Exercise sheet 1

1. Prove that homotopy is an equivalence relation
2. Prove that if Y is a convex subset of $\mathbb{R}^n$, then any two continuous functions from X to Y are homotopic. What if Y is star convex?
3. Prove that the relation of homotopy equivalence between two *spaces* is an equivalence relation.
4. Prove that the space $\mathbb{R}^n \setminus \{0\}$ is homotopically equivalent to S^{n-1}.
5. Prove that $\{1, 1/2, 1/3, \ldots\} \cup \{0\}$ can never be homotopically equivalent to $\{0, 1, 2, \ldots\}$ (both spaces are subsets of $\mathbb{R}$ and are given the subspace topology.)
6. Let $S_r := \{z \in \mathbb{C} \mid |z| < r\}$. Consider a polynomial $f(z) = a_0 + a_1 z + \cdots + a_n z^n$ of degree n, where $a_n \neq 0$. If r is such that f has has no root in S_r, then we can define the map $f_r : S_r \to \mathbb{C} \setminus \{0\}$, which is simply the restriction of f to S_r.
   a) If $f(z) = (2z-1)(3z-1)(z-4)(z-5)$, then prove that its restriction, f_1, to the unit circle is homotopic to the restriction of the map $z \to z^2(z-4)^2$. In general, prove that if f has k roots whose modulus is less than 1, then its restriction, f_1, to the unit circle is homotopic to the restriction of the polynomial $z^k(z-2)^{n-k}$. *(For all these parts, remember that the domain is $\mathbb{C} \setminus \{0\}$ and not just $\mathbb{C}$.)*
   b) Prove that if f does not have any root α, such that $|\alpha| \leq r$, then f_r is homotopic to a constant map.
   c) Prove that if r is large enough, then $f_r : S_r \to \mathbb{C} \setminus \{0\}$ is homotopic to the map $\theta_n : S_r \to \mathbb{C} \setminus \{0\}$, where $\theta_n(z) = z^n$. *(Hint: Of course, if r is large enough then f will not have a root in S_r. However, you have to choose a homotopy so that the throughout the homotopy there is no root in S_r. Note that if α is a root of f, then $-a_n\alpha^n = a_0 + a_1\alpha + \cdots a_{n-1}\alpha^{n-1}$. By taking the modulus on both sides, show that $|\alpha|$ must be less than $|a_0| + |a_1| + \cdots + |a_n|$ if α is a root such that $|\alpha| > 1$. How is this useful?)*
   d) Later in this course, will later prove that if $n > 0$ then θ_n is not homotopic to a constant map. How will that help you to deduce the fundamental theorem of algebra?
7. A map is said to be null-homotopic if it is homotopic to a constant map.

a) Prove that a space X is contractible (homotopically equivalent to a point) if and only if the identity map Id_X is null-homotopic.
b) Prove that if Y is contractible, then any continuous map $f : X \to Y$ is null-homotopic (and therefore, any two maps are homotopic).