



ECON526: Quantitative Economics with Data Science Applications

Applications of Linear Algebra and Eigenvalues

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Overview



Motivation and Materials

- In this lecture, we will cover some applications of the tools we developed in the previous lecture
- The goal is to build some useful tools to sharpen your intuition on linear algebra and eigenvalues/eigenvectors, and practice some basic coding

Packages

- Some additional material and references
 - [QuantEcon Python](#)
 - [QuantEcon DataScience](#)
 - [A First Course in Quantitative Economics with Python](#)

```
1 import numpy as np
2 import matplotlib.pyplot as plt
3 import scipy
4 from numpy.linalg import cond, matrix_rank, norm
5 from scipy.linalg import inv, solve, det, eig, lu, eigvals
6 from scipy.linalg import solve_triangular, eigvalsh, cholesky
```



Difference Equations

Linear Difference Equations as Iterative Maps

- Consider $A : \mathbb{R}^N \rightarrow \mathbb{R}^N$ as the linear map for the state $x_t \in \mathbb{R}^N$
- An example of a linear difference equation is

$$x_{t+1} = Ax_t$$

where

$$A \equiv \begin{bmatrix} 0.9 & 0.1 \\ 0.5 & 0.8 \end{bmatrix}$$

```
1 A = np.array([[0.9, 0.1], [0.5, 0.8]])
2 x_0 = np.array([1, 1])
3 x_1 = A @ x_0
4 print(f"x_1 = {x_1}, x_2 = {A @ x_1}")
```

$x_1 = [1. \quad 1.3], x_2 = [1.03 \quad 1.54]$

Iterating with $\rho(A) > 1$

Iterate $x_{t+1} = Ax_t$ from x_0 for $t = 100$

```
1 x_0 = np.array([1, 1])
2 t = 200
3 x_t = np.linalg.matrix_power(A, t) @ x_0
4 rho_A = np.max(np.abs(eigvals(A)))
5 print(f"rho(A) = {rho_A}")
6 print(f"x_{t} = {x_t}")
```

```
rho(A) = 1.079128784747792
x_200 = [3406689.32410673 6102361.18640516]
```

- Diverges to $x_\infty = [\infty \quad \infty]^T$
- $\rho = 1 + 0.079$ says in the worst case (i.e., $x_t \propto$ the eigenvector associated with $\lambda = 1.079$ eigenvalue), expands by **7.9%** on each iteration

Iterating with $\rho(A) < 1$

```
1 A = np.array([[0.6, 0.1], [0.5, 0.8]])
2 x_t = np.linalg.matrix_power(A, t) @ x_0
3 rho_A = np.max(np.abs(eigvals(A)))
4 print(f"rho(A) = {rho_A}")
5 print(f"x_{t} = {x_t}")
```

```
rho(A) = 0.9449489742783178
x_200 = [6.03450418e-06 2.08159603e-05]
```

- Converges to $x_\infty = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$

Iterating with $\rho(A) = 1$

- To make a matrix that has $\rho(A) = 1$ reverse eigendecomposition!
- Leave previous eigenvectors in Q , change Λ to force $\rho(A)$ directly

```
1 Q = np.array([[ -0.85065081, -0.52573111], [0.52573111, -0.85065081]])
2 print(f"check orthogonal: dot(x_1,x_2) approx 0: {np.dot(Q[:,0], Q[:,1])}")
3 Lambda = [1.0, 0.8] # choosing eigenvalue so max_n|lambda_n| = 1
4 A = Q @ np.diag(Lambda) @ inv(Q)
5 print(f"rho(A) = {np.max(np.abs(eigvals(A)))}")
6 print(f"x_{t} = {np.linalg.matrix_power(A, t) @ x_0}")
```

check orthogonal: dot(x_1,x_2) approx 0: 0.0

rho(A) = 1.0

x_200 = [0.27639321 -0.17082039]

Unemployment Dynamics

Dynamics of Employment without Population Growth

- Consider an economy where in a given year $\alpha = 5\%$ of employed workers lose job and $\phi = 10\%$ of unemployed workers find a job
- We start with $E_0 = 900,000$ employed workers, $U_0 = 100,000$ unemployed workers, and no birth or death. Dynamics for the year:

$$E_{t+1} = (1 - \alpha)E_t + \phi U_t$$

$$U_{t+1} = \alpha E_t + (1 - \phi)U_t$$

Write as Linear System

- Use matrices and vectors to write as a linear system

$$\underbrace{\begin{bmatrix} E_{t+1} \\ U_{t+1} \end{bmatrix}}_{X_{t+1}} = \underbrace{\begin{bmatrix} 1 - \alpha & \phi \\ \alpha & 1 - \phi \end{bmatrix}}_A \underbrace{\begin{bmatrix} E_t \\ U_t \end{bmatrix}}_{X_t}$$

Simulating

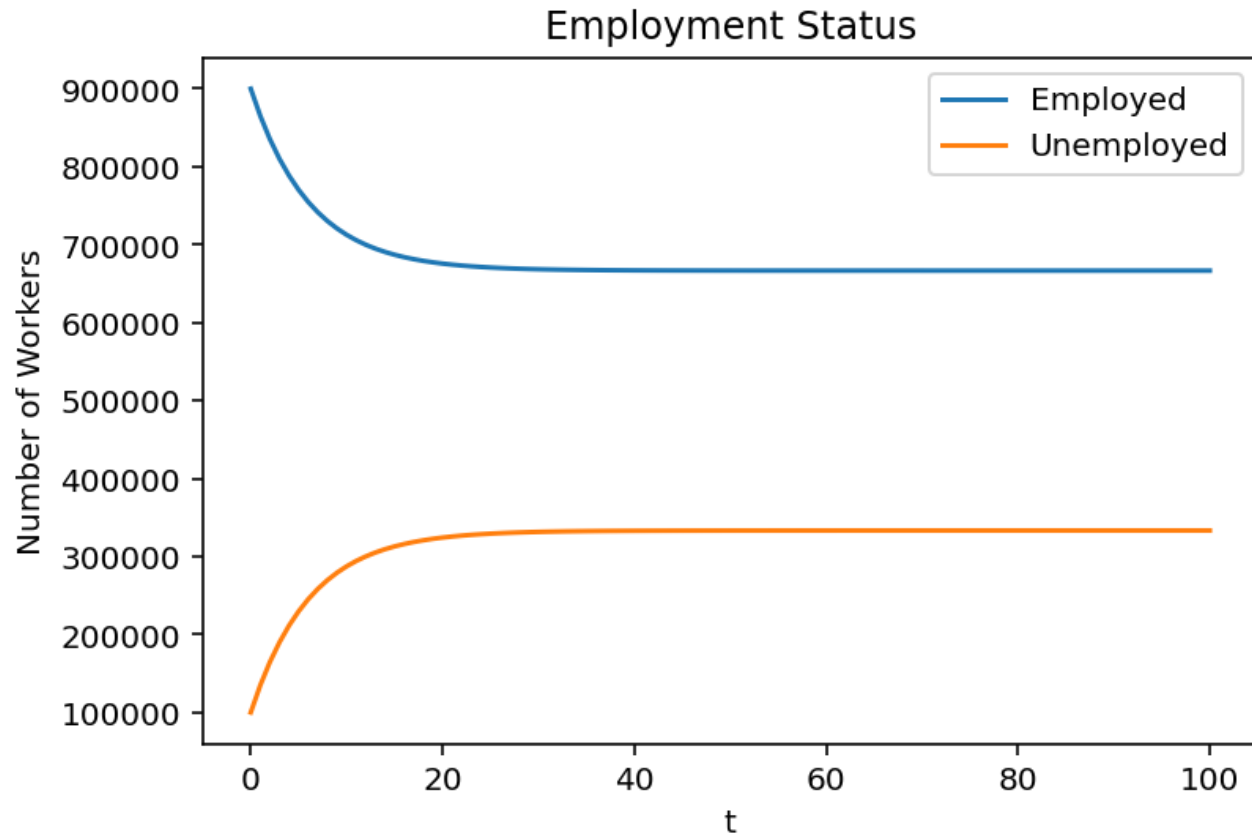
Simulate by iterating $X_{t+1} = AX_t$ from X_0 until $T = 100$

```
1 def simulate(A, X_0, T):
2     X = np.zeros((2, T+1))
3     X[:,0] = X_0
4     for t in range(T):
5         X[:,t+1] = A @ X[:,t]
6     return X
7 X_0 = np.array([900000, 100000])
8 A = np.array([[0.95, 0.1], [0.05, 0.9]])
9 T = 100
10 X = simulate(A, X_0, T)
11 print(f"X_{T} = {X[:,T]}")
```

```
X_100 = [666666.6870779  333333.31292209]
```

Dynamics of Unemployment

```
1 fig, ax = plt.subplots()
2 ax.plot(range(T+1), X.T,
3         label=["Employed", "Unemployed"])
4 ax.set(xlabel="t",
5        ylabel="Number of Workers",
6        title="Employment Status")
7 ax.legend()
8 plt.show()
```



Convergence to a Longrun Distribution

- Find \mathbf{X}_∞ by iterating $\mathbf{X}_{t+1} = \mathbf{A}\mathbf{X}_t$ many times from a \mathbf{X}_0 ?
 - Check if it has converged with $\mathbf{X}_\infty \approx \mathbf{A}\mathbf{X}_\infty$
 - Is \mathbf{X}_∞ the same from any \mathbf{X}_0 ? Will discuss “ergodicity” later
- Alternatively, note that this expression is the same as

$$\mathbf{1} \times \bar{\mathbf{X}} = \mathbf{A}\bar{\mathbf{X}}$$

- i.e, a $\lambda = 1$ where $\bar{\mathbf{X}}$ is the corresponding eigenvector of \mathbf{A}
- Is $\lambda = 1$ always an eigenvalue? (yes if all $\sum_{n=1}^N A_{ni} = 1$ for all i)
- Does $\bar{\mathbf{X}} = \mathbf{X}_\infty$? For any \mathbf{X}_0 ?
- Multiple eigenvalues with $\lambda = 1 \implies$ multiple $\bar{\mathbf{X}}$

Using the First Eigenvector for the Steady State

```
1 Lambda, Q = eig(A)
2 print(f"real eigenvalues = {np.real(Lambda)}")
3 print(f"eigenvectors in columns of =\n{Q}")
4 print(f"first eigenvalue = 1? \
5 {np.isclose(Lambda[0], 1.0)}")
6 X_bar = Q[:,0] / np.sum(Q[:,0]) * np.sum(X_0)
7 print(f"X_bar = {X_bar}\nX_{T} = {X[:,T]}")
```

```
real eigenvalues = [1.    0.85]
eigenvectors in columns of =
[[ 0.89442719 -0.70710678]
 [ 0.4472136   0.70710678]]
first eigenvalue = 1? True
X_bar = [666666.66666667 333333.33333333]
X_100 = [666666.6870779  333333.31292209]
```

Using the Second Eigenvalue for the Convergence Speed

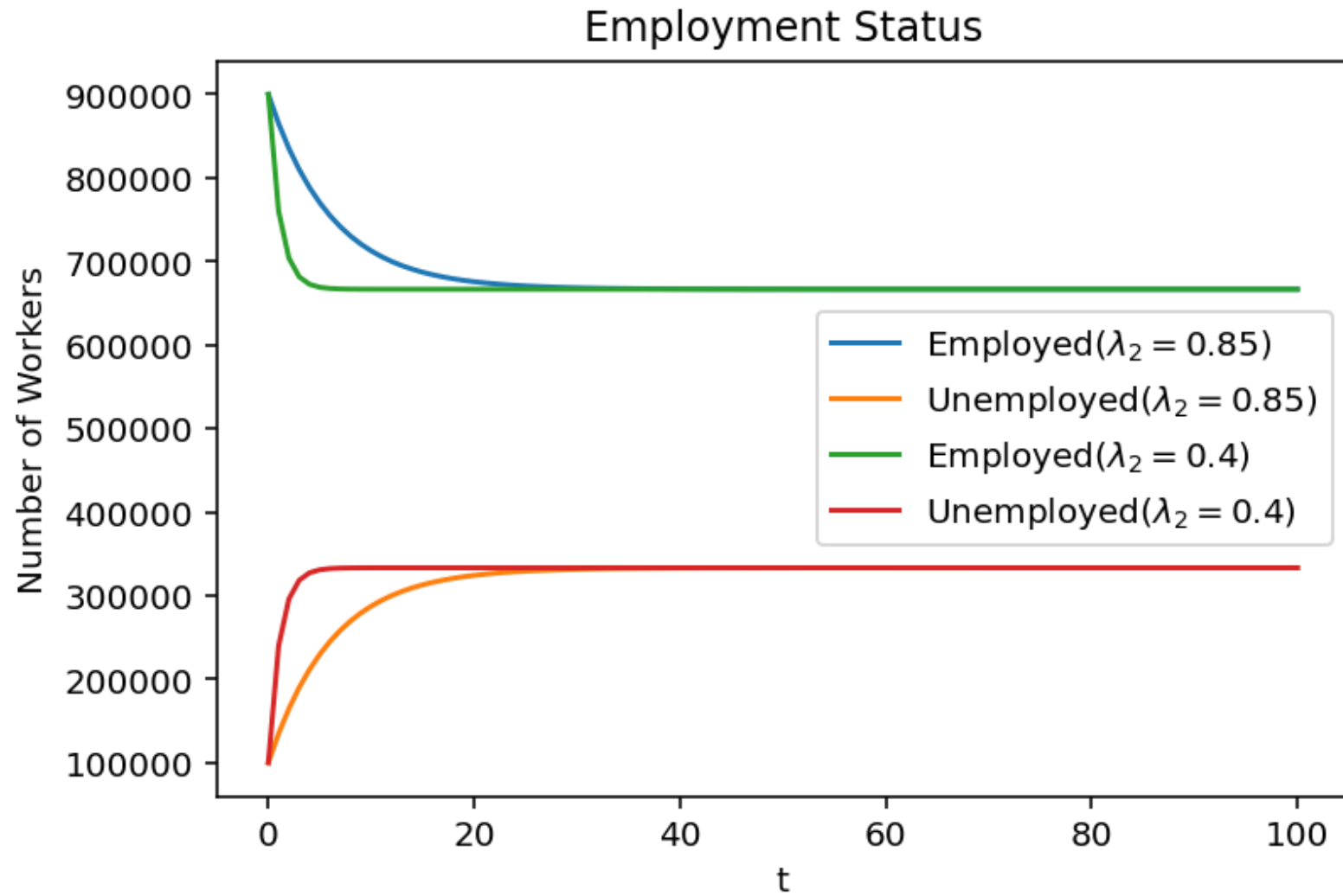
- The second largest ($\lambda_2 < 1$) provides information on the speed of convergence
 - 0 is instantaneous convergence here
 - 1 is no convergence here
- We will create a new matrix with the same steady state, different speed
 - To do this, build a new matrix with the same eigenvectors (in particular the same eigenvector associated with the $\lambda = 1$ eigenvalue)
 - But we will replace the eigenvalues $[1.0 \quad 0.85]$ with $[1.0 \quad 0.5]$
 - Then we will reconstruct A matrix and simulate again
- Intuitively we will see that the resulting A_{fast} implies α and ϕ which are larger by the same proportion

Simulating with Different Eigenvalues

```
1 Lambda_fast = np.array([1.0, 0.4])
2 A_fast = Q @ np.diag(Lambda_fast) @ inv(Q) # same eigenvectors
3 print("A_fast =\n", A_fast)
4 print(f"alpha_fast/alpha = {A_fast[1,0]/A[1,0]:.2g}, \
5 phi_fast/phi = {A_fast[0,1]/A[0,1]:.2g}")
6 X_fast = simulate(A_fast, X_0, T)
7 print(f"X_{T} = {X_fast[:,T]}")
```

```
A_fast =
[[0.8 0.4]
 [0.2 0.6]]
alpha_fast/alpha = 4, phi_fast/phi = 4
X_100 = [666666.66666667 333333.33333333]
```

Convergence Dynamics of Unemployment





Present Discounted Values

Geometric Series

- Assume dividends follow $y_{t+1} = G y_t$ for $t = 0, 1, \dots$ and y_0 is given

- $G > 0$, dividends are discounted at factor $\beta > 1$ then

$$p_t = \sum_{s=0}^{\infty} \beta^s y_{t+s} = \frac{y_t}{1-\beta G}$$

- More generally if $x_{t+1} = A x_t$, $x_t \in \mathbb{R}^N$, $y_t = G x_t$ and $A \in \mathbb{R}^{N \times N}$, then

$$p_t = y_t + \beta y_{t+1} + \beta^2 y_{t+2} + \dots = G x_t + \beta G A x_t + \beta G A A x_t + \dots$$

$$= \sum_{s=0}^{\infty} \beta^s A^s y_t$$

$$= G(I - \beta A)^{-1} x_t \quad , \text{ if } \rho(A) < 1/\beta$$

- where $\rho(A)$ is the spectral radius

Discounting and the Spectral Radius

- Intuitively, the spectral radius of A , the maximum scaling, must be less than discounting
- Intuition from univariate:
 - If $G \in \mathbb{R}^{1 \times 1}$ then $\text{eig}(G) = G$, so must have $|\beta G| < 1$

PDV Example

Here is an example with $1 < \rho(A) < 1/\beta$. Try with different A

```
1 beta = 0.9
2 A = np.array([[0.85, 0.1], [0.2, 0.9]])
3 G = np.array([[1.0, 1.0]]) # row vector
4 x_0 = np.array([1.0, 1.0])
5 p_t = G @ solve(np.eye(2) - beta * A, x_0)
6 #p_t = G @ inv(np.eye(2) - beta * A) @ x_0 # alternative
7 rho_A = np.max(np.abs(np.real(eigvals(A))))
8 print(f"p_t = {p_t[0]:.4g}, spectral radius = {rho_A:.4g}, 1/beta = {1/beta:.4g}")
```

p_t = 24.43, spectral radius = 1.019, 1/beta = 1.111

(Optional) Matrix Conditioning and Stability

Matrix Conditioning

- Poorly conditioned matrices can lead to inaccurate or wrong solutions
- Tends to happen when matrices are close to singular or when they have very different scales - so there will be times when you need to rescale your problems

```
1 eps = 1e-7
2 A = np.array([[1, 1], [1 + eps, 1]])
3 print(f"A =\n{A}")
4 print(f"A^{-1} =\n{inv(A)}")
```

```
A =
[[1.          1.          ]
 [1.00000001  1.          ]]
A^{-1} =
[[-9999999.99336215  9999999.99336215]
 [10000000.99336215 -9999999.99336215]]
```

Condition Numbers of Matrices

- $\det(A) \approx 0$ may say it is “almost” singular, but it is not scale-invariant
- $\text{cond}(A) \equiv \|A\| \cdot \|A^{-1}\|$ where $\|\cdot\|$ is the matrix norm - expensive to calculate in practice. Connected to eigenvalues $\text{cond}(A) = \left| \frac{\lambda_{\max}}{\lambda_{\min}} \right|$
- Scale free measure of numerical issues for a variety of matrix operations
- Intuition: if $\text{cond}(A) = K$, then $b \rightarrow b + \nabla b$ change in b amplifies to a $x \rightarrow x + K\nabla b$ error when solving $Ax = b$.
- See [Matlab Docs on inv](#) for example, where `inv` is a bad idea due to poor conditioning

```
1 print(f"condition(I) = {cond(np.eye(2))}")
2 print(f"condition(A) = {cond(A)}, condition(A^(-1)) = {cond(inv(A))}")
```

```
condition(I) = 1.0
condition(A) = 40000001.962777555, condition(A^(-1)) = 40000002.02779216
```

Example with Interpolation

- Consider fitting data $\mathbf{x} \in \mathbb{R}^{N+1}$ and $\mathbf{y} \in \mathbb{R}^{N+1}$ with an N -degree polynomial
- That is, find $\mathbf{c} \in \mathbb{R}^{N+1}$ such that

$$c_0 + c_1 x_1 + c_2 x_1^2 + \dots + c_N x_1^N = y_1$$
$$\dots = \dots$$

$$c_0 + c_1 x_N + c_2 x_N^2 + \dots + c_N x_N^N = y_N$$

- Which we can then use as $P(x) = \sum_{n=0}^N c_n x^n$ to interpolate between the points

Writing as a Linear System

- Define a matrix of all of the powers of the x values

$$A \equiv \begin{bmatrix} 1 & x_0 & x_0^2 & \dots & x_0^N \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_N & x_N^2 & \dots & x_N^N \end{bmatrix}$$

- Then solve for c as the solution (where A is invertible if x_n are unique)

$$Ac = y$$

Solving an Example

- Let's look at the numerical error here from the interpolation using the inf-norm, i.e., $\|x\|_\infty = \max_n |x_n|$

```
1 N = 5
2 x = np.linspace(0.0, 10.0, N + 1)
3 y = np.exp(x) # example function to interpolate
4 A = np.array([[x_i**n for n in range(N + 1)] for x_i in x]) # or np.vander
5 c = solve(A, y)
6 c_inv = inv(A) @ y
7 print(f"error = {norm(A @ c - y, np.inf)}, \
8 error using inv(A) = {norm(A @ c_inv - y, np.inf)}")
9 print(f"cond(A) = {cond(A)}")
```

```
error = 1.574562702444382e-11, error using inv(A) = 1.1932570487260818e-09
cond(A) = 564652.3214000753
```

Things Getting Poorly Conditioned Quickly

```
1 N = 10
2 x = np.linspace(0.0, 10.0, N + 1)
3 y = np.exp(x) # example function to interpolate
4 A = np.array([[x_i**n for n in range(N + 1)] for x_i in x]) # or np.vander
5 c = solve(A, y)
6 c_inv = inv(A) @ y # Solving with inv(A) instead of solve(A, y)
7 print(f"error = {norm(A @ c - y, np.inf)}, \
8 error using inv(A) = {norm(A @ c_inv - y, np.inf)}")
9 print(f"cond(A) = {cond(A)}")
```

```
error = 5.334186425898224e-10, error using inv(A) = 6.22717197984457e-06
cond(A) = 4462824600234.486
```

Matrix Inverses Fail Completely for $N = 20$

```
1 N = 20
2 x = np.linspace(0.0, 10.0, N + 1)
3 y = np.exp(x) # example function to interpolate
4 A = np.array([[x_i**n for n in range(N + 1)] for x_i in x]) # or np.vander
5 c = solve(A, y)
6 c_inv = inv(A) @ y # Solving with inv(A) instead of solve(A, y)
7 print(f"error = {norm(A @ c - y, np.inf)}, \
8 error using inv(A) = {norm(A @ c_inv - y, np.inf)}")
9 print(f"cond(A) = {cond(A):.4g}")
```

error = 6.784830475226045e-10, error using inv(A) = 31732.823760853855
cond(A) = 1.697e+24

Moral of this Story

- Use `solve`, which is faster and can often solve ill-conditioned problems. Rarely use `inv`, and only when you know the problem is well-conditioned
- Check conditioning of matrices when doing numerical work as an occasional diagnostic, as it is a good indicator of potential problems and collinearity
- For approximation, never use a monomial basis for polynomials
 - Prefer polynomials like Chebyshev, which are designed to be as orthogonal as possible

```
1 N = 40
2 x = np.linspace(-1, 1, N+1) # Or any other range of x values
3 A = np.array([[np.polynomial.Chebyshev.basis(n)(x_i) for n in range(N+1)] for x_i in x])
4 A_monomial = np.array([[x_i**n for n in range(N + 1)] for x_i in x]) # or np.vander
5 print(f"cond(A) = {cond(A):.4g}, cond(A_monomial) = {cond(A_monomial):.4g}")
```

`cond(A) = 3.64e+09, cond(A_monomial) = 2.926e+18`