Ex 3.2.5

(Positive Definite)

By definition, $d(x_1, x_2)$ is either 0 or 1. So $\forall x_1, x_2 \in X$, $d(x_1, x_2) \geq 0$.

Now we show $d(x_1, x_2) = 0 \Leftrightarrow x_1 = x_2$.

Suppose $d(x_1, x_2) = 0$. Then $x_1 = x_2$ by definition.

Suppose $x_1 = x_2$. Then $d(x_1, x_2) = 0$.

Now we show $d(x_1, x_2) = d(x_2, x_1)$. (Symmetry)

$x_1 = x_2 \Rightarrow d(x_1, x_2) = 0 = d(x_2, x_1)$

$x_1 \neq x_2 \Rightarrow d(x_1, x_2) = 1 = d(x_2, x_1)$

Now we show $d(x_1, x_2) \leq d(x_1, x_3) + d(x_3, x_2)$. (Triangle Inequality)

Case I: $x_1 = x_2$

i) $\therefore d(x_1, x_2) = 0 \leq d(x_1, x_3) \leq d(x_1, x_3) + d(x_3, x_2)$

Case II: $x_1 \neq x_2$

Then either $x_3 \neq x_1$ or $x_3 \neq x_2$

Then $d(x_1, x_3) = 1$ or $d(x_3, x_2) = 1$

$\therefore d(x_1, x_3) + d(x_3, x_2) \geq 1 = d(x_1, x_2)$

$\square$

Ex 3.2.7

$d_1(x,y) = \|x-y\|_1$

$= |x_1-y_1| + \dots + |x_n-y_n|$

Try to show $d(x_1, x_2) \le d(x_1, x_3) + d(x_2, x_3)$

$$\begin{aligned} d_1(x_1, x_2) &= \sum_{i=1}^{n} |x_{1i} - x_{2i}| \\ &\le \sum_{i=1}^{n} \left(|x_{1i} - x_{3i}| + |x_{3i} - x_{2i}|\right) \\ &= \sum_{i=1}^{n} |x_{1i} - x_{3i}| + \sum_{i=1}^{n} |x_{3i} - x_{2i}| \\ &= d(x_1, x_3) + d(x_2, x_3) \end{aligned}$$

3.2.10

$$d_\infty(x,y) = \max_{1 \le j \le n} |x_j - y_j|$$

if $d_\infty(x,y) = 0$ then $\max\limits_{1 \le j \le n} |x_j - y_j| = 0$

Then $\forall j$ $0 \le |x_j - y_j| \le \max\limits_{1 \le j \le n} |x_j - y_j| = 0$

So $x_j = y_j$

$x_j = y_j$ $\quad \max\limits_{1 \le j \le n} |x_j - y_j| = \max\limits_{1 \le j \le n} |0|$

$= 0$ (positive definite)

$d_\infty(x,y) = \max |x_j - y_j| = \max |y_j - x_j| = d_\infty(y,x)$

(symmetry)

$$
\begin{aligned}
d(x_1, x_2) &= \max_{1 \le j \le n} |x_j - y_j| \\
&= |x_{j_0} - y_{j_0}| \\
&\le |x_{j_0} - z_{j_0}| + |z_{j_0} - y_{j_0}| \\
&\le \max_{1 \le j \le n} |x_j - z_j| + \max_{1 \le j \le n} |z_j - y_j| \\
&= d(x,z) + d(z,y)
\end{aligned}
$$

(Triangle inequality)

3.3.5

$B^{d_p}_{r}(x_0) = \{x \in X, d_p(x, x_0) < r\}$

$B^{d_q}_{r}(x_0) = \{x \in X, d_q(x, x_0) < r\}$

Try to show: $B^{d_p}_{1}(0) \subseteq B^{d_q}_{1}(0)$

let $x \in B^{d_p}_{1}(0)$

Then $d_p(x, 0) < 1$

$$\left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} \leq 1$$

$$\sum_{i=1}^{n} |x_i|^p \leq 1$$

$$|x_i|^p \leq 1$$

$$|x_i| \leq 1$$

$$\therefore |x_i|^q \leq |x_i|^p$$

$$\sum_{i=1}^{n} |x_i|^q \leq \sum |x_i|^p \leq 1$$

$$\therefore x \in B^{d_q}_{1}(0)$$

3.3.6

$B_\varepsilon(0) = \{x \in \mathbb{R}^n \mid d_p(x,0) < \varepsilon\}$

$= \{x \in \mathbb{R}^n \mid (\sum |x_i|^p)^{1/p} < \varepsilon\}$

We have to show $\{x \in \mathbb{R}^n \mid (\sum |x_i|^p)^{1/p} < \varepsilon\} = \{\varepsilon \cdot x \mid x \in B_1(0)\}$

First, let $x \in \{x \in \mathbb{R}^n \mid (\sum |x_i|^p)^{1/p} < \varepsilon\}$

$x = (x_1, x_2, \dots, x_n)$

$= \varepsilon\left(\frac{x_1}{\varepsilon}, \dots, \frac{x_n}{\varepsilon}\right)$

$\therefore \sum \left(\left|\varepsilon \cdot \frac{x_i}{\varepsilon}\right|^p\right)^{1/p} < \varepsilon$

$= \sum \left(\varepsilon^p \cdot \left|\frac{x_i}{\varepsilon}\right|^p\right)^{1/p} < \varepsilon$

$\varepsilon \sum \left(\left|\frac{x_i}{\varepsilon}\right|^p\right)^{1/p} < \varepsilon$

$\sum \left(\left|\frac{x_i}{\varepsilon}\right|^p\right)^{1/p} < 1$

$\therefore x \in \{\varepsilon \cdot x \mid x \in B_1(0)\}$

let $y \in \{\varepsilon \cdot x \mid x \in B_1(0)\}$

then $y = \varepsilon \cdot x$ and $x \in B_1(0)$

then $\sum (|x_i|^p)^{1/p} < 1$

$y_i = \varepsilon \cdot x_i$

$(\sum |y_i|^p)^{1/p} = (\sum |\varepsilon x_i|^p)^{1/p}$

$= (\sum (|\varepsilon|^p \cdot |x_i|^p))^{1/p}$

$= (\varepsilon^p \sum |x_i|^p)^{1/p}$

$= \varepsilon (\sum |x_i|^p)^{1/p} < \varepsilon$

$\therefore y \in B_\varepsilon(0)$

3.3.7) let $(x,y) \in \mathbb{R}^2$ be s.t. $|x|+|y|>1$ & $\max\{|x|,|y|\}<1$

$f(p) = \|(x,y)\|_p = (|x|^p + |y|^p)^{1/p}$

Note that f is continuous on $[1,\infty)$.

We will try to apply the Intermediate Value Theorem so we need to find $f(\alpha) \leq 1$ and $f(\beta) > 1$.

$f(1) = \|x,y\|_1 = |x|+|y| > 1$

let $M = \max\{|x|,|y|\} < 1$

$$f(p) = (|x|^p + |y|^p)^{1/p} \leqslant (M^p + M^p)^{1/p}$$
$$= 2^{1/p} \cdot M$$

Since $\lim_{p\to\infty} 2^{1/p} M = M < 1$

$\exists\, q$ s.t. $2^{1/q} M < 1$

$f(q) \leqslant 2^{1/q} M \leqslant 1$

By I.V.T, $\exists\, p$ s.t. $f(p) = 1$

Ex 3.3.10

i) let $(x, y) \in$ first quadrant

let $B_r(x,y)$, $x>0$, $y>0$.

Choose $r = \min(x,y)$

$d((x,y), (z,w)) < r$

$|x-z| = \sqrt{(x-z)^2} \leqslant \sqrt{(x-z)^2 + (y-w)^2} < r$

$$\begin{aligned} z &= x - (x-z) \\ &\geqslant x - |x-z| \\ &> x - r \\ &\geqslant 0 \qquad \because r = \min(x,y) \end{aligned}$$

similarly, $w \geqslant 0$.

$\therefore (z,w) \in Q_1$

ii) Let (X, d) be a metric space equipped with the discrete metric.

let $A \subseteq X$

let $x, y \in A$

Consider $B_r(x)$, $0 < r < 1$

We have to show $B_r(x) \subseteq A$.

Notice that $B_r(x) = \{x\}$ and $x \in A$ $\therefore$ $B_r(x) \subseteq A$.

Ex 3.3.12

$$B_r(1) = \begin{cases} (1-r, 1] & \text{if } r < 2 \\ [-1, 1] & \text{if } r > 2 \\ (-1, 1] & \text{if } r = 2 \end{cases}$$