Let $\{a_1 , \cdots , a_g , b_1 , \cdots b_g\}$ be simple curves in X representing the generators of $\mathrm{H}_1(X, \mathbb{Z})$. Consider a basis $\omega_i \in \mathrm{H}^0(X, \Omega)$, where Ω is the holomorphic cotangent bundle of X, such that $\int_{a_i} \omega_j = \delta_{ij}$, $i , j = 1 , \cdots , g$. The $2g$ vectors $(\int_{a_i} \omega_1 , \cdots , \int_{a_i} \omega_g)$ and $(\int_{b_j} \omega_1 , \cdots , \int_{b_j} \omega_g)$, $1 \leq i , j \leq g$, of $\mathbb{C}^g$ are linearly independent over $\mathbb{R}$ and generate a lattice which will be denoted by Λ. Denote by $\mathrm{Jac}(X)$ the complex torus $\mathbb{C}^g/\Lambda$ of dimension g. For a fixed point p_0, define the map

$$\mu : X \longrightarrow \mathrm{Jac}(X) , \quad p \longmapsto (\int_{p_0}^{p} \omega_1 , \cdots , \int_{p_0}^{p} \omega_g) .$$

This map can be extended by linearity to the space of all divisors. When it is restricted to the space of degree 0 divisors, it is a bijection up to linear equivalence, implying that is isomorphic to $\mathrm{Pic}^0(X)$ [**?**]. The Jacobian of X is isomorphic to $\mathrm{Jac}(X) := \mathbb{C}^g/\Lambda$. A different choice of a_i and b_j will result in an isomorphic g dimensional complex torus.

Proof. Fix a real point $p \in X$. Consider a map

$$\theta : \mathrm{Sym}^g(X) \longrightarrow \mathrm{Pic}^0(X) , \ (x_1 , \cdots , x_g) \longmapsto [\sum_{i=1}^{g}(x_i - p)] .$$

Given an element $[D]$ of $\mathrm{Pic}^0(D)$, its inverse under θ is the set of effective divisors linearly equivalent to the divisor $D + g.p$. By the Riemann–Roch theorem, this linear system, which we will denote by $\mathcal{L}(D + g.p)$, is always non-empty, and therefore θ is surjective.

Now consider an element $[D] \in \mathbb{R}\mathrm{Pic}^0(X)$. Since the real part of the curve is non-empty, the line bundle $[D]$ cannot have a quaternionic structure since there are no real quaternionic bundles of odd rank. Therefore, the line bundle has a real structure, i.e., there is a lift $\widetilde{\sigma}$ of σ such that $\widetilde{\sigma} \circ \widetilde{\sigma}$ is the identity. For any holomorphic section s in $\mathcal{L}(D+g.p)$, the section $s+\widetilde{\sigma}(s)$ is fixed by $\widetilde{\sigma}$. The divisor (s), denoting the divisor of the zeros of this holomorphic section s is therefore in $\mathbb{R}\mathrm{Sym}^g(X)$. Therefore, θ restricted to $\mathbb{R}\mathrm{Sym}^g(X)$, surjects onto $\mathbb{R}\mathrm{Pic}^0(X)$. Hence $\mathrm{Sym}^g(X)$ and $\mathrm{Pic}^0(X)$ have the same number of components. □

Proof. Choose a $d \geq g$ so that d is even. Since d is even, there exists a point $(z_1 , z_2 , \cdots , z_d)$ in $\mathrm{Sym}^d(X)$ which is fixed by σ and $z_{2k} = z_{2k-1}$. Consider the map

$$\theta : \mathrm{Sym}^d(X) \longrightarrow \mathrm{Pic}^0(X) , \ (x_1 , \cdots , x_d) \longmapsto [\sum_{i=1}^{d}(x_i - z_i)] .$$

Once again, given an element $[D]$ of $\mathrm{Pic}^0(D)$, its inverse under θ is the set of effective divisors linearly equivalent to the divisor $D + \sum z_i$. Again, by the Riemann–Roch theorem, the linear system $\mathcal{L}(D + \sum z_i)$, is always non-empty, and therefore θ is surjective.

Now consider two elements $[D_1] , [D_2] \in \mathbb{R}\mathrm{Pic}^0(X)$ satisfying the condition that both of of them have a real structure, i.e., there is a lift $\widetilde{\sigma}$ of σ such that

$\widetilde{\sigma}\circ\widetilde{\sigma}$ is the identity map. As before, a holomorphic section of the form $s_i+\widetilde{\sigma}(s_i)$ in $[D_i]$ $(i=1,2)$ is fixed by $\widetilde{\sigma}$ and therefore, the divisors (s_i) associated to this holomorphic section are in $\mathbb{R}\mathrm{Sym}^g(X)$. The divisor (s_1) is of the form $(z_1,\sigma(z_1),z_2,\sigma(z_2),\ldots,z_{d/2},\sigma(z_{d/2})$ for some z_i that are not fixed by σ. Similarly, the divisor (s_2) is of the form $(w_1,\sigma(w_1),w_2,\sigma(w_2),\cdots,w_{d/2},\sigma(w_{d/2})$ for some w_i that are not fixed by σ. Paths joining z_i with w_i will induce a path joining (s_1) with (s_2). Therefore, any two degree 0 line bundles with a real structure can be connected by a path.

Note that the tensor product $L_1\otimes L_2$ of two quaternionic line bundles L_1 and L_2 is a real line bundle. Therefore, if there is a quaternionic line bundle $L\in\mathrm{Pic}^0(X)$, then the map $L'\longrightarrow L'\otimes L$ defines a bijection between real line bundles and quaternionic line bundles in $\mathrm{Pic}^0(X)$. If there exist line bundles $[D_1],[D_2]\in\mathbb{R}\mathrm{Pic}^0(X)$ with quaternionic structures, then by the previous paragraph, there is a path joining $[D_1]\otimes L$ and $[D_2]\otimes L$ which can be pulled back via the map $L'\longrightarrow L'\otimes L$ to a path joining $[D_1]$ and $[D_2]$. $\square$

Remark 1. Effective divisors will be seen later to correspond to line bundles with a holomorphic section.

Let $\{a_1,\cdots,a_g,b_1,\cdots b_g\}$ be simple curves in X representing the generators of $\mathrm{H}_1(X,\mathbb{Z})$. Consider a basis $\omega_i\in\mathrm{H}^0(X,\Omega)$, where Ω is the holomorphic cotangent bundle of X, such that $\int_{a_i}\omega_j=\delta_{ij}$, $i,j=1,\cdots,g$. The $2g$ vectors $(\int_{a_i}\omega_1,\cdots,\int_{a_i}\omega_g)$ and $(\int_{b_j}\omega_1,\cdots,\int_{b_j}\omega_g)$, $1\le i,j\le g$, of $\mathbb{C}^g$ are linearly independent over $\mathbb{R}$ and generate a lattice which will be denoted by Λ. Denote by $\mathrm{Jac}(X)$ the complex torus $\mathbb{C}^g/\Lambda$ of dimension g. For a fixed point p_0, define the map

$$\mu\,:\,X\longrightarrow\mathrm{Jac}(X)\,,\quad p\longmapsto(\int_{p_0}^{p}\omega_1,\cdots,\int_{p_0}^{p}\omega_g)\,.$$

This map can be extended by linearity to the space of all divisors. When it is restricted to the space of degree 0 divisors, it is a bijection up to linear equivalence, implying that is isomorphic to $\mathrm{Pic}^0(X)$ [**?**]. The Jacobian of X is isomorphic to $\mathrm{Jac}(X):=\mathbb{C}^g/\Lambda$. A different choice of a_i and b_j will result in an isomorphic g dimensional complex torus.

Proof. Fix a real point $p\in X$. Consider a map

$$\theta\,:\,\mathrm{Sym}^g(X)\longrightarrow\mathrm{Pic}^0(X)\,,\ (x_1,\cdots,x_g)\longmapsto[\sum_{i=1}^{g}(x_i-p)]\,.$$

Given an element $[D]$ of $\mathrm{Pic}^0(D)$, its inverse under θ is the set of effective divisors linearly equivalent to the divisor $D+g.p$. By the Riemann–Roch theorem, this linear system, which we will denote by $\mathcal{L}(D+g.p)$, is always non-empty, and therefore θ is surjective.

Now consider an element $[D]\in\mathbb{R}\mathrm{Pic}^0(X)$. Since the real part of the curve is non-empty, the line bundle $[D]$ cannot have a quaternionic structure since there are no real quaternionic bundles of odd rank. Therefore, the line bundle has a real structure, i.e., there is a lift $\widetilde{\sigma}$ of σ such that $\widetilde{\sigma} \circ \widetilde{\sigma}$ is the identity. For any holomorphic section s in $\mathcal{L}(D+g.p)$, the section $s+\widetilde{\sigma}(s)$ is fixed by $\widetilde{\sigma}$. The divisor (s), denoting the divisor of the zeros of this holomorphic section s is therefore in $\mathbb{R}\mathrm{Sym}^g(X)$. Therefore, θ restricted to $\mathbb{R}\mathrm{Sym}^g(X)$, surjects onto $\mathbb{R}\mathrm{Pic}^0(X)$. Hence $\mathrm{Sym}^g(X)$ and $\mathrm{Pic}^0(X)$ have the same number of components. □

Proof. Choose a $d \geq g$ so that d is even. Since d is even, there exists a point $(z_1, z_2, \cdots, z_d)$ in $\mathrm{Sym}^d(X)$ which is fixed by σ and $z_{2k} = z_{2k-1}$. Consider the map

$$\theta : \mathrm{Sym}^d(X) \longrightarrow \mathrm{Pic}^0(X), \ (x_1, \cdots, x_d) \longmapsto [\sum_{i=1}^{d}(x_i - z_i)].$$

Once again, given an element $[D]$ of $\mathrm{Pic}^0(D)$, its inverse under θ is the set of effective divisors linearly equivalent to the divisor $D + \sum z_i$. Again, by the Riemann–Roch theorem, the linear system $\mathcal{L}(D + \sum z_i)$, is always non-empty, and therefore θ is surjective.

Now consider two elements $[D_1], [D_2] \in \mathbb{R}\mathrm{Pic}^0(X)$ satisfying the condition that both of of them have a real structure, i.e., there is a lift $\widetilde{\sigma}$ of σ such that $\widetilde{\sigma} \circ \widetilde{\sigma}$ is the identity map. As before, a holomorphic section of the form $s_i + \widetilde{\sigma}(s_i)$ in $[D_i]$ $(i = 1, 2)$ is fixed by $\widetilde{\sigma}$ and therefore, the divisors (s_i) associated to this holomorphic section are in $\mathbb{R}\mathrm{Sym}^g(X)$. The divisor (s_1) is of the form $(z_1, \sigma(z_1), z_2, \sigma(z_2), \ldots, z_{d/2}, \sigma(z_{d/2})$ for some z_i that are not fixed by σ. Similarly, the divisor (s_2) is of the form $(w_1, \sigma(w_1), w_2, \sigma(w_2), \cdots, w_{d/2}, \sigma(w_{d/2})$ for some w_i that are not fixed by σ. Paths joining z_i with w_i will induce a path joining (s_1) with (s_2). Therefore, any two degree 0 line bundles with a real structure can be connected by a path.

Note that the tensor product $L_1 \otimes L_2$ of two quaternionic line bundles L_1 and L_2 is a real line bundle. Therefore, if there is a quaternionic line bundle $L \in \mathrm{Pic}^0(X)$, then the map $L' \longrightarrow L' \otimes L$ defines a bijection between real line bundles and quaternionic line bundles in $\mathrm{Pic}^0(X)$. If there exist line bundles $[D_1], [D_2] \in \mathbb{R}\mathrm{Pic}^0(X)$ with quaternionic structures, then by the previous paragraph, there is a path joining $[D_1] \otimes L$ and $[D_2] \otimes L$ which can be pulled back via the map $L' \longrightarrow L' \otimes L$ to a path joining $[D_1]$ and $[D_2]$. □

Proof. The map $u : \mathrm{Sym}^n(X) \to \mathrm{Pic}^0(X)$ $(n \geq 2g - 2)$ defined by $u(x_1 + x_2 + \ldots + x_n) = \mathcal{O}(\sum(x_i - p))$, where p is a real point of the curve, is a complex projective bundle, where the fibre of $\mathcal{O}(D)$ is the projectivised linear system $\mathbb{P}\mathcal{L}(D + n.p)$.

We now prove that the restriction of u to $\mathrm{Sym}^n(X)^\sigma$ surjects onto $\mathrm{Pic}^0(X)^\sigma$. Consider an element $\mathcal{O}(D) \in \mathrm{Pic}^0(X)^\sigma$. Since the real part of the curve is non-empty, the line bundle $\mathcal{O}(D)$ cannot have a quaternionic structure since there are no real quaternionic bundles of odd rank. Therefore, the line bundle has a real structure, i.e., there is a lift $\widetilde{\sigma}$ of σ such that $\widetilde{\sigma} \circ \widetilde{\sigma}$ is the identity. For any holomorphic section s in $\mathcal{L}(D + n.p)$, the section $s + \widetilde{\sigma}(s)$ is fixed by $\widetilde{\sigma}$. The divisor (s), denoting the divisor of the zeros of this holomorphic section s

is therefore in $\mathrm{Sym}^n(X)^\sigma$. Therefore, u restricted to $\mathrm{Sym}^n(X)^\sigma$, surjects onto $\mathrm{Pic}^0(X)^\sigma$.

The pull-back of a principal divisor (f) via the complex conjugation involution σ is the principal divisor $(\overline{\sigma^*(f)})$. Therefore if D is real, σ^* preserves the linear system $\mathcal{L}(D + n.p)$ and, therefore preserves the fibre of a $u(D)$ for a real D. Since u is surjective, as proved in the previous paragraph, the fibre of of the restriction of u to the real part is preserved by the involution.

Therefore, on restricting to the real part of $\mathrm{Sym}^n(X)$, u is a real projective bundle over the real part of $\mathrm{Pic}^0(X)$.

By [**?**], if X is an M-variety then $\mathrm{Pic}^0(X)$ is also an M-variety. The complex projective space is also an M-variety. Therefore, by the Leray-Hirsch theorem, $\mathrm{Sym}^n(X)$ is also an M-variety. $\square$