MTH 201, Curves and surfaces

Practice problem set 7

1. Let $f(x, y) = x\sin(x + 2y)$. Find f_x, f_y, f_{xy}, and f_{yx}.

2. Use the chain rule of partial derivatives to show that if $f : \mathbb{R} \to \mathbb{R}$ is a smooth function such that $y = f(x)$ satisfies $F(x, y) = 0$, where $F : \mathbb{R}^2 \to \mathbb{R}$ is also a smooth function, then $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{F_x}{F_y}$. This is called implicit differentiation.

3. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function and let (x_0, y_0) be a point in $\mathbb{R}^2$ and $\mathbf{v} = (v_1, v_2)$ a vector. Define $F = f(x_0 + v_1 t, y_0 + v_2 t)$. Use the chain rule for partial derivatives to show that $F'(0) = f_x(x_0, y_0)v_1 + f_y(x_0, y_0)v_2 = (f_x(x_0, y_0), f_y(x_0, y_0)) \cdot \mathbf{v}$. Note that $F'(0) = \lim\limits_{t \to 0} \dfrac{f(x_0 + tv_1, y_0 + tv_2) - f(x_0, y_0)}{t}$; it is called the directional derivative of f in the direction of $\mathbf{v}$ and is denoted by $f_{\mathbf{v}}$. This exercise shows that one can compute the directional derivative of f in the direction of any vector $\mathbf{v}$ if one knows the partial derivatives of f.

4. Which of the following surface patches are regular?

   a) $\sigma : \mathbb{R}^2 \to \mathbb{R}^3$, $\sigma(x, y) = (x, y, x + y)$
   b) $\sigma : \mathbb{R}^2 \to \mathbb{R}^3$, $\sigma(x, y) = (x, x^2, y^3)$

5. Show that each of the following subsets of $\mathbb{R}^3$ are smooth surfaces by finding enough regular surface patches to cover each surface.

   a) The cylinder, defined as $S := \{(x, y, z) \mid x^2 + y^2 = 1\}$
   b) The sphere, defined as $S := \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}$

6. Show that the surface patch, $\sigma(s, t) = ((a + b\cos(s))\cos(t), (a + b\cos(s))\sin(t), b\sin(s))$, where $b < a$, is regular. Can you imagine the surface that it is a surface patch of?

7. For an open subset U of $\mathbb{R}^2$, and a smooth map $f : U \to \mathbb{R}^3$, show that the set $S := \{(x, y, f(x, y)) \mid (x, y) \in \mathbb{R}^2\}$ is a smooth surface.

8. If U is an open subset of $\mathbb{R}^2$, then prove that the rank of the Jacobian of a map $f : U \to \mathbb{R}^3$ is 2 if and only if $\frac{\partial f}{\partial x} \times \frac{\partial f}{\partial y} \neq 0$

9. If $F : U \to \mathbb{R}$, where U an open subset of $\mathbb{R}^3$, is smooth, what condition do you need to impose on it so that the Jacobian of the map $f : U \to \mathbb{R}^3$, defined by $f(x, y, z) = (x, y, F(x, y, z))$, has non-zero determinant?