Poisson Confidence Interval: Wilson's Approach

Instructor: Chao Gao

Spring 2017

Consider i.i.d. observations $X_1, ..., X_n \sim \text{Poisson}(\lambda)$. The mean and the variance of the MLE $\hat{\lambda} = \bar{X}$ are λ and λ/n, respectively. Thus, by CLT,

$$\frac{\sqrt{n}(\hat{\lambda} - \lambda)}{\sqrt{\lambda}} \rightsquigarrow N(0,1).$$

The CLT implies that

$$\mathbb{P}\left(z_{\alpha/2} \leq \frac{\sqrt{n}(\hat{\lambda} - \lambda)}{\sqrt{\lambda}} \leq z_{1-\alpha/2}\right) \approx 1 - \alpha.$$

Since $-z_{\alpha} = z_{1-\alpha/2}$, we get $z_{\alpha}^2 = z_{1-\alpha/2}^2$. The above result can be equivalently written as

$$\mathbb{P}\left(\frac{n(\hat{\lambda} - \lambda)^2}{\lambda} \leq z_{1-\alpha/2}^2\right) \approx 1 - \alpha.$$

We need to solve the range of λ from

$$\frac{n(\hat{\lambda} - \lambda)^2}{\lambda} \leq z_{1-\alpha/2}^2.$$

After rearrangement, we get

$$n\lambda^2 - (2n\hat{\lambda} + z_{1-\alpha/2}^2)\lambda + n\hat{\lambda}^2 \leq 0.$$

The two endpoints of the interval determined by the above inequality can be solved from the equation

$$n\lambda^2 - (2n\hat{\lambda} + z_{1-\alpha/2}^2)\lambda + n\hat{\lambda}^2 = 0.$$

This gives the two roots

$$\lambda = \frac{2n\hat{\lambda} + z_{1-\alpha/2}^2 \pm \sqrt{(2n\hat{\lambda} + z_{1-\alpha/2}^2)^2 - 4n^2\hat{\lambda}^2}}{2n}.$$

An approximate $(1-\alpha)$-level confidence interval is

$$\left[\frac{2n\hat{\lambda} + z_{1-\alpha/2}^2 - \sqrt{(2n\hat{\lambda} + z_{1-\alpha/2}^2)^2 - 4n^2\hat{\lambda}^2}}{2n}, \frac{2n\hat{\lambda} + z_{1-\alpha/2}^2 + \sqrt{(2n\hat{\lambda} + z_{1-\alpha/2}^2)^2 - 4n^2\hat{\lambda}^2}}{2n}\right].$$