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REAL ANALYSIS SECOND WEEK

Exercise 1.6.35

1. Show the $\emptyset$ and $\mathbb{R}$ are the only subsets of $\mathbb{R}$ that are both open and closed.

   Proof. Let S be a non-empty open and closed set of $\mathbb{R}$. Fix $x_0 \in S$ and $S \neq \mathbb{R}$. Then, $\exists y \in \mathbb{R} \setminus S$. Without loss of generality we may assume $y > x_0$.

   Therefore we can form a set

   $$I = \{x \in \mathbb{R} | x > x_0, x \notin S\}$$

   By construction this set is bounded below by x_0 and not empty because $y > x_0$, $y \in I$. Therefore we can let $i = \inf I$

   greatest lower bound

   Suppose $i \in S$. Since S is open it contains the open interveral $(i - \epsilon, i + \epsilon)$ for $\epsilon > 0$. However, this interval contradicts that $i = \inf I$ because it implies a sequence $i_n > i$, $|i - i_n| < \frac{1}{n}$, where $i_n \in I$, which means $i_n \notin S$. Which is not possible because $[i, i + e) \subset S$

   Now suppose $i \notin S$. Since S is closed, S^c which means that it contains an open interval $(i - \epsilon, i + \epsilon)$, but this contradicts the definition $i = \inf I$ because then we can find $i - \frac{\epsilon}{2}$ that is in S^c.

   Therefore $S = \mathbb{R}$. ☐

2. Show that every non-empty open set in $\mathbb{R}$ can be written as a countable union of pairwise disjoint open intervals.

   Proof. Let S be an open interval in $\mathbb{R}$. We can choose and arbitrary $x \in \mathbb{Q}$ inside S and choose r and r' both greater than 0, such that $(x - r, x + r')$ covers the S. Since $\mathbb{Q}$ is countable, we can effectively cover any open interval in $\mathbb{R}$ by a countable union of pairwise disjoint open intervals.

   ☐

3. Show that an arbitrary union of open sets in $\mathbb{R}$ is open in $\mathbb{R}$.

   Proof. Suppose $\{A_i \subset \mathbb{R} | i \in I\}$ is an arbitrary collection of open sets.

   If $x \in \cup A_i$ then $x \in A_i$ for some $i \in I$. Since A_i is open, there $(x - \epsilon, x + \epsilon) \subset A_i$ for $\epsilon > 0$. Therefore,

   $$(x - \epsilon, x + \epsilon) \subset \bigcup_{i \in I} A_i$$

   since $A_i \subset \mathbb{R}$ we arrive at our conclusion: The arbitrary union of open sets, $\bigcup_{i \in I} A_i$ is open in $\mathbb{R}$. ☐

4. Show that a finite intersection of open sets in $\mathbb{R}$ is open in $\mathbb{R}$

Proof. Suppose $\{A_i \subset \mathbb{R} | i = 1, 2, ...n\}$ is a finite collection of open sets. If $x \in \bigcap_{i=1}^{n} A_i$ then $x \in A_i$ for every $1 \leq i \leq n$ Since A_i is open, there are $\epsilon_i > 0$ such that $(x - \epsilon_i, x + \epsilon_i) \subset A_i$

Let $\epsilon = min(\epsilon_1, ..., \epsilon_n) > 0$. This shows $(x - \epsilon, x + \epsilon) \subset \bigcap_{i=1}^{n} A_i$

□

5. Show, by example, that an infinite intersection of open sets is not necessarily open. The open interval $(-\frac{1}{n}, \frac{1}{n})$ is open $\forall n \in \mathbb{N}$ However, it's intersection

$$\bigcap_{n=1}^{\infty} (-\frac{1}{n}, \frac{1}{n}) = 0 = [0..0]$$

which is closed.

6. Show that an arbitrary intersection of closed sets in $\mathbb{R}$ is a closed set in $\mathbb{R}$.

Proof. Let $\{A_i \subset \mathbb{R}, i \in I\}$ be an arbitrary collection of closed sets. Let $\bigcap_{i \in I} A_i$ be the intersection of the closed sets. If this intersection is $\emptyset$ we are done. Supposing it is not, we'll continue.

By definition $\mathbb{R} \setminus \bigcap_{i \in I} A_i = \bigcup_{i \in I} (\mathbb{R} \setminus A_i)$. DeMorgan's Law

Since $\{A_i\} \subset \mathbb{R}$ are closed, $(\mathbb{R} \setminus A_i) \subset \mathbb{R}$ is made up of open sets. So we have an arbitrary union of open sets in $\mathbb{R}$ which we have already shown to be open. This means $\mathbb{R} \setminus \bigcap_{i \in I} A_i$ is also open. Therefore it's compliment $\bigcap_{i \in I} A_i$ is closed.

In part 3 of this exercise

□

7. Show that a finite union of closed sets in $\mathbb{R}$ is a closed set in $\mathbb{R}$

Proof. Let $\{A_i \subset \mathbb{R}, i = 1, 2, ..., n\}$ be a finite collection of some $n \in \mathbb{N}$ closed sets in $\mathbb{R}$. Let $\bigcup_{i=1}^{n} A_i$ be the union of the finite closed subsets.

Summoning DeMorgan's Law, once more

$$\mathbb{R} \setminus \bigcup_{i=1}^{n} A_i = \bigcap_{i=1}^{n} (\mathbb{R} \setminus A_i)$$

Since $\{A_i\} \subset \mathbb{R}$ are closed, $(\mathbb{R} \setminus A_i) \subset \mathbb{R}$ is made up of open sets. This means we have a a finite intersection of open sets, which we have already shown to be open. Therefore $\mathbb{R} \setminus \bigcup_{i=1}^{n} A_i$ is also open. Which allows us to conclude, the complement, $\bigcup_{i=1}^{n} A_i$ must be closed.

□

8. Show, by example, that an infinite union of closed sets in $\mathbb{R}$ is not necessarily closed in $\mathbb{R}$

   One example is $\bigcup_{n=1}^{\infty} [\frac{1}{n}, n]$ The union of this is open because you will never find a point of the union that lives at the boundary of the union. In more precise terms, there is always some $\epsilon > 0$ that you can add to some $x \in A_i$ where A_i is some closed sequence in the union such that $(x - \epsilon, x + \epsilon) \in A_i$.

   Also, consider that $\mathbb{R}$ can be constructed of an infinite union of closed sets. For example, each one point set of one point in $\mathbb{R}$, however we know that $\mathbb{R}$ is open.

Exercise 1.6.36

Show that a subset of $\mathbb{R}$ is closed if and only if it contains all its accumulation points.

Proof. "$\Rightarrow$" (Given in class)

Suppose S is closed and x is an accumulation point. We prove by contradiction. Suppose $x \notin S$ then $x \in S^c$ This means, $\exists x$ such that

$$(x - \epsilon, x + \epsilon) \subset S^c$$

That is to say $(x - \epsilon, x + \epsilon) \cap S = \emptyset$. Which contradicts that x is an accumulation point. □

Proof. "$\Leftarrow$" Suppose S contains all it's limits points and let S be open, prove by contradiction. Therefore, $\exists x \in S^c$ such that $(x - \epsilon, x + \epsilon)$ contains atleast one element of S, fix this value to $\bar{x} \in S^c$. Symbolically written as

$$(\bar{x} - \epsilon, \bar{x} + \epsilon) \cap S \neq \emptyset, \forall \epsilon > 0$$

For all $n \in \mathbb{N}$ let $x_n \in (\bar{x}, \frac{1}{n}) \cap S$

Notice, x_n is a sequence in S that converges to $\bar{x} \notin S$, meaning that $\bar{x}$ is an accumulation point of S that is not contained in the open S, so S must be closed. □

Exercise 1.6.37

1. The Cantor set is closed.

   Let $\mathcal{C} = \bigcap_{n\in\mathbb{N}} C_n$ Where C_n is a finite union union of closed sets. As, shown above the arbitrary intersection of closed sets, this is closed. exercise 1.6.35 no.6

2. The Cantor set consist of all numbers in the closed $[0, 1]$ whose ternary expansion consists of only 0s and 2s and may end infinitely in 2s.

   In every stage of constructing the Cantor set, the middle third of the of the set is removed, i.e. the numbers whose ternary decimal expansion notation has a 1 in the *nth* decimal place.

   For example in the first stage, all numbers with 1 in the first decimal place are removed, in the second all numbers 1 in the second decimal place and so on.

3. The Cantor set is uncountable

   Proof. Suppose $\mathcal{C}$ is countable.

   $$\mathcal{C} = \{x^1, x^2, x^3, x^4, ..., x^n\}$$

   for each x write

   $$x^i = 0.d_1^i, 0.d_2^i, 0.d_3^i, 0.d_4^i, ..., 0.d_m^i$$

   Let $(d_1, d_2, d_3, d_4, ...d_m)$ be the sequence that differs from the diagonal sequence of $d^i = d_1^i, d_2^i, d_3^i, d_4^i, ..., d_m^i$ such that

   $$d_m = \{^{2 \text{ if } d_m^n=0}_{0 \text{ if } d_m^n=2}\}$$

   The ternary expansion d_m, does not appear in the list x^n since $d_m \neq d_m^i$. So, $x = 0.d_1d_2d_3d_4...d_m$ is in $\mathcal{C}$ but no element in $\mathcal{C}$ has two different ternary expansions using only 0s and 2s. So x doesn't appear in the list above, which is a contradiction, so $\mathcal{C}$ is uncountable.

   □

4. Every point in the Cantor set is an accumulation point of the Cantor set.

   Every point of the Cantor set can be captured in an arbitrary union of closed sets. So for any $x \in \mathcal{C}$, you can write x in ternary as a limit of other numbers that have 0s or 2s in there ternary expansion.

Proof. Let $C_n = \bigcup S_m$ where S_m is a finite number of closed sets. For any $x \in S_m$ let $\bar{x}$ be the right end point of S_m, unless $x = \bar{x}$ then let $\bar{x}$ be the left endpoint. By construction, each subinterval of S_m has length $\frac{1}{3^n}$. Which means $|x - x_m| < \frac{1}{3^n}$.

This implies that the sequence of x_m converges to x and such is an accumulation point. Since we chose x to be arbitrary, this applies to all $x \in \mathcal{C}$. □

5. The compliment of the Cantor Set in $[0,1]$ is a dense subset of $[0,1]$

   In layman's terms, we delete the all the middle intervals at every stage of constructing the Cantor set. Which would mean the compliment is dense.

   Proof. Let, $\mathcal{C} = \bigcap\limits_{n \in \mathbb{N}} C_n$ where $C_n = \bigcup S_n$

   The length of intervals of S_n is $\frac{1}{3^n} = 3^{-n}$ Let $0 \leq a < b \leq 1$. Then interval $(a,b) \subset [0,1]$. Fix $n \in \mathbb{N}$ such that $3^{-n} < b - a$ so that the length of every interval in S_n is $3^{-n} < b - a$

   Therefore the intervals of S_n do not overlap any interval of length $(b-a)$. and no interval of length $b - a$ is contained in C_n. Likewise, since $C_n \subset \mathcal{C}$, no interval of length $b - a$ is contained in $\mathcal{C}$.

   Since lengths of a, b was arbitrary, there are no open intervals in $\mathcal{C}$, which means $\mathcal{C}$ is nowhere dense and $\mathcal{C}^c$ is dense. □

Exercise 1.6.42

Show that a compact subset of $\mathbb{R}$ is both closed and bounded.

Proof. Suppose A is a compact subset of R. $A \subset \bigcup\limits_{k=1}^{\infty} U_k$, where U_k are open sets.

By compactness of A, $\exists n \in \mathbb{N}$ such that $A \subset \bigcup\limits_{k=1}^{n} U_k$ Thus we can say A is bounded.

Suppose $A \neq \mathbb{R}$ and consider A^c and define it as $X = \mathbb{R} \setminus A$ and take any $x \in X$. For every $a \in A$ there are open sets $U_a = (a - \epsilon, a + \epsilon)$ and $V_a = (x - \epsilon, x + \epsilon)$, for some $\epsilon > 0$ such that $U_a \cap V_a = \emptyset$.

The sets $\{U_a | a \in A\}$ form an open cover over A and since A is compact, there are finitely many $m \in \mathbb{N}$ such that $A \subset \bigcup\limits_{j=1}^{m} U_{a_j}$, denoted U_A. Denote $V_A = \bigcap\limits_{j=0}^{m} V_{a_j}$. Then U_A and V_A are open and $U_A \cap V_A = \emptyset$.

Notice that $V_A \subset X = A^c$ and all $a \in A$ can be found between V_A. Since $x \in V_A$ and we chose x to be arbitrary, A^c is open, making A closed.

□

Exercise 1.6.46

A subset of $\mathbb{R}$ is compact if and only if it is sequentially compact.

Proof. "⇒" (Given in class) Suppose S is compact. Consider any infinite subsequence $(X_n) \subset S$. We want to show that (X_n) has a convergent subsequence whose limit is in S.

We cover any point of S by $(x-1, x+1)$. This gives us an open cover. Since S is compact, there is a finite cover. Therefore the sequence (X_n) is covered by finitely many intervals of length 2, hence there is an interval with infinitely many points in (x_n). Next, apply the theorem that every bounded sequence has a convergent subsequence.

We still need to show that the subsequence converges to a point x^* in S. Since x^* is an accumulation point and S is bounded and closed, S is closed and therefore $x \in S$

□

Proof. "⇐" Suppose S is sequentially compact. To show that S is also compact, we need to show that S is closed and bounded.

To see that S is closed, let's assume that it's not. Therefore, there exists an $x \notin S$ that is an accumulation point of S. But then there is an infinite sequence in S that converges to a point that isn't in S, which contradicts the definition of sequentially compact.

IF S is not bounded, there is an increasing sequence in S such that $a_n > n$. that is not bounded in S. Since this subsequence is not bounded, it does not converge to a point in S which contradicts that S is sequentially compact and therefore S is bounded.

Therefore S is compact. □

Exercise 1.9.6

1. If $N \in \mathbb{N}$ and $z \neq 1$, show that $S_N = \sum_{n=0}^{N} z^n = \frac{1-z^{N+1}}{1-z}$

   Proof. By induction. Base case. $N = 0$.

$$z^0 = \frac{1-z^{0+1}}{1-z}$$

$$1 = 1$$

Induction step.

$$\begin{aligned}\sum_{n=0}^{N} z^n &= z^0 + z^1 + ... + z^{N-1} + z^N \\ &= \frac{1 - z^{(N-1)+1}}{1-z} + z^N \\ &= \frac{1 - z^N + (1-z)z^N}{1-z} \\ &= \frac{1 - z^N + z^N - z^{N+1}}{1-z} \\ &= \frac{1 - z^{N+1}}{1-z}\end{aligned}$$

□

2. If $|z| < 1$, show that $\lim_{n\to\infty} z^n = 0$

   Let the sequence $(z_n)_{n\in\mathbb{N}}$ be defined as the sequence of numbers where $z_n = z^n$. If $|z| < 0$ its easy to see that for any $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for any $n > N$, $|z_n| < \epsilon$, which is to say the $\lim_{n\to\infty} z^n = 0$

3. If $|z| > 1$, show that $\lim_{n\to\infty} z^n$ does not exist.

   Defining $(z_n)_{n\in\mathbb{N}}$ as above, with the caveat that $|z| > 1$, this sequence is ever increasing, such that as n gets larger the distance between n and $n+1$ gets larger. Therefore, the limit does not exist.

Exercise 1.9.16

1. If $p \in \mathbb{R}$ and $p < 1$, show that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges.

   If $p < 1$, we have a series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ where our denominator n^p will be ever increasing. This is very similar to the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ which we know is divergent. If $h_n = \frac{1}{n}$ and $a_n = \frac{1}{n^p}$, when $p < 1$, it's easy to see that $0 \leq \frac{1}{n} \leq \frac{1}{n^p}$, so the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges.

A key idea hear is the contrapositive of the Theorem $\sum_{n=1}^{\infty} a_n$ converges then $\lim_{n\to\infty} a_n = 0$. The contrapositive can be worded to say, if the terms of the series a_n do not go to zero, then the series must diverge. Likewise the comparison test is useful, consider a divergent series of (b_n) If $0 \leq (b_n) \leq (a_n)$ the series of (a_n) is divergent.

2. If $p \in \mathbb{R}$ and $p > 1$, show that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges. We are given the hint...

$$\sum_{n=2}^{N} \frac{1}{n^p} \leq \int_1^N \frac{1}{x^p} dx = \frac{1}{p-1}(1 - \frac{1}{N^{p-1}})$$

   Proof. Let $R_N = \int_1^N \frac{1}{x^p} dx = \frac{1}{p-1}(1 - \frac{1}{N^{p-1}})$

   We say that the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if the sequence of partial sums $(S_N)_{N\in\mathbb{N}}$ The series converges if and only if a sequence is a

Cauchy sequence. Which is to say, given $\epsilon > 0$, there is a $N_\epsilon \in \mathbb{N}$ such that for $N, M > N_\epsilon$, (assuming $N > M$), then $|\sum_{k=m+1}^{n} a_n| < \epsilon$

$$|S_N - S_M| < e$$
$$|S_N - S_M| \leq |(1 + R_N) - (1 + R_M)| < \epsilon$$
$$|\frac{1}{p-1}(1 - \frac{1}{N^{p-1}}) - \frac{1}{p-1}(1 - \frac{1}{M^{p-1}})| < \epsilon$$
$$|\frac{1}{p-1}(1 - \frac{1}{N^{p-1}} - 1 - \frac{1}{M^{p-1}})| < \epsilon$$
$$|\frac{1}{p-1}(-\frac{1}{N^{p-1}} - \frac{1}{M^{p-1}})| < \epsilon$$

$\square$