Exercise sheet 2

1. Prove that if $r : X \to A$ is a retraction map from X to a subspace A, then the induced maps $r_* : \pi_1(X) \to \pi_1(A)$ is surjective and $i_* : \pi_1(A) \to \pi_1(X)$ is injective ($i : A \to X$ is the inclusion map)

2. Prove that $\pi_1(X, x_0)$ is trivial if and only if any two paths having the same end points, are path homotopic.

3. Prove that two base point change maps are equal if and only if the fundamental group of the space is abelian.

4. Prove that the fundamental groups of $\mathbb{R}^3$ minus a straight line is isomorphic to the fundamental group of a circle.

5. Let $\mathbb{R}^3_+ = \{(x, y, z) \in \mathbb{R}^3 \mid z \geq 0\}$. Note that its boundary, i.e. the subspace $\{(x, y, z) \in \mathbb{R}^3 \mid z = 0\}$, is a copy of $\mathbb{R}^2$. Prove that:

   a) The fundamental group of $\mathbb{R}^3_+$ is trivial.
   b) The fundamental group of $\mathbb{R}^3_+$ minus a straight line is trivial if the line lies on the boundary, and isomorphic to the fundamental group of a circle if it does not.
   c) The fundamental group of $\mathbb{R}^3_+$ minus a set A is trivial if A is a subset of the boundary.

6. Let A denote a *closed* subset of $\mathbb{R}^3$ that lies on the plane defined by $z = 0$, and which is homeomorphic to $\mathbb{R}$.

   a) Prove that the homeomorphism from A to $\mathbb{R}$ can be extended to a continuous map from the plane defined by $z = 0$ to $\mathbb{R}$ (Look up the statement of the Tietz extension theorem, if you have forgotten it).
   b) Prove that there exists a homeomorphism from $\mathbb{R}^3$ to $\mathbb{R}^3$ so that the image of A intersects every plane defined by $z = t$ in exactly one point, for each t.
   c) Part b) should help you to prove that there exists a homeomorphism from $\mathbb{R}^3$ to $\mathbb{R}^3$ that maps A to a straight line (in fact, the z-axis).
   d) Prove that the fundamental group of $\mathbb{R}^3 \setminus A$ is isomorphic to the fundamental group of the circle. *(When you study the Van Kampen theorem, you will see how this exercise and the previous one will help you to prove the Jordan Curve theorem)*

7. What is the fundamental group of $\mathbb{R}^n \setminus \{0\}$, for each $n = 1, 2, \ldots$?