

## Exercise sheet 4

1. Prove the following consequences of the Van Kampen theorem (try to prove each of them in detail using both versions of the theorem):
  - a. If  $U$  and  $V$  are simply connected open subsets of  $X$  that cover  $X$ , then prove that  $X$  is simply connected.
  - b. If  $U$  and  $V$  are open subsets of  $X$  that cover  $X$ , and  $V$  is simply connected, then prove that the inclusion map  $i : U \rightarrow X$  induces a surjection whose kernel is the smallest normal subgroup containing  $j_*(\pi_1(U \cap V))$ , where  $j : U \cap V \rightarrow U$ , is the inclusion map.
  - c. If  $U$  and  $V$  are open subsets of  $X$  that cover  $X$ , and  $U \cap V$  is simply connected, then prove that  $\pi_1(X)$  is isomorphic to  $\pi_1(U) * \pi_1(V)$ .
2. Use the previous question to compute the fundamental groups of:
  - a.  $S^n$ , for  $n \geq 2$ .
  - b. The projective plane realized as a closed 2-disk with the antipodal points on the boundary identified. (*Hint: Let  $U$  be the projective plane minus a point and  $V$  be a small disk containing that point. What does  $U$  deformation retract to? Find a loop,  $\gamma$ , representing a generator of its fundamental group. How does  $\gamma$  relate with a loop representating the generator of  $\pi_1(U \cap V)$ ?*)

to be updated