MTH 201, Curves and surfaces

Practice problem set 3

1. Compute the curvature function of the curve (after finding its arc length parametrization) parametrized by $\gamma(t) = (a\ \cos(t), a\ \sin(t), bt)$ for some fixed real numbers $a > 0$ and $b > 0$, $t \in \mathbb{R}$.

2. During the lecture, the formula for the curvature was expressed in terms of the unit speed (arc length) parametrization. However, finding the arc length as a formula may not be easy (let alone inverting it) because it is expressed as an integral (try deriving a formula for the arc length of an ellipse to see how tough it can be!). The following steps will rephrase the formula in terms of *any* parametrization that you use to define the curve, bypassing the need to find the arc length. For this exercise, let γ denote the original parametrization and $\tilde{\gamma}$ denote the unit speed parametrization. Therefore, $\tilde{\gamma}(t) = \gamma(s^{-1}(t))$, or equivalently, and perhaps more usefully for the computations below, $\tilde{\gamma}(s(t)) = \gamma(t)$.

   a) Compute $s''(t)$ in terms of $\dot{\gamma}(t)$ and $\ddot{\gamma}(t)$. (You might find it easier to first differentiate $(s'(t))^2$ because it can be expressed as a dot product). Can you see why this make sense intuitively? (Hint: interpret your answer as "$s''(t)$ is the magnitude of the component of the acceleration, $\ddot{\gamma}(t)$, in the direction of the velocity vector, $\dot{\gamma}(t)$". Compare this with the observation that for a *unit speed parametrization* $\tilde{\gamma}$, the acceleration vector, $\ddot{\tilde{\gamma}}(t)$, is orthogonal to the velocity vector, $\dot{\tilde{\gamma}}(t)$).
   b) Show that $\dot{\gamma}$ and $\dot{\tilde{\gamma}}$ are related by $\dot{\tilde{\gamma}}(s(t))s'(t) = \dot{\gamma}(t)$, or equivalently, $\dot{\tilde{\gamma}}(s(t)) = \frac{\dot{\gamma}(t)}{\|\dot{\gamma}(t)\|}$. Observe that this formula is confirming the intuitively obvious fact that if you reparametrize by the unit speed parametrization, the direction of the velocity vector at a given point on the curve does not change. Only its magnitude is scaled down to 1. Remember that a point on the curve represented by $\gamma(t)$ is represented by $\tilde{\gamma}(s(t))$ when you use the unit speed parametrization.
   c) Use the previous part to show that $\ddot{\tilde{\gamma}}(s(t)) = \frac{\ddot{\gamma}(t) - \dot{\tilde{\gamma}}(s(t))s''(t)}{(s'(t))^2}$
   d) Use a), b) to replace all occurences of s and $\tilde{\gamma}$ on the right hand side of c) to show that $\|\ddot{\tilde{\gamma}}(s(t))\| = \frac{\|\|\dot{\gamma}(t)\|^2\ddot{\gamma}(t) - \dot{\gamma}(t)(\dot{\gamma}(t)\cdot\ddot{\gamma}(t))\|}{\|\dot{\gamma}(t)\|^4}$. Note that $\|\ddot{\tilde{\gamma}}(s(t))\|$ is the curvature of the curve at the point $\tilde{\gamma}(s(t))$. The same point, in terms of the usual parametrization γ, is simply $\gamma(t)$ (because $\tilde{\gamma}(s(t)) = \gamma(t)$). Therefore, the curvature of a curve at the point $\gamma(t)$ on the curve may be computed directly by $\frac{\|\|\dot{\gamma}(t)\|^2\ddot{\gamma}(t) - \dot{\gamma}(t)(\dot{\gamma}(t)\cdot\ddot{\gamma}(t))\|}{\|\dot{\gamma}(t)\|^4}$,

even if γ may not be a unit speed parametrization. Can you see why this big formula reduces to $\|\ddot{\gamma}(t)\|$ is we assume that γ is a unit speed parametrization?

3. This exercise will help you to simplify the formula that you derived above.

   a) Recall the triple product identity $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$ for any three vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$. Can you recognize its right hand side in the the expression $\|\dot{\gamma}(t)\|^2\ddot{\gamma}(t) - \dot{\gamma}(t)(\dot{\gamma}(t) \cdot \ddot{\gamma}(t))$, which will enable you to rewrite it as $\dot{\gamma}(t) \times (\ddot{\gamma}(t) \times \dot{\gamma}(t))$? Do not forget that $\|\mathbf{v}\|^2$ can be written as a dot product!
   b) For orthogonal vectors $\mathbf{v}$ and $\mathbf{w}$, why is $\|\mathbf{v} \times \mathbf{w}\| = \|\mathbf{v}\|\|\mathbf{w}\|$? Use this to show that the curvature of a curve parametrized by γ can be computed at the point $\gamma(t)$ by $\frac{\|\ddot{\gamma}(t) \times \dot{\gamma}(t)\|}{\|\dot{\gamma}(t)\|^3}$. You would have had to use the fact that $\dot{\gamma}(t)$ is orthogonal to $\ddot{\gamma}(t) \times \dot{\gamma}(t)$. Why are they orthogonal?

4. Use the formula derived above to compute the curvatures of the curves parametrized by the following:

   a) $\gamma(t) = (t, \cosh(t))$.
   b) $\gamma(t) = (t^3 - t, t^2)$.
   c) $\gamma(t) = (\sin^3(t), \cos^3(t))$.