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REAL ANALYSIS SEVENTH WEEK

Exercise 3.5.9

Suppose that (X, d) and (X', d') are metric spaces and that $f : X \to X'$ is continuous. For each of the following statements, determine whether or not is true. If the assertion is true, prove it. If it is not true, give a counter example.

1. If A is an open subset of X, then $f(A)$ is an open subset of X';
   Not neccessarily true. Consider the constant function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = c$. Let A be an open subset of $\mathbb{R}$, then $f(A)$ is a closed subset of $\mathbb{R}$.

2. If A is a closed subset of X, then $f(A)$ is a closed subset of X';
   Not neccessarily true. Consider the function $f : \mathbb{R}_+ \to \mathbb{R}$, $f(x) = \frac{x}{x+1}$. If $A = [0, \infty)$ then $f(A) = [0, 1)$ which is not closed.

3. If B is a closed subset of X', then $f^{-1}(B)$ is a closed subset of X;
   True. First note that $f^{-1}(S^c) = (f^{-1}(S))^c$. Since $B \subset X'$ is closed, $B^c \subset X'$ is open. From Theorem 3.5.5. a function $f : X \to X'$ is continuous iff for any open set $V \in X'$, the set $f^{-1}(V)$ is open in X. Thefore, if B^c is open then $f^{-1}(B^c)$ is open so $f^{-1}(B^c) = (f^{-1}(B))^c$ then $((f^{-1}(B))^c)^c = (f^{-1}(B))$ is closed.

4. If A is a bounded subset of X, then $f(A)$ is a bounded subset of X';
   False, Consider $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = \frac{1}{x}$. Take the bonded subset, $A = (0, 1)$, however $\lim_{x \to 0_+} = \infty$. Therefore $\forall M > 0\ \exists \delta$ such that $|x| < \delta$ implies $|f(x)| > M$. In particular, $\forall n \in \mathbb{N}$, $\exists x_n$ such that $f|(x_n)| > n$ hence $f(A) = (1, \infty)$ is unbounded.

5. If B is a bounded subset of X', then $f^{-1}(B)$ is a bounded subset of X.
   False, define $f : \mathbb{R}_+ \to \mathbb{R}$ $f(x) = \frac{x}{x+1}$. Suppose $f^{-1}(B) = [0, \infty)$ then $f(f^{-1}(B)) = B = (0, 1)$ which is bounded.

6. If $A \subset X$ and x_0 is an isolated point of A, then x_0 is an isolated point of A;
   False. Consider the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$ and a subset of $\mathbb{R} \supset \{-1\} \cup [0, 2] = A$. Take the isolated point -1 in A and note that $f(-1) = 1$ which is not isolated since $f(A) = [0, 4]$.

7. If $A \subset X$, $x_0 \in X$ and $f(x_0)$ is an isolated point of $f(A)$, then x_0 is an isolated point of A;
   False. Consider, again, the constant function $f(x) = c$. Chose any $x \subset A(\subset X)$. Suppose A is open. $f(x) = c$ which is an isolated point since $\exists \epsilon$ such that $B_\epsilon(f(x)) \setminus f(x) \cap f(A) = \emptyset$. Since A is open, x is not an isolated point of A.

8. If $A \subset X$ and x_0 is an accumulation point of A, then x_0 is an accumulation point of $f(A)$.
   False, consider the same example as above. Let $x \in A$ and x is an accumulations, however $f(x)$ is an isolated point of $f(A)$.

9. If $A \subset X$, $x_0 \in X$, and $f(x_0)$ is an accumulation point of $f(A)$, then x_0 is an accumulation point of A.
   False. Consider the example used in item 6. Since $f(x_0) = 1$ is accumulation but x_0 can be 1 or -1. So x_0 is not necessarily an accumulation point.

Exercise 3.5.13

Let $X = [0, 1)$ with the induced metric from $\mathbb{R}$, and let $X' = \mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$ with the induced metric for $\mathbb{C}$. The function $f : X \to X'$, $f(x) = e^{2\pi i x}$ is a continuous bijection whose inverse is not continuous.

An alternative form of the function can be written $f(x) = \cos(2\pi x) + i \sin(2\pi x)$. Over the domain $X = [0, 1)$ this function is continuous, one to one and onto. Making it a continuous bijection. However, it's inverse function is not continuous since $X' = \mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$ is closed while X is not closed. Which is a corollary to Theorem 3.5.5, a function is continuous iff for any open set $V \subset X'$, the set $f^{-}1(V)$ is an open set in X.

Exercise 3.5.15

Let $X = R$ with the discrete metric, and let $X' = \mathbb{R}$ with the usual metric. Show that function $I : X \to X'$, $I(x) = x$ is a continuous bijection but is not a homeomorphism.

Proof. To show $I(x)$ is a continuous bijection we need to find that it is continuous, one to one, and onto. The identity function is continuous since give any $\epsilon > 0$ we can find a $\delta > 0$ such that $d'(x, y) < \delta$ implies $d'(I(x), I(y)) < \epsilon$. Let $\epsilon > 0$, we can choose $d = \epsilon + 1$ since $d(x, y) \leq 1$ for any $x, y \in X$, $d(x, y) < \delta$. Hence I is continuous. Since $I^{-1}(I(x)) = I(I^{-1}(x)) = x$ the function is one to one and onto. Therefore it is a continuous bijection.

Now we examine homeomorphism. A function is a homeomorphism if f is continuous, f is bijective, and f^{-1} is continuous. Since everything "disconnected" in discrete space, suspect that the function is not continuous. To see that the inverse function of this function is not continuous, choose any $\delta > 0$ and let $\epsilon = \frac{1}{2}$. $0 < d(x, y) < \delta$ implies $d'(x, y) = 1 > \epsilon$. Hence, we have shown that there exists an ϵ such that for any $\delta > 0$, $d'(x, y) > \epsilon$ and hence I^{-1} is not continuous. □

Exercise 3.5.23(sans isometry part)

In this exercise, we consider isometries from from $\mathbb{R}$ to itself in the usual metric.

1. Is $f(x) = x^3$ a bijection? A homeomorphism? An Isometry?

   Since, $f^{-1}(f(x))f^{-1}(x^3) = (x^3)^{\frac{1}{3}} = x$ and $f(f^{-1}(x)) = f(x^{\frac{1}{3}}) = (x^{\frac{1}{3}})^3 = x$, $f(x)$ is one to one. Since $\forall y \in \mathbb{R}'$ There exists $x \in \mathbb{R}$ such that $f(x) = y$ and so the function is onto, hence it is a bijection. Since the product of continuous functions is continuous, and $x * x * x = x^3$, $f(x)$ is continuous. Also not that f^{-1}

$$\begin{aligned} |\sqrt[3]{x} - \sqrt[3]{a}| &= |\sqrt[3]{x} - \sqrt[3]{a}| \times \frac{|x^{\frac{2}{3}} + \sqrt[3]{a}\sqrt[3]{x} + a^{\frac{2}{3}}|}{|x^{\frac{2}{3}} + \sqrt[3]{a}\sqrt[3]{x} + a^{\frac{2}{3}}|} \\ &= \frac{|x-a|}{|x^{\frac{2}{3}} + \sqrt[3]{a}\sqrt[3]{x} + a^{\frac{2}{3}}|} \leq \frac{|x-a|}{|ax|^{\frac{1}{3}}} \end{aligned}$$

   Assume $|x - a| < |a|$ then $|x| < 2|a|$ which implies $|ax| < 2|a|^2$ so $\frac{1}{2|a|^2} < \frac{1}{|ax|}$.

   Therefore $|x^{\frac{1}{3}} - a^{\frac{1}{3}}| < \frac{|x-a|}{|a|^{\frac{2}{3}}}$ whenever $|x - a| < |a|$ and $a \neq 0$. Pick $\delta = min\{|a|, \epsilon|a|^{\frac{2}{3}}\}$, or in the case that $a = 0$ pick $\delta = \epsilon^3$. These values for delta imply continuity.

   Hence, this function is a homeomorphism.

   This function is not an isometry since $d(x, y) \neq d'(f(x), f(y))$, $\forall x, y \in \mathbb{R}$. This can be verified by picking $x = 2$ and $y = 3$.

2. If $f(x) = x + \sin x$ a bijection? A homeomorphism? An isometry? First check to see if $f(x)$ is one to one an onto. Consider $sin(x) + x = sin(y) + y$, this can be reduced to $x = y$ so for any $x, y \in \mathbb{R}$ $f(x) = f(y)$ implies $x = y$. So it is one to one. The function is also continuous since sin is a continuous function and x is a continuous function. Like wise, the inverse function $f^{-1}(x) = cos(x) - x$ is continuous. Hence, it is a bijection and homeomorphism.

   To see that this function is not isometric. We can consider points $y = 2$ and $x = 4$ $d(x, y) = 2$, but $d'(f(x) + f(y)) = |(sin(2) + 2) - (sin(4) + 4)| \neq 2$.

Exercise 3.5.30

Define a sequence of functions $f_n : (0, 1) \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} \frac{1}{q^n}, & \text{if } x = \frac{p}{q} \text{ (reduced to lowest terms)} \\ 0, & \text{otherwise} \end{cases}$$

for $n \in \mathbb{N}$. Find the pointwise limit f of the sequence $(f_n)_{n\in\mathbb{N}}$ and show that $(f_n)_{n\in\mathbb{N}}$ converges to f uniformly.

Proof. For any $x \in (0,1)$, either $x = \frac{p}{q}$(reduced to lowest terms), in otherwords $x \in \mathbb{Q}$, or x is irrational.

When $x \in \mathbb{Q}$, $f_n(x) = \frac{1}{q^n}$. Since $\lim_{n\to\infty} \frac{1}{a^n} = 0$ when $a > 0$, the sequence $f_n(x)$ converges to 0. Likewise, when $x \notin \mathbb{Q}$, $f_n(x)$ converges to 0. Therefore, the pointwise limit of f can be defined $f(x) = 0$.

A sequence converges uniformly if given $\epsilon > 0$ $\exists N_\epsilon \in \mathbb{N}$ such that $sup_{x\in X}|f_n(x) - f(x)| < \epsilon$ whenever $n > N_\epsilon$.

Since $f(x) = 0$ for any $x \in (0,1)$ given any ϵ we can choose N_ϵ large enough such that either $|f_n(x)| < \epsilon$ when x is rational and when x is irrational $0 < \epsilon$. Also note that although $(0,1)$ contains infitiley many points, eventually, the value for $f_n(x) = \frac{1}{q^n}$ or $f_n(x) = 0$ becomes constant. In other words, putting aside the exponent n, there are only finitely many values either, $\frac{1}{q}$ or 0 in the range of the function. □

Exercise 3.5.33

Let $X = (0,\infty)$ and determine whether the following functions are uniformly continuous on X.

1. $f(x) = \frac{1}{x}$

   Suppose f is uniformly continous on $(0,\infty)$. Then given any $\epsilon > 0$, $\exists\delta$ such that if $x > 0$ and $y > 0$ then $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$.

   Set $x_n = \frac{1}{n}$ and $y_n = \frac{1}{2n}$. Then $|x_n - y_n| = \frac{1}{2n} < \delta$ for n large enough. Then, $|f(x_n) - f(y_n)| = \frac{|x_n - y_n|}{x_n y_n} = n > \epsilon$ for large n.

   Therefore, the fucntion is not uniformly continuous.

2. $f(x) = \sqrt{x}$

   Let $\delta = \epsilon^2$. If $|x - y| < \delta = \epsilon^2$ we have $|\sqrt{x} - \sqrt{y}|^2 \leq |\sqrt{x} - \sqrt{y}||\sqrt{x} + \sqrt{y}| = |x - y| < \epsilon^2$. Hence $|\sqrt{x} - \sqrt{y}| < \epsilon$. Thus f is continuous on the interval $(0,\infty)$.

3. $f(x) = \ln(x)$

   Suppose f is uniformly continous on $(0,\infty)$. Then given any $\epsilon > 0$, $\exists\delta$ such that if $x > 0$ and $y > 0$ then $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$.

   Set $x_n = \frac{1}{n}$ and $y_n = \frac{1}{2n}$. $|x_n - y_n| = \frac{1}{2n} < \delta$ for n large enough. Then, $|f(x_n) - f(y_n)| = \ln(\frac{2}{n^2}) > \epsilon$ for large n.

   Therefore, the fucntion is not uniformly continuous.

4. $f(x) = x\ln(x)$ Suppose f is uniformly continous on $(0, \infty)$. Then given any $\epsilon > 0$, $\exists\delta$ such that if $x > 0$ and $y > 0$ then $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$.

   Set $x_n = n$ and $y_n = n + 1$. $|x_n - y_n| = 1$ for any n, but $|f(x_n) - f(y_n)| = n\ln(\frac{1}{n} + 1) + \ln(n + 1) > \epsilon$ for large n.

   So f is not uniformly continous over $(0, \infty)$.