Exercise sheet 9

1. Realize the torus as a simplicial complex, K. Find subcomplexes K_1 and K_2 so that $|K_1|$ and $|K_2|$ are the cylinder, and $K_1 \cup K_2 = K$, while $|K_1 \cap K_2|$ is a circle.
   a. Compute all the homologies of $K_1 \cap K_2$ using the definition of homology.
   b. Compute all the homologies of K_1 and K_2 using the definition of homology. To represent the homology classes, find cycles induced by the inclusion map of the intersection. You will need to know this for the next part.
   c. Use the Mayer-Vietoris sequence to compute all the homologies of the torus.

2. This is an alternative way to compute the homologies of S^n. Each S^n can be realized as a simplicial complex K such that $K = K_1 \cup K_2$, where K_i are sub-complexes of K whose underlying spaces are each the closed ball of dimension n, and $K_1 \cap K_2$ is a subcomplex of K whose underlying space is S^{n-1}. Prove that K_1 and K_2 are acyclic. *(For example, if you realize S^n as the boundary of a simplex, you can realize K_1 and K_2 as cones.)* Use that observation along with the Mayer-Vietoris to prove that $\tilde{\mathrm{H}}_k(S^n) = \tilde{\mathrm{H}}_{k-1}(S^{n-1})$, thereby allowing you to inductively compute each homology of each S^n. Notice that reduced homology proved a lot more convenient.

3. Let $f, g : K \to L$ be simplicial maps and $D_k : C_k(K) \to C_{k+1}(L)$ is a map such that $\partial \circ D + D \circ \partial = f_\# - g_\#$.
   (a) Prove that $\alpha_k(c) := f_\#(c) - g_\#(c) - D_k(\partial(c))$ is a cycle for any $k+1$-chain c, and therefore even for an oriented simplex (treated as a chain).
   (b) Use the above to prove that if f and g are carried by an acyclic carrier Φ, then one can inductively define a chain homotopy between $f_\#$ and $g_\#$. The chain homotopy is also carried by Φ. *(Acyclicity allows you to express $\alpha_k(c)$ as a boundary of some chain. If you simply define D_{k+1} to be that chain, note that it satisfies the condition for it to be a chain homotopy. You merely have to check that by the definition of an acyclic carrier, that α_{k+1} is still in an acyclic subcomplex, thereby allowing you to proceed to the next level of induction.)*

4. If $f_1 : |K| \to |L|$ and $f_2 : |L| \to |M|$ are continuous maps, and $g_1 : K \to L$ and $g_2 : L \to M$ their respective simplicial approximates, then prove that $g_2 \circ g_1$ is a simplicial approximate of $f_2 \circ f_1$.

5. For a simplicial complex K,

    a. How can you realize $|K| \times [0, 1]$ as the polytope of a some simplicial complex L?
    b. The inclusion maps $i_0, i_1 : |K| \to |K| \times [0, 1]$ defined by $i_0(x) = (x, 0)$ and $i_1(x) = (x, 1)$ are simplicial maps (why?). Prove that their induced maps on the level of chains are chain homotopic. *(Hint: There is an obvious acyclic carrier for both maps.)*

6. Prove that the subdivision of a subdivision of a simplicial complex is a subdivision.

7. Consider a sudivision K' of a simplicial complex K

    a. Why is the set of points $|K|$ exactly the same set as $|K'|$?
    b. Why are the topologies of $|K|$ and $|K'|$ the same? *(Here is where you will use the fact that each simplex of K is a union of finitely many simplices of K').*
    c. Prove that if A is a subcomplex of K, then the simplices in K' that lie in A form a subdivision of A.