Exercise sheet 8

Probability and Statistics, MTH102

1. Let X denote a normal random variable with mean μ and variance σ^2. Remember that $\Phi(x)$ is the cumulative function for the *standard* normal distribution. Prove that $F_X(x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$

2. Let X denote a normal random variable with mean μ and variance σ^2. Show that
   (a) $P\{X > \mu + \sigma\} = 1 - \Phi(1)$
   (b) $P\{\mu < X < \mu + \sigma\} = \Phi(1) - \Phi(0)$
   (c) $P\{\mu - \sigma < X < \mu\} = \Phi(0) - \Phi(-1)$
   (d) $P\{\mu - 2\sigma < X < \mu - \sigma\} = \Phi(2) - \Phi(1)$
   (e) $P\{X < \mu - 2\sigma\} = \Phi(-2)$
   (f) In general, derive a formula for $P\{\mu - a\sigma \leq X \leq \mu + b\sigma\}$ in terms of Φ and observe that it does not depend on the parameters μ or σ.

3. If the average total marks of a particular class is 50%, and the standard deviation is 20%, then assuming that it is modelled by a normal distribution, estimate the percentage of students who will get an A grade if the cutoff for an A is 95%.

4. Show that if the cutoff for an A is set at the mean plus standard deviation for any course, then assuming that it is modelled by a normal distribution, the percentage of students earning an A will always be around $1 - \Phi(1)$

5. Consider a random variable with probability density function $f_X(x) = \lambda e^{-\lambda x}$ when $x \geq 0$ and 0 when x is negative.
   (a) Show that f_X is a valid density function
   (b) Find $\mathrm{E}[X]$ and $\mathrm{Var}(X)$
   (c) Show that X satisfies he property that $P\{X > a + b | X > a\} = P\{X > b\}$. That is that assuming that we know that X is bigger than a, then the probability that X is bigger than $a + b$ is the same as the probability that X is bigger than b.
   (d) Assume that the probability that a certain component of a laptop fails after a certain time follows an exponential distribibition with an average of 7 years. What is the probability that that component of your laptop will last longer than 1 *more* year? Note that because of the previous part, you need not know when the component was made.

(e) Assume that the duration a person spends on the phone follows an exponential probability distribition with an average of 20 minutes. You walk into a room and find someone on their phone (you do not know when they started). What is the probability that they will spend at least 2 minutes more on their phone?

6. Let X denote the uniform distribution on the interval (0, 1). Let Y denote the random variable satisfying $Y = X^3$. Find a probability density function for Y. (*Hint:* You will need to use the cumulative function and its interpetation as a probability)

7. It is known that for very large n the binomial distribution is approximated by the normal distribution with the same mean and variance (Unlike the Poisson, we do not need p to be relatively small, and also unlike Poisson this time it is continuous).

   (a) Use this to find the probability of getting more than 90% heads if the probability of getting a head is 0.7 in n tosses where n is very large. Leave your answer in terms of Φ.

   (b) In general, if X is binomial random variable with n very large, what do you think the following probability should approximately be in terms of Φ?

$$P\left\{a < \frac{X - np}{\sqrt{np(1-p)}} < b\right\}$$