

MTH 201, Curves and surfaces

Practice problem set 7

1. Let $f(x, y) = x \sin(x + 2y)$. Find f_x , f_y , f_{xy} , and f_{yx} .
2. Use the chain rule of partial derivatives to show that if $f : \mathbb{R} \rightarrow \mathbb{R}$ is a smooth function such that $y = f(x)$ satisfies $F(x, y) = 0$, where $F : \mathbb{R}^2 \rightarrow \mathbb{R}$ is also a smooth function, then $\frac{dy}{dx} = -\frac{F_x}{F_y}$. This is called implicit differentiation.
3. Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a smooth function and let (x_0, y_0) be a point in \mathbb{R}^2 and $\mathbf{v} = (v_1, v_2)$ a vector. Define $F = f(x_0 + v_1 t, y_0 + v_2 t)$. Use the chain rule for partial derivatives to show that $F'(0) = f_x(x_0, y_0)v_1 + f_y(x_0, y_0)v_2 = (f_x(x_0, y_0), f_y(x_0, y_0)) \cdot \mathbf{v}$. Note that $F'(0) = \lim_{t \rightarrow 0} \frac{f(x_0 + tv_1, y_0 + tv_2) - f(x_0, y_0)}{t}$; it is called the directional derivative of f in the direction of \mathbf{v} and is denoted by $f_{\mathbf{v}}$. This exercise shows that one can compute the directional derivative of f in the direction of any vector \mathbf{v} if one knows the partial derivatives of f .
4. Which of the following surface patches are regular?
 - a) $\sigma : \mathbb{R}^2 \rightarrow \mathbb{R}^3$, $\sigma(x, y) = (x, y, x + y)$
 - b) $\sigma : \mathbb{R}^2 \rightarrow \mathbb{R}^3$, $\sigma(x, y) = (x, x^2, y^3)$
5. Show that each of the following subsets of \mathbb{R}^3 are smooth surfaces by finding enough regular surface patches to cover each surface.
 - a) The cylinder, defined as $S := \{(x, y, z) \mid x^2 + y^2 = 1\}$
 - b) The sphere, defined as $S := \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}$
6. Show that the surface patch, $\sigma(s, t) = ((a + b \cos(s)) \cos(t), (a + b \cos(s)) \sin(t), b \sin(s))$, where $b < a$, is regular. Can you imagine the surface that it is a surface patch of?
7. For an open subset U of \mathbb{R}^2 , and a smooth map $f : U \rightarrow \mathbb{R}$, show that the set $S := \{(x, y, f(x, y)) \mid (x, y) \in U\}$ is a smooth surface.
8. If U is an open subset of \mathbb{R}^2 , then prove that the rank of the Jacobian of a map $f : U \rightarrow \mathbb{R}^3$ is 2 if and only if $\frac{\partial f}{\partial x} \times \frac{\partial f}{\partial y} \neq 0$.
9. If $F : U \rightarrow \mathbb{R}$, where U an open subset of \mathbb{R}^3 , is smooth, what condition do you need to impose on it so that the Jacobian of the map $f : U \rightarrow \mathbb{R}^3$, defined by $f(x, y, z) = (x, y, F(x, y, z))$, has non-zero determinant?