

Exercise sheet 1

Manifolds, MTH406

1. Prove that any open subset of a smooth manifold is smooth.
2. Prove that the product of smooth manifolds is smooth.
3. Prove that any chart $\phi : U \rightarrow \mathbb{R}^n$ on a smooth manifold is a smooth map.
4. Prove that if $F : M \rightarrow N$ and $G : N \rightarrow P$ are smooth maps between manifolds, then the composition $G \circ F$ is also smooth.
5. Prove that if X_p is a derivation at $p \in M$, then $X_p(c) = 0$ for any constant function c .
6. Given a point p on a smooth manifold M , let F_p denote the ideal of germs that vanish at p . Prove that the dual of the vector space F_p/F_p^2 is isomorphic to $T_p(M)$.
7. Given a smooth manifold M and a curve $\gamma : (-\epsilon, \epsilon) \rightarrow M$ passing through a point $p \in M$ (i.e. $\gamma(0) = p$), let $[\gamma]_p$ denote the equivalence class of paths under the following equivalence relation. $\gamma_1 \sim \gamma_2$ if and only if $\frac{d}{dt}(\phi \circ \gamma_1)|_{t=0} = \frac{d}{dt}(\phi \circ \gamma_2)|_{t=0}$.
 - (a) Prove that the equivalence relation is independent of the chart ϕ .
 - (b) Given an equivalence class of paths, $[\gamma]_p$, we can define a derivation, $X_{[\gamma]}(f) = \frac{d}{dt}f(\gamma(t))$.