Indian Institute of Technology Kanpur
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NONLINEAR DYNAMICAL SYSTEMS (MTH658A)
PROJECT REPORT

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200911

Topic

EXISTENCE OF SUPERCRITICAL HOPF BIFURCATION ON A TYPE-LORENZ SYSTEM

November 2023

Contents

Abstract

In this paper, a system of Lorenz-type ordinary differential equations is considered and, under some assumptions about the parameter space, the presence of the supercritical non-degenerate Hopf bifurcation is demonstrated. The technical tool used consists of the Central Manifold theorem, a well known formula to calculate the Lyapunov coefficient and Hopf's Theorem. For particular values of the parameters in the parameter space established in the main result of this work, a graph is presented that describes the evolution of the trajectories, obtained by means of numerical simulation.

Introduction

The Lorenz-Type systems are notable for having chaotic solutions for certain parameter values and initial conditions. These systems present a great variety of dynamic behaviors such as the presence of chaotic orbits, existence of homoclinic and heteroclinic orbits, presence of Hopf bifurcation, as well as the Lorenz attractor, among others. An interesting problem is determining the geometric structure of the Lorenz attractor for specific Lorenz-type systems. In particular, the Lorenz attractor is a set of chaotic solutions of the Lorenz system. In popular media the *butterfly effect* stems from the real-world implications of the Lorenz attractor, namely that several different initial chaotic conditions evolve in phase space in a way that never repeats, so all chaos is unpredictable.

A Hopf bifurcation is a critical point where, as a parameter changes, a system's stability switches and a periodic solution arises. More accurately, it is a local bifurcation in which a fixed point of a dynamical system loses stability. Under reasonably generic assumptions about the dynamical system, the fixed point becomes a small-amplitude limit cycle as the parameter changes. Hopf Bifurcation corresponds to the following situation: when the system parameter is varied and it crosses a critical value, the Jacobian, at equilibrium, has a pair of conjugate complex eigenvalues moving from the left half-plane to the right or vice versa, while the other eigenvalues remain fixed; at the time of crossing, the real parts of the two eigenvalues become zero, the stability of the equilibrium changes from stable to unstable, or from unstable to stable, thus giving rise to a limit cycle. When the limit cycle is stable, the Hopf Bifurcation is supercritical. When the limit cycle is unstable, the Hopf bifurcation is said to be subcritical.

In this paper, it is proposed the study of the Hopf Bifurcation for the Lorenz type system

$$\begin{aligned} \dot{x} &= a(y-x) \\ \dot{y} &= dy - xz \\ \dot{z} &= -bz + fx^2 + gxy \end{aligned} \tag{1}$$

when $a > 0$, $f \geq 0$, $g \geq 0$, $f + g > 0$, $b, d \in \mathbb{R}$, and $(x, y, z)^T \in \mathbb{R}^3$ represents the system state variable.

The system (1) is presented by Li and Ou in their article [2] of the year 2011; for this reason it is called Li-Ou system. The problem of determining whether the Hopf Bifurcation is nondegenerate, as well as the problem of distinguishing whether such a bifurcation is supercritical or subcritical, was left open. In this paper, the problem is addressed for the case $f > 0$ and it is proved the existence of the supercritical Hopf Bifurcation for this system, and a concrete example is modeled showing the Hopf bifurcation with the behavior of trajectories for a particular system.

Dynamics of the Li-Ou System

Symmetry

The following lemma shows that the Li-Ou system is symmetric with respect to the Z axis.

Lemma 1: *The system (1) is invariant under the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ for all $(x, y, z) \in \mathbb{R}^3$*

Proof follows from $F \circ T = T \circ F$ for the vector field F associated with system (1).

Equilibriums

The equilibrium points are obtained by solving

$$\begin{aligned} a(y-x) &= 0 \\ dy - xz &= 0 \\ -bz + fx^2 + gxy &= 0 \\ \implies x = y \ ,\ z &= d \\ \text{and}\ \ (f+g)x^2 &= bd \end{aligned}$$

This gives us three cases:

Case-1: $bd < 0$. The origin is the only equilibrium point of the system and is denoted by P_0.

Case-2: $bd = 0$. Equilibrium points are P_0 when $d = 0$ and $(0, 0, z)$ when $d \neq 0$.

Case-3: $bd > 0$. This system has three equilibrium points

$$P_0 = (0,0,0),\ P_1 = \left(\sqrt{\frac{bd}{f+g}}, \sqrt{\frac{bd}{f+g}}, d\right)\ \ ,\ P_2 = \left(-\sqrt{\frac{bd}{f+g}}, -\sqrt{\frac{bd}{f+g}}, d\right).$$

Only Case-3 is of our interest and we shall focus on $bd > 0$. In 2011, Li and Ou [2] showed that in P_0 the system (1) presents a dynamic without bifurcation, in contrast, at the equilibrium points P_1 and P_2 the system presents Hopf bifurcation. The results from [2] for the system (1), with $f = 0$, is stated in Theorem 1 and the result with $f > 0$, is stated in Theorem 2.

Theorem 1 *For the system (1) with parameters $a > 0$, $b > 0$, $g > 0$ and $f = 0$, the following statements hold:*
1) For $0 < d < \dfrac{a+b}{3}$ the equilibrium points P_1 and P_2 are stable

2) When $d > \dfrac{a+b}{3}$ *the equilibrium points* P_1 *and* P_2 *are unstable*

3) When $d = \dfrac{a+b}{3}$*, in each equilibrium* P_1 *and* P_2 *arises a periodic orbit with period* $T = \dfrac{2\pi}{\sqrt{ab}}$

Therefore the system presents a Hopf bifurcation at P_1 *and* P_2 *with bifurcation critical value* $d\_(0) := \dfrac{a+b}{3}$.

Theorem 2 *For the system (1) with parameters* $a > 0$*,* $b > 0$*,* $f > 0$*,* $g > 0$ *and* $f + g > 0$*, the following statements hold:*
1) For $0 < d < d\_(f)$ *the equilibrium points* P_1 *and* P_2 *are stable*

2) When $d > d\_(f)$ *the equilibrium points* P_1 *and* P_2 *are unstable*

3) For $d = d\_(f)$*, an orbit appears at each equilibrium* P_1 *and* P_2 *arises a periodic orbit with period* $T = \dfrac{2\pi}{\omega}$*, where* $\omega = \sqrt{\dfrac{2abd\_(f)}{a+b-d\_(f)}}$.

Therefore the system presents a Hopf bifurcation at P_1 *and* P_2 *with bifurcation critical value*

$$d\_(f) = \frac{3a(f+g)+(a+b)f}{2f} - \frac{\sqrt{9a^2(f+g)^2+2a(f+g)(a+b)f+(a+b)^2f^2}}{2f}.$$

Analysis of the Hopf bifurcation is very important in the study of the stability of the periodic orbits of a system. On the other hand, when the stability of a periodic orbit is not desired, it is possible to disturb the system in order to change its stability. This process is called stability control. In the Hopf bifurcation control, the information that provides the analysis of the bifurcation is considered primary information. A Hopf bifurcation analysis consists of determining whether the Hopf bifurcation is nondegenerate, and whether it is the case, to distinguish if it is supercritical or subcritical.

Nondegenerate Hopf Bifurcation

Case $f = 0$

This case was studied in Calderón-Saavedra, P.E., Alvarez-Mena, J. and Muñoz-Aguirre, E. (2018) *Tratamiento analítico de la bifurcación de hopf en una extensión del sistema de lü. Revista de Matemática: Teoría y Aplicaciones*, **25**, 29. There it is shown that the Hopf bifurcation is nondegenerate and supercritical in a specific region of parameters. The result is stated in Theorem 3.

Theorem 3 *When the parameters satisfy* $b > 0$*,* $a > b$*,* $d > \dfrac{a+b}{3}$*,* $g > 0$ *and* $f = 0$*, the system (1) presents a Hopf bifurcation nondegenerate and supercritical at equilibrium points* P_1 *and* P_2

Case $f > 0$

In this section we will work on two specific regions of system parameters, where the Hopf bifurcation is nondegenerate and supercritical.

Regions in the parameters space of the system are determined (1):

$$R_I : \{(a,b,d,f,g) \in \mathbb{R}^5 \;|a > 0, a = b, d = d\_(f), f = g = 1\}$$

$$R_{II} : \{(a,b,d,f,g) \in \mathbb{R}^5 \;|a > 0, a = b, d = d\_(f), f = 1, g = 2\}.$$

Theorem 4 *If the system parameters (1) are in the region R_I or the region R_{II}, the periodic orbits around P_1 and P_2 are stable. Therefore, the system (1) presents Hopf bifurcation non-degenerate and supercritical in R at equilibrium points P_1 and P_2.*

Proof: Under the conditions in the parameters, $a > 0, a = b, d = d\_(f), f > 0, g \geq 0$, Theorem 2 guarantees the existence of the Hopf bifurcation at equilibrium points P_1 and P_2. For analysis of the Hopf bifurcation at these equilibrium points, it proceeds as follows. By the symmetry of the system with respect to the z axis (Lemma 1), the critical point P_1 is analyzed and the results are extended to the point critical P_2. Using the well-known formula for the first Lyapunov coefficient it is determined that the first Lyapunov coefficient is negative at the equilibrium point P_1. Finally, it is concluded by the Hopf Theorem that the Hopf bifurcation is supercritical.

Jacobian matrix A of system (1) evaluated at equilibrium P_1 is

$$A = \begin{bmatrix} -a & a & 0 \\ -d\_(f) & d\_(f) & -h \\ (2f+g)h & gh & -a \end{bmatrix}, \ with \ \ h = \sqrt{\frac{ad\_(f)}{f+g}}$$

.

Solving the system $Aq = i\omega q$, the eigenvalue $\lambda_1 = i\omega$ is obtained, with eigenvector q. The adjoint eigenvector $p \in C^3$, satisfies the equation $A^T p = -\omega i p$. q and p are as follows

$$q = \begin{bmatrix} ah \\ ah + i \\ \omega^2 + (d\_(f) - a)\omega i \end{bmatrix} \qquad p = \begin{bmatrix} (\omega^2 + gh^2 - ad\_(f)) + (a + d\_(f))\omega i \\ ab - ai \\ -ah \end{bmatrix}$$

It is necessary to determine a vector parallel to p that satisfies the property $\langle p, q \rangle = 1$, hence, vector p is normalized:

$$p = \frac{1}{2ah\omega(d\_(f) - a)i} \begin{bmatrix} (\omega^2 + gh^2 - ad\_(f)) + (a + d\_(f))\omega i \\ ab - ai \\ -ah \end{bmatrix}.$$

In order to calculate the first Lyapunov coefficient, the equilibrium point must be transferred to the origin

$$P_1 = \left(\sqrt{\frac{ad\_(f)}{f+g}}, \sqrt{\frac{ad\_(f)}{f+g}}, d\_(f) \right) = (h, h, d\_(f)).$$

This is done by the transformation $Y = X - P_1$, where X satisfies the system (1). Then Y satisfies the system

$$\dot{Y} = \begin{bmatrix} -a & a & 0 \\ -d\_(f) & d\_(f) & -h \\ (2f+g)h & gh & -a \end{bmatrix} Y + y_1 K Y, \tag{2}$$

where y_1 is the first Y coordinate and K is the matrix

$$K = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ f & g & 0 \end{bmatrix}$$

It is observed that system (2) has the form $\dot{Y} = AY + F(Y)$, where A is the Jacobian matrix of the system evaluated at equilibrium P_1 and the nonlinear part

$$F(Y) = \begin{bmatrix} 0 \\ -y_1y_3 \\ fy_1^2 + gy_1y_2 \end{bmatrix}$$

.

Thus, $F(Y) = O(||Y||^2)$. On the other hand, the Taylor expansion of F in a neighborhood of $Y = 0$ is expressed by

$$F(Y) = \frac{1}{2}B(Y,Y) + \frac{1}{6}C(Y,Y,Y) + O(||Y||^4)$$

where $B(Y,Y)$, $C(Y,Y,Y)$ are multilinear vector functions with $Y \in \mathbb{R}^3$. To find an expression for multilinear vector functions B and C, the partial derivatives of the components functions (F_1, F_2, F_3) of F are used. The first component function is the zero function, so it does not contribute to the expressions that are searched for. The partial derivatives of F_2 are,

$$\begin{array}{llll}
\dfrac{\partial F_2}{\partial y_1} = -y_3 & \dfrac{\partial F_2}{\partial y_2} = 0 & \dfrac{\partial F_2}{\partial y_3} = -y_1 & \dfrac{\partial^2 F_2}{\partial y_1^2} = 0 \\
\dfrac{\partial^2 F_2}{\partial y_1 y_2} = 0 & \dfrac{\partial^2 F_2}{\partial y_1 y_3} = -1 & \dfrac{\partial^2 F_2}{\partial y_2 y_1} = 0 & \dfrac{\partial^2 F_2}{\partial y_2^2} = 0 \\
\dfrac{\partial^2 F_2}{\partial y_2 y_3} = 0 & \dfrac{\partial^2 F_2}{\partial y_3 y_1} = -1 & \dfrac{\partial^2 F_2}{\partial y_3 y_2} = 0 & \dfrac{\partial^2 F_2}{\partial y_3^2} = 0
\end{array}$$

Then the function B_2 is expressed in the form

$$B_2(q,q) = q^T \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} q = -2ah(\omega^2 + (d-a)\omega i).$$

The partial derivatives of F_3 are used,

$$\begin{array}{llll}
\dfrac{\partial F_3}{\partial y_1} = gy_2 + 2fy_1 & \dfrac{\partial F_3}{\partial y_2} = gy_1 & \dfrac{\partial F_3}{\partial y_3} = 0 & \dfrac{\partial^2 F_3}{\partial y_1^2} = 2f \\
\dfrac{\partial^2 F_3}{\partial y_1 y_2} = g & \dfrac{\partial^2 F_3}{\partial y_1 y_3} = 0 & \dfrac{\partial^2 F_3}{\partial y_2 y_1} = g & \dfrac{\partial^2 F_3}{\partial y_2^2} = 0 \\
\dfrac{\partial^2 F_3}{\partial y_2 y_3} = 0 & \dfrac{\partial^2 F_3}{\partial y_3 y_1} = 0 & \dfrac{\partial^2 F_3}{\partial y_3 y_2} = 0 & \dfrac{\partial^2 F_3}{\partial y_3^2} = 0
\end{array}$$

.

Then the function B_3 is expressed in the form

$$B_3(q,q) = q^T \begin{bmatrix} 2f & g & 0 \\ g & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} q = -2ah^2(af + ag + g\omega i).$$

By a similar process, the following expressions are obtained,

$$B(q,\bar{q}) = 2ah \begin{bmatrix} 0 \\ \omega^2 \\ ah(f+g) \end{bmatrix}, \quad B(\bar{q},\bar{q}) = 2ah \begin{bmatrix} 0 \\ -\omega^2 + (d\_(f) - a)\omega i \\ h(ag + af + g\omega i) \end{bmatrix}, \quad C(q,q,\bar{q}) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The Inverse matrix A^{-1} of A is given by

$$A^{-1} = \frac{1}{h}\begin{bmatrix} \dfrac{-gh^2+bd\_(f)}{2ah(f+g)} & \dfrac{-b}{2h(f+g)} & \dfrac{1}{2(f+g)} \\ \dfrac{bd\_(f)+h^2(2f+g)}{2ah(f+g)} & \dfrac{-b}{2h(f+g)} & \dfrac{1}{2(f+g)} \\ \dfrac{d}{a} & -1 & 0 \end{bmatrix}.$$

While the matrix $2i\omega I_3 - A$, and its inverse are written below,

$$2i\omega I_3 - A = \begin{bmatrix} 2i\omega + a & -a & 0 \\ d\_(f) & 2i\omega - d\_(f) & h \\ -h(2f+g) & -gh & 2i\omega + b \end{bmatrix}.$$

$$(2i\omega I_3 - A)^{-1} = \frac{1}{r}\begin{bmatrix} \dfrac{-4\omega^2 - bd\_(f) + h^2 g + 2(b - d\_(f))\omega i}{2} & \dfrac{ab + 2a\omega i}{2} & \dfrac{-ah}{2} \\ \dfrac{-[(bd\_(f) + h^2(2f+g)) + 2d\_(f))\omega i]}{2} & \dfrac{(ab - 4\omega^2 + 2(a+b)\omega i}{2} & \dfrac{-h(a+2\omega i)}{2} \\ h(-d\_(f)(f+g) + (2f+g)\omega i) & h(a(f+g) + g\omega i) & \omega[-2\omega + (a - d\_(f))i] \end{bmatrix}.$$

with $r = -6abd\_(f) - i2\omega(4\omega^2 + bd\_(f) - gh^2 - ab)$.

If the hypotheses $a = b, f = g = 1,\ d\_(f) = \dfrac{53}{100}a,\ h = \dfrac{51}{100}a$, and $\omega = \dfrac{849}{1000}a$, are considered, it is found that the Lyapunov coefficient in R_I is

$$l_1(0) = -1.7320a^2 < 0.$$

With the respective parameters, in the R_{II} region it is found that the Lyapunov coefficient is

$$l_1(0) = -1.528219575a^2 < 0.$$

Since $l_1(0)$ is negative in the regions R_I and R_{II}, as a consequence of Hopf Theorem, the periodic orbit that emerges in the point P_1 is stable. Therefore, in both parameter regions, the system (1) presents non-degenerate and supercritical Hopf bifurcation at equilibrium points P_1 and P_2.

Numerical Simulations

Figure 1 presents the phase portrait of system (1), offering a visual representation of the system's behavior over time. The depiction reveals chaotic dynamics, indicating the presence of complex, unpredictable trajectories within the system. The intricate patterns and irregularities observed in the phase portrait underscore the chaotic nature of the system's behavior.

Figure 2 and **Figure 3** showcase phase portraits corresponding to the dynamics of system (1) predicted by Theorem 1 and Theorem 2, respectively. These visual representations offer insights into the distinct behaviors and trajectories anticipated by the respective theorems.

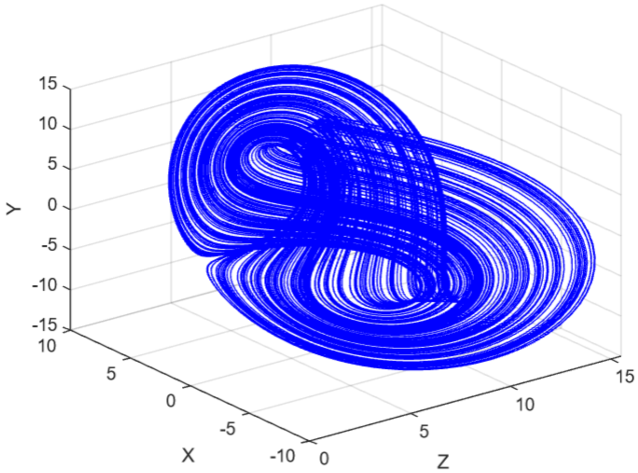

Figure 1: Phase portrait of System (1) with $(a, b, d, f, g) = (10, 3, 6, 1, 0)$. The figure depicts chaotic phenomenon in system (1).

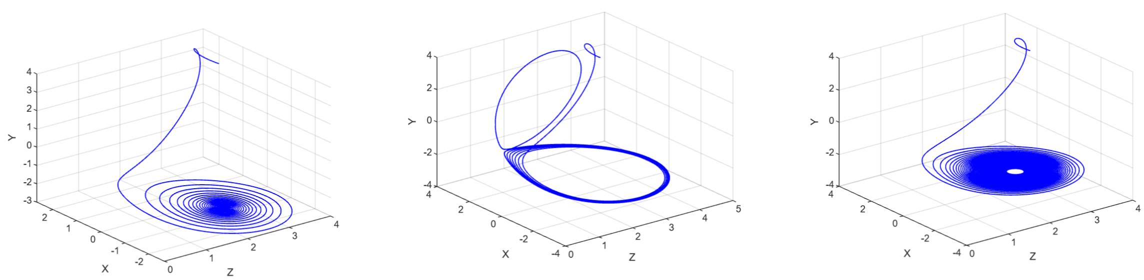

Figure 2: Phase portraits of system (1) with $(a, b, f, g) = (4, 2, 0, 1.2)$ for: (a) $d = 1.9$, (b) $d = 2.1$, (c) $d = 2$ (at this time, the critical value $d_0 = \dfrac{a+b}{3} = 2$)

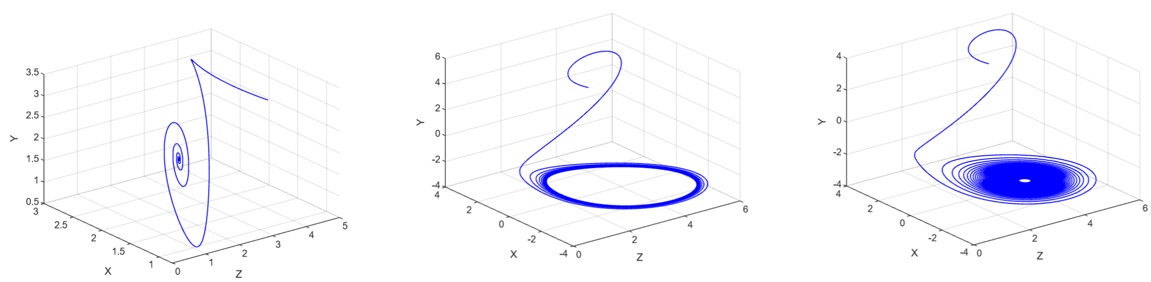

Figure 3: Phase portraits of system (1) with $(a, b, f, g) = (5, 5, 1, 2)$ for: (a) $d = 2$, (b) $d = 3$, (c) $d = 2.8779$ (the critical $d\_(f) = 2.8779$ at this time)

Figure 4 illustrates the Theorem 4, in which two particular trajectories are presented together with their graphs of the Li-Ou system, with values of the parameters $(a, b, d, f, g) = \left(1, 1, \dfrac{1439}{2500}, 1, 2\right)$. The chosen values verify the hypotheses, $b > 0, b = a, f > 0, g > 0, d = d\_(f)$ and clearly belong to the R_{II} region, therefore, it is verified what ensures the Theorem 4, the presence of a stable periodic orbit.

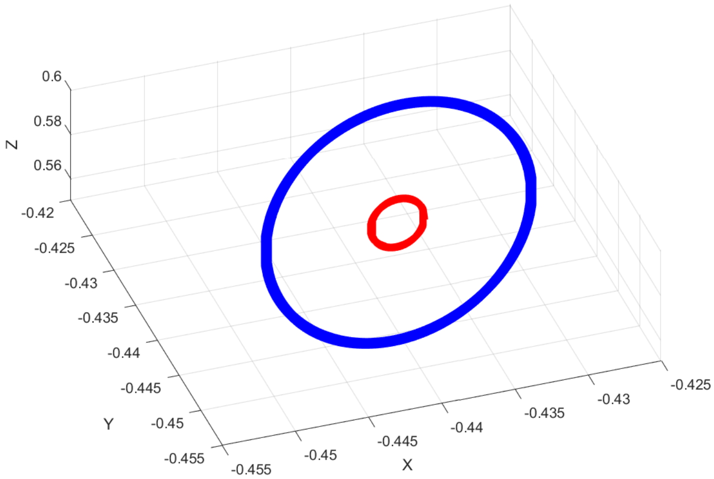

Figure 4: Stable limit cycle $(a, b, d, f, g) = \left(1, 1, \frac{1439}{2500}, 1, 2\right)$.

In Figure 4 it is represented in red in the graph of the orbit with initial condition

$$(x_0, y_0, z_0) = (-0.4473, -0.4544, 0.5863),$$

and it is represented in blue in the graph of the orbit with initial condition

$$(x_0, y_0, z_0) = (-0.4398, -0.4413, 0.5777).$$

Conclusion

Under some hypotheses in the parameters of the Li-Ou system, it is showed that the Hopf bifurcation, the existence of which has been known since the year 2011 [2], is non-degenerate and supercritical, Theorem 3. For this purpose, the symmetry of the system with respect to the z axis was used to reduce the analysis to only one critical point and the well known formula for the first Lyapunov coefficient. Theorem 4 is illustrated geometrically and graphically. For specific parameter values within the established parameter space outlined in the primary findings of this study, a graphical representation is provided. This graph illustrates the evolution of two trajectories, derived through numerical simulations.

References

[1] Muñoz-Aguirre, E., Alvarez-Mena, J., Calderón-Saavedra, P.E., Ramírez-Ortega, J. and Hernández- Zamora, F.G. (2023) *Existence of Supercritical Hopf Bifurcation on a Type-Lorenz System*, Journal of Applied Mathematics and Physics, **11**, 780-789.

[2] Li, X. and Ou, Q.J. (2011) *Dynamical Properties and Simulation of a New Lorenz-Like Chaotic System.*, Nonlinear Dynamical, **65**, 255-270.

[3] https://en.wikipedia.org/wiki/Lorenz_system

[4] https://en.wikipedia.org/wiki/Hopf_bifurcation