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STAT 245 HOMEWORK 2

1. *Cauchy Distribution*

Let X and Y be two independent $N(0,1)$ random variables. Show the distribution X/Y is the same as that of $X/|Y| = X/\sqrt{Y^2}$. This means that X/Y has a t_1-distribution which is also known as Cauchy-distribution.

First find $\frac{X}{Y} \sim$ Cauchy. We have $X, Y \sim N(0,1)$ and X and Y are independent. Let $W = \frac{X}{Y}$ then

$$\begin{aligned} f_W(w) &= \int_{-\infty}^{\infty} f_{X,Y}(wy, y)|y|dy \\ &= \int_{-\infty}^{\infty} \frac{1}{2\pi} e^{-\frac{w^2y^2+y^2}{2}} y dy \\ &= \frac{1}{\pi(1+w^2)}. \end{aligned}$$

Now let $Z = \frac{X}{-Y}$ then

$$\begin{aligned} f_Z(z) &= \int_{-\infty}^{\infty} f_{X,Y}(-zy, -y)| - y|dy \\ &= \int_{-\infty}^{\infty} \frac{1}{2\pi} e^{-\frac{z^2y^2+y^2}{2}} y dy \\ &= \frac{1}{\pi(1+z^2)}. \end{aligned}$$

Hence

$$\frac{X}{Y} = \frac{X}{-Y} \Rightarrow \frac{X}{|Y|} = \frac{X}{\sqrt{Y^2}} = \frac{X}{\pm Y}.$$

Alternatively, by symmetry of $N(0,1)$ over 0, $X = -X$. Which allows for the same conclusion.

2. *Bivariate normal distribution I*

Suppose (X, Y) has a bivariate normal distribution with expected values $\mathbb{E}(X) = 3$ and $\mathbb{E}(Y) = 1$, variances $\mathrm{Var}(X) = 9$ and $\mathrm{Var}(Y) = 16$, with correlation ρ. Let $W_a = 12 + aX + Y$ and $V = 19 + X + 2Y$.

(a) Fix $\rho = 1/3$ and find $a \in \mathbb{R}$ such that W_a and V are independent. Can you choose $p_0 \in (-1, 1)$ such that there does exist an $a \in \mathbb{R}$ making W_a and V independent? If yes, final all such p_0. If no, explain why not.

Since these are normally distributed variables we can use the Gaussain assumption, which says that normally distributed variables are independent if covariance is 0.

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\begin{aligned}
\mathrm{Cov}(W_a, V) &= \mathrm{Cov}(aX + Y, X + 2Y) \\
&= a\,\mathrm{Cov}(X, X) + \mathrm{Cov}(X, Y) + 2a\,\mathrm{Cov}(X, Y) + 2\,\mathrm{Cov}(Y, Y) \\
&= a\,\mathrm{Var}(X) + \mathrm{Cov}(X, Y)(1 + 2a) + 2\,\mathrm{Var}(Y) \\
&= 9a + 12\rho(1 + 2a) + 32.
\end{aligned}
$$

Fixing $\rho = 1/3$

$$\mathrm{Cov}(W_a, V) = 17a + 36$$

therefore when $a = -\frac{36}{17}$, $\mathrm{Cov}(W_a, V) = 0 \Rightarrow W, V$ independent.

Furthermore, solve

$$9a + 12\rho(1 + 2a) + 32 = 0$$

for a, then $a = -\frac{4(3\rho+8)}{24\rho+9}$ where $8\rho + 3 \neq 0$. Let $\rho = -\frac{3}{8}$ then there doesn't exists an $a \in \mathbb{R}$ such that X, Y are independent.

(b) Now fix $a = 1$ and find ρ such that W_a and V are independent. Can you choose $a_0 \in \mathbb{R}$ such that there does not exist a $\rho \in (-1, 1)$ making W_{a_o} and V independent? If yes, find all such a_0. If no, explain why not.

Fixing $a = 1$

$$\mathrm{Cov}(W, V) = 9 + 12\rho(3) + 32$$

and solving for 0, $\rho = -\frac{41}{36}$. Since $\rho \notin (-1, 1)$ choose $a_0 = a = 1$ then there does not exists $\rho \in (-1, 1)$ such that W_a, V are independent.

3. *Bivariate normal distribution II*

Let (X, Y) follow a bivariate normal distribution with $\mathbb{E}(X) = 5$ and $\mathbb{E}(Y) = 3$, variances $\mathrm{Var}(X) = 9$ and $\mathrm{Var}(Y) = 16$, and correlation $\rho = 0.4$. Find

(a) the conditional expectation $\mathbb{E}(X \mid Y = 8)$, In general, the conditional expectation for normal distributed variables is

$$E(X \mid Y) = u_Y + \rho\frac{\sigma_Y}{\sigma_X}(x - \mu_x).$$

Then

$$E[X \mid Y = 8] = 5 + .4\frac{3}{4}(8 - 3) = 6.5$$

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(b) the condtional variance $\mathrm{Var}(X \mid Y = 8)$. In general, the conditional variance for normal distributed random variables is

$$\mathrm{Var}[Y \mid X] = \sigma_Y^2(1 - \rho^2).$$

Then

$$\mathrm{Var}[X \mid Y = 8] = 9(1 - .4^2) = 7.56.$$

(c) The probability $\Pr(3 < X < 5)$.

Since $X \sim N(5, 9)$ we re-arrange the inequality

$$\begin{aligned} 3 <& X < 5 \\ \frac{3-5}{3} <& \frac{X-5}{3} < \frac{5-5}{3}. \end{aligned}$$

Now evaluate

$$\begin{aligned} \Pr(-\frac{2}{3} < Z < 0) &= \Pr(Z < 0) - \Pr(Z < -\frac{2}{3}) \\ &= .247 \end{aligned}$$

where $Z \sim N(0, 1)$.

(d) The conditional probablity $\Pr(3 < X < 5 \mid Y = 8)$.

Using facts from parts (a) and (b), $X \mid Y \sim N(6.5, 7.56)$. Then

$$\begin{aligned} \Pr(3 < X < 5 \mid Y = 8) &= \Pr(X < 5 \mid Y = 8) - \Pr(X < 3 \mid Y = 8) \\ &= \Pr\left(\frac{x - 6.5}{\sqrt{7.56}} < \frac{-1.5}{\sqrt{7.56}}\right) - \Pr\left(\frac{x - 6.5}{\sqrt{7.56}} < \frac{-3.5}{\sqrt{7.56}}\right) \\ &= \Pr\left(Z < \frac{-1.5}{\sqrt{7.56}}\right) - \Pr\left(Z < \frac{-3.5}{\sqrt{7.56}}\right) \\ &= .1911 \end{aligned}$$

where $Z \sim N(0, 1)$.

4. *Joint distributions*

Let X and Y be the scores of a Stat 245 student on midterm and final exam. We model these scores as

$$X = S + E_1, \ Y = S + E_2,$$

where S, E_1, E_2 are independent random variables distributed as $S \sim N(70, 49)$, $E_1, E_2 \sim N(0, 25)$. We think of S as a "skill" part of the score and E_1, E_2 as "luck" components.

(a) What is the joint distribution of (X, Y)?

The general form is

See Rice p.81

$$f_{X,Y} = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp\big(-\frac{1}{2(1-\rho^2)}\big[\frac{(y-\mu_Y)^2}{\sigma_x^2} + \frac{(x-\mu_X)^2}{\sigma_Y^2} - \frac{2\rho(x-\mu_X)(y-\mu_Y)}{\sigma_X\sigma_Y}\big]\big)$$

where

$$\rho = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\,\mathrm{Var}(Y)}}.$$

Compute

$$\begin{aligned}
\mathrm{Cov}(X,Y) &= \mathrm{Cov}(S+E_1, S+E_2) \\
&= \mathrm{Cov}(S,S) + \mathrm{Cov}(E_1,S) + \mathrm{Cov}(S,E_2) + \mathrm{Cov}(E_1,E_2) \\
&= \mathrm{Var}(S) + 0 + 0 + 0 \\
&= 49
\end{aligned}$$

in order to find

$$\rho = \frac{49}{74}.$$

Plug in the values to find the joint distrubution.

(b) Assume that a student recieved a midterm score that is one standard deviation below the midterm mean. What do you expect his/her final score to be?

Since $X = \mu_x - \sigma_x$

$$\begin{aligned}
E[Y \mid X] &= \mu_y + \rho\frac{\sigma_Y}{\sigma_X}(\mu_x - \sigma_x - \mu_x) \\
&= \mu_y + \rho\frac{\sigma_Y}{\sigma_X}(-\sigma_x) \\
&= \mu_y - \rho \cdot \sigma_Y \\
&= 70 - \frac{49}{74} \cdot \sqrt{49+25} \\
&= 64.3
\end{aligned}$$

5. *Mean square error when estimating a normal variance*

Let $X_1, ..., X_n$ be independent $N(\mu, \sigma^2)$ random variables. Let $\overline{X} = \frac{1}{n}\sum_{i=1}^n X_i$ be the sample mean. Consider two estimators of $sigma^2$, name the sample varaince

$$s^2 = \frac{1}{n-1}\sum_{i=1}^n (X_i - \overline{X})^2$$

and the MLE

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^n (X_i - \overline{X})^2.$$

The mean square error (MSE) measures how far on average these estimators are away from the "target" σ^2, where "away" is measured in square distance. The two MSE are defined as

$$\mathrm{MSE}(s^2) = E[(s^2 - \sigma^2)^2] \text{ and } \mathrm{MSE}(\hat{\sigma}^2) = E[(\hat{\sigma}^2 - \sigma^2)^2].$$

(a) Compute and compare $\text{MSE}(s^2)$ and $\text{MSE}(\hat{\sigma}^2)$.

First observe

$$\begin{aligned} E(\hat{\theta} - \theta) &= E(\hat{\theta}^2) + E(\theta^2) - 2\theta E(\hat{\theta}) \\ &= \text{Var}(\hat{\theta}) + [E(\hat{\theta})]^2 + \theta^2 - 2\theta E(\hat{\theta}) \\ &= \text{Var}(\hat{\theta}) + [E(\hat{\theta}) - \theta]^2. \end{aligned}$$

The second term is commonly reffered to as Bias^2, then

$$\text{Bais}(\hat{\theta}) = E(\hat{\theta}) - \theta.$$

To find $\text{MSE}(s^2)$ first find the variance by re-arranging s^2 such that

$$\frac{(n-1)s^2}{\sigma^2} = \frac{\sum_{i=1}^{n}(x_i - \overline{x})^2}{\sigma^2} \sim \chi^2_{n-1}.$$

Then

$$E\Big[\frac{(n-1)s^2}{\sigma^2} = n - 1 \Rightarrow E(s^2) = \sigma^2$$

and

$$\text{Var}\left[\frac{(n-1)s^2}{\sigma^2}\right] = 2(n-1) \Rightarrow \text{Var}(s^2) = \frac{2\sigma^4}{n-1}.$$

Now MSE can be calculated

$$\begin{aligned} \text{MSE}(s^2) &= \text{Var}(s^2) + \text{Bias}(s^2)^2 \\ &= \frac{2\sigma^4}{n-1} + (E(s^2) - \sigma^2)^2 \\ &= \frac{2\sigma^4}{n-1} \end{aligned}$$

which means s^2 is an unbiased estimator.

For $\hat{\sigma}^2$ observe

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X})^2 = \frac{n-1}{n}s^2$$

which can be used to find

$$\begin{aligned} \text{Var}(\hat{\sigma}^2) &= \frac{(n-1)^2}{n^2}\,\text{Var}(s^2) \\ &= \frac{(n-1)^2}{n^2}\frac{2\sigma^4}{n-1} \\ &= \frac{2(n-1)\sigma^4}{n^2}. \end{aligned}$$

Next

$$\begin{aligned} \text{Bias}(\hat{\sigma}^2 = E(\hat{\sigma}^2) - \sigma^2 & \\ &= E\Big(\frac{n-1}{n}s^2\Big) - \sigma^2 \\ &= \frac{n-1}{n}\sigma^2 - \sigma^2. \end{aligned}$$

The MSE is then

$$\mathrm{MSE}(\hat{\sigma}^2) = \frac{2n-1}{n^2}\sigma^4 + \left(\frac{n-1}{n}\sigma^2 - \sigma^2\right)^2 = \frac{2n-1}{n^2}\sigma^4.$$

Comparing the two

$$\mathrm{MSE}(\hat{\sigma}^2) < \frac{2n}{n^2}\sigma^4 < \frac{2\sigma^4}{n-1} = \mathrm{MSE}(s^2).$$

(b) Consider a general form of the estimator

$$\tilde{\sigma}^2 = c\sum_{i=1}^{n}(X_i - \overline{X})^2.$$

Find the best c such that $\mathrm{MSE}(\tilde{\sigma}^2) = E[(\tilde{\sigma}^2 - \sigma^2)^2]$ is minimized.

First observe

$$\tilde{\sigma}^2 = c\sum_{i=1}^{n}(X_i - \overline{X})^2 = c(n-1)s^2$$

and let $t = c(n-1)$. Then

$$E(\tilde{\sigma}^2) = tE(s^2) = t\sigma^2$$

and

$$\mathrm{Var}(\tilde{\sigma}^2) = t^2\,\mathrm{Var}(s^2) = \frac{2t^2}{n-1}\sigma^4.$$

Using the above facts

$$\begin{aligned}
\mathrm{MSE}(\tilde{\sigma}^2) &= \mathrm{Var}(\tilde{\sigma}^2) + (t\sigma^2 - \sigma^2)^2 \\
&= \mathrm{Var}(\tilde{\sigma}^2) + (t-1)^2\sigma^4 \\
&= t^2\,\mathrm{Var}(s^2) = \frac{2t^2}{n-1}\sigma^4 + (t-1)^2\sigma^4 \\
&= f(t)\sigma^4
\end{aligned}$$

where

$$f(t) = \frac{2t^2}{n-1} + (t-1)^2 = \frac{n+1}{n-1}t^2 - 2t + 1.$$

By differentiating, $f(t) = \frac{2}{n+1}$, its minimal value, when $t = \frac{n-1}{n+1}$. Hence the smallest value of $\mathrm{MSE}(\tilde{\sigma}^2) = \frac{2\sigma^4}{n+1}$ with

$$(n-1)c = t = \frac{n-1}{n+1}$$

which means $c = \frac{1}{n+1}$