Exercise sheet 2

1. Prove that the following are tautologies.
   a) $P \wedge Q \Rightarrow P$
   b) $((P \Rightarrow Q) \wedge (Q \Rightarrow R)) \Rightarrow (P \Rightarrow R)$
   c) $((P_0 \Rightarrow P_1) \wedge (P_1 \Rightarrow P_2) \wedge \ldots \wedge (P_{n-1} \Rightarrow P_n)) \Rightarrow (P_0 \Rightarrow P_n)$
   d) $(P \Rightarrow Q) \wedge (Q \Rightarrow P) \Rightarrow (P \Leftrightarrow Q)$
2. Consider the predicate $P(x, y, z) :=$ "$2x + 5y - 6z = 1$". Which of the following propositions are true:
   a) $\forall x \forall y \forall z P(x, y, z)$
   b) $\forall x \forall y \exists z P(x, y, z)$
   c) $\forall x \exists y \exists z P(x, y, z)$
   d) $\exists x \exists y \exists z P(x, y, z)$
3. Let $Q(x, y)$ denote the predicate "x is a question from Quiz y". Let $D(x)$ denote the predicate "x is difficult". Write expressions for the following statements using quantifiers, Q, and D:
   a) "Quiz 3 has exactly one difficult question"
   b) "Every quiz will have at least one difficult question"
   c) "For some quizzes, every question will be difficult"
   d) "For some quizzes, *at most one* question will be easy"
4. Which of these are true no matter what the predicate $P(x, y)$ is. For each of the others, find an example for which it does not hold.
   a) $\forall x \exists y P(x, y) \equiv \exists y \forall x P(x, y)$.
   b) $\forall x \forall y P(x, y) \equiv \forall y \forall x P(x, y)$.
   c) $\exists x \exists y P(x, y) \equiv \exists y \exists x P(x, y)$.
5. One can define a new quantifier $\exists! x P(x)$ to mean that "there exists a *unique* x such that $P(x)$ is true". Show that this can always be expressed by an expression involving only the usual two quantifiers, $\exists$, and $\forall$. *(Hint: you want to say that it exists **and** is unique. What does it mean to say that there is a unique x satisfying a predicate? Try considering its negation if that helps. You may assume that there is some notion of equality between the objects under consideration so that it makes sense to ask if "$a = b$" for any given a and b)*

To be completed