HOMEWORK 3 SOLUTIONS

1. (a) $\frac{(n-1)s^2}{\sigma^2} \sim \chi^2_{n-1}$.

$\therefore P\left[\chi^2_{n-1,1-\alpha/2} < \frac{(n-1)s^2}{\sigma^2} < \chi^2_{n-1,\alpha/2}\right] = 1-\alpha$.

$\therefore P\left[\frac{(n-1)s^2}{\chi^2_{n-1,\alpha/2}} < \sigma^2 < \frac{(n-1)s^2}{\chi^2_{n-1,1-\alpha/2}}\right] = 1-\alpha$.

$\therefore L_{\sigma^2}(\alpha) = \frac{(n-1)s^2}{\chi^2_{n-1,\alpha/2}}$, $U_{\sigma^2}(\alpha) = \frac{(n-1)s^2}{\chi^2_{n-1,1-\alpha/2}}$.

(b) $\frac{\sqrt{n}(\bar{X}-\mu)}{s} \sim t_{n-1}$

$\Rightarrow P\left[t_{n-1,1-\alpha/2} < \frac{\sqrt{n}(\bar{x}-\mu)}{s} < t_{n-1,\alpha/2}\right] = 1-\alpha$

$\therefore$ C.I. for μ : $\bar{X} - \frac{s}{\sqrt{n}} t_{n-1,\alpha/2}$, $\bar{X} - \frac{s}{\sqrt{n}} t_{n-1,1-\alpha/2}$.

(c) Since s^2 appear in both pairs of end points, $\therefore$ There is no reason to think that $(L_\mu(\alpha), U_\mu(\alpha))$ and $(L^2_\sigma(\alpha), U^2_\sigma(\alpha))$ are independent.

The event "$\mu \in [L_\mu(\alpha), U_\mu(\alpha)]$ and $\sigma^2 \in [L^2_\sigma(\alpha), U^2_\sigma(\alpha)]$" can be represented as a region in $\bar{x}-s$ plane bounded by the following four lines:

$L_1 : s = \sqrt{\sigma^2 \chi^2_{n-1,\alpha/2} / (n-1)}$

$L_2 : s = \sqrt{\sigma^2 \chi^2_{n-1,1-\alpha/2} / (n-1)}$

$L_3 : \bar{x} = \mu + \frac{s}{\sqrt{n}} t_{n-1,\alpha/2}$

$L_4 : \bar{x} = \mu + \frac{s}{\sqrt{n}} t_{n-1,1-\alpha/2}$.

$$P(\mu \in [L_\mu(\alpha), U_\mu(\alpha)],\ \sigma^2 \in [L_{\sigma^2}(\alpha), U^2_\sigma(\alpha)])$$
$$= 1 - (P(\mu \notin [L_\mu(\alpha), U_\mu(\alpha)] \text{ or } \sigma^2 \notin [L^2_\sigma(\alpha), U^2_\sigma(\alpha)]))$$
$$\geq 1 - P(\mu \notin [L_\mu(\alpha), U_\mu(\alpha)]) - P(\sigma^2 \notin [L^2_\sigma(\alpha), U^2_\sigma(\alpha)])$$
$$\geq 1 - 2\alpha.$$

(d) To obtain different C.I. choose $0 < a < b$ s.t

$$P\left[a < \frac{(n-1)s^2}{\sigma^2} < b\right] = 1-\alpha \longrightarrow (*)$$

$\therefore \left(\frac{(n-1)s^2}{b}, \frac{(n-1)s^2}{a}\right)$ is a $100(1-\alpha)\%$ CI for σ^2.

with length of the interval $(n-1)s^2\left(\frac{1}{a} - \frac{1}{b}\right)$

So we need to find the choice of a and b that minimizes $\frac{1}{a} - \frac{1}{b}$ subject to the constraint $(*)$

$(*)$ determines b as a function of a.

So, we can use implicit differentiation:

$$\frac{d}{da}\int_a^b f_{n-1}(x)\,dx = \frac{d}{da}(1-\alpha)$$

f_{n-1} is the density of χ^2_{n-1}

$$\Rightarrow -f_{n-1}(a) + f_{n-1}(b)\frac{db}{da} = 0$$

$$\Rightarrow \frac{db}{da} = \frac{f_{n-1}(a)}{f_{n-1}(b)}$$

Now, $$\frac{d}{da}\left(\frac{1}{a} - \frac{1}{b}\right) = -\frac{1}{a^2} + \frac{1}{b^2}\frac{f_{n-1}(a)}{f_{n-1}(b)}$$

This equals 0 when $a^2 f_{n-1}(a) = b^2 f_{n-1}(b)$

To verify ~~this~~ ~~we~~ that we have actually found a minimum, a second implicit differentiation yields

$$\frac{d^2b}{da^2} = \frac{f'_{n-1}(a)}{f_{n-1}(b)} - f'_{n-1}(b)\frac{f_{n-1}(a)}{f_{n-1}(b)}.$$

For the usual values of α, we have a must be small enough and b must be large enough so that f_{n-1} is increasing at a and decreasing at b, guaranteeing that $\frac{d^2b}{da^2} > 0$.

Consequently, an interval of minimum length will be found for any a, b such that

$$a^2 f_{n-1}(a) = b^2 f_{n-1}(b)$$

2. (a) Likelihood ratio

$$\Lambda = \frac{\left(\frac{1}{\sqrt{2\pi}}\right)^n \exp\left(-\sum_{i=1}^{n}(x_i-5)^2\right)}{\left(\frac{1}{\sqrt{2\pi}}\right)^n \exp\left(-\sum_{i=1}^{n}(x_i-10)^2\right)}$$

$$= \exp\left(-n(\bar{x}-5)^2 + n(\bar{x}-10)^2\right)$$

$$= \exp\left(-n\left(\bar{x}^2 + 25 - 10\bar{x} - \bar{x}^2 + 20\bar{x} - 100\right)\right)$$

$$= \exp\left(-n\left(-75 + 10\bar{x}\right)\right)$$

$$= \exp\left(-10n(\bar{x} - 7.5)\right)$$

$\Lambda > c \iff 10n(\bar{x}-7.5) < z$ for some z.

$\iff \bar{x} - 7.5 < z^*$ for some z^*.

Now, $\bar{x} \sim N(\theta, \frac{1}{n})$; under H_0, $\theta = 5$ & under H_1, $\theta = 10$.

$\therefore P_{H_0}(\bar{x} - 7.5 < z^*) = \alpha \iff P_{H_0}(\bar{x} - 5 < z^* + 7.5 - 5) = \alpha$

$\Leftrightarrow \quad P_{H_0}\left(\sqrt{n}(\bar{X} - 5) < \sqrt{n}(t^* + 2.5)\right) = \alpha .$

$\Leftrightarrow \quad \sqrt{n}(t^* + 2.5) = z_\alpha \qquad \left[\because (\bar{X} - 5) \sim N(0, \tfrac{1}{n}) \text{ under } H_0\right]$

$\uparrow$ α-th quantile of $N(0,1)$

$\Leftrightarrow \quad t^* = -2.5 + \frac{z_\alpha}{\sqrt{n}}$

$\therefore$ likelihood ratio test is $\mathbb{I}\{\Lambda > c\}$ or equivalently

$\mathbb{I}\{\bar{X} - 7.5 < t^*\}$ or equivalently

$\mathbb{I}\{\bar{X} < 5 + z_\alpha/\sqrt{n}\}$.

~~(b) Power of the test~~

We reject H_0 when Λ is small.

or, we accept H_0 when Λ is large, i.e. $\Lambda > c$.

i.e. we accept H_0 when $\bar{X} < 5 + \frac{z_\alpha}{\sqrt{n}}$.

(b) Power of the test: $P_{H_1}(\text{Reject } H_0)$

$$
\begin{aligned}
&= 1 - P_{H_1}(\text{Accept } H_0) \\
&= 1 - P_{\theta=10}\left(\bar{X} < 5 + \frac{z_\alpha}{\sqrt{n}}\right) \\
&= 1 - P_{\theta=10}\left(\bar{X} - 10 < -5 + \frac{z_\alpha}{\sqrt{n}}\right) \\
&= 1 - P_{\theta=10}\left(\sqrt{n}(\bar{X} - 10) < -5\sqrt{n} + z_\alpha\right) \\
&= 1 - \Phi(-5\sqrt{n} + z_\alpha)
\end{aligned}
$$

where Φ is standard Normal CDF

$[\because$ under H_1, $\sqrt{n}(\bar{X} - 10) \sim N(0,1)]$

Notice here as n increases, power increases.

(c). $H_0: \theta = 0$ vs $H_1: \theta = n^{-1/2}$.

Likelihood ratio $\Lambda = \dfrac{\exp\left(-\frac{1}{2}\sum_{i=1}^{n} x_i^2\right)}{\exp\left(-\frac{1}{2}\sum_{i=1}^{n}(x_i - n^{-1/2})^2\right)}$

$$= \exp\left(-\frac{1}{2}\sum_{i=1}^{n}\left(x_i^2 - x_i^2 + 2n^{-1/2}x_i + n^{-1}\right)\right)$$

$$= \exp\left(-\frac{1}{2}2n^{-1/2}\sum_{i=1}^{n}x_i - \frac{1}{2}n \cdot n^{-1}\right)$$

$$= \exp\left(-\frac{n\bar{x}}{\sqrt{n}} - \frac{1}{2}\right)$$

$$= \exp\left(-\sqrt{n}\bar{x} - \frac{1}{2}\right)$$

Reject H_0 when $\Lambda < c$

equivalently, when $\sqrt{n}\,\bar{x} > t^*$ for some t^*.

$P_{H_0}(\sqrt{n}\bar{x} > t^*) = \alpha \iff t^* = z_\alpha$ [$\because \sqrt{n}\bar{x} \sim N(0,1)$ under H_0]

$\Rightarrow$ Reject H_0 when $\bar{x} > z_\alpha/\sqrt{n}$

Power of the test: P_{H_1}(Reject H_0)

$$= P_{H_1}\left(\bar{x} > z_\alpha/\sqrt{n}\right)$$

$$= P_{H_1}\left(\bar{x} - \frac{1}{\sqrt{n}} > \frac{z_\alpha}{\sqrt{n}} - \frac{1}{\sqrt{n}}\right)$$

$$= P_{H_1}\left(\sqrt{n}\left(\bar{x} - \frac{1}{\sqrt{n}}\right) > z_\alpha - 1\right)$$

$$= 1 - P_{H_1}\left(\sqrt{n}\left(\bar{x} - \frac{1}{\sqrt{n}}\right) \leq z_\alpha - 1\right)$$

$= 1 - \Phi(z_\alpha - 1)$ where Φ is standard Normal CDF.

(d) Notice in (b), as n increases, power increases.
in (c), as n changes, power remains same!

3. $X_1, \ldots, X_n \sim N(\mu, \sigma^2)$.

μ unknown

$H_0: \sigma^2 = \sigma_0^2$ vs $H_1: \sigma^2 \neq \sigma_0^2$.

under H_0, $\dfrac{(n-1)s^2}{\sigma_0^2} \sim \chi^2_{n-1}$.

Reject H_0 when ~~$(n-1)s^2$~~ $\dfrac{s^2}{\sigma_0^2}$ is too small or too large.

equivalently reject H_0 when $\dfrac{s^2}{\sigma_0^2} < a$ OR $\dfrac{s^2}{\sigma_0^2} > b$

such that $P_{H_0}\left(\dfrac{s^2}{\sigma_0^2} < a\right) = P_{H_0}\left(\dfrac{s^2}{\sigma_0^2} > b\right) = \alpha/2$

$\therefore$ Reject when $\dfrac{(n-1)s^2}{\sigma_0^2} < \chi^2_{\alpha/2, n-1}$ OR $\dfrac{(n-1)s^2}{\sigma_0^2} > \chi^2_{1-\alpha/2, n-1}$

where $\chi^2_{\alpha/2, n-1}$ and $\chi^2_{1-\alpha/2, n-1}$ are ~~upper~~ $\alpha/2$-th and $(1-\alpha/2)$ th percentile of χ^2_{n-1} distribution

4. Let X = No. of heads. $\Rightarrow X \sim Bin(n, p)$

~~Reject H_0~~ where p = probability of heads., $n = 10$

$H_0: p = \frac{1}{2}$ vs $H_1: p \neq \frac{1}{2}$.

Reject H_0 if $X = 0$ or $X = 10$.

(a) significance level $= P_{H_0}(X=0) + P_{H_0}(X=10)$
$= (1-1/2)^{10} + (1/2)^{10} = 2 \cdot \left(\frac{1}{2}\right)^{10} = \frac{1}{2^9}$

(b) when $p = 0.1$, Power of the test $= P_{H_1}(X=0) + P_{H_1}(X=10)$
$= P_{p=0.1}(X=0) + P_{p=0.1}(X=10) = (1-0.1)^{10} + 0.1^{10} = 0.9^{10} + 0.1^{10}$

$\approx$