# Exercise sheet 1

1. Prove that homotopy is an equivalence relation
2. Prove that if $Y$ is a convex subset of $\mathbb{R}^n$, then any two continuous functions from $X$ to $Y$ are homotopic. What if Y is star convex?
3. Prove that the relation of homotopy equivalence between two *spaces* is an equivalence relation.
4. Prove that the space $\mathbb{R}^n \setminus \{0\}$ is homotopically equivalent to $S^{n-1}$.
5. Prove that $\{1, 2, \ldots, \} \cup \{0\}$ can never be homotopically equivalent to $\{0, 1, 2, \ldots\}$ (both spaces are subsets of $\mathbb{R}$ and are given the subspace topology.)
6. Let $S_r := \{z \in \mathbb{C} \mid |z| < r\}$. Consider a polynomial $f(z) = a_0 + a_1 z + \cdots + a_n z^n$ of degree $n$, where $a_n \neq 0$. If $r$ is such that $f$ has has no root in $S_r$, then we can define the map $f_r : S_r \to \mathbb{C} \setminus \{0\}$, which is simply the restriction of $f$ to $S_r$.

a) If $f(z) = (2z-1)(3z-1)(z-4)(z-5)$, then prove that its restriction, $f_1$, to the unit circle is homotopic to the restriction of the map $z \to z^2(z-4)^2$. In general, prove that if $f$ has $k$ roots whose modulus is less than 1, then its restriction, $f_1$, to the unit circle is homotopic to the restriction of the polynomial $z^k(z-2)^{n-k}$. *(For all these parts, remember that the domain is $\mathbb{C} \setminus \{0\}$ and not just $\mathbb{C}$.)*

b) Prove that if $f$ does not have any root $\alpha$, such that $|\alpha| \leq r$, then $f_r$ is homotopic to a constant map.

c) Prove that if $r$ is large enough, then $f_r : S_r \to \mathbb{C} \setminus \{0\}$ is homotopic to the map $\theta_n : S_r \to \mathbb{C} \setminus \{0\}$, where $\theta_n(z) = z^n$. *(Hint: Of course, if $r$ is large enough then $f$ will not have a root in $S_r$. However, you have to choose a homotopy so that the throughout the homotopy there is no root in $S_r$. Note that if $\alpha$ is a root of $f$, then $-a_n\alpha^n = a_0 + a_1\alpha + \cdots a_{n-1}\alpha^{n-1}$. By taking the modulus on both sides, show that $|\alpha|$ must be less than $|a_0| + |a_1| + \cdots + |a_n|$ if $\alpha$ is a root such that $|\alpha| > 1$. How is this useful?)*

d) Later in this course, will later prove that if $n > 0$ then $\theta_n$ is not homotopic to a constant map. How will that help you to deduce the fundamental theorem of algebra?

**to be updated**