

# MTH 201, Curves and surfaces

## Practice problem set 10

1. Compute the first fundamental form with respect to the surface patch  $\sigma(x, y) = (f(x)\cos(y), f(x)\sin(y), g(x))$ . Why do you think your answer does not involve  $g$ ?
2. Find a surface patch for each of the following surfaces, and compute the first fundamental form of that patch.
  - a) A sphere.  $\sigma(x, y) = (x, y, \sqrt{1 - x^2 - y^2})$
  - b) A generalized cylinder over a plane curve  $\gamma$ .  $\sigma(x, y) = \gamma(x) + y\delta(x)$ , where  $\delta$  is orthogonal to the plane that contains  $\gamma$ .
3. If  $\sigma$  is a regular surface patch of a surface,  $\tilde{\sigma} = \sigma \circ \Theta$ , another surface patch, where  $\Theta$  is a coordinate transformation, then show that their respective first fundamental forms  $E dx^2 + 2F dx dy + G dy^2$  and  $\tilde{E} d\tilde{x}^2 + 2\tilde{F} d\tilde{x} d\tilde{y} + \tilde{G} d\tilde{y}^2$  are related by

$$\begin{pmatrix} \tilde{E} & \tilde{F} \\ \tilde{F} & \tilde{G} \end{pmatrix} = (\text{Jac}(\Theta))^t \begin{pmatrix} E & F \\ F & G \end{pmatrix} \text{Jac}(\Theta)$$

4. Let  $\gamma$  be the unit speed parametrization of a regular *space* curve.
  - a) Prove that the surface patch  $\sigma(x, y) = \gamma(x) + y\dot{\gamma}(x)$  is regular, then the curvature  $\kappa$  is strictly positive.
  - b) Prove that if  $\gamma$  is a *plane* curve, then  $\sigma$  is a surface patch of a plane.
  - c) Compute the first fundamental form of  $\sigma$  ( $\gamma$  may be non-planar). Show that there exists a *plane* curve  $\tilde{\gamma}$  so that  $\tilde{\sigma}(x, y) = \tilde{\gamma}(x) + y\dot{\tilde{\gamma}}(x)$  has the same first fundamental form as  $\sigma$ .
5. Prove that a local diffeomorphism,  $f : S_1 \rightarrow S_2$ , is a local isometry if and only given *any* surface patch  $\sigma_1$  of  $S_1$ , its fundamental form is equal to the fundamental form of the surface patch  $f \circ \sigma$  of  $S_2$  by proving the following parts:
  - a)  $(f \circ \sigma)_x = D_p(f)(\sigma_x)$ . (Remember that  $\sigma_x$  can be interpreted as the derivative of some curve on the surface. That should help you to link it up with the definition of  $D_p(f)$ .)
  - b) Use part a) to prove that  $\langle (f \circ \sigma)_x, (f \circ \sigma)_x \rangle = f^* \langle \sigma_x, \sigma_x \rangle$
  - c) Using the fact that  $f$  is a local isometry if and only if  $\langle D_p(f)(\mathbf{v}), D_p(f)(\mathbf{w}) \rangle_p = \langle \mathbf{v}, \mathbf{w} \rangle_{f(p)}$ , and using the previous parts, prove that the first fundamental forms are the same.