Exercise sheet 2

1. Prove that if $r : X \to A$ is a retraction map from X to a subspace A, then the induced maps $r_* : \pi_1(X) \to \pi_1(A)$ is surjective and $i_* : \pi_1(A) \to \pi_1(X)$ is injective ($i : A \to X$ is the inclusion map)
2. Prove that $\pi_1(X, x_0)$ is trivial if and only if any two paths having the same end points, are path homotopic.
3. Prove that two base point change maps are equal if and only if the fundamental group of the space is abelian.
4. Prove that the fundamental groups of $\mathbb{R}^3$ minus a straight line is isomorphic to the fundamental group of a circle.
5. Let $\mathbb{R}^3_+ = \{(x, y, z) \in \mathbb{R}^3 \mid z \geq 0\}$. Note that its boundary, i.e. the subspace $\{(x, y, z) \in \mathbb{R}^3 \mid z = 0\}$, is a copy of $\mathbb{R}^2$. Prove that:
   a) The fundamental group of $\mathbb{R}^3_+$ is trivial.
   b) The fundamental group of $\mathbb{R}^3_+$ minus a straight line is trivial if the line lies on the boundary, and isomorphic to the fundamental group of a circle if it does not.
   c) The fundamental group of $\mathbb{R}^3_+$ minus a set A is trivial if A is a subset of the boundary.
6. What is the fundamental group of $\mathbb{R}^n \setminus \{0\}$, for each $n = 1, 2, \ldots$?

to be updated