Statistics 245: Homework 2 due April 18

When solving the problems below as well as future homework problems, give detailed derivations and arguments in order to receive credit. In your solution do not forget to include your name and the homework number. Please staple your pages together.

10 1. *(Cauchy distribution)* Let X and Y be two independent $N(0,1)$ random variables. Show that the distribution of X/Y is the same as that of $X/|Y| = X/\sqrt{Y^2}$. This means that X/Y has a t_1-distribution which is also known as Cauchy-distribution. (Hint: What is the distribution of $-X$?)

20 2. *(Bivariate normal distribution I)* Suppose (X,Y) has a bivariate normal distribution with expected values $\mathbb{E}[X] = 3$ and $\mathbb{E}[Y] = 1$, variances $\mathrm{var}[X] = 9$ and $\mathrm{var}[Y] = 16$, and correlation ρ. Let $W_a = 12 + aX + Y$ and $V = 19 + X + 2Y$.

   5+5 (a) Fix $\rho = 1/3$ and find $a \in \mathbb{R}$ such that W_a and V are independent. Can you choose $\rho_0 \in (-1,1)$ such that there does not exist an $a \in \mathbb{R}$ making W_a and V independent? If yes, find all such ρ_0. If no, explain why not.

   5+5 (b) Now fix $a = 1$ and find ρ such that W_a and V are independent. Can you choose $a_0 \in \mathbb{R}$ such that there does not exist a $\rho \in (-1,1)$ making W_{a_0} and V independent? If yes, find all such a_0. If no, explain why not.

20 3. *(Bivariate normal distribution II)* Let (X,Y) follow a bivariate normal distribution with $\mathbb{E}[X] = 5$ and $\mathbb{E}[Y] = 3$, variances $\mathrm{var}[X] = 9$ and $\mathrm{var}[Y] = 16$, and correlation $\rho = 0.4$. Find

   5 (a) the conditional expectation $\mathbb{E}[X \mid Y = 8]$,

   5 (b) the conditional variance $\mathrm{var}[X \mid Y = 8]$,

   5 (c) the probability $\mathbb{P}(3 < X < 5)$,

   5 (d) the conditional probability $\mathbb{P}(3 < X < 5 \mid Y = 8)$.

10 4. Let X and Y be the scores of a Stat 245 student on midterm and final exam. We model these scores as

$$X = S + E_1, \qquad Y = S + E_2,$$

where S, E_1, E_2 are independent random variables distributed as $S \sim N(70, 49)$, $E_1, E_2 \sim N(0, 25)$. We think of S as a "skill" part of the score and E_1, E_2 as "luck" components.

   5 (a) What is the joint distribution of (X,Y)?

   5 (b) Assume that a student received a midterm score that is one standard deviation below the midterm mean. What do you expect his/her final score to be? (Hint: find $\mathbb{E}(Y|X)$.)

15. 5. *(Mean square error when estimating a normal variance)* Let $X_1, \dots, X_n$ be independent $N(\mu, \sigma^2)$ random variables. Let $\bar{X} = \frac{1}{n}\sum_i X_i$ be the sample mean. Consider two estimators of σ^2, namely the sample variance

$$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2, \tag{1}$$

and the MLE

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2.$$

The mean square error (MSE) measures how far on average these estimators are away from the "target" σ^2, where "away" is measured in squared distance. The two MSE are defined as

$$\mathrm{MSE}(s^2) = \mathbb{E}[(s^2 - \sigma^2)^2] \quad \text{and} \quad \mathrm{MSE}(\hat{\sigma}^2) = \mathbb{E}[(\hat{\sigma}^2 - \sigma^2)^2].$$

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10 (a) Compute and compare $\mathrm{MSE}(s^2)$ and $\mathrm{MSE}(\hat{\sigma}^2)$. compare 2.

5. (b) Consider a general form of estimator

5 $$\tilde{\sigma}^2 = c\sum_{i=1}^{n}(X_i - \bar{X})^2.$$

Find the best c such that $\mathrm{MSE}(\tilde{\sigma}^2) = \mathbb{E}[(\tilde{\sigma}^2 - \sigma^2)^2]$ is minimized.

6. Let $X_1, \dots, X_n$ be iid distributed as $N(\mu, \mu^2)$, where $\mu \in \mathbb{R}$ is an unknown parameter.

15.

3 (a) Find pivotal(s) for μ.

3 (b) Find the MLE $\hat{\mu}$ of μ.

3 (c) The asymptotic distribution of MLE is given by the formula $\sqrt{n}(\hat{\mu} - \mu) \Rightarrow N(0, I(\mu)^{-1})$, where $I(\mu)$ is called Fisher information. Calculate $I(\mu)$ with the formula

$$I(\mu) = \int p_\mu(x)\left(\frac{\partial}{\partial\mu}\log p_\mu(x)\right)^2 dx,$$

where p_μ is the density of $N(\mu, \mu^2)$.

3 (d) Find a function g such that $\sqrt{n}[g(\hat{\mu}) - g(\mu)] \Rightarrow N(0, 1)$.

3 (e) Comment on confidence intervals for μ^2 constructed based on (a) and (b). Which one has smaller length?

10.

1. *(Cauchy distribution)*

$$\begin{aligned}
P(\frac{X}{Y} \leq c) &= P(\frac{X}{Y} \leq c \quad \text{and} \quad Y > 0) + P(\frac{X}{Y} \leq c \quad \text{and} \quad Y < 0) \\
&= P(\frac{X}{Y} \leq c \quad \text{and} \quad Y > 0) + P(\frac{-X}{Y} \leq c \quad \text{and} \quad Y < 0) \quad (\because X \sim -X \quad \text{and} \quad X \perp Y) \\
&= P(\frac{X}{Y} \leq c \quad \text{and} \quad Y > 0) + P(\frac{X}{-Y} \leq c \quad \text{and} \quad Y < 0) \\
&= P(\frac{X}{|Y|} \leq c) \qquad \text{where c is some constant}
\end{aligned}$$

Thus, we have shown that $\frac{X}{Y}$ and $\frac{X}{|Y|}$ have the same cumulative distribution function, which means that they have the same distribution.

2. *(Bivariate normal distribution I)*

(a) (i) Recall that Cov(X, Y)=0 iff X and Y are independent, when X and Y are bivariate normal.

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$$\begin{aligned}
Cov(W_a, V) &= Cov(12 + aX + Y, \quad 9 + X + 2Y) \\
&= aVarX + 2aCov(X,Y) + Cov(X,Y) + 2VarY \\
&= 9a + (2a+1) \cdot (12\rho) + 32 \qquad (\because \quad Cov(X,Y) = \rho \cdot \sigma_X \cdot \sigma_Y) \\
&= 9a + 4(1+2a) + 32 \\
&= 0 \qquad (\because W_a, Y \text{are independent})
\end{aligned}$$

Then, we solve for a, and have $a = -\frac{36}{17}$.

(ii) From $Cov(W_a, V) = 0$, we have,

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$$9a + (2a+1) \cdot (12\rho) + 32 = 0$$

We solve for a, and have $a = \frac{-(12\rho+32)}{24\rho+9}$. Let $\rho_0 = -\frac{3}{8}$. Then, there doesn't exist $a \in R$ to make W_a and V independent.

(b) (i) Now fix $a = 1$, and find ρ such that W_a and V are independent. From W_a and V being

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independent,

$$\begin{aligned} Cov(W_a, V) &= 9a + (2a+1)\cdot(12\rho) + 32 && (\because \quad part(a)) \\ &= 36\rho + 41 && (\because \quad a = 1) \\ &= 0 && (\because W_a, Y \text{are independent}) \end{aligned}$$

Solve for ρ, and we have $\rho = -\frac{41}{36}$. Note that ρ cannot be smaller than -1, so there does not exist ρ making W_a and Y independent.

(ii) From W_a and V being independent,

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$$\begin{aligned} Cov(W_a, V) &= 9a + (2a+1)\cdot(12\rho) + 32 && (\because \quad part(a)) \\ &= 0 && (\because W_a, Y \text{are independent}) \end{aligned}$$

Solve for ρ and we get $\rho = \frac{-(32+9a)}{12+24a}$. Let $a_0 = -\frac{1}{2}$. Then, there doesn't exist ρ making W_a and V independent. Moreover, for $-\frac{4}{3} \leq a_0 < -\frac{1}{2}$ and $-\frac{1}{2} < a_0 \leq \frac{4}{3}$, we have $\rho \geq 1$ and $\rho \leq -1$ respectively, to have W_a and V independent. Therefore, for $-\frac{4}{3} \leq a_0 \leq \frac{4}{3}$, there doesn't exist a $\rho \in (-1, 1)$ making W_a and V independent.

3. *(Bivariate normal distribution II)*

Let (X, Y) follow a bivariate normal distribution with $EX = \mu_X, EY = \mu_Y$ and $VarX = \sigma_X{}^2, VarY = \sigma_Y{}^2$ and $Corr(X,Y) = \rho$. Then, the joint density for X, Y and the marginal for Y are the following.

$$f_{X,Y} = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \cdot \exp\left[\frac{-1}{2(1-\rho^2)}\left(\left(\frac{X-\mu_X}{\sigma_X}\right)^2 - 2\rho\left(\frac{X-\mu_X}{\sigma_X}\right)\left(\frac{Y-\mu_Y}{\sigma_Y}\right) + \left(\frac{Y-\mu_Y}{\sigma_Y}\right)^2\right)\right]$$

$$f_Y = \frac{1}{\sigma_Y\sqrt{2\pi}} \cdot \exp\left[-\frac{1}{2}\cdot\frac{(Y-\mu_Y)^2}{\sigma_Y{}^2}\right]$$

Thus,

$$\begin{aligned} f_{X|Y}(x|y) &= \frac{f_{X,Y}(x,y)}{f_Y(y)} \\ &= \frac{1}{\sqrt{2\pi}\cdot\sigma_X\cdot\sqrt{1-\rho^2}} \cdot \exp\left[\frac{-1}{2(1-\rho^2)}\left(\left(\frac{X-\mu_X}{\sigma_X}\right)^2\right.\right. \\ &\qquad \left.\left. -2\rho\left(\frac{X-\mu_X}{\sigma_X}\right)\left(\frac{Y-\mu_Y}{\sigma_Y}\right) + \rho^2\cdot\left(\frac{Y-\mu_Y}{\sigma_Y}\right)^2\right)\right] \\ &= \frac{1}{\sqrt{2\pi}\cdot\sigma_X\cdot\sqrt{1-\rho^2}} \cdot \exp\left[-\frac{1}{2}\cdot\frac{\left(\frac{X-\mu_X}{\sigma_X} - \rho\cdot\frac{Y-\mu_Y}{\sigma_Y}\right)^2}{1-\rho^2}\right] \\ &= \frac{1}{\sqrt{2\pi\cdot\sigma_X{}^2\cdot(1-\rho^2)}} \cdot \exp\left[-\frac{1}{2}\cdot\frac{\left(X-\mu_X-\rho\cdot\frac{\sigma_X}{\sigma_Y}\cdot(Y-\mu_Y)\right)^2}{\sigma_X{}^2\cdot(1-\rho^2)}\right] \end{aligned}$$

Note: Can also derrive the conditinal mean and variance for MVN_2

$\mu_{1|2} = \mu_1 + \Sigma_{12}\Sigma_{22}^{-1}(X_2 - \mu_2)$ and use them.

$\Sigma_{1|2} = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{12}'$.

Therefore,

$$X|Y = y \quad \sim \quad N\left(\mu_X + \rho \cdot \frac{\sigma_X}{\sigma_Y} \cdot (y - \mu_Y), \qquad {\sigma_X}^2 \cdot (1 - \rho^2)\right)$$

(a)

$$\begin{aligned} E[X|Y = 8] &= \mu_X + \rho \cdot \frac{\sigma_X}{\sigma_Y} \cdot (8 - \mu_Y) \\ &= 5 + 0.4 \times \frac{3}{4} \cdot (8 - 3) \\ &= 6.5 \end{aligned}$$

(b)

$$\begin{aligned} Var[X|Y = 8] &= (1 - \rho^2) \cdot {\sigma_X}^2 \\ &= (1 - 0.4^2) \cdot 9 \\ &= 7.56 \end{aligned}$$

(c) Recall that $X \sim N(\mu_X, {\sigma_X}^2)$. Then,

$$\begin{aligned} P(3 < X < 5) &= P(\frac{3 - \mu_X}{\sigma_X} \leq z \leq \frac{5 - \mu_X}{\sigma_X}) \\ &= P(-\frac{2}{3} \leq z \leq 0) \\ &= 0.2486 \end{aligned}$$

(d) Given that $X|Y = 8 \quad \sim \quad N(6.5, 7.56)$,

$$\begin{aligned} P(3 < X < 5|Y = 8) &= P(\frac{3 - 6.5}{\sqrt{7.56}} \leq z \leq \frac{5 - 6.5}{\sqrt{7.56}} \quad |Y = 8) \\ &= P(-1.2730 \leq z \leq -0.5455|Y = 8) \\ &= 0.1892 \end{aligned}$$

4. *(Regression fallacy)*

(a) Let $E[S] = \mu_S, E[E_i] = \mu_{E_i}, VarS = {\sigma_S}^2$, and $VarE_i = {\sigma_{E_i}}^2$. Then, X and Y have a bivariate normal distribution in which the means are $EX = \mu_S + \mu_{E_1} = 70$ and $EY = \mu_S + \mu_{E_2} = 70$, and the variances are $VarX = {\sigma_S}^2 + {\sigma_{E_1}}^2 = 74$, $VarY = {\sigma_S}^2 + {\sigma_{E_2}}^2 = 74$, and $Cov(X, Y) = {\sigma_S}^2 = 49$. Because S, E_1 and E_2 are normal and independent, both X and Y are normal, and (X, Y) thus has a bivariate normal distribution with the following joint density.

$$f_{X,Y} = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \cdot \exp\left[\frac{-1}{2(1-\rho^2)}\left(\left(\frac{X-\mu_X}{\sigma_X}\right)^2 - 2\rho\left(\frac{X-\mu_X}{\sigma_X}\right)\left(\frac{Y-\mu_Y}{\sigma_Y}\right) + \left(\frac{Y-\mu_Y}{\sigma_Y}\right)^2\right)\right]$$

where $\mu_X = \mu_Y = 70, {\sigma_X}^2 = {\sigma_Y}^2 = 74,$ and $\rho = \frac{Cov(X,Y)}{\sigma_X \cdot \sigma_Y} = 0.66$

Note: Can also use matrix form of Multivariate Normal to solve this problem.

(b) Recall from Q.3 that

$$X|Y = y \quad \sim \quad N\left(\mu_X + \rho \cdot \frac{\sigma_X}{\sigma_Y} \cdot (y - \mu_Y), \qquad {\sigma_X}^2 \cdot (1 - \rho^2)\right)$$

By symmetry,

$$Y|X = x \quad \sim \quad N\left(\mu_Y + \rho \cdot \frac{\sigma_Y}{\sigma_X} \cdot (x - \mu_X), \qquad {\sigma_Y}^2 \cdot (1 - \rho^2)\right)$$

Given that $X = \mu_X - \sigma_X$,

$$\begin{aligned}
E[Y|X] &= \mu_Y + \rho \cdot \frac{\sigma_Y}{\sigma_X} \cdot (\mu_X - \sigma_X - \mu_X) \\
&= \mu_Y + \rho \cdot \frac{\sigma_Y}{\sigma_X} \cdot (-\sigma_X) \\
&= \mu_Y - \rho \cdot \sigma_Y \\
&= \mu_Y - \frac{{\sigma_S}^2}{\sqrt{{\sigma_S}^2 + {\sigma_{E_1}}^2} \cdot \sqrt{{\sigma_S}^2 + {\sigma_{E_2}}^2}} \cdot \sigma_Y \\
&= 70 - \left(\frac{49}{49 + 25}\right) \cdot \sqrt{49 + 25} \\
&= 64.3
\end{aligned}$$

~~(c) This is what is often called, "the regression toward the mean". When a student's midterm score is really high, the chances are that his/her final score moves back toward the mean. This is due to the characteristic of regression, not because of the praise.~~

5. *(Mean square error when estimationg a normal variance)*

(a). (i) Show that $MSE(\hat{\theta}) = Var\hat{\theta} + (Bias(\hat{\theta}))^2$

$$\begin{aligned}
MSE(\hat{\theta}) &= E[(\hat{\theta} - \theta)^2] \\
&= E[(\hat{\theta} - E\hat{\theta} + E\hat{\theta} - \theta)^2] \\
&= E[(\hat{\theta} - E\hat{\theta})^2 + (E\hat{\theta} - \theta)^2 + 2(\hat{\theta} - E\hat{\theta})(E\hat{\theta} - \theta)] \\
&= E(\hat{\theta} - E\hat{\theta})^2 + (E\hat{\theta} - \theta)^2 + 0 \\
&= Var\hat{\theta} + (Bias(\hat{\theta}))^2
\end{aligned}$$

(ii) Note that $\frac{(n-1)\cdot s^2}{\sigma^2} \sim \chi^2_{n-1}$ Then,

$$Var\left(\frac{(n-1)s^2}{\sigma^2}\right) = 2(n-1)$$

Thus,

$$Var(s^2) = \frac{2\sigma^4}{n-1} \quad \text{and} \quad Bias(s^2) = 0$$

Using $\hat{\sigma}^2 = \frac{n-1}{n} \cdot s^2$,

$$Var(\hat{\sigma}^2) = (\frac{n-1}{n})^2 \cdot \frac{2\sigma^4}{n-1} = \frac{2(n-1)\sigma^4}{n^2} \quad \text{and} \quad Bias(\hat{\sigma}^2) = -\frac{\sigma^2}{n}$$

(iii) Compute the MSE

Given the variances and the biases in (ii),

$$MSE(s^2) = Var(s^2) + (Bias(s^2))^2 = \frac{2\sigma^4}{n-1} + 0^2 = \frac{2\sigma^4}{n-1}$$

$$MSE(\hat{\sigma}^2) = Var(\hat{\sigma}^2) + (Bias(\hat{\sigma}^2))^2 = \frac{2(n-1)\sigma^4}{n^2} + \left(-\frac{\sigma^2}{n}\right)^2 = (\frac{2n-1}{n^2}) \cdot \sigma^4$$

We thus have,

$$MSE(\hat{\sigma}^2) = (\frac{2n-1}{n^2}) \cdot \sigma^4 < (\frac{2}{n-1}) \cdot \sigma^4 = MSE(s^2)$$

6. *(Inference about variance)*

Note that,

$$\frac{1}{\sigma^2} \cdot \sum_{i=1}^{n} (X_i - \overline{X})^2 \sim \chi^2_{n-1}$$

Given $s^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \overline{X})^2$,

$$\frac{(n-1) \cdot s^2}{\sigma^2} \sim \chi^2_{n-1}$$

We find c such that,

$$P\left(s^2 > c \cdot \sigma^2\right) = 0.05$$

$$\Longrightarrow P\left(\frac{(n-1)s^2}{\sigma^2} > c(n-1)\right) = 0.05$$

$$\Longrightarrow P\left(\chi^2_{n-1} > c(n-1)\right) = 0.05$$

$$\Longrightarrow P\left(\chi^2_9 > 9 \cdot c\right) = 0.05 \qquad (\because \quad n = 10)$$

Using R, we know $9 \cdot c = 16.919$, which leads to $c = 1.88$.

5. (b).

$$\tilde{\sigma}^2 = c \cdot \sum_{i=1}^{n} (X_i - \overline{X})^2$$

then:

$$Var(\tilde{\sigma}^2) = 2c^2(n-1)\sigma^4$$

$$Bias^2(\tilde{\sigma}^2) = (cn-c-1)^2 \cdot \sigma^4$$

$$\therefore MSE(\tilde{\sigma}^2) = Var(\tilde{\sigma}^2) + Bias^2($$

$$= \sigma^4[(n^2-1)c^2 + (2-2n)c + 1$$

$$= \sigma^4 \cdot (n^2-1) \cdot [(c - \frac{1}{n+1})^2 + (1 - \frac{1}{(n+1}$$

when $c = \frac{1}{n+1}$,

$MSE(\tilde{\sigma}^2)$ is minimized

Problem 6

(a) Note that

$$\frac{\frac{1}{n}\sum_{i=1}^{n} x_i - \mu}{\mu/\sqrt{n}} \sim N(0,1)$$

So the pivotal for μ is $\sqrt{n}\left(\frac{\bar{x}}{\mu} - 1\right)$.

Remark: Pivotal function for μ is not unique. E.g. $\sum_{i=1}^{n}\left(\frac{x_i - \bar{x}}{\mu}\right)^2$ is also pivotal (with distribution χ^2_{n-1}).

(b)

$$f(\mu|X_1, X_2, ...X_n) = \left(\frac{1}{\mu\sqrt{2\pi}}\right)^n e^{-\sum_{i=1}^{n}\frac{(x_i-\mu)^2}{2\mu^2}},$$

$$l(\mu|X_1, X_2, ...X_n) = -n\log(\mu) - \frac{n}{2}\log(2\pi) - \sum_{i=1}^{n}\frac{(x_i - \mu)^2}{2\mu^2}.$$

The score function is

$$\frac{\partial l}{\partial \mu} = -\frac{n}{\mu} + \sum_{i=1}^{n}\frac{x_i^2}{\mu^3} - \sum_{i=1}^{n}\frac{x_i}{\mu^2}.$$

Set it to zero, we have

$$n\mu^2 + \mu\sum_{i=1}^{n} x_i - \sum_{i=1}^{n} x_i^2 = 0 \qquad \hat{\mu} = \frac{-\sum_{i=1}^{n} X_i + \sqrt{(\sum_{i=1}^{n} X_i)^2 + 4n\sum X}}{2n}$$

So MLE $\hat{\mu}$ is the positive root of the equation above.

To derive the form of g, we try to use Taylor expansion in conjunction with asymptotic normality. If there is a function g such that $\sqrt{n}[g(\hat{\mu}) - g(\mu)] \to N(0,1)$, then by Taylor expansion, we have

$$\sqrt{n}g\prime(\mu)(\hat{\mu} - \mu) \approx \sqrt{n}(g(\hat{\mu}) - g(\mu)) \to N(0,1). \tag{1}$$

On the other hand, by asymptotic normality, we have

$$\sqrt{nI(\mu)}(\hat{\mu} - \mu) \to N(0,1) \tag{2}$$

where

(c)

$$\begin{aligned} I(\mu) &= -E\frac{\partial^2 l}{\partial \mu^2} \\ &= -\frac{1}{\mu^2} + \frac{3}{\mu^4}E(x^2) - \frac{2}{\mu^3}E(x) \\ &= -\frac{1}{\mu^2} + \frac{6}{\mu^2} - \frac{2}{\mu^2} \\ &= \frac{3}{\mu^2} \end{aligned}$$

is fisher information.

Compare LHS of (1) and (2) ,we have

(d)

$$\frac{dg}{du} = \frac{\sqrt{3}}{\mu},$$

which tells us

$$g(\mu) = \sqrt{3}\log \mu.$$

(e) From (a) we know that $\sqrt{n}(\frac{\bar{X}}{\mu} - 1) \sim N(0,1)$, so the CI with level α is

$$\begin{aligned} 1 - \alpha &= P\left(-z_{\alpha/2} < \sqrt{n}(\frac{\bar{X}}{\mu} - 1) < z_{\alpha/2}\right) \\ &= P\left(\frac{\bar{X}^2}{(1 + z_{\alpha/2}/\sqrt{n})^2} < \mu^2 < \frac{\bar{X}^2}{(1 - z_{\alpha/2}/\sqrt{n})^2}\right). \end{aligned}$$

The corresponding length with level $\alpha = 0.05$ is

$$\bar{X}^2\left(\frac{1}{(1 - 1.96/\sqrt{n})^2} - \frac{1}{(1 + 1.96/\sqrt{n})^2}\right). \tag{3}$$

From (b) we know that $\sqrt{3n}\log(\frac{\hat{\mu}}{\mu}) \sim N(0,1)$, so the CI with level α is

$$\begin{aligned} 1 - \alpha &= P\left(-z_{\alpha/2} < \sqrt{3n}\log(\frac{\hat{\mu}}{\mu}) < z_{\alpha/2}\right), \\ &= P\left(\hat{\mu}^2 e^{-2z_{\alpha/2}/\sqrt{3n}} < \mu^2 < \hat{\mu}^2 e^{2z_{\alpha/2}/\sqrt{3n}}\right). \end{aligned}$$

The corresponding length with $\alpha = 0.05$ is

$$\hat{\mu}^2\left(e^{2.26/\sqrt{n}} - e^{-2.26/\sqrt{n}}\right). \tag{4}$$

Compare (3) and (4). By consistency, $\bar{X} \to \mu$ and $\hat{\mu} \to \mu$ as $n \to \infty$. Moreover,

$$\frac{1}{(1 - 1.96/\sqrt{n})^2} - \frac{1}{(1 + 1.96/\sqrt{n})^2} \approx \frac{7.84}{\sqrt{n}},$$

$$e^{2.26/\sqrt{n}} - e^{-2.26/\sqrt{n}} \approx \frac{4.52}{\sqrt{n}}.$$

So the interval from part (b) has shorter length.