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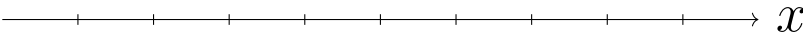
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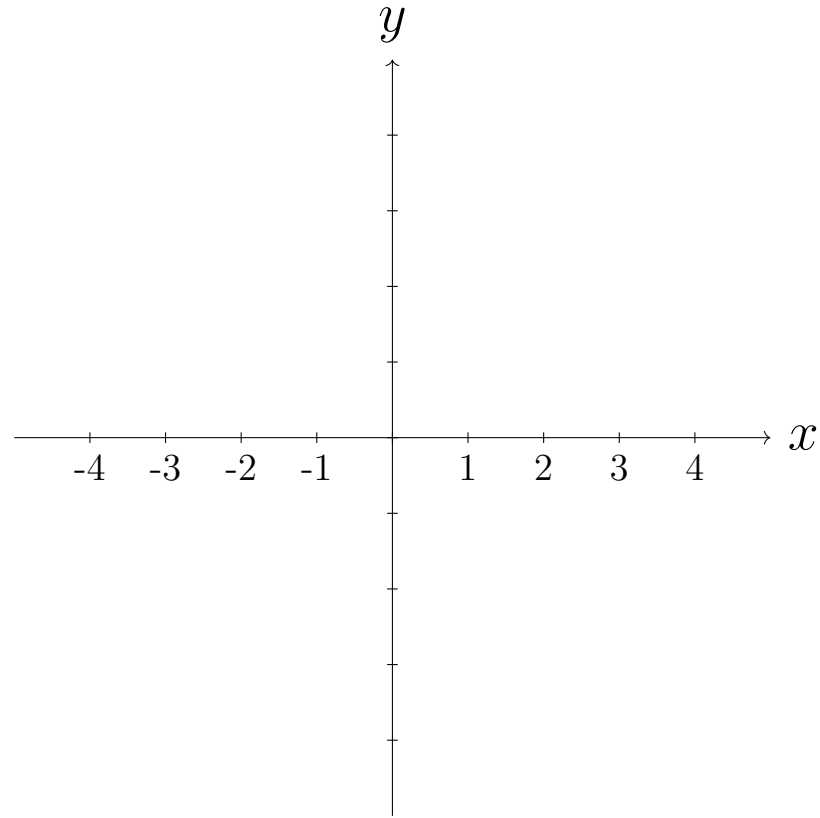
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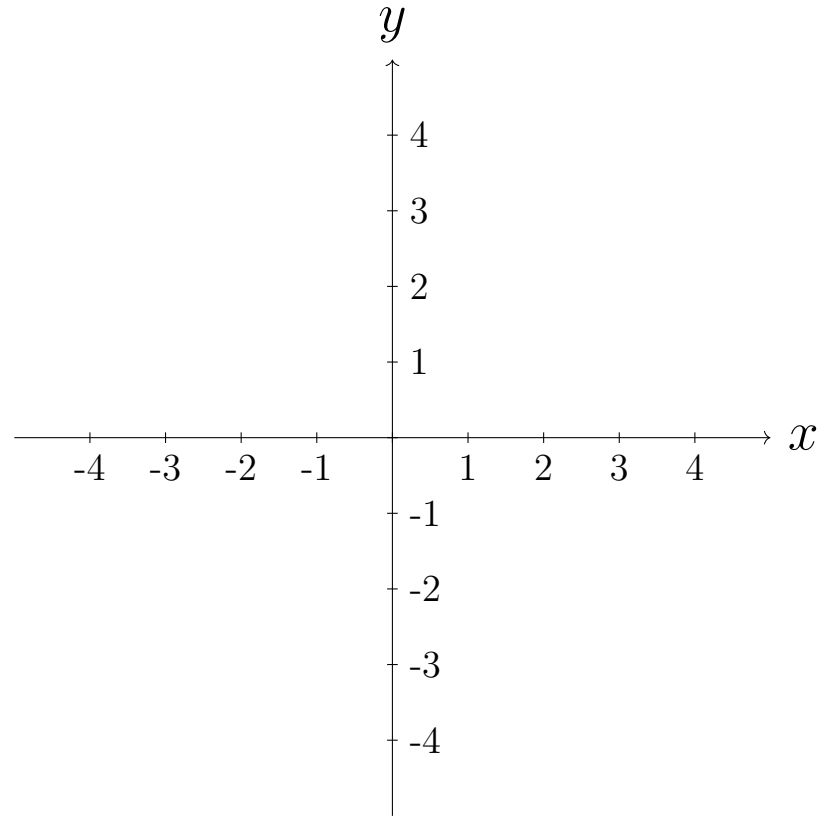
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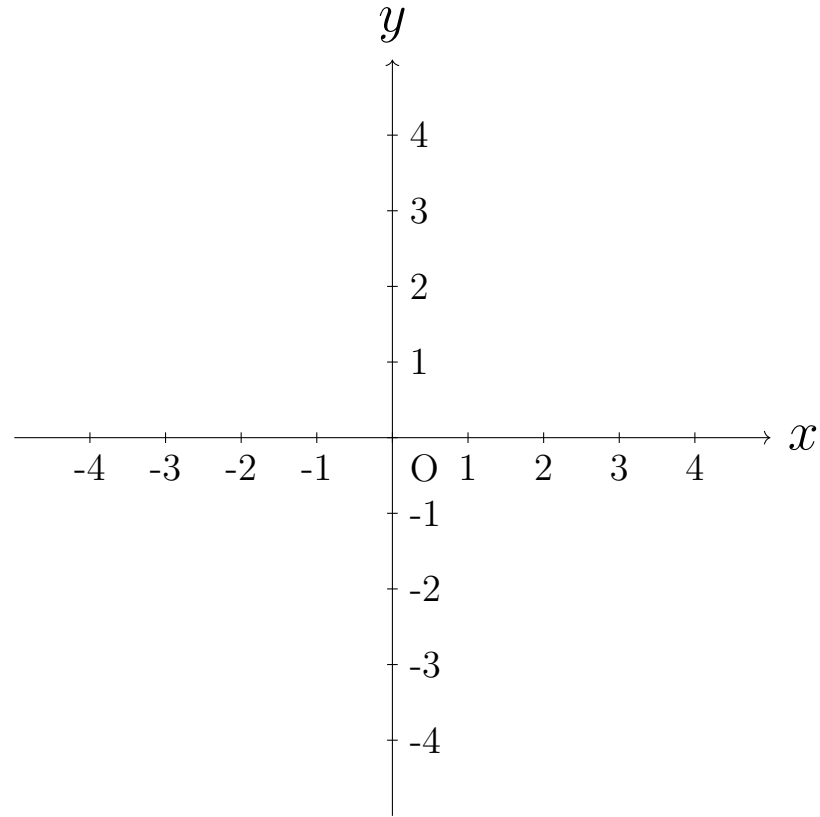




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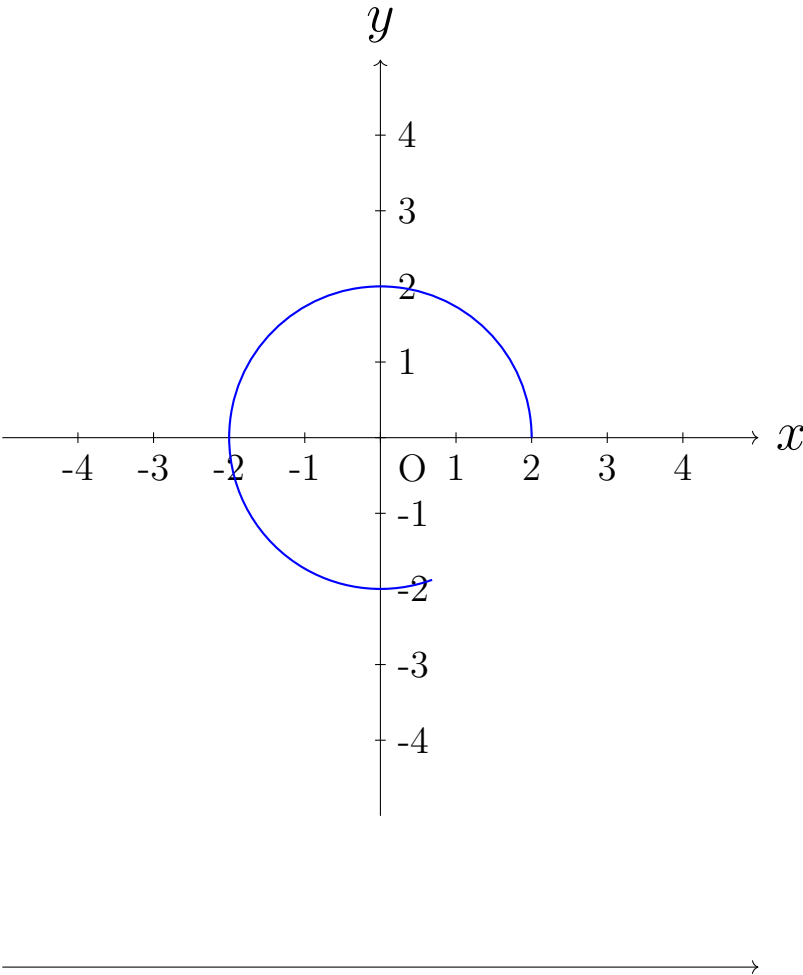
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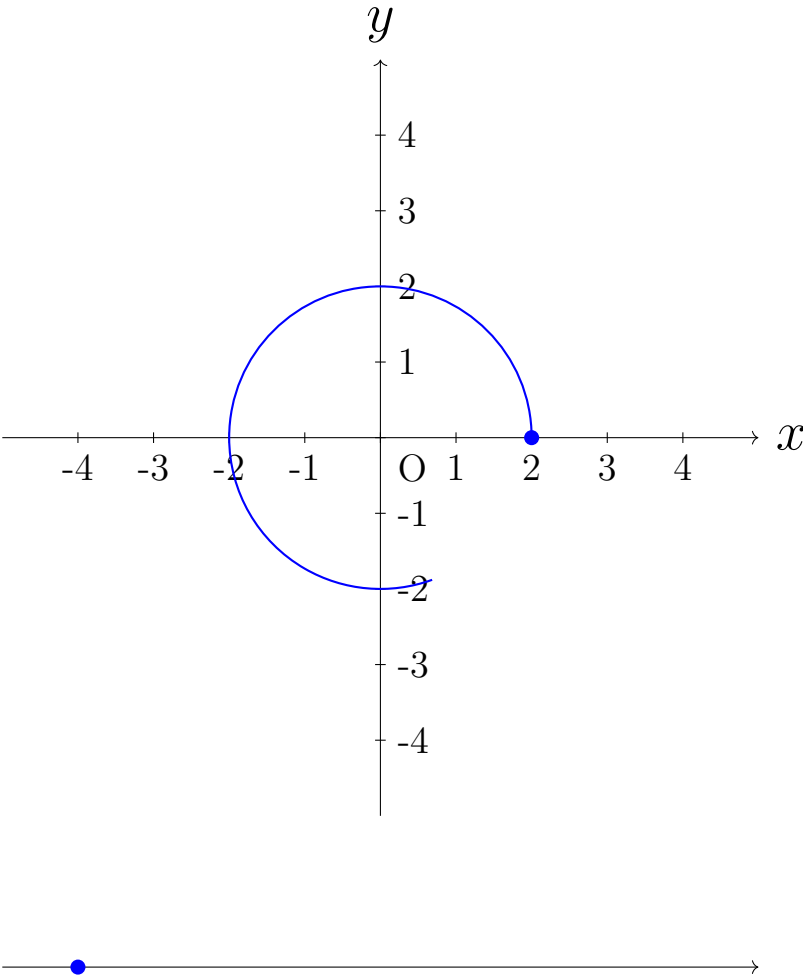
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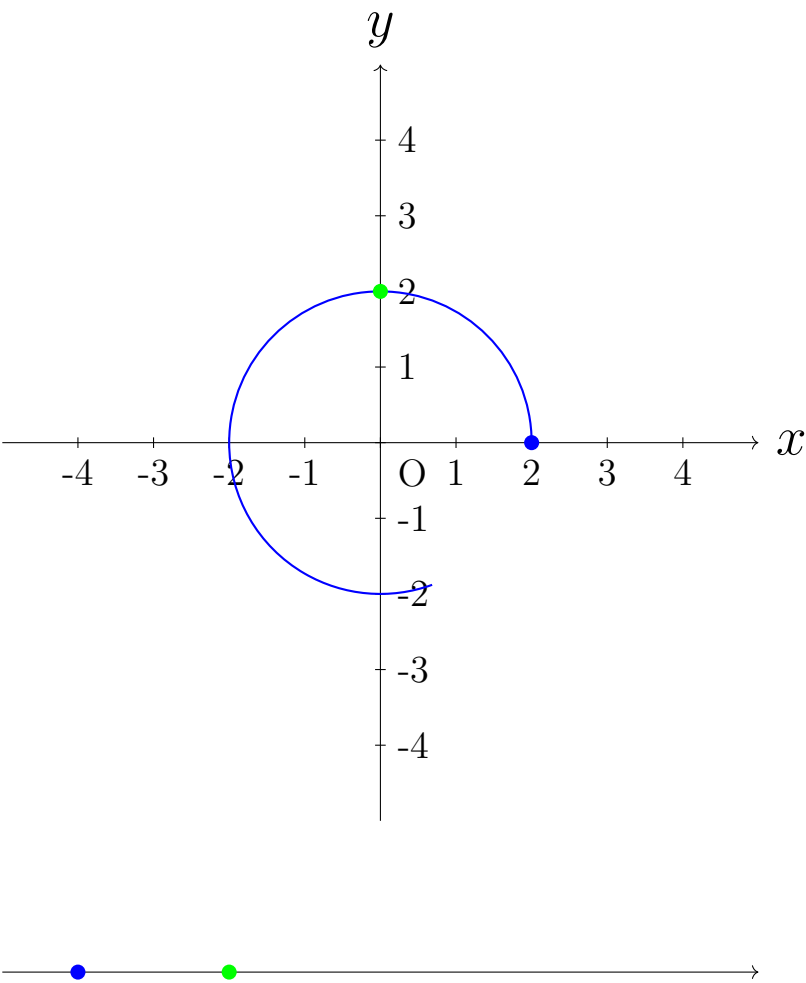
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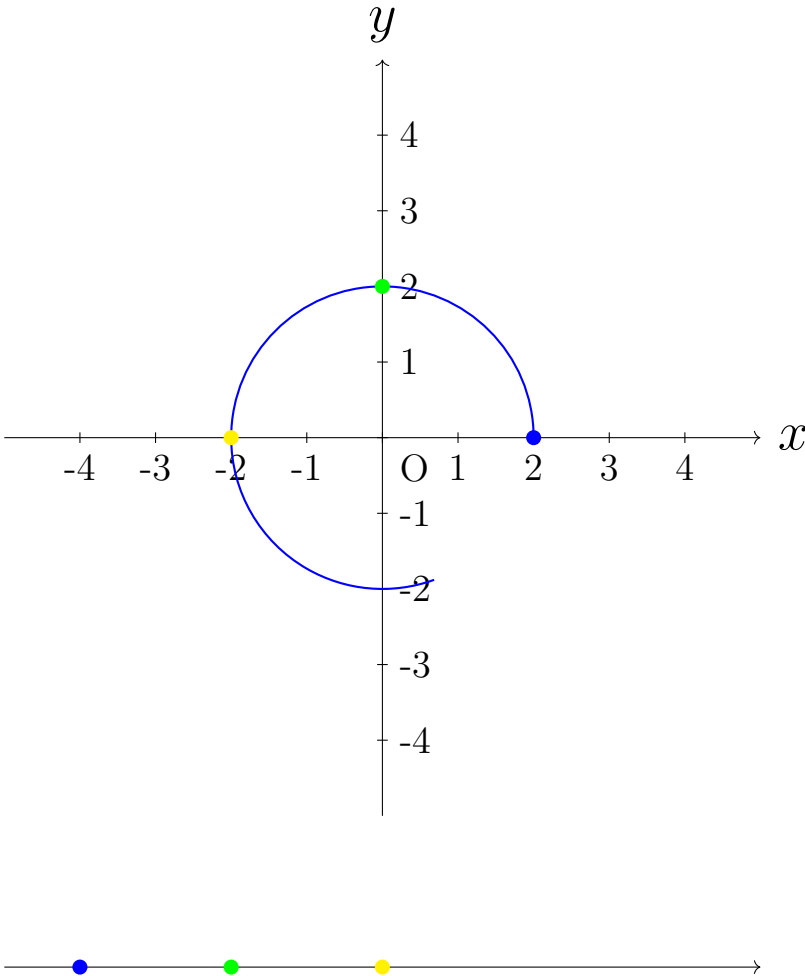
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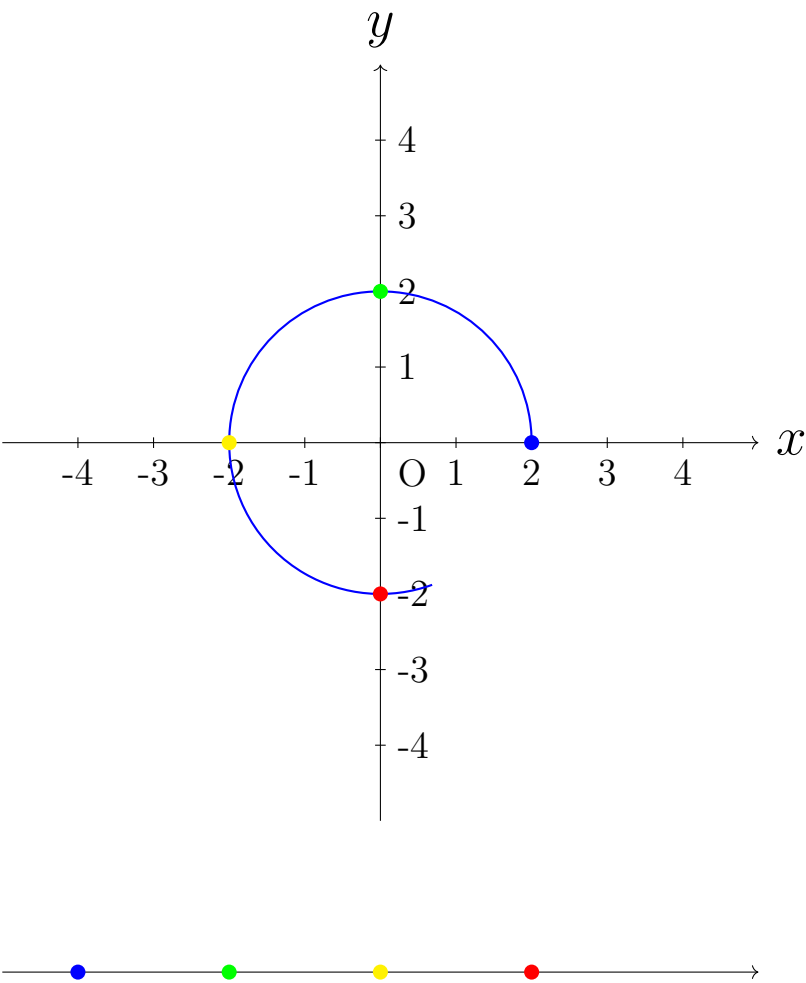
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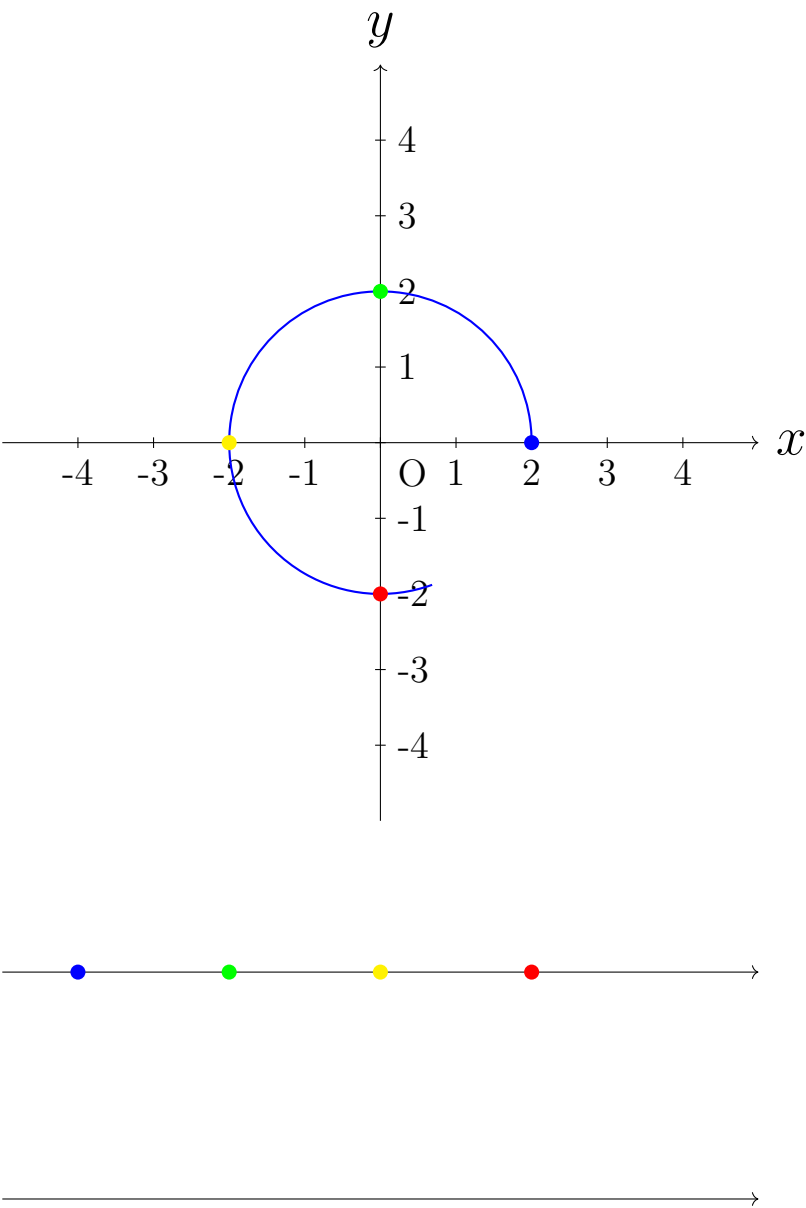
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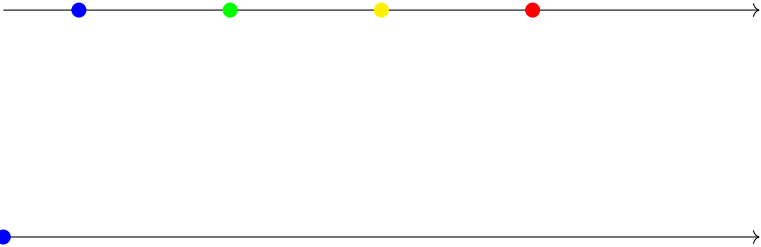
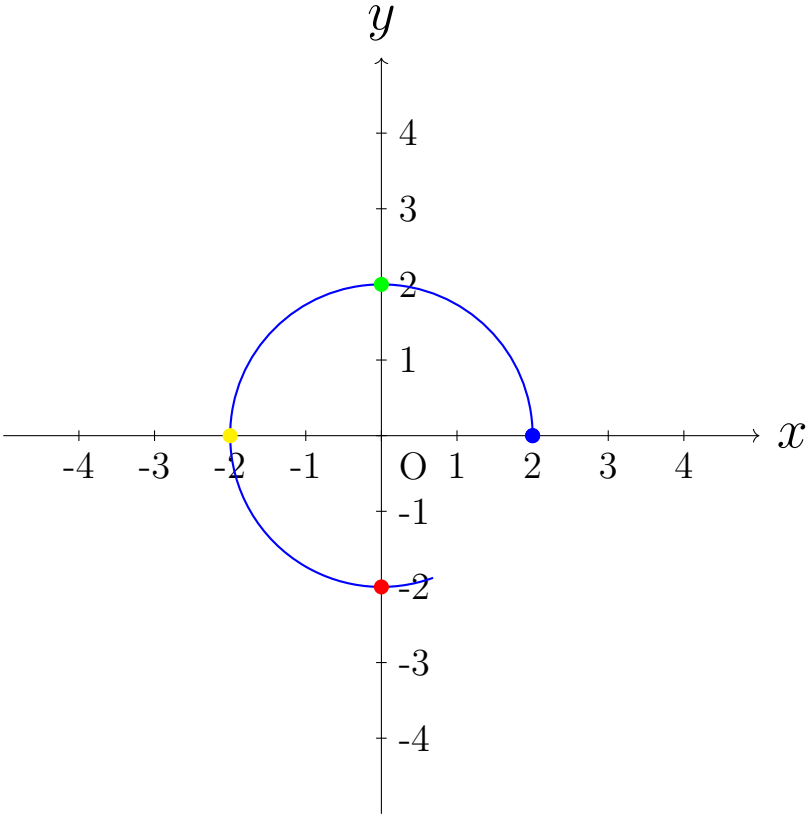
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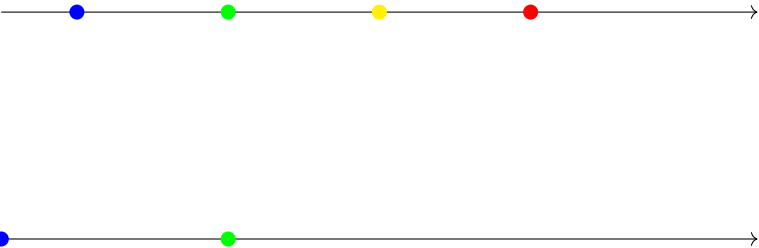
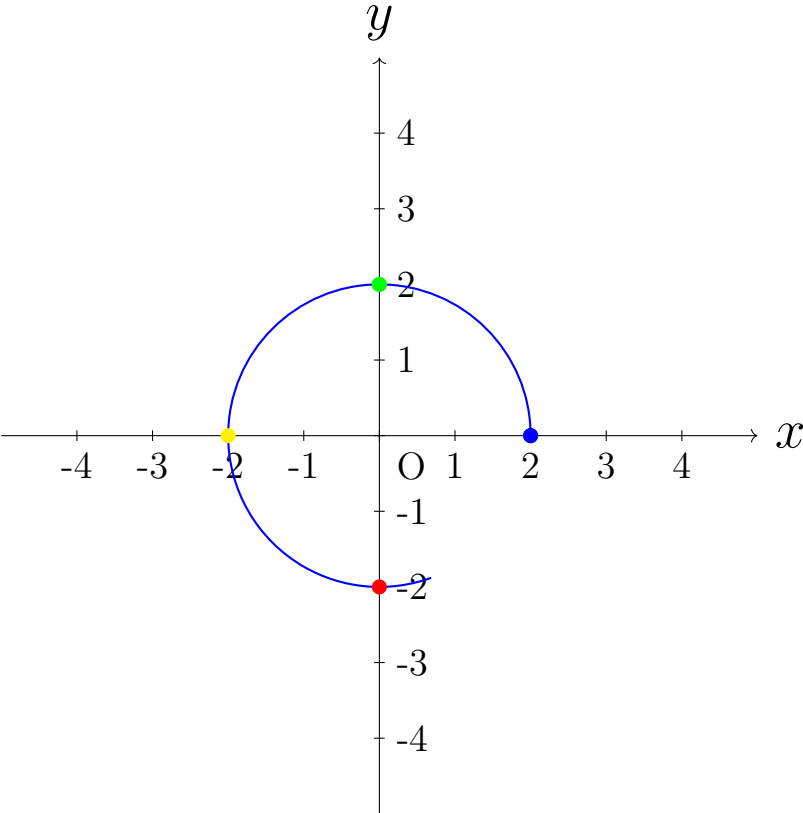




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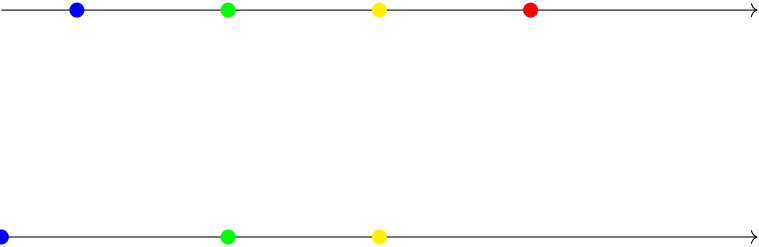
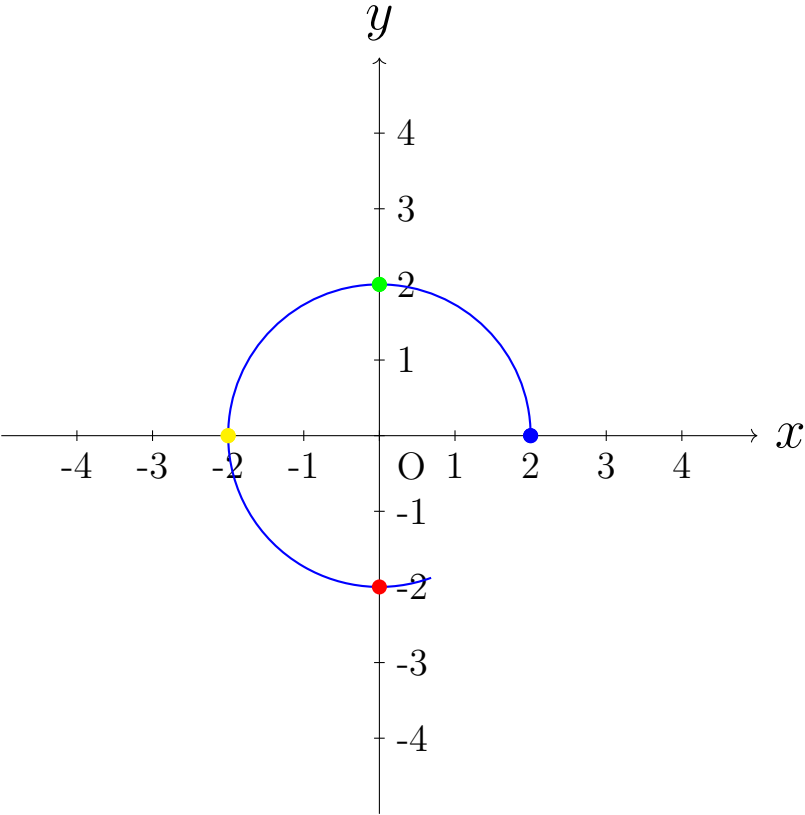
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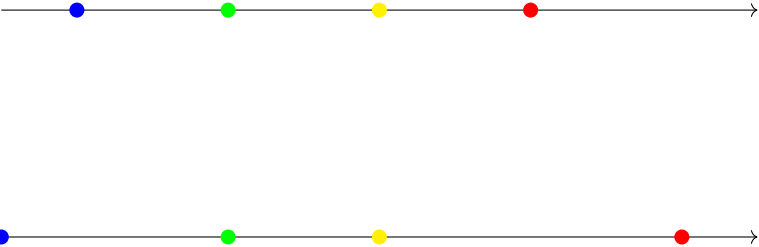
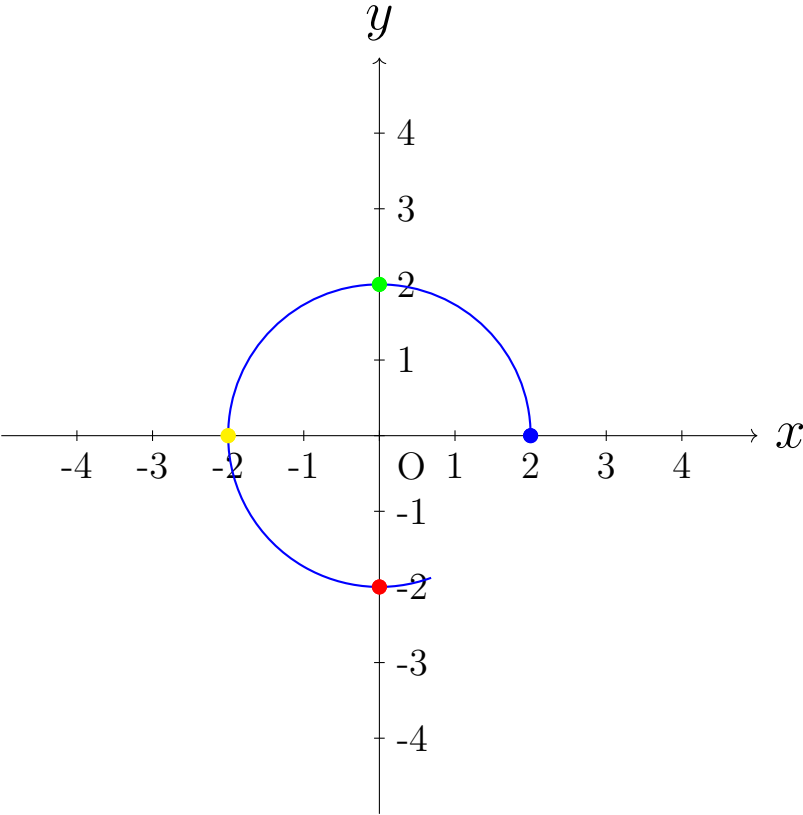
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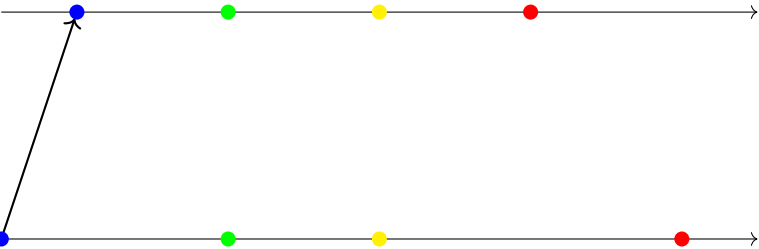
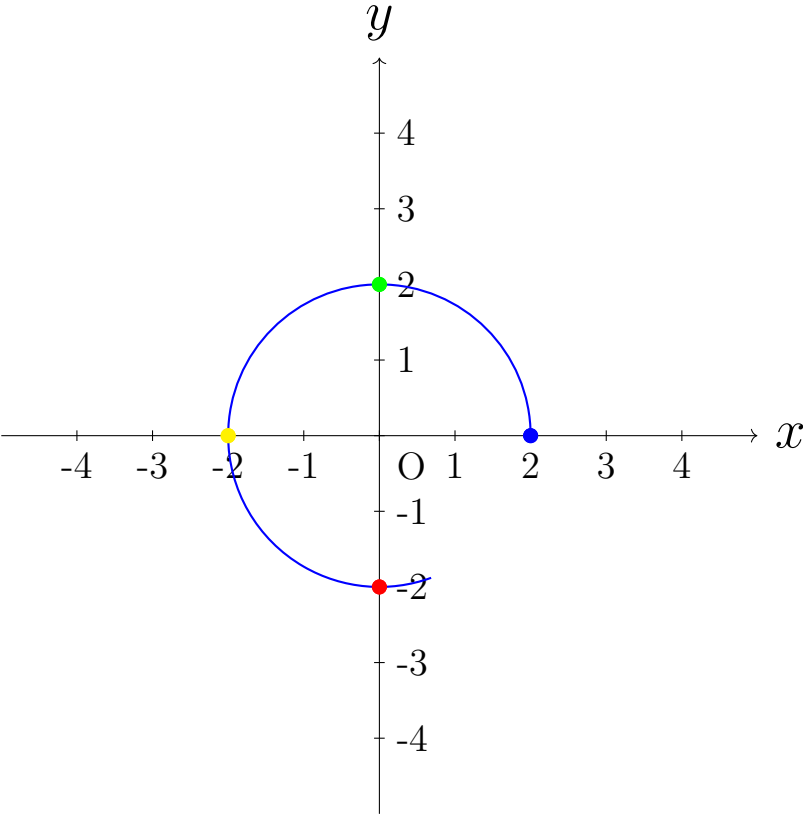
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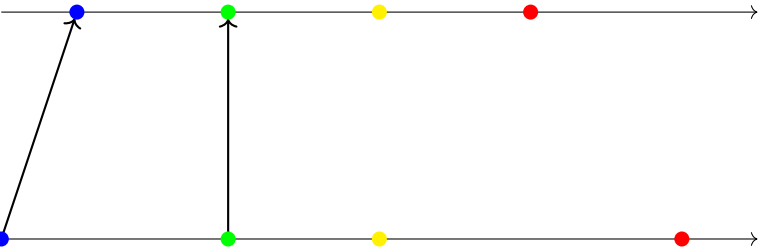
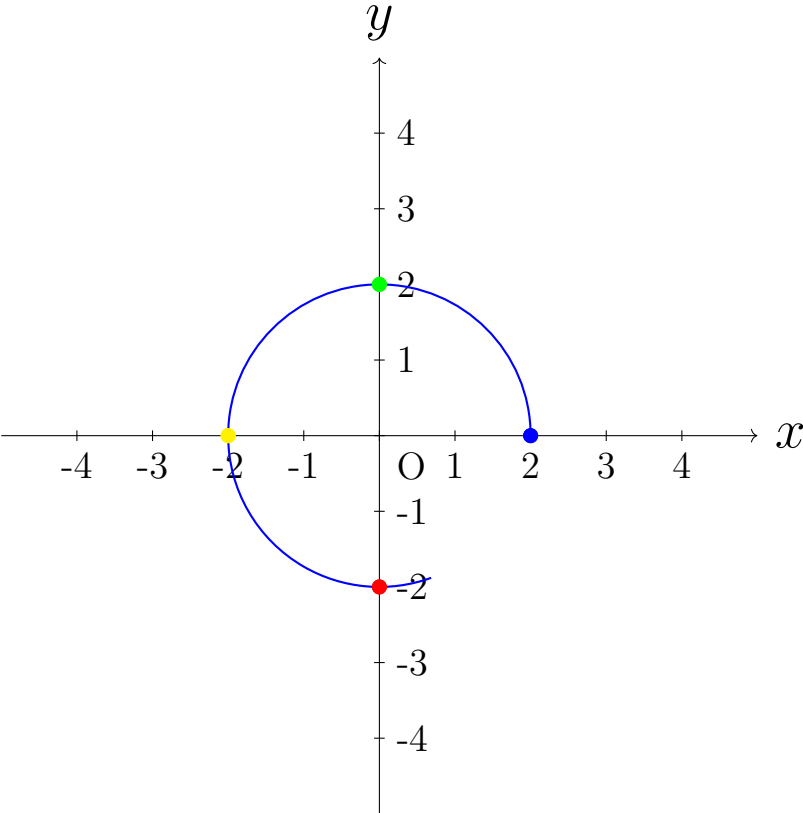
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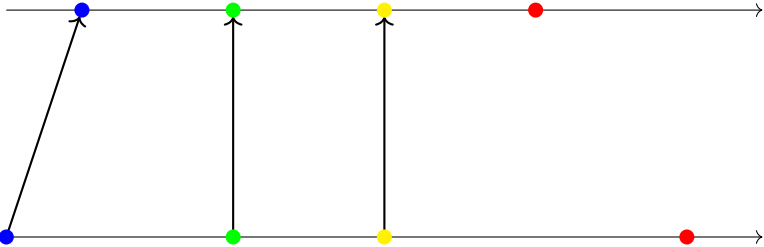
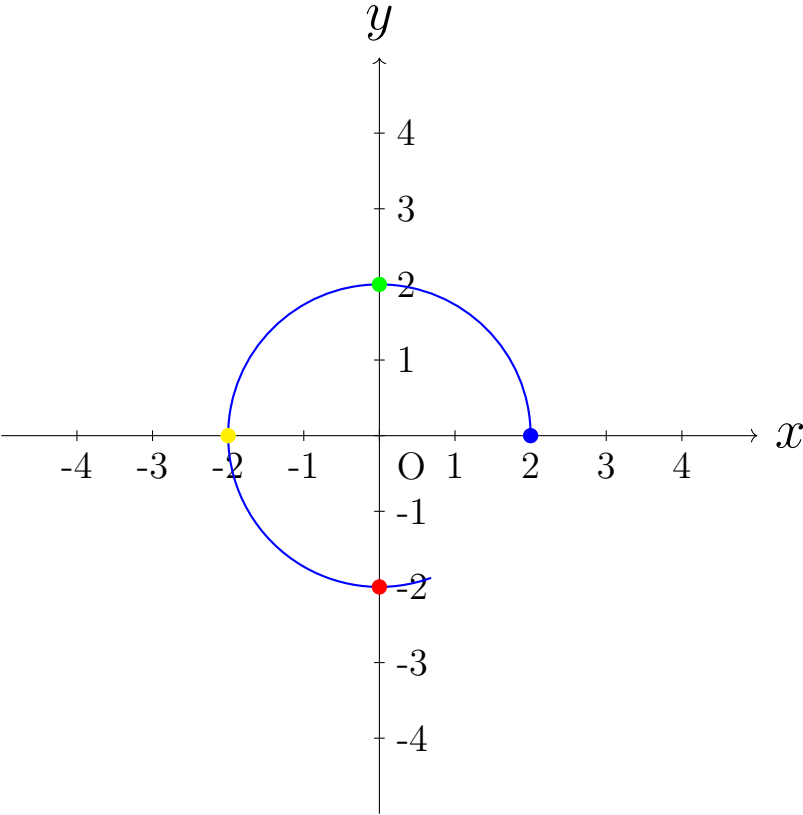
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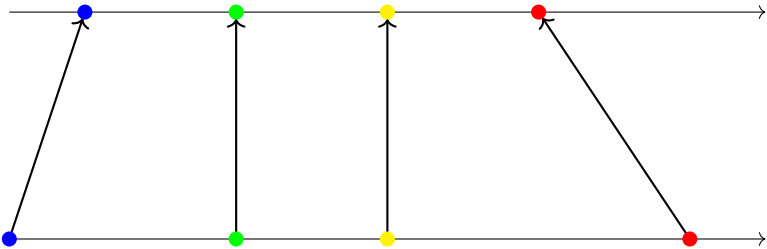
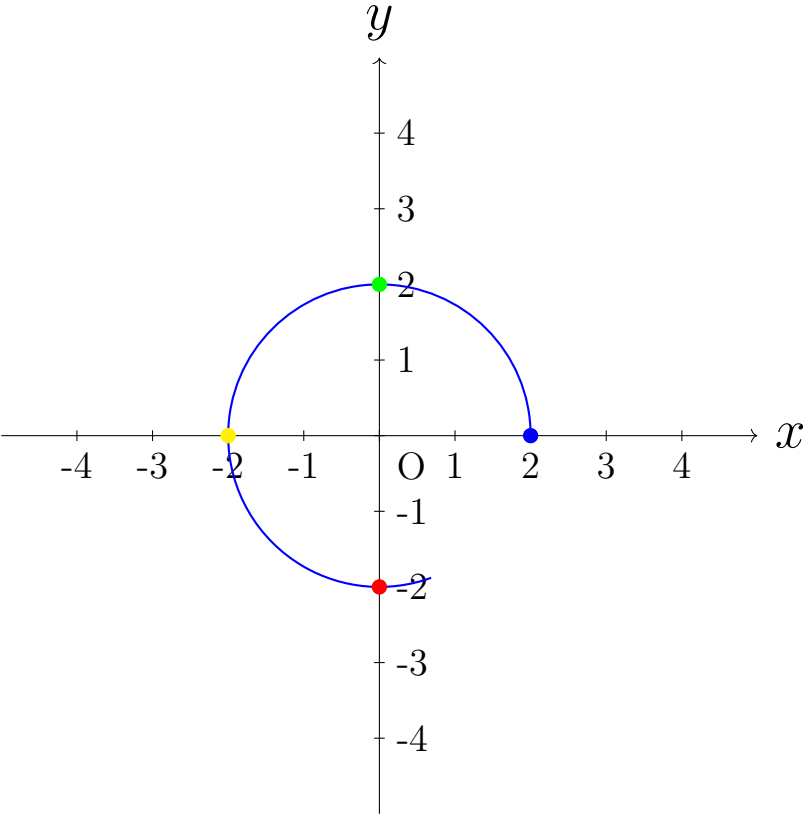
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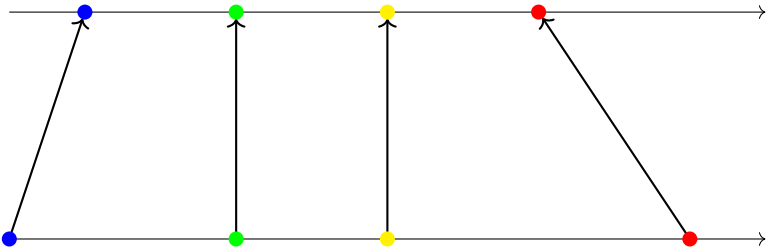
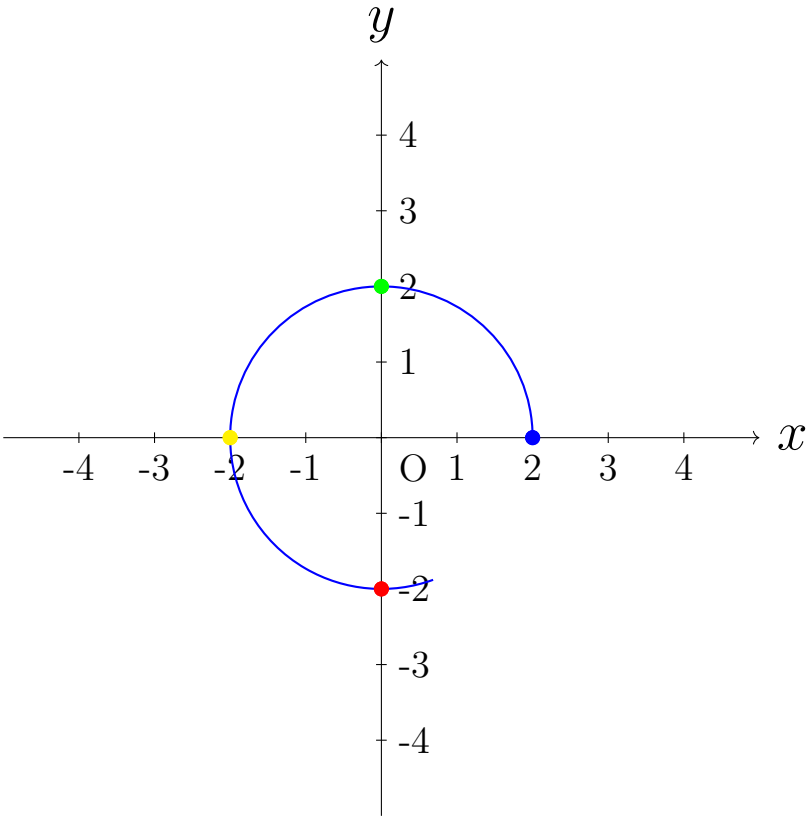
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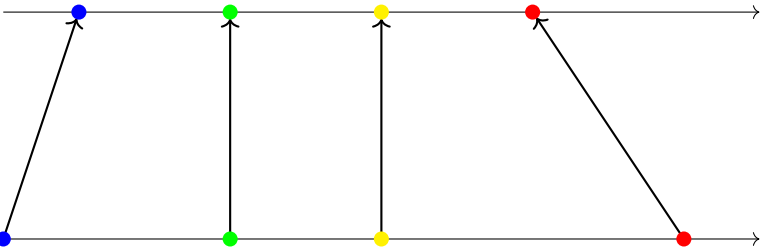
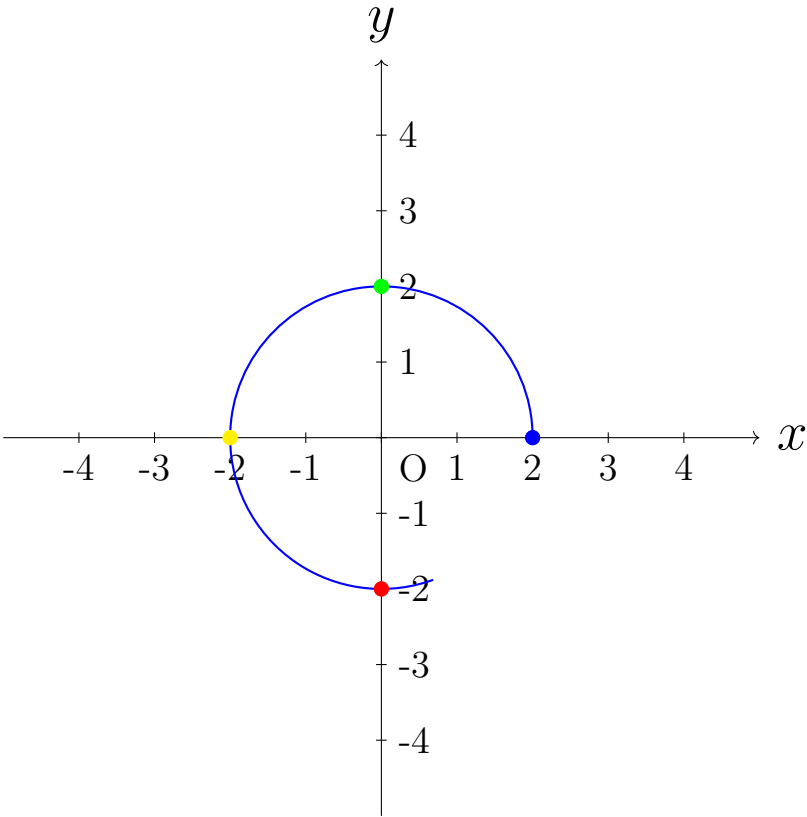
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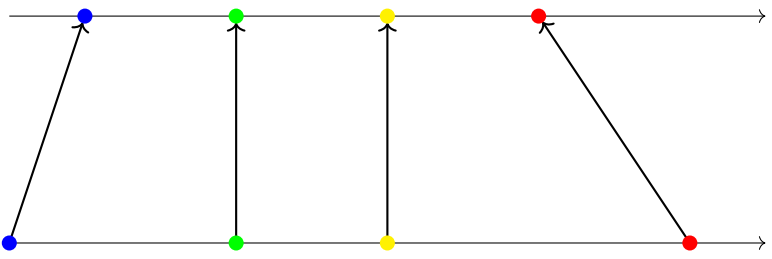
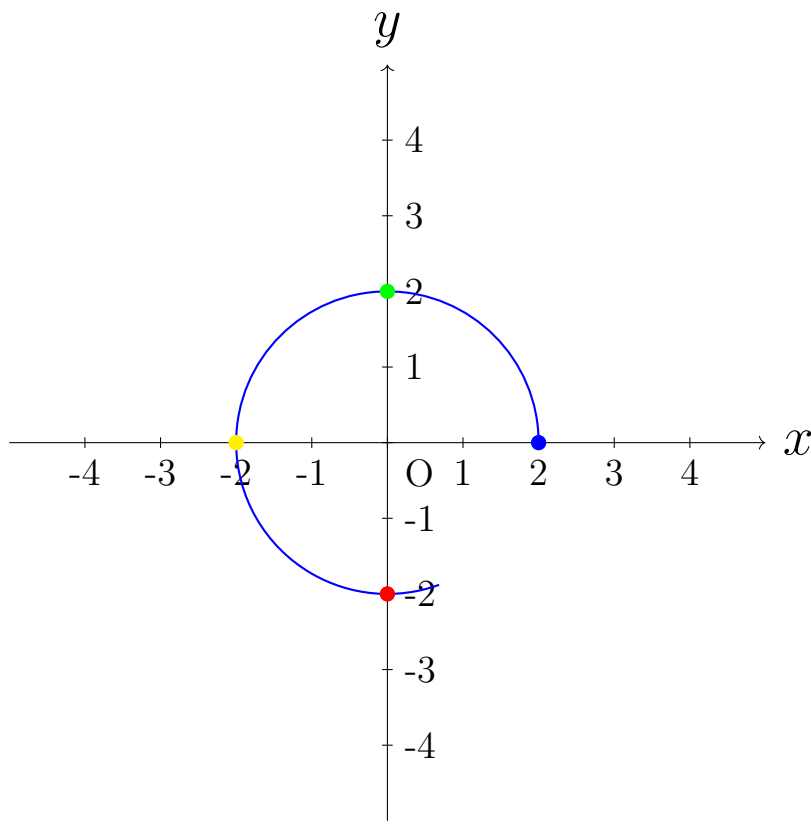
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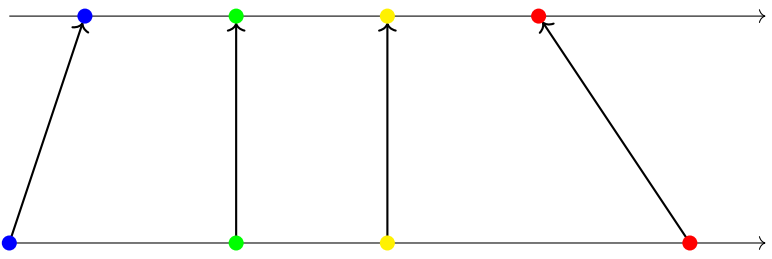
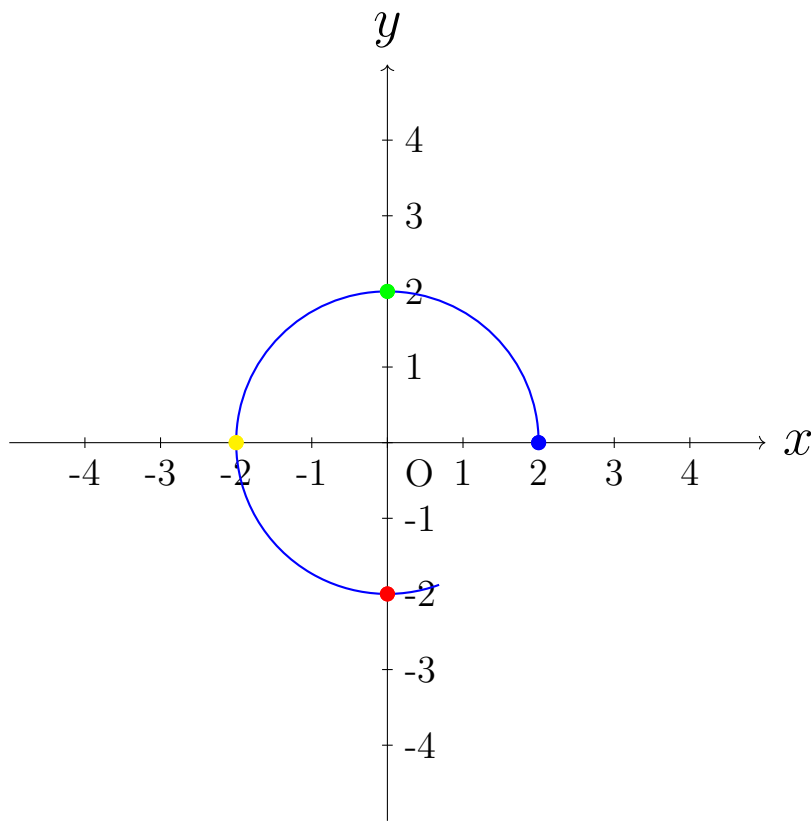
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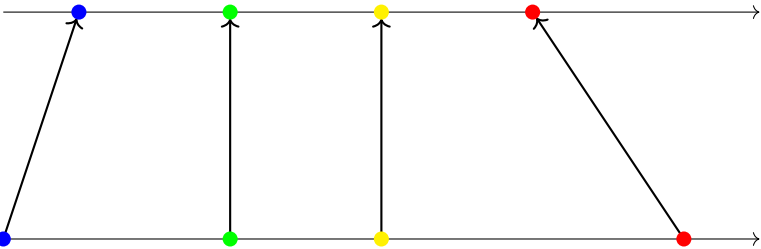
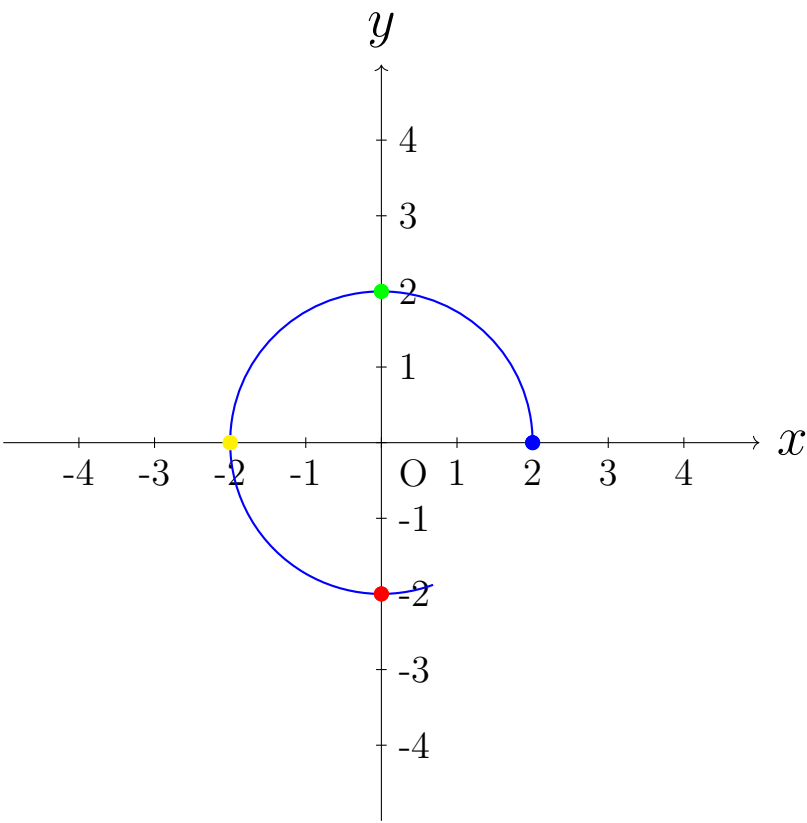
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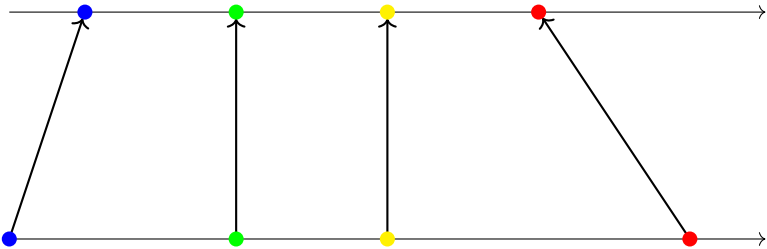
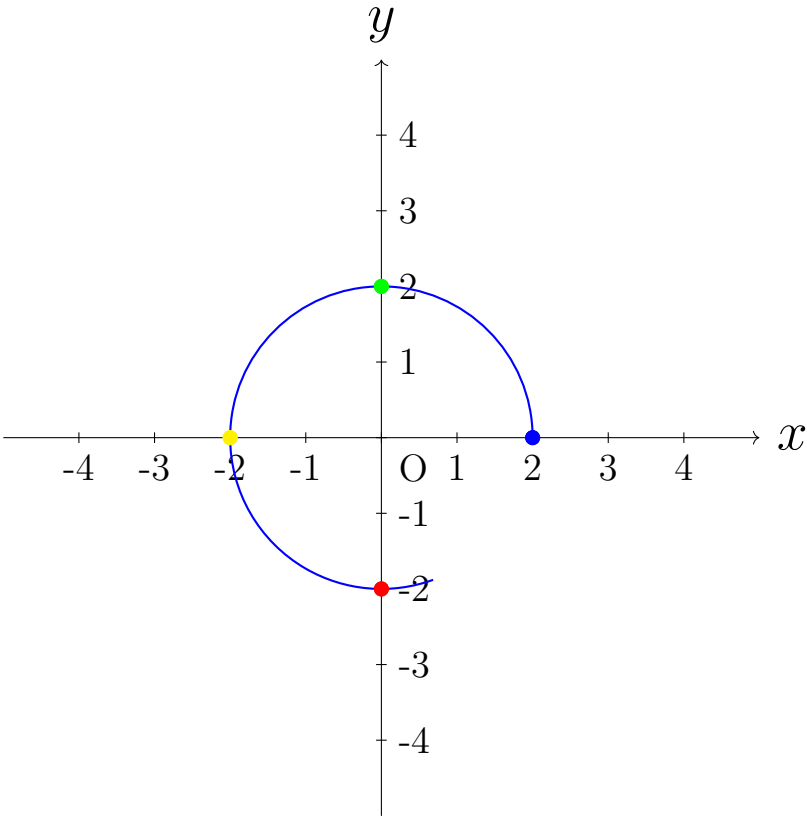
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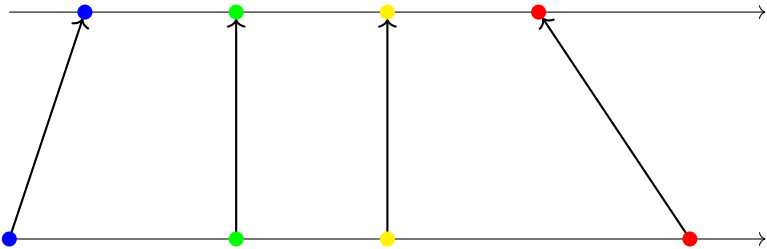
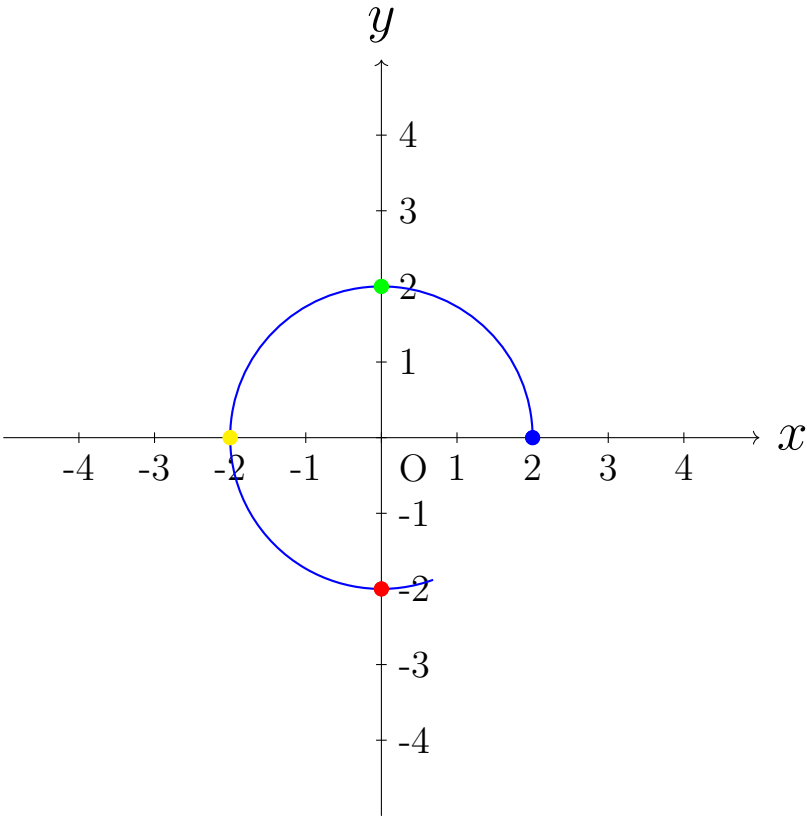
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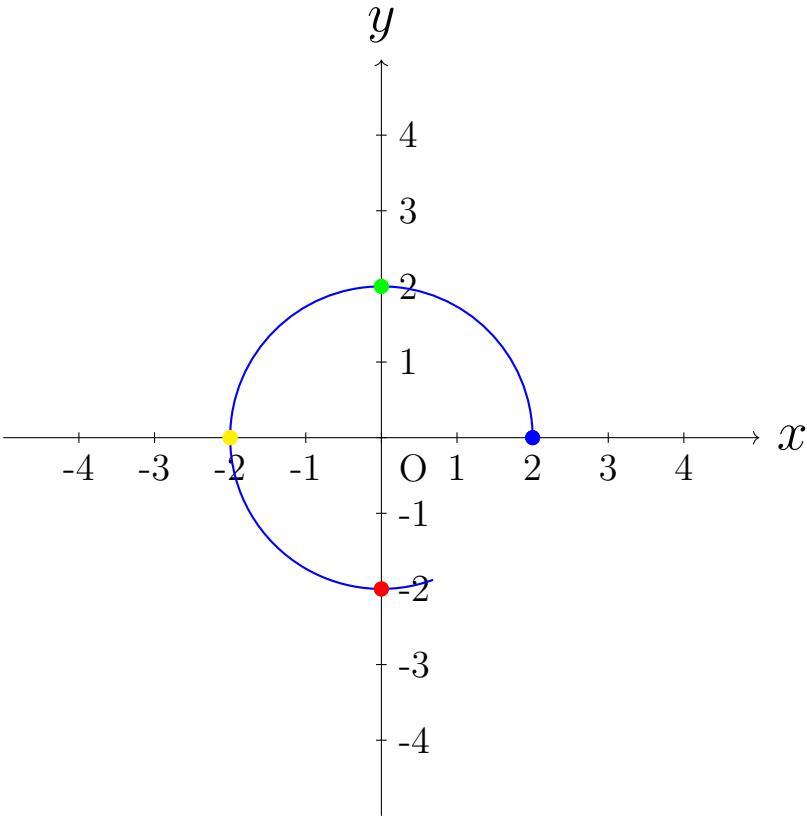
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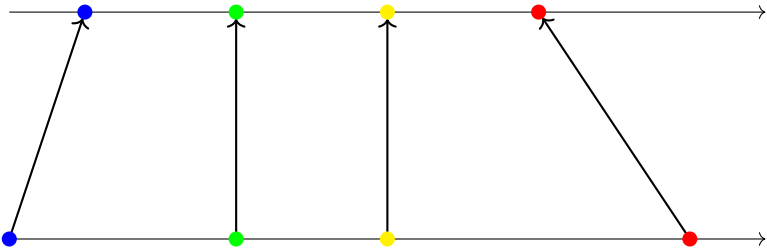
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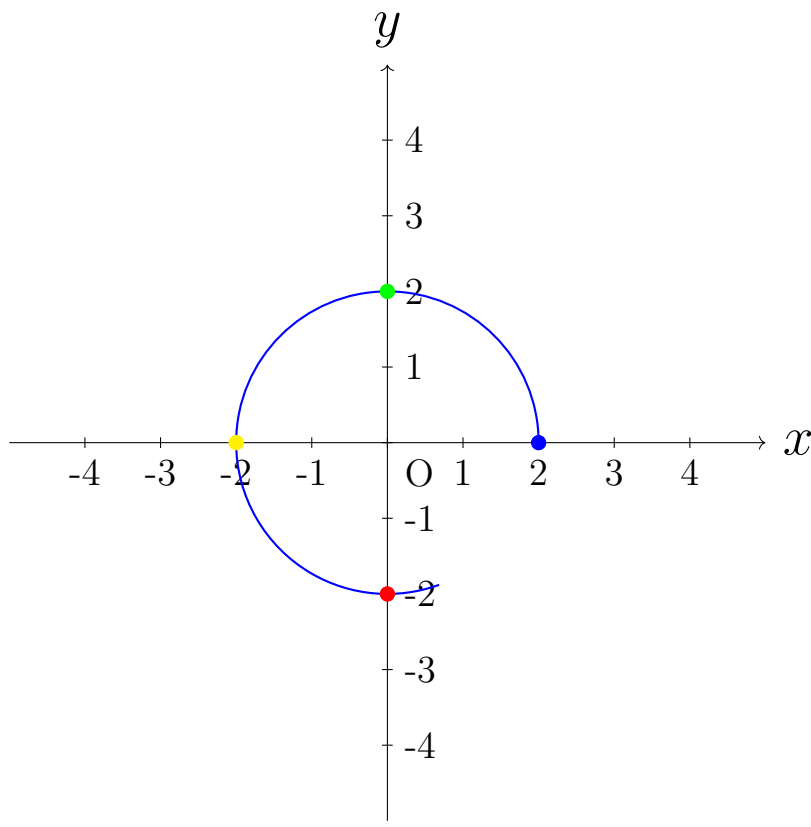
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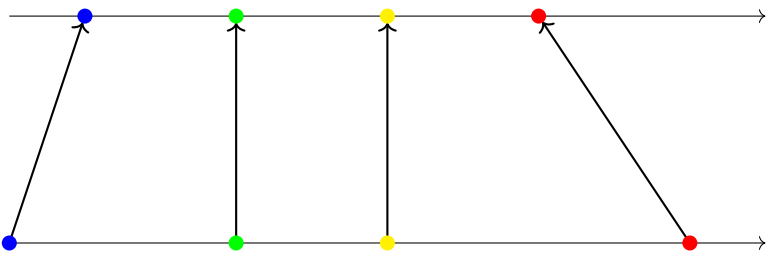
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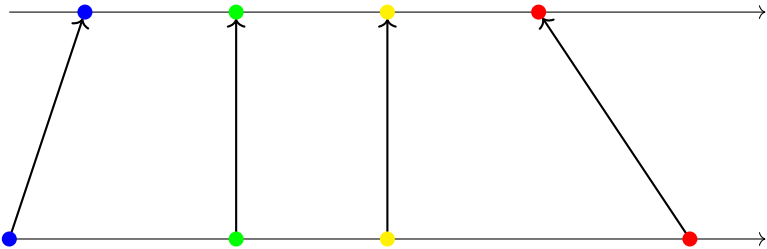
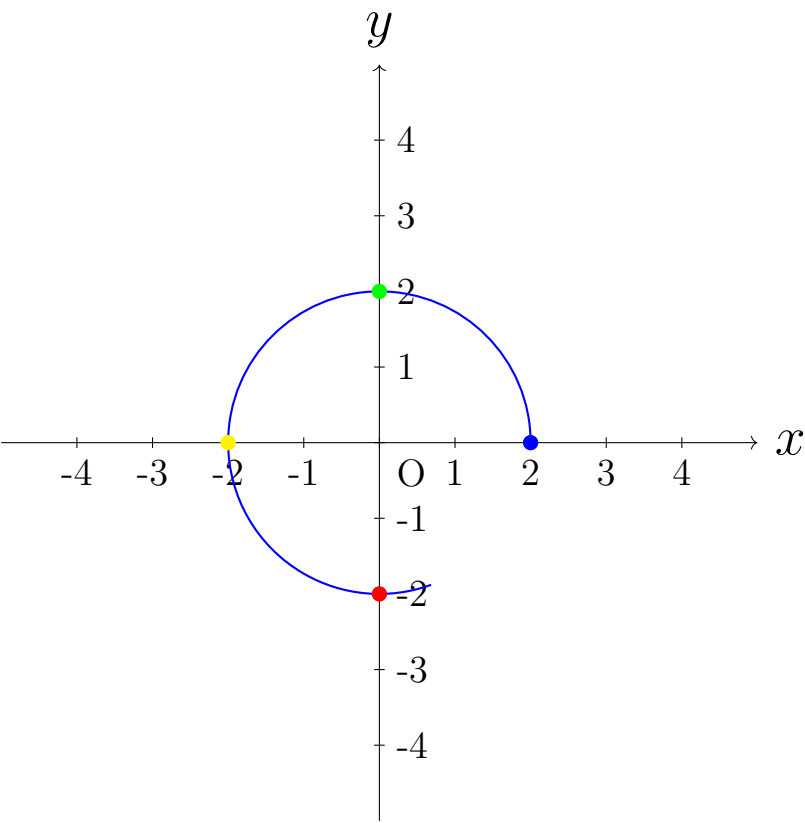
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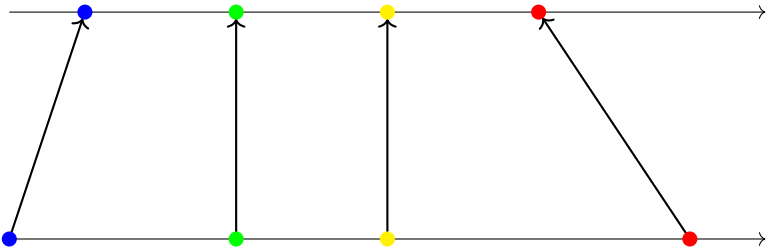
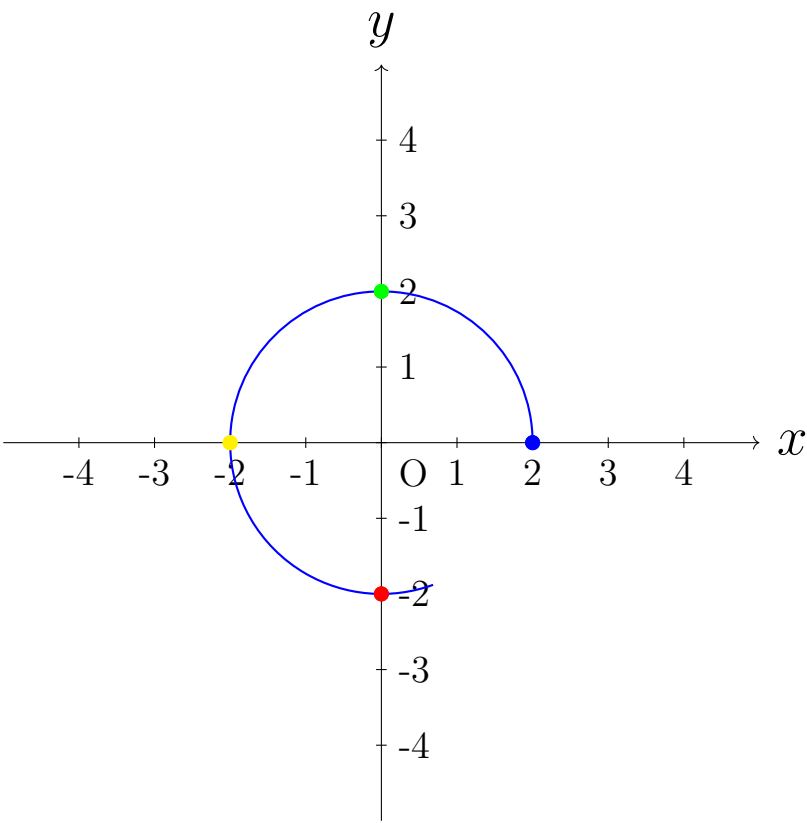
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## Example

$$\gamma : (-1, 1) \rightarrow \mathbb{R}^2$$

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So that  $\tilde{\gamma}(t)$

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$$\phi : (-1/2, 1/2) \rightarrow (-1, 1)$$

$$\phi(t) = 2t$$

$$\text{So that } \tilde{\gamma}(t) = \gamma(\phi(t))$$

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$$\gamma : (-1, 1) \rightarrow \mathbb{R}^2$$

$$\gamma(t) = (t, t)$$

$$\tilde{\gamma} : (-1/2, 1/2) \rightarrow \mathbb{R}^2.$$

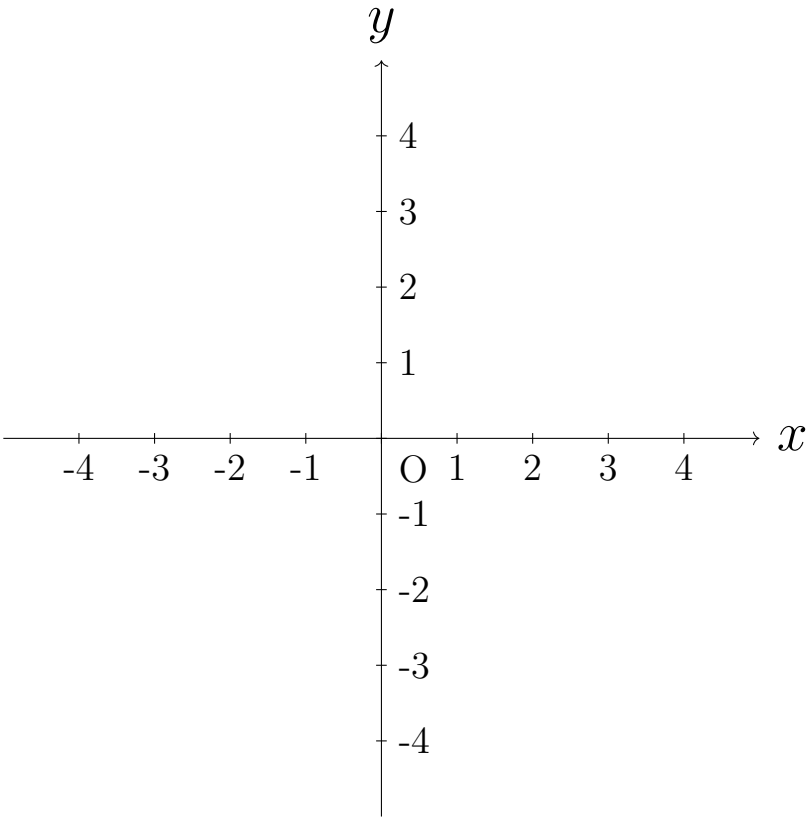
$$\tilde{\gamma}(t) = (2t, 2t)$$

$$\phi : (-1/2, 1/2) \rightarrow (-1, 1)$$

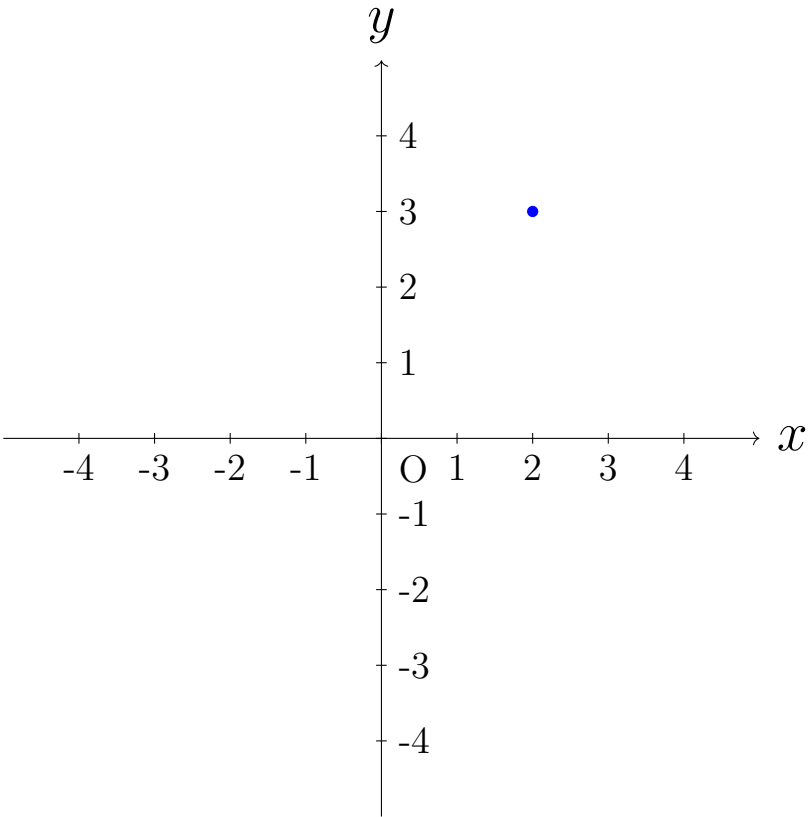
$$\phi(t) = 2t$$

$$\text{So that } \tilde{\gamma}(t) = \gamma(\phi(t)) = \gamma(2t) = (2t, 2t)$$

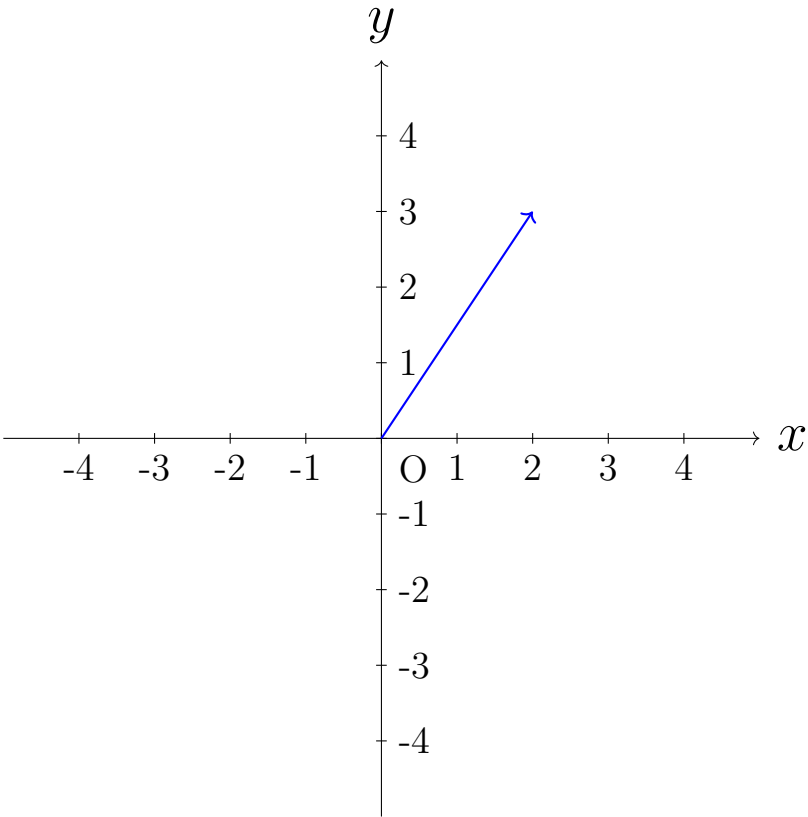
# Vectors



# Vectors

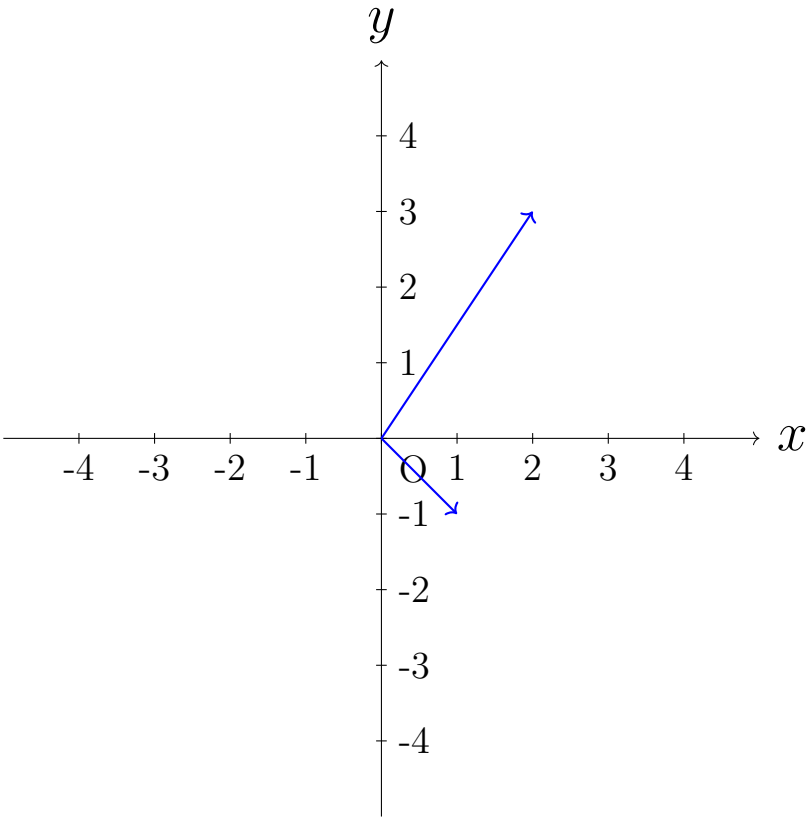


# Vectors



# Vectors

$$v = (2, 3)$$

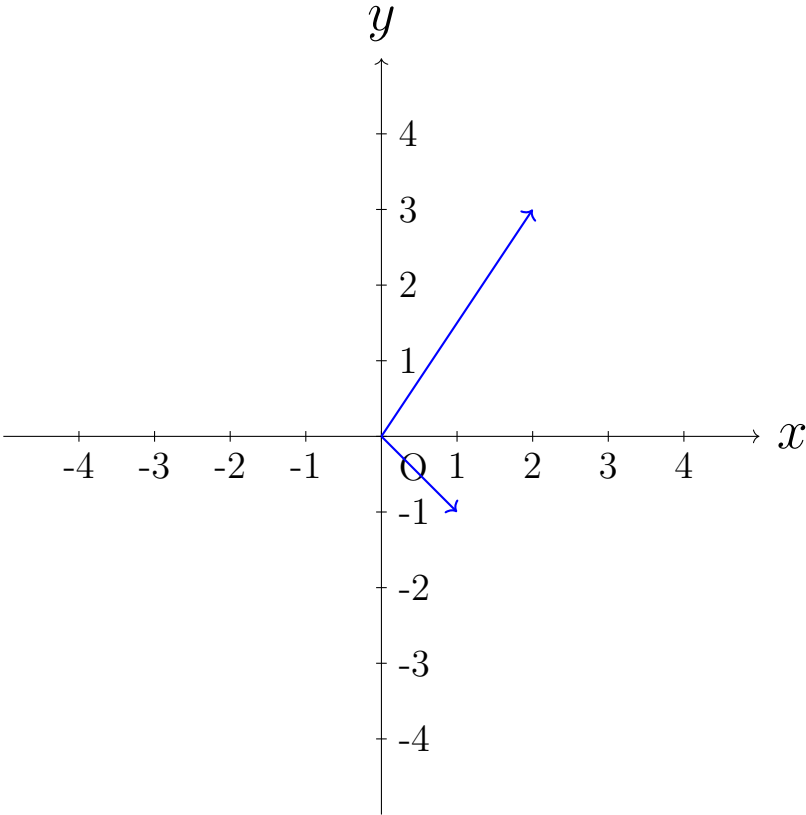




# Vectors

$$v = (2, 3)$$

$$w = (1, -1)$$

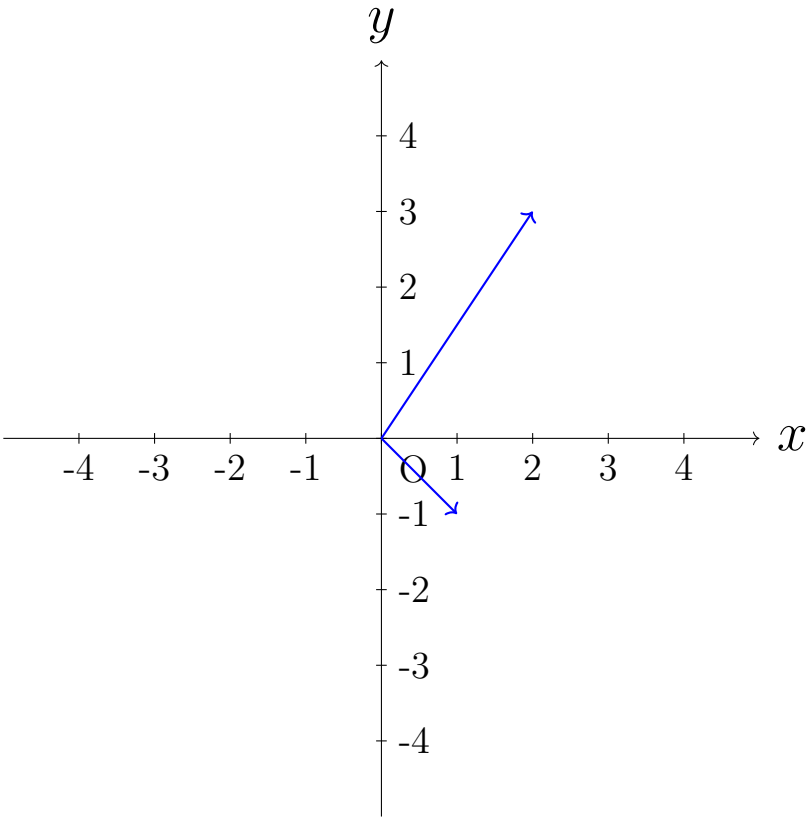


# Vectors

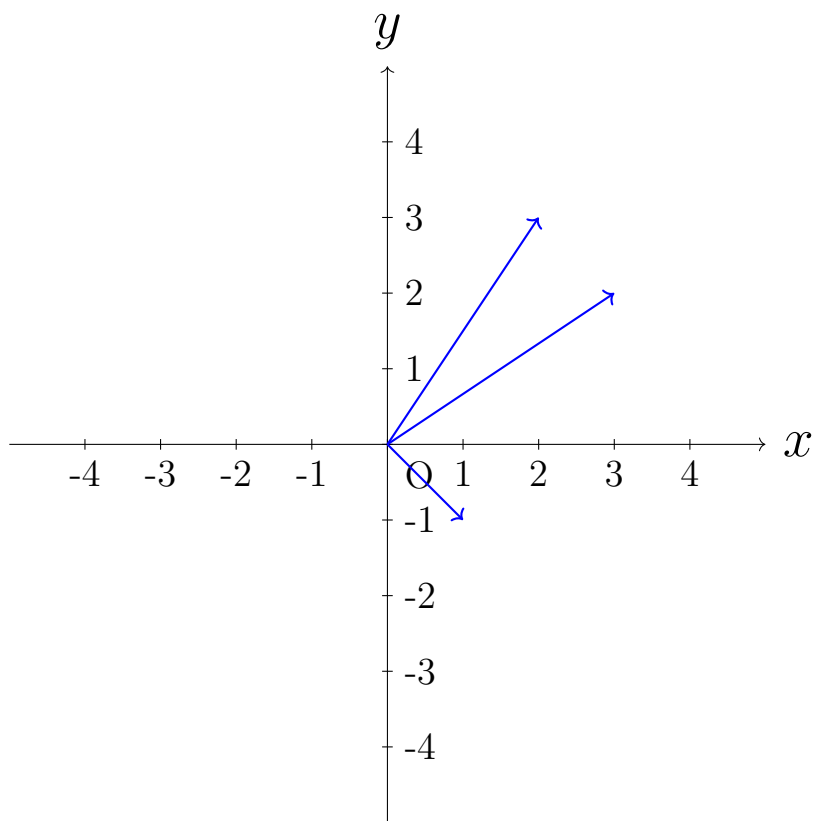
$$v = (2, 3)$$

$$w = (1, -1)$$

Vector addition :



# Vectors



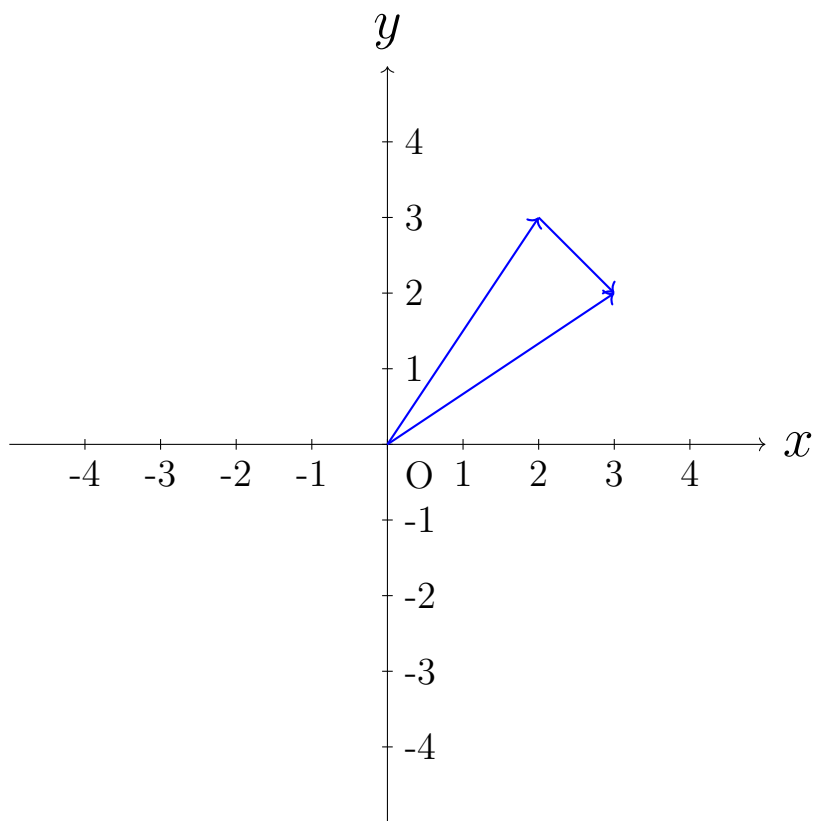
$$v = (2, 3)$$

$$w = (1, -1)$$

Vector addition :

$$v + w = (3, 2)$$

# Vectors



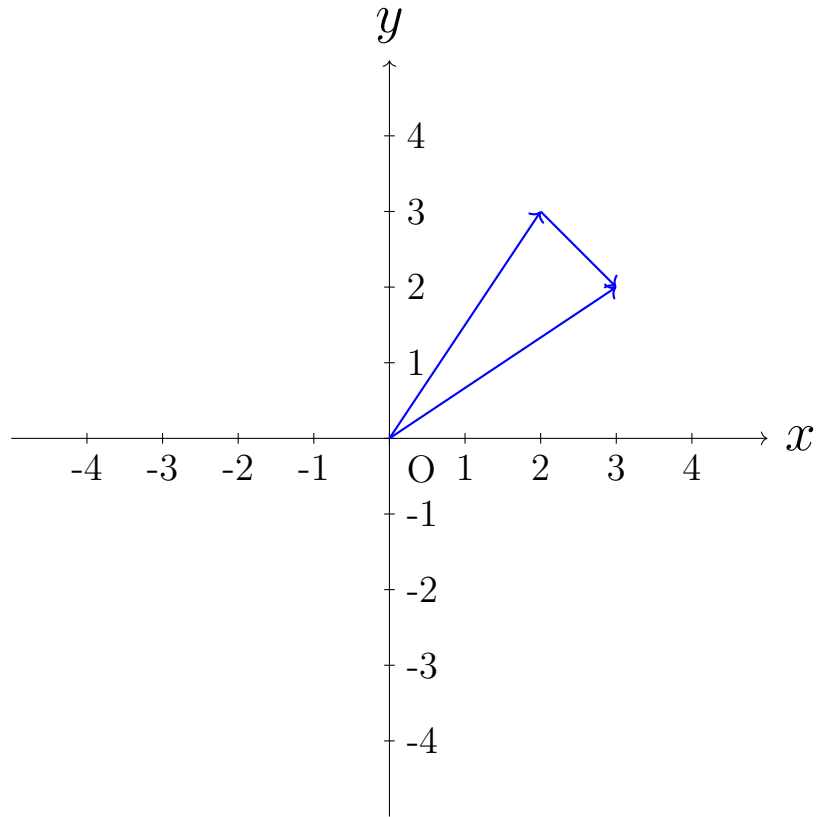
$$v = (2, 3)$$

$$w = (1, -1)$$

Vector addition :

$$v + w = (3, 2)$$

# Vectors



$$v = (2, 3)$$

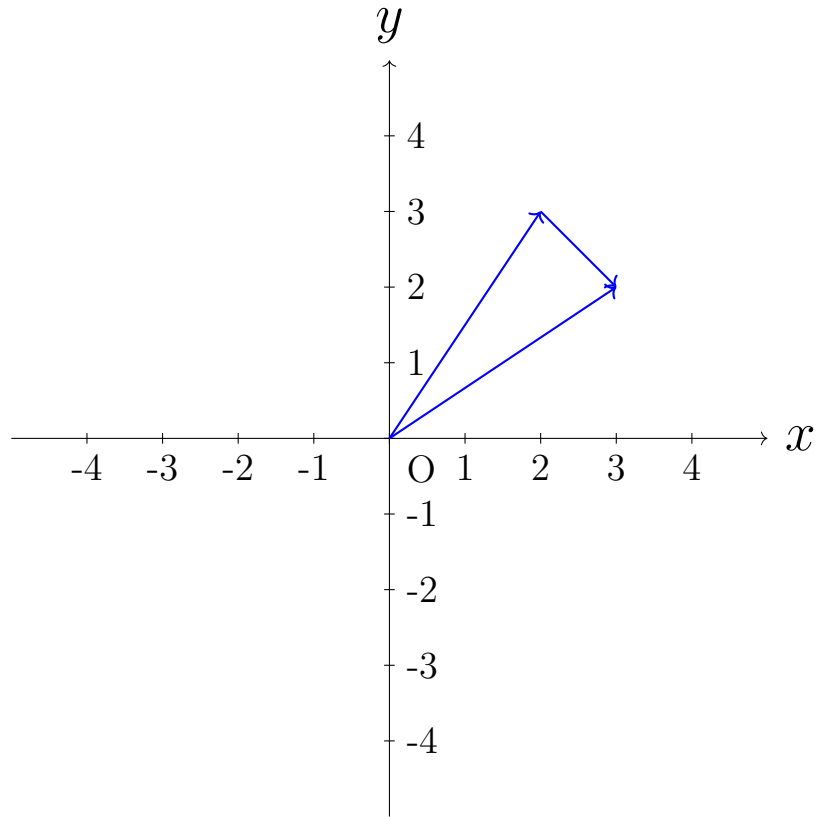
$$w = (1, -1)$$

Vector addition :

$$v + w = (3, 2)$$

In general:

# Vectors



$$v = (2, 3)$$

$$w = (1, -1)$$

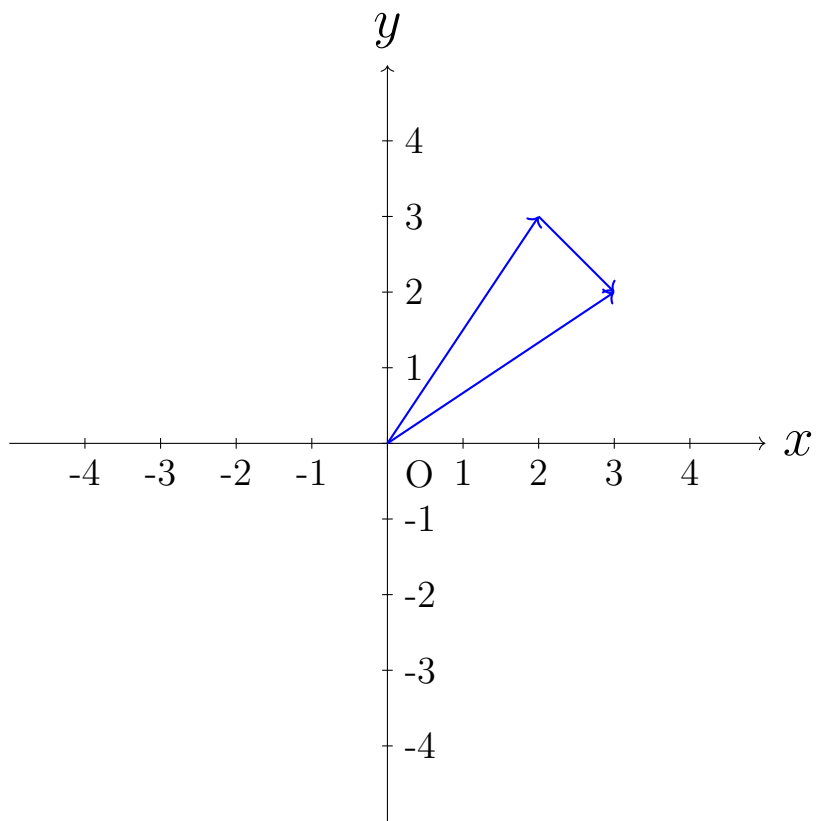
Vector addition :

$$v + w = (3, 2)$$

In general:

$$(x_1, y_1) + (x_2, y_2)$$

# Vectors



$$v = (2, 3)$$

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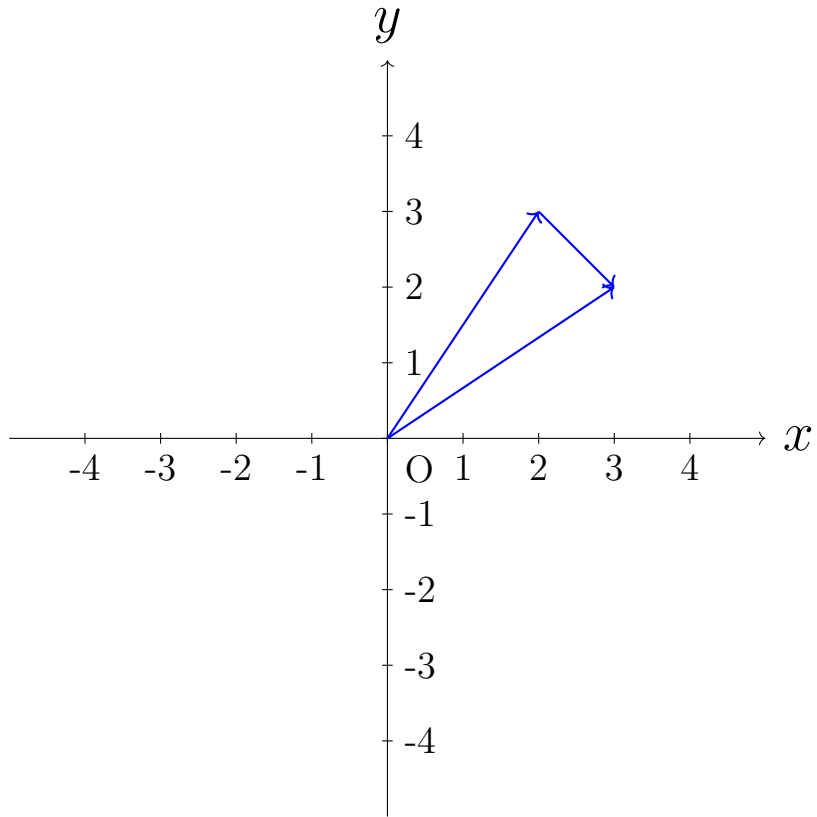
Vector addition :

$$v + w = (3, 2)$$

In general:

$$(x_1, y_1) + (x_2, y_2) := (x_1 + x_2, y_1 + y_2)$$

# Vectors



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Vector addition :

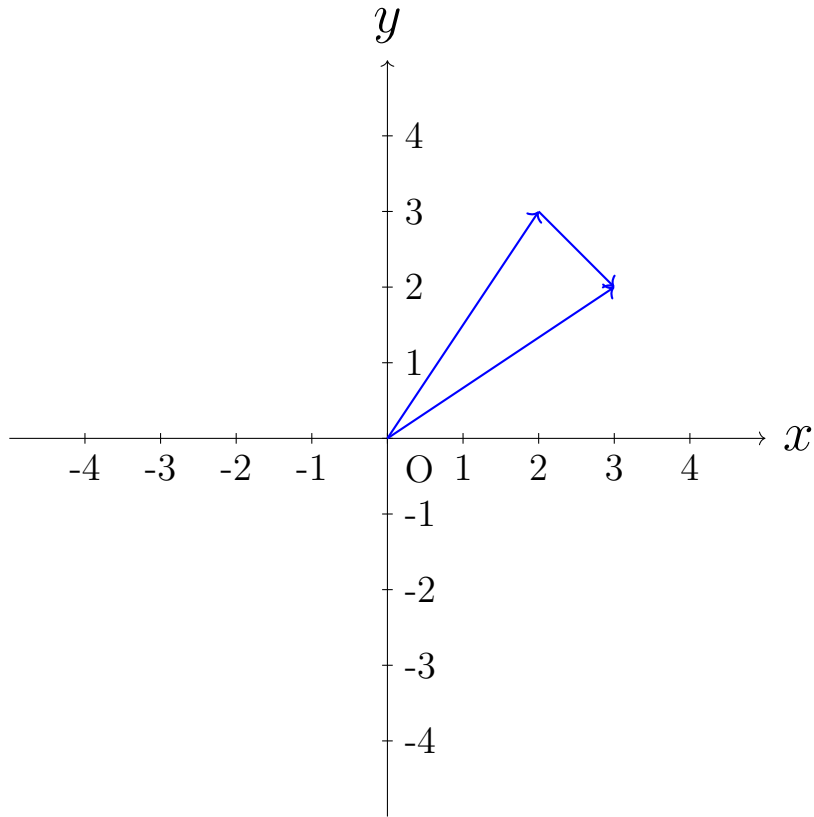
$$v + w = (3, 2)$$

In general:

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# Vectors



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Vector addition :

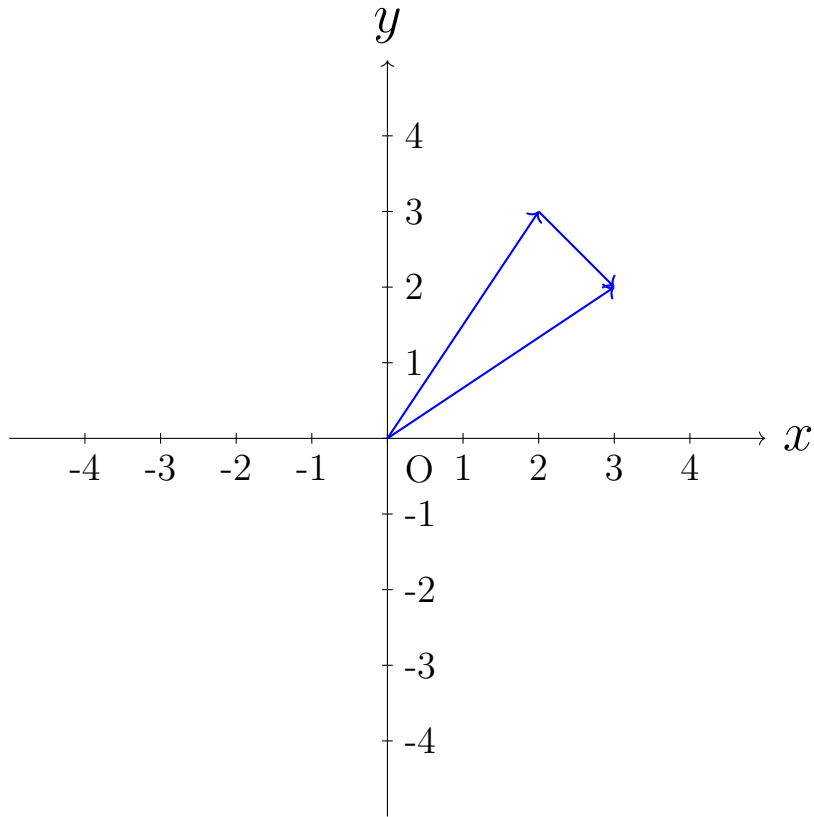
$$v + w = (3, 2)$$

In general:

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$$(x_1, y_1) - (x_2, y_2)$$

# Vectors



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Vector addition :

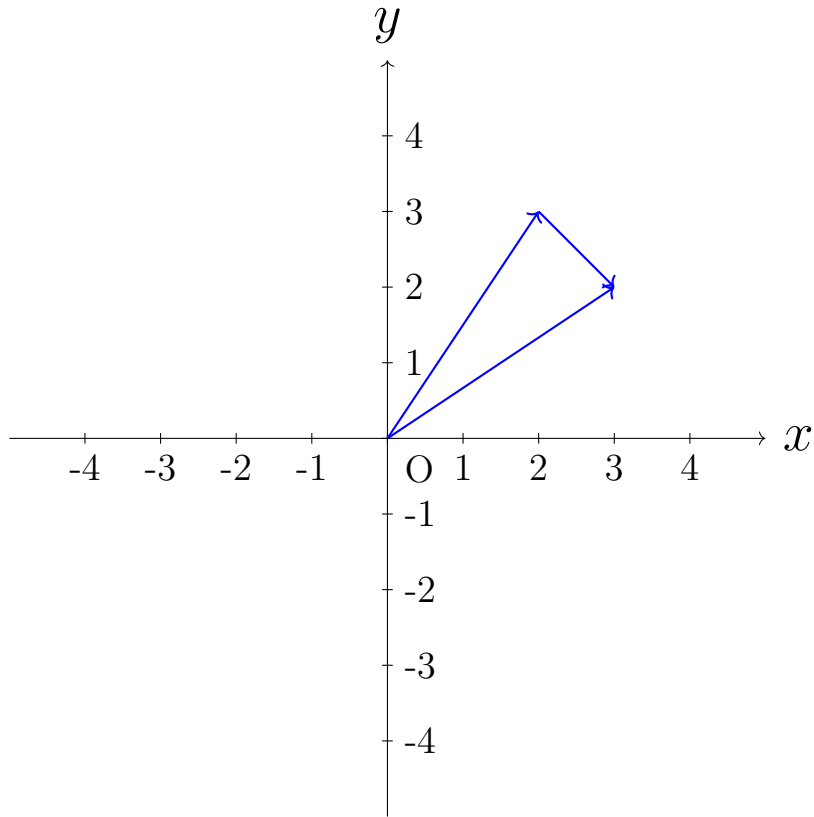
$$v + w = (3, 2)$$

In general:

$$(x_1, y_1) + (x_2, y_2) := (x_1 + x_2, y_1 + y_2)$$

$$(x_1, y_1) - (x_2, y_2) := (x_1 - x_2, y_1 - y_2)$$

# Vectors



$$v = (2, 3)$$

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Vector addition :

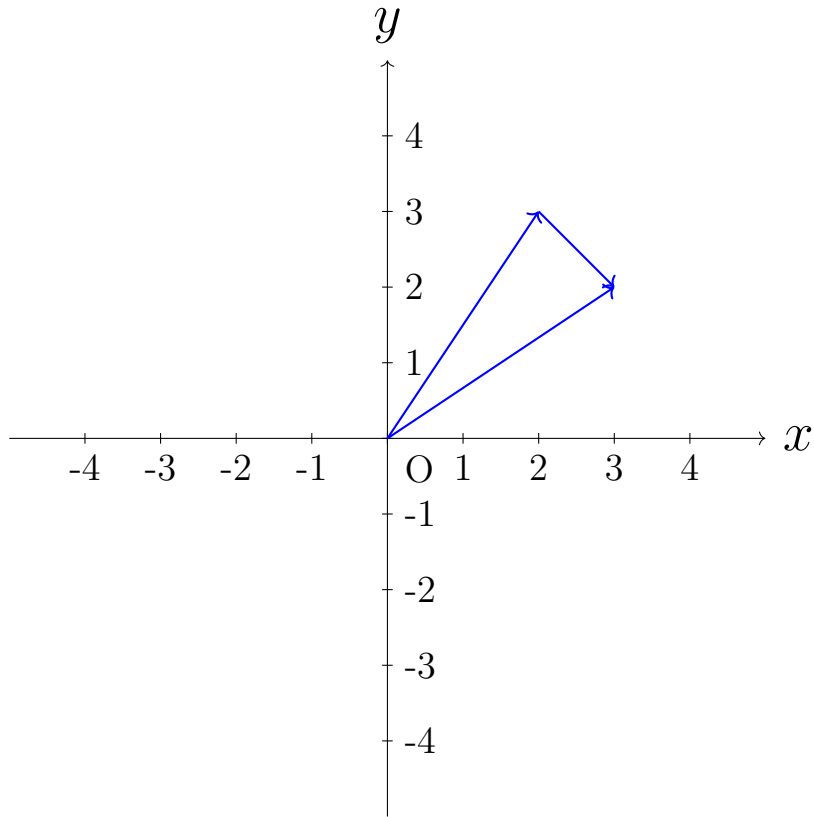
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# Vectors



$$v = (2, 3)$$

$$w = (1, -1)$$

Vector addition and subtraction :

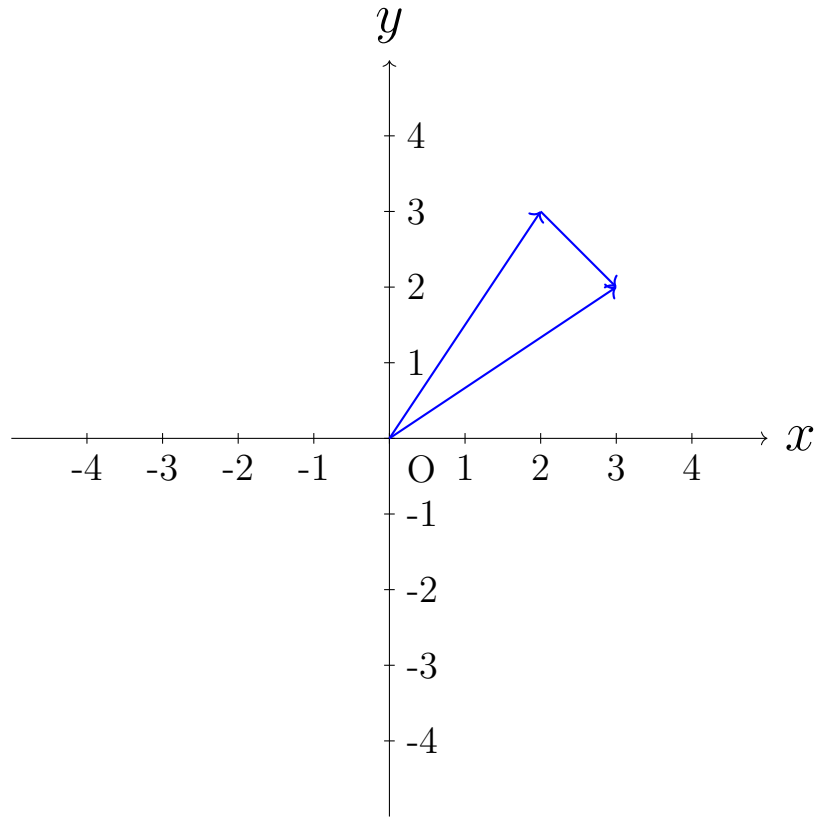
$$v + w = (3, 2)$$

In general:

$$(x_1, y_1) + (x_2, y_2) := (x_1 + x_2, y_1 + y_2)$$

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# Vectors



$$v = (2, 3)$$

$$w = (1, -1)$$

Vector addition and subtraction :

$$v + w = (3, 2)$$

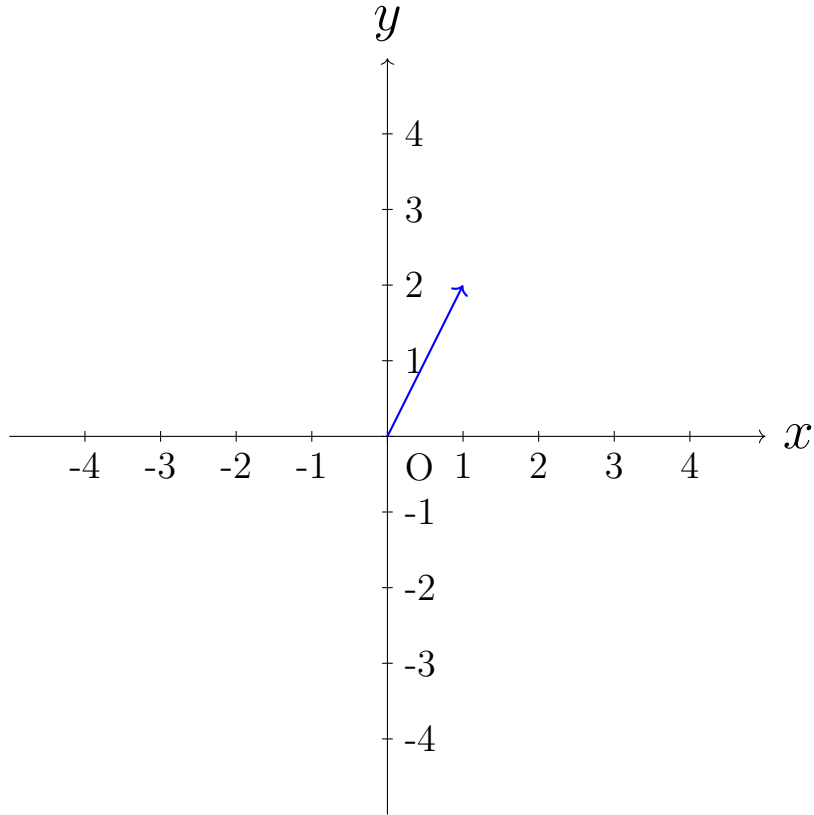
In general:

$$(x_1, y_1) + (x_2, y_2) := (x_1 + x_2, y_1 + y_2)$$

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Scalar multiplication:

# Vectors



$$v = (2, 3)$$

$$w = (1, -1)$$

Vector addition and subtraction :

$$v + w = (3, 2)$$

In general:

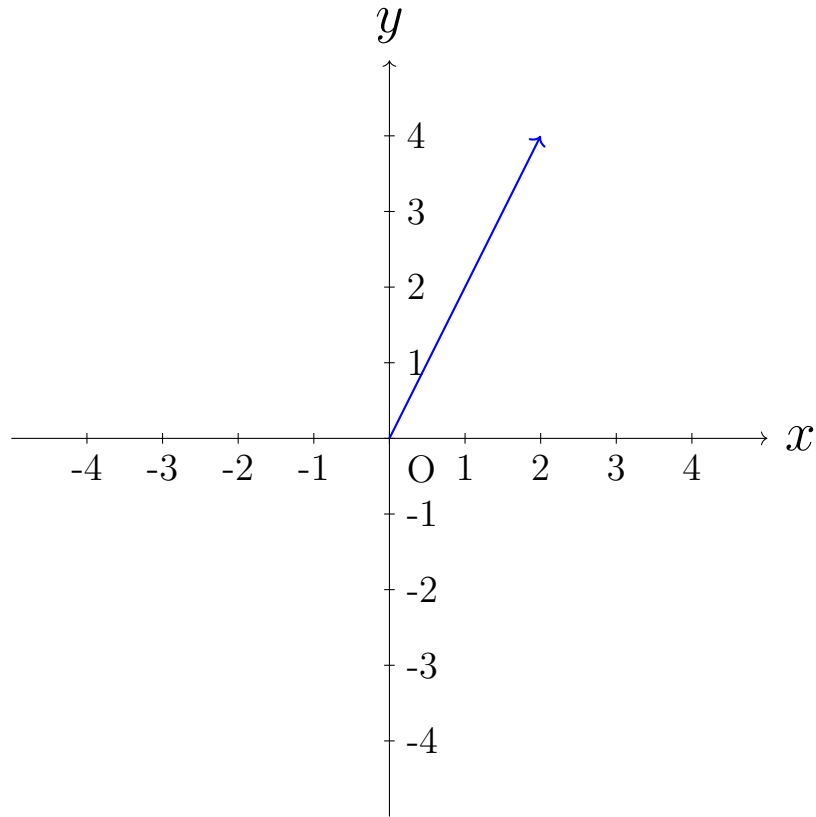
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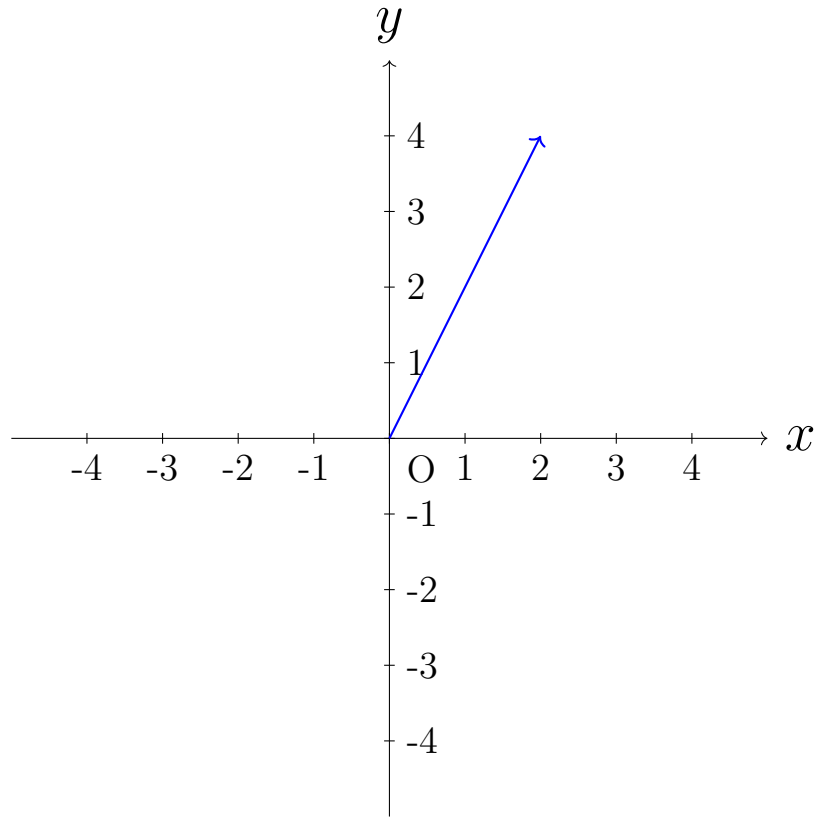
$$(x_1, y_1) - (x_2, y_2) := (x_1 - x_2, y_1 - y_2)$$

Scalar multiplication:

$$v := (1, 2)$$

$$2v$$

# Vectors



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$$w = (1, -1)$$

Vector addition and subtraction :

$$v + w = (3, 2)$$

In general:

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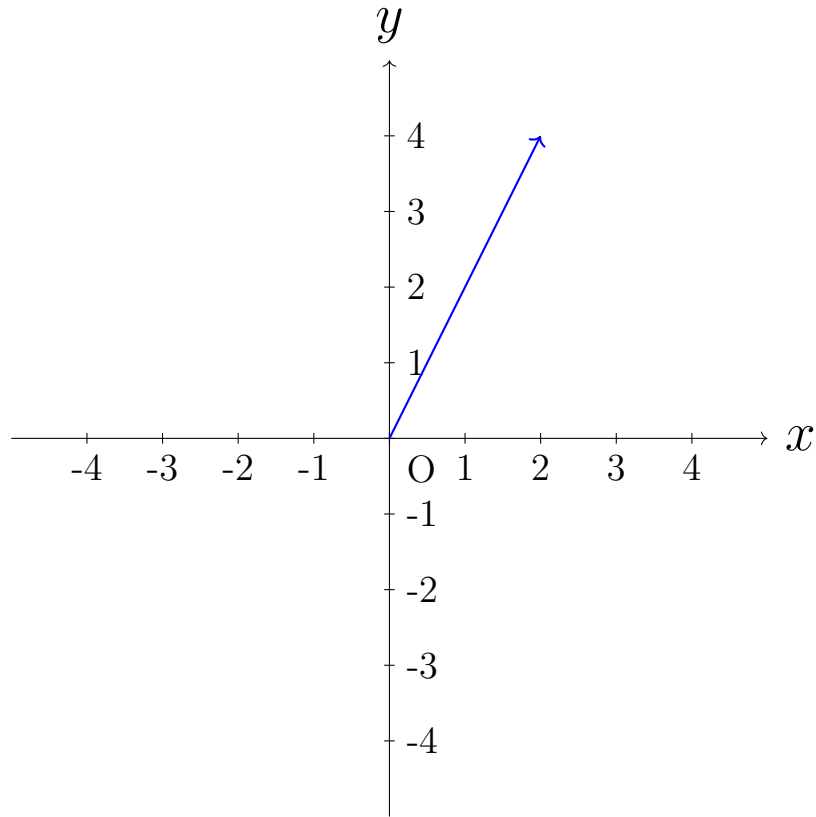
Scalar multiplication:

$$v := (1, 2)$$

$$2v = 2(1, 2)$$



# Vectors



$$v = (2, 3)$$

$$w = (1, -1)$$

Vector addition and subtraction :

$$v + w = (3, 2)$$

In general:

$$(x_1, y_1) + (x_2, y_2) := (x_1 + x_2, y_1 + y_2)$$

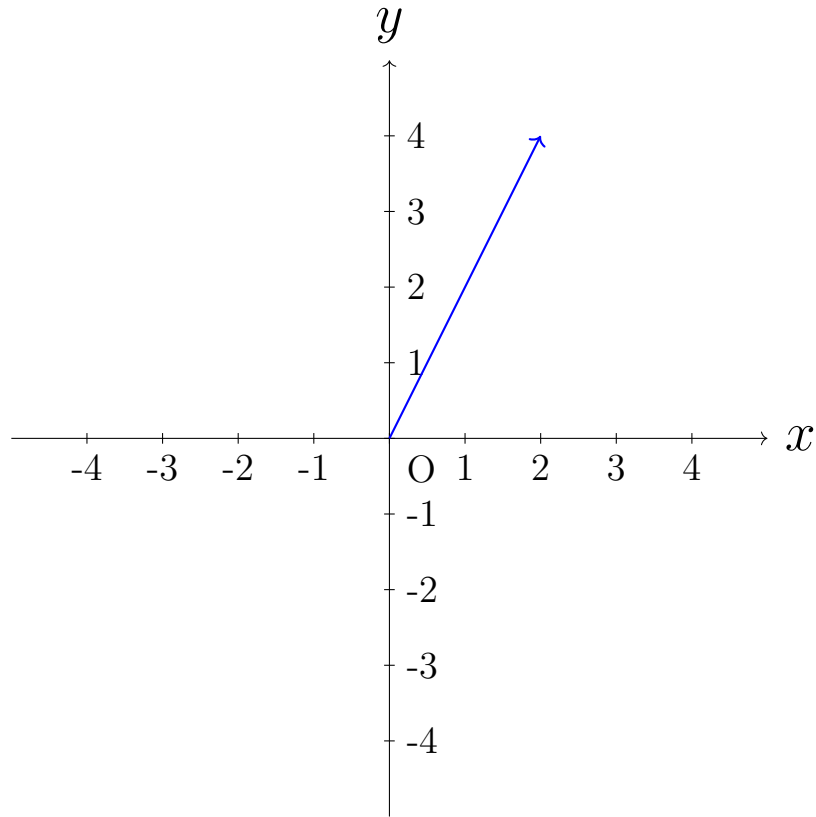
$$(x_1, y_1) - (x_2, y_2) := (x_1 - x_2, y_1 - y_2)$$

Scalar multiplication:

$$v := (1, 2)$$

$$2v = 2(1, 2) = (2, 4)$$

# Vectors



$$v = (2, 3)$$

$$w = (1, -1)$$

Vector addition and subtraction :

$$v + w = (3, 2)$$

In general:

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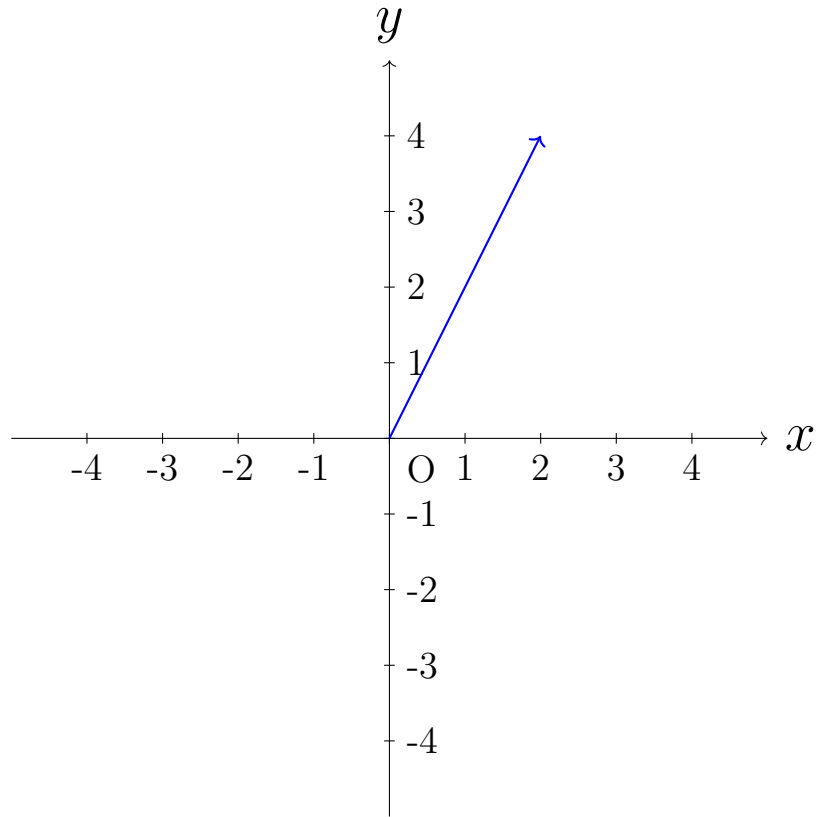
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Scalar multiplication:

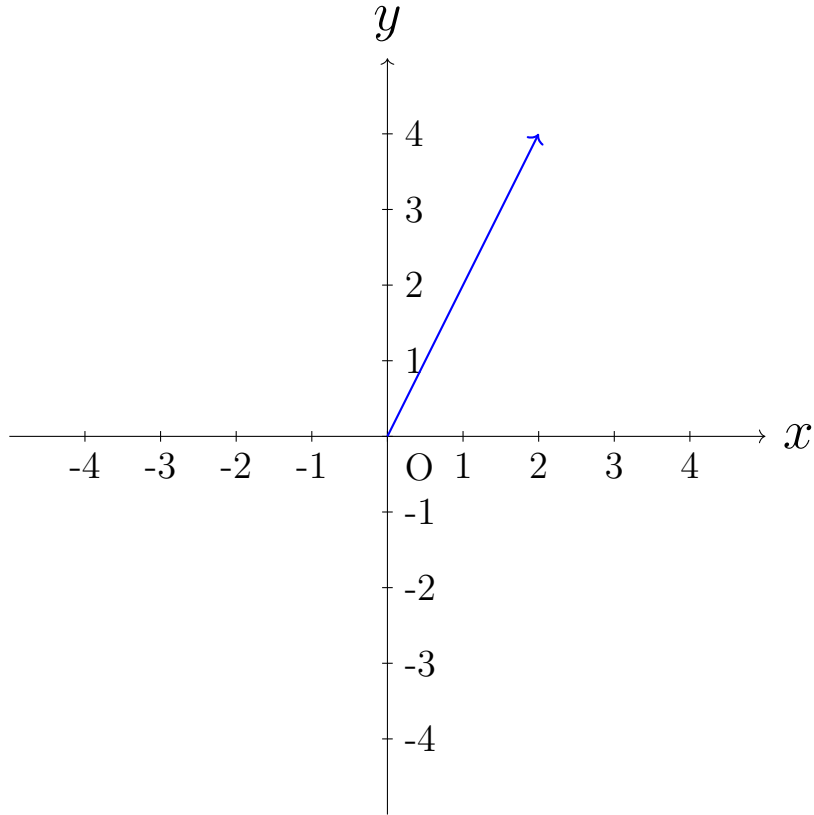
$$v := (1, 2)$$

$$2v = 2(1, 2) = (2, 4)$$

In general:

$$\lambda(x, y)$$

# Vectors



$$v = (2, 3)$$

$$w = (1, -1)$$

Vector addition and subtraction :

$$v + w = (3, 2)$$

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$$(x_1, y_1) - (x_2, y_2) := (x_1 - x_2, y_1 - y_2)$$

Scalar multiplication:

$$v := (1, 2)$$

$$2v = 2(1, 2) = (2, 4)$$

In general:

$$\lambda(x, y) := (\lambda x, \lambda y)$$

$$p := (2, 3),$$

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$$q := p + \mathbf{w} = (2, 3) + (1, 1) = (3, 4)$$

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(displacement of  $p$  by  $\mathbf{w}$ ).



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(displacement of  $p$  by  $\mathbf{w}$ ).

$$p := (2, 3) \text{ and } q = (3, 4),$$

$$\mathbf{v} = q - p$$

$p := (2, 3)$ ,  
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 $q := p + \mathbf{w} = (2, 3) + (1, 1) = (3, 4)$   
(displacement of  $p$  by  $\mathbf{w}$ ).

$p := (2, 3)$  and  $q = (3, 4)$ ,  
 $\mathbf{v} = q - p$  is the displacement

$$p := (2, 3),$$

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(displacement of  $p$  by  $\mathbf{w}$ ).

$$p := (2, 3) \text{ and } q = (3, 4),$$

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(displacement of  $p$  by  $\mathbf{w}$ ).

$$p := (2, 3) \text{ and } q = (3, 4),$$

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$$\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$$

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 $\mathbf{w} := (1, 1),$   
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$\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$  is a smooth parametrization.  
 $\gamma(t)$  is the *point* at  $t$

$$\begin{aligned}
p &:= (2, 3), \\
\mathbf{w} &:= (1, 1), \\
q &:= p + \mathbf{w} = (2, 3) + (1, 1) = (3, 4) \\
&\text{(displacement of } p \text{ by } \mathbf{w}\text{).}
\end{aligned}$$

$$\begin{aligned}
p &:= (2, 3) \text{ and } q = (3, 4), \\
\mathbf{v} &= q - p \text{ is the displacement that takes } p \text{ to } q
\end{aligned}$$

$$\begin{aligned}
\gamma : (\alpha, \beta) &\rightarrow \mathbb{R}^2 \text{ is a smooth parametrization.} \\
\gamma(t) &\text{ is the } \textit{point} \text{ at } t \\
\gamma(t + h) &\text{ is the } \textit{point} \text{ at } t + h
\end{aligned}$$



$$p := (2, 3),$$

$$\mathbf{w} := (1, 1),$$

$$q := p + \mathbf{w} = (2, 3) + (1, 1) = (3, 4)$$

(displacement of  $p$  by  $\mathbf{w}$ ).

$$p := (2, 3) \text{ and } q = (3, 4),$$

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$\gamma(t)$  is the *point* at  $t$

$\gamma(t + h)$  is the *point* at  $t + h$

$\gamma(t + h) - \gamma(t)$  is the displacement *vector* at  $t + h$

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**Definition.**  $\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$

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\end{aligned}$$

**Definition.**  $\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$  is a smooth parametrization.

$$\dot{\gamma}(t)$$

$$p := (2, 3),$$

$$\mathbf{w} := (1, 1),$$

$$q := p + \mathbf{w} = (2, 3) + (1, 1) = (3, 4)$$

(displacement of  $p$  by  $\mathbf{w}$ ).

$$p := (2, 3) \text{ and } q = (3, 4),$$

$\mathbf{v} = q - p$  is the displacement that takes  $p$  to  $q$

$\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$  is a smooth parametrization.

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**Definition.**  $\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$  is a smooth parametrization.

$$\dot{\gamma}(t) = \lim_{h \rightarrow 0}$$

$p := (2, 3)$ ,  
 $\mathbf{w} := (1, 1)$ ,  
 $q := p + \mathbf{w} = (2, 3) + (1, 1) = (3, 4)$   
(displacement of  $p$  by  $\mathbf{w}$ ).

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$\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$  is a smooth parametrization.

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**Definition.**  $\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$  is a smooth parametrization.

$$\dot{\gamma}(t) = \lim_{h \rightarrow 0} (1/h)(\gamma(t + h) - \gamma(t))$$

$p := (2, 3)$ ,  
 $\mathbf{w} := (1, 1)$ ,  
 $q := p + \mathbf{w} = (2, 3) + (1, 1) = (3, 4)$   
(displacement of  $p$  by  $\mathbf{w}$ ).

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**Definition.** A smooth parametrized curve,  $\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$ ,

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**Definition.** A smooth parametrized curve,  $\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$ , is called a **regular parametrized curve**

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**From now on, we will assume all parametrized curves to be regular**

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$$\dot{\tilde{\gamma}}(t) = (f'_1(\phi(t))\phi'(t), f'_2(\phi(t))\phi'(t))$$

$$\dot{\tilde{\gamma}}(t) = (f'_1(\phi(t)), f'_2(\phi(t)))\phi'(t)$$

$$\dot{\tilde{\gamma}}(t) = \dot{\gamma}(\phi(t))\phi'(t)$$

□

**Corollary.** *The tangent line is invariant under a reparametrization,  $\phi(t)$ .*

*Proof.*

$$\{\gamma(t) + k\dot{\gamma}(t)\}$$

□

**Lemma.** *If  $\tilde{\gamma}(t) = \gamma(\phi(t))$  is a reparametrization, then  $\dot{\tilde{\gamma}}(t) = \dot{\gamma}(\phi(t))\phi'(t)$*

*Proof.*

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*Proof.*

$$\{\gamma(t) + k\dot{\tilde{\gamma}}(t) \mid k \in \mathbb{R}\} = \{\gamma(t) + k\dot{\gamma}(\phi(t))\phi'(t)\}$$

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**Lemma.** *If  $\tilde{\gamma}(t) = \gamma(\phi(t))$  is a reparametrization, then  $\dot{\tilde{\gamma}}(t) = \dot{\gamma}(\phi(t))\phi'(t)$*

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□

**Corollary.** *The tangent line is invariant under a reparametrization,  $\phi(t)$ , if  $\phi'(t) \neq 0$*

*Proof.*

$$\begin{aligned}\{\gamma(t) + k\dot{\tilde{\gamma}}(t) \mid k \in \mathbb{R}\} &= \{\gamma(t) + k\dot{\gamma}(\phi(t))\phi'(t) \mid k \in \mathbb{R}\} \\ &= \{\gamma(t) + k\dot{\gamma}(\phi(t))\}\end{aligned}$$

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**Lemma.** *If  $\tilde{\gamma}(t) = \gamma(\phi(t))$  is a reparametrization, then  $\dot{\tilde{\gamma}}(t) = \dot{\gamma}(\phi(t))\phi'(t)$*

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□

**Lemma.** If  $\tilde{\gamma}(t) = \gamma(\phi(t))$  is a reparametrization, Note:  $\tilde{\gamma}(t)$  is the same point,  $p$ , as  $\gamma(\phi(t))$  then  $\dot{\tilde{\gamma}}(t) = \dot{\gamma}(\phi(t))\phi'(t)$

*Proof.*

$$\tilde{\gamma}(t) = \gamma(\phi(t))$$

$$\tilde{\gamma}(t) = (f_1(\phi(t)), f_2(\phi(t)))$$

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□

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$$\begin{aligned} \{\gamma(t) + k\dot{\tilde{\gamma}}(t) \mid k \in \mathbb{R}\} &= \{\gamma(t) + k\dot{\gamma}(\phi(t))\phi'(t) \mid k \in \mathbb{R}\} \\ &= \{\gamma(t) + k\dot{\gamma}(\phi(t)) \mid k \in \mathbb{R}\} \end{aligned}$$

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**Lemma.** *If  $\tilde{\gamma}(t) = \gamma(\phi(t))$  is a reparametrization,*

*then  $\dot{\tilde{\gamma}}(t) = \dot{\gamma}(\phi(t))\phi'(t)$*

Note:  $\tilde{\gamma}(t)$  is the same point,  $p$ , as  $\gamma(\phi(t))$

When using  $\tilde{\gamma}$ , the point  $p$  “appears at time  $t$ ”

*Proof.*

$$\begin{aligned}\tilde{\gamma}(t) &= \gamma(\phi(t)) \\ \tilde{\gamma}(t) &= (f_1(\phi(t)), f_2(\phi(t))) \\ \dot{\tilde{\gamma}}(t) &= (f'_1(\phi(t))\phi'(t), f'_2(\phi(t))\phi'(t)) \\ \dot{\tilde{\gamma}}(t) &= (f'_1(\phi(t)), f'_2(\phi(t)))\phi'(t) \\ \dot{\tilde{\gamma}}(t) &= \dot{\gamma}(\phi(t))\phi'(t)\end{aligned}$$

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*Proof.*

$$\begin{aligned}\{\gamma(t) + k\dot{\tilde{\gamma}}(t) \mid k \in \mathbb{R}\} &= \{\gamma(t) + k\dot{\gamma}(\phi(t))\phi'(t) \mid k \in \mathbb{R}\} \\ &= \{\gamma(t) + k\dot{\gamma}(\phi(t)) \mid k \in \mathbb{R}\}\end{aligned}$$

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Note:  $\tilde{\gamma}(t)$  is the same point,  $p$ , as  $\gamma(\phi(t))$

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When using  $\gamma$ , the point  $p$  “appears at time  $\phi(t)$ ”

$$\begin{aligned}\tilde{\gamma}(t) &= \gamma(\phi(t)) \\ \tilde{\gamma}(t) &= (f_1(\phi(t)), f_2(\phi(t))) \\ \dot{\tilde{\gamma}}(t) &= (f'_1(\phi(t))\phi'(t), f'_2(\phi(t))\phi'(t)) \\ \dot{\tilde{\gamma}}(t) &= (f'_1(\phi(t)), f'_2(\phi(t)))\phi'(t) \\ \dot{\tilde{\gamma}}(t) &= \dot{\gamma}(\phi(t))\phi'(t)\end{aligned}$$

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$$\begin{aligned}\{\gamma(t) + k\dot{\tilde{\gamma}}(t) \mid k \in \mathbb{R}\} &= \{\gamma(t) + k\dot{\gamma}(\phi(t))\phi'(t) \mid k \in \mathbb{R}\} \\ &= \{\gamma(t) + k\dot{\gamma}(\phi(t)) \mid k \in \mathbb{R}\}\end{aligned}$$

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**Lemma.** *If  $\tilde{\gamma}(t) = \gamma(\phi(t))$  is a reparametrization, then  $\dot{\tilde{\gamma}}(t) = \dot{\gamma}(\phi(t))\phi'(t)$*

*Proof.*

$$\tilde{\gamma}(t) = \gamma(\phi(t))$$

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When using  $\gamma$ , the point  $p$  “appears at time  $\phi(t)$ ”

So,  $\dot{\tilde{\gamma}}(t)$  and  $\dot{\gamma}(\phi(t))$  are velocity vectors at the same point  $p$

**Corollary.** *The tangent line is invariant under a reparametrization,  $\phi(t)$ , if  $\phi'(t) \neq 0$*

*Proof.*

$$\begin{aligned} \{\gamma(t) + k\dot{\tilde{\gamma}}(t) \mid k \in \mathbb{R}\} &= \{\gamma(t) + k\dot{\gamma}(\phi(t))\phi'(t) \mid k \in \mathbb{R}\} \\ &= \{\gamma(t) + k\dot{\gamma}(\phi(t)) \mid k \in \mathbb{R}\} \end{aligned}$$