Exercise sheet 2

1. Prove that the following are tautologies.
   a) $P \wedge Q \Rightarrow P$
   b) $((P \Rightarrow Q) \wedge (Q \Rightarrow R)) \Rightarrow (P \Rightarrow R)$
   c) $((P_0 \Rightarrow P_1) \wedge (P_1 \Rightarrow P_2) \wedge \ldots \wedge (P_{n-1} \Rightarrow P_n)) \Rightarrow (P_0 \Rightarrow P_n)$
   d) $(P \Rightarrow Q) \wedge (Q \Rightarrow P) \Rightarrow (P \Leftrightarrow Q)$
2. Consider the predicate $P(x, y, z) :=$ "$2x + 5y - 6z = 1$". Which of the following propositions are true:
   a) $\forall x \forall y \forall z P(x, y, z)$
   b) $\forall x \forall y \exists z P(x, y, z)$
   c) $\forall x \exists y \exists z P(x, y, z)$
   d) $\exists x \exists y \exists z P(x, y, z)$
3. Let $Q(x, y)$ denote the predicate "x is a question from Quiz y". Let $D(x)$ denote the predicate "x is difficult". Write expressions for the following statements using quantifiers, Q, and D:
   a) "Quiz 3 will have exactly one difficult question"
   b) "Every quiz will have at least one difficult question"
   c) "For some quizzes, every question will be difficult"
   d) "For some quizzes, *at most one* question will be easy"
4. Which of these are true no matter what the predicate $P(x, y)$ is. For each of the others, find an example for which it does not hold.
   a) $\forall x \exists y P(x, y) \equiv \exists y \forall x P(x, y)$.
   b) $\forall x \forall y P(x, y) \equiv \forall y \forall x P(x, y)$.
   c) $\exists x \exists y P(x, y) \equiv \exists y \exists x P(x, y)$.
5. One can define a new quantifier $\exists! x P(x)$ to mean that "there exists a *unique* x such that $P(x)$ is true". Show that this can always be expressed by an expression involving only the usual two quantifiers, $\exists$, and $\forall$. *(Hint: you want to say that it exists* ***and*** *is unique. What does it mean to say that there is a unique x satisfying a predicate? Try considering its negation if that helps. You may assume that there is some notion of equality between the objects under consideration so that it makes sense to ask if "a = b" for any given a and b)*

To be completed