1. *Find the Marginal Distribution p(x)*

Lab Assignment 3 Dai Li

— Normal distribution $x \sim N(\mu, \sigma^2)$

$$f(x|\mu,\sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}, \quad -\infty < x < \infty$$

Gamma distribution $x \sim Gamma(\alpha, \beta)$

$$f(x|\alpha,\beta) = \beta^{\alpha} \frac{1}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}, \quad x \geq 0, \ \alpha, \beta > 0$$

— Now, $x|\tau^2 \sim N(0, \frac{1}{\tau^2})$, $\tau^2 \sim Gamma(\frac{\nu}{2}, \frac{\nu}{2})$

So, $\pi(\tau^2) = (\frac{\nu}{2})^{\frac{\nu}{2}} \frac{1}{\Gamma(\frac{\nu}{2})} (\tau^2)^{(\frac{\nu}{2}-1)} e^{-\frac{\nu}{2}\tau^2}$

and, $f(x|0,\tau^2) = \frac{1}{\sqrt{2\pi(1/\tau^2)}} e^{-\frac{1}{2(1/\tau^2)}(x-0)^2}$

$$= \frac{\sqrt{\tau^2}}{\sqrt{2\pi}} e^{(-\frac{\tau^2}{2}x^2)}$$

~~The marginal distribution of x, p(x)~~

~~$p(x) = \int f(x|\tau^2)\pi(\tau^2)d\tau^2$~~

— Consequently, The marginal distribution of X, P(x)

~~$f(x, \tau^2|\nu)$~~

can be derived from

$$p(x) = \int p(x|\tau^2)\pi(\tau^2)d\tau^2$$

$$= \int \frac{\sqrt{\tau^2}}{\sqrt{2\pi}} e^{(-\frac{\tau^2}{2}x^2)} \cdot (\frac{\nu}{2})^{\frac{\nu}{2}} \frac{1}{\Gamma(\frac{\nu}{2})} (\tau^2)^{(\frac{\nu}{2}-1)} e^{-\frac{\nu}{2}\tau^2} d\tau^2$$

$$= \frac{(\nu/2)^{\frac{\nu}{2}}}{\Gamma(\frac{\nu}{2})\sqrt{2\pi}} \int_0^{\infty} (\tau^2)^{(\frac{\nu}{2}-\frac{1}{2})} \cdot e^{-\tau^2(\frac{1}{2}x^2 + \frac{1}{2}\nu)} d\tau^2$$

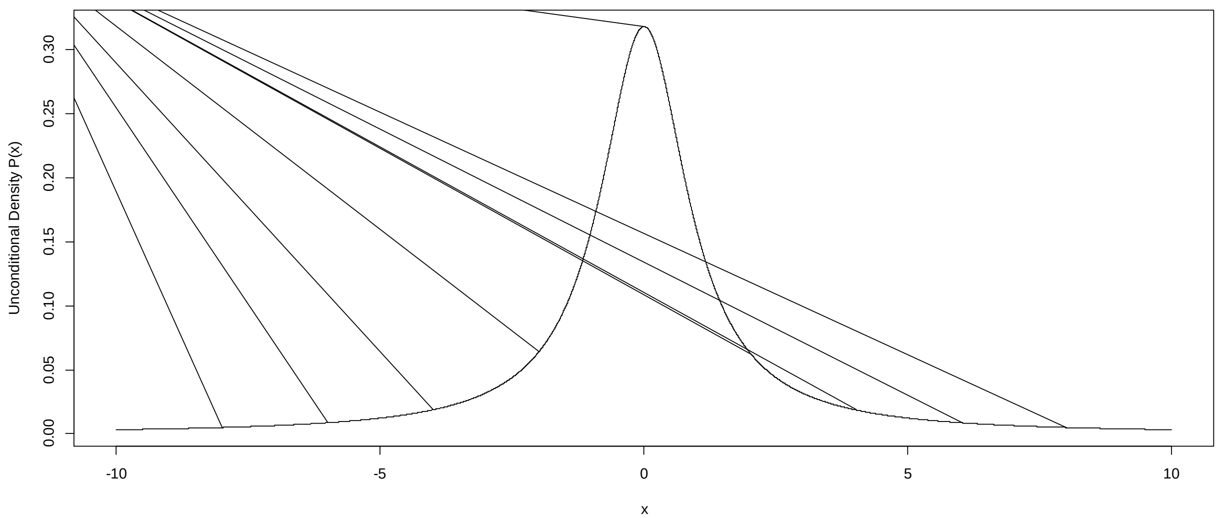
$$= \frac{(\nu/2)^{\frac{\nu}{2}}}{\Gamma(\frac{\nu}{2})\sqrt{2\pi}} \cdot \frac{\Gamma(\frac{\nu+1}{2})}{(\frac{1}{2}(x^2+\nu))^{\frac{\nu+1}{2}}}$$

$$= \frac{\Gamma(\frac{\nu+1}{2})}{\sqrt{\nu\pi}\,\Gamma(\frac{\nu}{2})} (1 + \frac{x^2}{\nu})^{-\frac{\nu+1}{2}}, \text{ which is}$$

The pdf of students' t distribution.

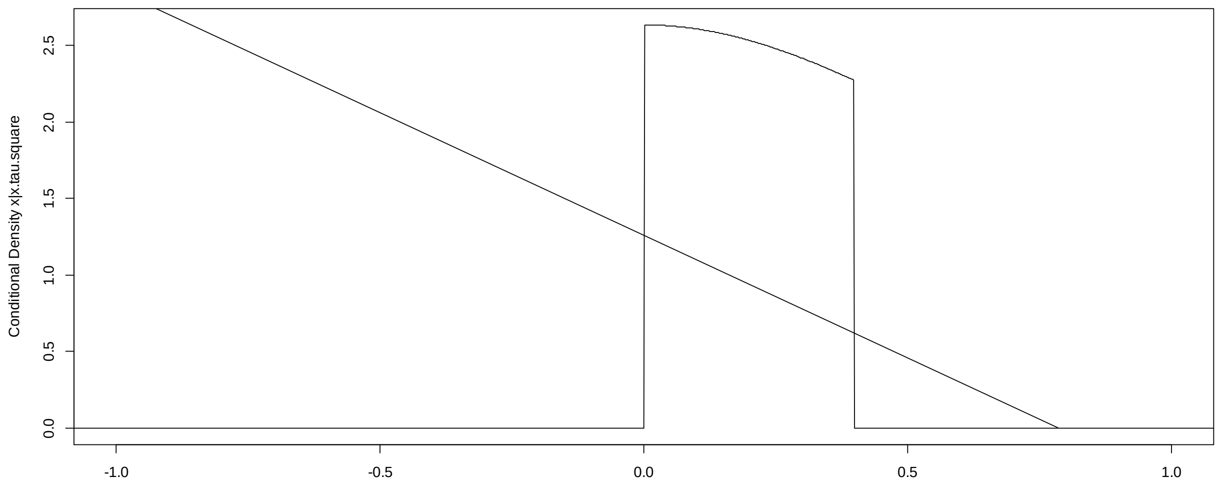
2. *Let nu=1. Get a sample of 10,000 from marginal distr. of x by drawing 10,000 tau^2's and then 10,000 x's given the tau^2's. Plot sample (either histogram or density is fine). Give two names for the actual marginal distribution p(x) when nu=1. Also, compute 2.5% & 97.5% percentile points of the distribution using the random samples and compare them to the theoretical values.*

The actual marginal distribution p(x) when nu=1 is plotted below:

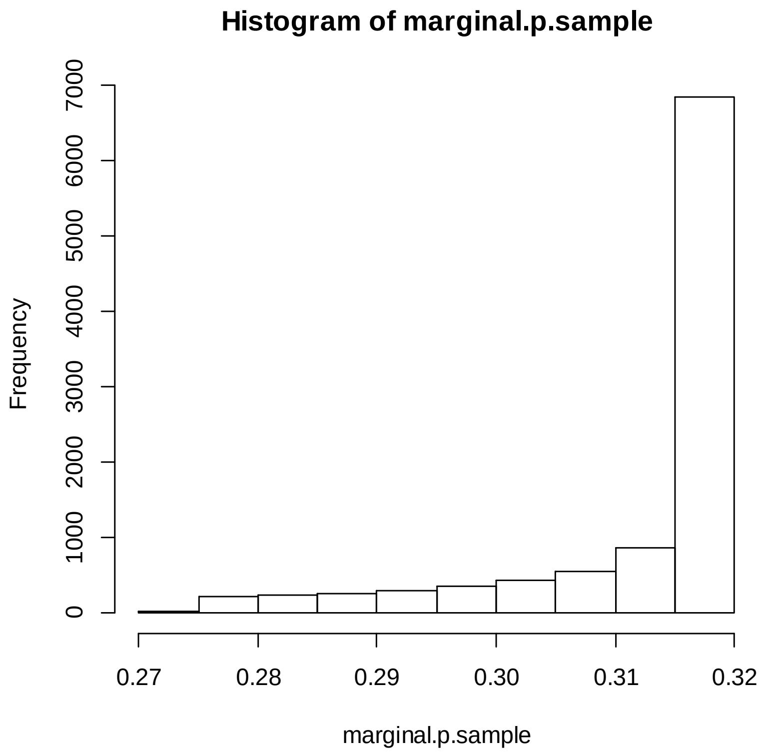


Which is a **<u>Students' T Distribution with parameter Nu=1</u>**. As, Nu=1, it is also a Cauchy Distribution with location parameter=0 and a scale parameter.

While using the 10000 sample points, we can draw out the theoretical conditional density of p(x|tau.square), which is plotted below:



Which is a Student's T distribution truncated to [min(x.tau.square.sample), max(x.tau.square.sample)], whose p(x)sample value histogram is also plotted below:



The 2.5% & 97.5% percentile points of theoretical p(x) are ***(-12.7062, 12.7062).*** While, the 2.5% & 97.5% percentile points of sampled p(x) are ***(0.009560445, 0.3877215).***

3*. Use Kolmogorov-Smirnov test (ks.test in R) to test whether your observed distribution is equal to a t(df=1). Report p-value. What is the conclusion of the test?*

Under null-hypothesis, p-value < 2.2e-16,

Using the sample points, D= 0.5853.

Conclusion of the test: R eject the hypothesis-***the observed distribution is not equal to a t(df=1).***

4*. Does the Central Limit Theorem hold for the mean of a sample from p(x) when nu=1? What about nu=2? nu=3? Why or why not? A quick explanation will do; an involved proof is NOT required.*

Given nu=1,2,or 3, there are finite value of X, so that there are finite value of mean E(x) and variance Var(x), and thus, the Central Limit Theorem hold for the sampled p(x).