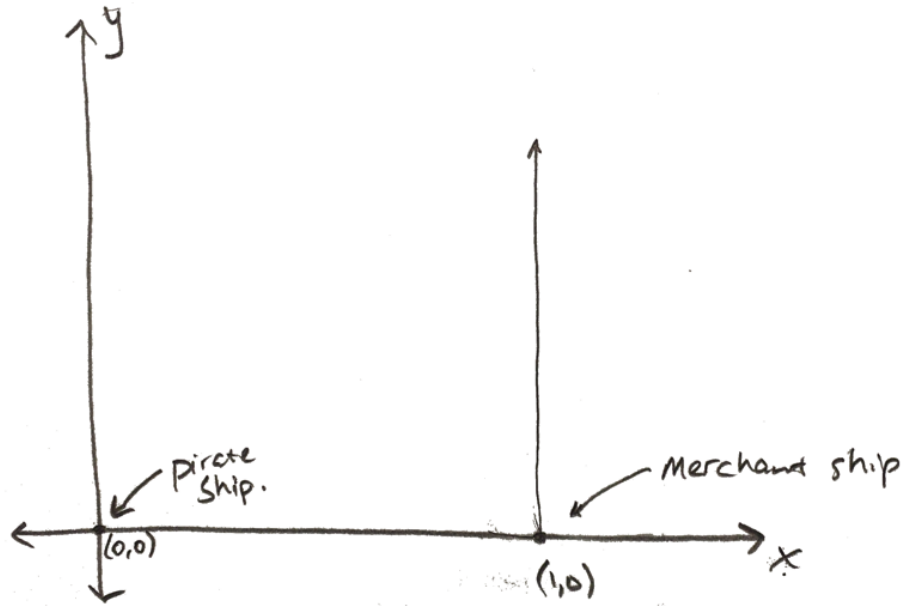


# Pursuit Curves

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1.



*Sketch of initial ship position and trajectory of merchant ship.*

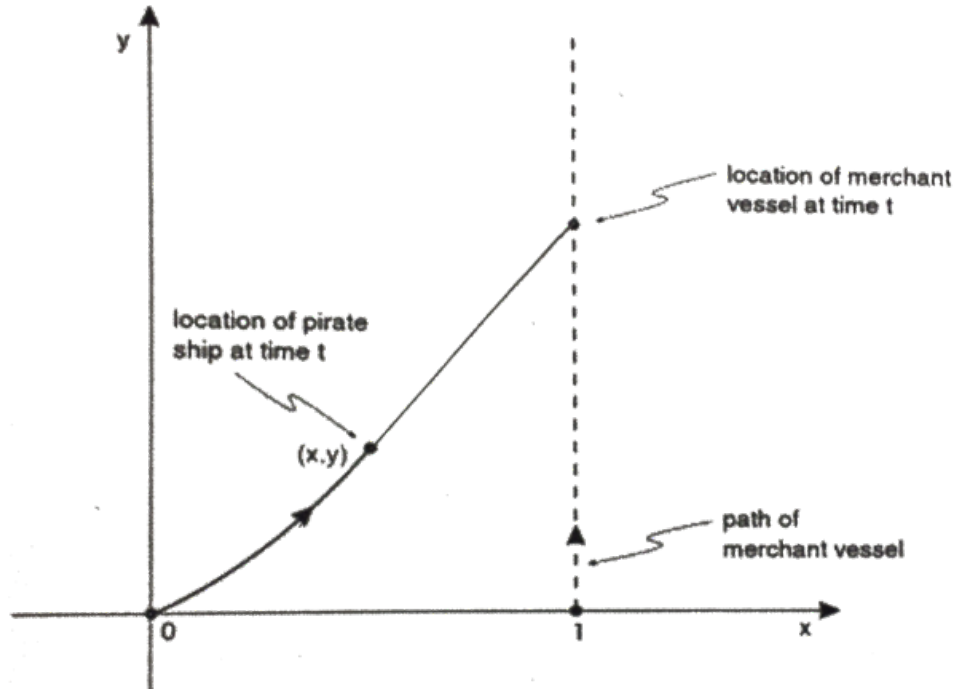
2.

Since both ships begin at the same  $y$  coordinate and the merchant ship is constantly traveling due North, it makes sense that the pirate ship's path will be a non-linear, square-looking one.

3.

The merchant ship will have traveled  $V_mt$  ...units... after  $t$  hours. At this time, the merchant ship's coordinates will be  $(1, V_mt)$ .

4.



5.

The slope of this tangent line between the pirate and merchant ships is:

$$\frac{y_p - y_m}{x_p - x_m} \rightarrow \frac{y_p - V_m t}{x_p - 1}$$

This is the instantaneous slope of the pirate ship, so:

$$\frac{dy}{dx} = \frac{y_p - V_m t}{x_p - 1}$$

Solving for  $t$ :

$$\frac{dy}{dx}(x_p - 1) = y_p - V_m t$$

$$\begin{aligned}\frac{dy}{dx}(x_p - 1) - y_p &= -V_m t \\ \frac{\frac{dy}{dx}(x_p - 1) - y_p}{-V_m} &= t\end{aligned}$$

$$t = \frac{-\frac{dy}{dx}(x_p - 1) + y_p}{V_m}$$

**6.**

We're given this equality, which makes enough sense:

$$V_p t = \int_0^x \sqrt{1 + \left(\frac{dy}{dz}\right)^2} dz$$

Solving for  $t$ :

$$t = \frac{1}{V_p} \int_0^x \sqrt{1 + \left(\frac{dy}{dz}\right)^2} dz$$

**7.**

Using the results from #5 and #6 as well as letting  $p(x) = \frac{dy}{dx}$ ,  
show that  $\frac{1}{V_p} \int_0^x \sqrt{1 + p^2(z)} dz = \frac{y}{V_m} - \frac{(x-1)}{V_m} \cdot p(x)$ .

First, we define  $p$ :

$$p(x) = \frac{dy}{dx}$$

Our conclusion from #6:

$$t = \frac{1}{V_p} \int_0^x \sqrt{1 + \left(\frac{dy}{dz}\right)^2} dz$$

becomes

$$t = \frac{1}{V_p} \int_0^x \sqrt{1 + p^2(z)} dz$$

Recalling our conclusion from #5:

$$t = \frac{-\frac{dy}{dx}(x_p - 1) + y_p}{V_m}$$

This can be reorganized to:

$$t = \frac{-p(x)(x - 1)}{V_m} + \frac{y}{V_m}$$

$$t = p(x) \cdot \frac{-(x - 1)}{V_m} + \frac{y}{V_m}$$

$$t = \frac{y}{V_m} - \frac{(x - 1)}{V_m} \cdot p(x)$$

So,

$$\frac{1}{V_p} \int_0^x \sqrt{1 + p^2(z)} dz = \frac{y}{V_m} - \frac{(x - 1)}{V_m} \cdot p(x)$$

**8.**

Using some ideas from the FTC, take a derivative in terms of  $x$  on both sides of the equation in #7 [...].

**Left-hand side:**

$$\frac{d}{dx} \left( \frac{1}{V_p} \int_0^x \sqrt{1 + p^2(z)} dz \right) \rightarrow \frac{1}{V_p} \cdot \frac{d}{dx} \left( \int_0^x \sqrt{1 + p^2(z)} dz \right)$$

Applying FTC<sub>2</sub> our LHS becomes:

$$\frac{1}{V_p} \cdot \sqrt{1 + p^2(x)}$$

**Now, the RHS:**

$$\frac{d}{dx} \left( \frac{y}{V_m} - \frac{x - 1}{V_m} \cdot p(x) \right)$$

$$\frac{1}{V_m} \left( \frac{dy}{dx} - \frac{d}{dx} \left( (x-1) \cdot p(x) \right) \right)$$

$$\frac{1}{V_m} \left( \frac{dy}{dx} - \left( p(x) + (x-1) \cdot p'(x) \right) \right) \rightarrow \frac{1}{V_m} \left( \frac{dy}{dx} - \left( \frac{dy}{dx} + (x-1) \cdot \frac{dp}{dx} \right) \right)$$

$$\frac{1}{V_m} \left( - (x-1) \cdot \frac{dp}{dx} \right)$$

$$- \frac{(x-1)}{V_m} \cdot \frac{dp}{dx}$$

Since we've just taken the derivative of both sides, they're still equal, and we get the desired equality:

$$\frac{1}{V_p} \sqrt{1 + p^2(x)} = - \frac{(x-1)}{V_m} \frac{dp}{dx}$$

Simplify this further by multiplying both sides by  $-V_m$ .

$$(-V_m) \frac{1}{V_p} \sqrt{1 + p^2(x)} = (-V_m) \frac{-(x-1)}{V_m} \frac{dp}{dx}$$

$$\frac{-V_m}{V_p} \sqrt{1 + p^2(x)} = (x-1) \frac{dp}{dx}$$

...and let  $n = \frac{V_m}{V_p}$ , the ratio of the merchant ship's velocity to the pirate ship's velocity.

$$-n \sqrt{1 + p^2(x)} = (x-1) \frac{dp}{dx}$$

**When**  $n > 1$  , the merchant ship is moving faster than the pirate ship.

**When**  $n = 1$  , the ships are traveling at the same velocity.

**When**  $n < 1$  , the pirate ship is moving faster than the merchant ship.

The case when  $n < 1$  is the most important to us, because the pirates will potentially be closing in on the merchants.

(Note that since the merchant ship is traveling in a "bee" line and the pirate ship is following some curve, the ratio  $n$  will have to be some amount *under* 1 in order for the pirate ship to actually get closer to the merchant ship.)

## 9.

We now have a differential equation that looks like:

$$(x - 1) \frac{dp}{dx} = -n \sqrt{1 + p^2(x)}$$

We use separation of variables to take the integrals:

$$\int \frac{1}{\sqrt{1 + p^2(x)}} dp = \int \frac{-n}{x - 1} dx$$

For the LHS, my computer gives me:

$$\int \frac{1}{\sqrt{1 + p^2(x)}} dp = \sinh^{-1}(p)$$

Per the definition of  $\sinh^{-1}$ :

$$\sinh^{-1}(p) = \ln(p + \sqrt{p^2 + 1})$$

For the RHS:

$$\begin{aligned} \int \frac{-n}{x - 1} dx &\rightarrow n \cdot \int \frac{1}{1 - x} dx \\ &= -n \cdot \ln(1 - x) \end{aligned}$$

Altogether:

$$\ln(p + \sqrt{p^2 + 1}) = -n \cdot \ln(1 - x)$$