Exercise sheet 1

- 1. Prove that homotopy is an equivalence relation
- 2. Prove that if Y is a convex subset of \mathbb{R}^n , then any two continuous functions from X to Y are homotopic. What if Y is star convex?
- 3. Prove that the relation of homotopy equivalence between two *spaces* is an equivalence relation.
- 4. Prove that the space $\mathbb{R}^n \setminus \{0\}$ is homotopically equivalent to S^{n-1} .
- 5. Prove that $\{1,2,\ldots,\}\cup\{0\}$ can never be homotopically equivalent to $\{0,1,2,\ldots\}$ (both spaces are subsets of $\mathbb R$ and are given the subspace topology.)
- 6. Let $S_r := \{z \in \mathbb{C} \mid |z| < r\}$. Consider a polynomial $f(z) = a_0 + a_1 z + \cdots + a_n z^n$ of degree n, where $a_n \neq 0$. If r is such that f has has no root in S_r , then we can define the map $f_r : S_r \to \mathbb{C} \setminus \{0\}$, which is simply the restriction of f to S_r .
- a) If f(z) = (2z-1)(3z-1)(z-4)(z-5), then prove that its restriction, f_1 , to the unit circle is homotopic to the restriction of the map $z \to z^2(z-4)^2$. In general, prove that if f has k roots whose modulus is less than 1, then its restriction, f_1 , to the unit circle is homotopic to the restriction of the polynomial $z^k(z-2)^{n-k}$. (For all these parts, remember that the domain is $\mathbb{C} \setminus \{0\}$ and not just \mathbb{C} .)
- b) Prove that if f does not have any root α , such that $|\alpha| \leq r$, then f_r is homotopic to a constant map.
- c) Prove that if r is large enough, then $f_r: S_r \to \mathbb{C} \setminus \{0\}$ is homotopic to the map $\theta_n: S_r \to \mathbb{C} \setminus \{0\}$, where $\theta_n(z) = z^n$. (Hint: Of course, if r is large enough then f will not have a root in S_r . However, you have to choose a homotopy so that the throughout the homotopy there is no root in S_r . Note that if α is a root of f, then $-a_n\alpha^n = a_0 + a_1\alpha + \cdots + a_{n-1}\alpha^{n-1}$. By taking the modulus on both sides, show that $|\alpha|$ must be less than $|a_0| + |a_1| + \cdots + |a_n|$ if α is a root such that $|\alpha| > 1$. How is this useful?)
- d) Later in this course, will later prove that if n > 0 then θ_n is not homotopic to a constant map. How will that help you to deduce the fundamental theorem of algebra?

to be updated