

JOE SEIDEL

# REAL ANALYSIS

## Section 1.5 Construction of the Real Numbers

### Exercise 1.5.1

Show that for any  $a, b \in \mathbb{Q}$ , we have  $||a| - |b|| \leq |a - b|$ .

*Proof.* Since  $a, b \in \mathbb{Q}$ ,

$$|a + b| \leq |a| + |b|$$

Absolute values on  $\mathbb{Q}$  satisfy the Triangle Inequality

So

$$|a| = |a - b + b| \leq |a - b| + |b|$$

$$|b| = |a + b - a| \leq |b - a| + |a|$$

These can be rewritten as

$$|a| - |b| \leq |a - b|$$

$$|b| - |a| \leq |b - a|$$

Since  $|a - b| = |b - a|$  and if  $t \geq x$  and  $t \geq -x$  then  $t \geq |x|$ , therefore

$$||a| - |b|| \leq |a - b|$$

□

### Exercise 1.5.5

If a sequence  $(a_k)_{k \in \mathbb{N}}$  converges in  $\mathbb{Q}$  show that  $(a_k)_{k \in \mathbb{N}}$  is a Cauchy sequence in  $\mathbb{Q}$ .

*Proof.* By definition if  $(a_k)_{k \in \mathbb{N}}$  converges in  $\mathbb{Q}$  given any rational number  $r > 0$  there exists an integer  $N$  such that if  $n \geq N$  then  $|a_n - a| < r$ .

Suppose  $(a_k)_{k \in \mathbb{N}}$  converges to  $a, a \in \mathbb{Q}$ . Let  $r > 0$ , since  $(a_k)_{k \in \mathbb{N}}$  converges to  $a$ ,  $\exists N$  such that  $\forall n \geq N, |a_n - a| < \frac{r}{2}$ .

Then  $\forall n, m > N$

$$|a_n - a_m| = |a_n - a + a - a_m| \leq |a_n - a| + |a - a_m|$$

Let  $n, m > N$

$$|a_n - a| < \frac{r}{2}$$

and

$$|a - a_m| = |a_m - a| < \frac{r}{2}$$

therefore

$$|a_n - a_m| < \frac{r}{2} + \frac{r}{2} = r$$

□

*Exercise 1.5.6*

Show that the limit of a convergent sequence is unique.

*Proof.* Suppose  $(a_k)_{k \in \mathbb{Q}}$  converges in  $\mathbb{Q}$  to  $L$  and  $M$ . Choose  $L$  and  $M, L \neq M$  and let  $r = \frac{|L-M|}{2}$ . Then  $\exists N_1 \in \mathbb{Z}$  such that if  $n \geq N_1$  then

$$|a_n - L| < r$$

and  $\exists N_2 \in \mathbb{Z}$  such that if  $n \geq N_2$  then

$$|a_n - M| < r$$

Let  $N = \max(N_1, N_2)$ . If  $n \geq N$  then

$$|L - M| = |L - a_n + a_n - M| \leq |L - a_n| + |a_n - M| < 2\left(\frac{|L - M|}{2}\right) = |L - M|$$

Reducing the above, we have  $|L - M| < |L - M|$  a contradiction,  
 $\Rightarrow \Leftarrow$ . Therefore,  $L = M$ .

□

*Exercise 1.5.9*

Show that the sum of two Cauchy sequences in  $\mathbb{Q}$  is a Cauchy sequence in  $\mathbb{Q}$ .

*Proof.* Let  $(a_k)_{k \in \mathbb{N}}$  and  $(b_k)_{k \in \mathbb{N}}$  be Cauchy sequences  $\mathbb{Q}$ . Let  $r > 0$ ,  $\exists N_1$  such that if  $n, m \geq N_1$  then

$$|a_n - a_m| < \frac{r}{2}$$

and  $\exists N_2$  such that if  $n, m \geq N_2$  then

$$|b_n - b_m| < \frac{r}{2}$$

Let  $N = \max(N_1, N_2)$ . If  $n, m \geq N$  then

$$|a_n - a_m| + |b_n - b_m| < \frac{r}{2} + \frac{r}{2} = r$$

$$|a_n - a_m + b_n - b_m| \leq |a_n - a_m| + |b_n - b_m|$$

Therefore

$$|(a_n + b_n) - (a_m + b_m)| < r$$

□

### Exercise 1.5.13

Show that if a Cauchy sequence  $(a_k)_{k \in \mathbb{N}}$  does not converge to 0, all the terms of the sequence eventually have the same sign.

**Lem.:** 1.5.12 Suppose  $(a_k)_{k \in \mathbb{N}} \in \mathcal{C} \setminus \mathcal{I}$ , then there exists a positive rational number  $r$  and an integer  $N$  such that  $|a_n| \geq r$  for all  $n \geq N$ .

Where  $\mathcal{C}$  denotes the set of all Cauchy sequences of rational numbers and  $\mathcal{I}$  denotes the set of all Cauchy sequences that converge to 0.

*Proof.* Suppose  $(a_k)_{k \in \mathbb{N}}$  is a Cauchy sequence that does not converge to 0. Therefore given any  $r > 0$  there exists an integer  $N$  such that if  $n, m \geq N$ , then  $|a_n - a_m| < r$ . From Lemma 1.5.2, we can choose  $r > 0$  and  $N$  such that  $|a_n| \geq r$  for all  $n \geq N$ .

Let  $r > 0$  and  $n, m \geq N$ . Therefore

$$|a_n - a_m| < r \leq |a_n|$$

Suppose  $a_n > 0$  and  $a_m < 0$

$$|a_n - (-a_m)| = |a_n + a_m| < |a_n| \Rightarrow \Leftarrow$$

In other words, they don't have the same sign.

Likewise, suppose  $a_n < 0$  and  $a_m > 0$

$$|a_n - a_m| = |a_m - a_n|$$

$$|a_m - (-a_n)| = |a_m + a_n| < |a_n| \Rightarrow \Leftarrow$$

Therefore, all terms must eventually be the same sign. □

### Exercise 1.5.15

Show that  $\sim$  defines an equivalence relation on  $\mathcal{C}$ . We need to show reflexivity, symmetry, and transitivity exist on Cauchy sequences that are equivalent.

$$(a_k) \sim (a_k)$$

For all  $a_n \in (a_k)$ ,  $|a_n - a_n| = 0$ . This we can say that  $(a_k - a_k)_{k \in \mathbb{N}}$  is in  $\mathcal{I}$ .

$$(a_k) \sim (b_k)$$

Suppose that  $(a_k)$  and  $(b_k)$  are equivalent and  $r > 0$ . For all  $N \in \mathbb{N}$  There exists  $|a_n - (b_n)| < r$  and  $|(b_n) - (a_n)| < r$  for  $n \geq N$

$$(a_k) \sim (b_k), (b_k) \sim (c_k) \Rightarrow (a_k) \sim (c_k)$$

Let  $r > 0$ ,  $\exists N_1 \in \mathbb{N}$  such that  $|a_n - b_n| < \frac{r}{2}$  for all  $n \geq N_1$  and  $\exists N_2 \in \mathbb{N}$  such that  $|b_n - c_n| < \frac{r}{2}$  for all  $n \geq N_2$ . This implies

$$|a_n - c_n| = |a_n - b_n + b_n - c_n| \leq |a_n - b_n| + |b_n - c_n| < \frac{r}{2} + \frac{r}{2} = r$$

for all  $n = \max(N_1, N_2)$

## Exercise 1.5.17

Show that  $\mathbf{R}$  is a commutative ring with  $1$ , with  $\mathcal{I}$  as the additive identity and  $[a_k]$  such that  $a_k = 1$  for all  $k$  as the multiplicative identity.

We know that if  $(a_k), (b_k)$  are Cauchy sequences,  $(a_n)_{n \in \mathbb{N}} + (a_n)_{n \in \mathbb{N}} = (a_n + b_n)_{n \in \mathbb{N}}$  and  $(a_n)_{n \in \mathbb{N}}(b_n)_{n \in \mathbb{N}} = (a_nb_n)_{n \in \mathbb{N}}$  are well-defined. Let  $[a_k]$  be an equivalence class, it easily follows that  $[a_k] + [a_k] = [a_k + a_k]$  and  $[a_k][a_k] = [a_ka_k]$

As examples, consider  $(a_k)_{k \in \mathbb{N}}$  and  $(a'_k)_{k \in \mathbb{N}}$  denoted as  $\{a_k\}$  and  $\{a'_k\}$ , respectively. Likewise,  $(b_k)_{k \in \mathbb{N}}$  and  $(b'_k)_{k \in \mathbb{N}}$  denoted as  $\{b_k\}$  and  $\{b'_k\}$

For addition, let  $\{a_k\} \sim \{a'_k\}$ ,  $\{b_k\} \sim \{b'_k\}$  and  $r > 0$ . Then,  $\exists N_1 \exists N_2$  in  $\mathbb{N}$  such that

$$|a_n - a'_n| < \frac{r}{2} \text{ for } n \geq N_1$$

and

$$|b_n - b'_n| < \frac{r}{2} \text{ for } n \geq N_2$$

This implies

$$|(a_n + b_n) - (a'_n + b'_n)| = |a_n - a'_n + b_n - b'_n| \geq |a_n - a'_n| - |b_n - b'_n| < \frac{r}{2} + \frac{r}{2}$$

for  $n \geq \max(N_1, N_2)$ .

If  $[i_k]$  is in  $\mathcal{I}$  then  $\{i_k\} + \{i_k\} = 0$  for all  $k \in \mathbb{N}$ . Then it easily follows  $[a_k] + [i_k] = [a_k]$

For multiplication, recall that  $\{a_k\}$ ,  $\{a'_k\}$ ,  $\{b_k\}$  and  $\{b'_k\}$  are bounded.  $\exists M > 0$  such that  $\{a_n\}$ ,  $\{a'_n\}$ ,  $\{b_n\}$ ,  $\{b'_n\} \leq M$  for all  $n \in \mathbb{N}$ .  $\exists N \in \mathbb{N}$  such that

$$|a_n - a'_n| < \frac{r}{2M} \text{ for } n \geq N_1$$

and

$$|b_n - b'_n| < \frac{r}{2M} \text{ for } n \geq N_2$$

This implies

$$\begin{aligned} 2|a_nb_n - a'_nb'_n| &= |(a_n - a'_n)(b_n + b'_n) + (a_n + a'_n)(b_n - b'_n)| \\ &\leq |(a_n - a'_n)(b_n + b'_n)| + |(a_n + a'_n)(b_n - b'_n)| \\ &= |a_n - a'_n||b_n + b'_n| + |a_n + a'_n||b_n - b'_n| \\ &\leq |a_n - a'_n|(|b_n| + |b'_n|) + (|a_n| + |a'_n|)|b_n - b'_n| \\ &< \frac{r}{2M}(2M) + \frac{r}{2M}(2M) \\ &= 2r \end{aligned}$$

Therefore  $|a_nb_n - a'_nb'_n| < r$  for all  $n = \max(N_1, N_2)$ . If  $[i_k]$  with  $i_k = 1$  for all  $k$  is the multiplicative identity it follows that  $[a_k][i_k] = [a_k]$

**Lem.: 1.5.8.** Let  $(a_k)_{k \in \mathbb{N}}$  be a Cauchy sequence of rational numbers. Then  $(a_k)_{k \in \mathbb{N}}$  is a bounded sequence

*Exercise 1.5.20*

Show that order relation, defined below is well-defined and makes  $\mathbf{R}$  and ordered field.

**Def.:** Let  $a = [a_k]$  and  $b = [b_k]$  be distinct elements of  $\mathbf{R}$ . We define  $a < b$  if  $[a_k] < [b_k]$  eventually and  $b > a$  if  $[b_k] > [a_k]$  eventually.

Consider  $a$  and  $b$  as denoted above. Let them be distinct elements of  $\mathbf{B}$ . They are not equivalent and therefore  $a - b$  is not in  $\mathcal{I}$ . We need to satisfy

(O1) **Trichotomy:** Since  $[a] - [b]$  is not in  $\mathcal{I}$ , by definition either  $a < b$  or  $b > a$ .

(O2) **Transitivity:** For sake of argument, let  $a < b$  and choose an additional arbitrary element of  $\mathbf{R}$   $[c_k] = c$ . Let  $b < c$ . Then  $a < c$