Exercise sheet 8

Probability and Statistics, MTH102

1. Let X denote a normal random variable with mean μ and variance σ^2. Remember that $\Phi(x)$ is the cumulative function for the *standard* normal distribution. Prove that $F_X(x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$

2. Let X denote a normal random variable with mean μ and variance σ^2. Show that

   (a) $P\{X > \mu + \sigma\} = 1 - \Phi(1)$

   (b) $P\{\mu < X < \mu + \sigma\} = \Phi(1) - \Phi(0)$

   (c) $P\{\mu - \sigma < X < \mu\} = \Phi(0) - \Phi(-1)$

   (d) $P\{\mu - 2\sigma < X < \mu - \sigma\} = \Phi(2) - \Phi(1)$

   (e) $P\{X < \mu - 2\sigma\} = \Phi(-2)$

   (f) In general, derive a formula for $P\{\mu - a\sigma \leq X \leq \mu + b\sigma\}$ in terms of Φ and observe that it does not depend on the parameters μ or σ.

3. If the average total marks of a particular class is 50%, and the standard deviation is 20%, then assuming that it is modelled by a normal distribution, estimate the percentage of students who will get an A grade if the cutoff for an A is 95%.

4. Show that if the cutoff for an A is set at the mean plus standard deviation for any course, then assuming that it is modelled by a normal distribution, the percentage of students earning an A will always be around $1 - \Phi(1)$

5. Consider a random variable with probability density function $f_X(x) = \lambda e^{-\lambda x}$ when $x \geq 0$ and 0 when x is negative.

   (a) Show that f_X is a valid density function

   (b) Find $\mathrm{E}[X]$ and $\mathrm{Var}(X)$

   (c) Show that X satisfies he property that $P\{X > a + b | X > a\} = P\{X > b\}$. That is that assuming that we know that X is bigger than a, then the probability that X is bigger than $a + b$ is the same as the probability that X is bigger than b.

   (d) Assume that the probability that a certain component of a laptop fails after a certain time follows an exponential distribition with an average of 7 years. What is the probability that that component of your laptop will last longer than 1 *more* year? Note that because of the previous part, you need not know when the component was made. All you know is it has lasted for some (uknown) time a until now.

(e) Assume that the duration a person spends on the phone follows an exponential probability distribition with an average of 20 minutes. You walk into a room and find someone on their phone (you do not know when they started. All you know is that the person is still spending time on their phone, and so since they have started they have spent at least some uknown time a on the phone.). What is the probability that they will spend at least 2 minutes more on their phone?

6. Let X denote the uniform distribution on the interval $(0, 1)$. Let Y denote the random variable satisfying $Y = X^3$. Find a probability density function for Y. (*Hint:* You will need to use the cumulative function and its interpetation as a probability)

7. It is known that for very large n the binomial distribution is approximated by the normal distribution with the same mean and variance (Unlike the Poisson, we do not need p to be relatively small, and also unlike Poisson this time it is continuous).

   (a) Use this to find the probability of getting more than 90% heads if the probability of getting a head is 0.7 in n tosses where n is very large. Leave your answer in terms of Φ.

   (b) In general, if X is binomial random variable with n very large, what do you think the following probability should approximately be in terms of Φ?

$$P\left\{a < \frac{X - np}{\sqrt{np(1-p)}} < b\right\}$$