Homework 0

(due in class on Tuesday, April 4 2017) When solving the problems below as well as future homework problems, please give detailed derivations and arguments in order to receive credit. In your solution do not forget to include your name and the homework number. Please staple your pages together.

1. *(Moments of Poisson distribution)* Let X be a random variable with a Poisson distribution. Find $E(X^4)$.

2. *(Poisson- and χ^2-tails)* For $\lambda > 0$, let X_λ be a discrete random variable with a Poisson distribution with expected value λ. For (integer) $d \in \mathbb{N}$, let Y_d be a continuous random variable with a χ^2-distribution with d degrees of freedom. In other words, the distribution of Y_d has the probability density function

$$f_{\chi^2_d}(y) = \frac{1}{2^{d/2}\Gamma(d/2)} y^{d/2-1} \exp(-y/2), \qquad y \geq 0,$$

   where $\Gamma(.)$ is the Gamma-function which satisfies $\Gamma(n) = (n-1)!$ for $n \in \mathbb{N}$.

   Show that for all $\lambda > 0$ and all (integer) $c \in \mathbb{N}$,

$$\mathbb{P}(X_\lambda \geq c+1) = \mathbb{P}(0 \leq Y_{2(c+1)} \leq 2\lambda) = \int_0^{2\lambda} f_{\chi^2_{2(c+1)}}(y)\, dy. \tag{1}$$

   (Hint: Treat the two probabilities in (1) as functions in λ, and compare the derivatives of the two functions.)

3. *(Approximations to Binomial probabilities)* Let X be distributed according to a Binomial(n, p) distribution. We are interested in the probability $\mathbb{P}(X = k)$ for

   (a) $n = 7$, $p = 0.3$, $k = 3$;

   (b) $n = 40$, $p = 0.4$, $k = 11$;

   (c) $n = 400$, $p = 0.0025$, $k = 2$.

   In each of these cases determine the exact Binomial probability, an approximation using the normal distribution (with or without continuity correction), and an approximation based on the Poisson distribution. In each case comment on the accuracy of the approximation. If approximations are good explain why.

4. *(Conditional distributions in Poisson process)* Let $(X_t)_{t \geq 0}$ be a Poisson process, and let

$$T_1 = \min\{t > 0 : X_t \geq 1\}$$

   be the time to the first event.

   (a) Find the conditional distribution of X_s given $X_t = n$ for fixed time points $t > s > 0$ and integer $n \in \mathbb{N}$.

(b) Show that the conditional distribution of T_1 given $X_t = 1$ is the uniform distribution on the interval $(0, t]$.
(Hint: Consider $\mathbb{P}(T_1 > s \mid X_t = 1)$ for $0 < s < t$.)

5. *(Data from Poisson process)* A detector counts the number of particles emitted from a radioactive source over the course of 10-second intervals. For 180 such 10-second intervals, the following counts were observed:

Count	# intervals
0	23
1	77
2	34
3	26
4	13
5	7

This table states, for example, that in 34 of the 10-second intervals a count of 2 was recorded. Sometimes, however, the detector did not function properly and recorded counts over intervals of length 20 seconds. This happened 20 times and the recorded counts are

Count	# intervals
0	2
1	4
2	9
3	5

Assume a Poisson process model for the particle emission process. Let $\lambda > 0$ (time unit = 1 sec.) be the unknown rate of the Poisson process.

(a) Formulate an appropriate likelihood function for the described scenario and derive the maximum likelihood estimator $\hat{\lambda}$ of the rate λ. Compute $\hat{\lambda}$ for the above data.

(b) What approximation to the distribution of $\hat{\lambda}$ does the central limit theorem suggest? (Note that the sum of all 200 counts has a Poisson distribution. What is its parameter?)