Hints / Solutions to Exercise sheet 2

Curves and Surfaces, MTH201

Question 1: For $\mathbf{v} : (\alpha, \beta) \to \mathbf{R}^2$ and $\mathbf{w} : (\alpha, \beta) \to \mathbf{R}^2$, show that $(\mathbf{v}(t).\mathbf{w}(t))' = \mathbf{v}'(t).\mathbf{w}(t) + \mathbf{v}(t).\mathbf{w}'(t)$.
Solution 1:
$\mathbf{v}(t) = (v_1(t), v_2(t))$
$\mathbf{w}(t) = (w_1(t), w_2(t))$
$\mathbf{v}(t).\mathbf{w}(t) = v_1(t)w_1(t) + v_2(t)w_2(t)$
$(\mathbf{v}(t).\mathbf{w}(t))' = (v_1(t)w_1(t))' + (v_2(t)w_2(t))'$ (by definition of differentiating a function to $\mathbb{R}^2$)
So,
$(\mathbf{v}(t).\mathbf{w}(t))' = v_1'(t)w_1(t) + v_1(t)w_1'(t) + v_2'(t)w_2(t) + v_2(t)w_2(t)'$
Rearranging,
$(\mathbf{v}(t).\mathbf{w}(t))' = (v_1'(t)w_1(t) + v_2'(t)w_2(t)) + (v_1(t)w_1'(t) + v_2(t)w_2(t)')$
$(\mathbf{v}(t).\mathbf{w}(t))' = (v_1'(t), v_2'(t)).(w_1(t), w_2(t)) + (v_1(t), v_2(t)).(w_1'(t).w_2(t)')$
$(\mathbf{v}(t).\mathbf{w}(t))' = \mathbf{v}'(t).\mathbf{w}(t) + \mathbf{v}(t).\mathbf{w}'(t)$

Question 2: If $\mathbf{n} : (\alpha, \beta) \to \mathbf{R}^2$ is such that $||\mathbf{n}(t)||$ is constant, then prove that $\dot{\mathbf{n}}(t)$ is either 0 or perpendicular to $\mathbf{n}(t)$.
Solution 2:
This question just generalizes what was seen $n(t).n(t) = C$
Differentiating,
$\dot{n}(t).n(t) + n(t).\dot{n}(t) = 0$
so, $2\dot{n}(t).n(t) = 0$
so, $\dot{n}(t).n(t) = 0$

Question 3: if we denote,

$$s_\alpha(t) := \int_{t_\alpha}^{t} ||\dot{\gamma}(u)||\mathrm{d}u$$

$$s_\beta(t) := \int_{t_\beta}^{t} ||\dot{\gamma}(u)||\mathrm{d}u$$

prove that $s_\beta(t) - s_\alpha(t)$ is a constant **(assume that $t_\alpha < t_\beta$)**.
Solution 3:
This exercise is just saying that if you start measuring the distance traced out by your parametrization at time t_β rather than time t_α, you only need to add

the distance covered from time t_α to time t_β We use the rule that,

$$\int_a^c f(t)\mathrm{d}t = \int_a^b f(t)\mathrm{d}t + \int_b^c f(t)\mathrm{d}t$$

and therefore,

$$\int_a^c f(t)\mathrm{d}t - \int_b^c f(t)\mathrm{d}t = \int_a^b f(t)\mathrm{d}t$$

So,

$$s_\alpha(t) := \int_{t_\alpha}^t ||\dot\gamma(u)||\mathrm{d}u$$

$$s_\beta(t) := \int_{t_\beta}^t ||\dot\gamma(u)||\mathrm{d}u$$

$$s_\beta(t) - s_\alpha(t) = \int_{t_\beta}^t ||\dot\gamma(u)||\mathrm{d}u - \int_{t_\alpha}^t ||\dot\gamma(u)||\mathrm{d}u = \int_{t_\alpha}^{t_\beta} ||\dot\gamma(u)||\mathrm{d}u$$

But the last integral is just a real number and does not depend on t so it is constant with respect to t.

Question 4: If $\gamma : (\alpha, \beta) \to \mathbb{R}^2$ is a smooth **and regular** parametrization, then show that $||\dot\gamma(t)|| : (\alpha, \beta) \to \mathbb{R}$ is smooth.
Solution 4:
We actually need to assume that γ is regular. Let $\gamma(t) = (x(t).y(t))$.
$\dot\gamma(t) = (\dot x(t).\dot y(t))$.
$\|\dot\gamma(t)\| = \sqrt{\dot x^2(t) + \dot y^2(t)}$.

$x(t)$ and $y(t)$ are smooth because that is the meaning of $\gamma(t)$ being smooth. Of course, even their derivatives are smooth, so $\dot x(t)$ and $\dot y(t)$ are smooth.

The squares of smooth functions are smooth, so $\dot x^2(t)$ and $\dot y^2(t)$ are smooth.

The sum of smooth functions is smooth, so $\dot x^2(t) + \dot y^2(t)$ is smooth.

We need to be careful about the square root function. Whenever $x > 0$, then if,

$$f(x) = \sqrt{x}$$

usin the rule for differentiating anything of the form x^n (in this case $x^{1/2}$),

$$f'(x) = \frac{1}{2\sqrt{x}}$$

Note that at $x = 0$, this is undefined, and indeed it is not differentiable at $x = 0$. So we need to ensure that we are taking the square root of something which is strictly positive. But $\dot x^2(t) + \dot y^2(t) > 0$ except when $\dot x(t)$ and $\dot y(t)$ are *both* 0, in which case $\dot\gamma(t) = 0$ for that t, but that cannot happen with a regular parametrization.