

Homework 3 (due Feb 9)

In your solution include your name and the homework number. Please staple your pages together. When solving the problems below, give detailed derivations in order to receive credit. Also, you may need to use **R** for certain problems.

1. Let X_1, \dots, X_n be independent $N(\mu, \sigma^2)$ random variables.
 - (a) Determine random variables $L_{\sigma^2}(\alpha)$ and $U_{\sigma^2}(\alpha)$ such that the interval $[L_{\sigma^2}(\alpha), U_{\sigma^2}(\alpha)]$ is a $(1 - \alpha)100\%$ confidence interval for σ^2 . In doing this ensure that

$$\mathbb{P}(L_{\sigma^2}(\alpha) > \sigma^2) = \mathbb{P}(U_{\sigma^2}(\alpha) < \sigma^2).$$

- (b) Find a $(1 - \alpha)100\%$ confidence interval for μ using t-statistic.
 - (c) What is the probability that both $\mu \in [L_\mu(\alpha), U_\mu(\alpha)]$ and $\sigma^2 \in [L_{\sigma^2}(\alpha), U_{\sigma^2}(\alpha)]$?
 - (d) Certainly there exist many confidence intervals for σ^2 , and the answer in (a) is one of them. Find the confidence interval which has minimum length.
2. Consider i.i.d. observations $X_1, \dots, X_n \sim p_\theta$. One needs to test

$$H_0 : \theta = \theta_0, \quad H_1 : \theta = \theta_1.$$

The likelihood ratio test is

$$\mathbb{I} \left\{ \prod_{i=1}^n \frac{p_{\theta_1}(X_i)}{p_{\theta_0}(X_i)} > C \right\},$$

where C is some number such that the Type-I error is α .

- (a) Find the α -level likelihood ratio test for $H_0 : \theta = 5$ vs $H_1 : \theta = 10$ with observations $X_1, \dots, X_n \sim N(\theta, 1)$.
 - (b) What is the power of the test when $n = 10, 50, 100$?
 - (c) Find the α -level likelihood ratio test for $H_0 : \theta = 0$ vs $H_1 : \theta = n^{-1/2}$ with observations $X_1, \dots, X_n \sim N(\theta, 1)$. What is the power of the test when $n = 10, 50, 100$?
 - (d) Discuss your discovery.
3. Consider i.i.d. observations $X_1, \dots, X_n \sim N(\mu, \sigma^2)$. The mean μ is unknown. Construct an α -level test for $H_0 : \sigma = \sigma_0^2$ vs $H_1 : \sigma \neq \sigma_0$.
4. Problem 1 on Page 362.