# Stat 201: Statistics I
# Chapter 11

Metropolitan State University

date

# Chapter 11
# Goodness-of-Fit and Contingency Tables

# Section 11.1
# Goodness-of-Fit

# Frequency distributions

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A frequency table is a list of the distribution of a sample drawn from a population.

A test can be conducted to see if the population a sample is drawn from has an expected distribution.

# Frequency distributions, example

**Example**

- A six-sided die is "fair" if the frequencies of each possible result of a roll (1 through 6) are equal.

  Given a sample of results from a number rolls of a particular die, a test could be conducted to test whether the die is "fair".

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  Given a sample of results from a number rolls of a particular die, a test could be conducted to test whether the die is "fair".

- M&Ms should have the following distribution of colors:

| Color | Blue | Brown | Green | Orange | Red | Yellow |
|---|---|---|---|---|---|---|
| Percent | 24% | 14% | 15% | 20% | 13% | 14% |

  Given a sample a M&Ms, a test could be conducted to test whether M&Ms really do have that distribution of colors.

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- $H_0$ : The frequency counts agree with the expected distribution.
- $H_a$ : The frequency counts do not agree with the expected distribution.
- Test statistic follows a $\chi^2$ (chi-squared) distribution with $k-1$ degrees of freedom

$$\chi^2 = \sum \frac{(O-E)^2}{E}$$

  where
  - $k$ is the number of classes or categories
  - $O$ is the observed count for each class or category, from sample
  - $E$ is the expected count for each class or category if the expected distribution is true

# Expected counts

The expected count for each class or category can be calculated by

$$E = P(c) \times n$$

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- $n$ is the sample size

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For expected uniform distributions, since $P(c) = 1/k$ where $k$ is the number of classes or categories, the expected count for each class is

$$E = \frac{1}{k} \times n = \frac{n}{k}$$

# Expected counts, example

**Example**

- Since a "fair" die has a uniform frequency distribution, the expected counts for each result for a sample of 100 die rolls is

$$E = \frac{n}{k} = \frac{100}{6} = 16.67$$

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## Example

- Since a “fair” die has a uniform frequency distribution, the expected counts for each result for a sample of 100 die rolls is

$$E = \frac{n}{k} = \frac{100}{6} = 16.67$$

- Blue M&Ms are expected to have a frequency of 24%. Thus, out of a sample of 150 M&Ms, the expected count for blue is

$$E = P(c) \times n = 0.24 \times 150 = 36$$

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- Critical value: If the calculated test statistic is greater than the critical value for the significance level and degrees of freedom ($\chi^2 > \chi^2_{\alpha,k-1}$), then reject the null hypothesis. Critical values can be found in Table A-4.

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For both methods, if conditions are not met to reject, fail to reject the null hypothesis.

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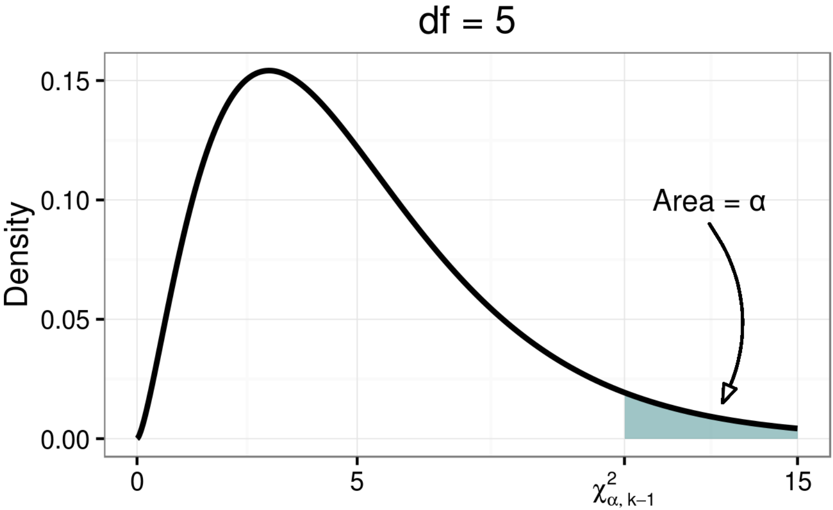
For both methods, if conditions are not met to reject, fail to reject the null hypothesis.

Note: Chi-square test statistics are always positive and chi-square tests are always one-sided. Large values of $\chi^2$ cause rejection of the null.

# Chi-square distribution



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- The sample is a simple random sample
- The sample data consists of frequency counts for each of the different categories
- For each class or category, the expected count is at least 5

# Goodness-of-fit tests in StatCrunch

- Stat $\rightarrow$ Goodness-of-fit $\rightarrow$ Chi-Square Test
- Select column that contains observed data
- Specify expected distribution:
  - For uniform distributions, select "All cells in equal proportion"
  - For non-uniform distributions, select the column which contains expected frequencies
- Leave default value of "Expected" for display
- Click "Compute!"
- The test statistic and p-value are found in "Chi-Square" and "P-value"

# Uniform goodness-of-fit test, example

## Example

To determine if there is evidence is is not a die is “fair”, roll the die 40 times and perform a goodness-of-fit test on the results.

A “fair” die will have a uniform frequency distribution, so each result has a probability of $1/6$ (16.67%).

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- Requirements: The expected count for each result is $E = \frac{1}{6} \times 40 = 6.667 > 5$

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  $H_a$ : The frequency count of at least one result differs from the others
- Requirements: The expected count for each result is
  $E = \frac{1}{6} \times 40 = 6.667 > 5$
- Find test statistic $\chi^2$, p-value and report decision

# Group work

- Complete question 1.

# Section 11.2
# Contingency Tables

# Contingency tables

Recall, a **contingency table** is a two dimensional table (rows and columns) displaying frequency counts of classes or categories of two factors for a single sample.

# Contingency tables, example

## Example

Recall the cancer screening example. A sample of 1000 randomly selected people where given a new screening test for a particular kind of of cancer. Each subject either has cancer or doesn't, and either tested positive or tested negative.

| | Test Result | |
|---|---|---|
| Diagnosis | Positive | Negative |
| Cancer | 74 | 13 |
| No cancer | 26 | 887 |

# Contingency tables, example

**Example**

Recall the example of the school distract attempting to reduce the rate of teen drivers who text or email. The school distract created an educational program that was attend by about half the students. Afterwards, a survey was taken of a sample of teen drivers. Each teen driver either attended the program or didn't, and either texted or emailed while driving or didn't.

| | Texted or emailed? | |
|---|---|---|
| Attended program? | Yes | No |
| Yes | 62 | 150 |
| No | 59 | 114 |

# Independence in contingency tables

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An important question that can be asked about data in contingency tables is whether the two factors are independent.

Factors are independent if the value of one factor does not impact the value of the other factor. In other words, if the probability of being in a category of one factor does not change depending on the category of the second factor, for all categories of both factors, then the factors are independent

# Independence in contingency tables

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- For the cancer screening example, if the probability of testing positive is the same regardless of whether the subject has cancer or not, then the test results and cancer status are independent.

# Independence in contingency tables

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- For the cancer screening example, if the probability of testing positive is the same regardless of whether the subject has cancer or not, then the test results and cancer status are independent.
- For the teen driver example, if the probability of a teen driver texting or emailing is the same regardless of whether they attended the educational program or not, then texting or emailing and program attendance are independent.

# Test for independence

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- $H_0$ : The two factors are independent
- $H_a$ : The two factors are dependent
- Test statistic follows a $\chi^2$ (chi-squared) distribution with $(r-1)\times(c-1)$ degrees of freedom

$$\chi^2 = \sum \frac{(O-E)^2}{E}$$

where

- $r$ is the number of rows and $c$ is the number of columns
- $O$ is the observed count for each table cell, from sample
- $E$ is the expected count for each table cell if the factors are independent

# Expected counts

Like with goodness-of-fit tests, the expected count for each cell is the probability for that cell under the null hypothesis times the sample size.

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Recall, if events are independent that the probability of both being true is the product of the probabilities of both.

$$P(A \text{ and } B) = P(A) \times P(B)$$

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$$P(A \text{ and } B) = P(A) \times P(B)$$

Thus, if $A$ is an event of one factor and $B$ is an event of the other factor, the expected count for the cell of $A$ and $B$ is

$$E = P(A) \times P(B) \times n$$

# Expected counts, cont.

The probability of an event of one factor is the marginal probability, the total count for the row or column divided by the total sample size.

| | Factor 2 | | |
|---|---|---|---|
| Factor 1 | B | $\sim$B | Total |
| A | # (A and B) | # (A and $\sim$B) | # A |
| $\sim$A | # ($\sim$A and B) | # ($\sim$A and $\sim$B) | # $\sim$A |
| Total | # B | # $\sim$B | n |

$$P(A) = \frac{\#A}{n} \qquad P(B) = \frac{\#B}{n}$$

# Expected counts, cont.

Thus, the expected count for the cell of A and B is

$$E_{A,B} = P(A) \times P(B) \times n = \frac{\#A}{n} \times \frac{\#B}{n} \times n$$

# Expected counts, cont.

Thus, the expected count for the cell of A and B is

$$E_{A,B} = P(A) \times P(B) \times n = \frac{\#A}{n} \times \frac{\#B}{n} \times n$$

After some algebra, a simpler formula for expected count is

$$E_{A,B} = \frac{\#A \times \#B}{n}$$

# Expected counts, example

## Example

| | Test Result | | |
|---|---|---|---|
| Diagnosis | Positive | Negative | Total |
| Cancer | 74 | 13 | 87 |
| No cancer | 26 | 887 | 913 |
| Total | 100 | 900 | 1000 |

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- $E_{+,\text{cancer}} = \dfrac{100 \times 87}{1000} = 8.7$
- $E_{-,\text{cancer}} = \dfrac{900 \times 87}{1000} = 78.3$

# Expected counts, example

## Example

| Diagnosis | Test Result: Positive | Negative | Total |
|---|---|---|---|
| Cancer | 74 | 13 | 87 |
| | (8.7) | (78.3) | |
| No cancer | 26 | 887 | 913 |
| | (91.3) | (821.7) | |
| Total | 100 | 900 | 1000 |

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For both methods, if conditions are not met to reject, fail to reject the null hypothesis.

# Requirements for tests for independence

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- For every cell, the expected count is at least 5

# Tests for independence in StatCrunch

- Stat → Tables → Contingency → With Summary
- Select the columns that contain observed data
- Select the column that contains row labels
- If desired, select calculated values to be displayed ("Expected count" can be useful)
- Leave "Hypothesis tests" on default value of "Chi-Square test for independence"
- Click "Compute!"
- In the "Chi-Square test" table, the test statistic and p-value are found in "Value" and "P-value"

# Test for independence, example

## Example

Recall the cancer screening data:

| | Test Result | | |
|---|---|---|---|
| Diagnosis | Positive | Negative | Total |
| Cancer | 74 (8.7) | 13 (78.3) | 87 |
| No cancer | 26 (91.3) | 887 (821.7) | 913 |
| Total | 100 | 900 | 1000 |

Test whether cancer diagnosis has an effect on the screening test result at $\alpha = 0.01$ level of significance.

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- $H_0$ : Test result and cancer diagnosis are independent
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- $\chi^2 = 596.47717$
  $p < 0.0001$

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- $\chi^2 = 596.47717$
  $p < 0.0001$
- $p < 0.0001 < 0.01 = \alpha$. Reject null hypothesis.

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- $H_0$ : Test result and cancer diagnosis are independent
  $H_a$ : Test result and cancer diagnosis are dependent (test result is associated with cancer diagnosis)
- Requirements: The smallest expected count, for cancer and positive test, is 8.7 which is larger than 5
- $\chi^2 = 596.47717$
  $p < 0.0001$
- $p < 0.0001 < 0.01 = \alpha$. Reject null hypothesis.
- There is evidence that test results and cancer diagnosis are associated.

# Test for independence, example

## Example

Recall the teen driver data:

| | Texted or emailed? | |
|---|---|---|
| Attended program? | Yes | No |
| Yes | 62 | 150 |
| No | 59 | 114 |

Test whether texting or emailing while driving is associated with program attendance at $\alpha = 0.05$ level of significance.

# Test for independence, example

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**Example**

- $H_0$ : Texting or emailing and program attendance are independent
  $H_a$ : Texting or emailing and program attendance are dependent
  (texting or emailing is associated with program attendance)

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- $H_0$ : Texting or emailing and program attendance are independent
  $H_a$ : Texting or emailing and program attendance are dependent (texting or emailing is associated with program attendance)
- Requirements: We don't have expected counts yet, but the sample size is in the hundreds and no outcome appears very unlikely. We can confirm when we have expected counts.

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- $H_0$ : Texting or emailing and program attendance are independent
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- $\chi^2 = 1.0435298$
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- $p = 0.307 > 0.05 = \alpha$. Fail to reject null hypothesis.

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- $\chi^2 = 1.0435298$
  $p = 0.307$
- $p = 0.307 > 0.05 = \alpha$. Fail to reject null hypothesis.
- There is no evidence that texting or emailing while driving and program attendence are associated.

# Test for independence vs. proportion test

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# Test for independence vs. proportion test

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# Test for independence vs. proportion test

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- The results were $z = -1.0215331$, $p = 0.1535$.
- Remember, the p-value for a two-sided test with the same data is twice that of a one-sided test. So, a two-sided proportion test would have a p-value of $p = 0.1535 \times 2 = 0.307$.

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- The test for independence, while conducted as a one-sided test, is actually a two-sided test in that is does not distinguish between observed values that are lower or higher than expected values.

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- Remember, the p-value for a two-sided test with the same data is twice that of a one-sided test. So, a two-sided proportion test would have a p-value of $p = 0.1535 \times 2 = 0.307$.
- The test for independence, while conducted as a one-sided test, is actually a two-sided test in that is does not distinguish between observed values that are lower or higher than expected values.
- Thus, the test for independence gave us identical results as the equivalent proportion test.

## Group work

- Complete questions 2 and 3.