

# MTH 201, Curves and surfaces

## Practice problem set 8

1. Show that the set  $S = \{(x, y, z) \mid 2x^2 + y^2 - z^2 = 1\}$  is a surface. Do not find an explicit surface patch but use the theorem for level sets (i.e. defined by  $\{(x, y, z) \mid F(x, y, z) = 0\}$ )
2.  $\sigma(x, y) = ((1+x\sin(y/2))\cos(y), (1+x\sin(y/2))\sin(y), \cos(y/2))$  be a surface patch defined when  $-1/2 < x < 1/2$  and  $-\pi < y < \pi$ .
  - a) Compute  $\mathbf{n}(x, y) := \sigma_x(x, y) \times \sigma_y(x, y)$ .
  - b) Show that  $\lim_{y \rightarrow \pi} \mathbf{n}(0, y) = - \lim_{y \rightarrow -\pi} \mathbf{n}(0, y)$
3. Let  $\sigma(x, y) = (f(x)\cos(y), f(x)\sin(y), g(x))$ . Compute  $\sigma_x \times \sigma_y$ . When is it zero?
4. If  $\sigma$  is a surface patch and  $\gamma(t) = \sigma(x(t), y(t))$ , where  $x : \mathbb{R} \rightarrow \mathbb{R}$  and  $y : \mathbb{R} \rightarrow \mathbb{R}$  are smooth functions, then prove that  $\dot{\gamma} = \dot{x}\sigma_x + \dot{y}\sigma_y$ .
5. If  $\sigma$  is a surface patch and  $\gamma(t) = \sigma(a + \alpha t, b + \beta t)$ , then prove that  $\dot{\gamma} = \alpha\sigma_x + \beta\sigma_y$  at  $t = 0$ .
6. Recall the definition of a smooth function  $f : S \rightarrow \mathbb{R}$ . Prove that if  $\pi(x, y, z) = x$  is restricted to the surface  $S$ , then it is a smooth function from  $S$  to  $\mathbb{R}$ .
7. Recall the definition of  $D_p(f) : T_p(S) \rightarrow T_p(S)$ , where  $T_p(S)$  denotes the tangent space at  $p$  of a surface  $S$ , and prove the following:
  - a)  $D_p(\text{id}_S) = \text{id}_{T_p(S)}$ . ( $\text{id}_A : A \rightarrow A$  is the identity map on  $A$ ).
  - b) If  $f : S_1 \rightarrow S_2$  and  $g : S_2 \rightarrow S_3$  are smooth maps then  $D_p(g \circ f) = D_{f(p)}(g) \circ D_p(f)$ .
  - c) If  $f$  is a diffeomorphism, then  $D_p(f)$  is invertible.
8. Consider the set  $S := \{(x, y, z) \mid F(x, y, z) = 0\}$ . Define  $f : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  by  $f(x, y, z) = (x, y, F(x, y, z))$ 
  - a) Prove that if  $F_z \neq 0$  for some  $p = (x_0, y_0, z_0)$  in  $S$ , then there exists some open neighbourhood  $U$  around  $p$  and an open neighbourhood  $V$  around  $f(p)$  so that  $f : U \rightarrow V$  has a smooth inverse  $g : V \rightarrow U$ .
  - b) Prove that the image of  $V \cap \{(x, y, z) \mid z = 0\}$  under  $g$  is in  $S$ . Use this observation to define a regular surface patch of  $S$  around  $p$ . To check regularity you will need to examine the coordinates of  $g$ .

- c) Can you find a condition to impose on  $F$  so that  $S$  is a smooth surface?
9. Let  $\sigma : U \rightarrow S \cap W$  (here,  $W$  is open in  $\mathbb{R}^3$ ) be a regular surface patch around a point  $p = \sigma(x_0, y_0)$  and define  $\pi_1(x, y, z) = (y, z)$ ,  $\pi_2(x, y, z) = (x, z)$ ,  $\pi_3(x, y, z) = (x, y)$ .
- Prove that for some  $i = 1, 2, 3$ ,  $\pi_i$  is injective when restricted to  $\sigma(U')$ , where  $U'$  is an open subset (possibly different from  $U$ ) containing  $(x_0, y_0)$ . In fact, prove this as a consequence of the fact that  $\pi_i \circ \sigma$  is a diffeomorphism when restricted to  $U'$ .
  - Let  $\gamma : (a, b) \rightarrow W \cap S \subset \mathbb{R}^3$  parametrize a curve lying on the intersection of the surface  $S$  with an open subset  $W$ . Let  $\tilde{\gamma} : (a, b) \rightarrow U \subset \mathbb{R}^2$  denote the curve in  $U$  such that  $\sigma \circ \tilde{\gamma} = \gamma$ . Use the previous part to show that  $\tilde{\gamma}$  is smooth.
  - Let  $\sigma' : U' \rightarrow W \cap S$  denote another surface patch. Why is  $\sigma^{-1} \circ \sigma' = (\pi_i \circ \sigma)^{-1} \circ (\pi_i \circ \sigma')$ ? Use that to prove that  $\sigma^{-1} \circ \sigma'$  is smooth. Why could it not be done directly?