Exercise sheet 8

1. Recall that we defined $C(v) = \cup_{w \sim v} \mathrm{St}(w)$.
   (a) Why is $C(v)$ path connected?
   (b) Why is $C(v)$ open?
   (c) Why is $C(v) \cap C(w) = \emptyset$ if $v \nsim w$?
   (d) Why is $C(v) = C(w)$ if $v \sim w$?
   (e) Why are they the components of the underlying space of the simplicial complex?
   (f) Prove that if v and w are vertices that belong to $C(u)$ then there exists then there exists a 1-chain c so that $\partial c = v - w$.
2. Prove that $\partial_{n+1} \circ \partial_n = 0$.
3. Realize the klein bottle as a simplicial complex and compute its homology.
4. Prove that if $\sum n_i \sigma_i \in Z_n(K)$ is such that σ_0 has an $n-1$ face that is not shared by any other σ_i where $i \neq 0$, then $n_0 = 0$.
5. Prove that if $\sum n_i \sigma_i \in Z_n(K)$ is such that each $n-1$ face of each σ_i is the face of exactly two distinct σ_j, then $n_i = \pm n_0$ for each i.
6. Prove that an n-chain homologous to an n-cycle is also an n-cycle.
7. Prove that the augmentation map is a homomorphism.
8. Prove that $\epsilon(c) = 0$ is a necessary condition for the zero chain c to be a boundary of some 1-chain. Is it also sufficient?
9. Prove that for any 0-simplex v of a simplicial complex, nv, where $n \in \mathbb{Z}$, can never be a boundary of a 1-chain.
10. Derive an expression for the $\partial([w, \sigma])$ in terms of $\partial(\sigma)$ for any simplex σ. Use that to extend the expression to $\partial([w, c])$ for any chain c. (Note that in dimension 0 your expression for $\partial([w, c])$ should involve the augmentation map.)
11. Use the above to prove that any chain in a cone is homologous to a chain which has non-zero coefficient only for the simplices that contain the vertex of the cone as a vertex of the simplex. Then prove that any such chain can only be a cycle if all of its coefficients are zero, thereby proving that a cone is acyclic (all reduced homologies are 0).

12. The previous question may be used to compute the homology of a boundary of an n-simplex, $\mathrm{Bd}(\sigma)$, for some simplex σ, which can be realized as a simplicial complex (how?):

   (a) Prove that the $\mathrm{H}_i(\mathrm{Bd}(\sigma)) = \mathrm{H}_i(\sigma)$ for $i \leq n-1$. Here we are using the same notation for the simplex or its boundary treated as a simplicial complex. *(Hint: between the simplex, treated as a simplicial complex, and its boundary, also treated as a simplicial complex, there is only one simplex extra)*

   (b) Prove that if any $n-1$-chain of $\mathrm{Bd}(\sigma)$ is a cycle then all the coefficients must be the same. Since it cannot be a boundary (why?), this tells us that $\mathrm{H}_{n-1}(\mathrm{Bd}(\sigma)) = \mathbb{Z}$

13. Prove that for any simplicial map $f : K \to L$, $\partial \circ f_{\#} = f_{\#} \circ \partial$, where $f_{\#} : C(K) \to C(L)$ is the map induced by f. *(You will have to split it into cases depending on the dimension*)

14. Prove that for any simplicial map $f : K \to L$, $f_* : \mathrm{H}_n(K) \to \mathrm{H}_n(L)$ is well defined.