ECON526: Quantitative Economics with Data Science Applications

Applications of Linear Algebra

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Overview

Motivation and Materials

- In this lecture, we will cover some applications of the tools we developed in the previous lecture
- The goal is to build some useful tools to sharpen your intuition on linear algebra and eigenvalues/eigenvectors, and practice some basic coding
- We introduce scikit-learn, a package for old-school (i.e. not deep learning or neural networks) ML and data analysis
    - Introduces "unsupervised learning" (i.e., tools to interpret data structure without any forecasts/predictions)
- Some additional material and references
    - QuantEcon Python
    - QuantEcon DataScience
    - A First Course in Quantitative Economics with Python

Packages

```python
import numpy as np
import matplotlib.pyplot as plt
import scipy
from numpy.linalg import cond, matrix_rank, norm
from scipy.linalg import inv, solve, det, eig, lu, eigvals
from scipy.linalg import solve_triangular, eigvalsh, cholesky
```

New Packages for Data Science and ML

```python
import seaborn as sns
import pandas as pd
from sklearn.decomposition import PCA
from sklearn.cluster import KMeans
```

Difference Equations

Linear Difference Equations as Iterative Maps

- Consider $A : \mathbb{R}^N \to \mathbb{R}^N$ as the linear map for the state $x_t \in \mathbb{R}^N$
- An example of a linear difference equation is

$$x_{t+1} = Ax_t$$

where

$$A \equiv \begin{bmatrix} 0.9 & 0.1 \\ 0.5 & 0.8 \end{bmatrix}$$

```python
A = np.array([[0.9, 0.1], [0.5, 0.8]])
x_0 = np.array([1, 1])
x_1 = A @ x_0
print(f"x_1 = {x_1}, x_2 = {A @ x_1}")
```

```
x_1 = [1.  1.3], x_2 = [1.03 1.54]
```

Iterating with $\rho(A) > 1$

Iterate $x_{t+1} = Ax_t$ from x_0 for $t = 100$

```python
x_0 = np.array([1, 1])
t = 200
print(f"rho(A) = {np.max(np.abs(eigvals(A)))}")
print(f"x_{t} = {np.linalg.matrix_power(A, t) @ x_0}")
```

```
rho(A) = 1.079128784747792
x_200 = [3406689.32410673 6102361.18640516]
```

- Diverges to $x_\infty = \begin{bmatrix} \infty & \infty \end{bmatrix}^T$
- $\rho = 1 + 0.079$ says in the worst case (i.e., $x_t \propto$ the eigenvector associated with $\lambda = 1.079$ eigenvalue), expands by 7.9% on each iteration

Iterating with $\rho(A) < 1$

```python
A = np.array([[0.6, 0.1], [0.5, 0.8]])
print(f"rho(A) = {np.max(np.abs(eigvals(A)))}")
print(f"x_{t} = {np.linalg.matrix_power(A, t) @ x_0}")
```

```
rho(A) = 0.9449489742783178
x_200 = [6.03450418e-06 2.08159603e-05]
```

- Converges to $x_\infty = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$

Iterating with $\rho(A) = 1$

- To make a matrix that has $\rho(A) = 1$ reverse eigendecomposition!
- Leave previous eigenvectors in Q, change Λ to force $\rho(A)$ directly

```python
Q = np.array([[-0.85065081, -0.52573111], [0.52573111, -0.85065081]])
print(f"check orthogonal: dot(x_1,x_2) approx 0: {np.dot(Q[:,0], Q[:,1])}")
Lambda = [1.0, 0.8]  # choosing eigenvalue so max_n|lambda_n| = 1
A = Q @ np.diag(Lambda) @ inv(Q)
print(f"rho(A) = {np.max(np.abs(eigvals(A)))}")
print(f"x_{t} = {np.linalg.matrix_power(A, t) @ x_0}")
```

```
check orthogonal: dot(x_1,x_2) approx 0: 0.0
rho(A) = 1.0
x_200 = [ 0.27639321 -0.17082039]
```

Unemployment Dynamics

Dynamics of Employment without Population Growth

- Consider an economy where in a given year $\alpha = 5\%$ of employed workers lose job and $\phi = 10\%$ of unemployed workers find a job
- We start with $E_0 = 900,000$ employed workers, $U_0 = 100,000$ unemployed workers, and no birth or death. Dynamics for the year:

$$E_{t+1} = (1-\alpha)E_t + \phi U_t$$
$$U_{t+1} = \alpha E_t + (1-\phi)U_t$$

- Can write this as a matrix equation

$$\underbrace{\begin{bmatrix} E_{t+1} \\ U_{t+1} \end{bmatrix}}_{X_{t+1}} = \underbrace{\begin{bmatrix} 1-\alpha & \phi \\ \alpha & 1-\phi \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} E_t \\ U_t \end{bmatrix}}_{X_t}$$

Simulating

Simulate by iterating $X_{t+1} = AX_t$ from X_0 until $T = 100$

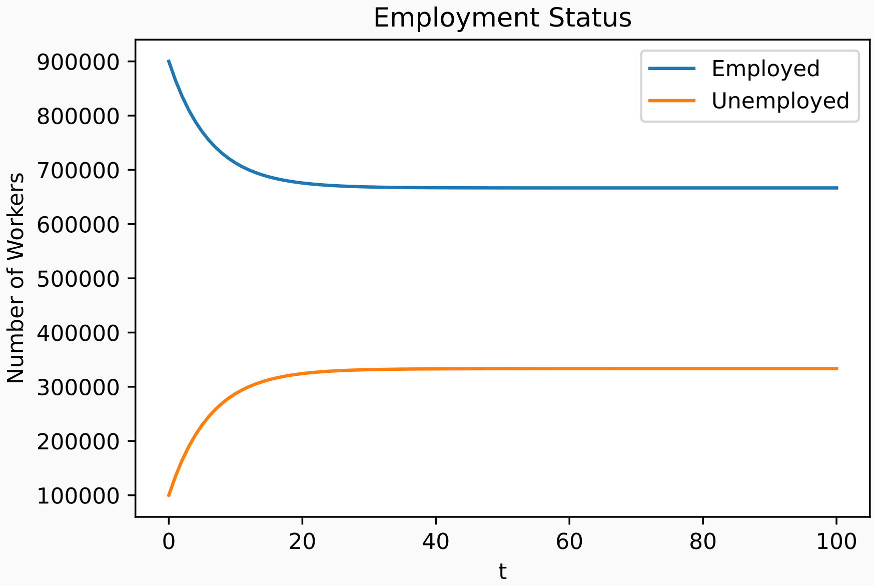
```python
def simulate(A, X_0, T):
    X = np.zeros((2, T+1))
    X[:,0] = X_0
    for t in range(T):
        X[:,t+1] = A @ X[:,t]
    return X
X_0 = np.array([900000, 100000])
A = np.array([[0.95, 0.1], [0.05, 0.9]])
T = 100
X = simulate(A, X_0, T)
print(f"X_{T} = {X[:,T]}")
```

```
X_100 = [666666.6870779  333333.31292209]
```

Plotting Code

```python
fig, ax = plt.subplots(figsize=(6, 4))
ax.plot(range(T+1), X.T, label=["Employed", "Unemployed"])
ax.set(xlabel="t", ylabel="Number of Workers", title="Employment Status")
ax.legend()
plt.show()
```

Dynamics of Unemployment



Convergence to a Longrun Distribution

- Find X_∞ by iterating $X_{t+1} = AX_t$ many times from a X_0?
    - Check if it has converged with $X_\infty \approx AX_\infty$
    - Is X_∞ the same from any X_0? Will discuss "ergodicity" later
- Alternatively, note that this expression is the same as

$$1 \times \bar{X} = A\bar{X}$$

- i.e, a $\lambda = 1$ where $\bar{X}$ is the corresponding eigenvector of A
- Is $\lambda = 1$ always an eigenvalue? (yes if all $\sum_{n=1}^{N} A_{ni} = 1$ for all i)
- Does $\bar{X} = X_\infty$? For any X_0?
- Multiple eigenvalues with $\lambda = 1 \implies$ multiple $\bar{X}$

Using the First Eigenvector for the Steady State

```python
Lambda, Q = eig(A)
print(f"real eigenvalues = {np.real(Lambda)}")
print(f"eigenvectors are column-by-column in Q =\n{Q}")
print(f"first eigenvalue = 1? {np.isclose(Lambda[0], 1.0)}")
X_bar = Q[:,0] / np.sum(Q[:,0]) * np.sum(X_0)
print(f"X_bar = {X_bar}\nX_{T} = {X[:,T]}")
```

```
real eigenvalues = [1.   0.85]
eigenvectors are column-by-column in Q =
[[ 0.89442719 -0.70710678]
 [ 0.4472136   0.70710678]]
first eigenvalue = 1? True
X_bar = [666666.66666667 333333.33333333]
X_100 = [666666.6870779  333333.31292209]
```

Using the Second Eigenvalue for the Convergence Speed

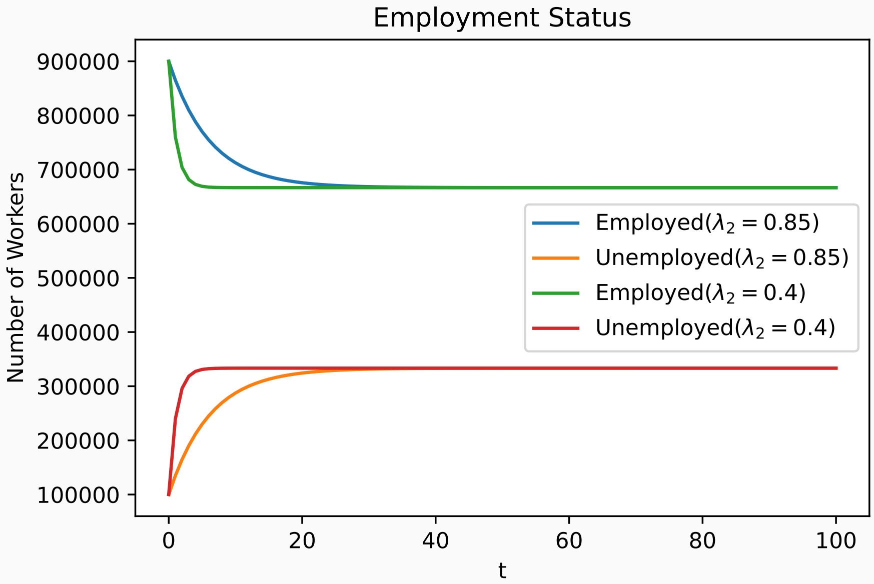
- The second largest $(\lambda_2 < 1)$ provides information on the speed of convergence
  - 0 is instantaneous convergence here
  - 1 is no convergence here
- We will create a new matrix with the same steady state, different speed
  - To do this, build a new matrix with the same eigenvectors (in particular the same eigenvector associated with the $\lambda = 1$ eigenvalue)
  - But we will replace the eigenvalues $\begin{bmatrix}1.0 & 0.85\end{bmatrix}$ with $\begin{bmatrix}1.0 & 0.5\end{bmatrix}$
  - Then we will reconstruct A matrix and simulate again
- Intuitively we will see the that the resulting A_{fast} implies α and ϕ which are larger by the same proportion

Simulating with Different Eigenvalues

```python
Lambda_fast = np.array([1.0, 0.4])
A_fast = Q @ np.diag(Lambda_fast) @ inv(Q) # same eigenvectors
print("A_fast =\n", A_fast)
print(f"alpha_fast/alpha = {A_fast[1,0]/A[1,0]:.2g}, \
phi_fast/phi = {A_fast[0,1]/A[0,1]:.2g}")
X_fast = simulate(A_fast, X_0, T)
print(f"X_{T} = {X_fast[:,T]}")
```

```
A_fast =
 [[0.8 0.4]
 [0.2 0.6]]
alpha_fast/alpha = 4, phi_fast/phi = 4
X_100 = [666666.66666667 333333.33333333]
```

Dynamics of Unemployment For Difference Convergence Speeds



Present Discounted Values

Geometric Series

- Assume dividends follow $y_{t+1} = G y_t$ for $t = 0, 1, \ldots$ and y_0 is given
- $G > 0$, dividends are discounted at factor $\beta > 1$ then $p_t = \sum_{s=0}^{\infty} \beta^s y_{t+s} = \frac{y_t}{1-\beta G}$
- More generally if $x_{t+1} = A x_t$, $x_t \in \mathbb{R}^N$, $y_t = G x_t$ and $A \in \mathbb{R}^{N\times N}$, then

$$\begin{aligned}
p_t &= y_t + \beta y_{t+1} + \beta^2 y_{t+2} + \ldots = G x_t + \beta G A x_t + \beta G A A x_t + \ldots \\
&= \sum_{s=0}^{\infty} \beta^s A^s y_t \\
&= G(I - \beta A)^{-1} x_t \quad , \text{ if } \rho(A) < 1/\beta
\end{aligned}$$

- i.e., spectral radius of A, the maximum scaling, must be less than discounting
- Intuition from univariate: of $G \in \mathbb{R}^{1\times 1}$ then $\text{eig}(G) = G$, so must have $|\beta G| < 1$

PDV Example

Here is an example with $1 < \rho(A) < 1/\beta$. Try with different A

```python
beta = 0.9
A = np.array([[0.85, 0.1], [0.2, 0.9]])
G = np.array([[1.0, 1.0]]) # row vector
x_0 = np.array([1.0, 1.0])
p_t = G @ solve(np.eye(2) - beta * A, x_0)
#p_t = G @ inv(np.eye(2) - beta * A) @ x_0 # alternative
rho_A = np.max(np.abs(np.real(eigvals(A))))
print(f"p_t = {p_t[0]:.4g}, spectral radius = {rho_A:.4g}, 1/beta = {1/beta:.
```

```
p_t = 24.43, spectral radius = 1.019, 1/beta = 1.111
```

Latent Variables

Features, Labels, and Latents

- Data science and ML often use different terminology than economists:
  - **Features** are economists **explanatory or independent variables**. They have the key source of variation to make predictions and conduct counterfactuals
  - **Labels** correspond to economists **observables or dependent variables**
  - **Latent Variables** are **unobserved variables**, typically sources of heterogeneity or which may drive both the dependent and independent variables
- Economists will use theory and experience to transform data (i.e., what ML people call "feature engineering") for better explanatory power or map to theoretical models
- ML refers to methods using only **features** as **unsupervised learning**. The structure of the underlying data can teach you about its data generating process
- Key: uncover and interpret latent variables using statistics coupled with assumptions from economic theory. There is theory beyond all interpretation

Principle Components and Factor Analysis

- Another application of eigenvalues is dimension reduction, which simplifies **features** by uncovering **latent** variables. Unsupervised
- One technique is Principle Components Analysis (PCA), which uncovers latent variables that capture the primary directions of variation in the underlying data
  - May allow mapping data into a lower-dimensional, uncorrelated set of features
  - Uses Singular Value Decomposition (SVD) - a generalization of eigendecomposition to non-square matrices
- Given a matrix $X \in \mathbb{R}^{N \times M}$, can we find a lower-dimensional representation $Z \in \mathbb{R}^{N \times L}$ for $L < M$ that captures the most variation in X?
- The goal is to invert the X data to find the Z—and provide a mapping to reduce the dimensionality for future data.
  - One of many methods. Many algorithms in ML and econometrics have similar goals

Singular Value Decomposition

PCA can be interpreted with an eigendecomposition, but can be more confusing than just using the SVD directly. An SVD for any $X \in \mathbb{R}^{N \times M}$ is:

$$X = U\Sigma V^T$$

- $\Sigma \in \mathbb{R}^{N \times M}$ where
  - The diagonal elements are called singular values, and there are zeros everywhere else. If $M < N$ then there M singular values $(\sigma_1, \ldots \sigma_M)$
  - Those singular values are also the square roots of the eigenvalues of XX^T (or $X^T X$)
  - The number of non-zero singular values is the rank of the matrix X
- $U \in \mathbb{R}^{N \times N}$ and $V \in \mathbb{R}^{M \times M}$ are orthogonal matrices
  - U is the set of eigenvectors of XX^T and V is the set of eigenvectors of $X^T X$
- Many applications of SVD (e.g., least squares, checking rank), in part because it is especially "numerically stable" (i.e., not sensitive to the roundoff errors we talked about previously)

Decoming the Data

A key result is that we can decompose the data into a sum of outer products of the eigenvectors and singular values. Assume ordered so that $\sigma_1 \geq \sigma_2 \geq ... \geq \sigma_M$:

$$X = U\Sigma V^T = \sum_{m=1}^{M} \sigma_m u_m v_m^T$$

Where here we assumed that the rank$(X) \leq M$ and

- $u_m \in \mathbb{R}^N$ is the m-th column of U and $v_m \in \mathbb{R}^M$ is the mth column of V
- So $u_m v_m^T$ is an $N \times M$ matrix but you can show that it is rank-1. i.e., you can decompose it into the product of two vectors.
- Intuition: rank r if it can be decomposed into the sum of r rank-1 matrices
  - Alternatively, can interpret rank of an $N \times M$ matrix is 3 if can find a $A \in \mathbb{R}^{N\times 3}$ and $B \in \mathbb{R}^{3\times M}$ such that $X = AB$
- Remember: this works for **any** matrix $X \in \mathbb{R}^{N\times M}$

Dimension Reduction

- Frequently $\sigma_1 \gg \sigma_M$ and the σ_m may decay quickly, so we can approximate X with fewer terms by truncating the sum at $L < M$.

$$X \approx \sum_{m=1}^{L} \sigma_m u_m v_m^T$$

- Note that if the data is actually lower-dimensional in a suitable space (e.g., rank$(X) = L < M)$ then $\sigma_m = 0$ for $L < m \leq M$, so the truncated sum is exact
- Can prove that if we truncate at $L < M$, this is the best rank L approximation to X according to some formal criteria.
    - Intuitively, finds directions of the data that capture the most variation in the covariance matrix
    - Can prove it is the solution to the optimization problem to explain the most variation in the data with the lowest dimensionality

Creating a Dataset with Latent Factors

Create a dataset with two latent factors, the first dominating

```python
N = 50 # number of observations
L, M = 2, 3 # number of latent and observed factors
Z = np.random.randn(N, L) # latent factors
F = np.array([[1.0, 0.05], # X_1 = Z_1 + 0.05 Z_2
              [2.0, 0.0], # X_2 = 2 Z_1
              [3.0, 0.1]]) # X_3 = 3 Z_1 + 0.1 Z_2
X = Z @ F.T + 0.1 * np.random.randn(N, M) # added noise
```

PCA Without Dimension Reduction

- See QuantEcon SVD for coding yourself. We will use the sklearn package
- The explained variance is the fraction of the variance explained by each factor

```
pca = PCA(n_components=3)
pca.fit(X)
with np.printoptions(precision=4, suppress=True, threshold=5):
  print(f"Singular Values (sqrt eigenvalues):\n{pca.singular_values_}")
  print(f"Explained Variance (ordered):\n{pca.explained_variance_ratio_}")
```

```
Singular Values (sqrt eigenvalues):
[22.6551  0.8492  0.5553]
Explained Variance (ordered):
[0.998  0.0014 0.0006]
```

Dimension Reduction with PCA

```python
pca = PCA(n_components=2) # one less, and correctly specified
Z_hat = pca.fit_transform(X) # transformed by dropping last factor
# Scale and sign may not match due to indeterminacy
print(f"Correlation of Z_1 to Z_hat_1 = {np.corrcoef(Z.T, Z_hat.T)[0,2]}")
print(f"Correlation of Z_2 to Z_hat_2 = {np.corrcoef(Z.T, Z_hat.T)[1,3]}")
```

```
Correlation of Z_1 to Z_hat_1 = -0.9990001214107445
Correlation of Z_2 to Z_hat_2 = 0.590890491037916
```

Interpreting the Results

- The first factor in the decomposition is nearly perfectly (positive or negatively) correlated with the more important latent factor
    - The sign could have gone either way. The key is the shared information
    - How could you have known the sign is indeterminate?
- The 2nd factor has a good but not great correlation with the 2nd latent. Why?
- The variance decomposition that gave a 3rd factor with non-zero variance
    - In our process, there are only two latent variables. Why didn't it figure it out?
- How could you have changed the DGP to make this **less** successful?
- **Warning:** have just scratched the surface to build some intuition. Many missing details and caveats (e.g., you may want to rescale your data, make sure everything is de-meaned if implementing yourself, etc.)

Auto-Encoders and Dimensionality Reduction

- General class of problems which they call auto-encoders in ML/data science
    - Function f, the encoder, maps X to a latent space Z, which may be lower-dimensional
    - Function g, the decoder, maps points in the latent space Z back to X
    - θ_e and θ_d are parameters for f and g which we are trying to find
- Then the goal is to find the θ_e and θ_d parameters for our encoder, f, and decoder, g, where for as many X as possible we have

$$g(f(x;\theta_e);\theta_d) \approx x$$

- If $z = f(x;\theta_e)$ may be lower-dimensional, but may be useful regardless
- In more advanced machine learning examples, intuition seems to come up frequently. Related to embeddings, which come up with NLP, networks, etc.

PCA as a Linear Auto-Encoder

$$\min_{\theta_e,\theta_d} \mathbb{E}||g(f(x;\theta_e);\theta_d) - x||_2^2, \quad \text{If we had distribution for } x$$

$$\min_{\theta_e,\theta_d} \frac{1}{N}\sum_{n=1}^{N}||g(f(x_n;\theta_e);\theta_d) - x_n||_2^2, \quad \text{Using data as empirical distribution}$$

PCA is a linear encoder and decoder, where $f(x) = W^T x$ and $g(z) = Wz$ where $W \in \mathbb{R}^{M\times L}$. If $\hat{x} \approx WW^T x$, "reconstruction error" is $||\hat{x} - x||_2^2$.

$$\min_{W} \frac{1}{N}\sum_{n=1}^{N}||W \overbrace{W^T x_n}^{z_n = f(x_n;W)} - x_n||_2^2, \quad \text{with } W^T W = I$$

Can show the solution gives equivalent to PCA! For fixed L latent space size, W are the first L columns of V from the SVD (sorted by size of singular values)

Discrete Latent Variables

Clustering and Discrete Latent Variables

- PCA was a way to uncover continuous latent variables or find low-dimensional continuous approximations
- But latent variables may be discrete (e.g., types of people, firms)
- Hidden discrete variables require assigning observations to groups
- Clustering lets you take a set of observations with (potentially) variables (i.e., features) and try to assign a discrete latent variable to each observation
    - Sometimes, we know the number of groups from theory, usually, we do not
    - While some are statistical and probabilistic, most methods assign a single latent type rather than a distribution
    - Choosing the number of groups to assign to is a challenge that requires theory and regularization - which we will avoid here
    - Instead, just as with PCA we will choose the number of groups ad-hoc rather than in a disciplined way

Partitioning Sets

- Let $X \in \mathbb{R}^{N \times M}$ with $x_1, \dots x_N \in \mathbb{R}^M$ the individual observations
- Assume that each x_n has a latent discrete $k \in \{1, \dots K\}$ then we can assign each observation to one group
  - $\mathbf{S} \equiv \{S_1, \dots, S_K\}$ where each $n = 1, \dots N$ is in exactly one S_k (i.e. a partition)
- The goal is to find the partition which is the most likely to assign each x_n the correct latent variable k
- An alternative interpretation is to think of this as a dimension-reduction technique that reduces complicated data into a low-dimensional discrete variable
- In economics, we will sometimes cluster on some observations to reduce the dimension, then leave others continuous

k-means Clustering

- If theory suggests that $n \in S_k$ with similar latent variables should have similar x_n
  - Group observations that are close or similar to each other
  - As always in linear algebra, close suggests using a norm. The Euclidean norm in the M dimensional feature space is a good baseline
- The objective of k-means is to choose the partition $\mathbf{S}$ which minimizes the norm between observations within each group (normalized by group size $|S_k|$):

$$\min_{\mathbf{S}} \sum_{k=1}^{K} \frac{1}{|S_k|} \sum_{x_n, x_{n'} \in S_k} ||x_n - x_{n'}||_2^2$$

- Using standard Euclidean norm between two elements in S_k

$$||x_n - x_{n'}||_2^2 = \sum_{m=1}^{M} (x_{nm} - x_{n'm})^2$$

k-means Objective Function

- Can prove that the previous objective is equivalent to minimizing the sum of the squared distances from the group k's mean

$$\min_{\mathbf{S}} \sum_{k=1}^{K} \sum_{n \in S_k} ||x_n - \bar{x}_k||_2^2$$

- Where the mean of group k is standard, and across all m features

$$\bar{x}_k \equiv \frac{1}{|S_k|} \sum_{x_n \in S_k} x_n$$

- Careful with using wildly different scales (i.e. $\bar{x}_k$ may be dominated by one feature)

Generating Data with Latent Groups

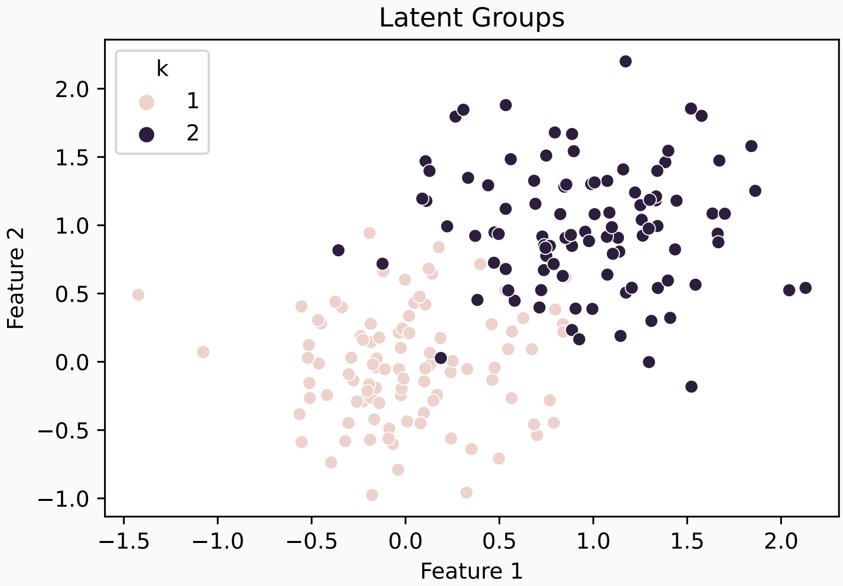
Generate data with 2 features and 2 latent groups and see how k-means does

```
mu_1 = np.array([0.0, 0.0]) # mean of k=1
mu_2 = np.array([1.0, 1.0]) # mean of k=2
sigma = np.array([[0.2, 0], [0, 0.2]]) # use same variance
N = 100 # observations
X_1 = np.random.multivariate_normal(mu_1, sigma, N)
X_2 = np.random.multivariate_normal(mu_2, sigma, N)
df_1 = pd.DataFrame({"f1": X_1[:, 0], "f2": X_1[:, 1], "k": 1})
df_2 = pd.DataFrame({"f1": X_2[:, 0], "f2": X_2[:, 1], "k": 2})
df = pd.concat([df_1, df_2], ignore_index=True)
```

Plotting Code with Seaborn

```python
fig, ax = plt.subplots(figsize=(6, 4))
sns.scatterplot(data=df, x="f1", y="f2", hue="k", ax=ax)
ax.set(xlabel="Feature 1", ylabel="Feature 2", title="Latent Groups")
plt.show()
```

Plot of Features and Latents



k-means to Recover the Latent Groups

- Run k-means with 2 clusters and check the results
- If correlation is close to 1 then succesfully recovered the latent groups
- If the correlation is close to -1 then it was succesful. The latent groups $\hat{k}$ numbers are ordered arbitrarily, just as k was

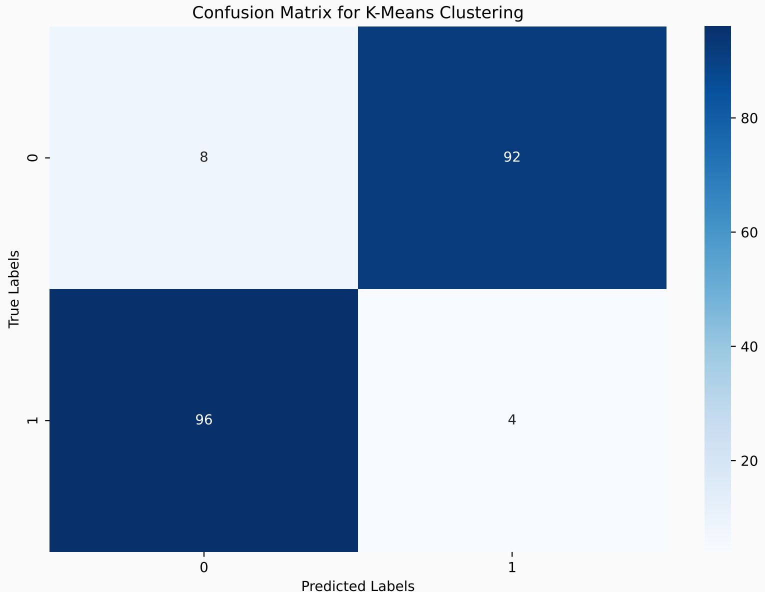
```
kmeans = KMeans(n_clusters=2, random_state=0)
k_hat = kmeans.fit_predict(df[["f1", "f2"]])
df["k_hat"] = k_hat + 1
corr = df["k"].corr(df["k_hat"])
print(f"Correlation between k and k_hat:{corr:.2f}")
```

```
Correlation between k and k_hat:-0.88
```

Confusion Matrix

```python
from sklearn.metrics import confusion_matrix

# compute confusion matrix
cm = confusion_matrix(df["k"], df["k_hat"])

# plot confusion matrix
sns.heatmap(cm, annot=True, cmap='Blues')
plt.xlabel('Predicted k')
plt.ylabel('True k')
plt.title('Confusion Matrix for K-Means Clustering')
plt.show()
```

Confusion Matrix



Potentially Swap $\hat{k}$ and Compare

Label ordering arbitary, so “confusion matrix might require reordering to compare

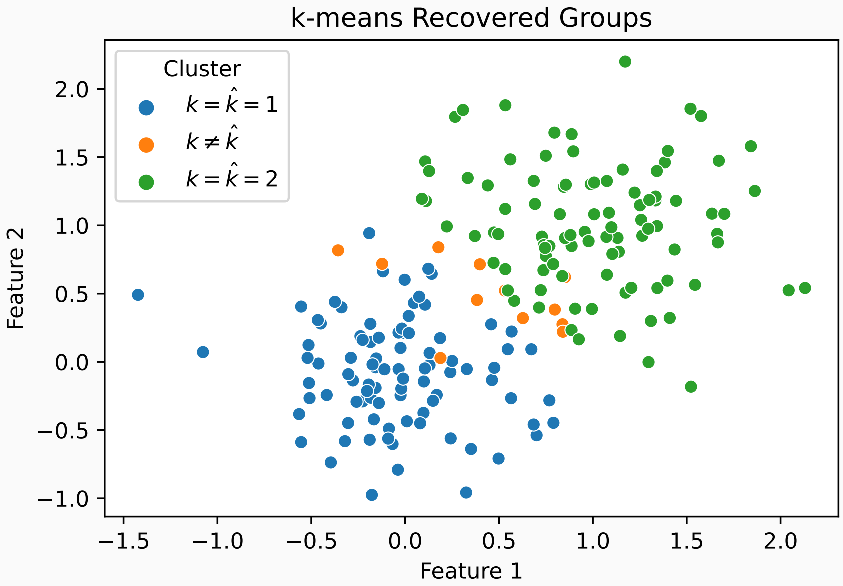
```python
if df['k'].corr(df['k_hat']) < 0.5:
  df['k_hat'] = df['k_hat'].replace({1: 2, 2: 1})
  print(f"Correlation now {df['k'].corr(df['k_hat'])}")

df['Cluster'] = df.apply(lambda x: rf"$k=\hat{{k}}={{{x['k']:.0g}}}$"
                         if x['k'] == x['k_hat'] else r'$k \neq \hat{k}$',
                         axis=1)
```

```
Correlation now 0.8807048459279798
```

Plotting the Uncovered Latent Groups

```python
fig, ax = plt.subplots(figsize=(6, 4))
sns.scatterplot(data=df, x="f1", y="f2", hue="Cluster", ax=ax)
ax.set(xlabel="Feature 1", ylabel="Feature 2",\
       title="k-means Recovered Groups")
plt.show()
```

Plotting the Uncovered Latent Groups



(Optional) Matrix Conditioning and Stability

Matrix Conditioning

- Poorly conditioned matrices can lead to inaccurate or wrong solutions
- Tends to happen when matrices are close to singular or when they have very different scales - so there will be times when you need to rescale your problems

```python
eps = 1e-7
A = np.array([[1, 1], [1 + eps, 1]])
print(f"A =\n{A}")
print(f"A^{-1} =\n{inv(A)}")
```

```
A =
[[1.        1.        ]
 [1.0000001 1.        ]]
A^-1 =
[[-9999999.99336215  9999999.99336215]
 [10000000.99336215 -9999999.99336215]]
```

Condition Numbers of Matrices

- $\det(A) \approx 0$ may say it is “almost” singular, but it is not scale-invariant
- cond$(A) \equiv ||A|| \cdot ||A^{-1}||$ where $|| \cdot ||$ is the matrix norm - expensive to calculate in practice. Connected to eigenvalues cond$(A) = |\frac{\lambda_{max}}{\lambda_{min}}|$
- Scale free measure of numerical issues for a variety of matrix operations
- Intuition: if cond$(A) = K$, then $b \to b + \nabla b$ change in b amplifies to a $x \to x + K\nabla b$ error when solving $Ax = b$.
- See Matlab Docs on inv for example, where `inv` is a bad idea due to poor conditioning

```python
print(f"condition(I) = {cond(np.eye(2))}")
print(f"condition(A) = {cond(A)}, condition(A^(-1)) = {cond(inv(A))}")
```

```
condition(I) = 1.0
condition(A) = 40000001.962777555, condition(A^(-1)) = 40000002.02779216
```

Example with Interpolation

- Consider fitting data $x \in \mathbb{R}^{N+1}$ and $y \in \mathbb{R}^{N+1}$ with an N-degree polynomial
- That is, find $c \in \mathbb{R}^{N+1}$ such that

$$\begin{aligned} c_0 + c_1 x_1 + c_2 x_1^2 + \ldots + c_N x_1^N &= y_1 \\ \ldots &= \ldots \\ c_0 + c_1 x_N + c_2 x_N^2 + \ldots + c_N x_N^N &= y_N \end{aligned}$$

- Which we can then use as $P(x) = \sum_{n=0}^{N} c_n x^n$ to interpolate between the points

Writing as a Linear System

- Define a matrix of all of the powers of the x values

$$A \equiv \begin{bmatrix} 1 & x_0 & x_0^2 & \dots & x_0^N \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_N & x_N^2 & \dots & x_N^N \end{bmatrix}$$

- Then solve for c as the solution to,

$$Ac = y$$

- Which we can solve using our tools. As long as x_n are unique, it is A is invertible
- Let's look at the numerical error here from the interpolation using the inf-norm, i.e., $||x||_\infty = \max_n |x_n|$

Solving an Example

```python
N = 5
x = np.linspace(0.0, 10.0, N + 1)
y = np.exp(x)  # example function to interpolate
A = np.array([[x_i**n for n in range(N + 1)] for x_i in x])  # or np.vander
c = solve(A, y)
c_inv = inv(A) @ y
print(f"error = {norm(A @ c - y, np.inf)}, \
error using inv(A) = {norm(A @ c_inv - y, np.inf)}")
print(f"cond(A) = {cond(A)}")
```

```
error = 1.574562702444382e-11, error using inv(A) = 1.1932570487260818e-09
cond(A) = 564652.3214000753
```

Things Getting Poorly Conditioned Quickly

```python
N = 10
x = np.linspace(0.0, 10.0, N + 1)
y = np.exp(x)  # example function to interpolate
A = np.array([[x_i**n for n in range(N + 1)] for x_i in x])  # or np.vander
c = solve(A, y)
c_inv = inv(A) @ y # Solving with inv(A) instead of solve(A, y)
print(f"error = {norm(A @ c - y, np.inf)}, \
error using inv(A) = {norm(A @ c_inv - y, np.inf)}")
print(f"cond(A) = {cond(A)}")
```

```
error = 5.334186425898224e-10, error using inv(A) = 6.227197984457e-06
cond(A) = 4462824600234.486
```

Matrix Inverses Fail Completely for $N = 20$

```python
N = 20
x = np.linspace(0.0, 10.0, N + 1)
y = np.exp(x)  # example function to interpolate
A = np.array([[x_i**n for n in range(N + 1)] for x_i in x])  # or np.vander
c = solve(A, y)
c_inv = inv(A) @ y # Solving with inv(A) instead of solve(A, y)
print(f"error = {norm(A @ c - y, np.inf)}, \
error using inv(A) = {norm(A @ c_inv - y, np.inf)}")
print(f"cond(A) = {cond(A):.4g}")
```

```
error = 6.784830475226045e-10, error using inv(A) = 31732.823760853855
cond(A) = 1.697e+24
```

Moral of this Story

- Use `solve`, which is faster and can often solve ill-conditioned problems. Rarely use `inv`, and only when you know the problem is well-conditioned
- Check conditioning of matrices when doing numerical work as an occasional diagnostic, as it is a good indicator of potential problems and collinearity
- For approximation, never use a monomial basis for polynomials
    - Prefer polynomials like Chebyshev, which are designed to be as orthogonal as possible

```python
N = 40
x = np.linspace(-1, 1, N+1)  # Or any other range of x values
A = np.array([[np.polynomial.Chebyshev.basis(n)(x_i) for n in range(N+1)] for
A_monomial = np.array([[x_i**n for n in range(N + 1)] for x_i in x])  # or np
print(f"cond(A) = {cond(A):.4g}, cond(A_monimial) = {cond(A_monomial):.4g}")
```

```
cond(A) = 3.64e+09, cond(A_monimial) = 2.926e+18
```

(Optional) Factors within a Porfolio Model

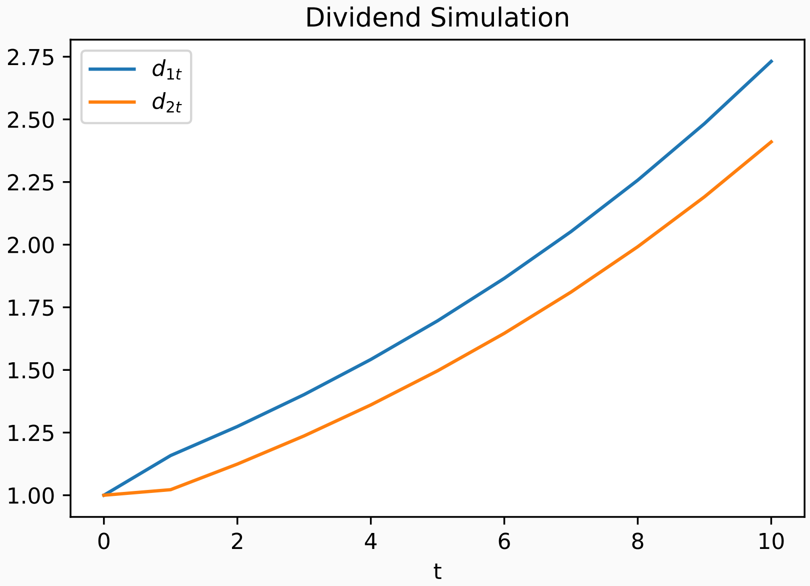
A Portfolio Example

- Two assets pay dividends $d_t \equiv \begin{bmatrix} d_{1t} & d_{2t} \end{bmatrix}^T$ following $d_{t+1} = A\, d_t$ from d_0
- Porfolio has $G \equiv \begin{bmatrix} G_1 & G_2 \end{bmatrix}$ shares of each asset and you discount at rate β

```python
A = np.array([[0.6619469, 0.49646018],[0.5840708, 0.4380531]])
G = np.array([[10.0, 4.0]])
d_0 = np.array([1.0, 1.0])
T, beta = 10, 0.9
p_0 = G @ solve(np.eye(2) - beta * A, d_0)
d = simulate(A, d_0, T)
y = G @ d # total dividends from portfolio
print(f"Portfolio value at t=0 is {p_0[0]:.5g}, total dividends at time {T} i
```

```
Portfolio value at t=0 is 1424.5, total dividends at time 10 is 36.955
```

Dividends Seem to Grow at a Similar Rate?



Digging Deeper

- Let's do an eigendecomposition to analyze the factors

```python
Lambda, Q = eig(A)
print(np.real(Lambda))
```

```
[ 1.10000000e+00 -2.65486733e-09]
```

- The first eigenvector is 1.1, but the second is very close to zero!
    - (In fact, I rigged it to be zero by constructing from a Λ, so this is all numerical copy/paste errors)
- Suggests that maybe only one latent factor driving both d_{1t} and d_{2t}?
- Of course, you may have noticed that the columns in the matrix looked collinear, which was another clue.

Evolution Matrix is Very Simple with $\lambda_2 = 0$

If we stack columns $Q \equiv \begin{bmatrix} q_1 & q_2 \end{bmatrix}$ then,

$$A = Q\Lambda Q^{-1} = Q \begin{bmatrix} \lambda_1 & 0 \\ 0 & 0 \end{bmatrix} Q^{-1} = \lambda_1 q_1 q_1^{-1}$$

```python
lambda_1 = np.real(Lambda[0])
q_1 = np.reshape(Q[:,0], (2,1))
q_1_inv = np.reshape(inv(Q)[0,:], (1,2))
norm(A - lambda_1 * q_1 @ q_1_inv) # pretty close to zero!
```

```
2.663274500543771e-09
```

Transforming to the Latent State

- Recall: $A = Q\Lambda Q^{-1}$ can be interpreted as:
  - Transformation to latent space, scaling, transform back
- We can demonstrate this in our example:
  - Transforming d_0 to ℓ_0 using q_1^{-1}
  - Evolving ℓ_t from ℓ_0 with $\ell_{t+1} = \lambda_1 \ell_t$, or $\ell_t = \lambda_1^t \ell_0$
  - Transforming back with q_1
  - Checking if it aligns with the d_t

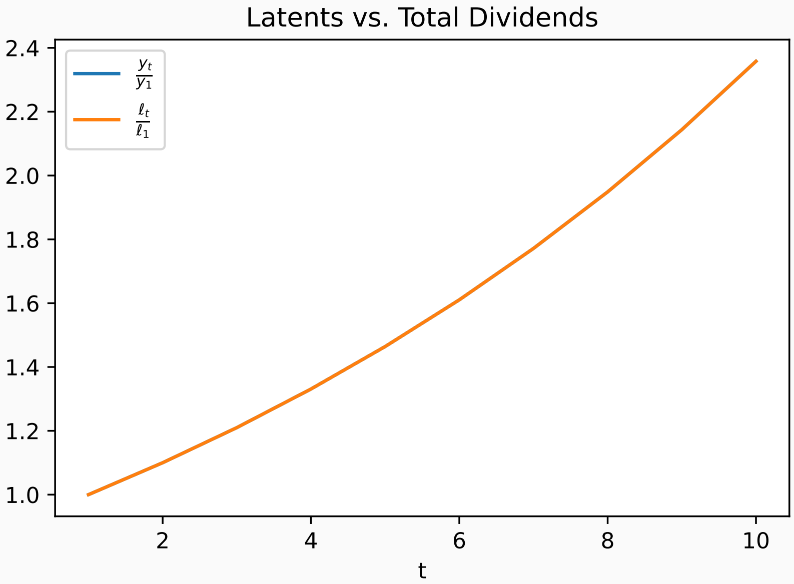
Implementation

```python
l_0 = lambda_1 * q_1_inv @ d_0 # latent space
l = l_0 * np.power(lambda_1, np.arange(0, T)) # powers
d_hat = q_1 * l # back to original space
# Missing d_0 since doing A * d_0 iterations
print(f"norm = {norm(d[:,1:] - d_hat)}")
y_hat = G @ d_hat
```

```
norm = 2.3494410875961204e-10
```

Let's see if these line up perfectly

Total Dividends and the Latent Variable


Latents vs. Total Dividends
$\frac{y_t}{y_1}$
$\frac{\ell_t}{\ell_1}$
t