

MTH 201, Curves and surfaces

Practice problem set 1

1. Let $\gamma_1 : (a, b) \rightarrow \mathbb{R}^3$ and $\gamma_2 : (a, b) \rightarrow \mathbb{R}^3$ be two smooth functions. Prove the following
 - a) $\frac{d}{dt}(\gamma_1(t) + \gamma_2(t)) = \frac{d}{dt}\gamma_1(t) + \frac{d}{dt}\gamma_2(t)$
 - b) $\frac{d}{dt}(\gamma_1(t) - \gamma_2(t)) = \frac{d}{dt}\gamma_1(t) - \frac{d}{dt}\gamma_2(t)$
 - c) $\frac{d}{dt}(c\gamma_1(t)) = c\frac{d}{dt}\gamma_1(t)$ where c is some real number.
 - d) $\frac{d}{dt}(f(t)\gamma_1(t)) = f'(t)\gamma_1(t) + f(t)\frac{d}{dt}\gamma_1(t)$ where $f : \mathbb{R} \rightarrow \mathbb{R}$ is a smooth *real valued* function.
 - e) $\frac{d}{dt}\gamma_1(f(t)) = f'(t)\frac{d}{dt}\gamma_1(f(t))$ where $f : \mathbb{R} \rightarrow \mathbb{R}$ is a smooth *real valued* function. This will be referred to as the “chain rule”, below.
 - f) $\frac{d}{dt}(\gamma_1(t) \cdot \gamma_2(t)) = \frac{d}{dt}\gamma_1(t) \cdot \gamma_2(t) + \gamma_1(t) \cdot \frac{d}{dt}\gamma_2(t)$. The symbol \cdot refers to the “dot product”.
1. Consider the parametrization $\gamma(t) = (a \cos(t), a \sin(t), bt)$ for some fixed real numbers $a > 0$ and $b > 0$.
 - a) What curve does this parametrization define? Can you imagine its shape?
 - b) Compute the speed of the parametrization, i.e. $\|\dot{\gamma}(t)\|$, for each t .
 - c) Which of these points, $(a, 0, 0)$, $(a, a, 0)$, $(-a, 0, \pi b)$, $(a, 0, 2\pi b)$, lie on the curve described by γ ?
 - d) Can you think of a “unit speed” reparametrization, $\tilde{\gamma}$, describing the same curve, i.e. so that $\|\frac{d\tilde{\gamma}}{dt}\| = 1$? You must also describe the reparametrization map ϕ so that $\tilde{\gamma}(t) = \gamma(\phi(t))$.
 - e) Compute the arc length function $s(t) = \int_0^t \|\dot{\gamma}(u)\| du$. Use this to compute the arc length of the part of the curve between the points $(a, 0, 0)$ and $(a, 0, 2\pi b)$ (after, of course, verifying that they do indeed lie on the curve).
 - f) Compute $\|\frac{d}{dt}\gamma(s^{-1}(t))\|$.
2. Consider a line segment between two points $p = (x_1, x_2, x_3)$ and $q = (y_1, y_2, y_3)$ in \mathbb{R}^3 .
 - a) Find a parametrization of the line segment.
 - b) Use the parametrization to show that the arc length of the line segment is equal to $\|p - q\|$.
3. This exercise will help you to prove that the line segment joining two points

on the curve parametrized by some parametrization, $\gamma : (a, b) \rightarrow \mathbb{R}^3$, is the shortest curve joining the two points $p = \gamma(t_0)$ and $q = \gamma(t_1)$. Note that by the previous exercise, the length of the line segment between points p and q is $\|\gamma(t_1) - \gamma(t_0)\|$.

- a) Prove the Cauchy-Schwartz inequality, i.e. $|\mathbf{v} \cdot \mathbf{w}| \leq \|\mathbf{v}\| \|\mathbf{w}\|$ for any pair of vectors \mathbf{v} and \mathbf{w} (this follows easily from the definition of the dot product $\mathbf{v} \cdot \mathbf{w} = \|\mathbf{v}\| \|\mathbf{w}\| \cos(\theta)$ where θ is the angle between the two vectors.).
 - b) Show that for any vector \mathbf{v} , we can figure out its norm by taking the dot product with the unit vector in the same direction, i.e. $\|\mathbf{v}\| = \mathbf{v} \cdot \frac{\mathbf{v}}{\|\mathbf{v}\|}$.
 - c) Use the second fundamental theorem of calculus to show that $(\gamma(t_1) - \gamma(t_0)) \cdot \mathbf{v} = \int_{t_0}^{t_1} \dot{\gamma}(t) \cdot \mathbf{v} dt$.
 - d) By the previous two parts, the distance between points $\gamma(t_0)$ and $\gamma(t_1)$ on the curve is $\|\gamma(t_1) - \gamma(t_0)\| = \int_{t_0}^{t_1} \dot{\gamma}(t) \cdot \frac{\gamma(t_1) - \gamma(t_0)}{\|\gamma(t_1) - \gamma(t_0)\|} dt$ (Why?). This will allow you to use Cauchy-Schwartz inequality to relate this integral with integral defining the arc length and complete the proof of the inequality, $\|\gamma(t_1) - \gamma(t_0)\| \leq \int_{t_0}^{t_1} \|\dot{\gamma}(t)\| dt$.
4. Consider the arc-length function $s(t) = \int_{t_0}^t \|\dot{\gamma}(u)\| du$, where γ is some parametrization of a curve. Further, assume that $\dot{\gamma}(t) \neq 0$ for any t in the domain of γ (while solving the parts of this exercise below, can you identify the parts of your argument that need this assumption?).
- a) Use the second fundamental theorem of calculus to show that $s'(t) = \|\dot{\gamma}(t)\|$ and therefore that $s'(t) > 0$.
 - b) Use the solution to the previous part to compute the derivative of $s^{-1}(t)$, which is the inverse of $s(t)$ (in the next lecture, we will see why this inverse exists and why it is smooth).
 - c) Let $\tilde{\gamma}(t) := \gamma(s^{-1}(t))$, and use the chain rule and the previous parts to show that $\|\frac{d}{dt} \tilde{\gamma}(t)\| = 1$. In other words, you can use the arc-length function to reparametrize γ to a “unit speed” parametrization.
5. Prove that if $\gamma : (a, b) \rightarrow \mathbb{R}^3$ is a parametrization of a curve such that $\|\dot{\gamma}(t)\| = 1$ for any t in the interval (a, b) , then $s(t) = t$, where $s(t)$ is the arc length function $s(t) = \int_0^t \|\dot{\gamma}(u)\| du$. Observe that $s(t) = t$ means that the arc-length of the curve until a point $\gamma(t)$ is equal to the parameter t and therefore such a γ is also called an “arc length parametrization”.
6. Use the chain rule to show that if $\|\frac{d}{dt} \gamma(\phi(t))\| = 1$ then $\dot{\gamma}(\phi(t)) \neq 0$. Therefore, we can only hope to reparametrize γ to obtain a unit speed parametrization if $\dot{\gamma}(t) \neq 0$ for any t that is in the interval on which γ is defined.
7. Show that if $\dot{\gamma}(t) \neq 0$ then $\frac{d}{dt} \gamma(\phi(t)) \neq 0$. Therefore, the property that $\dot{\gamma}(t) \neq 0$ for any t in the interval on which it is defined, does not change on reparametrizing