1. Find the normalizing constant analytically

For any normal distribution, we have the probability density integrated to 1, which is

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(\theta-\mu)^2}{2\sigma^2}} \mathrm{d}\theta = 1$$

Make σ=1,

$$\int_{-\infty}^{\infty} e^{\frac{-(\theta-\mu)^2}{2}} \mathrm{d}\theta = \sqrt{2\pi}$$

Now Consider

$$\pi(\theta) \propto e^{\frac{-\theta^2}{2}} + 0.5 * e^{\frac{-(\theta-3)^2}{2}}$$

Let the normalizing constant be c, we then have

$$\pi(\theta) = c\{e^{\frac{-\theta^2}{2}} + 0.5 * e^{\frac{-(\theta-3)^2}{2}}\}$$

Since

$$\int \pi(\theta) d\theta = 1$$

We have,

$$\int \pi(\theta) d\theta = \int c\{e^{\frac{-\theta^2}{2}} + 0.5 * e^{\frac{-(\theta-3)^2}{2}}\} d\theta = c \int e^{\frac{-\theta^2}{2}} d\theta + 0.5 * c \int e^{\frac{-(\theta-3)^2}{2}} d\theta$$

$$c * \left(1 + \frac{1}{2}\right) * \sqrt{2\pi} = 1$$

Consequently,

$$c = \frac{1}{\left(1 + \frac{1}{2}\right) * \sqrt{2\pi}} = \frac{2}{3 * \sqrt{2\pi}}$$

2. Metropolis Hasting Sampling- Choose such σ_{cand} that the acceptance probability is very close to 45%.

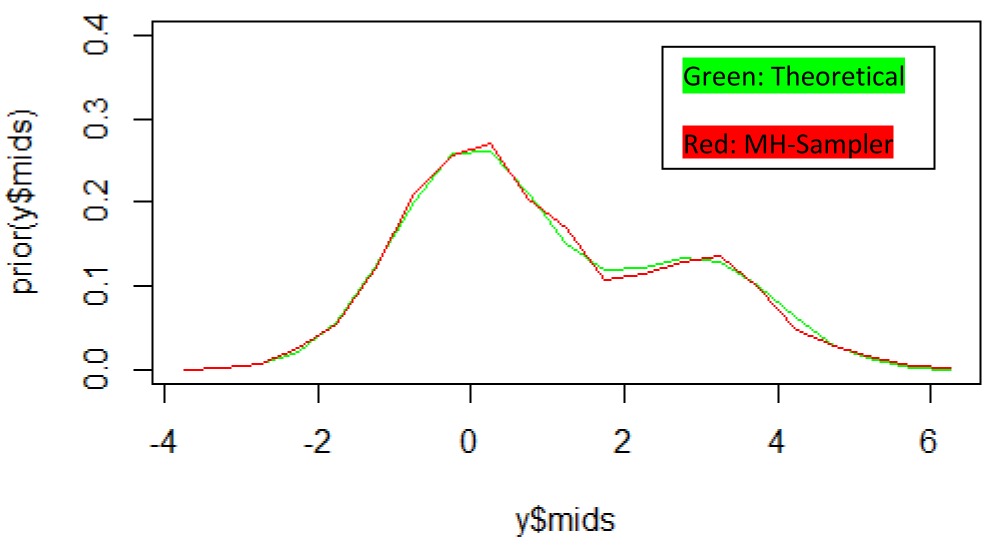
$$\theta^* \sim N(\theta^* | \theta, \sigma cand)$$

$$r = \frac{\pi(\theta^*)}{\pi(\theta^{(t-1)})}$$

Using the attached Code we can calculate the acceptance rate.

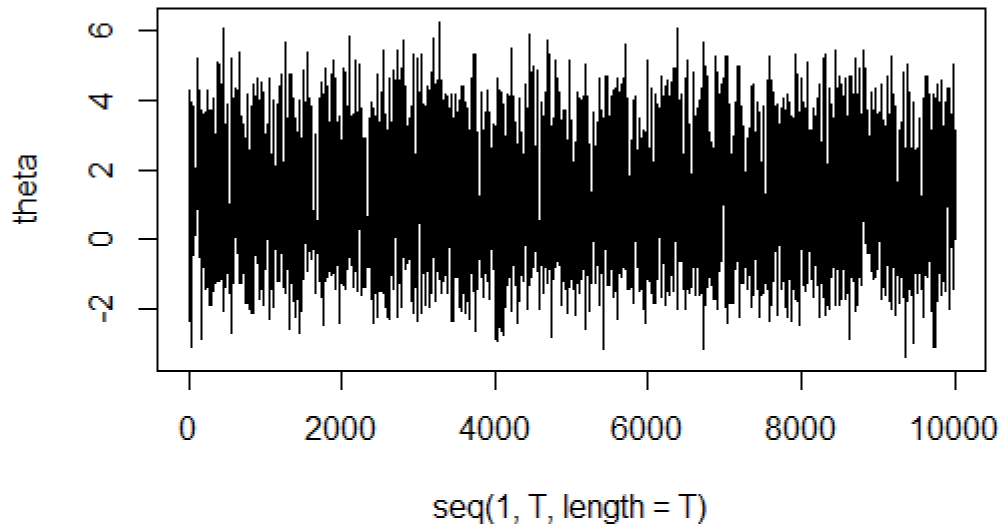
σ_{cand}	1	3	4	5	100
Accept. Rate	79.4%	51.58%	44.46%	36.7%	2.14%

We know, when σ_{cand}=4, the acceptance is most close to 45%. The corresponding plot of theoretical and sampled density is:

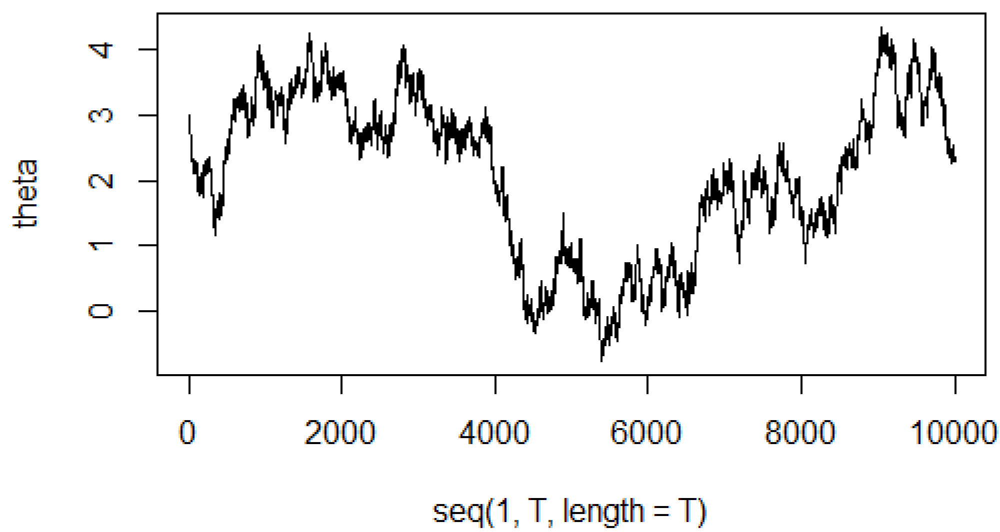


3. In theory, what's wrong with using σ_{cand}=0.05? Is there anything wrong in practice? Do your answers change for σ_{cand}=8? What about for σ_{cand}=100? Please be specific.

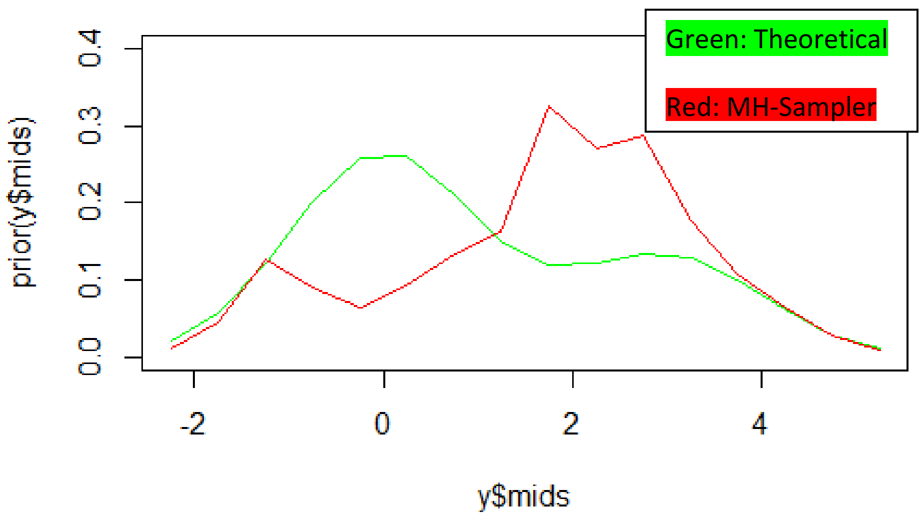
The trace plot of theta when $\sigma_{cand}=4$ is shown below:



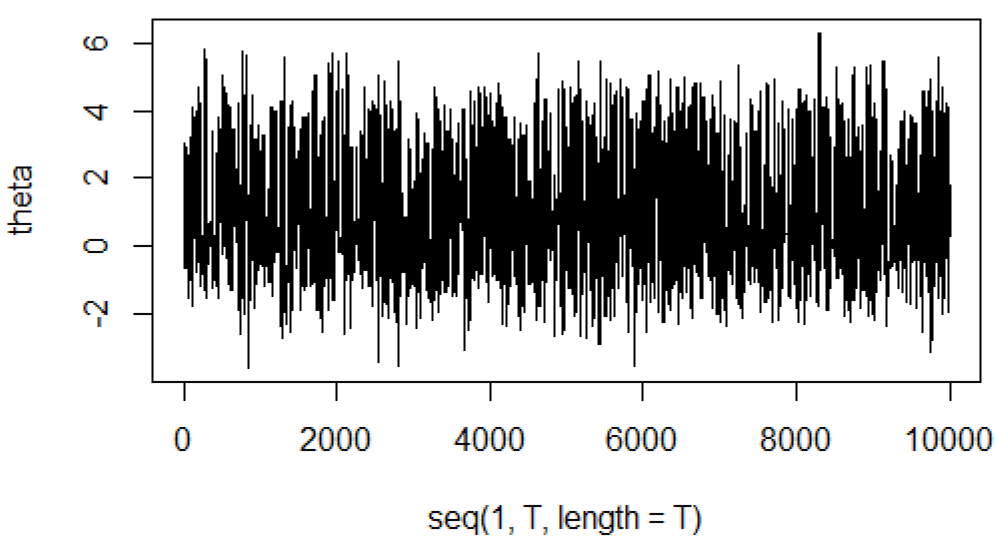
Compare to the trace plot of theta when $\sigma_{cand}=0.05$, which is:

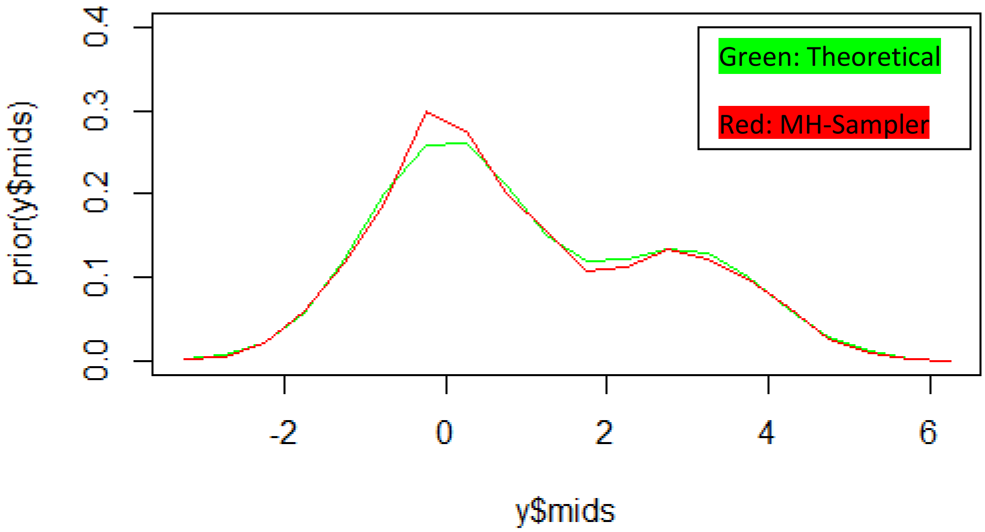


We can see, σ_{cand} is too small, when acceptance rate is high as 99.9%, resulted in a lot of waste in computation. More importantly, the sample result is bad, as shown in the below plot:

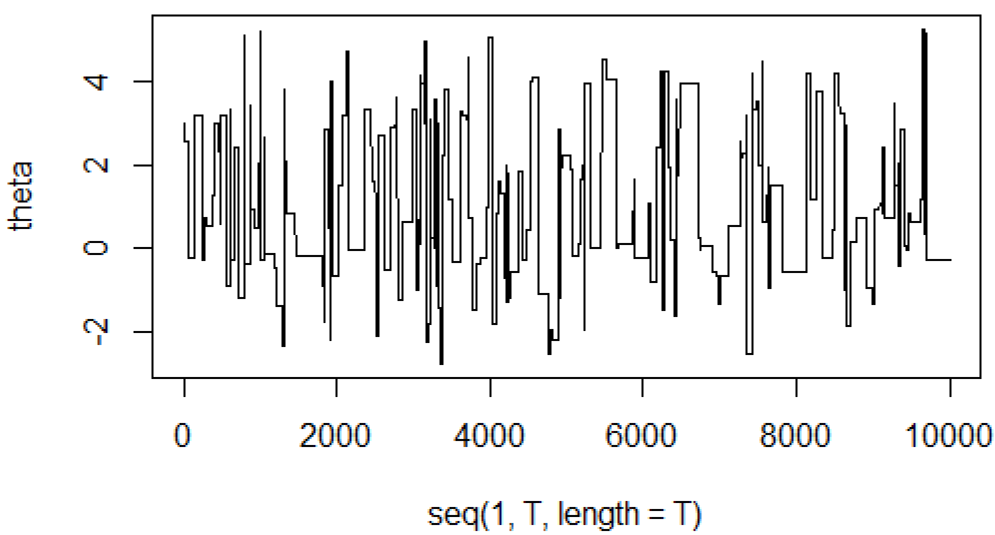


When σ_{cand}=8, acceptance is 25.5%, which has a reasonable trace plot and simulation of density of theta.

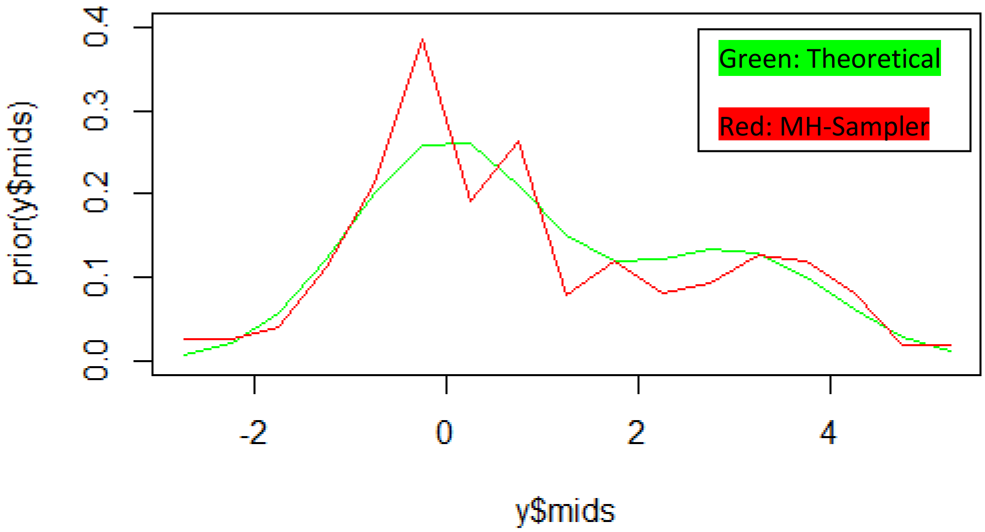




However, when σ_{cand}=100, the acceptance rate is only 2.5%, which resulted in a trace plot of theta like below:



The consequence is that the sampled density plot does not represent the theoretical well enough.


Green: Theoretical
Red: MH-Sampler
prior(y$mids)
0.0
0.1
0.2
0.3
0.4
-2
0
2
4
y$mids

#######Code#############

#######2##########

```
T=10001;sigma.cand=4
theta=matrix(nrow=1,ncol=T)
theta.star=matrix(nrow=1,ncol=T)
r=matrix(nrow=1,ncol=T)
theta[1]=3
acceptance.count=0

prior <- function(theta) {
  z <- 2/(3*sqrt(2*pi))*(exp(-1*0.5*theta^2)+0.5*exp(-1*0.5*(theta-3)^2))
  return (z)
}

for (i in 2:T) {
  theta.star[i]<-rnorm(1,theta[i-1],sigma.cand)
  r[i] <- prior(theta.star[i])/prior(theta[i-1])
  u <- runif(1,0,1)
  if (u< min(1,r[i])) {theta[i] <- theta.star[i]
  acceptance.count=acceptance.count+1}
  else {theta[i] <- theta[i-1]}
}

Acceptance.Rate <- acceptance.count/(T-1)

y <- hist(theta,plot="false")

plot(y$mids,prior(y$mids),type="l",col="green",ylim=c(0,0.4))
lines(y$mids,y$density,type="l",col="red")
```

##############3########

```
###Trace Plot
plot(seq(1,T,length=T),theta,type="l")
```