1.6.35) Show that $\emptyset$ and R are the only subsets of R that

i) are both open and closed in R.

Consider $X \subseteq R$. Let $X \neq R$, be non-empty, open, and closed.
let $x \in X$. $\exists\, y \notin X$ s.t. $y > x$. let us assume $y > x$ WLOG

let $S = \{ s \in R \mid s > x,\ s \notin X \}$.
$y \in S$ so S is non-empty.
$y > x\ \forall x \in X$ $\therefore$ S is bounded below $\Rightarrow$ inf exists.

Then $\forall x > 0$, $\exists\, s_0 \in S$ s.t. inf $S + \varepsilon > s_0$

Case 1: $\operatorname{Inf} S \in X$. Then since X is open,
$\exists\, \varepsilon > 0$ s.t $(\operatorname{Inf} S - \varepsilon, \operatorname{Inf} S + \varepsilon) \subseteq X$.
But $\operatorname{Inf} S < \operatorname{Inf} S + \varepsilon < s\ \ \forall s \in S$ which contradicts.

Case 2: $\operatorname{Inf} S \notin X$. Then since X is closed, R/X must be open. $\exists\, \varepsilon > 0$ s.t. $(\operatorname{Inf} S - \varepsilon, \operatorname{Inf} S + \varepsilon) \subseteq R/X$.
but $\operatorname{Inf} S < \operatorname{Inf} S + \varepsilon < s\ \ \forall s \in S$ which contradicts.

a contradiction because there doesn't exist 2 unique infs of the same set.

1.6.35 ii) let A be an open set in $\mathbb{R}$.
Define $\sim$ on A: $x \sim y$ if $[\min\{x,y\}, \max\{x,y\}] \subseteq A$

Note also that $\sim$ is an equivalence relation because it satisfies symmetry, reflexitivity, and transitivity

Then A is a union of pairwise disjoint classes

let I be an equivalent class of $\sim$

Case I: I is bounded above & below
$\therefore$ Inf I and sup I exist
We will show that $I = (\inf I, \sup I)$
First we show $I \subseteq (\inf I, \sup I)$
let $a \in I$. Then $\inf I \leq a \leq \sup I$
If $a = \inf I$, $\forall x \in I$, $(x-\varepsilon, x+\varepsilon) \subseteq A$
then $\exists\, \varepsilon > 0$, s.t. $(a-\varepsilon, a+\varepsilon) \subseteq A$
Then $[a-\varepsilon/2, a] \subseteq (a-\varepsilon, a+\varepsilon) \subset A$

$\therefore a - \frac{\varepsilon}{2} \sim a \Rightarrow a - \frac{\varepsilon}{2} \in [a] = I$

$$a - \varepsilon/2 \geq \inf I$$

$$a \geq \inf I + \varepsilon/2 > \operatorname{Inf} I$$

Now we show $(\inf I, \sup I) \subseteq I$
Similarly, if $a = \sup I$, $\exists\, \varepsilon > 0$ s.t. $(a-\varepsilon, a+\varepsilon) \subseteq A$
Then $[a-\varepsilon/2, a] \subseteq (a-\varepsilon, a+\varepsilon) \subseteq A$
$\therefore a - \frac{\varepsilon}{2} \sim a \Rightarrow a - \frac{\varepsilon}{2} \in [a] = I$

$$a - \varepsilon/2 \leq \sup I$$

$$a \leq \sup I - \frac{\varepsilon}{2} < \sup I$$

$\therefore$ $\operatorname{Inf} < a < \sup I$

Conl
ow
A

er 1.6.35. ii) Continued from previous page

We will now show $(\inf I, \sup I) \subseteq I$. Let $a \in (\inf I, \sup I)$

Then $\inf I < a < \sup I$

$\exists\, b \in I$ s.t. $b < a$

$\exists\, c \in I$ s.t. $c > a$

$b < a < c$

$\therefore\ b \sim c \Rightarrow [b,c] \subseteq A$

$\Rightarrow [b,a] \subseteq A$

$\Rightarrow b \sim a$

$\Rightarrow a \in [b] = I$

$\therefore (\inf I, \sup I) \subseteq I \Rightarrow I$ is an open interval

Case II : I is bounded above but not below

We will show that $I = (-\infty, \sup I)$

To show $I \subseteq (-\infty, \sup I)$

let $a \in I$ $\therefore a < \sup I$ which follows in the same exact manner as case I.

$\therefore\ I \subseteq (-\infty, \sup I)$

Now we will show $(-\infty, \sup I) \subseteq I$.

let $a \in (-\infty, \sup I)$

$a < \sup I \Rightarrow \exists\, c \in I$ s.t. $a < c$

Since I is not bounded below, $\exists\, b \in I$, $b < a$

Then we have $b < a < c$ which again follows in the same exact manner as case I.

Case III - I is bounded below but not above

Case IV - I is unbounded below and above.

Case III and IV are similar in proofs to case I and II

1.6.35 ii) Continued

Now we will show countability.

$A = \bigcup_{\alpha \in B} I_\alpha$ (B is an index set) where I_α is an open interval for each $\alpha \in B$

We want to show that B is countable.

For each $\alpha \in B$, I_α contains some rational number q_α (since Q is dense in $\mathbb{R}$).

Define $f : B \to Q$ by $f(\alpha) = q_\alpha$

If $\alpha \neq \beta$ then since $I_\alpha \cap I_\beta = \emptyset$, clearly $q_\alpha \neq q_\beta$

$\therefore$ f is 1-1.

Since Q is countable, so is B

1.6.35)

iii) Show that an arbitrary union of open sets in R is open in R.

let $O = \cup_{\alpha \in A} O_\alpha$ where A is an indexing set.

let $x \in O_\alpha$ for some α. By definition of an open set,
$\forall x \in O_\alpha$, $\exists\, \varepsilon > 0$ s.t $(x-\varepsilon, x+\varepsilon) \subseteq O_\alpha$ but $O_\alpha \subseteq O$
$\therefore (x-\varepsilon, x+\varepsilon) \subseteq O$ so O is open.

1.6-35

iv) Show that a finite intersection of open sets in R is open in R.

let $\{O\}_N = \{O_1, O_2, \ldots, O_r\}$ i.e. a collection of finite open sets.

let $x \in O_1 \cap O_2 \cap \ldots \cap O_r$

$\forall x \in O_1 \;\exists\; \varepsilon_1 > 0$ s.t $(x-\varepsilon_1, x+\varepsilon_1) \subseteq O_1$

In the same way, $\forall x \in O_2 \;\exists\; \varepsilon_2 > 0$ s.t $(x-\varepsilon_2, x+\varepsilon_2) \subseteq O_2$

Continuing this logic for $x \in O_3$, $x \in O_4, \ldots, x \in O_r$.

$\forall x \in O_r$, $\exists\; \varepsilon_r > 0$ s.t. $(x-\varepsilon_r, x+\varepsilon_r) \subseteq O_r$

let $\varepsilon = \min\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_r\}$. Then $(x-\varepsilon, x+\varepsilon)$ is contained in each set of $\{O_n\}$.

Then $\forall x \in O_1 \cap O_2 \cap \ldots \cap O_r$, we have a $\varepsilon > 0$ s.t. $(x-\varepsilon, x+\varepsilon) \subseteq O_1 \cap O_2 \cap \ldots \cap O_r$

$\therefore O_1 \cap O_2 \cap \ldots \cap O_r$ is open.

1.6.35)

v) let $O_n = \{(-1/n, 1/n) \mid n \in N\}$.

Each O_n is an open set but $\bigcap_{n=1}^{\infty} O_n = \{0\}$ which is not open.

Consider $\varepsilon = 1$, $(0-1, 0+1) = (-1, 1) \notin \{0\}$.
Hence, $\{0\}$ is not open.

1.6.35)

vi) let $\{O\}_{N \in I}$ be a collection of closed sets indexed by I.

From De Morgan's laws:

$$R \setminus \bigcap_{i \in I} O_i = \bigcup_{i \in I} (R \setminus O_i)$$

Since each O_i is closed, each $(R \setminus O_i)$ must be open as it is the complement.

Then $\bigcup_{i \in I} (R \setminus O_i)$ is a union of open sets which is also open according to 1.6.35 iii, as proved earlier.

$$\bigcap_{i \in I} O_i = \left(R \setminus \bigcap_{i \in I} O_i\right)^c = \left(\bigcup_{i \in I} (R \setminus O_i)\right)^c$$

By definition, the complement of an open set must be closed $\therefore \bigcap_{i \in I} O_i$ is closed.

6.35)

vii) Show that a finite union of closed sets in R is a closed set in R.

let $\{O_n\} = \{O_1, O_2, \ldots, O_r\}$, a finite collection of closed sets.

By De Morgan's laws,

$$R \setminus \bigcup_{i=1}^{n} (O_i) = \bigcap_{i=1}^{n} (R \setminus O_i)$$

Since each O_i is closed, each $R \setminus O_i$, the complement of O_i must be open.

We have $\bigcap_{i=1}^{n} (R \setminus O_i)$ is a finite intersection of open sets which is open according to 1.6.35 iv.

$$\bigcup_{i=1}^{n} (O_i) = \left(R \setminus \bigcup_{i=1}^{n} (O_i)\right)^c = \left(\bigcap_{i=1}^{n} R \setminus O_i\right)^c$$

Since the complement of an open set is closed, $\bigcup_{i=1}^{n} (O_i)$ must be closed.

1.6.35

viii. Consider a subset of $\mathbb{R}$ say $(0,1) = \bigcup_{i \in (0,1)} \{r_i\}$

That is to say, $(0,1)$ can be constructed by the union of single point sets which are closed. But $(0,1)$ is clearly not closed and thus a counterexample.

1.6.36) Show that a subset of R is closed iff it contains all its accumulation points.

We proved the forward direction in class.
Now for the other direction,
let $F \subseteq R$ and $S = R / F$. Assume F contains all its accumulation points.
let $x \in R$ be an accumulation point of F.
Then x is not a accumulation point of S.

$\exists \varepsilon > 0$ s.t. there does not exist $y \in F$ s.t. $y \neq x$ and $y \in (x-\varepsilon, x+\varepsilon)$. So for this ε, either $y = x$ or $y \notin (x-\varepsilon, x+\varepsilon)$ $\forall y \in F$.
But $y \neq x$ because $x \in F$ and $y \in S$.
$\therefore (x-\varepsilon, x+\varepsilon) \subseteq R$

1.6.37) Show that the cantor set is closed.

i) At each stage, the Cantor set contains a finite union of closed intervals which is closed according to the result of 1.6.35 vii.
The Cantor set is the intersection of these stages
An arbitrary intersection of closed sets is closed according to 1.6.35 vi $\therefore$ The Cantor set is closed $\square$

1.6.37)

ii) Consider $C_n = \bigcap_{k=1}^{n} C_k$ where $C_1, C_2, \ldots, C_k$ represent each stage of construction of the Cantor set

Consider C_1, by removing the middle third, we remove all such numbers that consist of a 1 on the first decimal point.
Similarly C_2, we remove all such numbers that consist of a 1 on the second decimal point.
If we iterate this process infinitely, through the stages of $C_1, C_2, C_3, \ldots$, then all the remaining numbers will not have a 1 on any decimal point
QED.

1.6.37 iii) Show that the Cantor set is uncountable.

Assume, for contradiction, that the Cantor set is countable
Then there is a bijection $f: \mathbb{N} \to C$.
Consider;

let $f(1) = 0.a_{11}a_{12}a_{13}\dots$ where $a_{ij} \in \{0,2\}$

$f(2) = 0.a_{21}a_{22}a_{23}\dots$

$\vdots$

$f(n) = 0.a_{n_1}a_{n_2}a_{n_3}\dots$

Consider; $0.b_1 b_2 b_3 \dots \in C$

where $b_i = \begin{cases} 0 & \text{if } a_{ii} = 2 \\ 2 & \text{if } a_{ii} = 0 \end{cases}$

$b_i \notin \{f(1), f(2), \dots, f(n)\}$ as b_i has at least one digit to $f(i) \in \{f(1), f(2), \dots, f(n)\}$

$\therefore f$ is not onto which is a contradiction.

1.6.37

iv) Show that every point in the Cantor

let $C =$ Cantor Set

let $a \in C$. let $\varepsilon > 0$.

$a = 0.a_1 a_2 a_3 \ldots$ where $a_i \in \{0, 2\}$

Note that $\lim_{n \to \infty} \frac{2}{3^n} = 0$.

$\forall \varepsilon > 0$, $\exists N$ s.t. $\frac{2}{3^N} < \varepsilon$.

Choose $b \in C$ by switching the N^{th} digit of a from 0 to 2 or vice versa.

$\therefore |a - b| = \frac{2}{3^N} < \varepsilon$ $\square$

1.6.37 v) A is dense in $[0,1]$ means $\forall x \in [0,1], \forall \varepsilon > 0,$
$\exists a \in A$ s.t. $a \in (x-\varepsilon, x+\varepsilon)$

We want to show that C^c is dense in $[0,1]$.
Suppose not, for contradiction.
Then $\exists x \in [0,1], \exists \varepsilon > 0$ s.t. $\forall a \in C^c, a \notin (x-\varepsilon, x+\varepsilon)$

$a \in C^c \Rightarrow a \notin (x-\varepsilon, x+\varepsilon)$
$a \in (x-\varepsilon, x+\varepsilon) \Rightarrow a \in C$
$(x-\varepsilon, x+\varepsilon) \subseteq C_1 \subseteq C_n$, which is a contradiction.
The length of C_n at the nth stage is $\left(\frac{2}{3}\right)^n$.

Then for some large n, we have $\left(\frac{2}{3}\right)^n < \varepsilon$

thus $(x-\varepsilon, x+\varepsilon)$ cannot be a subset of C_n, which leads to the mentioned contradiction. $\square$

1.6.42 Show that a compact subset of R is both closed and bounded.

First, we will show boundedness.
let $K \subseteq R$.
let $S_n = (-n, n)$. By the archimedean principle and the fact that S_n is open, $\{S_n\}_{n \in N} = \bigcup_{n \in N} S_n = R$

Hence, $\{S_n\}_{n \in N}$ is an open cover of R and thus also an open cover of K. By compactness, $\exists \{S_{n_1}, S_{n_2}, \dots S_{n_r}\}$ that is a finite subcover of S. But because $\{S_n\}$ are nested intervals, $K \subseteq \bigcup_{k=1}^{r} S_{n_k} \subset S_{n_r} = (-r, r)$

$\therefore$ K is bounded

Now we will show that K is closed, by contradiction.
Assume K is not closed $\Rightarrow \exists$ a limit point x of K s.t. $x \notin K$.
$\forall y \in K$, $y \neq x$, $\exists$ an open interval (I_y) s.t. $y \in I_y$ but $x \notin I_y$

$\{I_y \mid y \in K\}$ is an open cover of K.

K is compact so $\exists$ finite subcover $\{I_{y_1}, I_{y_2}, \dots, I_{y_n}\}$
$K \subseteq \bigcup_{j=1}^{n} I_{y_j} \subseteq \overline{\bigcup_{j=1}^{n} I_{y_j}}$ (The Closure of this set)

$x \notin \overline{I_{y_j}}$, $\forall j$ so $x \in (\cup \overline{I_{y_j}})^c$

Due to open set property,
$\exists \varepsilon > 0$ s.t. $(x-\varepsilon, x+\varepsilon) \subseteq (\cup \overline{I_{y_j}})^c$

Thus there is a contradiction.

6.46) compact $\Leftrightarrow$ sequentially compact

Let us first prove compact $\Rightarrow$ sequentially compact

let (a_n) be an infinite sequence in $S \subseteq R$

S is compact $\Rightarrow$ S is closed and bounded by 1.6.42.

$\therefore$ $A = \{a_n \mid n \in N\}$ is bounded and infinite

Then A has an accumulation point in R

$\therefore$ $\exists$ x that is an accumulation point of S.

But S is closed so $x \in S$.

$\exists\, a_{n_1} \in (x-1, x+1)$

$\exists\, a_{n_2} \in (x-\frac{1}{2}, x+\frac{1}{2})$, $n_1 < n_2$

$\vdots$

$\exists\, a_{n_k} \in (x-\frac{1}{k}, x+\frac{1}{k})$, $n_1 < n_2 < n_3 < n_4 < \dots$

subsequence $(a_{n_1}, a_{n_2}, \dots)$ converges to $x \in S$.

Now, let us prove sequentially compact $\Rightarrow$ compact

let $\{O\}_{\alpha \in I}$ be an open cover of S.

We must show that S is closed and bounded.

If S is not closed, then $\exists$ an accumulation point x, $x \notin S$.

$\exists\, (a_n)$ in S s.t. $a_n \to x$ but $x \notin S$ which contradicts the definition of sequentially compact.

If S is not bounded, then there is an increasing sequence in S s.t. $a_n > n$. But here we have no convergent subsequence.

$\therefore$ S is closed & bounded $\Rightarrow$ S is compact

$\square$

1.9.6 i) If $N \in \mathbb{N}$ and $z \neq 1$, show that $S_N = \sum_{n=0}^{N} z^n = \frac{1 - z^{N+1}}{1-z}$

$S_N = 1 + z + z^2 + \dots + z^{n-1}$

$z \cdot S_N = z + z^2 + z^3 + \dots + z^{n-1} + z^n$

$S_N - z S_N = 1 - z^n$

$S_N (1 - z) = 1 - z^n$

$S_N = \frac{1 - z^n}{1 - z}$

1.9.6 ii) If $|z| < 1$, show that $\lim_{n\to\infty} z^n = 0$

We have $-1 < z < 1$.

Case I : $z = 0$. This is trivial as $\lim_{n\to\infty} 0 = 0$.

Case II : $0 < |z| < 1$.
we have $0 < |z|^{n+1} < |z|^n$ then clearly the sequence (z^n) is monotonically decreasing and has 0 as a lower bound.

Using lemma 1.6.14, Every bounded monotonic sequence in $\mathbb{R}$ converges to an element in $\mathbb{R}$.
Thus $(|z|^n)$ has a limit $k \in \mathbb{R}$.

$$k = \lim_{n\to\infty} |z|^{n+1} = \lim_{n\to\infty} |z|\cdot|z|^n = |z|\cdot \lim_{n\to\infty} |z|^n = |z|\cdot k$$

$|z|\cdot k = k \Rightarrow k(|z| - 1) = 0$ but $z \neq 1$ so $k = 0$

We have $\forall \varepsilon > 0$, $\exists N$ s.t. $\forall n \geq N$, $\left||z|^n - 0\right| < \varepsilon$
$|z^n| < \varepsilon$ $\square$.

1.9.6

iii) If $|z| > 1$, show that $\lim_{n \to \infty} z^n$ does not exist

Assume, for contradiction, that the $\lim_{n \to \infty} z^n$ exists.

Then $\forall \varepsilon > 0, \exists N$ s.t. $\forall n \geq N$, $|z^n - L| < \varepsilon$

Now, $\varepsilon > |z^n - L| \geq |z^n| - L$

Lemma $\forall x > 0$, $(1+x)^n \geq 1 + nx$

let $P(n)$ be the lemma above.

Consider $P(1)$; $LHS = (1+x)^1 = 1 + x$

$RHS = 1 + 1(x) = 1 + x$

$LHS \geq RHS$ $\therefore$ $P(1)$ is true.

Assume $P(k)$ is true. Consider $P(k+1)$;

$$(1+x)^k (1+x) \geq (1+kx)(1+x)$$

$$(1+x)^{k+1} \geq 1 + kx + x + kx^2$$

$$> 1 + kx + x = 1 + (k+1)x$$

$\therefore$ $P(k+1)$ is true

$P(1)$ is true and $P(k)$ implies $P(k+1)$ is true thus $P(n)$ is true $\forall n > 1$.

Now, continuing the proof we started before our lemma proof,

$\varepsilon + |L| > |z^n| = (1 + (|z| - 1))^n \geq 1 + n(|z| - 1)$

$\varepsilon + |L| - 1 > n(|z| - 1)$. This contradicts the Archimedean principle

1.9.16)

i.) If $p \in R$ and $p < 1$, show that $\sum_{n=1}^{\infty} 1/n^{p}$ diverges.

We begin by showing the Harmonic series $(p=1)$ diverges.

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \dots$$

$$> 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \dots$$

$$= 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \dots$$

For each $n \in N$ and $q \in N$, we have chosen a $r \in N$ s.t.

$$\left(\frac{1}{2}\right)^{q} \leq \left(\frac{1}{2}\right)^{r} \leq \left(\frac{1}{n}\right).$$

Following from above, we have

$$= 1 + \left(\frac{1}{2} + \frac{1}{2}\right) + \dots + \frac{1}{2} + \dots$$

$$= 1 + 1 + 1 + \dots \dots$$

By the archimedean principle, this series diverges.

Now we apply the comparison test,

if $p < 1$, then $\sum_{n=1}^{\infty} \frac{1}{n} < \sum_{n=1}^{\infty} \frac{1}{n^{p}}$.

Since the harmonic series diverges, then $\sum_{n=1}^{\infty} 1/n^{p}$ must also diverge.

1.9.16)

ii) If $p \in \mathbb{R}$ and $p > 1$, show that $\sum_{n=1}^{\infty} 1/n^p$ converges.

$$\sum_{n=2}^{N} 1/n^p \leq \int_1^N 1/x^p \, dx = \frac{1}{(p-1)} \cdot \left(1 - \frac{1}{N^{p-1}}\right)$$

$$1 + \sum_{n=2}^{N} 1/n^p \leq 1 + \frac{1}{p-1}\left(1 - \frac{1}{N^{p-1}}\right)$$

$$S_N \leq 1 + \frac{1}{p-1}\left(1 - \frac{1}{N^{p-1}}\right)$$

$$< 1 + \frac{1}{p-1}$$

Since S_N is bounded and monotonically increasing then $\sum_{n=1}^{\infty} 1/n^p$ converges. $\square$