ \frac{2}{5} & \frac{4}{5} & \frac{6}{5} \end{vmatrix}$$

$= -2i + 2j + 0k - 2l = (-2, 2, 0, -2)$

iii) $\tilde{v}_1 = \frac{1}{\sqrt{3}}(1, 1, -1, 0)$

$\tilde{v}_2 = \frac{1}{\sqrt{14/9}}\left(\frac{2}{3}, -\frac{1}{3}, \frac{1}{3}, -1\right)$

$\tilde{v}_3 = \frac{1}{\sqrt{12/5}}\left(\frac{2}{5}, \frac{4}{5}, \frac{6}{5}, \frac{2}{5}\right)$

$\tilde{v}_4 = \frac{1}{\sqrt{12}}(2, 2, 0, -2)$