Exercise sheet 3

1. Prove that the fundamental groups of homotopically equivalent spaces are isomorphic.
2. Prove that a map $f : S^1 \to X$ is null homotopic if and only if it extends to a continuous map from the closed disc to X.
3. Prove that the if $f : S^1 \to S^1$ satisfies the property that $f(-x) = -f(x)$, then the induced map f_* is a non-trivial homomorphism. Here we are treating S^1 as the subspace of unit complex numbers.
4. The projective plane may be defined as a space obtained by identifying each point in the 2-sphere with its antipodal point.
   a) Prove that the quotient map is a covering map.
   b) Use part a) to deduce the fundamental group of the projective plane.
5. Prove that $\pi_1(X, x_0) \times \pi_1(Y, y_0) \cong \pi_1(X \times Y, (x_0, y_0))$.
6. Compute the fundamental group of the torus, $S^1 \times S^1$.
7. Consider the map $i : S^1 \to S^1 \times S^1$ defined by $i(x) = (x, 1)$. Prove that i_* is an injective map between the respective fundamental groups.