Statistics 245: Homework 2 due April 18

When solving the problems below as well as future homework problems, give detailed derivations and arguments in order to receive credit. In your solution do not forget to include your name and the homework number. Please staple your pages together.

1. *(Cauchy distribution)* Let X and Y be two independent $N(0,1)$ random variables. Show that the distribution of X/Y is the same as that of $X/|Y| = X/\sqrt{Y^2}$. This means that X/Y has a t_1-distribution which is also known as Cauchy-distribution. (Hint: What is the distribution of $-X$?)

2. *(Bivariate normal distribution I)* Suppose (X, Y) has a bivariate normal distribution with expected values $\mathbb{E}[X] = 3$ and $\mathbb{E}[Y] = 1$, variances $\mathrm{var[X]} = 9$ and $\mathrm{var[Y]} = 16$, and correlation ρ. Let $W_a = 12 + aX + Y$ and $V = 19 + X + 2Y$.

   (a) Fix $\rho = 1/3$ and find $a \in \mathbb{R}$ such that W_a and V are independent. Can you choose $\rho_0 \in (-1, 1)$ such that there does not exist an $a \in \mathbb{R}$ making W_a and V independent? If yes, find all such ρ_0. If no, explain why not.

   (b) Now fix $a = 1$ and find ρ such that W_a and V are independent. Can you choose $a_0 \in \mathbb{R}$ such that there does not exist a $\rho \in (-1, 1)$ making W_{a_0} and V independent? If yes, find all such a_0. If no, explain why not.

3. *(Bivariate normal distribution II)* Let (X, Y) follow a bivariate normal distribution with $\mathbb{E}[X] = 5$ and $\mathbb{E}[Y] = 3$, variances $\mathrm{var[X]} = 9$ and $\mathrm{var[Y]} = 16$, and correlation $\rho = 0.4$. Find

   (a) the conditional expectation $\mathbb{E}[X \mid Y = 8]$,

   (b) the conditional variance $\mathrm{var[X \mid Y = 8]}$,

   (c) the probability $\mathbb{P}(3 < X < 5)$,

   (d) the conditional probability $\mathbb{P}(3 < X < 5 \mid Y = 8)$.

4. Let X and Y be the scores of a Stat 245 student on midterm and final exam. We model these scores as
$$X = S + E_1, \qquad Y = S + E_2,$$
where S, E_1, E_2 are independent random variables distributed as $S \sim N(70, 49)$, $E_1, E_2 \sim N(0, 25)$. We think of S as a "skill" part of the score and E_1, E_2 as "luck" components.

   (a) What is the joint distribution of (X, Y)?

   (b) Assume that a student received a midterm score that is one standard deviation below the midterm mean. What do you expect his/her final score to be? (Hint: find $\mathbb{E}(Y|X)$.)

5. *(Mean square error when estimating a normal variance)* Let $X_1, \ldots, X_n$ be independent $N(\mu, \sigma^2)$ random variables. Let $\bar{X} = \frac{1}{n}\sum_i X_i$ be the sample mean. Consider two estimators of σ^2, namely the sample variance

$$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2, \tag{1}$$

and the MLE

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2.$$

The mean square error (MSE) measures how far on average these estimators are away from the "target" σ^2, where "away" is measured in squared distance. The two MSE are defined as

$$\mathrm{MSE}(s^2) = \mathbb{E}[(s^2 - \sigma^2)^2] \qquad \text{and} \qquad \mathrm{MSE}(\hat{\sigma}^2) = \mathbb{E}[(\hat{\sigma}^2 - \sigma^2)^2].$$

(a) Compute and compare $\mathrm{MSE}(s^2)$ and $\mathrm{MSE}(\hat{\sigma}^2)$.

(b) Consider a general form of estimator

$$\tilde{\sigma}^2 = c\sum_{i=1}^{n}(X_i - \bar{X})^2.$$

Find the best c such that $\mathrm{MSE}(\tilde{\sigma}^2) = \mathbb{E}[(\tilde{\sigma}^2 - \sigma^2)^2]$ is minimized.

6. Let $X_1, \ldots, X_n$ be iid distributed as $N(\mu, \mu^2)$, where $\mu \in \mathbb{R}$ is an unknown parameter.

(a) Find pivotal(s) for μ.

(b) Find the MLE $\hat{\mu}$ of μ.

(c) The asymptotic distribution of MLE is given by the formula $\sqrt{n}(\hat{\mu} - \mu) \Rightarrow N(0, I(\mu)^{-1})$, where $I(\mu)$ is called Fisher information. Calculate $I(\mu)$ with the formula

$$I(\mu) = \int p_\mu(x)\left(\frac{\partial}{\partial \mu}\log p_\mu(x)\right)^2 dx,$$

where p_μ is the density of $N(\mu, \mu^2)$.

(d) Find a function g such that $\sqrt{n}[g(\hat{\mu}) - g(\mu)] \Rightarrow N(0, 1)$.

(e) Comment on confidence intervals for μ^2 constructed based on (a) and (b). Which one has smaller length?