Exercise sheet 4

1. Prove the following consequences of the Van Kampen theorem (try to prove each of them in detail using both versions of the theorem):

   a. If U and V are simply connected open subsets of X that cover X, then prove that X is simply connected.
   b. If U and V are open subsets of X that cover X, and V is simply connected, then prove that the inclusion map $i : U \to X$ induces a surjection whose kernel is the smallest normal subgroup containing $j_*(\pi_1(U \cap V))$, where $j : U \cap V \to U$, is the inclusion map.
   c. If U and V are open subsets of X that cover X, and $U \cap V$ is simply connected, then prove that $\pi_1(X)$ is isomorphic to $\pi_1(U) * \pi_1(V)$.

2. Use the previous question to compute the fundamental groups of:

   a. S^n, for $n \geq 2$.
   b. The projective plane realized as a closed 2-disk with the antipodal points on the boundary identified. *(Hint: Let U be the projective plane minus a point and V be a small disk containing that point. What does U deformation retract to? Find a loop, γ, representing a generator of its fundamental group. How does γ relate with a loop representating the generator of $\pi_1(U \cap V)$?)*
   c. The connected sum of two projective planes.

3. Can you construct a space whose fundamental group is $\mathbb{Z}/n\mathbb{Z}$ for any given natural number n? *(Hint: try generalizing the projective plane example)* How about constructing a space whose fundamental group is any given abelian group?

4. This exercise is for those among you who know what a manifold is: Prove that the fundamental group of a manifold of dimension greater than 3, remains unchanged if you delete a point from the manifold.

to be updated