Grading Reference

STAT 245, Spring 2014

Jakub Tucholski

Homework 0

Total: 100 pt

(20) 1 Question 1: Moments of Poisson Distribution

First, we should find the value of $\mathrm{E}(X^2)$. We can do this by calculating $\mathrm{Var}(X)$:

$$\mathrm{Var}(X) = \mathrm{E}(X^2) - E(X)^2$$

$$\mathrm{Var}(X) = \sum_{k=0}^{\infty} k^2 e^{-\lambda} \frac{\lambda^k}{k!} - \lambda^2$$

Now we can work some algebra magic:

$$\mathrm{E}(X^2) - \lambda = \mathrm{E}(X(X-1)) = \sum_{k=0}^{\infty} k(k-1) e^{-\lambda} \frac{\lambda^k}{k!}$$

$$\mathrm{E}(X^2) - \lambda = e^{-\lambda}\lambda^2 \sum_{k=2}^{\infty} \frac{\lambda^{k-2}}{(k-2)!}$$

$$\mathrm{E}(X^2) = \lambda^2 + \lambda$$

$$\mathrm{Var}(X) = \mathrm{E}(X^2) - E(X)^2 = \lambda + \lambda^2 - \lambda^2 = \lambda$$

Now, we can use this to calculate $\mathrm{E}(X^3)$:

$$\mathrm{E}(X^3) - 3\mathrm{E}(X^2) + 2\mathrm{E}(X) = \mathrm{E}(X^3 - 3x^2 + 2x) = \mathrm{E}[(x^2 - x)(x-2)] = \mathrm{E}[(x)(x-1)(x-2)]$$

$$\mathrm{E}[(x)(x-1)(x-2)] = \sum_{k=0}^{\infty} k(k-1)(k-2) e^{-\lambda} \frac{\lambda^k}{k!}$$

$$\mathrm{E}[(x)(x-1)(x-2)] = e^{-\lambda}\lambda^3 \sum_{k=3}^{\infty} \frac{\lambda^{k-3}}{(k-3)!}$$

$$\mathrm{E}[(x)(x-1)(x-2)] = \lambda^3$$

$$\mathrm{E}[(x)(x-1)(x-2)] = \mathrm{E}(X^3) - 3\mathrm{E}(X^2) + 2\mathrm{E}(X) = \lambda^3$$

$$\mathrm{E}(X^3) - 3(\lambda^2 - \lambda) + 2\lambda = \lambda^3$$

$$\mathrm{E}(X^3) = \lambda^3 + 3\lambda^2 + \lambda$$

Let's do the same for $\mathrm{E}(X^4)$:

$$\mathrm{E}(X^4) - 6\mathrm{E}(X^3) + 11\mathrm{E}(X^2) - 6\mathrm{E}(X) = \mathrm{E}[x(x-1)(x-2)(x-3)]$$

$$\mathrm{E}[x(x-1)(x-2)(x-3)] = \sum_{k=0}^{\infty} k(k-1)(k-2)(k-3) e^{-\lambda} \frac{\lambda^k}{k!}$$

$$\mathrm{E}[x(x-1)(x-2)(x-3)] = \lambda^4 e^{-\lambda} \sum_{k=4}^{\infty} \frac{\lambda^{k-4}}{(k-4)!} = \lambda^4 e^{-\lambda} e^{\lambda}$$

$$\mathrm{E}[x(x-1)(x-2)(x-3)] = \lambda^4$$

$$\mathrm{E}(X^4) = \lambda^4 + 6\lambda^3 + 7\lambda^2 + \lambda$$

(20)

2 Question 2: Poisson and χ^2-Tails

We want to show that:

$$P(X_\lambda \geq c+1) = P(0 \leq Y_{2(c+1)} \leq 2\lambda) = \int_0^{2\lambda} f_{\chi^2_{2(c+1)}}(y)dy$$

The last two terms are trivially equal. Let's first work out the left side:

$$P(X_\lambda \geq c+1) = 1 - P(X_\lambda \leqslant c) = 1 - \sum_{i=0}^{c} \frac{\lambda^i e^{-\lambda}}{i!}$$

Now let's take the derivative with respect to λ: $\frac{d}{d\lambda} P(X_\lambda \geq c+1) = e^{-\lambda} - \sum_{i=1}^{c} e^{-\lambda} \cdot \lambda^{i-1} \cdot \frac{i-\lambda}{i!}$

[redacted] $= e^{-\lambda} - e^{-\lambda}\left(1 - \frac{\lambda^c}{c!}\right) = e^{-\lambda} \cdot \frac{\lambda^c}{c!}$

$= 2 f_{\chi^2_{2(c+1)}}(2\lambda) = \frac{d}{d\lambda} \int_0^{2\lambda} f_{\chi^2_{2(c+1)}}(y)dy$

(20)

3 Question 3: Approximations to Binomial Probabilities

for $\lambda = 0$.

$P(X_\lambda \geq c+1) = 0$.

$P(0 \leq Y_{2(c+1)} \leq 2\lambda) = 0$.

so $P(X_\lambda \geq c+1) = P(0 \leq Y_{2(c+1)} \leq 2\lambda)$

3.1 Part A

$$n = 7, p = 0.3, k = 3$$

$$P(k=3) = \binom{7}{3} 0.3^3 0.7^4 = 0.2269$$

Normal Approximation:

$$Z = \frac{k - np}{\sqrt{np(1-p)}}$$

$$\cancel{Z = \frac{3-(7)(0.3)}{\sqrt{(7)(0.3)(0.7)}} = 0.7423}$$

$$\cancel{P(Z \leq 0.7423) = 0.7718}$$

$Z_1 = \frac{2.5 - 7 \times 0.3}{\sqrt{7 \times 0.3 \times 0.7}} = 0.3288$

$Z_2 = \frac{3.5 - 7 \times 0.3}{\sqrt{7 \times 0.3 \times 0.7}} = 1.1547$

$P(Z_1 \leq Z \leq Z_2) = 0.2466$

Poisson Approximation:

$$P(X) \approx \frac{e^{-np}(np)^k}{k!}$$

$$P(3) \approx \frac{e^{-7(0.3)}(7(0.3))^3}{3!}$$

$$P(3) \approx 0.1890$$

3.2 Part B

$$n = 40, p = 0.4, k = 11$$

$$P(k=11) = \binom{40}{11} 0.4^{11} 0.6^{29} = 0.03573$$

Normal Approximation:

$$Z = \frac{k - \mu}{\sigma}$$

$$Z = \frac{k - np}{\sqrt{np(1-p)}}$$

$$Z = \frac{11 - (40)(0.4)}{\sqrt{(40)(0.4)(0.6)}} = -1.6137$$
$$P(Z \leq -1.6137) = 0.0533$$

$$z_1 = \frac{10.5 - 40 \times 0.4}{\sqrt{40 \times 0.4 \times 0.6}} = -1.7751$$
$$z_2 = \frac{11.5 - 40 \times 0.4}{\sqrt{40 \times 0.4 \times 0.6}} = -1.4523$$
$$P(z_1 \leq z \leq z_2) = 0.0353$$

Poisson Approximation:

$$P(X) \approx \frac{e^{-np}(np)^k}{k!}$$
$$P(11) \approx \frac{e^{-40(0.4)}(40(0.4))^{11}}{11!}$$
$$P(11) \approx 0.0496$$

3.3 Part C

$$n = 400, p = 0.0025, k = 2$$
$$P(k = 2) = \binom{400}{2} 0.0025^2 0.9975^{398} = 0.1842$$

Normal Approximation:

$$Z = \frac{k - np}{\sqrt{np(1-p)}}$$
$$Z = \frac{2 - (400)(0.0025)}{\sqrt{(400)(0.0025)(0.9975)}} = 1.0012$$
$$P(Z \leq 1.0012) = 0.8416$$

$$z_1 = \frac{1.5 - 400 \times 0.0025}{\sqrt{400 \times 0.0025 \times 0.9975}} = 0.5006$$
$$z_2 = \frac{2.5 - 400 \times 0.0025}{\sqrt{400 \times 0.0025 \times 0.9975}} = 1.5019$$
$$P(z_1 \leq z \leq z_2) = 0.2418$$

Poisson Approximation:

$$P(X) \approx \frac{e^{-np}(np)^k}{k!}$$
$$P(11) \approx \frac{e^{-400(0.0025)}(400(0.0025))^2}{2!}$$
$$P(11) \approx 0.1839$$

(20) 4 Question 4: Conditional Distributions in Poisson Process

4.1 Part A

We're going to use the fact that a Poisson process has a constant rate of emission on equilength intervals. Let's assume that $X \sim \text{Poisson}(\lambda)$ for a time unit of 1, and use Bayes Rule to set the problem up

$$P(X_s = k | X_t = n) = \frac{P(X_t = n \mid X_s = k)P(X_s = k)}{P(X_t = n)}$$

For the first term in the numerator, since $t > s > 0$, we know that:

$$P(X_t = n \mid X_s = k) = P(X_{t-s} = n - k) = \frac{e^{-(t-s)\lambda}((t-s)\lambda)^{n-k}}{(n-k)!}$$

$$P(X_s = k) = \frac{e^{-s\lambda}(s\lambda)^k}{k!}$$
$$P(X_t = n) = \frac{e^{-t\lambda}(t\lambda)^n}{n!}$$

Now we can plug this all in and combine it:

$$P(X_s = k|X_t = n) = \frac{\left(\frac{e^{-(t-s)\lambda}((t-s)\lambda)^{n-k}}{(n-k)!}\right)\left(\frac{e^{-s\lambda}(s\lambda)^k}{k!}\right)}{\frac{e^{-t\lambda}(t\lambda)^n}{n!}}$$

$$P(X_s = k|X_t = n) = \binom{n}{k}\left(\frac{((t-s)\lambda)^{n-k}(s\lambda)^k}{(t\lambda)^n}\right)$$

$$P(X_s = k|X_t = n) = \binom{n}{k}\left(\frac{(t-s)^{n-k}s^k}{t^n}\right)$$

4.2 Part B

$$P(T_1 \leq s \mid X_t = 1) = P(X_s = 1, X_t - X_s = 0 \mid X_t = 1) = \frac{P(X_s = 1, X_t - X_s = 0)}{P(X_t = 1)} = \frac{P(X_s = 1)P(X_t - X_s = 0)}{P(X_t = 1)}$$

Now we can use the Poisson distribution to get:

$$P(T_1 \leq s \mid X_1 = 1) = \frac{e^{-rs}se^{-r(t-s)}}{e^{-rt}t} = \frac{s}{t},\ 0 < s \leq t$$

(20) 5 Question 5: Data from Poisson Process

Table 1: 10 Second Counts

Count	# intervals
0	23
1	77
2	34
3	26
4	13
5	7

Table 2: 20 Second Counts

Count	# intervals
0	2
1	4
2	9
3	5

(a). Given this data, we can effectively construct a maximum likelihood estimator for $\hat{\lambda}$:

$$L = \prod_{i=1}^{180} \frac{e^{-10\lambda}(10\lambda)^{y_i}}{y_i!} \times \prod_{j=1}^{20} \frac{e^{-20\lambda}(20\lambda)^{z_j}}{z_j!}$$

Now we transform this into a loglikelihood function:

$$\ln L = -180(10\lambda) + \sum_{i=1}^{180} y_i \ln(10\lambda) - \sum_{j=1}^{180} \ln(y_i!) + -20(20\lambda) + \sum_{j=1}^{20} z_j \ln(20\lambda) - \sum_{j=1}^{20} \ln(z_j!)$$

$$\ln L = -180(10\lambda)+\sum_{i=1}^{180} y_i \ln(10)+\sum_{i=1}^{180} y_i \ln(\lambda)-\sum_{j=1}^{180} \ln(y_i!)+-20(20\lambda)+\sum_{j=1}^{20} z_j \ln(20)+\sum_{j=1}^{20} z_j \ln(\lambda)-\sum_{j=1}^{20} \ln(z_j!)$$

Now we can differentiate the equation with respect to λ, set the derivative to 0 and solve for $\hat{\lambda}$:

$$\frac{\partial \ln L}{\partial \lambda} = -180(10) + \frac{1}{\lambda}\sum_{i=1}^{180} y_i - 20(20) + \frac{1}{\lambda}\sum_{j=1}^{20} z_j$$

$$2200\ [redacted] = \frac{1}{\lambda}\sum_{i=1}^{180} y_i + \frac{1}{\lambda}\sum_{j=1}^{20} z_j \qquad \hat{\lambda} = \frac{\sum_{i=1}^{180} y_i + \sum_{j=1}^{20} z_j}{2200}$$

$$3800 = \frac{1}{\lambda}[23(0) + 77(1) + 34(2) + 26(3) + 13(4) + 7(5)] + \frac{1}{\lambda}[2(0) + 4(1) + 9(2) + 5(3)]$$

$$3800 = \frac{310}{\lambda} + \frac{37}{\lambda} = \frac{347}{\lambda}$$

$$\cancel{\lambda = \frac{347}{3800}} \qquad \hat{\lambda} = \frac{347}{2200} = 0.1577$$

$$\cancel{\lambda = 0.09132}$$

for the above data.

(b). We know $y_i \sim \text{Poisson}(10\lambda)$ $1 \le i \le 180$ and $z_j \sim \text{Poisson}(20\lambda)$ $1 \le j \le 20$. Moreover, y_i's, z_j's are independent.

Thus $\sum_{i=1}^{180} y_i + \sum_{j=1}^{20} z_j \sim \text{Poisson}(180 \times 10 + 20 \times 20\lambda) = \text{Poisson}(2200\lambda)$

Hence $\hat{\lambda} \sim \frac{\text{Poisson}(2200\lambda)}{2200}$

By CLT, $\hat{\lambda}$ approximately follows [redacted] $N(\lambda, \frac{\lambda}{2200})$.