Exercise sheet 7

1. Prove that $\{v_0, v_1, \dots v_n\}$ are geometrically independent if and only if $\{v_1 - v_0, v_2 - v_0, \dots, v_n - v_0\}$ are linearly independent.

2. Prove that a simplex spanned by vertices $\{v_0, v_1, \dots, v_n\}$ is the smallest convex set generated by the vertices.

3. Prove that the distance of any point x of a simplex from its vertices never exceeds the distance between any two vertices. *(Hint: Rephrase it as follows, which is easy to prove: a closed ball (convex set) around x that is large enough to contain all the vertices must contain the entire simplex.)*

4. Prove that the diameter of a simplex is equal to the maximum distance between any pair of vertices. *(Hint: the intersection of balls centered around each vertex is a convex set)*

5. If K is a finite simplicial complex, then prove that the diameter of a star is bounded by twice the maximum distance between any pair of vertices of K.

6. Prove that the representation of a point of a simplex in terms of its barycentric coordinates is unique.

7. Let $x = \sum t_i v_i$ be a point of the simplex σ spanned by $\{v_0, v_1, \dots v_n\}$
   (a) Prove that x is in the interior of σ if and only if all the t_i's are strictly positive.
   (b) Prove that x lies in the interior of exactly one sub-simplex (not necessarily proper) of σ. x has positive barycentric coordinates with precisely those vertices.

8. Recall how we extend the barycentric coordinates of x to include all the vertices of the simplicial complex and not just the vertices of the simplex which contains x: choose a simplex that x belongs to, writing its coordinates in terms of the vertices of that simplex, and setting the coordinates with respect to other vertices as 0. However if x belongs to a face of a simplex, it may belong to at least three simplices: the face itself, the simplex, or another simplex which intersects the given one along the given face. Why do we get the same barycentric coordinates irrespective of which simplex we choose?

9. Prove that if x has positive barycentric coordinates (extended to the entire complex) with respect to a set of vertices $v_0, v_1, \dots v_n$, if and only if $\{v_0, v_1, \dots v_n\}$ span a simplex in the simplicial complex whose *interior* contains x.

10. If $\{v_0, v_1, \dots, v_n\}$ span a simplex, prove that the set $\cap_{i=0}^{n} \mathrm{St}(v_i)$ is the interior of the simplex.

11. Express the condition $x \in \mathrm{St}(v)$ in terms of the barycentric coordinates of x (extended to all the vertices of the complex).

12. Prove that $\{v_0, v_1, \dots v_n\}$ span a simplex if and only if $\cap \mathrm{St}(v_i)$ is non-empty.

13. Prove that a set map $f : K^{(0)} \to L^{(0)}$ can be *uniquely* extended to a continuous map $f' : |K| \to |L|$ that has a linear restriction to each simplex, if f maps the set vertices of any simplex in K to vertices of *some* simplex in L (not necessarily of the same dimension). This justifies the definition of a simplicial map.

14. Let $f : |K| \to |L|$ be a continuous map and $g : K^{(0)} \to L^{(0)}$ be a set map such that $f(\mathrm{St}(v)) \subset \mathrm{St}(g(v))$ for each vertex v.

    (a) Prove that if $x \in |K|$ has a positive barycentric coordinate with respect to a vertex w then $f(x)$ has a positive barycentric coordinate with respect to $g(w)$.

    (b) Prove that if x lies in the interior of a simplex spanned by $\{v_0, \dots, v_n\}$, then $f(x)$ lies in the interior of a simplex whose vertices contain the set $\{g(v_0), \dots, g(v_n)\}$. Note that this proves that if $\{v_0, \dots, v_n\}$ spans a simplex in K then $\{g(v_0), \dots, g(v_n)\}$ spans a simplex in L.

    (c) Prove that if there is another set map $g' : K^{(0)} \to L^{(0)}$ such that $f(\mathrm{St}(v)) \subset \mathrm{St}(g'(v))$ and $\{v_0, \dots, v_n\}$ spans a simplex σ then $\{g(v_0), \dots, g(v_n), g'(v_0), \dots, g'(v_n)\}$ spans a simplex σ' in L *(Hint: To identify the simplex σ', pick a point in the interior of σ; what are the barycentric coordinates of its image?).*