# Pursuit Curves

David Grant

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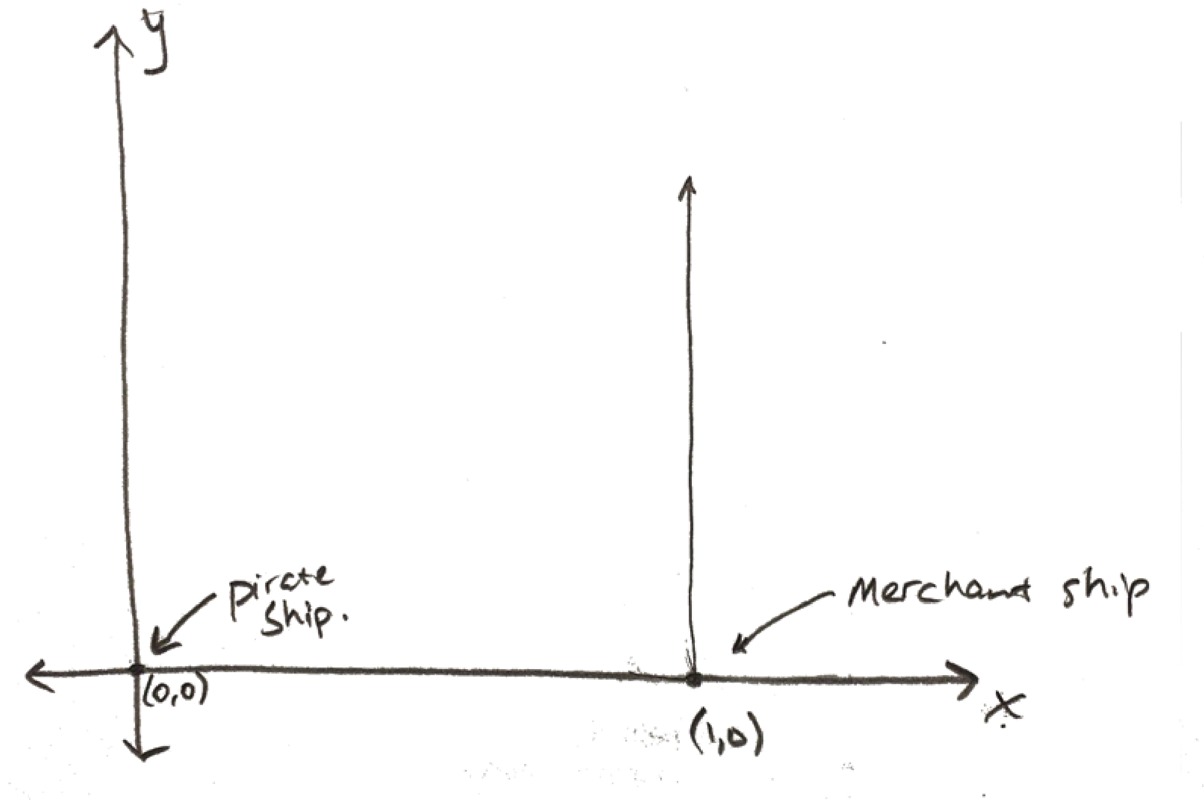
## 1.




*Sketch of initial ship position and trajectory of merchant ship.*

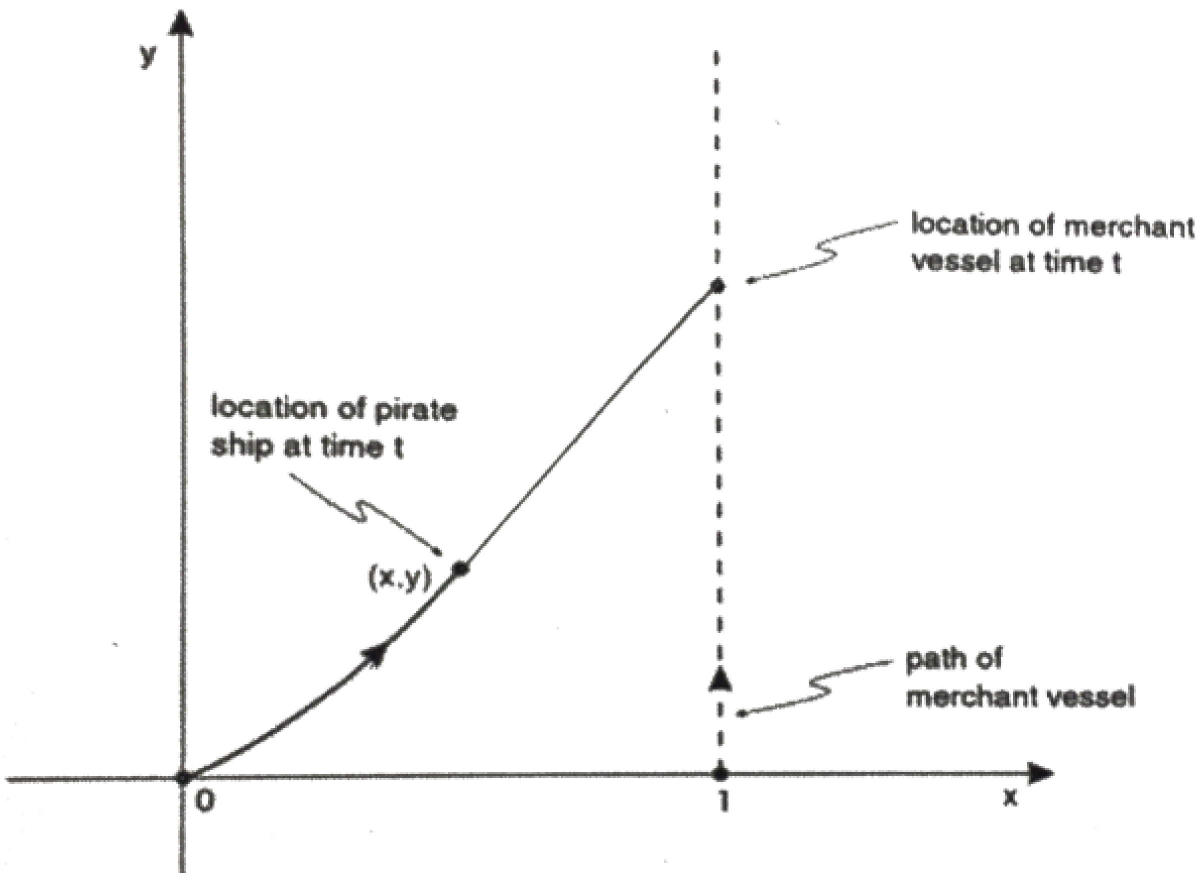
## 2.

Since both ships begin at the same $y$ coordinate and the merchant ship is constantly traveling due north, it makes sense that the pirate ship's path will be a non-linear, square-looking one.

## 3.

The merchant ship will have traveled $V_m t$ *...units...* after $t$ hours. At this time, the merchant ship's coordinates will be $(1, V_m t)$.

## 4.




## 5.

The slope of this tangent line between the pirate and merchant ships is:

$$\frac{y_p - y_m}{x_p - x_m} \to \frac{y_p - V_m t}{x_p - 1}$$

This is the instantaneous slope of the pirate ship, so:

$$\frac{dy}{dx} = \frac{y_p - V_m t}{x_p - 1}$$

Solving for $t$:

$$\frac{dy}{dx}(x_p - 1) = y_p - V_m t$$
$$\frac{dy}{dx}(x_p - 1) - y_p = -V_m t$$

$$\frac{\frac{dy}{dx}(x_p - 1) - y_p}{-V_m} = t$$

$$t = \frac{-\frac{dy}{dx}(x_p - 1) + y_p}{V_m}$$

## 6.

We're given this equality, which makes enough sense:

$$V_p t = \int_0^x \sqrt{1 + \left(\frac{dy}{dz}\right)^2} dz$$

Solving for $t$:

$$t = \frac{1}{V_p} \int_0^x \sqrt{1 + \left(\frac{dy}{dz}\right)^2} dz$$

## 7.

Using the results from #5 and #6 as well as letting $p(x) = \frac{dy}{dx}$, show that $\frac{1}{V_p} \int_0^x \sqrt{1 + p^2(z)} dz = \frac{y}{V_m} - \frac{(x-1)}{V_m} \cdot p(x)$.

First, we define $p$:

$$p(x) = \frac{dy}{dx}$$

Our conclusion from #6:

$$t = \frac{1}{V_p} \int_0^x \sqrt{1 + \left(\frac{dy}{dz}\right)^2} dz$$

becomes

$$t = \frac{1}{V_p} \int_0^x \sqrt{1 + p^2(z)} dz$$

Recalling our conclusion from #5:

$$t = \frac{-\frac{dy}{dx}(x_p - 1) + y_p}{V_m}$$

This can be reorganized to:

$$t = \frac{-p(x)(x-1)}{V_m} + \frac{y}{V_m}$$

$$t = p(x) \cdot \frac{-(x-1)}{V_m} + \frac{y}{V_m}$$

$$t = \frac{y}{V_m} - \frac{(x-1)}{V_m} \cdot p(x)$$

So,

$$\frac{1}{V_p} \int_0^x \sqrt{1+p^2(z)}dz = \frac{y}{V_m} - \frac{(x-1)}{V_m} \cdot p(x)$$

## 8.

> Using some ideas from the FTC, take a derivative in terms of $x$ on both sides of the equation in #7 [...].

**Left-hand side:**

$$\frac{d}{dx}\Big(\frac{1}{V_p} \int_0^x \sqrt{1+p^2(z)}dz\Big) \to \frac{1}{V_p} \cdot \frac{d}{dx}\Big(\int_0^x \sqrt{1+p^2(z)}dz\Big)$$

Applying $\text{FTC}_2$ our LHS becomes:

$$\frac{1}{V_p} \cdot \sqrt{1+p^2(x)}$$

**Now, the RHS:**

$$\frac{d}{dx}\Big(\frac{y}{V_m} - \frac{x-1}{V_m} \cdot p(x)\Big)$$

$$\frac{1}{V_m}\left(\frac{dy}{dx} - \frac{d}{dx}\Big((x-1) \cdot p(x)\Big)\right)$$

$$\frac{1}{V_m}\left(\frac{dy}{dx} - \Big(p(x) + (x-1) \cdot p'(x)\Big)\right) \to \frac{1}{V_m}\left(\frac{dy}{dx} - \Big(\frac{dy}{dx} + (x-1) \cdot \frac{dp}{dx}\Big)\right)$$

$$\frac{1}{V_m}\Big(-(x-1) \cdot \frac{dp}{dx}\Big)$$

$$-\frac{(x-1)}{V_m} \cdot \frac{dp}{dx}$$

Since we've just taken the derivative of both sides, they're still equal, and we get the desired equality:

$$\frac{1}{V_p}\sqrt{1+p^2(x)} = -\frac{(x-1)}{V_m}\frac{dp}{dx}$$

Simplify this further by multiplying both sides by $-V_m$.

$$(-V_m)\frac{1}{V_p}\sqrt{1+p^2(x)} = (-V_m)\frac{-(x-1)}{V_m}\frac{dp}{dx}$$

$$\frac{-V_m}{V_p}\sqrt{1+p^2(x)} = (x-1)\frac{dp}{dx}$$

...and let $n = \frac{Vm}{Vp}$, the ratio of the merchant ship's velocity to the pirate ship's velocity.

$$-n\sqrt{1+p^2(x)} = (x-1)\frac{dp}{dx}$$

**When** $n > 1$ , the merchant ship is moving faster than the pirate ship.

**When** $n = 1$ , the ships are traveling at the same velocity.

**When** $n < 1$ , the pirate ship is moving faster than the merchant ship.

The case when $n < 1$ is the most important to us, because the pirates will potentially be closing in on the merchants.

(Note that since the merchant ship is traveling in a "bee" line and the pirate ship is following some curve, the ratio $n$ will have to be some amount *under* 1 in order for the pirate ship to actually get closer to the merchant ship.)

## 9.

We now have a differential equation that looks like:

$$(x-1)\frac{dp}{dx} = -n\sqrt{1+p^2(x)}$$

We use separation of variables to take the integrals:

$$\int \frac{1}{\sqrt{1+p^2(x)}}dp = \int \frac{-n}{x-1}dx$$

For the LHS, my computer gives me:

$$\int \frac{1}{\sqrt{1+p^2(x)}}dp = sinh^{-1}(p) + C$$

Per the definition of $sinh^{-1}$:

$$sinh^{-1}(p) = ln(p + \sqrt{p^2+1})$$

For the RHS:

$$\int \frac{-n}{x-1}dx \rightarrow n \cdot \int \frac{1}{1-x}dx$$

$$= -n \cdot ln(1-x) + C$$

Altogether:

$$ln(p + \sqrt{p^2+1}) + C = -n \cdot ln(1-x)$$

## 10.

When $t = 0$, $x$ is also 0. At this time, the instantaneous value of $p$, which was defined as $\frac{dy}{dx}$, will be $\frac{0}{something}$. That is, at this time the pirate ship will be proceeding due east directly at the merchant ship. $p_x$ will be changing and $p_y$ will not.

Recalling our previous equation:

$$ln(p + \sqrt{p^2+1}) + C = -n \cdot ln(1-x)$$

We can plug in some values since we know that when $x = 0$, $p$ is also 0:

$$ln(0 + \sqrt{0^2+1}) + C = -n \cdot ln(1-0)$$

$$ln(1) + C = -n \cdot ln(1)$$

$$C = 0$$

## 11.

Rearranging using various natural log rules:

$$ln(p + \sqrt{p^2+1}) = -n \cdot ln(1-x)$$

$$ln(p + \sqrt{p^2+1}) = -ln((1-x)^n)$$

$$ln(p + \sqrt{p^2+1}) + ln((1-x)^n) = 0$$

$$ln\Big((p + \sqrt{p^2+1})(1-x)^n\Big) = 0$$

$$e^{ln((p+\sqrt{p^2+1})(1-x)^n)} = e^0$$

$$(p + \sqrt{p^2+1})(1-x)^n = 1$$

## 12.

Letting $q = (1-x)^{-n}$:

$$(p + \sqrt{p^2+1})\frac{1}{q} = 1$$

$$p + \sqrt{p^2+1} = q$$

$$\sqrt{p^2+1} = q - p$$

$$p^2 + 1 = (q-p)^2$$

$$p^2 + 1 = q^2 + p^2 - 2qp$$

$$p^2 - p^2 = q^2 - 2qp - 1$$

$$2qp = q^2 - 1$$

$$p = \frac{q^2 - 1}{2q}$$

$$p = \frac{1}{2}(q - \frac{1}{q})$$

Since $p = \frac{dy}{dx}$ and $q = (1-x)^{-n}$:

$$\frac{dy}{dx} = \frac{1}{2}\Big((1-x)^{-n} - (1-x)^{n}\Big)$$

## 13.

Integrate our result from #12 in terms of $x$:

$$\int \frac{dy}{dx} dx = \int \frac{1}{2}\Big((1-x)^{-n} - (1-x)^{n}\Big)dx$$

LHS:

$$\int \frac{dy}{dx} dx = y$$

RHS:

$$\int \frac{1}{2}\Big((1-x)^{-n} - (1-x)^{n}\Big)dx$$

$$\frac{1}{2}\Big(\int (1-x)^{-n}dx - \int (1-x)^{n}dx\Big)$$

Integrate the first term, $\int(1-x)^{-n}dx$. Let $u = 1-x$, $du = -dx$, $dx = -du$:

$$-\int u^{-n}du = \frac{-u^{1-n}}{1-n}$$

Back-substituting for $u$:

$$\frac{-(1-x)^{1-n}}{1-n}$$

Integrate the second term, $\int(1-x)^{n}dx$. Again, let $u = 1-x$, $du = -dx$, $dx = -du$:

$$-\int u^{n}du = \frac{-u^{n+1}}{1+n}$$

Back-substituting for $u$:

$$\frac{-(1-x)^{n+1}}{1+n}$$

Our RHS expression becomes:

$$\frac{1}{2}\Big(\frac{(1-x)^{n+1}}{1+n} - \frac{(1-x)^{1-n}}{1-n}\Big)$$

Factor a $(1-x)$ out of the interior:

$$\frac{1}{2}(1-x)\Big(\frac{(1-x)^{n}}{1+n} - \frac{(1-x)^{-n}}{1-n}\Big)$$

Altogether:

$$y = \frac{1}{2}(1-x)\left(\frac{(1-x)^{n}}{1+n} - \frac{(1-x)^{-n}}{1-n}\right) + C$$

## 14.

Solve for $C$, given that $y = 0$ when $x = 0$.

$$y = \frac{1}{2}(1-x)\left(\frac{(1-x)^{n}}{1+n} - \frac{(1-x)^{-n}}{1-n}\right) + C$$

$$0 = \frac{1}{2}\left(\frac{(1)^{n}}{1+n} - \frac{(1)^{-n}}{1-n}\right) + C$$

$$C = -\frac{1}{2}\left(\frac{1}{1+n} - \frac{1}{1-n}\right)$$

$$C = -\frac{1}{2}\left(\frac{-2n}{1-n^2}\right)$$

$$C = \frac{n}{1-n^2}$$

## 15.

Now we have a complete equation for the pirate ship's position:

$$y = \frac{1}{2}(1-x)\left(\frac{(1-x)^n}{1+n} - \frac{(1-x)^{-n}}{1-n}\right) + \frac{n}{1-n^2}$$

Cooking up a function in Mathematica:

```
Pursuit[x_, n_] := 1/2 (1 - x) ((1 - x)^n/(1 + n) - (1 - x)^-n/(1 - n)) + n/(1 - n^2)
```

We can graph a bunch of different values of $n$. (See figure 1.)

## 16.

As $n$ approaches 1, the merchant ship is able to get further out in front of the pirate ship. The pirate ship is forced to turn north-bound more and more in order to chase the merchant.

Contrasted with lower $n$ values (i.e., 0.1), where the pirate ship is going significantly faster than the merchant ship, and is able to speed mostly due east to intercept the merchant.

When I plot a few graphs where $n > 1$, I get pirate ship trajectories that do not end, even for very high values of $y$. This makes sense, as the pirate ship is forever chasing a ship that it is outpaced by. (See figure 2.)

```
Plot[{Pursuit[x, .10], Pursuit[x, .40], Pursuit[x, .70], Pursuit[x, .95] }, {x, 0, 1},
 ImageSize → Large, PlotRange → {0, 2}, PlotLegends → "Expressions" ]
```

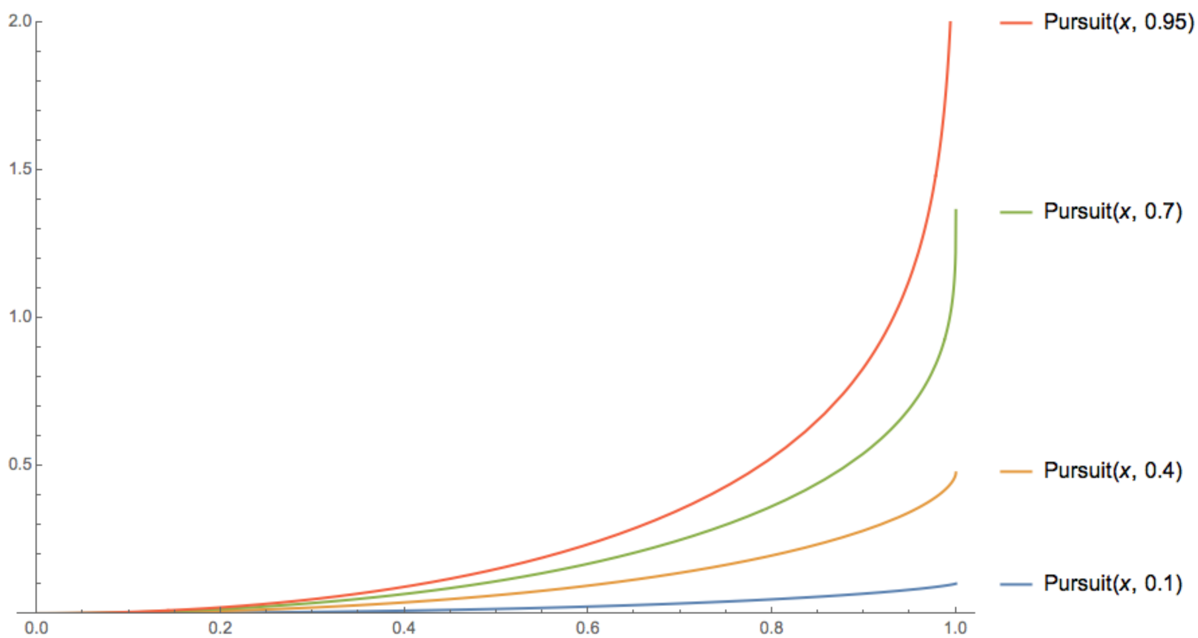



Figure 1: Graph of scenarios where $n < 1$.

```
Plot[{Pursuit[x, .70], Pursuit[x, .95], Pursuit[x, 1.1], Pursuit[x, 1.25], Pursuit[x, 1.6] }
 {x, 0, 1}, ImageSize → Large, PlotRange → {0, 7}, PlotLegends → "Expressions" ]
```

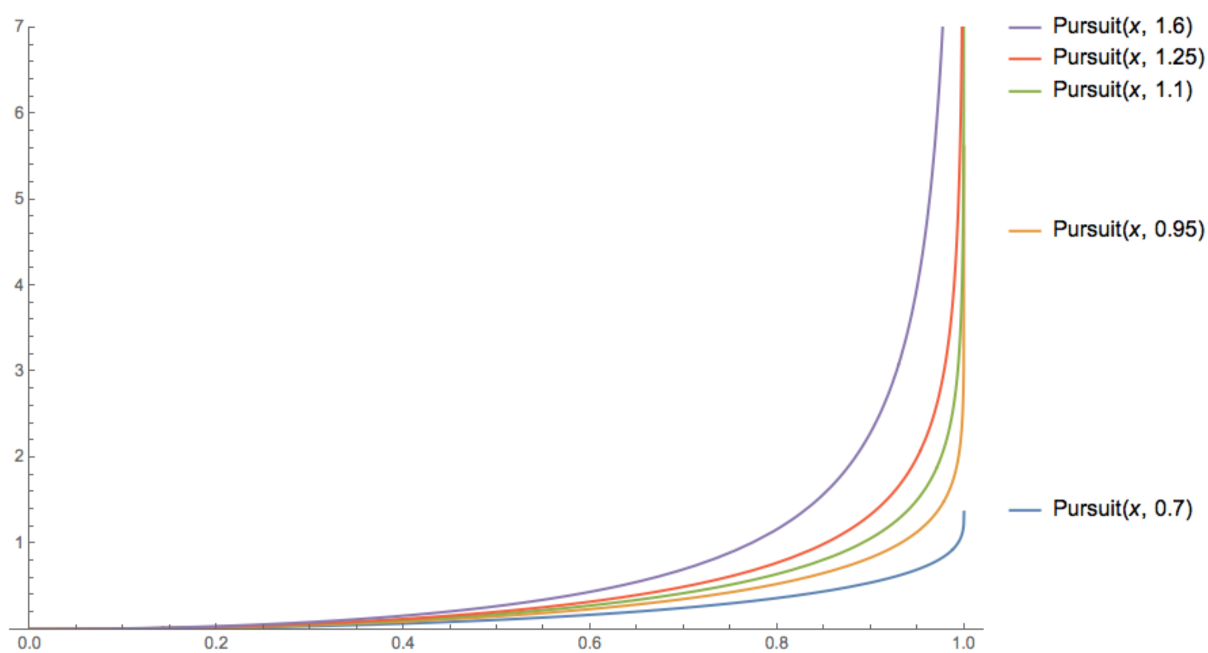



Figure 2: Graph of scenarios where $n > 1$.