Exercise sheet 5

Curves and Surfaces, MTH201

Additional exercises

NOTE: These exercises repeat many of the concepts / exrecises covered earlier and are meant for you to identify gaps in your understanding. They are not exhaustive and the mid-semester examination will not be restricted to these questions.

Let $S \subset \mathbb{R}^3$ be a part of a surface and $\sigma : U \to S$ be a regular surface patch.

1. For each of the surface patches below, identify the surface that they (partially) cover:

   (a) $\sigma : \mathbb{R}^2 \to \mathbb{R}^3$, $\sigma(x, y) = (x, y, 0)$.

   (b) $\sigma : \mathbb{R}^2 \to \mathbb{R}^3$, $\sigma(x, y) = (x, y, x + y)$.

   (c) $\sigma : \mathbb{R}^2 \to \mathbb{R}^3$, $\sigma(x, y) = (\cos(x), \sin(x), y)$.

   (d) $\sigma : \mathbb{R}^2 \to \mathbb{R}^3$, $\sigma(x, y) = (x, y, \sqrt{r^2 - x^2 - y^2})$.

   (e) $\sigma : \mathbb{R}^2 \to \mathbb{R}^3$, $\sigma(x, y) = (x, y, \sqrt{r^2 - x^2 + y^2})$.

2. Consider a $\gamma : (a, b) \to S \subset \mathbb{R}^3$ parametrizing a curve that lies on the part of the surface covered by the surface patch. In other words, for each t, $\gamma(t)$ must, be in the image of σ, i.e. there is some $x(t)$, and $y(t)$ in U, so that $\gamma(t) = \sigma(x(t), y(t))$. Assuming that $x(t)$ and $y(t)$ are smooth,

   (a) Consider the part of the surface covered by $\sigma : \mathbb{R}^2 \to \mathbb{R}^3$, $\sigma(x, y) = (\cos(x), \sin(x), y)$ and consider the curve $\gamma(t) = (0, 0, t)$. Note that it lies on the surface. Write it in the form, $\gamma(t) = \sigma(x(t), y(t))$ by finding suitable functions $x(t)$ and $y(t)$. Do the same for the curve $\gamma_2(t) = (\cos(t), -\sin(t), 0)$ which also lies on the surface.

   (b) Show that

   $$\dot{\gamma}(t_0) = x'(t_0)\sigma_x(x(t_0), y(t_0)) + y'(t_0)\sigma_y(x(t_0), y(t_0))$$

3. Show that $\sigma_x(x_0, y_0)$ and $\sigma_y(x_0, y_0)$ are each velocity vectors of curves that lie on the surface. Why are they linearly independent?

4. Why do the previous two exercises show that $\sigma_x(x_0, y_0)$ and $\sigma_y(x_0, y_0)$ are a basis for the tangent vectors?

5. Consider a point p on the part of the surface covered by a surface patch. Therefore, it is of the form $p = \sigma(x_0, y_0)$ for some x_0 an y_0. Consider $\hat{n}(p) = \sigma_x(x_0, y_0) \times \sigma_y(x_0, y_0)$ which is a vector in $\mathbb{R}^3$ based at p.

(a) Is it a tangent vector? Why or why not?

(b) Why is its dot product with $\sigma_x(x_0, y_0)$ and $\sigma_y(x_0, y_0)$ zero?

(c) Why is its dot product with *any* tangent vector (of the surface at p) zero?

6. Compute $\hat{n}(p)$ for any point p on a sphere.