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REAL ANALYSIS

Section 1.5 Construction of the Real Numbers

Exercise 1.5.1

Show that for any $a, b \in \mathbb{Q}$, we have $||a| - |b|| \leq |a - b|$.

Proof. Since $a, b \in \mathbb{Q}$,

$$|a + b| \leq |a| + |b|$$

Absolute values on $\mathbb{Q}$ satisfy the Triangle Inequality

So

$$|a| = |a - b + b| \leq |a - b| + |b|$$

$$|b| = |a + b - a| \leq |b - a| + |a|$$

These can be rewritten as

$$|a| - |b| \leq |a - b|$$

$$|b| - |a| \leq |b - a|$$

Since $|a - b| = |b - a|$ and if $t \geq x$ and $t \geq -x$ then $t \geq |x|$, therefore

$$||a| - |b|| \leq |a - b|$$

□

Exercise 1.5.5

If a sequence $(a_k)_{k \in \mathbb{N}}$ converges in $\mathbb{Q}$ show that $(a_k)_{k \in \mathbb{N}}$ is a Cauchy sequence in $\mathbb{Q}$.

Proof. By definition if $(a_k)_{k \in \mathbb{N}}$ converges in $\mathbb{Q}$ given any rational number $r > 0$ there exists an integer N such that if $n \geq N$ then $|a_n - a| < r$.

Suppose $(a_k)_{k \in \mathbb{N}}$ converges to $a, a \in \mathbb{Q}$. Let $r > 0$, since $(a_k)_{k \in \mathbb{N}}$ converges to a, $\exists N$ such that $\forall n \geq N$, $|a_n - a| < \frac{r}{2}$.

Then $\forall n, m > N$

$$|a_n - a_m| = |a_n - a + a - a_m| \leq |a_n - a| + |a - a_m|$$

Since $n > N$ and $m > N$

$$|a_n - a| < \frac{r}{2}$$

and

$$|a - a_m| = |a_m - a| < \frac{r}{2}$$

therefore

$$|a_n - a_m| < \frac{r}{2} + \frac{r}{2} = r$$

□

Exercise 1.5.6

Show that the limit of a convergent sequence is unique.

Proof. Suppose $(a_k)_{k \in mathbbQ}$ converges in $\mathbb{Q}$ to L and M. Choose L and $M, L \neq M$ and let $r = \frac{|L-M|}{2}$. Then $\exists N_1 \in \mathbb{Z}$ such that if $n > N_1$ then

$$|a_n - L| < r$$

and $\exists N_2 \in \mathbb{Z}$ such that if $n > N_2$ then

$$|a_n - M| < r$$

Let $N = max(N_1, N_2)$. If $n > N$ then

$$|L - M| = |l - a_n + a_n - M| \leq |L - a_n| + |a_n - M| < 2(\frac{L-M}{2}) = L - M$$

Reducing the above, we have $|L - M| < L - M$ ☠, a contradiction. Therefore, $L = M$.

□

Exercise 1.5.9

Show that the sum of two Cauchy sequences in $\mathbb{Q}$ is a Cauchy sequence in $\mathbb{Q}$.

Proof. Let $(a_k)_{k \in \mathbb{N}}$ and $(b_k)_{k \in \mathbb{N}}$ be Cauchy sequences $\mathbb{Q}$ and $r > 0$. Let $r > 0$, $\exists N_1$ such that if $n, m \geq N_1$ then

$$|a_n - a_m| < \frac{r}{2}$$

and $\exists N_2$ such that if $n, m \geq N_2$ then

$$|b_n - b_m| < \frac{r}{2}$$

Let $N = max(N_1, N_2)$ such that if $n, m > N$ then

$$|a_n - a_m| + |b_n - b_m| < \frac{r}{2} + \frac{r}{2} = r$$

$$|a_n - a_m + b_n - b_m| \leq |a_n - a_m| + |b_n - b_m|$$

Therefore

$$|(a_n + b_n) - (a_m + b_m)| < r$$

□

Exercise 1.5.13

Show that if a Cauchy sequence $(a_k)_{k\in\mathbb{N}}$ does not converge to 0, all the terms of the sequence eventually have the same sign.

Lem.: *1.5.12 Suppose $(a_k)_{k\in\mathbb{N}} \in \mathcal{C} \setminus \mathcal{I}$, then there exists a positive rational number r and an integer N such that $|a_n| \geq r$ for all $n \geq N$.*

Where $\mathcal{C}$ denotes the set of all Cauchy sequences of rational numbers and $\mathcal{I}$ denotes the set of all Cauchy sequences that converge to 0.

Proof. Suppose $(a_k)_{k\in\mathbb{N}}$ is a Cauchy sequence that does not converge to 0. Therefore given any $r > 0$ there exists an integer N such that if $n, m \geq N$, then $|a_n - a_m| < r$. From Lemma 1.5.2, we can choose $r > 0$ and N such that $|a_n| \geq r$ for all $n \geq N$.

Let $r > 0$ and $n, m \geq N$. Therefore

$$|a_n - a_m| < r \leq |a_n|$$

Suppose $a_n > 0$ and $a_m < 0$

In other words, they don't have the same sign.

$$|a_n - (-a_m)| = |a_n + a_m| < |a_n|$$ ☠

Likewise, suppose $a_n < 0$ and $a_m > 0$

$$|a_n - a_m| = |a_m - a_n|$$

$$|a_m - (-a_n)| = |a_m + a_n| < |a_n|$$ ☠

Therefore, all terms must eventually be the same sign.

□