MTH 201, Curves and surfaces

Practice problem set 11

Recall that $\mathbf{n}(x, y)$ is the standard unit normal at $\sigma(x, y)$ with respect to some surface patch $\sigma(x, y)$. With respect to this same surface patch, $\sigma_x(x, y)$ and $\sigma_y(x, y)$ form a basis for the tangent space at $\sigma(x, y)$. Together, the three vectors form a basis for the all vectors at $\sigma(x, y)$ (not necessarily only the ones tangent to the surface). Be careful, only $\mathbf{n}$ is a unit vector and is orthogonal to the other two vectors.

Recall the following definitions: $E(x, y) := \sigma_x(x, y) \cdot \sigma_x(x, y)$, $F(x, y) := \sigma_x(x, y) \cdot \sigma_y(x, y)$, $G(x, y) := \sigma_y(x, y) \cdot \sigma_y(x, y)$, and that,

$L(x, y) := \sigma_{xx}(x, y) \cdot \mathbf{n}(x, y)$, $M(x, y) := \sigma_{xy}(x, y) \cdot \mathbf{n}$, $N(x, y) := \sigma_{yy}(x, y) \cdot \mathbf{n}$,

1. The following parts will attempt to find the derivatives of the above basis vector valued functions, i.e. $\mathbf{n}_x$, $\mathbf{n}_y$, σ_{xx}, σ_{xy}, and σ_{yy}, as linear combinations of the basis vectors $\mathbf{n}$, σ_x, and σ_y. Observe that all the coefficients will be in terms of E, F, G, L, M, and N.
   a) Why are $\mathbf{n}_x$ and $\mathbf{n}_y$ linear combinations of merely σ_x and σ_y? Therefore, $\mathbf{n}_x = \alpha\sigma_x + \beta\sigma_y$ and $\mathbf{n}_y = \gamma\sigma_x + \delta\sigma_y$. Prove that the coefficients, $\alpha, \beta, \gamma, \delta$, can be computed in terms of E, F, G, L, M, and N. (Take the dot product of each equation with σ_x and then with σ_y to get four linear equations involving E, F, G, L, M, and N. You now have four equations and four unknowns. You will need to use the fact that $\mathbf{n}$ is orthogonal to σ_x)
   b) Show that the respective coefficients of $\mathbf{n}$, when writing σ_{xx}, σ_{xy}, and σ_{yy}, as a linear combination of σ_x, σ_y, and $\mathbf{n}$, are L, M, and N, respectively. Thefore,

   $$\sigma_{xx} = \Gamma^1_{11}\sigma_x + \Gamma^2_{11}\sigma_y + L\mathbf{n}$$

   $$\sigma_{xy} = \Gamma^1_{12}\sigma_x + \Gamma^2_{12}\sigma_y + M\mathbf{n}$$

   $$\sigma_{yy} = \Gamma^1_{22}\sigma_x + \Gamma^2_{22}\sigma_y + N\mathbf{n}$$

   Prove that each Γ^k_{ij} can be expressed entirely in terms of E, F, and G. (To find enough linear relations, take the dot product with σ_x, and then with σ_y. Play with the derivatives of $\sigma_x \cdot \sigma_y, \sigma_x \cdot \sigma_y$, and $\sigma_x \cdot \sigma_y$, which are just E, F, and G, to rewrite terms such as $\sigma_{xy} \cdot \sigma_x$ in terms of E, F, and G.)

2. Consider a regular curve on a surface parametrized by a unit speed parametrization, $\gamma(t) = \sigma(x(t), y(t))$, where σ is a surface patch of the surface.
   a) Compute the vector $\dot{\gamma}(t)$ as a linear combination of σ_x and σ_y (You have done this in a previous problem set!).
   b) Compute the vector $\ddot{\gamma}(t)$ as a linear combination of σ_x, σ_y, and $\mathbf{n}$ (The previous exercise should help you to replace the σ_{xx} etc in your calculations.).
   c) An alternative basis, instead of $\{\sigma_x, \sigma_y, \mathbf{n}\}$, is the set (taking $\mathbf{T} := \dot{\gamma}$), $\{\mathbf{T}, \mathbf{n}, \mathbf{n} \times \mathbf{T}\}$. Why is $\ddot{\gamma}$ a linear combination of only $\mathbf{n}$ and $\mathbf{n} \times \mathbf{T}$? Therefore $\ddot{\gamma} = \kappa_n \mathbf{n} + \kappa_g \mathbf{n} \times \mathbf{T}$, for some $\kappa_n = \ddot{\gamma} \cdot \mathbf{n}$ and $\kappa_g = \ddot{\gamma} \cdot (\mathbf{n} \times \mathbf{T})$. Show that k_n depends only on L, M, and N, and that k_g depends only on E, F, and G. (Use the previous parts and the linearity of the dot and cross products, wherever applicable, to write everything in terms of σ_x, σ_y, and $\mathbf{n}$. Somewhere, you will get a term like $\mathbf{n} \cdot (\sigma_x \times \sigma_y)$, but that can be written in terms of E, F, and G. Do you see why?)
3. Recall the definition of $\mathcal{W}_{p,S}$, which can be interpreted as a linear map from $T_p(S)$ to itself.
   a) Prove that $\mathcal{W}_{p,S}(\sigma_x) = -\mathbf{n}_x$ and $\mathcal{W}_{p,S}(\sigma_y) = -\mathbf{n}_y$. (Recall the definition of $D_p(f)$. What curve represents σ_x? Why is the derivative of the image of that curve under the Gauss map $\mathcal{G}_S$ equal to $\mathbf{n}_x$?)
   b) $\mathcal{W}_{p,S}(\sigma_x)$ and $\mathcal{W}_{p,S}(\sigma_y)$ are both tangent vectors of the surface (why?) and therefore is some linear combination of only σ_x and σ_y, i.e. $-\mathbf{n}_x = \alpha\sigma_x + \beta\sigma_y$ and $-\mathbf{n}_y = \gamma\sigma_x + \delta\sigma_y$. This is another way of seeing that $\mathbf{n}_x$ and $\mathbf{n}_y$ can be written as linear combinations of only σ_x and σ_y (proved in 1 (a)). Use part a) of the first question to find a matrix representation for $\mathcal{W}_{p,S}$.
4. Recall the definition $\langle v, w\rangle'_p := \langle \mathcal{W}_{p,S} v, w\rangle$
   a) $\langle \sigma_x, \sigma_x\rangle' = L$, $\langle \sigma_x, \sigma_y\rangle' = M$, $\langle \sigma_y, \sigma_y\rangle' = N$ (Relate, for example, $\mathbf{n}_x \cdot \sigma_x$ with L by using the fact that $\mathbf{n} \cdot \sigma_x$ is orthogonal.)
   b) $\langle v, w\rangle'$ is symmetric.
   c) Show that if we express any vector tangent to the surface in terms of the basis σ_x and σ_y as $v = \alpha\sigma_x + \beta\sigma_y$, then $\langle v, v\rangle' = L\alpha^2 + 2M\alpha\beta + N\beta^2$.
   d) Show that for a unit speed parametrization γ of a regular curve on the surface, κ_n (defined in question 2) is $\langle \dot{\gamma}, \dot{\gamma}\rangle'$. (You can use part (a) of the previous question to relate it with $\mathcal{W}_{p,S}$). This will be an alternative proof for why κ_n depends only on L, M, and N.