1. Report a good choice for single-parameter likelihood for the number of cars you sell in one day (x_i), and an informative conjugate prior for the parameter.

A good choice for single-parameter likelihood for the number of cars I sell in one day (x_i) is **<u>Poisson likelihood</u>**, which indicates a discrete probability distribution that expresses the probability of a given number of events occurring in a fixed interval of time.

$$x_i \sim Poisson(\theta), \lambda = 1$$

An informative conjugate prior for the parameter would be a Gamma Prior,

$$\theta \sim \mathrm{Gamma}(\alpha, \beta)$$

, whose variance$\sigma^2 = \frac{\alpha}{\beta^2}$, mean $\mu = \frac{\alpha}{\beta}$

To make the variance =1, and mean=1 because of our primary estimate of 1 car/day, we can get α=1, β=1.

Consequently, the prior is

$$\theta \sim \mathrm{Gamma}(1,1)$$

, which is an informative prior as $\int g(\theta)d\theta<\infty$ as required.

2. After one week, you sell 4 cars. Report the posterior distribution of your parameter.

When we evaluate from a time interval of one week, which is 5 days, according to the summation property of gamma distribution[1], we have the 5-day prior

$$\theta \sim \mathrm{Gamma}(5,1)$$

Because $y = \sum_{i=1}^{5} x_i = 4$, according to the property "Sum of independent Poisson random variables is Poisson", [2]we know for one week, total car sales number y

$$y = \sum x_i \sim \sum Poisson(\theta)\,, \lambda = 5$$

Thus, for the conjugate gamma prior and Poisson likelihood, we have posterior in weekly view:

$$\pi(\theta|y) \sim \mathrm{Gamma}(y_1 + 5, n) = \mathrm{Gamma}(9,2)$$

[1] http://en.wikipedia.org/wiki/Gamma_distribution#Summation
[2] http://www.proofwiki.org/wiki/Sum_of_independent_Poisson_random_variables_is_Poisson

3. Use the posterior predictive distribution to calculate (or simulate) the probability that you will match or exceed Ronald Aylmer's 2-week performance. Report this probability (rounded to nearest percent). Assume that your ability to sell cars remains constant throughout your employment.

In order to match or exceed Ronald Aylmer's 2-week performance, which is 10 cars in 2 week, I have to sell no less than 10-4=6 cars in week 2, which means $y_2>=6|y_1=4$.

The Posterior Predictive Distribution is:

$$\mathrm{p}(y_2|y^n) = \int p(y_2|\theta)p(\theta|y^n)d\theta = \int \frac{(5\lambda)^y}{y!}e^{-5\lambda}\frac{\beta^\alpha}{\Gamma(\alpha)}\lambda^{\alpha-1}e^{-\beta\lambda}d\lambda$$

$$= \frac{5^y}{y!}\frac{\beta^\alpha}{\Gamma(\alpha)}\int \lambda^{y+\alpha-1}e^{-(\beta+5)\lambda}d\lambda$$

$$= \frac{1}{y!}\frac{\Gamma(y+\alpha)}{\Gamma(\alpha)}(\frac{5}{\beta+5})^y(\frac{\beta}{\beta+5})^\alpha = \mathrm{Negative-Binomial}(\mathrm{y}; \mathrm{r}=\alpha, \mathrm{p}=\frac{5}{\beta+5})$$

Where, $p(y_2|\theta) \sim \mathrm{Poisson}(\theta)$ and $p(\theta|y^n) \sim Gamma(9,2)$, α=9,β=2. And

Let $\mathrm{g}(\theta) = 1(\theta \geq 6)$, according to the Law of Large Numbers,

$$\frac{1}{s}\sum_{i=1}^{s} 1(\theta i \geq 6) \rightarrow E[1(\theta \geq 6)|y^n] = P(\theta \geq 6|y^n)$$

The empirical distribution of $\{\theta 1, \theta 2, \ldots, \theta i\} \rightarrow p(\theta|y^n)$

Consequently, using Monte Carlo Simulation, we first generate y(i) using rgamma function in R, then compute the g(y(i)), and finally use the expectation of g(y(i) to calculate the probability of $y_2>=6$.

When S=100, p=12%;

When S=1000, p=18.3%

When S=10000, p=15.71%

When S=100000, p=15.393%

When S=500000, p=15.4178%

When S=1000000, p= 15.5311%

Consequently, we get the final result of **a probability 15%** to match or exceed Ronald's 2-week performance.

In fact, I also calculated the theoretical probability of the question, which can be acquired through the integral $\int g(\theta)p(\theta|y^n)d\theta$, and the result is 15.5027%.

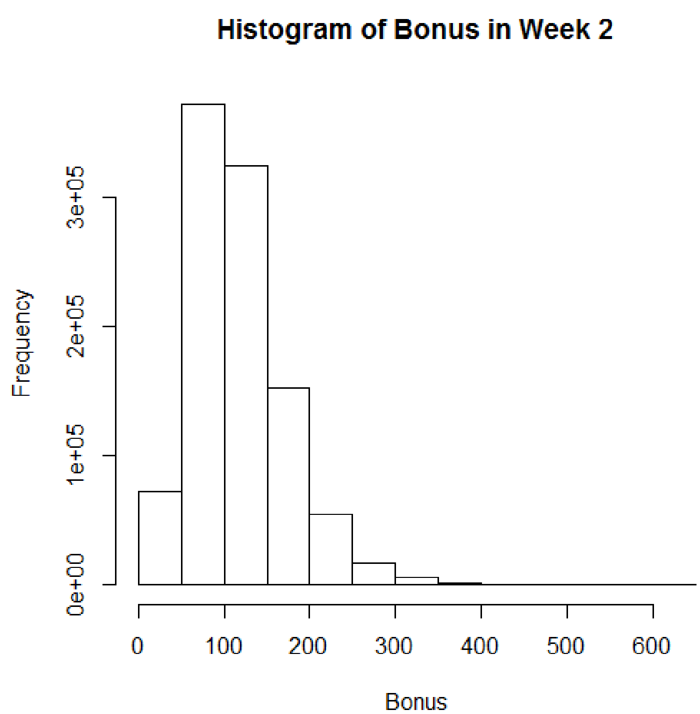
4. What is the probability that your bonus after your second week of work, b2, will exceed $100, given that you sold 4 cars in week 1? Report this to the nearest percent. Plot the distribution of b2 as a histogram, not a density.

Given the fact that I sold 4 cars in week 1, I have a posterior of $\mathrm{Gamma}(9,2)$, which is the same as in problem 3.

In order that

$$b_2 = (2\pi y_2)^{\sqrt{2}} > 100$$

There has to be a y with $y_2 > \frac{100^{1/\sqrt{2}}}{2\pi} \approx 4.13$. Because y has to be integer, y_2>=5. The result is shown in the following histogram and the corresponding probability of getting >100 bonus is 33%.



```
#################Code#####################


######problem 3###########
theta=seq(0,10,length=11);alpha=9;beta=2;S=1000000;
posterior <- dgamma(theta,alpha,beta)

expectation <- function(x){
z=dgamma(x,9,2)*as.numeric(x>=6)
return (z)
}
theoretical=integrate(expectation,lower=0,upper=Inf)

expectation <- sum(as.numeric(rgamma(S,alpha,beta)>=6))/S
print(expectation)

#######Problem 4############
threshold <- ceiling(100^(1/sqrt(2))/(2*pi))
temp <- rgamma(S,alpha,beta)
expectation2 <- sum(as.numeric(temp>=threshold))/S
print(expectation2)
hist((2*pi*temp)^(sqrt(2)),breaks=10,xlab="Bonus",main="Histogram of Bonus in Week 2")
```