Homework 4 (due May 16)

In your solution include your name and the homework number. Please staple your pages together. When solving the problems below, give detailed derivations in order to receive credit.

1. *(Sample distribution: simple derivation)* Consider independent observations $y_1, ..., y_n \sim N(\beta_0, \sigma^2)$. The MLE of β_0 is $\hat{\beta}_0 = \bar{y}$. The residue is $\hat{e}_i = y_i - \bar{y}$ for $i = 1, ..., n$. In class, we learned $\bar{y} \sim N\left(\beta_0, \frac{\sigma^2}{n}\right)$ and $\sum_{i=1}^n \hat{e}_i^2/\sigma^2 \sim \chi^2_{n-1}$. In order to obtain a t-distribution, we need the independence between $\bar{y}$ and $\sum_{i=1}^n \hat{e}_i^2$. Here is a simple way to do it:

   (a) Calculate $\mathsf{Cov}(\bar{y}, \hat{e}_i)$.

   (b) Can you claim the independence between $\bar{y}$ and $\sum_{i=1}^n \hat{e}_i^2$?

2. *(Residue)* Consider independent observations $y_i \sim N(\beta_0 + \beta_1 x_i, \sigma^2)$ for $i = 1, ..., n$. For the LSE $\hat{\beta}_0$ and $\hat{\beta}_1$, define the residue $\hat{e}_i = y_i - (\hat{\beta}_0 + \hat{\beta}_1 x_i)$ for $i = 1, ..., n$.

   (a) Calculate $\mathbb{E}\left(\frac{1}{n-2}\sum_{i=1}^n \hat{e}_i^2\right)$.

   (b) Calculate $\mathsf{Cov}(\hat{\beta}_1, \hat{e}_i)$.

   (c) Can you claim the independence between $\hat{\beta}_1$ and $\sum_{i=1}^n \hat{e}_i^2$?

   (d) Are $\hat{\beta}_0$ and $\sum_{i=1}^n \hat{e}_i^2$ independent?

3. *(Joint distribution of $(\hat{\beta}_0, \hat{\beta}_1)$)* Consider independent observations $y_i \sim N(\beta_0 + \beta_1 x_i, \sigma^2)$ for $i = 1, ..., n$. For the LSE $\hat{\beta}_0$ and $\hat{\beta}_1$, we derived $\mathsf{Var}(\hat{\beta}_0)$ and $\mathsf{Var}(\hat{\beta}_1)$ in the class.

   (a) Calculate $\mathsf{Cov}(\hat{\beta}_0, \hat{\beta}_1)$.

   (b) What is the joint distribution of $(\hat{\beta}_0, \hat{\beta}_1)$?

4. *(Linear regression without slope)* Consider independent observations $y_i \sim N(\beta_1 x_i, \sigma^2)$ for $i = 1, ..., n$.

   (a) Find the MLE for β_1, denoted as $\hat{\beta}_1$.

   (b) Find $\mathbb{E}(\hat{\beta}_1)$.

   (c) Find $\mathsf{Var}(\hat{\beta}_1)$.

   (d) What is the distribution of $\hat{\beta}_1$?

5. *(Linear regression with centered covariates)* Some people like to center their x_i before applying regression. This leads to the model $y_i \sim N(\beta_0 + \beta_1(x_i - \bar{x}), \sigma^2)$ independently for $i = 1, ..., n$.

   (a) Find the MLE for β_0, β_1, denoted as $\hat{\beta}_0, \hat{\beta}_1$.

   (b) Find the expectations of $\hat{\beta}_0, \hat{\beta}_1$.

(c) Find the variances of $\hat{\beta}_0, \hat{\beta}_1$.

6. *(Check the matrix formula)* For multivariate linear regression with model $y \sim N(X\beta, \sigma^2 I_n)$, we showed in class that the MLE is $\hat{\beta} = (X^T X)^{-1} X^T y$. It has distribution $\hat{\beta} \sim N(\beta, \sigma^2 (X^T X)^{-1})$. Now consider the simple case of $p = 2$ so that

$$\beta = \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix}, \quad X = \begin{pmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{pmatrix}, \quad y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}.$$

   (a) For $p = 2$, work out the formula $(X^T X)^{-1} X^T y$.

   (b) For $p = 2$, work out the formula $\sigma^2 (X^T X)^{-1}$.

   (c) Do these formulas give you the same answers that we learned for $p = 2$ in class?

7. *(LSE=MLE)* For $y \sim N(X\beta, \sigma^2 I_n)$, write down the likelihood function of y. Show that maximizing the likelihood function is equivalent to minimizing $\|y - X\beta\|^2$.