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REAL ANALYSIS SECOND WEEK

Exercise 1.6.35

1. Show the $\emptyset$ and $\mathbb{R}$ are the only subsets of $\mathbb{R}$ that are both open and closed.

   Proof. Let S be a non-empty open and closed set of $\mathbb{R}$. Fix $x_0 \in S$ and $S \neq \mathbb{R}$. Then, $\exists y \in \mathbb{R} \setminus S$. Without loss of generality we may assume $y > x_0$.

   Therefore we can form a set

   $$I = \{x \in \mathbb{R} | x > x_0, x \notin S\}$$

   By construction this set is bounded below by x_0 and not empty because $y > x_0$, therefore $y \in I$. Therefore we can let $i = \inf I$

   greatest lower bound

   Suppose $i \in S$. Since S is open it contains the open interveral $(i - \epsilon, i + \epsilon)$ for $\epsilon > 0$. However, this interval contradicts that $i = \inf I$ because it implies a sequence $i_n > i$, $|i - i_n| < \frac{1}{n}$, where $i_n \in I$, which means $i_n \notin S$. Which is not possible because $[i, i + e) \subset S$

   Now suppose $i \notin S$. Since S is closed, S^c which means that it contains an open interval $(i - \epsilon, i + \epsilon)$, but this contradicts the definition $i = \inf I$ because then we can find $i - \frac{\epsilon}{2}$ that is in S^c.

   Therefore $S = \mathbb{R}$. □

2. Show that every non-empty open set in $\mathbb{R}$ can be written as a countable union of pairwise disjoint open intervals.

   Proof. Let $U \subset \mathbb{R}$. Let $\mathcal{O}$ be the set of open intervals that are a subsection of U. For $I, J \in \mathcal{O}$ define $I \sim J$ iff there are

   $$I_0 = I, I_1, I_2, ..., I_n = J \in \mathcal{O}$$

   Such that $I_k \cap I_{k+1} \neq \emptyset$ for $k = 0, ..., n-1$. Then $\sim$ defines an equivalence relation on $\mathcal{O}$. For $I \in \mathcal{O}$ let $[I]$ be the $\sim$ of I. Then $\{\cup[I]$ for $I \in \mathcal{O}\}$ is decomposition of U into pairwise disjoint intervals. By construction, these intervals are countable.

   □

3. Show that an arbitrary union of open sets in $\mathbb{R}$ is open in $\mathbb{R}$.

   Proof. Suppose $\{A_i \subset \mathbb{R} | i \in I\}$ is an arbitrary collection of open sets.

If $x \in \cup A_i$ then $x \in A_i$ for some $i \in I$. Since A_i is open, there $(x-\epsilon, x+\epsilon) \subset A_i$ for $\epsilon > 0$. Therefore,

$$(x-\epsilon, x+\epsilon) \subset \bigcup_{i \in I} A_i$$

since $A_i \subset \mathbb{R}$ we arrive at our conclusion: The arbitrary union of open sets, $\bigcup\limits_{i \in I} A_i$ is open in $\mathbb{R}$. □

4. Show that a finite intersection of open sets in $\mathbb{R}$ is open in $\mathbb{R}$

Proof. Suppose $\{A_i \subset \mathbb{R} | i = 1, 2, ...n\}$ is a finite collection of open sets. If $x \in \bigcap\limits_{i=1}^{n} A_i$ then $x \in A_i$ for every $1 \leq i \leq n$ Since A_i is open, there are $\epsilon_i > 0$ such that $(x - \epsilon_i, x + \epsilon_i) \subset A_i$

Let $\epsilon = min(\epsilon_1, ..., \epsilon_n) > 0$. This shows $(x-\epsilon, x+\epsilon) \subset \bigcap\limits_{i=1}^{n} A_i$

□

5. Show, by example, that an infinite intersection of open sets is not necessarily open. The open interval $(-\frac{1}{n}, \frac{1}{n})$ is open $\forall n \in \mathbb{N}$ However, it's intersection

$$\bigcap_{n=1}^{\infty} (-\frac{1}{n}, \frac{1}{n}) = 0 = [0..0]$$

which is closed. I

6. Show that an arbitrary intersection of closed sets in $\mathbb{R}$ is a closed set in $\mathbb{R}$.

Proof. Let $\{A_i \subset \mathbb{R}, i \in I\}$ be an arbitrary collection of closed sets. Let $\bigcap\limits_{i \in I} A_i$ be the intersection of the closed sets. If this intersection is $\emptyset$ we are done. Supposing it is not, we'll continue.

By definition $\mathbb{R} \setminus \bigcap\limits_{i \in I} A_i = \bigcup\limits_{i \in I} (\mathbb{R} \setminus A_i)$. DeMorgan's Law

Since $\{A_i\} \subset \mathbb{R}$ are closed, $(\mathbb{R} \setminus A_i) \subset \mathbb{R}$ is made up of open sets. So we have an arbitrary union of open sets in $\mathbb{R}$ which we have already shown to be open. This means $\mathbb{R} \setminus \bigcap\limits_{i \in I} A_i$ is also open. Therefore it's compliment $\bigcap\limits_{i \in I} A_i$ is closed.

□

7. Show that a finite union of closed sets in $\mathbb{R}$ is a closed set in $\mathbb{R}$

Proof. Let $\{A_i \subset \mathbb{R}, i = 1, 2, ..., n\}$ be a finite collection of some $n \in \mathbb{N}$ closed sets in $\mathbb{R}$. Let $\bigcup\limits_{i=1}^{n} A_i$ be the union of the finite closed subsets.

Summoning DeMorgan's Law, once more

$$\mathbb{R} \setminus \bigcup_{i=1}^{n} A_i = \bigcap_{i=1}^{n} (\mathbb{R} \setminus A_i)$$

Since $\{A_i\} \subset \mathbb{R}$ are closed, $(\mathbb{R} \setminus A_i) \subset \mathbb{R}$ is made up of open sets. This means we have a a finite intersection of open sets, which we have already shown to be open. Therefore $\mathbb{R} \setminus \bigcup\limits_{i=1}^{n} A_i$ is also open. Which allows us to conclude, the complement, $\bigcup\limits_{i=1}^{n} A_i$ must be closed.

$\square$

8. Show, by example, that an infinite union of closed sets in $\mathbb{R}$ is not necessarily closed in $\mathbb{R}$

   One example is $\bigcup\limits_{n=1}^{\infty} [\frac{1}{n}, n]$ The union of this is open because you will never find a point of the union that lives at the boundary of the union.

   Also, consider that $\mathbb{R}$ can be constructed of an infinite union of closed sets. For example, each one point set of one point in $\mathbb{R}$, however we know that $\mathbb{R}$ is not closed.

Exercise 1.6.36

Show that a subset of $\mathbb{R}$ is closed iff it contains all its accumulation points.

Proof. "$\Rightarrow$" (Given in class)

Suppose S is closed and x is an accumulation point. We prove by contradiction. Suppose $x \notin S$ then $x \in S^c$ This means, $\exists x$ such that

$$(x - \epsilon, x + \epsilon) \subset S^c$$

That is to say $(x - \epsilon, x + \epsilon) \cap S = \emptyset$. Which contradicts that x is an accumulation point. $\square$

Proof. "$\Leftarrow$" Suppose S contains all it's limits points and let S be open, prove by contradiction. Therefore, $\exists x \in S^c$ such that $(x - \epsilon, x + \epsilon)$ contains atleast one element of S. Symbolically written as

$$(x - \epsilon, x + \epsilon) \cap S \neq \emptyset, \forall \epsilon > 0$$

For all $n \in \mathbb{N}$ let $x_n \in (x, \frac{1}{n}) \cap S$

Notice, x_n is a sequence in S that converges to $x \notin S$, meaning that x is an accumulation point of S that is not contained in the open S, so S must be closed. □

Exercise 1.6.42

Show that a compact subset of $\mathbb{R}$ is both closed and compact.

Proof. Suppose A is a compact subset of R. $A \subset \bigcup\limits_{k=1}^{\infty} U_k$, where U_k are open sets.

By compactness of A, $\exists n \in \mathbb{N}$ such that $A \subset \bigcup\limits_{k=1}^{n} U_k$ Thus we can say A is bounded.

Consider A^c and define it as $X = \mathbb{R} \setminus A$ and take any $x \in X$ For every $a \in A$ there are open sets $U_a = (a - \epsilon, a + \epsilon)$ and $V_a = (x - \epsilon, x + \epsilon)$, for some $\epsilon > 0$ such that $U_a \cap V_a = \emptyset$.

The sets $\{U_a | a \in A\}$ form an open cover over A and since A is compact, there are finitely many points, $m \in \mathbb{N}$ such that $A \subset \bigcup\limits_{j=1}^{m} U_{a_j}$, denote this as U_A and $V_A = \bigcap\limits_{j=0}^{m} V_{a_j}$. Then U_A and V_A are open and $U_A \cap V_A = \emptyset$.

Notice that $V_A \subset X = A^c$ and since $x \in V_A$ and we chose x to be arbitrary, A^c is open, making A closed.

□