

Exercise sheet 1

1. Prove that homotopy is an equivalence relation
2. Prove that if Y is a convex subset of \mathbb{R}^n , then any two continuous functions from X to Y are homotopic. What if Y is star convex?
3. Prove that the relation of homotopy equivalence between two *spaces* is an equivalence relation.
4. Prove that the space $\mathbb{R}^n \setminus \{0\}$ is homotopically equivalent to S^{n-1} .
5. Prove that $\{1, 2, \dots\} \cup \{0\}$ can never be homotopically equivalent to $\{0, 1, 2, \dots\}$ (both spaces are subsets of \mathbb{R} and are given the subspace topology.)
6. Let $S_r := \{z \in \mathbb{C} \mid |z| < r\}$. Consider a polynomial $f(z) = a_0 + a_1z + \dots + a_nz^n$ of degree n , where $a_n \neq 0$. If r is such that f has no root in S_r , then we can define the map $f_r : S_r \rightarrow \mathbb{C} \setminus \{0\}$, which is simply the restriction of f to S_r .
 - a) If $f(z) = (2z - 1)(3z - 1)(z - 4)(z - 5)$, then prove that its restriction, f_1 , to the unit circle is homotopic to the restriction of the map $z \rightarrow z^2(z - 4)^2$. In general, prove that if f has k roots whose modulus is less than 1, then its restriction, f_1 , to the unit circle is homotopic to the restriction of the polynomial $z^k(z - 2)^{n-k}$. (For all these parts, remember that the domain is $\mathbb{C} \setminus \{0\}$ and not just \mathbb{C} .)
 - a) Prove that if f does not have any root α , such that $|\alpha| \leq r$, then f_r is homotopic to a constant map.
 - b) Prove that if r is large enough, then $f_r : S_r \rightarrow \mathbb{C} \setminus \{0\}$ is homotopic to the map $\theta_n : S_r \rightarrow \mathbb{C} \setminus \{0\}$, where $\theta_n(z) = z^n$. (Hint: Of course, if r is large enough then f will not have a root in S_r . However, you have to choose a homotopy so that throughout the homotopy there is no root in S_r . Note that if α is a root of f , then $-a_n\alpha^n = a_0 + a_1\alpha + \dots + a_{n-1}\alpha^{n-1}$. By taking the modulus on both sides, show that $|\alpha|$ must be less than $|a_0| + |a_1| + \dots + |a_n|$ if α is a root such that $|\alpha| > 1$. How is this useful?)
 - c) Later in this course, will later prove that if $n > 0$ then θ_n is not homotopic to a constant map. How will that help you to deduce the fundamental theorem of algebra?

to be updated