# MTH 201, Curves and surfaces

## Practice problem set 5

1. Consider a (plane) curve parametrized by $\gamma : (a, b) \to \mathbf{R}^2$ and a point on that curve $p = \gamma(t_0)$. We will find a circle which best approximates the curve at $p$, in the sense defined below:

   a) Prove that if a circle is tangent to the curve defined by $\gamma$ at $p$ ("tangent" means that the circle touches the curve and the circle's tangent line and the curve's tangent line are the same at p), then its center must lie on the line containing the vector $\mathbf{N}_s(t)$. For this and the part below you may assume that a normal line of a circle contains its center.

   b) For some real number $r$, let $C_r$ denote the circle of radius $|r|$, with its center at the point $p + r\mathbf{N}_s(t)$. Why is it tangent to the curve at $p$? Note that $C_r$ divides the plane into an interior and exterior component and $r$ may be negative, in which case the center is in a direction opposite to $\mathbf{N}_s(t)$.

   c) Prove that a point $\gamma(t)$ avoids the interior component of $C_r$ if and only if $d(t) := \|\gamma(t) - (p + r\mathbf{N}(t))\|^2 \geq r^2$ and avoids the exterior component if and only if $d(t) \leq r^2$ (it always intersects the circle at $p$, so at $t_0$ you get $r^2$). The square is only to allow us to express it as a dot product. Since $d(t)$ always positive, taking the square is harmless.

   d) We say that $C_r$ is too small if, at least in the vicinity of $p$, every point on the curve defined by $\gamma$ avoids the interior of $C_r$, i.e. there is an $\epsilon$ so that for any $t$ inside the interval $(t_0 - \epsilon, t_0 + \epsilon)$, $\gamma(t)$ avoids the interior of $C_r$. Use the previous part to rewrite this in terms of the function $d(t)$, which is defined above. Why does that mean that $d$ has a local minimum at $t_0$? Remember that a function has a local minimum at $t_0$ if for $t$ *in the vicinity* of $t_0$, $f(t) \geq f(t_0)$

   e) We say that $C_r$ is too big if, at least in the vicinity of $p$, every point on the curve defined by $\gamma$ avoids the exterior of $C_r$, i.e. there is an $\epsilon$ so that for any $t$ inside the interval $(t_0 - \epsilon, t_0 + \epsilon)$, $\gamma(t)$ avoids the exterior of $C_r$. Use the previous part to rewrite this in terms of the function $d(t)$, which is defined above. Why does that mean that $d$ has a local maximum at $t_0$?

   f) Prove that no matter what $r$ is, $d'(t_0) = 0$. (By now you should be in the habit of expressing such derivatives in terms of that orthonormal basis $\mathbf{N}_s(t)$ and $\mathbf{T}(t)$ so that you can easily identify which coefficients cancel).

g) Remember that a function $f$ has a local maximum at $t_0$ if $f'(t_0) = 0$ and $f''(t_0) < 0$; it has a local minimum at $t_0$ if $f'(t_0) = 0$ and $f''(t_0) > 0$. Compute $d''(t)$ and use parts d) and e) above to show that $C_r$ would be too big if $r > 1/\kappa(t_0)$ and too small if $r < 1/\kappa(t_0)$. Therefore, a circle of radius $1/\kappa(t_0)$ may be thought of as best approximating the curve at $p$. Such a circle is called an *osculating circle* and its radius is $1/\kappa(t_0)$ is called the radius of curvature.

**Space curves**

2. Prove that if $f : (a, b) \to \mathbb{R}^3$ and $g : (a, b) \to \mathbb{R}^3$ are two vector valued functions and $h(t) = f(t) \times g(t)$ then $\dot{h}(t) = \dot{f}(t) \times g(t) + f(t) \times \dot{g}(t)$.

3. Prove that for a space curve parametrized by a *unit-speed parametrization*, $\gamma : (a, b) \to \mathbb{R}^3$, $\mathbf{N}(t) := \frac{\ddot{\gamma}(t)}{\kappa(t)} = \frac{\dot{T}(t)}{\|T(t)\|}$ is a unit vector which is orthogonal to the unit tangent vector $\mathbf{T}(t) = \frac{\dot{\gamma}(t)}{\|\dot{\gamma}(t)\|}$. Note, here $\kappa$ is the curvature and *not* the signed curvature, which only makes sense for plane curves. Note also that all this makes sense only if $\gamma$ is regular and $\kappa(t) \neq 0$ (it appears in the denominator!)

4. Consider the vector $\mathbf{B}(t) := \mathbf{T}(t) \times \mathbf{N}(t)$. What is $\|\mathbf{B}(t)\|$? Why does the set $\{\mathbf{T}(t), \mathbf{N}(t), \mathbf{B}(t)\}$ form an orthonormal basis of $\mathbb{R}^3$?

5. You know the cross product of two vectors written in terms of a basis if you know the cross products of the respective basis elements. Therefore, it will be useful to know $\mathbf{B}(t) \times \mathbf{T}(t)$ and $\mathbf{N}(t) \times \mathbf{B}(t)$ (by the previous part, you already know $\mathbf{T}(t) \times \mathbf{N}(t) = \mathbf{B}(t)$ by the definition of $\mathbf{B}$). What are they? Be careful of the order and the resulting sign!

6. Show that if $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ form an orthonormal basis in $\mathbb{R}^3$ and if a vector $\mathbf{w}$ is orthogonal to both $\mathbf{v}_1$ and $\mathbf{v}_2$, it must be a scalar multiple of $\mathbf{v}_3$.

7. Show that $\dot{\mathbf{B}}(t)$ is a scalar multiple of $\mathbf{N}(t)$.

8. Since any vector valued function can be written as a linear combination of $\mathbf{T}(t)$, $\mathbf{N}(t)$, and $\mathbf{B}(t)$, we can know its derivative if we know $\dot{\mathbf{T}}(t)$, $\dot{\mathbf{N}}(t)$, and $\dot{\mathbf{B}}(t)$. Express the following vector valued functions in terms of the basis elements $\mathbf{T}(t)$, $\mathbf{N}(t)$, and $\mathbf{B}(t)$ so that your coefficients involve either $\kappa(t)$ or $\tau(t)$ (Try to use 2. and 5. You will need to use 5. more than once.)

   a) $\dot{\mathbf{T}}(t)$

   b) $\dot{\mathbf{N}}(t)$

   c) $\dot{\mathbf{B}}(t)$