

## Exercise sheet 8

Probability and Statistics, MTH102

1. Let  $X$  denote a normal random variable with mean  $\mu$  and variance  $\sigma^2$ . Remember that  $\Phi(x)$  is the cumulative function for the *standard* normal distribution. Prove that  $F_X(x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$
2. Let  $X$  denote a normal random variable with mean  $\mu$  and variance  $\sigma^2$ . Show that
  - (a)  $P\{X > \mu + \sigma\} = 1 - \Phi(1)$
  - (b)  $P\{\mu < X < \mu + \sigma\} = \Phi(1) - \Phi(0)$
  - (c)  $P\{\mu - \sigma < X < \mu\} = \Phi(0) - \Phi(-1)$
  - (d)  $P\{\mu - 2\sigma < X < \mu - \sigma\} = \Phi(2) - \Phi(1)$
  - (e)  $P\{X < \mu - 2\sigma\} = \Phi(-2)$
  - (f) In general, derive a formula for  $P\{\mu - a\sigma \leq X \leq \mu + b\sigma\}$  in terms of  $\Phi$  and observe that it does not depend on the parameters  $\mu$  or  $\sigma$ .
3. If the average total marks of a particular class is 50%, and the standard deviation is 20%, then assuming that it is modelled by a normal distribution, estimate the percentage of students who will get an A grade if the cutoff for an A is 95%.
4. Show that if the cutoff for an A is set at the mean plus standard deviation for any course, then assuming that it is modelled by a normal distribution, the percentage of students earning an A will always be around  $1 - \Phi(1)$
5. Consider a random variable with probability density function  $f_X(x) = \lambda e^{-\lambda x}$  when  $x \geq 0$  and 0 when  $x$  is negative.
  - (a) Show that  $f_X$  is a valid density function
  - (b) Find  $E[X]$  and  $\text{Var}(X)$
6. Let  $X$  denote the uniform distribution on the interval  $(0, 1)$ . Let  $Y$  denote the random variable satisfying  $Y = X^3$ . Find a probability density function for  $Y$ . (*Hint:* You will need to use the cumulative function and its interpretation as a probability)
7. It is known that for very large  $n$  the binomial distribution is approximated by the normal distribution with the same mean and variance (Unlike the Poisson, we do not need  $p$  to be relatively small, and also unlike Poisson this time it is continuous).

- (a) Use this to find the probability of getting more than 90% heads if the probability of getting a head is 0.7 in  $n$  tosses where  $n$  is very large. Leave your answer in terms of  $\Phi$ .
- (b) In general, if  $X$  is binomial random variable with  $n$  very large, what do you think the following probability should approximately be in terms of  $\Phi$ ?

$$P \left\{ a < \frac{X - np}{\sqrt{np(1-p)}} < b \right\}$$