Stat 201: Statistics I Week 11



July 31, 2017

Chapter 11 Goodness-of-Fit and Contingency Tables

Section 11.2 Goodness-of-Fit

Frequency distributions

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A frequency table is a list of the distribution of a sample drawn from a population.

A test can be conducted to see if the population a sample is drawn from has an expected distribution.

Frequency distributions, example

Example

• A six-sided die is "fair" if the frequencies of each possible result of a roll (1 through 6) are equal.

Given a sample of results from a number rolls of a particular die, a test could be conducted to test whether the die is "fair".

Frequency distributions, example

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M&Ms should have the following distribution of colors:

Color	Blue	Brown	Green	Orange	Red	Yellow
Percent	24	14	16	20	13	14

Given a sample a M&Ms, a test could be conducted to test whether M&Ms really do have that distribution of colors.

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A **goodness-of-fit test** is an hypothesis test which tests whether an observed frequency distribution matches, or fits, an expected distribution.

- ullet H_0 : The frequency counts agree with the expected distribution.
- H_a : The frequency counts do not agree with the expected distribution.
- \bullet Test statistic follows a χ^2 (chi-squared) distribution with k-1 degrees of freedom

$$\chi^2 = \sum \frac{(O-E)^2}{E}$$

where

- ullet k is the number of classes or categories
- ullet O is the observed count for each class or category, from sample
- ullet is the expected count for each class or category if the expected distribution is true

Expected counts

The expected count for each class or category can be calculated by

$$E = P(c) \times n$$

where

- P(c) is the probability of class or category c
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For expected uniform distributions, since P(c)=1/k where k is the number of classes or categories, the expected count for each class is

$$E = \frac{1}{k} \times n = \frac{n}{k}$$

Expected counts, example

Example

• Since a "fair" die has a uniform frequency distribution, the expected counts for each result for a sample of 100 die rolls is

$$E = \frac{n}{k} = \frac{100}{6} = 16.67$$

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• Blue M&Ms are expected to have a frequency of 24%. Thus, out of a sample of 150 M&Ms, the expected count for blue is

$$E = P(c) \times n = 0.24 \times 150 = 36$$

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For both methods, if conditions are not met to reject, fail to reject the null hypothesis.

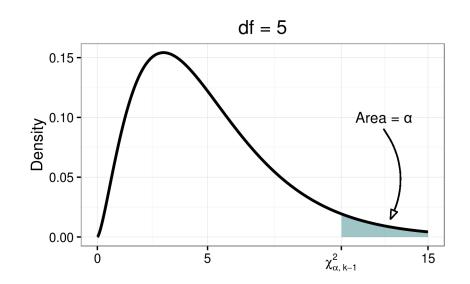
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Note: Chi-square test statistics are always positive and chi-square tests are always one-sided. Large values of χ^2 cause rejection of the null.

Chi-square distribution



Requirements for goodness-of-fit tests

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- For each class or category, the expected count is at least 5

Goodness-of-fit tests in StatCrunch

See "STATCRUNCH Tutorial for Chi-square Goodness-of-fit test"

- ullet Stat o Goodness-of-fit o Chi-Square Test
- Select column that contains observed data
- Specify expected distribution:
 - For uniform distributions, select "All cells in equal proportion"
 - For non-uniform distributions, select the column which contains expected frequencies
- Leave default value of "Expected" for display
- Click "Compute!"
- The test statistic and p-value are found in "Chi-Square" and "P-value"

Example

To determine if there is evidence is is not a die is "fair", roll the die 40 times and perform a goodness-of-fit test on the results.

A "fair" die will have a uniform frequency distribution, so each result has a probability of 1/6 (16.67%).

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- \bullet Find test statistic χ^2 , p-value and report decision

Example

Test whether M&M colors have the frequency distribution that is claimed by the company.

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- \bullet Requirements: The least frequent color is red at 13%. The number of M&Ms counted, n, must be large enough so that $E=0.13\times n\geq 5$ or $n\geq 38.46$

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Section 11.3 Contingency Tables

Contingency tables

Recall, a **contingency table** is a two dimensional table (rows and columns) displaying frequency counts of classes or categories of two factors for a single sample.

Contingency tables, example

Example

Recall the cancer screening example. A sample of 1000 randomly selected people where given a new screening test for a particular kind of of cancer. Each subject either has cancer or doesn't, and either tested positive or tested negative.

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Diagnosis	Positive	Negative
Cancer	74	13
No cancer	26	887

Contingency tables, example

Example

Recall the example of the school distract attempting to reduce the rate of teen drivers who text or email. The school distract created an educational program that was attend by about half the students. Afterwards, a survey was taken of a sample of teen drivers. Each teen driver either attended the program or didn't, and either texted or emailed while driving or didn't.

T		• 1	1 12
Texted	or	email	led (

Attended program?	Yes	No
Yes	62	150
No	59	114

An important question that can be asked about data in contingency tables is whether the two factors are independent.

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Factors are independent if the value of one factor does not impact the value of the other factor. In other words, if the probability of being in a category of one factor does not change depending on the category of the second factor, for all categories of both factors, then the factors are independent

Example

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- For the cancer screening example, if the probability of testing
 positive is the same regardless of whether the subject has cancer
 or not, then the test results and cancer status are independent.
- For the teen driver example, if the probability of a teen driver texting or emailing is the same regardless of whether they attended the educational program or not, then texting or emailing and program attendance are independent.

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- ullet H_0 : The two factors are independent
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- Test statistic follows a χ^2 (chi-squared) distribution with $(r-1)\times(c-1)$ degrees of freedom

$$\chi^2 = \sum \frac{(O-E)^2}{E}$$

where

- ullet r is the number of rows and c is the number of columns
- O is the observed count for each table cell, from sample
- E is the expected count for each table cell if the factors are independent

Expected counts

Like with goodness-of-fit tests, the expected count for each cell is the probability for that cell under the null hypothesis times the sample size.

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Recall, if events are independent that the probability of both being true is the product of the probabilities of both.

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$$P(A \text{ and } B) = P(A) \times P(B)$$

Thus, if A is an event of one factor and B is an event of the other factor, the expected count for the cell of A and B is

$$E = P(A) \times P(B) \times n$$

Expected counts, cont.

The probability of an event of one factor is the marginal probability, the total count for the row or column divided by the total sample size.

Factor 2

Factor 1	В	∼B	Total
А	# (A and B)	$\#$ (A and \sim B)	# A
\sim A	$\#$ (\sim A and B)	$\#$ (\sim A and \sim B)	# ~A
Total	# B	# ∼B	n

$$P(A) = \frac{\#A}{n} \qquad P(B) = \frac{\#B}{n}$$

Expected counts, cont.

Thus, the expected count for the cell of A and B is

$$E_{A,B} = P(A) \times P(B) \times n = \frac{\#A}{n} \times \frac{\#B}{n} \times n$$

Expected counts, cont.

Thus, the expected count for the cell of A and B is

$$E_{A,B} = P(A) \times P(B) \times n = \frac{\#A}{n} \times \frac{\#B}{n} \times n$$

After some algebra, a simpler formula for expected count is

$$E_{A,B} = \frac{\#A \times \#B}{n}$$

Example

Test Result

Diagnosis	Positive	Negative	Total
Cancer	74	13	87
No cancer	26	887	913
Total	100	900	1000

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$$\begin{split} \bullet \ E_{+, \mathrm{cancer}} &= \frac{100 \times 87}{1000} = 8.7 \\ \bullet \ E_{-, \mathrm{cancer}} &= \frac{900 \times 87}{1000} = 78.3 \end{split}$$

Example

Test Result

Diagnosis	Positive Negative		Total
Cancer	74 13		87
	(8.7)	(78.3)	
No cancer	26	887	913
	(91.3)	(821.7)	
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Tests for independence in StatCrunch

See "STATCRUNCH Tutorial for Chi-square test of independence"

- Stat \rightarrow Tables \rightarrow Contingency \rightarrow With Summary
- Select the columns that contain observed data
- Select the column that contains row labels
- If desired, select calculated values to be displayed ("Expected count" can be useful)
- Leave "Hypothesis tests" on default value of "Chi-Square test for independence"
- Click "Compute!"
- In the "Chi-Square test" table, the test statistic and p-value are found in "Value" and "P-value"

Example

Recall the cancer screening data:

Test Result

Diagnosis	Positive Negative		Total
Cancer	74 13		87
	(8.7)	(78.3)	
No cancer	26	887	913
	(91.3)	(821.7)	
Total	100	900	1000

Test whether cancer diagnosis has an effect on the screening test result at $\alpha=0.01$ level of significance.



Example

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- $\chi^2 = 596.47717$ p < 0.0001
- $p < 0.0001 < 0.01 = \alpha$. Reject null hypothesis.
- There is evidence that test results and cancer diagnosis are associated.

Example

Recall the teen driver data:

	lexted or emailed?	
Attended program?	Yes	No
Yes	62	150
No	59	114

Test whether texting or emailing while driving is associated with program attendance at $\alpha=0.05$ level of significance.



Example

• H_0 : Texting or emailing and program attendance are independent H_a : Texting or emailing and program attendance are dependent (texting or emailing is associated with program attendance)

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- $\chi^2 = 1.0435298$ p = 0.307
- $p=0.307>0.05=\alpha$. Fail to reject null hypothesis.
- There is no evidence that texting or emailing while driving and program attendence are associated.

Recall, we performed a test with this data in Chapter 9. Testing the alternative hypothesis that the proportion of teen drivers who texted or emailed while driving was lower than for teens who attended the educational program than those who did not.

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The test for independence, while conducted as a one-sided test, is actually a two-sided test in that is does not distinguish between observed values that are lower or higher than expected values.

Thus, the test for independence gave us identical results as the equivalent proportion test.

Example

The Tortilla and Cheese Organization (TACO) thinks that preferences for types of tacos are the same for men and women. They conduct a survey and collect the following data:

Type of taco

Gender	Beef	Pork	Chicken	Fish
Men	105	34	56	27
Women	83	29	75	35

Test the claim the taco preference is the same for men and women at lpha=0.05 level of significance.



Example

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- $\chi^2 = 6.7592767$ p = 0.08
- $p = 0.08 > 0.05 = \alpha$. Fail to reject null hypothesis.
- There is no evidence to reject the claim that taco preferences are the same for men and women.