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# REAL ANALYSIS
# EIGHTH WEEK

*Question 1*

Prove that the $\ell^p$ norm on $\mathbb{R}^2$ is equivalent to the $\ell^\infty$ norm for all $p \geq 1$.

*Proof.* The $\ell^p$ norm of $x$ is $\|x\|_p = (|x_1|^p + |x_2|^p)^{\frac{1}{p}}$ $\forall x = (x_1, x_2) \in \mathbb{R}^2$. The $\ell^\infty$ norm of $x$ is $\|x\|_\infty = max\{|x_1|, |x_2|\}$. Suppose, without loss of generality, $\|x\|_\infty = |x_1|$, i.e. $|x_1| \geq |x_2|$.

First

$$\|x\|_\infty = |x_1| = (|x_1|^p)^{\frac{1}{p}} \leq (|x_1|^p + |x_2|^p)^{\frac{1}{p}} = \|x\|_p$$

which implies $\|x\|_\infty \leq \|x\|_p$. Next

$$\|x\|_p = (|x_1|^p + |x_2|^p)^{\frac{1}{p}} \leq (|x_1|^p + |x_1|^p)^{\frac{1}{p}} = 2^{\frac{1}{p}} |x_1|^{\frac{1}{p}} = 2^{\frac{1}{p}} \|x\|_\infty$$

So

$$2^{\frac{1}{p}} \|x\|_\infty \leq \|x\|_\infty \leq \|x\|_p$$

□

*Question 2*

Suppose $f : X \to X'$ is a bijection (one-to-one and onto) and continuous where $X(\subset \mathbb{R}$ is compact and $X' \subset \mathbb{R}$. Prove that $f$ is in fact a homeomorphism.

*Proof.* It remains to show $f^{-1}$ is continuous. We need to show for any open set $U \subset X$, $(f^{-1})^{-1}(U) = f(U)$ is open in $X'$. Equivently for any closed set $V \subset X$, $(f^{-1})^{-1}(V) = f(V)$ is closed in $X'$.

Since $X$ is compact, we have any closed subset $V(\subset X)$ is compact (midterm 1). Next $f$ is continuous implies that $f$ maps compact sets to compact sets. So $f(V)$ is compact. Since $X'$ is bounded in $\mathbb{R}$, $f(V)$ as a compact set in $\mathbb{R}$ is bounded and closed. This shows that for any closed set $V \subset X$, its image $f(V)$ is closed. This shows $f^{-1}$ is continuous, so $f$ is a homeomorphism. □

*Question 3*

Show the sequence

$$\{\cos^n x \,|\, x \in [-\frac{\pi}{2}, \frac{\pi}{2}]\}$$

does not converge uniformly.

*Proof.* A simple proof is by the Dini Theorem. If $\cos^n x$ for $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ converges uniformly, by Dini theorem, the limiting function should be continuous. However, the pointwise limit is

$$\cos^n x \to f(x) = \begin{cases} 1, & \text{if } x = 0 \\ 0, & \text{if } x \neq 0 \end{cases}$$

discontinous. $\square$

*Proof.* Or we can show $\exists\epsilon$ such that for all $n$, there exists $x_n$ such that $|\cos^n x_n - f(x_n)| > \epsilon$. We choose $\epsilon = \frac{1}{2}$ Since $f(x) = 0$ for $x \neq 0$ it is enough to find $x_n$ satisfying $\cos x_n > (\frac{1}{2})^{\frac{1}{n}}$. Since $0 < (\frac{1}{2})^{\frac{1}{n}} < 1$, such $x_n$ always exists.

$\square$

## *Question 4*

Find the closure, interior and boundary of the following sets.

1. The interval $(0, 1)$ as a subset of $\mathbb{C}$.
   Closure $[0, 1] \subset \mathbb{C}$. Interior $\emptyset$. Boundary $[0, 1] \subset \mathbb{C}$
2. The set of rational numbers $\mathbb{Q}$ as a subset of $\mathbb{R}$
   Closure $\mathbb{R}$. Interior $\emptyset$. Boundary $\mathbb{R}$.
3. The Cantor set as a subset of $\mathbb{R}$
   Closure Cantor set. Interor $\emptyset$. Boundary Cantor set.

## *Question 6*

Consider a metric space $(X, d)$. Suppose both two sets $S_1, S_2 \subset X$ are open and dense in $X$. Prove that $S_1 \cap S_2$ is open and dense in $X$.

*Proof.* The intersection of finite open sets is open, so $S_1 \cap S_2$ is open. To show $S_1 cap S_2$ is dense, we consider any nonempty open set $U \subset X$. Since $S_1$ is dense, we have $S_1 \cap U \neq \emptyset$. Pick $x \in S_1 \cap U$, since $S_1 \cap U$ is open, we have that $\exists\epsilon$ such that $B_\epsilon(x) \subset S_1 \cap U$. Since $S_2$ is dense, we get $B_\epsilon \cap S_2 \neq \emptyset$. This implies $B_\epsilon(x) \cap S_2 \subset S_1 \cap S_2 \cap U$. Hence $S_1 \cap S_2$ is dense. $\square$

## *Question 7*

We introduce the metric $d(x, y) = \frac{|x-y|}{1+|x-y|}$ on $\mathbb{R}$. Show that $\mathbb{R}$ is complete under this metric. You do not need to prove that $d$ is a metric.

*Proof.* It is enough to show any Cauchy sequence has a limit in $\mathbb{R}$. Supposed $\{x_n\}$ is a Cauchy sequence in the new metric, i.e. $\forall\epsilon\ \exists N$ such that when $m, n > N$ we have

$$d(x_m, x_n) = \frac{|x_m - x_n|}{1 + |x_m - x_n|} < \epsilon$$

.

Then we have $|x_m - x_n| < \epsilon + |x_m - x_n|\epsilon$ for $\epsilon < \frac{1}{2}$, we have $|x_m - x_n| < \frac{\epsilon}{1-\epsilon} < 2\epsilon$. This implies $\forall\epsilon < \frac{1}{2}\ \exists N$ such that when

$m, n > N$ we have $|x_m - x_n| < 2\epsilon$. Hence $\{x_n\}$ is a Cauchy sequence in $\mathbb{R}$ in the usual metric. $\mathbb{R}$ is complete in this metric, so $\exists x \in \mathbb{R}$ such that $\lim_{n\to\infty} x_n = x$, in the usual metric.

Furthermore, $d(x_n, x) = \frac{|x_n - x|}{1+|x_n+x|} < |x_n - x|$. Therefore $\{x_n\}$ converges to $x$ also in the new metric.

$\square$