

## Exercise sheet 4

Manifolds, MTH406

1. Prove that two vector fields  $X$  and  $Y$  on a smooth manifold  $M$  are equal if and only if  $X(f) = Y(f)$  for every continuous function  $f : M \rightarrow \mathbb{R}$
  2. Prove that given a smooth vector field  $X$  on a smooth manifold  $M$  so that  $X_p \neq 0$  for some point  $p \in M$ , there exists a chart  $\phi : U \rightarrow \mathbb{R}^n$  for some open neighbourhood  $U$  of  $p$  so that  $X|_U = \frac{\partial}{\partial x_1}$ . Here  $x_1$  denotes the first coordinate function of the chart.
  3. Consider a Lie Group, i.e. a smooth manifold  $G$  which is also a group and the multiplication and inverse is smooth, i.e. the map  $\mu : G \times G \rightarrow G$  defined by  $\mu(x, y) = xy$  and the map  $\iota : G \rightarrow G$  defined by  $\iota(x) = x^{-1}$  are smooth. Let  $l_g : G \rightarrow G$  denote the (smooth) map  $l_g(x) = gx$ . Define a vector field  $X$  on  $G$  to be left invariant if  $l_{g*}(X) = X$ .
- (a) Prove that there is a natural bijective correspondence between the tangent space at the identity element,  $T_e(G)$ , and left invariant vector fields on  $G$ .