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STAT 245
HOMEWORK 0

1. *Moments of Poisson Distribution*

Let X be a random variable with a Poisson distribution. Find $E(X^4)$.

One way to this using the formula

$$E(X^4) = \sum_k x^4 \Pr\{X = k\}.$$

Observe

Stirling numbers of the second kind

$$x^4 = x + 7x(x-1) + 6x(x-1)(x-2) + x(x-1)(x-2)(x-3).$$

We can derive each summation term individually.

$$\begin{aligned}
E(X^4) &= \sum_x x^4 \Pr\{X = x\} \\
&= \sum_x x \Pr\{X = x\} + \sum_x 7x(x-1) \Pr\{X = x\} \\
&+ \sum_x 6x(x-1)(x-2) \Pr\{X = x\} \\
&+ \sum_x x(x-1)(x-2)(x-3) \Pr\{X = x\}.
\end{aligned}$$

$$\sum_x x \Pr\{X = x\} = \lambda$$

$$\begin{aligned}
\sum_x 7x(x-1) \Pr\{X = x\} &= \sum_x 7x(x-1) e^{-\lambda} \frac{\lambda^x}{x!} \\
&= 7e^{-\lambda} \sum_x x(x-1) \frac{\lambda^x}{x(x-1)(x-2)!} \\
&= 7e^{-\lambda} \lambda^2 \sum_x \frac{\lambda^{x-2}}{(x-2)!} \\
&= 7e^{-\lambda} \lambda^2 e^{\lambda} \\
&= 7\lambda^2
\end{aligned}$$

Using a similiar method, find

$$\sum_x 6x(x-1)(x-2) \Pr\{X = x\} = 6\lambda^3$$

and

$$\sum_x x(x-1)(x-2)(x-3) \Pr\{X = x\} = \lambda^4.$$

Combining the results

$$E(X^4) = \lambda + 7\lambda^2 + 6\lambda^3 + \lambda^3.$$

2. *Poisson- and χ^2-tails*

For $\lambda > 0$, let X_λ be a discrete random variable with a Poisson distribution with expected value λ. For (integer) $d \in \mathbb{N}$, let Y_d by a continuous random variable with a χ^2-distribution with d degrees of freedom. In other words, the distribution of Y_d has the probability density function

$$f_{\chi^2_d}(y) = \frac{1}{2^{\frac{d}{2}}\Gamma(\frac{d}{2})} y^{\frac{d}{2}-1} e^{\frac{-y}{2}}, /y \geq 0$$

where $\Gamma(.)$ is the Gamma-function which satisfies $\Gamma(n) = (n-1)!$ for $n \in \mathbb{N}$. Show that for all $\lambda > 0$ and all (integer) $c \in \mathbb{N}$,

$$\Pr\{X_\lambda \geq c+1\} = \Pr\{0 \leq Y_{2(c+1)} \leq 2\lambda\} = \int_0^{2\lambda} f_{\chi^2_{2(c+1)}}(y)dy.$$

First observe

$$\Pr\{X_\lambda \geq c+1\} = 1 - \sum_{k=0}^{c} e^{-\lambda}\frac{\lambda^k}{k!}$$

whose derivative is just Poisson density

$$\frac{d}{d\lambda}\Pr\{X_\lambda \geq c+1\} = \Pr\{X = c\} = e^{-\lambda}\frac{\lambda^c}{c!}.$$

Next observe

$$\Pr\{0 \leq Y_{2(c+1)} \leq 2\lambda\} = \int_0^{2\lambda} \frac{1}{2^{c+1}c!} y^c e^{-\frac{y}{2}}$$

whose derivative is

$$\begin{aligned}\frac{\partial}{\partial\lambda}\Pr\{0 \leq Y_{2(c+1)} \leq 2\lambda\} &= \frac{\partial}{\partial\lambda}\frac{\Gamma(c+1) - \Gamma(c+1,\lambda)}{c!} \\ &= e^{-\lambda}\frac{\lambda^c}{c!}\end{aligned}$$

Hence $\frac{\partial}{\partial\lambda}\Pr\{0 \leq Y_{2(c+1)} \leq 2\lambda\} = \frac{\partial}{\partial\lambda}\Pr\{X_\lambda \geq c+1\}$ which implies

$$\Pr\{X_\lambda \geq c+1\} = \Pr\{0 \leq Y_{2(c+1)} \leq 2\lambda\}.$$

3. *Approximation to Binomial probabilities*

Let X be distributed according to a Binomial(n, p) distribution. We are interested in the probability $\Pr(X = k)$ for

1. $n = 7,\ p = 0.3,\ k = 3;$

The binomial probability

$$\Pr\{X = 3\} = \binom{7}{3}.3^3(1-.3)^4 = .2268.$$

The Normal Distribution approximation with $E(X) = np$ and $\mathrm{Var}(X) = np(1-p)$.

P.187 in Rice has a nice explanation if this

$$\begin{aligned}\Pr\{X \geq 3\} &= \Pr\{\frac{X-2.1}{\sqrt{1.47}} \leq \frac{3-2.1}{\sqrt{1.47}}\} \\ &\approx 1 - \Phi(0.742) \\ &= .249\end{aligned}$$

The Poisson approximation $\lambda = np$.

$$\Pr\{X = 3\} = e^{-2.1}\frac{2.1^3}{3!} = .189$$

2. $n = 40,\ p = 0.4,\ k = 11;$

   Binomial $=$ .0357, Normal approximation $=$ 0.035, and Poisson $=$ 0.0495.

3. $n = 400,\ p = .0025,\ k = 2;$

   Binomial $=$ .18417, Normal approximation $=$ .2419, and Poisson $=$ .1839.

The Poisson is a good approximation when p is small and n is large. The Normal is a good approximation when n is large and p is close to $\frac{1}{2}$, i.e. the binomial distribution is symmetric.

4. *Conditional distributions in Poisson process*

Let $(X_t)_{t\geq 0}$ be a Possion process, and let

$$T_1 = \min\{t > 0 :\ X_t \geq 1\}$$

be the time to the first event.

1. Find the conditional distibution of X_s given $X_t = n$ for fixed time points $t > s > 0$ and integer $n \in \mathbb{N}$.

   The coniditional distributio is given

$$\Pr(X_s \mid X_t = n) = \frac{\Pr(X_s \cap X_t)}{\Pr(X_t = n)}.$$

   Note that

$$\Pr(X_s \cap X_t) = \Pr(X_s = x \cap X_t - X_s = n - x)$$

where X_s and $X_t - X_s$ are independent of each other.

Then

$$
\begin{aligned}
\Pr(X_s \mid X_t = n) &= \frac{e^{-\lambda s}\frac{(\lambda s)^x}{x!} \cdot e^{-\lambda(t-s)}\frac{[\lambda(t-s)]^{n-x}}{(n-x)!}}{e^{-\lambda t}\frac{(\lambda t)^n}{n!}} \\
&= \frac{n!}{x!(n-x)!} \cdot \frac{e^{-\lambda s}e^{-\lambda(t-s)}(\lambda s)^x[(t-s)\lambda]^{n-x}}{e^{-\lambda t}(\lambda t)^n} \\
&= \binom{n}{x}\frac{s^x(t-s)^{n-x}}{t^x \cdot t^{n-x}} \\
&= \binom{n}{x}\left(\frac{s}{t}\right)^x\left(1-\frac{s}{t}\right)^{n-x}
\end{aligned}
$$

2. Show that the conditional distribution of T_1 given $X_t = 1$ is the uniform distribution on the interval $(0, t]$.

   Consider $\Pr\{T_1 > s \mid X_t = 1\}$ for $0 < s < t$.

$$
\begin{aligned}
\Pr\{T_1 > s \mid X_t = 1\} &= \frac{\Pr\{0 \text{ events in } (0,s] \cap 1 \text{ event in } (s,t]\}}{\Pr\{X_t = 1\}} \\
&= \frac{e^{-\lambda s} \cdot e^{-\lambda(t-s)}\lambda(t-s)}{e^{-\lambda t}\lambda t} \\
&= \frac{t-s}{t}
\end{aligned}
$$

   Then

$$\Pr\{T_1 \leq s \mid X_t = 1\} = 1 - \Pr\{T_1 > s \mid X_t = 1\} = \frac{s}{t}.$$

   Taking the deritive of the above equation results in

$$\Pr\{T_1 = s \mid X_t = 1\} = \frac{1}{t}.$$

5. Data from Poisson process

A detector counts the number of particles emmoted from a radioactive source over the couse of 10-second intervals. For 180 such 10-second intervals, the following counts were observed:

Count	# Itervals
0	23
1	77
2	34
3	26
4	13
5	7

This table states, for example, that in 34 of the 10-second intervals a count of 2 was recorded. Sometimes, however, the detector did not function properly and recorded counts over intervals of length 20 seconds. This happened 20 times and recorded counts are

Count	# Itervals
0	2
1	4
2	9
3	5

Assume a Poisson process model for the particle emission process. Let $\lambda > 0$ (time unit = 1 sec.) be the unknown rate of the Poisson process.

1. Formulate an appropriate likelihood function for the described scenario and derive the maximum likelihood estimator for $\hat{\lambda}$ of the rate λ. Compute $\hat{\lambda}$ for the above data.

   First observe
   $$\Pr\{X = x\} = e^{-\lambda}\frac{\lambda^x}{x!}$$
   then
   $$\text{lik}(\lambda) = \prod_{i=1}^{n} e^{-\lambda}\frac{\lambda_i^x}{x_i!}$$
   and the log is
   $$\begin{aligned} l(\lambda) &= \sum_{i=1}^{n}(-\lambda + X_i \log\lambda - \log(X_i) \\ &= -n\lambda + \log\lambda\sum_{i=1}^{n} X_i - \sum_{i=1}^{n}\log X_i!. \end{aligned}$$

   Setting $l'(\lambda)$ to zero gives
   $$l'(\lambda) = -n + \frac{1}{\lambda}\sum i = 1^n X_i = 0$$

$$\hat{\lambda} = \frac{1}{n}\sum_{i=1}^{n} X_i = \overline{X}.$$

That data observes 347 occurances over 2200 seconds which can be use to compute

$$\hat{\lambda} = \frac{347}{2200} = .1577.$$

The above method works, but can be consicely stated and found by deriving $\hat{\lambda}$ beginging with

$$\text{Lik}(\theta) = \prod_{i=1}^{180} e^{-10\lambda}\frac{(10\lambda)^{y_i}}{y_i!} \cdot \prod_{j=1}^{20} e^{-20\lambda}\frac{(20\lambda)^{z_j}}{z_j!}$$

2. What approximation to the distribution of $\hat{\lambda}$ does the central limit theorem suggest?

Explanation of this in Rice p.262

Let

$$S = X_1 + X_2$$

and $\hat{\lambda} = \frac{s}{n}$ is a random variable.

$$\Pr\{\hat{\lambda} = v\} = \Pr\{s = nv\} = e^{-n\lambda_0}\frac{(n\lambda_0)^{nv}}{(nv)!}$$

for v such that nv is a nonnegative integer.

Since $S \sim \text{Pois}(n\lambda_0)$

$$E(\hat{\lambda}) = frac1nE(S) = \lambda_0$$

and

$$\text{Var}(\hat{\lambda}) = \frac{1}{n^2}\text{Var}(S) = \frac{\lambda_0}{n}.$$

Since $E(\hat{\lambda}) = \lambda_0$, $\hat{\lambda}$ is unbiased and centered at λ_0 with standard error

$$\sigma_{\hat{\lambda}} = \sqrt{\frac{\lambda_0}{n}}.$$

The standard error can be estimated

$$s_{\hat{\lambda}} = \sqrt{\frac{\hat{\lambda}}{n}}$$

therefore $\hat{\lambda} \sim N(\lambda_0, s_{\hat{\lambda}})$.