

# MTH 201, Curves and surfaces

## Practice problem set 7

1. Let  $f(x, y) = x \sin(x + 2y)$ . Find  $f_x$ ,  $f_y$ ,  $f_{xy}$ , and  $f_{yx}$ .
2. Use the chain rule of partial derivatives to show that if  $f : \mathbb{R} \rightarrow \mathbb{R}$  is a smooth function such that  $y = f(x)$  satisfies  $F(x, y) = 0$ , where  $F : \mathbb{R}^2 \rightarrow \mathbb{R}$  is also a smooth function, then  $\frac{dy}{dx} = -\frac{F_x}{F_y}$ . This is called implicit differentiation.
3. Let  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  be a smooth function and let  $(x_0, y_0)$  be a point in  $\mathbb{R}^2$  and  $\mathbf{v} = (v_1, v_2)$  a vector. Define  $F = f(x_0 + v_1 t, y_0 + v_2 t)$ . Use the chain rule for partial derivatives to show that  $F'(0) = f_x(x_0, y_0)v_1 + f_y(x_0, y_0)v_2 = (f_x(x_0, y_0), f_y(x_0, y_0)) \cdot \mathbf{v}$ . Note that  $F'(0) = \lim_{t \rightarrow 0} \frac{f(x_0 + tv_1, y_0 + tv_2) - f(x_0, y_0)}{t}$ ; it is called the directional derivative of  $f$  in the direction of  $\mathbf{v}$  and is denoted by  $f_{\mathbf{v}}$ . This exercise shows that one can compute the directional derivative of  $f$  in the direction of any vector  $\mathbf{v}$  if one knows the partial derivatives of  $f$ .
4. Which of the following surface patches are regular?
  - a)  $\sigma : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ ,  $\sigma(x, y) = (x, y, x + y)$
  - b)  $\sigma : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ ,  $\sigma(x, y) = (x, x^2, y^3)$
5. Show that each of the following subsets of  $\mathbb{R}^3$  are smooth surfaces by finding enough regular surface patches to cover each surface.
  - a) The cylinder, defined as  $S := \{(x, y, z) \mid x^2 + y^2 = 1\}$
  - b) The sphere, defined as  $S := \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}$
6. Show that the surface patch,  $\sigma(s, t) = ((a + b \cos(s)) \cos(t), (a + b \cos(s)) \sin(t), b \sin(s))$ , where  $b < a$ , is regular. Can you imagine the surface that it is a surface patch of?
7. For an open subset  $U$  of  $\mathbb{R}^2$ , and a smooth map  $f : U \rightarrow \mathbb{R}^3$ , show that the set  $S := \{(x, y, f(x, y)) \mid (x, y) \in \mathbb{R}^2\}$  is a smooth surface.
8. If  $U$  is an open subset of  $\mathbb{R}^2$ , then prove that the determinant of the Jacobian of a map  $f : U \rightarrow \mathbb{R}^3$  is non-zero if and only if  $\frac{\partial f}{\partial x} \times \frac{\partial f}{\partial y} \neq 0$ .
9. If  $F : U \rightarrow \mathbb{R}$ , where  $U$  an open subset of  $\mathbb{R}^3$ , is smooth, what condition do you need to impose on it so that the Jacobian of the map  $f : U \rightarrow \mathbb{R}^3$ , defined by  $f(x, y, z) = (x, y, F(x, y, z))$ , has non-zero determinant?