Exercise sheet 4

1. Prove the following consequences of the Van Kampen theorem (try to prove each of them in detail using both versions of the theorem):

   a. If U and V are simply connected open subsets of X that cover X, then prove that X is simply connected.
   b. If U and V are open subsets of X that cover X, and V is simply connected, then prove that the inclusion map $i : U \to X$ induces a surjection whose kernel is the smallest normal subgroup containing $j_*(\pi_1(U \cap V))$, where $j : U \cap V \to U$, is the inclusion map.
   c. If U and V are open subsets of X that cover X, and $U \cap V$ is simply connected, then prove that $\pi_1(X)$ is isomorphic to $\pi_1(U) * \pi_1(V)$.

2. Use the previous question to compute the fundamental groups of:

   a. S^n, for $n \geq 2$.
   b. The projective plane realized as a closed 2-disk with the antipodal points on the boundary identified. *(Hint: Let U be the projective plane minus a point and V be a small disk containing that point. What does U deformation retract to? Find a loop, γ, representing a generator of its fundamental group. How does γ relate with a loop representating the generator of $\pi_1(U \cap V)$?)*
   c. The connected sum of two projective planes.

3. Can you construct a space whose fundamental group is $\mathbb{Z}/n\mathbb{Z}$ for any given natural number n? *(Hint: try generalizing the projective plane example)* How about constructing a space whose fundamental group is any given abelian group?

4. A space is said to be an n dimensional manifold if each point on it has an open neighbourhood that is homeomorphic to $\mathbb{R}^n$. Prove that the fundamental group of a manifold of dimension greater than or equal to 3 remains unchanged if you delete a point from the manifold.

5. Along with problem 6 from exercise set 2, this should help you to prove that if A is a closed subset of $\mathbb{R}^2$ which is homeomorphic to $\mathbb{R}$, then $\mathbb{R}^2 \setminus A$ is disconnected:

   a. Let $\mathbb{R}^3_{+\epsilon}$ denote the subspace $\{(x, y, z) \in \mathbb{R}^3 \mid z > -\epsilon\}$ and $\mathbb{R}^3_{-\epsilon}$ denote the subspace $\{(x, y, z) \in \mathbb{R}^3 \mid z < \epsilon\}$. Prove that $U := \mathbb{R}^3_{+\epsilon} \setminus (A \times (-\epsilon, 0])$ deformation retracts onto $\mathbb{R}^3_+ \setminus A$, $V := \mathbb{R}^3_{-\epsilon} \setminus (A \times [0, \epsilon))$

deformation retracts onto $\mathbb{R}^3_-$ and $U \cap V$ deformation retracts onto $\mathbb{R}^2 \setminus A$.

b. Prove that if $\mathbb{R}^2 \setminus A$ is connected then so is $U \cap V$.

c. Prove that if we assume that $\mathbb{R}^2 \setminus A$ is connected, then the fundamental group of $\mathbb{R}^3 \setminus A$ must be trivial. Use exercise 6 from exercise set 2. Why does this lead to a contradiction?

6. Ues the previous exercise to prove that if C is a closed subspace of S^2 which is homeomorphic to S^1, then $S^2 \setminus C$ is disconnected. *(Hint: S^2 minus a single point is homeomorphic to $\mathbb{R}^2$. By removing C, you have removed a lot more than one point!)*