

JOE SEIDEL

REAL ANALYSIS FIRST WEEK

Section 1.5 Construction of the Real Numbers

Exercise 1.5.1

Show that for any $a, b \in \mathbb{Q}$, we have $||a| - |b|| \leq |a - b|$.

Proof. Since $a, b \in \mathbb{Q}$,

$$|a + b| \leq |a| + |b|$$

Absolute values on \mathbb{Q} satisfy the Triangle Inequality

So

$$|a| = |a - b + b| \leq |a - b| + |b|$$

$$|b| = |a + b - a| \leq |b - a| + |a|$$

These can be rewritten as

$$|a| - |b| \leq |a - b|$$

$$|b| - |a| \leq |b - a|$$

Since $|a - b| = |b - a|$ and if $t \geq x$ and $t \geq -x$ then $t \geq |x|$, therefore

$$||a| - |b|| \leq |a - b|$$

□

Exercise 1.5.5

If a sequence $(a_k)_{k \in \mathbb{N}}$ converges in \mathbb{Q} show that $(a_k)_{k \in \mathbb{N}}$ is a Cauchy sequence in \mathbb{Q} .

Proof. By definition if $(a_k)_{k \in \mathbb{N}}$ converges in \mathbb{Q} given any rational number $r > 0$ there exists an integer N such that if $n \geq N$ then $|a_n - a| < r$.

Suppose $(a_k)_{k \in \mathbb{N}}$ converges to $a, a \in \mathbb{Q}$. Let $r > 0$, since $(a_k)_{k \in \mathbb{N}}$ converges to a , $\exists N$ such that $\forall n \geq N, |a_n - a| < \frac{r}{2}$.

Then $\forall n, m > N$

$$|a_n - a_m| = |a_n - a + a - a_m| \leq |a_n - a| + |a - a_m|$$

Let $n, m > N$

$$|a_n - a| < \frac{r}{2}$$

and

$$|a - a_m| = |a_m - a| < \frac{r}{2}$$

therefore

$$|a_n - a_m| < \frac{r}{2} + \frac{r}{2} = r$$

□

Exercise 1.5.6

Show that the limit of a convergent sequence is unique.

Proof. Suppose $(a_k)_{k \in \mathbb{Q}}$ converges in \mathbb{Q} to L and M . Choose L and $M, L \neq M$ and let $r = \frac{|L-M|}{2}$. Then $\exists N_1 \in \mathbb{N}$ such that if $n \geq N_1$ then

$$|a_n - L| < r$$

and $\exists N_2 \in \mathbb{N}$ such that if $n \geq N_2$ then

$$|a_n - M| < r$$

Let $N = \max(N_1, N_2)$. If $n \geq N$ then

$$|L - M| = |L - a_n + a_n - M| \leq |L - a_n| + |a_n - M| < 2\left(\frac{|L - M|}{2}\right) = |L - M|$$

Reducing the above, we have $|L - M| < |L - M|$ a contradiction,
 $\Rightarrow \Leftarrow$. Therefore, $L = M$.

□

Exercise 1.5.9

Show that the sum of two Cauchy sequences in \mathbb{Q} is a Cauchy sequence in \mathbb{Q} .

Proof. Let $(a_k)_{k \in \mathbb{N}}$ and $(b_k)_{k \in \mathbb{N}}$ be Cauchy sequences \mathbb{Q} . Let $r > 0$,
 $\exists N_1 \in \mathbb{N}$ such that if $n, m \geq N_1$ then

$$|a_n - a_m| < \frac{r}{2}$$

and $\exists N_2 \in \mathbb{N}$ such that if $n, m \geq N_2$ then

$$|b_n - b_m| < \frac{r}{2}$$

Let $N = \max(N_1, N_2)$ and choose $n, m \geq N$. This implies

$$|a_n - a_m| + |b_n - b_m| < \frac{r}{2} + \frac{r}{2} = r$$

$$|a_n - a_m + b_n - b_m| \leq |a_n - a_m| + |b_n - b_m|$$

Therefore

$$|(a_n + b_n) - (a_m + b_m)| < r$$

□

Exercise 1.5.13

Show that if a Cauchy sequence $(a_k)_{k \in \mathbb{N}}$ does not converge to 0, all the terms of the sequence eventually have the same sign.

Lem.: 1.5.12 Suppose $(a_k)_{k \in \mathbb{N}} \in \mathcal{C} \setminus \mathcal{I}$, then there exists a positive rational number r and an integer N such that $|a_n| \geq r$ for all $n \geq N$.

Proof. Suppose $(a_k)_{k \in \mathbb{N}}$ is a Cauchy sequence that does not converge to 0. Therefore given any $r > 0$ there exists an integer N such that if $n, m \geq N$, then $|a_n - a_m| < r$. From Lemma 1.5.2, we can choose $r > 0$ and N such that $|a_n| \geq r$ for all $n \geq N$.

Let $r > 0$ and $n, m \geq N$. Therefore

$$|a_n - a_m| < r \leq |a_n|$$

Suppose $a_n > 0$ and $a_m < 0$

$$|a_n - (-a_m)| = |a_n + a_m| < |a_n| \Rightarrow \Leftarrow$$

Likewise, suppose $a_n < 0$ and $a_m > 0$

$$|a_n - a_m| = |a_m - a_n|$$

$$|a_m - (-a_n)| = |a_m + a_n| < |a_n| \Rightarrow \Leftarrow$$

Therefore, all terms must eventually be the same sign.

□

Where \mathcal{C} denotes the set of all Cauchy sequences of rational numbers and \mathcal{I} denotes the set of all Cauchy sequences that converge to 0.

In other words, they don't have the same sign.

Exercise 1.5.15

Show that \sim defines an equivalence relation on \mathcal{C} . We need to show reflexivity, symmetry, and transitivity exist on Cauchy sequences that are equivalent.

$$(a_k)_{k \in \mathbb{N}} \sim (a_k)_{k \in \mathbb{N}}$$

For all $a_n \in (a_k)_{k \in \mathbb{N}}$, $|a_n - a_n| = 0$. Thus we can say that $(a_k - a_k)_{k \in \mathbb{N}}$ is in \mathcal{I} .

$$(a_k)_{k \in \mathbb{N}} \sim (b_k)_{k \in \mathbb{N}}$$

Suppose that $(a_k)_{k \in \mathbb{N}}$ and $(b_k)_{k \in \mathbb{N}}$ are equivalent and $r > 0$. Then, there exist $N \in \mathbb{N}$, such that exists $|a_n - b_n| < r$ and $|b_n - a_n| < r$ for $n \geq N$ and $(b_k - a_k)_{k \in \mathbb{N}}$ in \mathcal{I} .

$$(a_k)_{k \in \mathbb{N}} \sim (b_k)_{k \in \mathbb{N}}, (b_k)_{k \in \mathbb{N}} \sim (c_k)_{k \in \mathbb{N}} \Rightarrow (a_k)_{k \in \mathbb{N}} \sim (c_k)_{k \in \mathbb{N}}$$

Let $r > 0$, $\exists N_1 \in \mathbb{N}$ such that $|a_n - b_n| < \frac{r}{2}$ for all $n \geq N_1$ and $\exists N_2 \in \mathbb{N}$ such that $|b_n - c_n| < \frac{r}{2}$ for all $n \geq N_2$. This implies

$$|a_n - c_n| = |a_n - b_n + b_n - c_n| \leq |a_n - b_n| + |b_n - c_n| < \frac{r}{2} + \frac{r}{2} = r$$

for all $n = \max(N_1, N_2)$. Therefore, $(a_k - c_k)_{k \in \mathbb{N}}$ is also in \mathcal{I} .

Exercise 1.5.17

Show that \mathbf{R} is a commutative ring with 1 , with \mathcal{I} as the additive identity and $[a_k]$ such that $a_k = 1$ for all k as the multiplicative identity.

We know that if $(a_k), (b_k)$ are Cauchy sequences, $(a_n)_{n \in \mathbb{N}} + (a_n)_{n \in \mathbb{N}} = (a_n + b_n)_{n \in \mathbb{N}}$ and $(a_n)_{n \in \mathbb{N}}(b_n)_{n \in \mathbb{N}} = (a_nb_n)_{n \in \mathbb{N}}$ are well-defined.

Let $[a_k]$ be an equivalence class, addition and multiplication are defined as follows $[a_k] + [a_k] = [a_k + a_k]$ and $[a_k][a_k] = [a_ka_k]$

As examples, consider $(a_k)_{k \in \mathbb{N}}$ and $(a'_k)_{k \in \mathbb{N}}$ denoted as $\{a_k\}$ and $\{a'_k\}$, respectively. Likewise, $(b_k)_{k \in \mathbb{N}}$ and $(b'_k)_{k \in \mathbb{N}}$ denoted as $\{b_k\}$ and $\{b'_k\}$

For addition, let $\{a_k\} \sim \{a'_k\}$, $\{b_k\} \sim \{b'_k\}$ and $r > 0$. Then, $\exists N_1 \exists N_2$ in \mathbb{N} such that

$$|a_n - a'_n| < \frac{r}{2} \text{ for } n \geq N_1$$

and

$$|b_n - b'_n| < \frac{r}{2} \text{ for } n \geq N_2$$

This implies

$$|(a_n + b_n) - (a'_n + b'_n)| = |a_n - a'_n + b_n - b'_n| \leq |a_n - a'_n| + |b_n - b'_n| < \frac{r}{2} + \frac{r}{2}$$

for $n \geq \max(N_1, N_2)$. Therefore $[a_k + b_k]$ is in \mathcal{I}

It follows, if $[i_k]$ is \mathcal{I} then $[a_k] + [i_k] = [a_k]$

For multiplication, recall that $\{a_k\}$, $\{a'_k\}$, $\{b_k\}$ and $\{b'_k\}$ are bounded. $\exists M > 0$ such that $\{a_n\}$, $\{a'_n\}$, $\{b_n\}$, $\{b'_n\} \leq M$ for all $n \geq N \in \mathbb{N}$ such that

$$|a_n - a'_n| < \frac{r}{2M} \text{ for } n \geq N_1$$

and

$$|b_n - b'_n| < \frac{r}{2M} \text{ for } n \geq N_2$$

Lem.: 1.5.8. Let $(a_k)_{k \in \mathbb{N}}$ be a Cauchy sequence of rational numbers. Then $(a_k)_{k \in \mathbb{N}}$ is a bounded sequence

$$\begin{aligned} 2|a_nb_n - a'_nb'_n| &= |(a_n - a'_n)(b_n + b'_n) + (a_n + a'_n)(b_n - b'_n)| \\ &\leq |(a_n - a'_n)(b_n + b'_n)| + |(a_n + a'_n)(b_n - b'_n)| \\ &= |a_n - a'_n||b_n + b'_n| + |a_n + a'_n||b_n - b'_n| \\ &\leq |a_n - a'_n|(|b_n| + |b'_n|) + (|a_n| + |a'_n|)|b_n - b'_n| \\ &< \frac{r}{2M}(2M) + \frac{r}{2M}(2M) \\ &= 2r \end{aligned}$$

Therefore $|a_nb_n - a'_nb'_n| < r$ for all $n \geq \max(N_1, N_2)$.

If $[i_k]$ with $i_k = 1$ for all k is the multiplicative identity it follows that $[a_k][i_k] = [a_k]$

Exercise 1.5.20

Show that order relation, defined below is well-defined and makes \mathbf{R} and ordered field.

Def.: Let $a = [a_k]$ and $b = [b_k]$ be distinct elements of \mathbf{R} . We define $a < b$ if $a_k < b_k$ eventually and $b < a$ if $b_k < a_k$ eventually.

Let $r > 0$, then there exists $n, m \geq N_1 \in \mathbb{N}$ such that

$$|c_n - c_m| = |(a_n - b_n) - (a_m - b_m)| < r$$

Since $[c_k]$ is not in \mathcal{I} We can eventually find an $n > N_2 \in \mathbb{N}$ such that $|c_n| > r$ Therefore

$$|a_n - b_n| > r > 0$$

We also know, from exercise 1.5.13 that all terms at this point in the sequence need to have the same sign and it's easy to see that $a_n \neq b_n$. So it follows that either $a_k > b_k$ or $a_k < b_k$, eventually.

We can apply the above to the Order Axioms.

1. (O1) **Trichotomy:** Since $[a] - [b]$ is not in \mathcal{I} , by definition either $a_k < b_k$ or $b_k > a_k$, eventually.
2. (O2) **Transitivity:** For sake of argument, let $a_k < b_k$, eventually, and choose an additional arbitrary element of \mathbf{R} $[c_k]$. Let $b_k < c_k$. Then $a_k < c_k$, eventually
3. (O3) **Addition:** Let $a_k < b_k$ and choose $[c]$ to be in \mathcal{I} it easily follows that $a_k + c_k < b_k + c_k$, eventually
4. (O4) **Multiplication:** $a_k < b_k$ and let $[c_k]$ be the multiplicative identity $c_k = 1$ for all $k \in \mathbb{N}$, then $a_k c_k < b_k c_k$, eventually

Therefore order relation is well-defined and makes \mathbf{R} and ordered field.

Exercise 1.6.11

Find a bounded sequence of real numbers that is not convergent.

Define $(a_k)_{k \in \mathbb{N}} = (-1)^k$, this sequence is bounded $[-1, 1]$. It is clear that $\{1, -1, 1, -1, \dots\}$ does not converge.

Exercise 1.6.16

Prove Lemma 1.5.15

Lem.: Lemma 1.5.15 Every bounded sequence in \mathbb{R} has a convergent subsequence

Let $(a_k)_{k \in \mathbb{N}}$ be a bounded sequence in \mathbb{R} .

Lem.: Lemma 1.6.13: every bounded sequence in \mathbb{R} has a monotonic subsequence.

Lem.: Lemma 1.6.14: Every bounded monotonic sequence in \mathbb{R} converges to an element in \mathbb{R} .

If $(a_k)_{k \in \mathbb{N}}$ does not have a monotonically increasing subsequence, $\exists n_1 \in \mathbb{N}$ such that $a_{n_1} > a_k$ for $k > n_1$. It follows that since $(a_k)_{k > n_1}$ is not monotonically increasing, there exists $a_{n_2} > a_k$ for $k > n_2$ and $a_{n_1} > a_{n_2}$. This process can be repeated over the set $(a_k)_{k \in \mathbb{N}}$ to create a strictly monotonic decreasing set $(a_{n_1}, a_{n_2}, \dots, a_{n_k})$.

Alternatively, if $(a_k)_{k \in \mathbb{N}}$ does not have a strictly monotonic decreasing subsequence. We say $a_{n_1} < a_k$ for $k \geq n_1$. Repeating steps above to form a set $(a_{n_1}, a_{n_2}, \dots, a_{n_k})$. Which is monotonic increasing.

Since (a_k) is bounded, (a_{k_j}) is bounded and we can apply Lemma 1.6.14 (a_{k_j}) converges to an element in \mathbb{R} .

Lem.: Every bounded monotonic sequence in \mathbb{R} and conclude converges to an element in \mathbb{R}

Exercise 1.6.20

Show that if $\limsup_{k \rightarrow \infty} (a_k) = \liminf_{k \rightarrow \infty} (a_k)$, then $(a_k)_{k \in \mathbb{N}}$ is convergent, and $\lim_{k \rightarrow \infty} (a_k) = \limsup_{k \rightarrow \infty} (a_k) = \liminf_{k \rightarrow \infty} (a_k)$.

It first helps to rewrite the definition of limit supremum and limit infimum.

$$\limsup_{k \rightarrow \infty} (a_k) = \lim_{n \rightarrow \infty} (b_n), \text{ where } b_n = \sup\{a_k | k \geq n\}$$

$$\liminf_{k \rightarrow \infty} (a_k) = \lim_{n \rightarrow \infty} (c_n), \text{ where } c_n = \inf\{a_k | k \geq n\}$$

These definitions combined with the information that the limit supremum of (a_k) being equal the the limit infimum of (a_k) imply that our sequence (a_n) converges to the same points $\forall n \geq k$. This allows us to write

$$\lim_{k \rightarrow \infty} (b_k) = \lim_{k \rightarrow \infty} (c_k).$$

Notice that c_n increases as $n \rightarrow \infty$ and b_n decreases as $n \rightarrow \infty$. Likewise, Since $c_n \leq a_n \leq b_n$ we can apply the Sandwich theorem and say $\lim_{n \rightarrow \infty} a_n = a$

Since c_k and b_k are items in $(a_k)_{k \in \mathbb{N}}$ and equal we can refer to the above as

$$\lim_{k \rightarrow \infty} (a_k)$$

Which implies $(a_k)_{k \in \mathbb{N}}$ converges and finally...

$$\lim_{k \rightarrow \infty} (a_k) = \limsup_{k \rightarrow \infty} (a_k) = \liminf_{k \rightarrow \infty} (a_k)$$