Exercise sheet 6

Probability and Statistics, MTH102

1. For a random variable X which only takes positive values, show that $aP\{X \geq a\} \leq E[X]$ (*Hint*: why is $E[X] = \sum x_i p(x_i) \geq \sum_{i \text{ so that } x_i \geq a} ap(x_i)$? How do you use that?)

2. Note that if $|X - E[X]| \geq k$ then $(X - E[X])^2 \geq k^2$.

   (a) Use this observation and the previous exercise to show that $P\{|X - E[X]| \geq k\} \leq \frac{\mathrm{Var}(X)}{k^2}$

   (b) Show that if the variance is 0, then the probability that X is exactly the expectation is 1. (*Hint*: Show that the probability that the $|X - E[X]|$ is bigger than any non-zero number (however small!) is 0.)

3. Let X_1, X_2, $\ldots X_n$ denote independent and identical distributions. Since they are identical, $E[X_1] = E[X_2] = \cdots = E[X_n]$, and let us denote this common expectation by μ. Similarly, $\mathrm{Var}(X_1) = \mathrm{Var}(X_2) = \cdots = \mathrm{Var}(X_n)$, and we denote the common variance by σ^2. Use the above to show that $P\{|\frac{X_1+X_2+\cdots+X_n}{n} - \mu| \geq k\} \leq \frac{\sigma^2}{k^2 n}$

4. Someone claims that a particular coin is biased toward getting a head with probability 0.7. Suppose you allow an error of at most 0.01, apply the previous exercise to estimate the number of times you should toss the coin to be 95% sure that the claim is true. (*Hint:* Assuming that the probability of getting a head is 0.7, what should the expected value be? If you performed the experiment n times, and the average number of heads differed from 0.7 by at most 0.01, what does the previous question tell you about its probability?)