Exercise sheet 3

Manifolds, MTH406

1. Consider a smooth frame $s_1, s_2, \dots, s_k$ of a smooth vector bundle $\pi : E \to B$ over an open set $U \subset B$. Prove that any other section $s(p) = \lambda_1(p)s_1(p) + \cdots + \lambda_k(p)s_k(p)$ over U is a smooth vector field if and only if $\lambda_i : U \to \mathbb{R}$ are smooth functions.

2. Consider a (not-necessarily smooth) section $s : M \to T(M)$ such that $\pi \circ s = Id$, where $T(M)$ denotes the tangent bundle and $\pi : T(M) \to M$ the natural projection. Prove that the following are equivalent (and, therefore,they are all equivalent definitions of a smooth vector field):

   (a) $s : M \to T(M)$ is a smooth map (considering the manifold structure on $T(M)$).

   (b) Given a chart $\phi : U \to \mathbb{R}^n$, let x_i be the coordinate functions, i.e. $\phi(p) = (x_1(p), x_2(p), \dots, x_n(p))$. Consider the smooth (local) frame on U given by $\frac{\partial}{\partial y_i}$. Let $\lambda_i : U \to \mathbb{R}$ define smooth functions so that $s(p) = \Sigma_i \lambda_i(p) \frac{\partial}{\partial y_i}$, then λ_i are smooth.

   (c) Since $s(p)$ is a tangent vector, it is a derivation. So given a smooth function $f : M \to \mathbb{R}$, $s(p)(f) \in \mathbb{R}$. Therefore, we get a map $p \to s(p)(f)$ which is smooth for every (global!) smooth function $f : M \to \mathbb{R}$ (Be careful, we are only considering global functions f, not local ones).

3. Prove that (global) vector fields on a smooth manifold form a vector space.

4. Let (M) denote the set of functions $X : C^\infty(M) \to C^\infty(M)$ which is linear and satisfies Leibnitz rule. Prove that this forms a vector space under point wise addition and scalar multiplication and prove that this vector space is isomorphic to the vector space of vector fields on M.