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STAT 245 HOMEWORK 3

Confidence intervals

Let $X_1, ..., X_n$ be independent $N(\mu, \sigma^2)$ random variables.

(a) Determine random variables $L_{\sigma^2}(\alpha)$ and $U_{\sigma^2}(\alpha)$ such that the interval $[L_{\sigma^2}(\alpha), U_{\sigma^2}(\alpha)$ is a $(1-\alpha)$ confidence interval for σ^2. In doing this ensure that

$$\Pr(L_{\sigma^2} > \sigma^2) = \Pr(U_{\sigma^2} < \sigma^2)$$

Consider two estimators for σ^2

$$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2$$

and

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X})^2.$$

Since

$$\frac{(n-1)s^2}{\sigma^2} \sim \chi^2_{n-1}.$$

then

$$\frac{n\hat{\sigma}^2}{\sigma^2} \sim \chi^2_{n-1}$$

and

$$\Pr(\chi^2_{n-1}(1-\alpha/2) \leq \frac{n\hat{\sigma}^2}{\sigma^2} \leq \chi^2_{n-1}(\alpha/2)) = 1-\alpha$$

where $\chi^2_m(\alpha)$ denotes the point beyond which the chi-squared distribution with m degrees of freedom has probabilty α. With some manipulation

$$\Pr\left(\frac{n\hat{\sigma}^2}{\chi^2_{n-1}(\alpha/2)} \leq \sigma^2 \leq \frac{n\hat{\sigma}^2}{\chi^2_{n-1}(1-\alpha/2)}\right) = 1-\alpha$$

hence the interval for σ^2 is

$$\left[\frac{n\hat{\sigma}^2}{\chi^2_{n-1}(\alpha/2)}, \frac{n\hat{\sigma}^2}{\chi^2_{n-1}(1-\alpha/2)}\right].$$

(b) Find $(1-\alpha)100\%$ confidence interval for μ using t-statistic.

We can assume that $\hat{\mu} \sim N(\mu, \frac{s^2}{n})$, approximately from CLT. Since

$$\frac{\sqrt{n}(\overline{X} - \mu)}{s} \sim t_{n-1}$$

where s^2 is as defined in part (a) then we can build an interval for $\overline{X}$ without knowing σ^2. Let $t_m(\alpha)$ denote that point beyond which the t-distribution with m degrees of freedom has probabilty α. By symmetry of t

$$\Pr(-t_{n-1}(\alpha/2) \leq \frac{\sqrt{n}(\overline{X}-\mu)}{s} \leq t_{n-1}(\alpha/2)) = 1-\alpha$$

which can be manipulated to

$$\Pr(\overline{X} - \frac{s}{\sqrt{n}}t_{n-1}(\alpha/2) \leq \mu \leq \overline{X} + \frac{s}{\sqrt{n}}t_{n-1}(\alpha/2)) = 1-\alpha$$

forming the interval

$$\left[\overline{X} \pm \frac{s}{\sqrt{n}}t_{n-1}(\alpha/2)\right].$$

(c) What is the probability that both $\mu \in [L_\mu(\alpha), U_\mu(\alpha)]$ and $\sigma^2 \in [L_{\sigma^2}(\alpha), U_{\sigma^2}(\alpha)]$.

Since s^2 appears in both pairs of endpoints, there is no reason to think that the intervals are independent. The event that $\mu \in [L_\mu(\alpha), U_\mu(\alpha)]$ and $\sigma^2 \in [L_{\sigma^2}(\alpha), U_{\sigma^2}(\alpha)]$ can be represented as the region in $\overline{X} - s$ plane bound by the following lines

$$\begin{aligned}
L_1 &: s = \sqrt{\sigma^2\chi^2_{n-1}(\alpha/2)/(n-1)} \\
L_2 &: s = \sqrt{\sigma^2\chi^2_{n-1}(1-\alpha/2)/(n-1)} \\
L_3 &: \overline{X} = \mu + \frac{s}{\sqrt{n}}t_{n-1}(\alpha/2) \\
L_4 &: \overline{X} = \mu + \frac{s}{\sqrt{n}}t_{n-1}(1-\alpha/2).
\end{aligned}$$

Then

$$\begin{aligned}
&\Pr(\mu \in [L_\mu(\alpha), U_\mu(\alpha)], \sigma^2 \in [L_{\sigma^2}(\alpha), U_{\sigma^2}(\alpha)]) \\
&\quad = 1 - \Pr(\mu \notin [L_\mu(\alpha), U_\mu(\alpha)] \text{ or } \sigma^2 \notin [L_{\sigma^2}(\alpha), U_{\sigma^2}(\alpha)]) \\
&\quad \geq 1 - \Pr(\mu \in [L_\mu(\alpha), U_\mu(\alpha)]) - \Pr(\sigma^2 \in [L_{\sigma^2}(\alpha), U_{\sigma^2}(\alpha)]) \\
&\quad \geq 1 - 2\alpha
\end{aligned}$$

(d) Certainly there exist many confidence intervals for σ^2, and the answer in (a) is one of them. Find the confidence interval which has minimum length.

Since the Chi-squared distribution is not symmetric for small values of m, you need find points on the distribution a, b such that $f(a) = f(b)$ and $F(b) - F(a) = 0$. There is some R code in the directory that will solve if degress of freedom are provided.

To obtain a different confidence interval choose $0 < a < b$ such that

$$\Pr(a < \frac{(n-1)s^2}{\sigma^2} < b) = 1 - \alpha \to (*).$$

Therefore

$$\Big(\frac{(n-1)s^2}{b}, \frac{(n-1)s^2}{a}\Big)$$

is a $100(1-\alpha)\%$ confidence interval for σ^2 with length of

$$(n-1)s^2(\frac{1}{a} - \frac{1}{b}).$$

We need to find a choice a and b that minimizes $\frac{1}{a} - \frac{1}{b}$ subject to the constraint $(*)$. Since $(*)$ determines b as a function of a we can use implicit differentiation:

$$\frac{d}{da}\int_a^b f_{n-1}(x)dx = \frac{d}{da}(1-\alpha)$$

where f_{n-1} is the density χ^2_{n-1}. Then

$$-f_{n-1}(a) + f_{n-1}(b)\frac{db}{da} = 0 \Rightarrow \frac{db}{da} = \frac{f_{n-1}(a)}{f_{n-1}(b)}.$$

Now

$$\frac{d}{da}(\frac{1}{a} - \frac{1}{b}) = -\frac{1}{a^2} + \frac{1}{b^2}\frac{f_{n-1}(a)}{f_{n-1}(b)}$$

which equals 0 when $a^2 f_{n-1}(a) = b^2 f_{n-1}(b)$. This result can be verified as a minumum thus an interval of mininum length when a, b such that

$$a^2 f_{n-1}(a) = b^2 f_{n-1}(b)$$

2. *Likelihood ratio test*

Consider i.i.d. observations $X_1, ..., X_n \sim p_\theta$. One needs to test

$$H_0: \ \theta = \theta_0 \ , \ H_1: \ \theta = \theta_1.$$

The likelihood ratio test is

$$\mathbb{I}\Big\{\prod_{i=1}^n \frac{p_{\theta_1}(X_i)}{p_{\theta_0}(X_i)} > C\Big\},$$

where C is some number such that the Type-I error is α.

(a) Find the α-level likelihood ratio test for $H_0: \ \theta = 5$ vs $H_1: \ \theta = 10$ with observations $X_1, ..., X_n simN(\theta, 1)$.

Reject H_0 when $\overline{X} > C$, e.g. large values of $\overline{X}$ should be considered evidence for H_1.

Stigler 6-6 derives where this idea comes from.

Let

$$\begin{aligned}\alpha &= \Pr(\overline{X} > C \mid H_0) \\ &= \Pr(\frac{\sqrt{n}(\overline{X} - \mu_0)}{\sigma_0} > \frac{\sqrt{n}(C - \mu_0)}{\sigma_0} \mid H_0) \\ &= \Pr(Z > \frac{\sqrt{n}(C - \mu_0)}{\sigma_0}\end{aligned}$$

Where $Z \sim N(0,1)$. To find C, observe

$$z_{1-\alpha} = \frac{\sqrt{n}(C-\mu_0)}{\sigma_0} \Rightarrow C = \mu_0 + z_{1-\alpha}(\frac{\sigma_0}{\sqrt{n}}).$$

We know $\mu_0 = 5$ and $\sigma_0 = \sigma = 1$ so

$$C = 5 + z_{1-\alpha}\frac{1}{\sqrt{n}}.$$

The test will reject H_0 when $\overline{X} > C$ for observed $X_1, ..., X_n$ and given α.

(b) What is the power of the test when $n = 10, 50, 100$? Power is

$$\pi = \Pr(\overline{X} > C \mid H_1) = 1 - \beta$$

where

$$\begin{aligned}
\beta &= \Pr(\overline{X} \le C \mid H_1) \\
&= \Pr(\overline{X} - \mu_1 \le C - \mu_1) \\
&= \Pr(Z \le (5 + z_{1-\alpha}\frac{1}{\sqrt{n}}) - 10)) \\
&= \Pr(Z \le -5 + z_{1-\alpha}(\frac{1}{\sqrt{n}}))
\end{aligned}$$

Where $\beta \to pnorm(-5)$ as $n \to \infty$ so Power increases as n increases.

(c) Find the α-level likelihood ratio test for $H_0 : \theta = 0$ vs $H_1 : \theta = n^{-1/2}$ with observations $X_1, ..., X_n \sim N(\theta, 1)$, What is the power of the test when $n = 10, 50, 100$?

Observe, where $n^{-1/2} = \frac{1}{\sqrt{n}}$ we reject H_0 when $\overline{X} > C$ where

$$C = z_{1-\alpha}(\frac{1}{\sqrt{n}}).$$

The power of this test is $1 - \beta$ where

$$\begin{aligned}
\beta &= \Pr(\overline{X} \le C \mid H_1) \\
&= \Pr(Z \le \frac{z1-\alpha}{\sqrt{n}} - \frac{1}{\sqrt{n}}) \\
&= \Pr(Z \le \frac{1}{\sqrt{n}}(z_{1-\alpha} - 1)).
\end{aligned}$$

As $n \to \infty$ $\beta \to .5$.

(d) Discuse your discovery.

Power and H_1 both depend on n smaple size. When n increases, power decreases to .5 because it becomes harder to discern H_0 from H_1.

3. *Chi-squared*

Consider i.i.d. observations $X_1, ..., X_n \sim N(\mu, \sigma^2)$. The mean μ is unknown. Construct an α-level test for $H_0 : \sigma^2 = \sigma_0^2$ vs $H_1 : \sigma^2 \neq \sigma_0^2$.

The test statistic will be

$$\frac{(n-1)s^2}{\sigma_0^2} \sim \chi^2_{n-1}.$$

Reject H_0 if $\chi^2_{n-1} \in W$ where $W = [0, z_1) \cup (z_2, \infty]$ and $z_1, z_2 \in \mathbb{R}$. To determine critical values, z_1, z_2 use Chi-squared table with $n-1$ degrees of freedom to satisfy

$$\alpha = \Pr(\frac{(n-1)s^2}{\sigma_0^2} \in W)$$

alternatively

$$1 - \alpha = \Pr(\frac{(n-1)s^2}{\sigma_0^2} \notin W)$$

which is similar to confidence interval found in question 1.

$$1 - \alpha = \Pr\left(\frac{(n-1)s^2}{\chi^2_{n-1}(\alpha/2)} \leq \sigma_0^2 \leq \frac{(n-1)s^2}{\chi^2_{n-1}(1-\alpha/2)}\right).$$

Which can be written as confidence interval

$$\left[\frac{(n-1)s^2}{\chi^2_{n-1}(\alpha/2)}, \frac{(n-1)s^2}{\chi^2_{n-1}(1-\alpha/2)}\right].$$

Therefore, our test rejects H_0 if σ_0^2 not in the CI.

Specifically stated, reject the null when

$$\frac{(n-1)s^2}{\sigma_0^2} < \chi^2_{n-1}(\alpha/2) \text{ or } \frac{(n-1)s^2}{\sigma_0^2} > \chi^2_{n-1}(1-\alpha/2)$$

where $\chi^2_{n-1}(a)$ is the a percentile of of a Chi-squared distribution with $n-1$ degress of freedom.

4. *Problem 1 on Page 362*

A coin is thrown independently 10 times to test the hypothesis that the probability of heads is $\frac{1}{2}$ versus the alternative that the probability is not $\frac{1}{2}$. The test rejects if either 0 or 10 heads are observed.

(a) What is the significance level of the test?

$$\begin{aligned}
\alpha &= \Pr(\text{Reject } H_0 \mid H_0) \\
&= \Pr(X = 0 \text{ or } X = 10 \mid H_0) \\
&= \Pr(X = 0) + \Pr(X = 10) \\
&= \binom{10}{0} .5^0 (.5)^{10} + \binom{10}{10} .5^{10} (1 - .5)^0 \\
&= .00195
\end{aligned}$$

(b) If in fact the probability of heads is .1, what is the power of the test?

$$\begin{aligned}
\pi = 1 - \beta &= \Pr(\text{Reject } H_0 \mid p = .1) \\
&= \Pr(X = 0 \text{ or } X = 10 \mid p = .1) \\
&= \binom{10}{0} .1^0 (1 - .1)^{10} + \binom{10}{10} .1^{10} (1 - .1)^0 \\
&= .349
\end{aligned}$$