Exercise sheet 5

Curves and Surfaces, MTH201

1. Compute $\ddot{\mathbf{N}}_s(t)$ as a linear combination of $\mathbf{T}(t)$ and $\mathbf{N}_s(t)$.

2. For a regular plane curve parametrized by $\gamma(t)$, the curve parametrized by $\gamma_c(t) := \gamma(t) + c\mathbf{N}_s(t)$ for some fixed number c, is said to be "parallel to the curve parametrized by $\gamma(t)$".

   (a) What is the curve parallel to a circle of radius r?

   (b) Prove that the $\dot{\gamma}_c(t)$ is a scalar multiple of $\dot{\gamma}(t)$.

   (c) Compute the signed curvature of $\gamma_c(t)$ in terms of the signed curvature fuction, $k(t)$, for γ. You will need to assume that $k(t) \neq 1/c$. Hint: Just as in the previous exercise, it may be useful to express $\ddot{\gamma}_c(t)$ in terms of $\mathbf{N}_s(t)$ and $\mathbf{T}(t)$, where $\mathbf{N}_s(t)$ and $\mathbf{T}(t)$ are the unit normal and unit tangent vectors, respectively, of $\gamma(t)$ and compute the coefficients by taking the dot product with appropriate vectors.

3. If a curve parametrized by γ has signed curvature function $\kappa_s(t)$, what is the signed curvature of the curve parametrizaed by $c\gamma(t)$, where c is some constant?

4. Prove that for a space curve parametrized by a *unit-speed parametrization*, $\gamma : (a,b) \to \mathbb{R}^3$, $\mathbf{N}(t) := \frac{\ddot{\gamma}(t)}{\kappa(t)} = \frac{\dot{T}(t)}{\|\dot{T}(t)\|}$ is a unit vector which is orthogonal to the unit tangent vector $\mathbf{T}(t) = \frac{\dot{\gamma}(t)}{\|\dot{\gamma}(t)\|}$. Note, here κ is the curvature and *not* the signed curvature, which only makes sense for plane curves. Note also that all this makes sense only if γ is regular and $\kappa(t) \neq 0$ (it appears in the denominator!)

5. Consider a (plane) curve parametrized by unit speed parametrization $\gamma : (a,b) \to \mathbf{R}^2$ and a point on that curve $p = \gamma(t_0)$. We will find a circle which best approximates the curve at p, in the sense defined below. This will give another perspective on curvature. To solve this exercise, you need to be familiar with using derivatives to find out local maxima or minima.

   (a) Prove that if a circle is tangent to the curve defined by γ at p ("tangent" means that the circle touches the curve and the circle's tangent line and the curve's tangent line are the same at p), then its center must lie on the line containing the vector $\mathbf{N}_s(t)$. For this and the part below you may assume that a normal line of a circle contains its center.

(b) For some real number r, let C_r denote the circle of radius $|r|$, with its center at the point $p + r\mathbf{N}_s(t)$. Why is it tangent to the curve at p? Note that C_r divides the plane into an interior and exterior component and r may be negative, in which case the center is in a direction opposite to $\mathbf{N}_s(t)$.

(c) Prove that a point $\gamma(t)$ avoids the interior component of C_r if and only if $d(t) := \|\gamma(t) - (p + r\mathbf{N}(t))\|^2 \geq r^2$ and avoids the exterior component if and only if $d(t) \leq r^2$ (it always intersects the circle at p, so at t_0 you get r^2). The square is only to allow us to express it as a dot product. Since $d(t)$ always positive, taking the square is harmless.

(d) We say that C_r is too small if, at least in the vicinity of p, every point on the curve defined by γ avoids the interior of C_r, i.e. there is an ϵ so that for any t inside the interval $(t_0 - \epsilon, t_0 + \epsilon)$, $\gamma(t)$ avoids the interior of C_r. Use the previous part to rewrite this in terms of the function $d(t)$, which is defined above. Why does that mean that d has a local minimum at t_0? Remember that a function has a local minimum at t_0 if for t *in the vicinity* of t_0, $f(t) \geq f(t_0)$

(e) We say that C_r is too big if, at least in the vicinity of p, every point on the curve defined by γ avoids the exterior of C_r, i.e. there is an ϵ so that for any t inside the interval $(t_0 - \epsilon, t_0 + \epsilon)$, $\gamma(t)$ avoids the exterior of C_r. Use the previous part to rewrite this in terms of the function $d(t)$, which is defined above. Why does that mean that d has a local maximum at t_0?

(f) Prove that no matter what r is, $d'(t_0) = 0$. (By now you should be in the habit of expressing such derivatives in terms of that orthonormal basis $\mathbf{N}_s(t)$ and $\mathbf{T}(t)$ so that you can easily identify which coefficients cancel).

(g) Remember that a function f has a local maximum at t_0 if $f'(t_0) = 0$ and $f''(t_0) < 0$; it has a local minimum at t_0 if $f'(t_0) = 0$ and $f''(t_0) > 0$. Compute $d''(t)$ and use parts d) and e) above to show that C_r would be too big if $r > 1/\kappa(t_0)$ and too small if $r < 1/\kappa(t_0)$. Therefore, a circle of radius $1/\kappa(t_0)$ may be thought of as best approximating the curve at p. Such a circle is called an *osculating circle* and its radius is $1/\kappa(t_0)$ is called the radius of curvature.

6. Let $\gamma : (\alpha, \beta) \to \mathbb{R}^2$ be a regular *unit speed* parametrization of **space** curves. As with plane curves, we can define $\mathbf{T}(t) := \dot{\gamma}(t)$.

For a plane curve, there are only two unit vectors normal to $\mathbf{T}(t)$ for a given t, but now there are infinitely many (why?). So for space curves, we choose the "normal" in the direction of the acceleration:

(a) Assume that the (ordinary, *not* signed) curvature, $\kappa(t) \neq 0$ for all t. Show that $\mathbf{N}(t) := \frac{\ddot{\gamma}(t)}{\kappa(t)} = \frac{\dot{\mathbf{T}}}{\|\dot{\mathbf{T}}\|}$ is a non-zero unit vector orthogonal to $\mathbf{T}(t)$. Why is $\mathbf{N}(t)$ smooth?

(b) Define $\mathbf{B}(t) := \mathbf{T}(t) \times \mathbf{N}(t)$. Prove that $\mathbf{B}(t)$ is perpendicular to both $\mathbf{T}(t)$ and $\mathbf{N}(t)$. Why is it smooth?

(c) By the previous part, $\{\mathbf{T}(t), \mathbf{N}(t), \mathbf{B}(t)\}$ form an orthonormal basis. So,

$$\dot{\mathbf{T}}(t) = x_T(t)\mathbf{T}(t) + y_T(t)\mathbf{N}(t) + z_T(t)\mathbf{B}(t)$$

for some $x_T(t), y_T(t), z_T(t)$. What are these coefficients, $x_T(t), y_T(t), z_T(t)$? This should be straightforward (follows from part 1).

(d) Similarly,

$$\dot{\mathbf{N}}(t) = x_N(t)\mathbf{T}(t) + y_N(t)\mathbf{N}(t) + z_N(t)\mathbf{B}(t)$$

for some $x_N(t), y_N(t), z_N(t)$. What are the coefficients, $x_N(t)$ and $y_N(t)$? $z_N(t)$ will need to be done later. (*Hint:* As with plane curves, figuring out the coefficient involves taking an appropriate dot product. Sometimes, product rule may help you to shift the derivative and relate it with a known dot product. Remember that unit vector fields are orthgonal to their derivatives!)

(e) Similarly,

$$\dot{\mathbf{B}}(t) = x_B(t)\mathbf{T}(t) + y_B(t)\mathbf{N}(t) + z_B(t)\mathbf{B}(t)$$

for some $x_B(t), y_B(t), z_B(t)$. Show that $x_B(t) = 0$ and $z_B(t) = 0$. In other words $\dot{\mathbf{B}}(t)$ is always a scalar multiple of $\mathbf{N}(t)$ (which is denoted $y_B(t)$ above).

(f) Show that $y_B(t) = -z_N(t)$. So the only two unknown coefficients are negatives of each other! Let us denote $-y_B(t)$ by $\tau(t)$ (the negative sign is only a convention and simplifies some notation later). $\tau(t)$ is a new term that cannot be written in terms of known terms like the curvature etc and is called the "torsion" at t. We have shown that the derivatives of $\mathbf{T}(t)$, $\mathbf{N}(t)$, and $\mathbf{B}(t)$ can be written in terms of the basis $\{\mathbf{T}(t), \mathbf{N}(t), \mathbf{B}(t)\}$ and the coefficients depend only on the curvature or the torsion.

7. Prove that the torsion $\tau(t)$, defined in the previous question, of a curve is the constant 0 if and only if the curve lies on a plane.

8. If $\mathbf{v}_1(t), \mathbf{v}_2(t), \mathbf{v}_3(t)$, denote some vector fields which are *not necessarily* $\mathbf{T}(t), \mathbf{N}(t), \mathbf{B}(t)$, but nevertheless satisfy the same equations:

$$\begin{aligned}
\dot{\mathbf{v}_1} &= \kappa(t)\mathbf{v}_2(t) \\
\dot{\mathbf{v}_2} &= -\kappa(t)\mathbf{v}_1(t) + \tau(t)\mathbf{v}_3(t) \\
\dot{\mathbf{v}_3} &= -\tau(t)\mathbf{v}_2(t)
\end{aligned}$$

Show that $\mathbf{v}_i(t).\mathbf{v}_j(t)$ are constant for any i, j (*Hint:* Product rule, of course!). So, the angles and magnitudes remain the same for all t. We will see the significance of this exercise during the lecture.

9. Let $\mathbf{v}(t)$ denote a unit vector field. Prove that there is always a unit speed parametrization, γ so that $\dot{\gamma}(t) = \mathbf{v}(t)$.

10. Consider two parametrizations, $\gamma_1 : (\alpha, \beta) \to \mathbb{R}^3$ and $\gamma_2 : (\alpha, \beta) \to \mathbb{R}^3$ (note that they have the same domains). Denote the unit tangent, unit normal, and unit binormal of γ_1 by $\mathbf{T}_1(t)$, $\mathbf{N}_1(t)$, and $\mathbf{B}_1(t)$. Similarly, denote the unit tangent, unit normal, and unit binormal of γ_2 by $\mathbf{T}_2(t)$, $\mathbf{N}_2(t)$, and $\mathbf{B}_2(t)$. Assume also that both parametrizations have exactly the same curvature and torsion at t, i.e. $\kappa(t)$ and $\tau(t)$ and t. Show that the expression $\mathbf{T}_1(t).\mathbf{T}_2(t) + \mathbf{N}_1(t).\mathbf{N}_2(t) + \mathbf{B}_1(t).\mathbf{B}_2(t)$ is constant.