City Location Choice and Productivity

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1 Model of Production

1.1 Production Funciton

$$Y_{s|c} = T_{cs} \prod_k Q_{sk}^{\omega_{sk}} \tag{1}$$

The economy consists of $s \in S$ sectors which employ occupations/tasks according to a Cobb-Douglas production function. ω_{sk} is the weight that each occupation takes for every sector, where $\sum_k \omega_{sk} = 1$ for each sector s. T_{cs} is some exogenous productivity associated with a city-sector pair (Detriot and automanufacturing, for example).

1.2 MRP

$$MRP_{csk}(\nu) = p_s T_{cs} \omega_{sk} z_k(\nu) \tag{2}$$

The world consists of some continuum of householdslds $\nu \in [0, 1]$. Households exhibit some productivity for a range of occupations/tasks, denoted $k \in K$. For worker ν, their productivity in k is $z_k(\nu)$. Each worker therefore faces a marginal revenue product associated with being employed in occupation k within sector s in city c.

p_s is the price of the good produced by sector s, which we assume is freely traded and priced under perfect competition. That is, the price of s is identical in all cities. We assume that when households choose a city c they are randomly assigned to a sector s depending on the employment shares within that city accruing to sector s: ϕ_{cs}. Notice that ϕ_{cs} is related to T_{cs}, and for now we make the simplifying assumption that:

$$\phi_{cs} = \frac{T_{cs}}{\sum_s T_{cs}} \tag{3}$$

The marginal revenue product of being employed in occupation k in city c for household ν is therefore the following:

$$MRP_{ck}(\nu) = z_k(\nu) \sum_s p_{cs}\phi_{cs}T_{cs}\omega_{sk} \tag{4}$$

Finally, we separate T_{cs} into a city-specific component which applies to all sectors, T_c, and an idiosyncratic component that is city-sector specific, $\tilde{T}_{cs}$. We can therefore re-write our expected marginal revenue product of worker ν working in occupation k in city c as:

$$MRP_{ck}(\nu) = z_k(\nu)T_cB_{ck} \tag{5}$$

where $B_{ck} = \sum_s p_s\phi_{cs}\tilde{T}_{cs}\omega_{sk}$ and captures the employment structure in city c and how attractive this structure is to a worker in occupation k.

2 Max Stable Multivariate Fréchet Distribution

We assume that the joint distribution of productivity across cities is given by

$$P[Z^{\star}_{1k}(\nu) < z, \ldots, Z^{\star}_{Nk}(\nu) < z] = exp[-\sum_{c}^{N}(T^{\star}_{ck}z(\nu)^{-\theta})^{\frac{1}{1-\rho_k}}]^{1-\rho_k} \tag{6}$$

Where $Tck^{\star}$ is the scale parameter for city c and occupation k, this represents a city's absolute advantage for occupation k. $\theta > 0$ is the shape parameter, characaterizing the tail behavior of the distribution. ρ_k is the occupation specific correlation parameter, which dictates the extent to which productivity draws are correlated across cities for occupation k.

A household's schedule of productivities is characterized by a vector of draws from the Fréchet distribution for each occupation k in each city c. The realized productivity a hosuehold of type ν has in city c is however the occupation that maximizes the producticity that particular household has in the city. This is given by the following:

$$Z_c(\nu) = \max_k \{Z^{\star}_{ck}(\nu)\} \tag{7}$$

Unlike sequential games where households might pick a city before picking an occupation, this schedule of productivity already determines the ideal occupation for a household in a city. The joint probability of all productivities being less than some value z for all cities is then given by the following:

$$P[Z_1(\nu) < z, \ldots, Z_N(\nu) < z] = exp\{-\sum_k [\sum_c^N (T^{\star}_{ck} Z(\nu)^{-\theta})^{\frac{1}{1-\rho_k}}]^{1-\rho_k}\} \tag{8}$$

The joint probability above is a max-stable multivariate Fréchet distribution with a cross-nested CES correlation function. This distribution is a generalization of the Fréchet distribution to the multivariate case, and is used to model the joint distribution of extreme values. This is simillar to the GEV distribution, but with the added feature of a correlation function that allows for the dependence of extreme values across occupations.

A houehold of type ν has realised productivity that is hence characterised by the maxmimum productivity draw across all cities scaled by the inverse of that city's price index Φ_o and is gievn by the following:

$$Z(\nu) = \max_{c=1,\ldots,N} \left\{ \frac{Z_c(\nu)}{\Phi_o} \right\} \tag{9}$$

2.1 Correlation Function

With the case of a cross-nested CES function, we can approximate a correlation funciton. Assuming that productivity is distributed max-stable multivariate Fréchet, with scale parameter $T^{\star}_{ck}$, shape parameter θ, and correlation parameter ρ_k, we can assume the following correlation function:

$$G(Z_1^{-\theta}, \ldots, Z_N^{-\theta}) = \sum_k [\sum_c^N (T^{\star}_{ck} Z_c^{-\theta})^{\frac{1}{1-\rho_k}}]^{1-\rho_k} \tag{10}$$

$$G_c(Z_1^{-\theta}, \ldots, Z_N^{-\theta}) = \frac{\partial G(Z_1^{-\theta}, \ldots, Z_N^{-\theta})}{\partial Z_c^{-\theta}} \tag{11}$$

$$\gamma = \Gamma(\frac{\theta - 1}{\theta}) \tag{12}$$

Where G_c is the derivative of the correlation function with respect to city c. This correlation function is a generalization of the CES correlation function to the multivariate case, and is used to model the dependence of extreme values across cities. The Γ function which takes in the ratio of $\theta-1$ to θ gives us the substitutability between cities. Productivity is related to γ by the following funciton:

$$Z_c^{-\theta} = (\gamma \Phi_c)^{-\theta} T^{\star}_c \tag{13}$$

Where $T^{\star}_c$ is the city's absolute advantage that applies to all occupations and is given by $T^{\star}_c = \sum_k T^{\star}_{ck}$.

2.2 Choice Shares

Under perfect competition, wages are equal to marginal revenue product and is a direct function of productivity. As established in equation (5), the productivity for a worker ν in city c is given by the maximum productivity draw across all occupations. All we have to do is intergrate over all houshold types to get the productivity index for city c:

$$Z_c = \max_k \left\{ \int_0^1 Z^{\star}_{ck}(\nu) d\nu \right\} \tag{14}$$

If productivity is distributed max-stable multivariate Fréchet and a continiously differentiable correlation function, then city c's choice shares is given by the following:

$$\pi_c = \frac{Z_c^{-\theta} G_c(Z_1^{-\theta}, \ldots, Z_N^{-\theta})}{G(Z_1^{-\theta}, \ldots, Z_N^{-\theta})} \tag{15}$$

First, the share of location choices for households has the same form as choice probabilities in GEV discrete choice models, with $Z_c^{-\theta}$ replacing choice specific utility. Second, as in EK, the share of location choices of city c equals the probability that a worker is most productive in city c. Finally, the location choice share in each city is determined by the ratio of the expected productivity in that city to the expected productivity in all cities.

$$\pi_c = \frac{Z_c^{-\theta}}{\sum_c^N Z_c^{-\theta}} \tag{16}$$