JOE SEIDEL

# REAL ANALYSIS SEVENTH WEEK

*Exercise 3.5.9*

Suppose that $(X, d)$ and $(X', d')$ are metric spaces and that $f : X \to X'$ is continuous. For each of the following statements, determine whether or not is true. If the assertion is true, prove it. If it is not true, give a counter example.

1. If $A$ is an open subset of $X$, then $f(A)$ is an open subset of $X'$;
   Not neccessarily true. Consider the constant function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = c$. Let $A$ be an open subset of $\mathbb{R}$, then $f(A)$ is a closed subset of $\mathbb{R}$.

2. If $A$ is a closed subset of $X$, then $f(A)$ is a closed subset of $X'$;
   Not neccessarily true. Consider the function $f : \mathbb{R}_+ \to \mathbb{R}$, $f(x) = \frac{x}{x+1}$. If $A = [0, \infty)$ then $f(A) = [0, 1)$ which is not closed.

3. If $B$ is a closed subset of $X'$, then $f^{-1}(B)$ is a closed subset of $X$;
   True. First note that $f^{-1}(S^c) = (f^{-1}(S))^c$. Since $B \subset X'$ is closed, $B^c \subset X'$ is open. From Theorem 3.5.5. a function $f : X \to X'$ is continuous iff for any open set $V \in X'$, the set $f^{-1}(V)$ is open in X. Thefore, if $B^c$ is open then $f^{-1}(B^c)$ is open so $f^{-1}(B^c) = (f^{-1}(B))^c$ then $((f^{-1}(B))^c)^c = (f^{-1}(B))$ is closed.

4. If $A$ is a bounded subset of $X$, then $f(A)$ is a bounded subset of $X'$;
   False, Consider $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = \frac{1}{x}$. Take the bonded subset, $A = (0, 1)$, however $\lim_{x \to 0_+} = \infty$. Therefore $\forall M > 0$ $\exists \delta$ such that $|x| < \delta$ implies $|f(x)| > M$. In particular, $\forall n \in \mathbb{N}$, $\exists x_n$ such that $f|(x_n)| > n$ hence $f(A) = (1, \infty)$ is unbounded.

5. If $B$ is a bounded subset of $X'$, then $f^{-1}(B)$ is a bounded subset of $X$.
   False, define $f : \mathbb{R}_+ \to \mathbb{R}$ $f(x) = \frac{x}{x+1}$. Suppose $f^{-1}(B) = [0, \infty)$ then $f(f^{-1}(B)) = B = (0, 1)$ which is bounded.

6. If $A \subset X$ and $x_0$ is an isolated point of $A$, then $x_0$ is an isolated point of $A$;
   False. Consider the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$ and a subset of $\mathbb{R} \supset \{-1\} \cup [0, 2] = A$. Take the isolated point $-1$ in $A$ and note that $f(-1) = 1$ which is not isolated since $f(A) = [0, 4]$.

7. If $A \subset X$, $x_0 \in X$and $f(x_0)$ is an isolated point of $f(A)$, then $x_0$ is an isolated point of $A$;
   False. Consider, again, the constant function $f(x) = c$. Chose any $x \subset A(\subset X)$. Suppose $A$ is open. $f(x) = c$ which is an isolated point since $\exists \epsilon$ such that $B_\epsilon(f(x)) \setminus f(x) \cap f(A) = \emptyset$. Since $A$ is open, $x$ is not an isolated point of $A$.

8. If $A \subset X$ and $x_0$ is an accumulation point of $A$, then $x_0$ is an accumulation point of $f(A)$.
   False, consider the same example as above. Let $x \in A$ and $x$ is an accumulations, however $f(x)$ is an isolated point of $f(A)$.

9. If $A \subset X$, $x_0 \in X$, and $f(x_0)$ is an accumulation point of $f(A)$, then $x_0$ is in accumulation point of $A$.
   False. Consider the example used in item 6. Since $f(x_0) = 1$ is accumulation but $x_0$ can be 1 or $-1$. So $x_0$ is not necessarily an accumulation point.

*Exercise 3.5.13*

Let $X = [0, 1)$ with the induced metric from $\mathbb{R}$, and let $X' = \mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$ with the induced metric for $\mathbb{C}$. The function $f : X \to X'$, $f(x) = e^{2\pi i x}$ is a continuous bijection whose inverse is not continuous.

An alternative form of the function can be written $f(x) = \cos(2\pi x) + i \sin(2\pi x)$. Over the domain $X = [0, 1)$ this function is continuous, one to one and onto. Making it a continuous bijection. However, it's inverse function is not continuous since $X' = \mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$ is closed while $X$ is not closed. Which is a corollary to Theorem 3.5.5, a function is continuous iff for any open set $V \subset X'$, the set $f^{-}1(V)$ is an open set in $X$.

*Exercise 3.5.15*

Let $X = R$ with the discrete metric, and let $X' = \mathbb{R}$ with the usual metric. Show that function $I : X \to X'$, $I(x) = x$ is a continuous bijection but is not a homeomorphism.

*Proof.* To show $I(x)$ is a continuous bijection we need to find that it is continuous, one to one, and onto. The identity function is continuous since give any $\epsilon > 0$ we can find a $\delta > 0$ such that $d'(x, y) < \delta$ implies $d'(I(x), I(y)) < \epsilon$. Let $\epsilon > 0$, we can choose $d = \epsilon + 1$ since $d(x, y) \leq 1$ for any $x, y \in X$, $d(x, y) < \delta$. Hence $I$ is continuous. Since $I^{-1}(I(x)) = I(I^{-1}(x)) = x$ the function is one to one and onto. Therefore it is a continuous bijection.

Now we examine homeomorphism. A function is a homeomorphism if $f$ is continuous, $f$ is bijective, and $f^{-1}$ is continuous. Since everything "disconnected" in discrete space, suspect that the function is not continuous. To see that the inverse function of this function is not continuous, choose any $\delta > 0$ and let $\epsilon = \frac{1}{2}$. $0 < d(x, y) < \delta$ implies $d'(x, y) = 1 > \epsilon$. Hence, we have shown that there exists an $\epsilon$ such that for any $\delta > 0$, $d'(x, y) > \epsilon$ and hence $I^{-1}$ is not continuous. $\square$

*Exercise 3.5.23(sans isometry part)*

In this exercise, we consider isometries from from $\mathbb{R}$ to itself in the usual metric.

1. Is $f(x) = x^3$ a bijection? A homeomorphism? An Isometry?

   Since, $f^{-1}(f(x))f^{-1}(x^3) = (x^3)^{\frac{1}{3}} = x$ and $f(f^{-1}(x)) = f(x^{\frac{1}{3}}) = (x^{\frac{1}{3}})^3 = x$, $f(x)$ is one to one. Since $\forall x \in \mathbb{R}$ $f(x) \in \mathbb{R}$ so the function is onto, hence it is a bijection. Since the product of continuous functions is continuous, and $x * x * x = x^3$, $f(x)$ is continuous. Also not that $f^{-1}$ is defined on all points in $\mathbb{R}$ so the function is a homeomorphism.

   This function is not an isometry since $d(x,y) \neq d'(f(x), f(y))$, $\forall x, y \in \mathbb{R}$

2. If $f(x) = x + \sin x$ a bijection? A homeomorphism? An isometry?

*Exercise 3.5.30*

*Exercise 3.5.33*