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STAT 245
HOMEWORK 5

8. Common Variance, different mean

Consider $X_1, ..., X_n \sim N(\mu_1, \sigma^2)$ and $X_{n+1}, ..., X_{2n} \sim N(\mu_2, \sigma^2)$. Everything is independent here.

1. Fine the MLE of σ^2, denoted $\hat{\sigma}^2$.

   The likelihood function $L(\sigma^2)$ is

$$L(\sigma^2) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp[\frac{-(x_i - \mu_1)^2}{2\sigma^2}] \prod_{i=n+1}^{2n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp[\frac{-(x_i - \mu_2)^2}{2\sigma^2}]$$

   after some organization is equivently

$$L(\sigma^2) = \left(\frac{1}{2\pi\sigma^2}\right)^n \exp[-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i - \mu_1)^2 - \frac{1}{2\sigma^2}\sum_{i=n+1}^{2n}(x_i - \mu_2)^2].$$

   Taking the log,

$$l(\sigma^2) = -\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i - \mu_1)^2 - \frac{1}{2\sigma^2}\sum_{i=n+1}^{2n}(x_i - \mu_2)^2 + n\log(2\pi\sigma^2)$$

   setting $l(\sigma^2) = 0$

$$\hat{\sigma}^2 = \frac{1}{2n}[\sum_{i=1}^{n}(x_i - \hat{\mu}_1)^2 + \sum_{i=n+1}^{2n}(x_i - \hat{\mu}_2)^2]$$

2. Based on $\hat{\sigma}^2$, construct and exact confidence interval of σ^2.

   We know that

$$\frac{\sum_{i=1}^{n}(x_i - \hat{\mu}_1)^2}{\sigma^2} \sim \chi^2_{n-1}$$

   and

$$\frac{\sum_{i=n+1}^{2n}(x_i - \hat{\mu}_2)^2}{\sigma^2} \sim \chi^2_{n-1}.$$

   These facts imply

$$\frac{2n\hat{\sigma}^2}{\sigma^2} \sim \chi^2_{2n-2}$$

   which is a pivotal and can be used to construct the confidence interval.

9. Poisson Regression

Consider $y_i \sim \text{Poisson}(\beta_1 x_1)$ independetly for $i = 1, ..., n$.

1. What is the distribution of $\sum_{i=1}^n y_i$?

   In Stat 244, we've shown

$$\sum_{i=1}^n y_i \sim \text{Poisson}(\beta_1 \sum_{i=1}^n x_i).$$

2. Consider the estimator $\hat{\beta}_1 = \frac{\bar{y}}{\bar{x}}$, find $\mathbb{E}(\hat{\beta}_1)$ and $\text{Var}(\hat{\beta}_1)$.

$$\mathbb{E}(\hat{\beta}_1) = \frac{\mathbb{E}(\sum_{i=1}^n y_i)}{\sum_{i=1}^n x_i} = \frac{\beta_1 \sum_{i=1}^n x_i}{\sum_{i=1}^n x_i} = \beta_1$$

$$\begin{aligned}\text{Var}(\hat{\beta}_1) &= \frac{\text{Var}(\sum_{i=1}^n y_i)}{(\sum_{i=1}^n x_i)^2} \\ &= \frac{\beta_1 \sum_{i=1}^n x_i}{(\sum_{i=1}^n x_i)^2} = \frac{\beta_1}{\sum_{i=1}^n x_i}\end{aligned}$$

3. Find the asymptotic distribution of $\sqrt{n}(\hat{\beta}_1 - \beta_1)$. (Hint: it's normal just find the mean and variance).

   First recall

$$\frac{\hat{\beta}_1 - \beta_1}{\sqrt{\frac{\beta_1}{\sum_{i=1}^n x_i}}} \rightsquigarrow N(0, 1)$$

   which means

$$\sqrt{n}(\hat{\beta}_1 - \beta_1) = \sqrt{\frac{n\beta_1}{\sum_{i=1}^n x_i}} \frac{\hat{\beta}_1 - \beta_1}{\sqrt{\frac{\beta_1}{\sum_{i=1}^n x_i}}} \rightsquigarrow N(0, \frac{\beta_1}{\bar{x}}).$$

4. Find a transformation g such that the asymptotic distribution of $\sqrt{n}(g(\hat{\beta}_1) - g(\beta_1))$ does not depend on β_1.

   The delta method tells us that

$$\sqrt{n}(g(\hat{\beta}_1) - g(\beta_1)) \rightsquigarrow N(0, |g'(\beta_1)|^2 \frac{\beta_1}{\bar{x}})$$

   therefore we want $|g'(\beta_1)|^2 = 1$. This implies we should let

$$g'(\beta_1) = \frac{1}{\sqrt{\beta_1}}.$$

   Then

$$g(\beta_1) = \int g'(\beta_1) \propto \sqrt{\beta_1}$$

5. Is $\hat{\beta_1}$ the MLE?

   A a log likelihood transformation set equal to 0 will conlude the answer is YES.