

Dicussion before the lecture

Reparametrization example by “shifting the phase”

$$\dot{\tilde{\gamma}}(t) = \dot{\gamma}(\phi(t))\phi'(t)$$

Dicussion before the lecture

Reparametrization example by “shifting the phase”

$$\dot{\tilde{\gamma}}(t) = \dot{\gamma}(\phi(t))\phi'(t)$$

$$\gamma(t) := (\cos(t), \sin(t))$$

Dicussion before the lecture

Reparametrization example by “shifting the phase”

$$\dot{\tilde{\gamma}}(t) = \dot{\gamma}(\phi(t))\phi'(t)$$

$$\gamma(t) := (\cos(t), \sin(t))$$

$$\tilde{\gamma}(t) := (\cos(t + \pi/2), \sin(t + \pi/2))$$

Dicussion before the lecture

Reparametrization example by “shifting the phase”

$$\dot{\tilde{\gamma}}(t) = \dot{\gamma}(\phi(t))\phi'(t)$$

$$\gamma(t) := (\cos(t), \sin(t))$$

$$\tilde{\gamma}(t) := (\cos(t + \pi/2), \sin(t + \pi/2))$$

$$\phi(t) = t + \pi/2$$

Dicussion before the lecture

Reparametrization example by “shifting the phase”

$$\dot{\tilde{\gamma}}(t) = \dot{\gamma}(\phi(t))\phi'(t)$$

$$\gamma(t) := (\cos(t), \sin(t))$$

$$\tilde{\gamma}(t) := (\cos(t + \pi/2), \sin(t + \pi/2))$$

$$\phi(t) = t + \pi/2$$

$$\tilde{\gamma}(t) := \gamma(\phi(t))$$

Regular parametrization

Definition.

The point $\gamma(t)$

Regular parametrization

Definition.

The point $\gamma(t)$ of γ :

Regular parametrization

Definition.

The point $\gamma(t)$ of $\gamma : (\alpha, \beta) \rightarrow$

Regular parametrization

Definition.

The point $\gamma(t)$ of $\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$

Regular parametrization

Definition.

The point $\gamma(t)$ of $\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$ is regular point

Regular parametrization

Definition.

The point $\gamma(t)$ of $\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$ is regular point if $\dot{\gamma}(t) \neq 0$ and singular if $\dot{\gamma}(t) = 0$.

Regular parametrization

Definition.

The point $\gamma(t)$ of $\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$ is regular point if $\dot{\gamma}(t) \neq 0$ and singular if $\dot{\gamma}(t) = 0$. The parametrization γ is called regular

Regular parametrization

Definition.

The point $\gamma(t)$ of $\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$ is regular point if $\dot{\gamma}(t) \neq 0$ and singular if $\dot{\gamma}(t) = 0$. The parametrization γ is called regular if $\dot{\gamma}(t) \neq 0$

Regular parametrization

Definition.

The point $\gamma(t)$ of $\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$ is regular point if $\dot{\gamma}(t) \neq 0$ and singular if $\dot{\gamma}(t) = 0$. The parametrization γ is called regular if $\dot{\gamma}(t) \neq 0$ for *every* $t \in (\alpha, \beta)$.

Regular parametrization

Definition.

The point $\gamma(t)$ of $\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$ is regular point if $\dot{\gamma}(t) \neq 0$ and singular if $\dot{\gamma}(t) = 0$. The parametrization γ is called regular if $\dot{\gamma}(t) \neq 0$ for *every* $t \in (\alpha, \beta)$.

Example. $\gamma(t) = (t^3, t^3)$ is singular at 0 because $\dot{\gamma}(t) = (3t^2, 3t^2)$.

Lemma.

If ϕ

Regular parametrization

Definition.

The point $\gamma(t)$ of $\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$ is regular point if $\dot{\gamma}(t) \neq 0$ and singular if $\dot{\gamma}(t) = 0$. The parametrization γ is called regular if $\dot{\gamma}(t) \neq 0$ for *every* $t \in (\alpha, \beta)$.

Example. $\gamma(t) = (t^3, t^3)$ is singular at 0 because $\dot{\gamma}(t) = (3t^2, 3t^2)$.

Lemma.

If $\phi : (\tilde{\alpha}, \tilde{\beta}) \rightarrow$

Regular parametrization

Definition.

The point $\gamma(t)$ of $\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$ is regular point if $\dot{\gamma}(t) \neq 0$ and singular if $\dot{\gamma}(t) = 0$. The parametrization γ is called regular if $\dot{\gamma}(t) \neq 0$ for *every* $t \in (\alpha, \beta)$.

Example. $\gamma(t) = (t^3, t^3)$ is singular at 0 because $\dot{\gamma}(t) = (3t^2, 3t^2)$.

Lemma.

If $\phi : (\tilde{\alpha}, \tilde{\beta}) \rightarrow (\alpha, \beta)$

Regular parametrization

Definition.

The point $\gamma(t)$ of $\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$ is regular point if $\dot{\gamma}(t) \neq 0$ and singular if $\dot{\gamma}(t) = 0$. The parametrization γ is called regular if $\dot{\gamma}(t) \neq 0$ for *every* $t \in (\alpha, \beta)$.

Example. $\gamma(t) = (t^3, t^3)$ is singular at 0 because $\dot{\gamma}(t) = (3t^2, 3t^2)$.

Lemma.

If $\phi : (\tilde{\alpha}, \tilde{\beta}) \rightarrow (\alpha, \beta)$ is invertible

Regular parametrization

Definition.

The point $\gamma(t)$ of $\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$ is regular point if $\dot{\gamma}(t) \neq 0$ and singular if $\dot{\gamma}(t) = 0$. The parametrization γ is called regular if $\dot{\gamma}(t) \neq 0$ for *every* $t \in (\alpha, \beta)$.

Example. $\gamma(t) = (t^3, t^3)$ is singular at 0 because $\dot{\gamma}(t) = (3t^2, 3t^2)$.

Lemma.

If $\phi : (\tilde{\alpha}, \tilde{\beta}) \rightarrow (\alpha, \beta)$ is invertible with inverse ψ

Regular parametrization

Definition.

The point $\gamma(t)$ of $\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$ is regular point if $\dot{\gamma}(t) \neq 0$ and singular if $\dot{\gamma}(t) = 0$. The parametrization γ is called regular if $\dot{\gamma}(t) \neq 0$ for *every* $t \in (\alpha, \beta)$.

Example. $\gamma(t) = (t^3, t^3)$ is singular at 0 because $\dot{\gamma}(t) = (3t^2, 3t^2)$.

Lemma.

If $\phi : (\tilde{\alpha}, \tilde{\beta}) \rightarrow (\alpha, \beta)$ is invertible with inverse $\psi : (\alpha, \beta) \rightarrow$

Regular parametrization

Definition.

The point $\gamma(t)$ of $\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$ is regular point if $\dot{\gamma}(t) \neq 0$ and singular if $\dot{\gamma}(t) = 0$. The parametrization γ is called regular if $\dot{\gamma}(t) \neq 0$ for *every* $t \in (\alpha, \beta)$.

Example. $\gamma(t) = (t^3, t^3)$ is singular at 0 because $\dot{\gamma}(t) = (3t^2, 3t^2)$.

Lemma.

If $\phi : (\tilde{\alpha}, \tilde{\beta}) \rightarrow (\alpha, \beta)$ is invertible with inverse $\psi : (\alpha, \beta) \rightarrow (\tilde{\alpha}, \tilde{\beta})$

Regular parametrization

Definition.

The point $\gamma(t)$ of $\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$ is regular point if $\dot{\gamma}(t) \neq 0$ and singular if $\dot{\gamma}(t) = 0$. The parametrization γ is called regular if $\dot{\gamma}(t) \neq 0$ for *every* $t \in (\alpha, \beta)$.

Example. $\gamma(t) = (t^3, t^3)$ is singular at 0 because $\dot{\gamma}(t) = (3t^2, 3t^2)$.

Lemma.

If $\phi : (\tilde{\alpha}, \tilde{\beta}) \rightarrow (\alpha, \beta)$ is invertible with inverse $\psi : (\alpha, \beta) \rightarrow (\tilde{\alpha}, \tilde{\beta})$, then $\phi'(t) \neq 0$

Regular parametrization

Definition.

The point $\gamma(t)$ of $\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$ is regular point if $\dot{\gamma}(t) \neq 0$ and singular if $\dot{\gamma}(t) = 0$. The parametrization γ is called regular if $\dot{\gamma}(t) \neq 0$ for *every* $t \in (\alpha, \beta)$.

Example. $\gamma(t) = (t^3, t^3)$ is singular at 0 because $\dot{\gamma}(t) = (3t^2, 3t^2)$.

Lemma.

If $\phi : (\tilde{\alpha}, \tilde{\beta}) \rightarrow (\alpha, \beta)$ is invertible with inverse $\psi : (\alpha, \beta) \rightarrow (\tilde{\alpha}, \tilde{\beta})$, then $\phi'(t) \neq 0$ for all $t \in (\tilde{\alpha}, \tilde{\beta})$

Regular parametrization

Definition.

The point $\gamma(t)$ of $\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$ is regular point if $\dot{\gamma}(t) \neq 0$ and singular if $\dot{\gamma}(t) = 0$. The parametrization γ is called regular if $\dot{\gamma}(t) \neq 0$ for *every* $t \in (\alpha, \beta)$.

Example. $\gamma(t) = (t^3, t^3)$ is singular at 0 because $\dot{\gamma}(t) = (3t^2, 3t^2)$.

Lemma.

If $\phi : (\tilde{\alpha}, \tilde{\beta}) \rightarrow (\alpha, \beta)$ is invertible with inverse $\psi : (\alpha, \beta) \rightarrow (\tilde{\alpha}, \tilde{\beta})$, then $\phi'(t) \neq 0$ for all $t \in (\tilde{\alpha}, \tilde{\beta})$ and $\psi'(t) \neq 0$

Regular parametrization

Definition.

The point $\gamma(t)$ of $\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$ is regular point if $\dot{\gamma}(t) \neq 0$ and singular if $\dot{\gamma}(t) = 0$. The parametrization γ is called regular if $\dot{\gamma}(t) \neq 0$ for *every* $t \in (\alpha, \beta)$.

Example. $\gamma(t) = (t^3, t^3)$ is singular at 0 because $\dot{\gamma}(t) = (3t^2, 3t^2)$.

Lemma.

If $\phi : (\tilde{\alpha}, \tilde{\beta}) \rightarrow (\alpha, \beta)$ is invertible with inverse $\psi : (\alpha, \beta) \rightarrow (\tilde{\alpha}, \tilde{\beta})$, then $\phi'(t) \neq 0$ for all $t \in (\tilde{\alpha}, \tilde{\beta})$ and $\psi'(t) \neq 0$ for all $t \in (\alpha, \beta)$.

Regular parametrization

Definition.

The point $\gamma(t)$ of $\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$ is regular point if $\dot{\gamma}(t) \neq 0$ and singular if $\dot{\gamma}(t) = 0$. The parametrization γ is called regular if $\dot{\gamma}(t) \neq 0$ for *every* $t \in (\alpha, \beta)$.

Example. $\gamma(t) = (t^3, t^3)$ is singular at 0 because $\dot{\gamma}(t) = (3t^2, 3t^2)$.

Lemma.

If $\phi : (\tilde{\alpha}, \tilde{\beta}) \rightarrow (\alpha, \beta)$ is invertible with inverse $\psi : (\alpha, \beta) \rightarrow (\tilde{\alpha}, \tilde{\beta})$, then $\phi'(t) \neq 0$ for all $t \in (\tilde{\alpha}, \tilde{\beta})$ and $\psi'(t) \neq 0$ for all $t \in (\alpha, \beta)$.

Proof.

$$\psi(\phi(t)) = t$$

□

Regular parametrization

Definition.

The point $\gamma(t)$ of $\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$ is regular point if $\dot{\gamma}(t) \neq 0$ and singular if $\dot{\gamma}(t) = 0$. The parametrization γ is called regular if $\dot{\gamma}(t) \neq 0$ for *every* $t \in (\alpha, \beta)$.

Example. $\gamma(t) = (t^3, t^3)$ is singular at 0 because $\dot{\gamma}(t) = (3t^2, 3t^2)$.

Lemma.

If $\phi : (\tilde{\alpha}, \tilde{\beta}) \rightarrow (\alpha, \beta)$ is invertible with inverse $\psi : (\alpha, \beta) \rightarrow (\tilde{\alpha}, \tilde{\beta})$, then $\phi'(t) \neq 0$ for all $t \in (\tilde{\alpha}, \tilde{\beta})$ and $\psi'(t) \neq 0$ for all $t \in (\alpha, \beta)$.

Proof.

$\psi(\phi(t)) = t$ for *each* t

□

Regular parametrization

Definition.

The point $\gamma(t)$ of $\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$ is regular point if $\dot{\gamma}(t) \neq 0$ and singular if $\dot{\gamma}(t) = 0$. The parametrization γ is called regular if $\dot{\gamma}(t) \neq 0$ for *every* $t \in (\alpha, \beta)$.

Example. $\gamma(t) = (t^3, t^3)$ is singular at 0 because $\dot{\gamma}(t) = (3t^2, 3t^2)$.

Lemma.

If $\phi : (\tilde{\alpha}, \tilde{\beta}) \rightarrow (\alpha, \beta)$ is invertible with inverse $\psi : (\alpha, \beta) \rightarrow (\tilde{\alpha}, \tilde{\beta})$, then $\phi'(t) \neq 0$ for all $t \in (\tilde{\alpha}, \tilde{\beta})$ and $\psi'(t) \neq 0$ for all $t \in (\alpha, \beta)$.

Proof.

$$\psi(\phi(t)) = t \text{ for each } t$$

$$\psi'(\phi(t))\phi'(t) = 1$$

□

Regular parametrization

Definition.

The point $\gamma(t)$ of $\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$ is regular point if $\dot{\gamma}(t) \neq 0$ and singular if $\dot{\gamma}(t) = 0$. The parametrization γ is called regular if $\dot{\gamma}(t) \neq 0$ for *every* $t \in (\alpha, \beta)$.

Example. $\gamma(t) = (t^3, t^3)$ is singular at 0 because $\dot{\gamma}(t) = (3t^2, 3t^2)$.

Lemma.

If $\phi : (\tilde{\alpha}, \tilde{\beta}) \rightarrow (\alpha, \beta)$ is invertible with inverse $\psi : (\alpha, \beta) \rightarrow (\tilde{\alpha}, \tilde{\beta})$, then $\phi'(t) \neq 0$ for all $t \in (\tilde{\alpha}, \tilde{\beta})$ and $\psi'(t) \neq 0$ for all $t \in (\alpha, \beta)$.

Proof.

$\psi(\phi(t)) = t$ for each t

$\psi'(\phi(t))\phi'(t) = 1$ for each t

□

Regular parametrization

Definition.

The point $\gamma(t)$ of $\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$ is regular point if $\dot{\gamma}(t) \neq 0$ and singular if $\dot{\gamma}(t) = 0$. The parametrization γ is called regular if $\dot{\gamma}(t) \neq 0$ for *every* $t \in (\alpha, \beta)$.

Example. $\gamma(t) = (t^3, t^3)$ is singular at 0 because $\dot{\gamma}(t) = (3t^2, 3t^2)$.

Lemma.

If $\phi : (\tilde{\alpha}, \tilde{\beta}) \rightarrow (\alpha, \beta)$ is invertible with inverse $\psi : (\alpha, \beta) \rightarrow (\tilde{\alpha}, \tilde{\beta})$, then $\phi'(t) \neq 0$ for all $t \in (\tilde{\alpha}, \tilde{\beta})$ and $\psi'(t) \neq 0$ for all $t \in (\alpha, \beta)$.

Proof.

$\psi(\phi(t)) = t$ for each t

$\psi'(\phi(t))\phi'(t) = 1$ for each t

So, $\psi'(\phi(t)) \neq 0$ and $\phi'(t) \neq 0$

□

Regular parametrization

Definition.

The point $\gamma(t)$ of $\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$ is regular point if $\dot{\gamma}(t) \neq 0$ and singular if $\dot{\gamma}(t) = 0$. The parametrization γ is called regular if $\dot{\gamma}(t) \neq 0$ for *every* $t \in (\alpha, \beta)$.

Example. $\gamma(t) = (t^3, t^3)$ is singular at 0 because $\dot{\gamma}(t) = (3t^2, 3t^2)$.

Lemma.

If $\phi : (\tilde{\alpha}, \tilde{\beta}) \rightarrow (\alpha, \beta)$ is invertible with inverse $\psi : (\alpha, \beta) \rightarrow (\tilde{\alpha}, \tilde{\beta})$, then $\phi'(t) \neq 0$ for all $t \in (\tilde{\alpha}, \tilde{\beta})$ and $\psi'(t) \neq 0$ for all $t \in (\alpha, \beta)$.

Proof.

$\psi(\phi(t)) = t$ for each t

$\psi'(\phi(t))\phi'(t) = 1$ for each t

So, $\psi'(\phi(t)) \neq 0$ and $\phi'(t) \neq 0$ for each t □

Regular parametrization

Definition.

The point $\gamma(t)$ of $\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$ is regular point if $\dot{\gamma}(t) \neq 0$ and singular if $\dot{\gamma}(t) = 0$. The parametrization γ is called regular if $\dot{\gamma}(t) \neq 0$ for *every* $t \in (\alpha, \beta)$.

Example. $\gamma(t) = (t^3, t^3)$ is singular at 0 because $\dot{\gamma}(t) = (3t^2, 3t^2)$.

Lemma.

If $\phi : (\tilde{\alpha}, \tilde{\beta}) \rightarrow (\alpha, \beta)$ is invertible with inverse $\psi : (\alpha, \beta) \rightarrow (\tilde{\alpha}, \tilde{\beta})$, then $\phi'(t) \neq 0$ for all $t \in (\tilde{\alpha}, \tilde{\beta})$ and $\psi'(t) \neq 0$ for all $t \in (\alpha, \beta)$.

Proof.

$\psi(\phi(t)) = t$ for *each* t

$\psi'(\phi(t))\phi'(t) = 1$ for *each* t

So, $\psi'(\phi(t)) \neq 0$ and $\phi'(t) \neq 0$ for *each* t

So, $\psi'(t) \neq 0$ and $\phi'(t) \neq 0$

□

Regular parametrization

Definition.

The point $\gamma(t)$ of $\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$ is regular point if $\dot{\gamma}(t) \neq 0$ and singular if $\dot{\gamma}(t) = 0$. The parametrization γ is called regular if $\dot{\gamma}(t) \neq 0$ for *every* $t \in (\alpha, \beta)$.

Example. $\gamma(t) = (t^3, t^3)$ is singular at 0 because $\dot{\gamma}(t) = (3t^2, 3t^2)$.

Lemma.

If $\phi : (\tilde{\alpha}, \tilde{\beta}) \rightarrow (\alpha, \beta)$ is invertible with inverse $\psi : (\alpha, \beta) \rightarrow (\tilde{\alpha}, \tilde{\beta})$, then $\phi'(t) \neq 0$ for all $t \in (\tilde{\alpha}, \tilde{\beta})$ and $\psi'(t) \neq 0$ for all $t \in (\alpha, \beta)$.

Proof.

$\psi(\phi(t)) = t$ for each t

$\psi'(\phi(t))\phi'(t) = 1$ for each t

So, $\psi'(\phi(t)) \neq 0$ and $\phi'(t) \neq 0$ for each t

So, $\psi'(t) \neq 0$ and $\phi'(t) \neq 0$ for each t

□

Regular parametrization

Definition.

The point $\gamma(t)$ of $\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$ is regular point if $\dot{\gamma}(t) \neq 0$ and singular if $\dot{\gamma}(t) = 0$. The parametrization γ is called regular if $\dot{\gamma}(t) \neq 0$ for *every* $t \in (\alpha, \beta)$.

Example. $\gamma(t) = (t^3, t^3)$ is singular at 0 because $\dot{\gamma}(t) = (3t^2, 3t^2)$.

Lemma.

If $\phi : (\tilde{\alpha}, \tilde{\beta}) \rightarrow (\alpha, \beta)$ is invertible with inverse $\psi : (\alpha, \beta) \rightarrow (\tilde{\alpha}, \tilde{\beta})$, then $\phi'(t) \neq 0$ for all $t \in (\tilde{\alpha}, \tilde{\beta})$ and $\psi'(t) \neq 0$ for all $t \in (\alpha, \beta)$.

Proof.

$\psi(\phi(t)) = t$ for *each* t

$\psi'(\phi(t))\phi'(t) = 1$ for *each* t

So, $\psi'(\phi(t)) \neq 0$ and $\phi'(t) \neq 0$ for *each* t

So, $\psi'(t) \neq 0$ and $\phi'(t) \neq 0$ for *each* t because ϕ is bijective. \square

Regular parametrization

Proposition. *A reparametrization of a regular parametrization*

Definition.

The point $\gamma(t)$ of $\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$ is regular point if $\dot{\gamma}(t) \neq 0$ and singular if $\dot{\gamma}(t) = 0$. The parametrization γ is called regular if $\dot{\gamma}(t) \neq 0$ for *every* $t \in (\alpha, \beta)$.

Example. $\gamma(t) = (t^3, t^3)$ is singular at 0 because $\dot{\gamma}(t) = (3t^2, 3t^2)$.

Lemma.

If $\phi : (\tilde{\alpha}, \tilde{\beta}) \rightarrow (\alpha, \beta)$ is invertible with inverse $\psi : (\alpha, \beta) \rightarrow (\tilde{\alpha}, \tilde{\beta})$, then $\phi'(t) \neq 0$ for all $t \in (\tilde{\alpha}, \tilde{\beta})$ and $\psi'(t) \neq 0$ for all $t \in (\alpha, \beta)$.

Proof.

$\psi(\phi(t)) = t$ for each t

$\psi'(\phi(t))\phi'(t) = 1$ for each t

So, $\psi'(\phi(t)) \neq 0$ and $\phi'(t) \neq 0$ for each t

So, $\psi'(t) \neq 0$ and $\phi'(t) \neq 0$ for each t because ϕ is bijective. \square

Regular parametrization

Proposition. *A reparametrization of a regular parametrization is regular.*

Definition.

The point $\gamma(t)$ of $\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$ is regular point if $\dot{\gamma}(t) \neq 0$ and singular if $\dot{\gamma}(t) = 0$. The parametrization $\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$ is called regular if $\dot{\gamma}(t) \neq 0$ for every $t \in (\alpha, \beta)$. □

Example. $\gamma(t) = (t^3, t^3)$ is singular at 0 because $\dot{\gamma}(t) = (3t^2, 3t^2)$.

Lemma.

If $\phi : (\tilde{\alpha}, \tilde{\beta}) \rightarrow (\alpha, \beta)$ is invertible with inverse $\psi : (\alpha, \beta) \rightarrow (\tilde{\alpha}, \tilde{\beta})$, then $\phi'(t) \neq 0$ for all $t \in (\tilde{\alpha}, \tilde{\beta})$ and $\psi'(t) \neq 0$ for all $t \in (\alpha, \beta)$.

Proof.

$\psi(\phi(t)) = t$ for each t

$\psi'(\phi(t))\phi'(t) = 1$ for each t

So, $\psi'(\phi(t)) \neq 0$ and $\phi'(t) \neq 0$ for each t

So, $\psi'(t) \neq 0$ and $\phi'(t) \neq 0$ for each t because ϕ is bijective. □

Regular parametrization

Proposition. *A reparametrization of a regular parametrization is regular.*

Definition.

The point $\gamma(t)$ of $\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$ is regular point if *Proof.*

$\dot{\gamma}(t) \neq 0$ and singular if $\dot{\gamma}(t) = 0$. The parametrization $\gamma : (\alpha, \beta) \rightarrow$
 γ is called regular if $\dot{\gamma}(t) \neq 0$ for *every* $t \in (\alpha, \beta)$. □

Example. $\gamma(t) = (t^3, t^3)$ is singular at 0 because $\dot{\gamma}(t) = (3t^2, 3t^2)$.

Lemma.

If $\phi : (\tilde{\alpha}, \tilde{\beta}) \rightarrow (\alpha, \beta)$ is invertible with inverse $\psi : (\alpha, \beta) \rightarrow (\tilde{\alpha}, \tilde{\beta})$, then $\phi'(t) \neq 0$ for all $t \in (\tilde{\alpha}, \tilde{\beta})$ and $\psi'(t) \neq 0$ for all $t \in (\alpha, \beta)$.

Proof.

$\psi(\phi(t)) = t$ for each t

$\psi'(\phi(t))\phi'(t) = 1$ for each t

So, $\psi'(\phi(t)) \neq 0$ and $\phi'(t) \neq 0$ for each t

So, $\psi'(t) \neq 0$ and $\phi'(t) \neq 0$ for each t because ϕ is bijective. □

Regular parametrization

Proposition. *A reparametrization of a regular parametrization is regular.*

Definition.

The point $\gamma(t)$ of $\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$ is regular point if *Proof.*

$\dot{\gamma}(t) \neq 0$ and singular if $\dot{\gamma}(t) = 0$. The parametrization $\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$ □
 γ is called regular if $\dot{\gamma}(t) \neq 0$ for *every* $t \in (\alpha, \beta)$.

Example. $\gamma(t) = (t^3, t^3)$ is singular at 0 because $\dot{\gamma}(t) = (3t^2, 3t^2)$.

Lemma.

If $\phi : (\tilde{\alpha}, \tilde{\beta}) \rightarrow (\alpha, \beta)$ is invertible with inverse $\psi : (\alpha, \beta) \rightarrow (\tilde{\alpha}, \tilde{\beta})$, then $\phi'(t) \neq 0$ for all $t \in (\tilde{\alpha}, \tilde{\beta})$ and $\psi'(t) \neq 0$ for all $t \in (\alpha, \beta)$.

Proof.

$\psi(\phi(t)) = t$ for each t

$\psi'(\phi(t))\phi'(t) = 1$ for each t

So, $\psi'(\phi(t)) \neq 0$ and $\phi'(t) \neq 0$ for each t

So, $\psi'(t) \neq 0$ and $\phi'(t) \neq 0$ for each t because ϕ is bijective. □

Regular parametrization

Proposition. *A reparametrization of a regular parametrization is regular.*

Definition.

The point $\gamma(t)$ of $\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$ is regular point if *Proof.*

$\dot{\gamma}(t) \neq 0$ and singular if $\dot{\gamma}(t) = 0$. The parametrization $\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$

γ is called regular if $\dot{\gamma}(t) \neq 0$ for every $t \in (\alpha, \beta)$. $\tilde{\gamma} :$ □

Example. $\gamma(t) = (t^3, t^3)$ is singular at 0 because $\dot{\gamma}(t) = (3t^2, 3t^2)$.

Lemma.

If $\phi : (\tilde{\alpha}, \tilde{\beta}) \rightarrow (\alpha, \beta)$ is invertible with inverse $\psi : (\alpha, \beta) \rightarrow (\tilde{\alpha}, \tilde{\beta})$, then $\phi'(t) \neq 0$ for all $t \in (\tilde{\alpha}, \tilde{\beta})$ and $\psi'(t) \neq 0$ for all $t \in (\alpha, \beta)$.

Proof.

$\psi(\phi(t)) = t$ for each t

$\psi'(\phi(t))\phi'(t) = 1$ for each t

So, $\psi'(\phi(t)) \neq 0$ and $\phi'(t) \neq 0$ for each t

So, $\psi'(t) \neq 0$ and $\phi'(t) \neq 0$ for each t because ϕ is bijective. □

Regular parametrization

Proposition. *A reparametrization of a regular parametrization is regular.*

Definition.

The point $\gamma(t)$ of $\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$ is regular point if *Proof.*

$\dot{\gamma}(t) \neq 0$ and singular if $\dot{\gamma}(t) = 0$. The parametrization $\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$
 γ is called regular if $\dot{\gamma}(t) \neq 0$ for *every* $t \in (\alpha, \beta)$. $\tilde{\gamma} : (\tilde{\alpha}, \tilde{\beta}) \rightarrow$ □

Example. $\gamma(t) = (t^3, t^3)$ is singular at 0 because $\dot{\gamma}(t) = (3t^2, 3t^2)$.

Lemma.

If $\phi : (\tilde{\alpha}, \tilde{\beta}) \rightarrow (\alpha, \beta)$ is invertible with inverse $\psi : (\alpha, \beta) \rightarrow (\tilde{\alpha}, \tilde{\beta})$, then $\phi'(t) \neq 0$ for all $t \in (\tilde{\alpha}, \tilde{\beta})$ and $\psi'(t) \neq 0$ for all $t \in (\alpha, \beta)$.

Proof.

$\psi(\phi(t)) = t$ for each t

$\psi'(\phi(t))\phi'(t) = 1$ for each t

So, $\psi'(\phi(t)) \neq 0$ and $\phi'(t) \neq 0$ for each t

So, $\psi'(t) \neq 0$ and $\phi'(t) \neq 0$ for each t because ϕ is bijective. □

Regular parametrization

Proposition. *A reparametrization of a regular parametrization is regular.*

Definition.

The point $\gamma(t)$ of $\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$ is regular point if *Proof.*

$\dot{\gamma}(t) \neq 0$ and singular if $\dot{\gamma}(t) = 0$. The parametrization $\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$
 γ is called regular if $\dot{\gamma}(t) \neq 0$ for every $t \in (\alpha, \beta)$. $\tilde{\gamma} : (\tilde{\alpha}, \tilde{\beta}) \rightarrow \mathbb{R}^2$

□

Example. $\gamma(t) = (t^3, t^3)$ is singular at 0 because $\dot{\gamma}(t) = (3t^2, 3t^2)$.

Lemma.

If $\phi : (\tilde{\alpha}, \tilde{\beta}) \rightarrow (\alpha, \beta)$ is invertible with inverse $\psi : (\alpha, \beta) \rightarrow (\tilde{\alpha}, \tilde{\beta})$, then $\phi'(t) \neq 0$ for all $t \in (\tilde{\alpha}, \tilde{\beta})$ and $\psi'(t) \neq 0$ for all $t \in (\alpha, \beta)$.

Proof.

$\psi(\phi(t)) = t$ for each t

$\psi'(\phi(t))\phi'(t) = 1$ for each t

So, $\psi'(\phi(t)) \neq 0$ and $\phi'(t) \neq 0$ for each t

So, $\psi'(t) \neq 0$ and $\phi'(t) \neq 0$ for each t because ϕ is bijective. □

Regular parametrization

Definition.

The point $\gamma(t)$ of $\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$ is regular point if $\dot{\gamma}(t) \neq 0$ and singular if $\dot{\gamma}(t) = 0$. The parametrization γ is called regular if $\dot{\gamma}(t) \neq 0$ for every $t \in (\alpha, \beta)$.

Example. $\gamma(t) = (t^3, t^3)$ is singular at 0 because $\dot{\gamma}(t) = (3t^2, 3t^2)$.

Lemma.

If $\phi : (\tilde{\alpha}, \tilde{\beta}) \rightarrow (\alpha, \beta)$ is invertible with inverse $\psi : (\alpha, \beta) \rightarrow (\tilde{\alpha}, \tilde{\beta})$, then $\phi'(t) \neq 0$ for all $t \in (\tilde{\alpha}, \tilde{\beta})$ and $\psi'(t) \neq 0$ for all $t \in (\alpha, \beta)$.

Proof.

$\psi(\phi(t)) = t$ for each t
 $\psi'(\phi(t))\phi'(t) = 1$ for each t
So, $\psi'(\phi(t)) \neq 0$ and $\phi'(t) \neq 0$ for each t
So, $\psi'(t) \neq 0$ and $\phi'(t) \neq 0$ for each t because ϕ is bijective. □

Proposition. A reparametrization of a regular parametrization is regular.

Proof.

$$\begin{aligned} \gamma &: (\alpha, \beta) \rightarrow \mathbb{R}^2 \\ \tilde{\gamma} &: (\tilde{\alpha}, \tilde{\beta}) \rightarrow \mathbb{R}^2 \\ \tilde{\gamma}(\tilde{t}) \end{aligned}$$

□

Regular parametrization

Definition.

The point $\gamma(t)$ of $\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$ is regular point if $\dot{\gamma}(t) \neq 0$ and singular if $\dot{\gamma}(t) = 0$. The parametrization γ is called regular if $\dot{\gamma}(t) \neq 0$ for every $t \in (\alpha, \beta)$.

Example. $\gamma(t) = (t^3, t^3)$ is singular at 0 because $\dot{\gamma}(t) = (3t^2, 3t^2)$.

Lemma.

If $\phi : (\tilde{\alpha}, \tilde{\beta}) \rightarrow (\alpha, \beta)$ is invertible with inverse $\psi : (\alpha, \beta) \rightarrow (\tilde{\alpha}, \tilde{\beta})$, then $\phi'(t) \neq 0$ for all $t \in (\tilde{\alpha}, \tilde{\beta})$ and $\psi'(t) \neq 0$ for all $t \in (\alpha, \beta)$.

Proof.

$\psi(\phi(t)) = t$ for each t
 $\psi'(\phi(t))\phi'(t) = 1$ for each t
So, $\psi'(\phi(t)) \neq 0$ and $\phi'(t) \neq 0$ for each t
So, $\psi'(t) \neq 0$ and $\phi'(t) \neq 0$ for each t because ϕ is bijective. □

Proposition. A reparametrization of a regular parametrization is regular.

Proof.

$$\begin{aligned} \gamma &: (\alpha, \beta) \rightarrow \mathbb{R}^2 \\ \tilde{\gamma} &: (\tilde{\alpha}, \tilde{\beta}) \rightarrow \mathbb{R}^2 \\ \tilde{\gamma}(\tilde{t}) &= \gamma(\phi(\tilde{t})) \end{aligned}$$

□

Regular parametrization

Definition.

The point $\gamma(t)$ of $\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$ is regular point if $\dot{\gamma}(t) \neq 0$ and singular if $\dot{\gamma}(t) = 0$. The parametrization γ is called regular if $\dot{\gamma}(t) \neq 0$ for every $t \in (\alpha, \beta)$.

Example. $\gamma(t) = (t^3, t^3)$ is singular at 0 because $\dot{\gamma}(t) = (3t^2, 3t^2)$.

Lemma.

If $\phi : (\tilde{\alpha}, \tilde{\beta}) \rightarrow (\alpha, \beta)$ is invertible with inverse $\psi : (\alpha, \beta) \rightarrow (\tilde{\alpha}, \tilde{\beta})$, then $\phi'(t) \neq 0$ for all $t \in (\tilde{\alpha}, \tilde{\beta})$ and $\psi'(t) \neq 0$ for all $t \in (\alpha, \beta)$.

Proof.

$\psi(\phi(t)) = t$ for each t
 $\psi'(\phi(t))\phi'(t) = 1$ for each t
So, $\psi'(\phi(t)) \neq 0$ and $\phi'(t) \neq 0$ for each t
So, $\psi'(t) \neq 0$ and $\phi'(t) \neq 0$ for each t because ϕ is bijective. □

Proposition. A reparametrization of a regular parametrization is regular.

Proof.

$$\begin{aligned} \gamma &: (\alpha, \beta) \rightarrow \mathbb{R}^2 \\ \tilde{\gamma} &: (\tilde{\alpha}, \tilde{\beta}) \rightarrow \mathbb{R}^2 \\ \tilde{\gamma}(\tilde{t}) &= \gamma(\phi(\tilde{t})) \\ \tilde{\gamma}'(\tilde{t}) &= \gamma'(\phi(\tilde{t})) \end{aligned}$$

□

Regular parametrization

Definition.

The point $\gamma(t)$ of $\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$ is regular point if $\dot{\gamma}(t) \neq 0$ and singular if $\dot{\gamma}(t) = 0$. The parametrization γ is called regular if $\dot{\gamma}(t) \neq 0$ for every $t \in (\alpha, \beta)$.

Example. $\gamma(t) = (t^3, t^3)$ is singular at 0 because $\dot{\gamma}(t) = (3t^2, 3t^2)$.

Lemma.

If $\phi : (\tilde{\alpha}, \tilde{\beta}) \rightarrow (\alpha, \beta)$ is invertible with inverse $\psi : (\alpha, \beta) \rightarrow (\tilde{\alpha}, \tilde{\beta})$, then $\phi'(t) \neq 0$ for all $t \in (\tilde{\alpha}, \tilde{\beta})$ and $\psi'(t) \neq 0$ for all $t \in (\alpha, \beta)$.

Proof.

$\psi(\phi(t)) = t$ for each t
 $\psi'(\phi(t))\phi'(t) = 1$ for each t
So, $\psi'(\phi(t)) \neq 0$ and $\phi'(t) \neq 0$ for each t
So, $\psi'(t) \neq 0$ and $\phi'(t) \neq 0$ for each t because ϕ is bijective. □

Proposition. A reparametrization of a regular parametrization is regular.

Proof.

$$\begin{aligned} \gamma &: (\alpha, \beta) \rightarrow \mathbb{R}^2 \\ \tilde{\gamma} &: (\tilde{\alpha}, \tilde{\beta}) \rightarrow \mathbb{R}^2 \\ \tilde{\gamma}(\tilde{t}) &= \gamma(\phi(\tilde{t})) \\ \tilde{\gamma}'(\tilde{t}) &= \gamma'(\phi(\tilde{t}))\phi'(\tilde{t}) \end{aligned}$$

□

Regular parametrization

Definition.

The point $\gamma(t)$ of $\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$ is regular point if $\dot{\gamma}(t) \neq 0$ and singular if $\dot{\gamma}(t) = 0$. The parametrization γ is called regular if $\dot{\gamma}(t) \neq 0$ for every $t \in (\alpha, \beta)$.

Example. $\gamma(t) = (t^3, t^3)$ is singular at 0 because $\dot{\gamma}(t) = (3t^2, 3t^2)$.

Lemma.

If $\phi : (\tilde{\alpha}, \tilde{\beta}) \rightarrow (\alpha, \beta)$ is invertible with inverse $\psi : (\alpha, \beta) \rightarrow (\tilde{\alpha}, \tilde{\beta})$, then $\phi'(t) \neq 0$ for all $t \in (\tilde{\alpha}, \tilde{\beta})$ and $\psi'(t) \neq 0$ for all $t \in (\alpha, \beta)$.

Proof.

$\psi(\phi(t)) = t$ for each t
 $\psi'(\phi(t))\phi'(t) = 1$ for each t
So, $\psi'(\phi(t)) \neq 0$ and $\phi'(t) \neq 0$ for each t
So, $\psi'(t) \neq 0$ and $\phi'(t) \neq 0$ for each t because ϕ is bijective. □

Proposition. A reparametrization of a regular parametrization is regular.

Proof.

$$\begin{aligned} \gamma &: (\alpha, \beta) \rightarrow \mathbb{R}^2 \\ \tilde{\gamma} &: (\tilde{\alpha}, \tilde{\beta}) \rightarrow \mathbb{R}^2 \\ \tilde{\gamma}(\tilde{t}) &= \gamma(\phi(\tilde{t})) \\ \tilde{\gamma}'(\tilde{t}) &= \gamma'(\phi(\tilde{t}))\phi'(\tilde{t}) \\ \text{But, } \gamma'(t) &\neq 0 \end{aligned}$$

□

Regular parametrization

Definition.

The point $\gamma(t)$ of $\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$ is regular point if $\dot{\gamma}(t) \neq 0$ and singular if $\dot{\gamma}(t) = 0$. The parametrization γ is called regular if $\dot{\gamma}(t) \neq 0$ for every $t \in (\alpha, \beta)$.

Example. $\gamma(t) = (t^3, t^3)$ is singular at 0 because $\dot{\gamma}(t) = (3t^2, 3t^2)$.

Lemma.

If $\phi : (\tilde{\alpha}, \tilde{\beta}) \rightarrow (\alpha, \beta)$ is invertible with inverse $\psi : (\alpha, \beta) \rightarrow (\tilde{\alpha}, \tilde{\beta})$, then $\phi'(t) \neq 0$ for all $t \in (\tilde{\alpha}, \tilde{\beta})$ and $\psi'(t) \neq 0$ for all $t \in (\alpha, \beta)$.

Proof.

$\psi(\phi(t)) = t$ for each t
 $\psi'(\phi(t))\phi'(t) = 1$ for each t
So, $\psi'(\phi(t)) \neq 0$ and $\phi'(t) \neq 0$ for each t
So, $\psi'(t) \neq 0$ and $\phi'(t) \neq 0$ for each t because ϕ is bijective. □

Proposition. A reparametrization of a regular parametrization is regular.

Proof.

$\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$
 $\tilde{\gamma} : (\tilde{\alpha}, \tilde{\beta}) \rightarrow \mathbb{R}^2$
 $\tilde{\gamma}(\tilde{t}) = \gamma(\phi(\tilde{t}))$
 $\tilde{\gamma}'(\tilde{t}) = \gamma'(\phi(\tilde{t}))\phi'(\tilde{t})$
But, $\gamma'(t) \neq 0$ for all t , □

Regular parametrization

Definition.

The point $\gamma(t)$ of $\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$ is regular point if $\dot{\gamma}(t) \neq 0$ and singular if $\dot{\gamma}(t) = 0$. The parametrization γ is called regular if $\dot{\gamma}(t) \neq 0$ for every $t \in (\alpha, \beta)$.

Example. $\gamma(t) = (t^3, t^3)$ is singular at 0 because $\dot{\gamma}(t) = (3t^2, 3t^2)$.

Lemma.

If $\phi : (\tilde{\alpha}, \tilde{\beta}) \rightarrow (\alpha, \beta)$ is invertible with inverse $\psi : (\alpha, \beta) \rightarrow (\tilde{\alpha}, \tilde{\beta})$, then $\phi'(t) \neq 0$ for all $t \in (\tilde{\alpha}, \tilde{\beta})$ and $\psi'(t) \neq 0$ for all $t \in (\alpha, \beta)$.

Proof.

$\psi(\phi(t)) = t$ for each t
 $\psi'(\phi(t))\phi'(t) = 1$ for each t
So, $\psi'(\phi(t)) \neq 0$ and $\phi'(t) \neq 0$ for each t
So, $\psi'(t) \neq 0$ and $\phi'(t) \neq 0$ for each t because ϕ is bijective. □

Proposition. A reparametrization of a regular parametrization is regular.

Proof.

$\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$
 $\tilde{\gamma} : (\tilde{\alpha}, \tilde{\beta}) \rightarrow \mathbb{R}^2$
 $\tilde{\gamma}(\tilde{t}) = \gamma(\phi(\tilde{t}))$
 $\tilde{\gamma}'(\tilde{t}) = \gamma'(\phi(\tilde{t}))\phi'(\tilde{t})$
But, $\gamma'(t) \neq 0$ for all t , therefore, even for $\phi(\tilde{t})$ □

Regular parametrization

Definition.

The point $\gamma(t)$ of $\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$ is regular point if $\dot{\gamma}(t) \neq 0$ and singular if $\dot{\gamma}(t) = 0$. The parametrization γ is called regular if $\dot{\gamma}(t) \neq 0$ for every $t \in (\alpha, \beta)$.

Example. $\gamma(t) = (t^3, t^3)$ is singular at 0 because $\dot{\gamma}(t) = (3t^2, 3t^2)$.

Lemma.

If $\phi : (\tilde{\alpha}, \tilde{\beta}) \rightarrow (\alpha, \beta)$ is invertible with inverse $\psi : (\alpha, \beta) \rightarrow (\tilde{\alpha}, \tilde{\beta})$, then $\phi'(t) \neq 0$ for all $t \in (\tilde{\alpha}, \tilde{\beta})$ and $\psi'(t) \neq 0$ for all $t \in (\alpha, \beta)$.

Proof.

$\psi(\phi(t)) = t$ for each t
 $\psi'(\phi(t))\phi'(t) = 1$ for each t
So, $\psi'(\phi(t)) \neq 0$ and $\phi'(t) \neq 0$ for each t
So, $\psi'(t) \neq 0$ and $\phi'(t) \neq 0$ for each t because ϕ is bijective. □

Proposition. A reparametrization of a regular parametrization is regular.

Proof.

$\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$
 $\tilde{\gamma} : (\tilde{\alpha}, \tilde{\beta}) \rightarrow \mathbb{R}^2$
 $\tilde{\gamma}(\tilde{t}) = \gamma(\phi(\tilde{t}))$
 $\tilde{\gamma}'(\tilde{t}) = \gamma'(\phi(\tilde{t}))\phi'(\tilde{t})$
But, $\gamma'(t) \neq 0$ for all t , therefore, even for $\phi(\tilde{t})$ and, $\phi'(\tilde{t}) \neq 0$ □

Regular parametrization

Definition.

The point $\gamma(t)$ of $\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$ is regular point if $\dot{\gamma}(t) \neq 0$ and singular if $\dot{\gamma}(t) = 0$. The parametrization γ is called regular if $\dot{\gamma}(t) \neq 0$ for every $t \in (\alpha, \beta)$.

Example. $\gamma(t) = (t^3, t^3)$ is singular at 0 because $\dot{\gamma}(t) = (3t^2, 3t^2)$.

Lemma.

If $\phi : (\tilde{\alpha}, \tilde{\beta}) \rightarrow (\alpha, \beta)$ is invertible with inverse $\psi : (\alpha, \beta) \rightarrow (\tilde{\alpha}, \tilde{\beta})$, then $\phi'(t) \neq 0$ for all $t \in (\tilde{\alpha}, \tilde{\beta})$ and $\psi'(t) \neq 0$ for all $t \in (\alpha, \beta)$.

Proof.

$\psi(\phi(t)) = t$ for each t
 $\psi'(\phi(t))\phi'(t) = 1$ for each t
So, $\psi'(\phi(t)) \neq 0$ and $\phi'(t) \neq 0$ for each t
So, $\psi'(t) \neq 0$ and $\phi'(t) \neq 0$ for each t because ϕ is bijective. □

Proposition. A reparametrization of a regular parametrization is regular.

Proof.

$\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$
 $\tilde{\gamma} : (\tilde{\alpha}, \tilde{\beta}) \rightarrow \mathbb{R}^2$
 $\tilde{\gamma}(\tilde{t}) = \gamma(\phi(\tilde{t}))$
 $\tilde{\gamma}'(\tilde{t}) = \gamma'(\phi(\tilde{t}))\phi'(\tilde{t})$
But, $\gamma'(t) \neq 0$ for all t , therefore, even for $\phi(\tilde{t})$
and, $\phi'(\tilde{t}) \neq 0$ for all \tilde{t} □

Regular parametrization

Definition.

The point $\gamma(t)$ of $\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$ is regular point if $\dot{\gamma}(t) \neq 0$ and singular if $\dot{\gamma}(t) = 0$. The parametrization γ is called regular if $\dot{\gamma}(t) \neq 0$ for every $t \in (\alpha, \beta)$.

Example. $\gamma(t) = (t^3, t^3)$ is singular at 0 because $\dot{\gamma}(t) = (3t^2, 3t^2)$.

Lemma.

If $\phi : (\tilde{\alpha}, \tilde{\beta}) \rightarrow (\alpha, \beta)$ is invertible with inverse $\psi : (\alpha, \beta) \rightarrow (\tilde{\alpha}, \tilde{\beta})$, then $\phi'(t) \neq 0$ for all $t \in (\tilde{\alpha}, \tilde{\beta})$ and $\psi'(t) \neq 0$ for all $t \in (\alpha, \beta)$.

Proof.

$\psi(\phi(t)) = t$ for each t
 $\psi'(\phi(t))\phi'(t) = 1$ for each t
So, $\psi'(\phi(t)) \neq 0$ and $\phi'(t) \neq 0$ for each t
So, $\psi'(t) \neq 0$ and $\phi'(t) \neq 0$ for each t because ϕ is bijective. □

Proposition. A reparametrization of a regular parametrization is regular.

Proof.

$\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$
 $\tilde{\gamma} : (\tilde{\alpha}, \tilde{\beta}) \rightarrow \mathbb{R}^2$
 $\tilde{\gamma}(\tilde{t}) = \gamma(\phi(\tilde{t}))$
 $\tilde{\gamma}'(\tilde{t}) = \gamma'(\phi(\tilde{t}))\phi'(\tilde{t})$
But, $\gamma'(t) \neq 0$ for all t , therefore, even for $\phi(\tilde{t})$
and, $\phi'(\tilde{t}) \neq 0$ for all \tilde{t}

□

Inner product:

Inner product:

$$v = (2, 3)$$

Inner product:

$$v = (2, 3)$$

$$w = (2, 1)$$

Inner product:

$$v = (2, 3)$$

$$w = (2, 1)$$

$$v.w$$

Inner product:

$$v = (2, 3)$$

$$w = (2, 1)$$

$$v.w := (2, 3).(2, 1)$$

Inner product:

$$v = (2, 3)$$

$$w = (2, 1)$$

$$v.w := (2, 3).(2, 1) = 2 \times 2 + 3 \times 1$$

Inner product:

$$v = (2, 3)$$

$$w = (2, 1)$$

$$v.w := (2, 3).(2, 1) = 2 \times 2 + 3 \times 1 = 7$$

Inner product:

$$v = (2, 3)$$

$$w = (2, 1)$$

$$v.w := (2, 3).(2, 1) = 2 \times 2 + 3 \times 1 = 7$$

In general:

Inner product:

$$v = (2, 3)$$

$$w = (2, 1)$$

$$v.w := (2, 3).(2, 1) = 2 \times 2 + 3 \times 1 = 7$$

In general:

$$(x_1, y_1).(x_2, y_2) := x_1x_2 + y_1y_2$$

Inner product:

$$v = (2, 3)$$

$$w = (2, 1)$$

$$v.w := (2, 3).(2, 1) = 2 \times 2 + 3 \times 1 = 7$$

In general:

$$(x_1, y_1).(x_2, y_2) := x_1x_2 + y_1y_2$$

Exercise. For $\mathbf{v} : (\alpha, \beta) \rightarrow \mathbf{R}^2$

Inner product:

$$v = (2, 3)$$

$$w = (2, 1)$$

$$v.w := (2, 3).(2, 1) = 2 \times 2 + 3 \times 1 = 7$$

In general:

$$(x_1, y_1).(x_2, y_2) := x_1x_2 + y_1y_2$$

Exercise. For $\mathbf{v} : (\alpha, \beta) \rightarrow \mathbf{R}^2$ and $\mathbf{w} : (\alpha, \beta) \rightarrow \mathbf{R}^2$,

Inner product:

$$v = (2, 3)$$

$$w = (2, 1)$$

$$v.w := (2, 3).(2, 1) = 2 \times 2 + 3 \times 1 = 7$$

In general:

$$(x_1, y_1).(x_2, y_2) := x_1x_2 + y_1y_2$$

Exercise. For $\mathbf{v} : (\alpha, \beta) \rightarrow \mathbf{R}^2$ and $\mathbf{w} : (\alpha, \beta) \rightarrow \mathbf{R}^2$,
show that $(\mathbf{v}(t).\mathbf{w}(t))' = \mathbf{v}'(t).\mathbf{w}(t) + \mathbf{v}(t).\mathbf{w}'(t)$

Inner product:

$$v = (2, 3)$$

$$w = (2, 1)$$

$$v.w := (2, 3).(2, 1) = 2 \times 2 + 3 \times 1 = 7$$

In general:

$$(x_1, y_1).(x_2, y_2) := x_1x_2 + y_1y_2$$

Exercise. For $\mathbf{v} : (\alpha, \beta) \rightarrow \mathbf{R}^2$ and $\mathbf{w} : (\alpha, \beta) \rightarrow \mathbf{R}^2$,
show that $(\mathbf{v}(t).\mathbf{w}(t))' = \mathbf{v}'(t).\mathbf{w}(t) + \mathbf{v}(t).\mathbf{w}'(t)$

$$v = (x, y)$$

Inner product:

$$v = (2, 3)$$

$$w = (2, 1)$$

$$v.w := (2, 3).(2, 1) = 2 \times 2 + 3 \times 1 = 7$$

In general:

$$(x_1, y_1).(x_2, y_2) := x_1x_2 + y_1y_2$$

Exercise. For $\mathbf{v} : (\alpha, \beta) \rightarrow \mathbf{R}^2$ and $\mathbf{w} : (\alpha, \beta) \rightarrow \mathbf{R}^2$,
show that $(\mathbf{v}(t).\mathbf{w}(t))' = \mathbf{v}'(t).\mathbf{w}(t) + \mathbf{v}(t).\mathbf{w}'(t)$

$$v = (x, y)$$

$$v.v$$

Inner product:

$$v = (2, 3)$$

$$w = (2, 1)$$

$$v.w := (2, 3).(2, 1) = 2 \times 2 + 3 \times 1 = 7$$

In general:

$$(x_1, y_1).(x_2, y_2) := x_1x_2 + y_1y_2$$

Exercise. For $\mathbf{v} : (\alpha, \beta) \rightarrow \mathbf{R}^2$ and $\mathbf{w} : (\alpha, \beta) \rightarrow \mathbf{R}^2$,
show that $(\mathbf{v}(t).\mathbf{w}(t))' = \mathbf{v}'(t).\mathbf{w}(t) + \mathbf{v}(t).\mathbf{w}'(t)$

$$v = (x, y)$$

$$v.v = (x, y).(x, y) = x^2 + y^2$$

Inner product:

$$v = (2, 3)$$

$$w = (2, 1)$$

$$v.w := (2, 3).(2, 1) = 2 \times 2 + 3 \times 1 = 7$$

In general:

$$(x_1, y_1).(x_2, y_2) := x_1x_2 + y_1y_2$$

Exercise. For $\mathbf{v} : (\alpha, \beta) \rightarrow \mathbf{R}^2$ and $\mathbf{w} : (\alpha, \beta) \rightarrow \mathbf{R}^2$,
show that $(\mathbf{v}(t).\mathbf{w}(t))' = \mathbf{v}'(t).\mathbf{w}(t) + \mathbf{v}(t).\mathbf{w}'(t)$

$$v = (x, y)$$

$$v.v = (x, y).(x, y) = x^2 + y^2$$

$$||(x, y)|| = \sqrt{x^2 + y^2}$$

Inner product:

$$v = (2, 3)$$

$$w = (2, 1)$$

$$v.w := (2, 3).(2, 1) = 2 \times 2 + 3 \times 1 = 7$$

In general:

$$(x_1, y_1).(x_2, y_2) := x_1x_2 + y_1y_2$$

Exercise. For $\mathbf{v} : (\alpha, \beta) \rightarrow \mathbf{R}^2$ and $\mathbf{w} : (\alpha, \beta) \rightarrow \mathbf{R}^2$,
show that $(\mathbf{v}(t).\mathbf{w}(t))' = \mathbf{v}'(t).\mathbf{w}(t) + \mathbf{v}(t).\mathbf{w}'(t)$

$$v = (x, y)$$

$$v.v = (x, y).(x, y) = x^2 + y^2$$

$$||(x, y)|| = \sqrt{x^2 + y^2}$$

$$v = (x, y)$$

Inner product:

$$v = (2, 3)$$

$$w = (2, 1)$$

$$v.w := (2, 3).(2, 1) = 2 \times 2 + 3 \times 1 = 7$$

In general:

$$(x_1, y_1).(x_2, y_2) := x_1x_2 + y_1y_2$$

Exercise. For $\mathbf{v} : (\alpha, \beta) \rightarrow \mathbf{R}^2$ and $\mathbf{w} : (\alpha, \beta) \rightarrow \mathbf{R}^2$,
show that $(\mathbf{v}(t).\mathbf{w}(t))' = \mathbf{v}'(t).\mathbf{w}(t) + \mathbf{v}(t).\mathbf{w}'(t)$

$$v = (x, y)$$

$$v.v = (x, y).(x, y) = x^2 + y^2$$

$$||(x, y)|| = \sqrt{x^2 + y^2}$$

$$v = (x, y)$$

$$v.w =$$

Inner product:

$$v = (2, 3)$$

$$w = (2, 1)$$

$$v.w := (2, 3).(2, 1) = 2 \times 2 + 3 \times 1 = 7$$

In general:

$$(x_1, y_1).(x_2, y_2) := x_1x_2 + y_1y_2$$

Exercise. For $\mathbf{v} : (\alpha, \beta) \rightarrow \mathbf{R}^2$ and $\mathbf{w} : (\alpha, \beta) \rightarrow \mathbf{R}^2$,
show that $(\mathbf{v}(t).\mathbf{w}(t))' = \mathbf{v}'(t).\mathbf{w}(t) + \mathbf{v}(t).\mathbf{w}'(t)$

$$v = (x, y)$$

$$v.v = (x, y).(x, y) = x^2 + y^2$$

$$||(x, y)|| = \sqrt{x^2 + y^2}$$

$$v = (x, y)$$

$$v.w = ||v||||w|| \cos(\theta) \text{ where,}$$

θ is the angle between v and w

Inner product:

$$v = (2, 3)$$

$$w = (2, 1)$$

$$v.w := (2, 3).(2, 1) = 2 \times 2 + 3 \times 1 = 7$$

In general:

$$(x_1, y_1).(x_2, y_2) := x_1x_2 + y_1y_2$$

Exercise. For $\mathbf{v} : (\alpha, \beta) \rightarrow \mathbf{R}^2$ and $\mathbf{w} : (\alpha, \beta) \rightarrow \mathbf{R}^2$, show that $(\mathbf{v}(t).\mathbf{w}(t))' = \mathbf{v}'(t).\mathbf{w}(t) + \mathbf{v}(t).\mathbf{w}'(t)$

$$v = (x, y)$$

$$v.v = (x, y).(x, y) = x^2 + y^2$$

$$||(x, y)|| = \sqrt{x^2 + y^2}$$

$$v = (x, y)$$

$$v.w = ||v||||w|| \cos(\theta) \text{ where,}$$

θ is the angle between v and w

Definition. $\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$ is a unit speed parametrization

Inner product:

$$v = (2, 3)$$

$$w = (2, 1)$$

$$v.w := (2, 3).(2, 1) = 2 \times 2 + 3 \times 1 = 7$$

In general:

$$(x_1, y_1).(x_2, y_2) := x_1x_2 + y_1y_2$$

Exercise. For $\mathbf{v} : (\alpha, \beta) \rightarrow \mathbf{R}^2$ and $\mathbf{w} : (\alpha, \beta) \rightarrow \mathbf{R}^2$, show that $(\mathbf{v}(t).\mathbf{w}(t))' = \mathbf{v}'(t).\mathbf{w}(t) + \mathbf{v}(t).\mathbf{w}'(t)$

$$v = (x, y)$$

$$v.v = (x, y).(x, y) = x^2 + y^2$$

$$||(x, y)|| = \sqrt{x^2 + y^2}$$

$$v = (x, y)$$

$$v.w = ||v||||w|| \cos(\theta) \text{ where,}$$

θ is the angle between v and w

Definition. $\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$ is a unit speed parametrization if $||\dot{\gamma}(t)|| = 1$

Inner product:

$$v = (2, 3)$$

$$w = (2, 1)$$

$$v.w := (2, 3).(2, 1) = 2 \times 2 + 3 \times 1 = 7$$

In general:

$$(x_1, y_1).(x_2, y_2) := x_1x_2 + y_1y_2$$

Exercise. For $\mathbf{v} : (\alpha, \beta) \rightarrow \mathbf{R}^2$ and $\mathbf{w} : (\alpha, \beta) \rightarrow \mathbf{R}^2$, show that $(\mathbf{v}(t).\mathbf{w}(t))' = \mathbf{v}'(t).\mathbf{w}(t) + \mathbf{v}(t).\mathbf{w}'(t)$

$$v = (x, y)$$

$$v.v = (x, y).(x, y) = x^2 + y^2$$

$$||(x, y)|| = \sqrt{x^2 + y^2}$$

$$v = (x, y)$$

$$v.w = ||v||||w|| \cos(\theta) \text{ where,}$$

θ is the angle between v and w

Definition. $\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$ is a unit speed parametrization if $||\dot{\gamma}(t)|| = 1$ for each $t \in (\alpha, \beta)$

Inner product:

$$v = (2, 3)$$

$$w = (2, 1)$$

$$v.w := (2, 3).(2, 1) = 2 \times 2 + 3 \times 1 = 7$$

In general:

$$(x_1, y_1).(x_2, y_2) := x_1x_2 + y_1y_2$$

Exercise. For $\mathbf{v} : (\alpha, \beta) \rightarrow \mathbf{R}^2$ and $\mathbf{w} : (\alpha, \beta) \rightarrow \mathbf{R}^2$, show that $(\mathbf{v}(t).\mathbf{w}(t))' = \mathbf{v}'(t).\mathbf{w}(t) + \mathbf{v}(t).\mathbf{w}'(t)$

$$v = (x, y)$$

$$v.v = (x, y).(x, y) = x^2 + y^2$$

$$||(x, y)|| = \sqrt{x^2 + y^2}$$

$$v = (x, y)$$

$$v.w = ||v||||w|| \cos(\theta) \text{ where,}$$

θ is the angle between v and w

Definition. $\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$ is a unit speed parametrization if $||\dot{\gamma}(t)|| = 1$ for each $t \in (\alpha, \beta)$

Theorem. *If $\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$ is a unit speed parametrization*

Inner product:

$$v = (2, 3)$$

$$w = (2, 1)$$

$$v.w := (2, 3).(2, 1) = 2 \times 2 + 3 \times 1 = 7$$

In general:

$$(x_1, y_1).(x_2, y_2) := x_1x_2 + y_1y_2$$

Exercise. For $\mathbf{v} : (\alpha, \beta) \rightarrow \mathbf{R}^2$ and $\mathbf{w} : (\alpha, \beta) \rightarrow \mathbf{R}^2$, show that $(\mathbf{v}(t).\mathbf{w}(t))' = \mathbf{v}'(t).\mathbf{w}(t) + \mathbf{v}(t).\mathbf{w}'(t)$

$$v = (x, y)$$

$$v.v = (x, y).(x, y) = x^2 + y^2$$

$$||(x, y)|| = \sqrt{x^2 + y^2}$$

$$v = (x, y)$$

$$v.w = ||v||||w||\cos(\theta) \text{ where,}$$

θ is the angle between v and w

Definition. $\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$ is a unit speed parametrization if $||\dot{\gamma}(t)|| = 1$ for each $t \in (\alpha, \beta)$

Theorem. If $\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$ is a unit speed parametrization, then $\dot{\gamma}(t).\ddot{\gamma}(t) = 0$

Inner product:

$$v = (2, 3)$$

$$w = (2, 1)$$

$$v.w := (2, 3).(2, 1) = 2 \times 2 + 3 \times 1 = 7$$

In general:

$$(x_1, y_1).(x_2, y_2) := x_1x_2 + y_1y_2$$

Exercise. For $\mathbf{v} : (\alpha, \beta) \rightarrow \mathbf{R}^2$ and $\mathbf{w} : (\alpha, \beta) \rightarrow \mathbf{R}^2$,
show that $(\mathbf{v}(t).\mathbf{w}(t))' = \mathbf{v}'(t).\mathbf{w}(t) + \mathbf{v}(t).\mathbf{w}'(t)$

$$v = (x, y)$$

$$v.v = (x, y).(x, y) = x^2 + y^2$$

$$||(x, y)|| = \sqrt{x^2 + y^2}$$

$$v = (x, y)$$

$$v.w = ||v||||w||\cos(\theta) \text{ where,}$$

θ is the angle between v and w

Definition. $\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$ is a unit speed parametrization if $||\dot{\gamma}(t)|| = 1$ for each $t \in (\alpha, \beta)$

Theorem. If $\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$ is a unit speed parametrization, then $\dot{\gamma}(t).\ddot{\gamma}(t) = 0$ for all $t \in (\alpha, \beta)$.

Inner product:

$$v = (2, 3)$$

$$w = (2, 1)$$

$$v.w := (2, 3).(2, 1) = 2 \times 2 + 3 \times 1 = 7$$

In general:

$$(x_1, y_1).(x_2, y_2) := x_1x_2 + y_1y_2$$

Exercise. For $\mathbf{v} : (\alpha, \beta) \rightarrow \mathbf{R}^2$ and $\mathbf{w} : (\alpha, \beta) \rightarrow \mathbf{R}^2$, show that $(\mathbf{v}(t).\mathbf{w}(t))' = \mathbf{v}'(t).\mathbf{w}(t) + \mathbf{v}(t).\mathbf{w}'(t)$

$$v = (x, y)$$

$$v.v = (x, y).(x, y) = x^2 + y^2$$

$$||(x, y)|| = \sqrt{x^2 + y^2}$$

$$v = (x, y)$$

$$v.w = ||v||||w|| \cos(\theta) \text{ where,}$$

θ is the angle between v and w

Definition. $\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$ is a unit speed parametrization if $||\dot{\gamma}(t)|| = 1$ for each $t \in (\alpha, \beta)$

Theorem. If $\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$ is a unit speed parametrization, then $\dot{\gamma}(t).\ddot{\gamma}(t) = 0$ for all $t \in (\alpha, \beta)$.

Proof. $||\dot{\gamma}(t)|| = 1$

□

Inner product:

$$v = (2, 3)$$

$$w = (2, 1)$$

$$v.w := (2, 3).(2, 1) = 2 \times 2 + 3 \times 1 = 7$$

In general:

$$(x_1, y_1).(x_2, y_2) := x_1x_2 + y_1y_2$$

Exercise. For $\mathbf{v} : (\alpha, \beta) \rightarrow \mathbf{R}^2$ and $\mathbf{w} : (\alpha, \beta) \rightarrow \mathbf{R}^2$, $\dot{\gamma}(t).\dot{\gamma}(t) = 1$
show that $(\mathbf{v}(t).\mathbf{w}(t))' = \mathbf{v}'(t).\mathbf{w}(t) + \mathbf{v}(t).\mathbf{w}'(t)$

$$v = (x, y)$$

$$v.v = (x, y).(x, y) = x^2 + y^2$$

$$||(x, y)|| = \sqrt{x^2 + y^2}$$

$$v = (x, y)$$

$$v.w = ||v||||w|| \cos(\theta) \text{ where,}$$

θ is the angle between v and w

Definition. $\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$ is a unit speed parametrization if $||\dot{\gamma}(t)|| = 1$ for each $t \in (\alpha, \beta)$

Theorem. If $\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$ is a unit speed parametrization, then $\dot{\gamma}(t).\ddot{\gamma}(t) = 0$ for all $t \in (\alpha, \beta)$.

Proof. $||\dot{\gamma}(t)|| = 1$

□

Inner product:

$$v = (2, 3)$$

$$w = (2, 1)$$

$$v.w := (2, 3).(2, 1) = 2 \times 2 + 3 \times 1 = 7$$

In general:

$$(x_1, y_1).(x_2, y_2) := x_1x_2 + y_1y_2$$

Exercise. For $\mathbf{v} : (\alpha, \beta) \rightarrow \mathbf{R}^2$ and $\mathbf{w} : (\alpha, \beta) \rightarrow \mathbf{R}^2$, $\dot{\gamma}(t).\dot{\gamma}(t) = 1$

show that $(\mathbf{v}(t).\mathbf{w}(t))' = \mathbf{v}'(t).\mathbf{w}(t) + \mathbf{v}(t).\mathbf{w}'(t)$ $(\dot{\gamma}(t).\dot{\gamma}(t))' = 0$

Definition. $\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$ is a unit speed parametrization if $||\dot{\gamma}(t)|| = 1$ for each $t \in (\alpha, \beta)$

Theorem. If $\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$ is a unit speed parametrization, then $\dot{\gamma}(t).\ddot{\gamma}(t) = 0$ for all $t \in (\alpha, \beta)$.

Proof. $||\dot{\gamma}(t)|| = 1$

□

$$v = (x, y)$$

$$v.v = (x, y).(x, y) = x^2 + y^2$$

$$||(x, y)|| = \sqrt{x^2 + y^2}$$

$$v = (x, y)$$

$$v.w = ||v||||w||\cos(\theta) \text{ where,}$$

θ is the angle between v and w

Inner product:

$$v = (2, 3)$$

$$w = (2, 1)$$

$$v.w := (2, 3).(2, 1) = 2 \times 2 + 3 \times 1 = 7$$

In general:

$$(x_1, y_1).(x_2, y_2) := x_1x_2 + y_1y_2$$

Exercise. For $\mathbf{v} : (\alpha, \beta) \rightarrow \mathbf{R}^2$ and $\mathbf{w} : (\alpha, \beta) \rightarrow \mathbf{R}^2$, show that $(\mathbf{v}(t).\mathbf{w}(t))' = \mathbf{v}'(t).\mathbf{w}(t) + \mathbf{v}(t).\mathbf{w}'(t)$

$$\begin{aligned} v &= (x, y) \\ v.v &= (x, y).(x, y) = x^2 + y^2 \\ ||(x, y)|| &= \sqrt{x^2 + y^2} \end{aligned}$$

$$\begin{aligned} v &= (x, y) \\ v.w &= ||v||||w||\cos(\theta) \text{ where,} \\ \theta &\text{ is the angle between } v \text{ and } w \end{aligned}$$

Definition. $\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$ is a unit speed parametrization if $||\dot{\gamma}(t)|| = 1$ for each $t \in (\alpha, \beta)$

Theorem. If $\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$ is a unit speed parametrization, then $\dot{\gamma}(t).\ddot{\gamma}(t) = 0$ for all $t \in (\alpha, \beta)$.

$$\begin{aligned} \textit{Proof. } ||\dot{\gamma}(t)|| &= 1 \\ \dot{\gamma}(t).\dot{\gamma}(t) &= 1 \\ (\dot{\gamma}(t).\dot{\gamma}(t))' &= 0 \\ \ddot{\gamma}(t).\dot{\gamma}(t) + \dot{\gamma}(t).\ddot{\gamma}(t) &= 0 \end{aligned}$$

□

Inner product:

$$v = (2, 3)$$

$$w = (2, 1)$$

$$v.w := (2, 3).(2, 1) = 2 \times 2 + 3 \times 1 = 7$$

In general:

$$(x_1, y_1).(x_2, y_2) := x_1x_2 + y_1y_2$$

Exercise. For $\mathbf{v} : (\alpha, \beta) \rightarrow \mathbf{R}^2$ and $\mathbf{w} : (\alpha, \beta) \rightarrow \mathbf{R}^2$, show that $(\mathbf{v}(t).\mathbf{w}(t))' = \mathbf{v}'(t).\mathbf{w}(t) + \mathbf{v}(t).\mathbf{w}'(t)$

$$\begin{aligned} v &= (x, y) \\ v.v &= (x, y).(x, y) = x^2 + y^2 \\ ||(x, y)|| &= \sqrt{x^2 + y^2} \end{aligned}$$

$$\begin{aligned} v &= (x, y) \\ v.w &= ||v||||w||\cos(\theta) \text{ where,} \\ \theta &\text{ is the angle between } v \text{ and } w \end{aligned}$$

Definition. $\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$ is a unit speed parametrization if $||\dot{\gamma}(t)|| = 1$ for each $t \in (\alpha, \beta)$

Theorem. If $\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$ is a unit speed parametrization, then $\dot{\gamma}(t).\ddot{\gamma}(t) = 0$ for all $t \in (\alpha, \beta)$.

$$\begin{aligned} \textit{Proof. } ||\dot{\gamma}(t)|| &= 1 \\ \dot{\gamma}(t).\dot{\gamma}(t) &= 1 \\ (\dot{\gamma}(t).\dot{\gamma}(t))' &= 0 \\ \ddot{\gamma}(t).\dot{\gamma}(t) + \dot{\gamma}(t).\ddot{\gamma}(t) &= 0 \\ 2\ddot{\gamma}(t).\dot{\gamma}(t) &= 0 \end{aligned}$$

□

Inner product:

$$v = (2, 3)$$

$$w = (2, 1)$$

$$v.w := (2, 3).(2, 1) = 2 \times 2 + 3 \times 1 = 7$$

In general:

$$(x_1, y_1).(x_2, y_2) := x_1x_2 + y_1y_2$$

Exercise. For $\mathbf{v} : (\alpha, \beta) \rightarrow \mathbf{R}^2$ and $\mathbf{w} : (\alpha, \beta) \rightarrow \mathbf{R}^2$, show that $(\mathbf{v}(t).\mathbf{w}(t))' = \mathbf{v}'(t).\mathbf{w}(t) + \mathbf{v}(t).\mathbf{w}'(t)$

$$v = (x, y)$$

$$v.v = (x, y).(x, y) = x^2 + y^2$$

$$||(x, y)|| = \sqrt{x^2 + y^2}$$

$$v = (x, y)$$

$$v.w = ||v||||w|| \cos(\theta) \text{ where,}$$

θ is the angle between v and w

Definition. $\gamma : (\alpha, \beta) \rightarrow \mathbf{R}^2$ is a unit speed parametrization if $||\dot{\gamma}(t)|| = 1$ for each $t \in (\alpha, \beta)$

Theorem. If $\gamma : (\alpha, \beta) \rightarrow \mathbf{R}^2$ is a unit speed parametrization, then $\dot{\gamma}(t).\ddot{\gamma}(t) = 0$ for all $t \in (\alpha, \beta)$.

Proof. $||\dot{\gamma}(t)|| = 1$

$$\dot{\gamma}(t).\dot{\gamma}(t) = 1$$

$$(\dot{\gamma}(t).\dot{\gamma}(t))' = 0$$

$$\ddot{\gamma}(t).\dot{\gamma}(t) + \dot{\gamma}(t).\ddot{\gamma}(t) = 0$$

$$2\ddot{\gamma}(t).\dot{\gamma}(t) = 0$$

$$\ddot{\gamma}(t).\dot{\gamma}(t) = 0$$

□

Inner product:

$$v = (2, 3)$$

$$w = (2, 1)$$

$$v.w := (2, 3).(2, 1) = 2 \times 2 + 3 \times 1 = 7$$

In general:

$$(x_1, y_1).(x_2, y_2) := x_1x_2 + y_1y_2$$

Exercise. For $\mathbf{v} : (\alpha, \beta) \rightarrow \mathbf{R}^2$ and $\mathbf{w} : (\alpha, \beta) \rightarrow \mathbf{R}^2$, show that $(\mathbf{v}(t).\mathbf{w}(t))' = \mathbf{v}'(t).\mathbf{w}(t) + \mathbf{v}(t).\mathbf{w}'(t)$

$$v = (x, y)$$

$$v.v = (x, y).(x, y) = x^2 + y^2$$

$$||(x, y)|| = \sqrt{x^2 + y^2}$$

$$v = (x, y)$$

$$v.w = ||v||||w|| \cos(\theta) \text{ where,}$$

θ is the angle between v and w

Definition. $\gamma : (\alpha, \beta) \rightarrow \mathbf{R}^2$ is a unit speed parametrization if $||\dot{\gamma}(t)|| = 1$ for each $t \in (\alpha, \beta)$

Theorem. If $\gamma : (\alpha, \beta) \rightarrow \mathbf{R}^2$ is a unit speed parametrization, then $\dot{\gamma}(t).\ddot{\gamma}(t) = 0$ for all $t \in (\alpha, \beta)$.

Proof. $||\dot{\gamma}(t)|| = 1$

$$\dot{\gamma}(t).\dot{\gamma}(t) = 1$$

$$(\dot{\gamma}(t).\dot{\gamma}(t))' = 0$$

$$\ddot{\gamma}(t).\dot{\gamma}(t) + \dot{\gamma}(t).\ddot{\gamma}(t) = 0$$

$$2\ddot{\gamma}(t).\dot{\gamma}(t) = 0$$

$$\ddot{\gamma}(t).\dot{\gamma}(t) = 0$$

□

Dicussion after the lecture

$$\ddot{\tilde{\gamma}}(t) = \ddot{\gamma}(\phi(t))\phi'(t)\phi'(t)$$

Dicussion after the lecture

$$\ddot{\tilde{\gamma}}(t) = \ddot{\gamma}(\phi(t))\phi'(t)\phi'(t) + \dot{\gamma}(\phi(t))\phi''(t)$$

Dicussion after the lecture

$$\ddot{\tilde{\gamma}}(t) = \ddot{\gamma}(\phi(t))\phi'(t)\phi'(t) + \dot{\gamma}(\phi(t))\phi''(t)$$

(may change the direction of acceleration)

Dicussion after the lecture

$$\ddot{\gamma}(t) = \ddot{\gamma}(\phi(t))\phi'(t)\phi'(t) + \dot{\gamma}(\phi(t))\phi''(t)$$

(may change the direction of acceleration)

If the direction of velocity does not change
with t

Dicussion after the lecture

$$\ddot{\gamma}(t) = \ddot{\gamma}(\phi(t))\phi'(t)\phi'(t) + \dot{\gamma}(\phi(t))\phi''(t)$$

(may change the direction of acceleration)

If the direction of velocity does not change
with t

$$\dot{\gamma}(t) =$$

Dicussion after the lecture

$$\ddot{\gamma}(t) = \ddot{\gamma}(\phi(t))\phi'(t)\phi'(t) + \dot{\gamma}(\phi(t))\phi''(t)$$

(may change the direction of acceleration)

If the direction of velocity does not change
with t

$$\dot{\gamma}(t) = f(t)\mathbf{v} =$$

Dicussion after the lecture

$$\ddot{\gamma}(t) = \ddot{\gamma}(\phi(t))\phi'(t)\phi'(t) + \dot{\gamma}(\phi(t))\phi''(t)$$

(may change the direction of acceleration)

**If the direction of velocity does not change
with t**

$$\dot{\gamma}(t) = f(t)\mathbf{v} = (f(t)v_1, f(t)v_2)$$

Dicussion after the lecture

$$\ddot{\gamma}(t) = \ddot{\gamma}(\phi(t))\phi'(t)\phi'(t) + \dot{\gamma}(\phi(t))\phi''(t)$$

(may change the direction of acceleration)

**If the direction of velocity does not change
with t**

$$\dot{\gamma}(t) = f(t)\mathbf{v} = (f(t)v_1, f(t)v_2)$$
$$\ddot{\gamma}(t) =$$

Dicussion after the lecture

$$\ddot{\gamma}(t) = \ddot{\gamma}(\phi(t))\phi'(t)\phi'(t) + \dot{\gamma}(\phi(t))\phi''(t)$$

(may change the direction of acceleration)

**If the direction of velocity does not change
with t**

$$\dot{\gamma}(t) = f(t)\mathbf{v} = (f(t)v_1, f(t)v_2)$$
$$\ddot{\gamma}(t) = (f'(t)v_1, f'(t)v_2) =$$

Dicussion after the lecture

$$\ddot{\gamma}(t) = \ddot{\gamma}(\phi(t))\phi'(t)\phi'(t) + \dot{\gamma}(\phi(t))\phi''(t)$$

(may change the direction of acceleration)

**If the direction of velocity does not change
with t**

$$\begin{aligned}\dot{\gamma}(t) &= f(t)\mathbf{v} = (f(t)v_1, f(t)v_2) \\ \ddot{\gamma}(t) &= (f'(t)v_1, f'(t)v_2) = f'(t)\mathbf{v}\end{aligned}$$

Dicussion after the lecture

$$\ddot{\gamma}(t) = \ddot{\gamma}(\phi(t))\phi'(t)\phi'(t) + \dot{\gamma}(\phi(t))\phi''(t)$$

(may change the direction of acceleration)

**If the direction of velocity does not change
with t**

$$\dot{\gamma}(t) = f(t)\mathbf{v} = (f(t)v_1, f(t)v_2)$$

$$\ddot{\gamma}(t) = (f'(t)v_1, f'(t)v_2) = f'(t)\mathbf{v}$$

Direction of acceleration also unchanged