1. Derive the joint distribution **π**(y, N, **β**).

Considering N and β are independent, thus

$$\pi(\mathrm{y},\mathrm{N},\beta) = \pi(\mathrm{y}|\mathrm{N},\beta)\pi(\mathrm{N},\beta) = \pi(\mathrm{y}|\mathrm{N},\beta)\pi(\mathrm{N})\pi(\beta)$$

Consequently,

$$\pi(\mathrm{y},\mathrm{N},\beta) = \binom{N}{y}\beta^{y}(1-\beta)^{N-y}\frac{\lambda^{N}}{N!}e^{-\lambda}$$

2. Derive full conditional distribution **π** (N|**β**, y) and **π** (**β** |N, y).

To get π (β |N, y), we just get rid of all the terms irrelevant to β, which is

$$\pi(\beta|\mathrm{y},\mathrm{N}) \propto \cancel{\binom{N}{y}}\beta^{y}(1-\beta)^{N-y}\cancel{\frac{\lambda^{N}}{N!}e^{-\lambda}} = \beta^{y}(1-\beta)^{N-y}$$

Normalizing it, we can get

$$\pi(\beta|\mathrm{y},\mathrm{N}) = \frac{1}{B(y+1,N-y+1)}\beta^{y}(1-\beta)^{N-y}\sim Normal(y+1,N-y+1)$$

For π(N|β, y), we get rid of all the terms irrelevant to N, which is

$$\pi(\beta|\mathrm{y},\mathrm{N}) \propto \frac{\cancel{N!}}{\cancel{y!}(N-y)!}(1-\beta)^{N-y}\frac{\lambda^{N}}{\cancel{N!}}\cancel{e^{-\lambda}} = \frac{(1-\beta)^{N-y}}{(N-y)!}\lambda^{N}$$

$$\pi(\beta|\mathrm{y},\mathrm{N}) \propto \frac{(1-\beta)^{N-y}}{(N-y)!}\lambda^{N} \propto \frac{[\lambda(1-\beta)]^{N-y}}{(N-y)!}$$

Normalizing it, we can get

$$\pi(\beta|\mathrm{y},\mathrm{N}) = \frac{[\lambda(1-\beta)]^{N-y}}{(N-y)!}e^{-[\lambda(1-\beta)]}\sim \mathrm{Poission}[\lambda(1-\beta)]$$

3. Use these to sample (using Gibbs sampling) from the joint posterior (**β**, N | y) using a starting value of (**β**(1), N(1)))=(0.05,50).

$$\pi(\beta,\mathrm{N}|\mathrm{y})\pi(\mathrm{y}) = \pi(\mathrm{y}|\beta,\mathrm{N})\pi(\beta,\mathrm{N}) = \pi(\mathrm{y}|\mathrm{N},\beta)\pi(\mathrm{N})\pi(\beta)$$

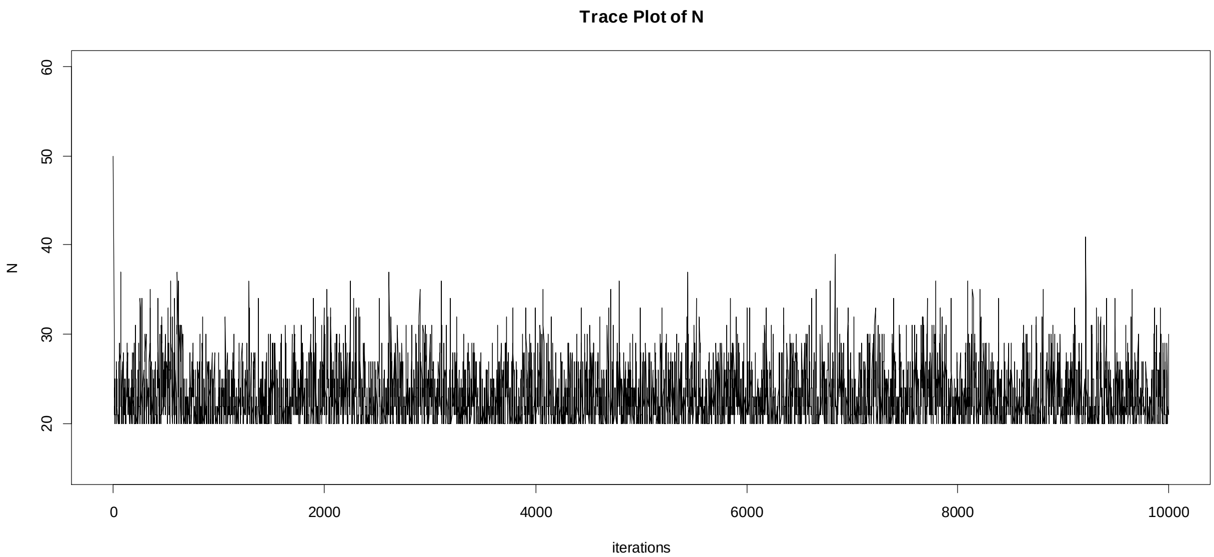
$$\pi(\beta,\mathrm{N}|\mathrm{y}) = \frac{\pi(\mathrm{y}|\mathrm{N},\beta)\pi(\mathrm{N})\pi(\beta)}{\pi(\mathrm{y})}$$

$$\pi(\beta,\mathrm{N}|\mathrm{y}) \propto \frac{\pi(\mathrm{y}|\mathrm{N},\beta)\pi(\mathrm{N})\pi(\beta)}{\pi(\mathrm{y})} \propto \pi(\mathrm{y},\mathrm{N},\beta) = \binom{N}{y}\beta^{y}(1-\beta)^{N-y}\frac{\lambda^{N}}{N!}e^{-\lambda}$$

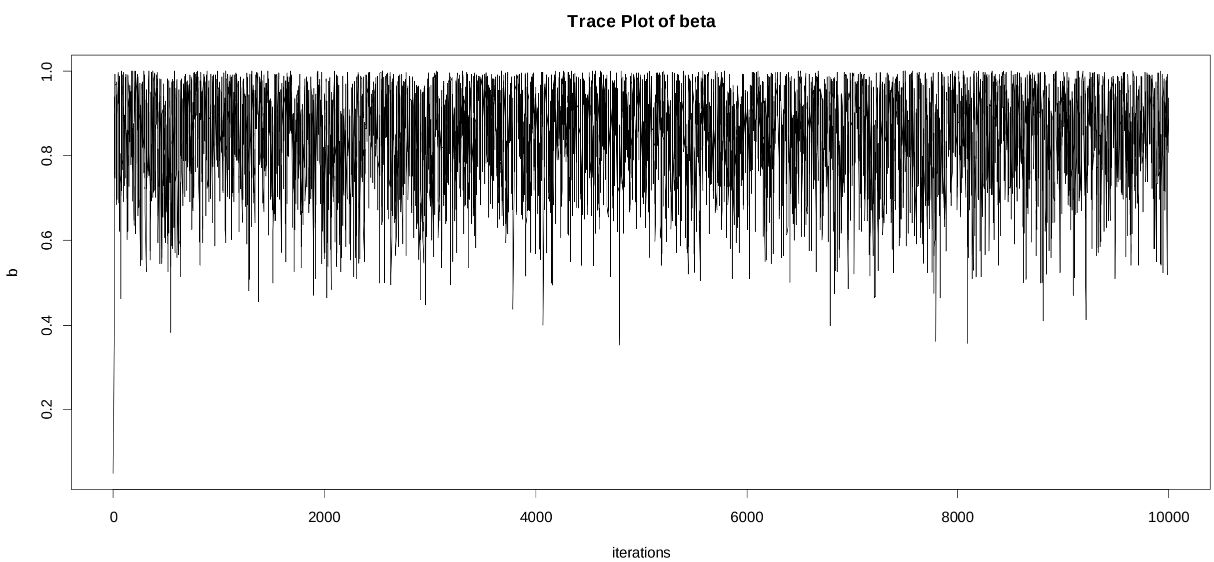
The sampling process is recorded in the R code attached.

4. Show trace plots for **β** and N.

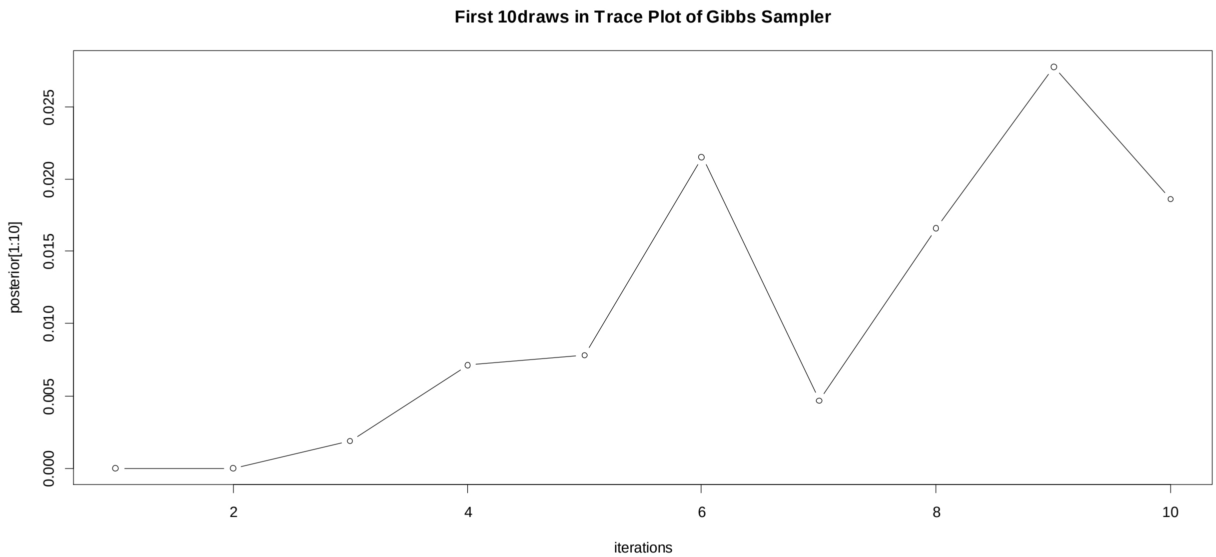
Trace Plot for N:




Trace Plot to β:



5. Show the 2D trace plot for the first 10 draws of the Gibbs sampler, (**β** (1:10), N(1:10)). Show both the points and the connecting lines.




6. Give the central 90% posterior credible interval for **β**, accurate to (and rounded to) the nearest 1% for both upper and lower limits.

The central 90% posterior credible interval for β is (0.64, 0.99). Calculation is in attached code.

7. What is the probability that exactly 20 people were polled? Base your answer on at least 10,000 draws (post-burn-in), and round to the nearest one tenth of 1%.

The probability that exactly 20 people were polled is achieved when N=20, which is 36.4%, after doing 100 burn-in steps and 10000 post-burn-in steps.