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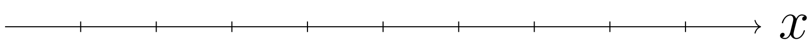

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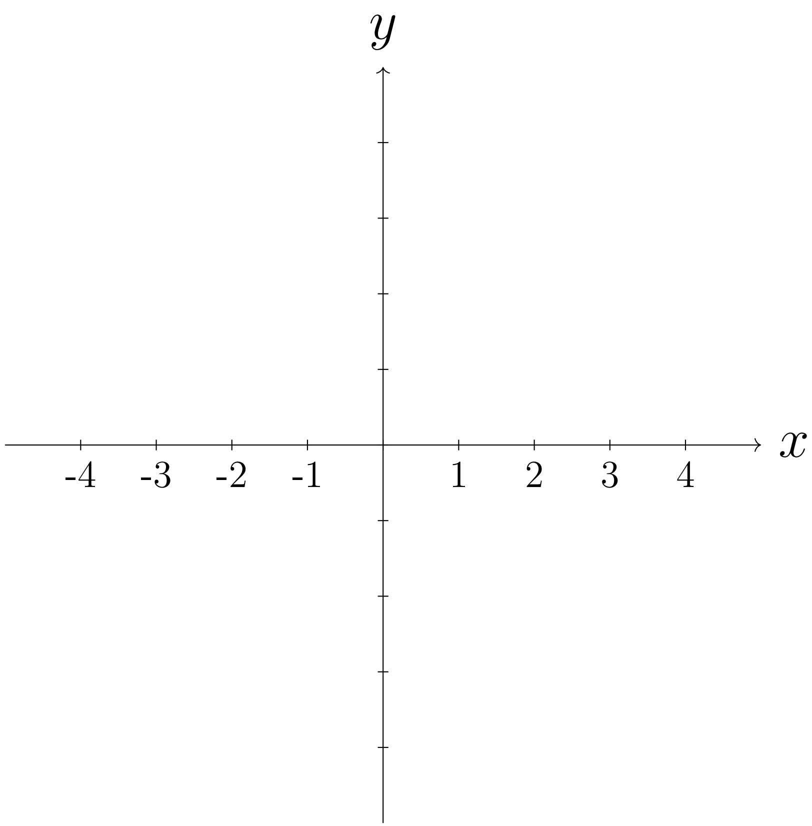

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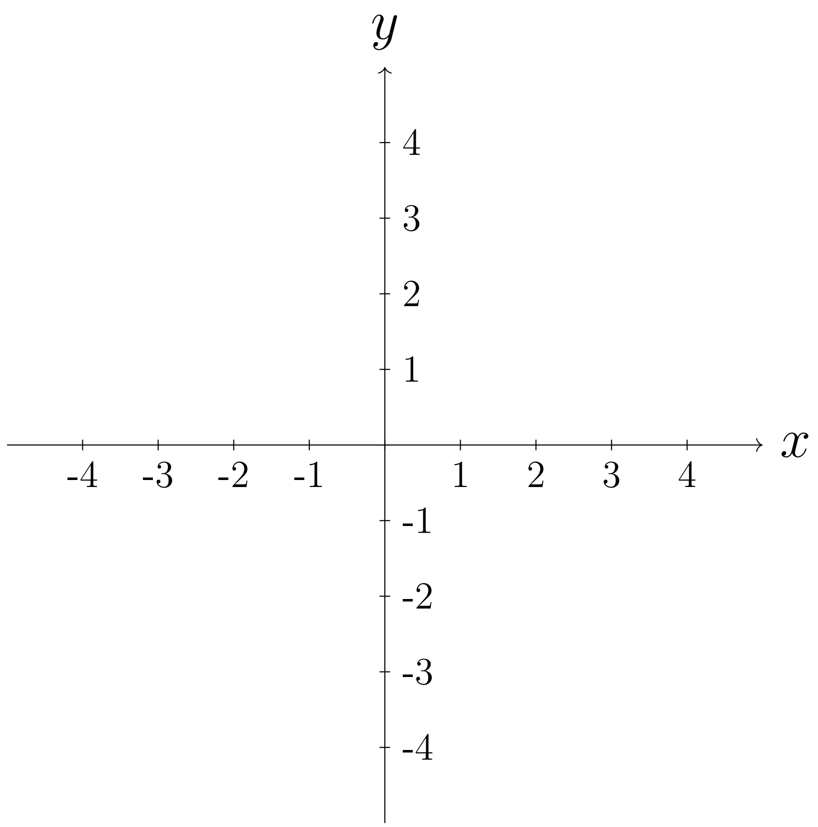

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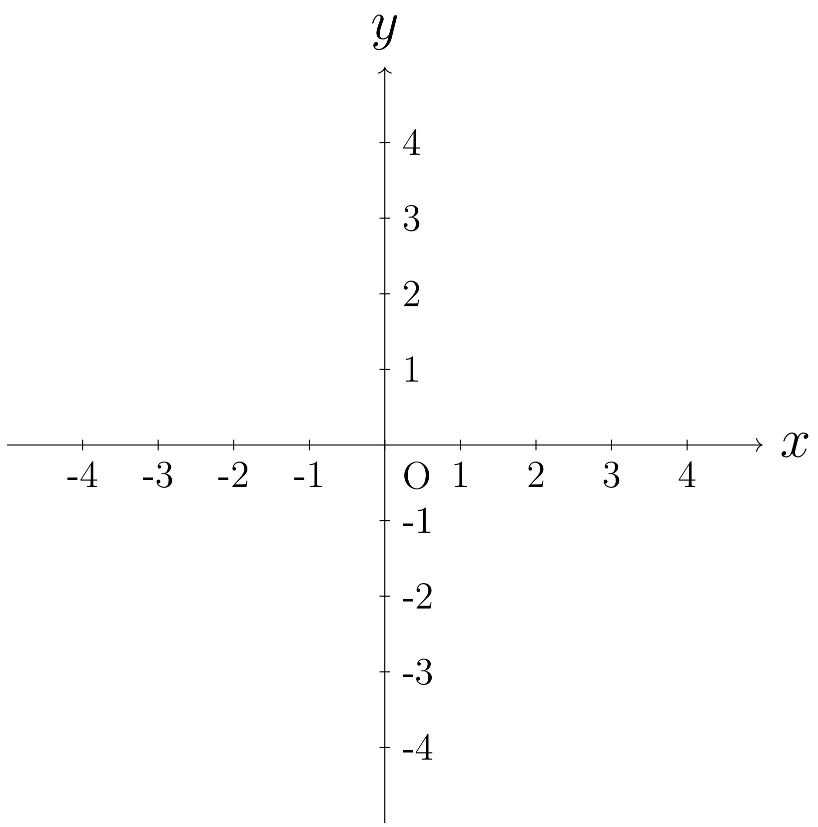

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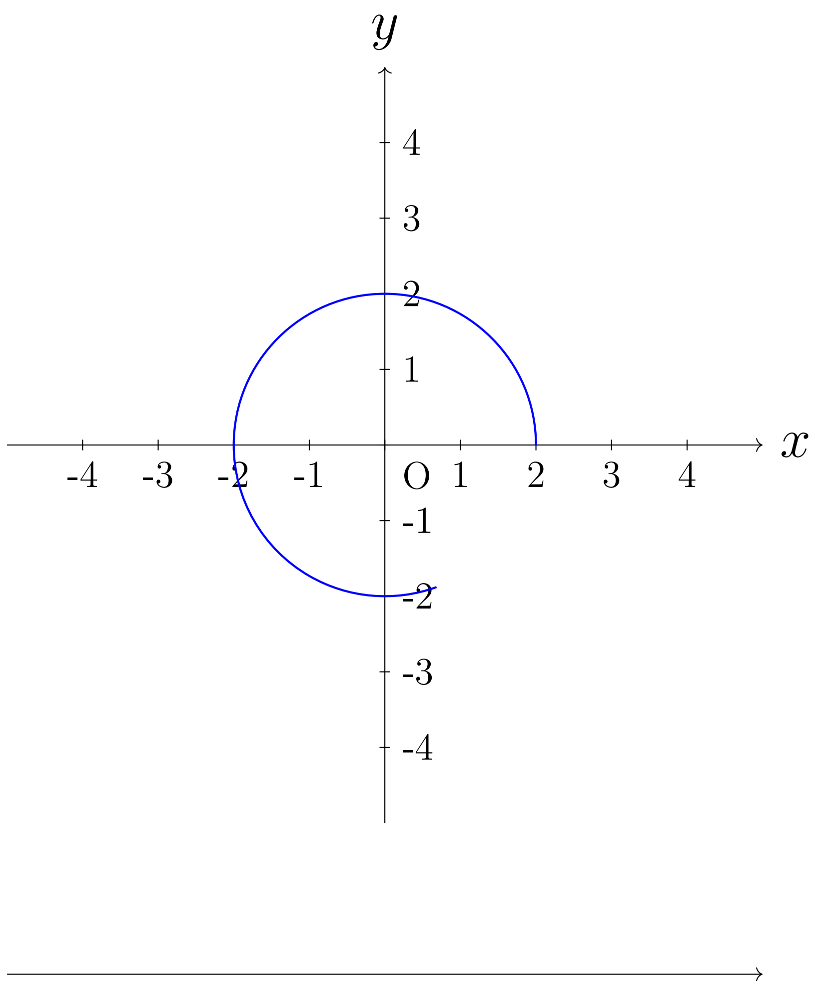

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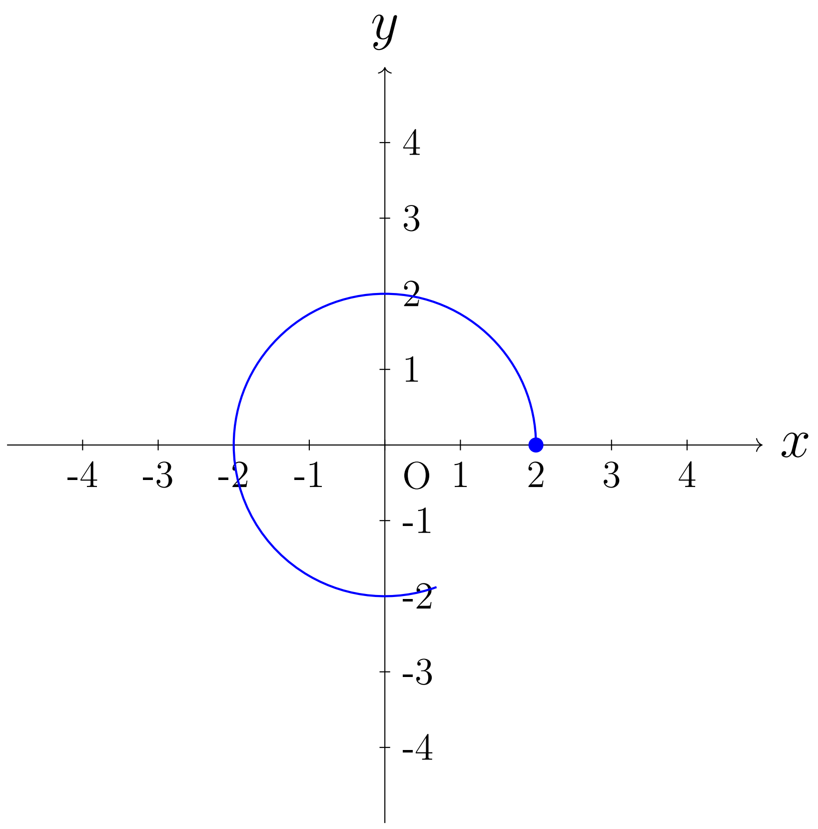

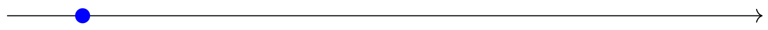

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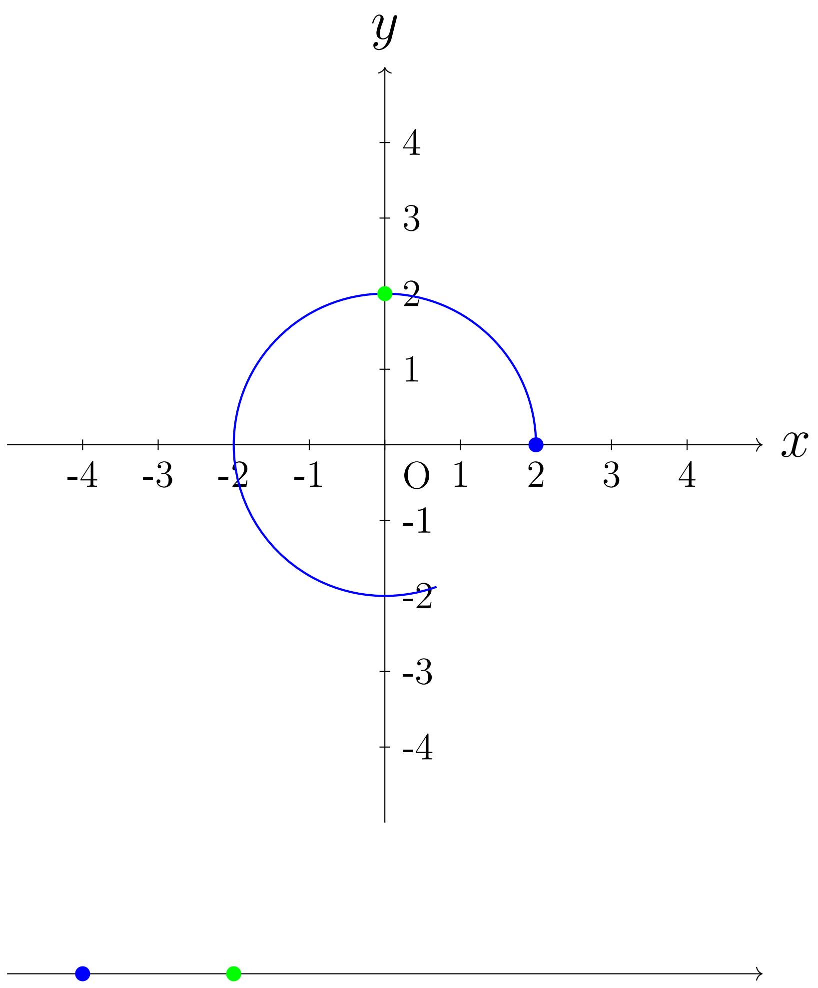

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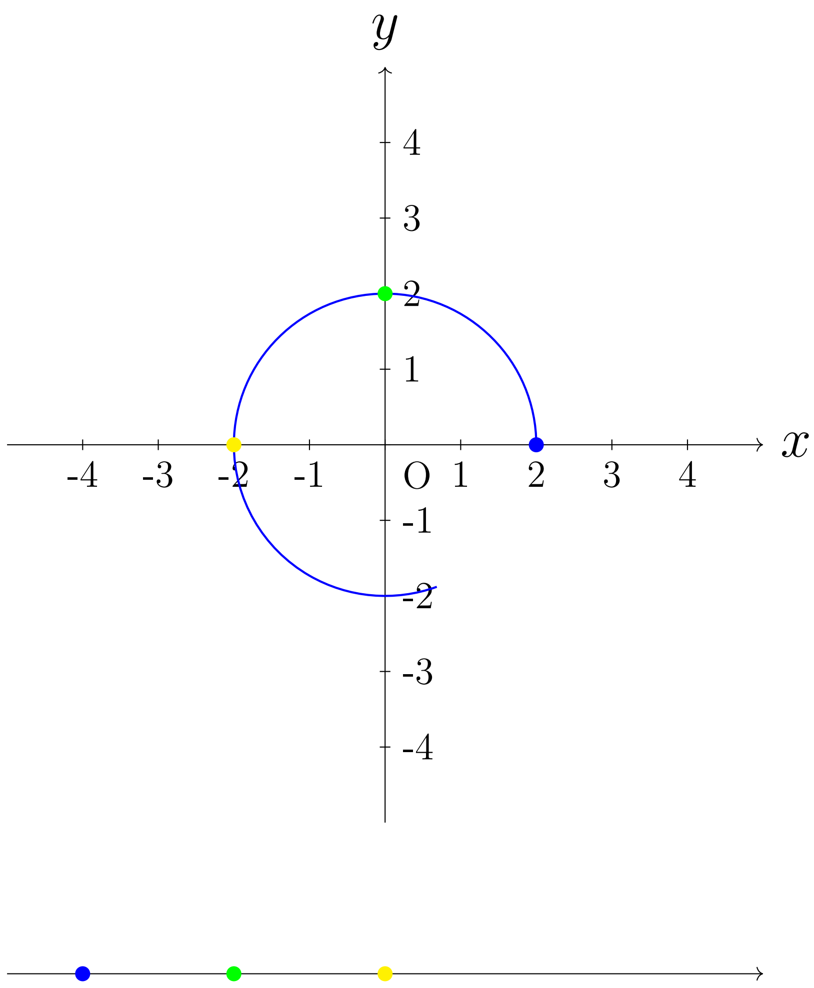

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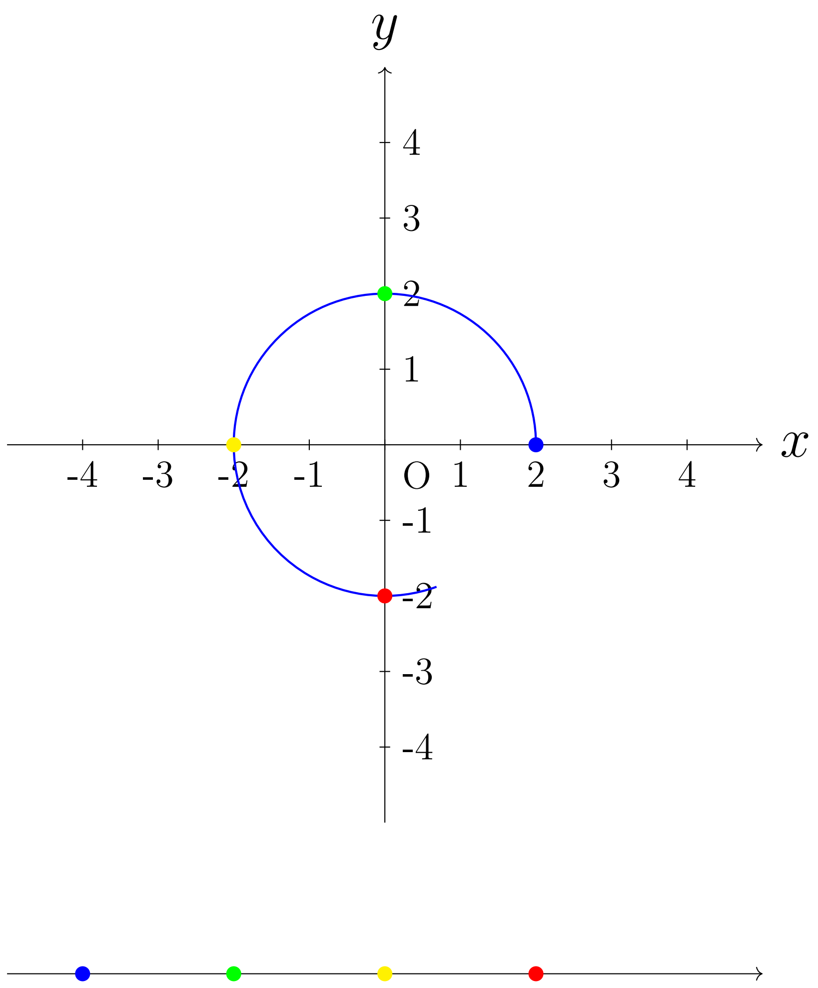

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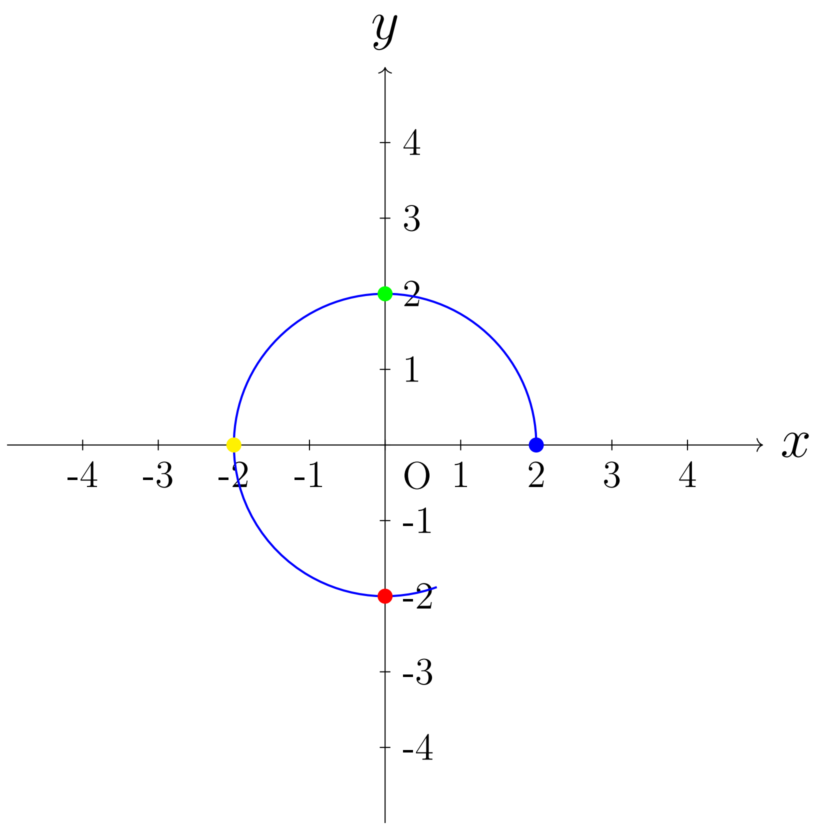

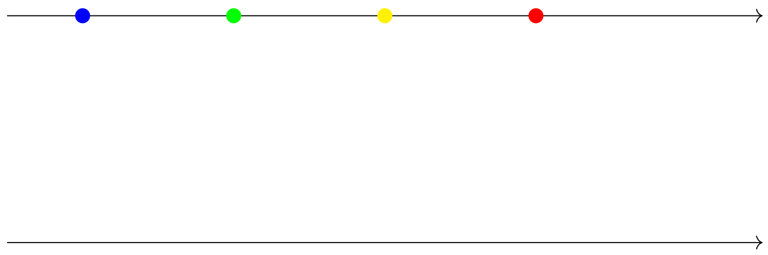

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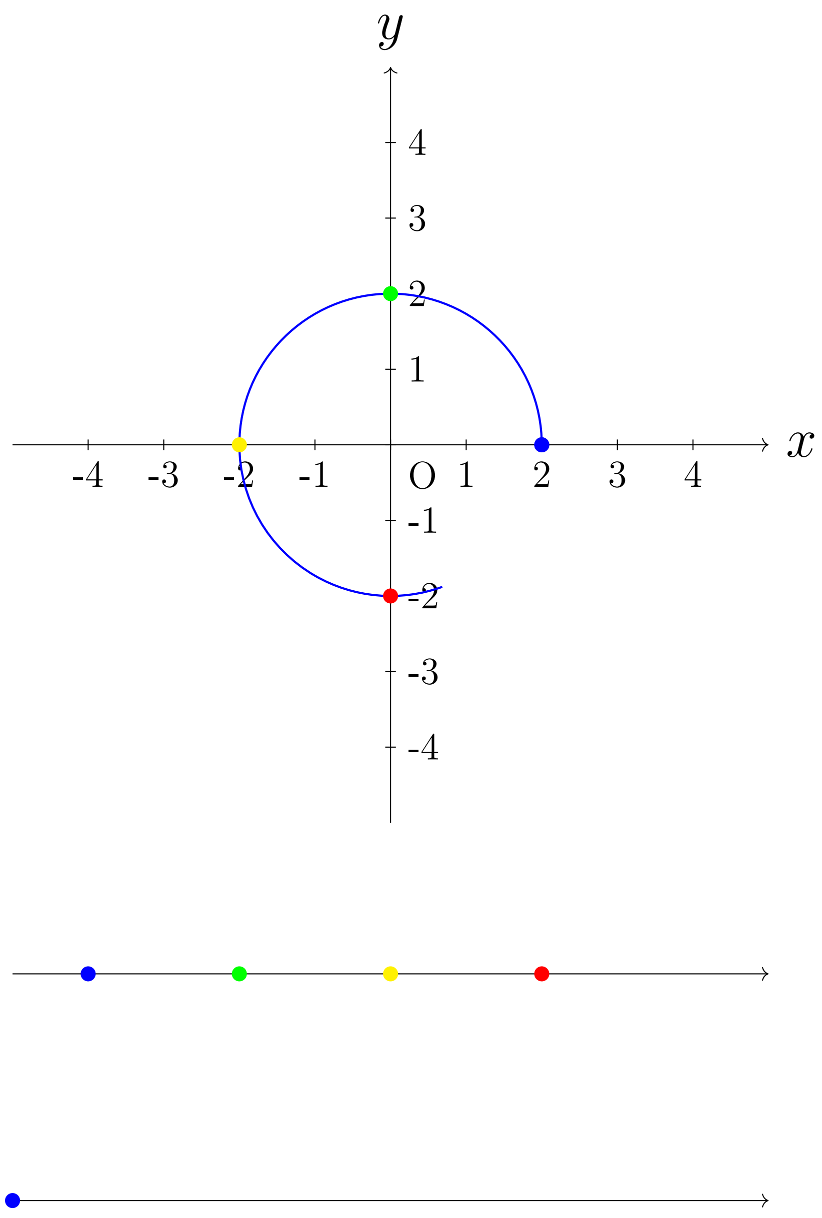

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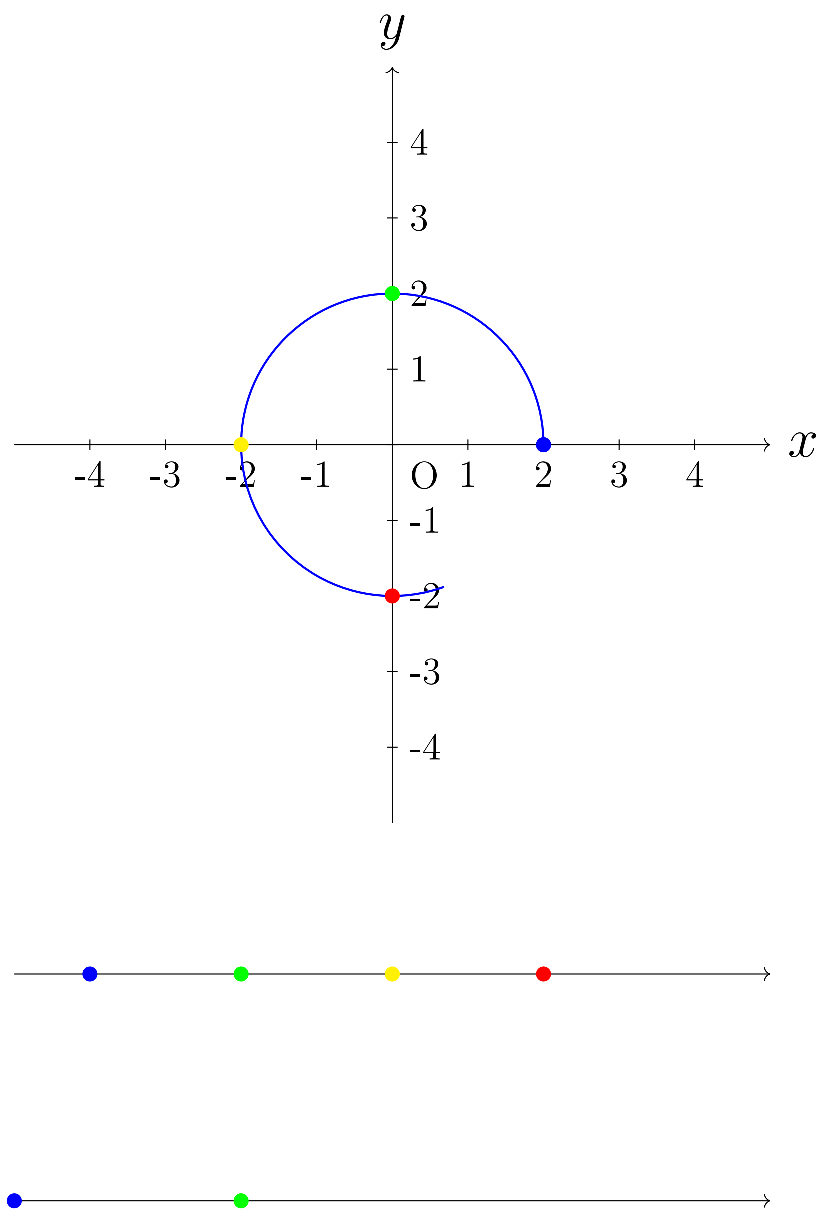

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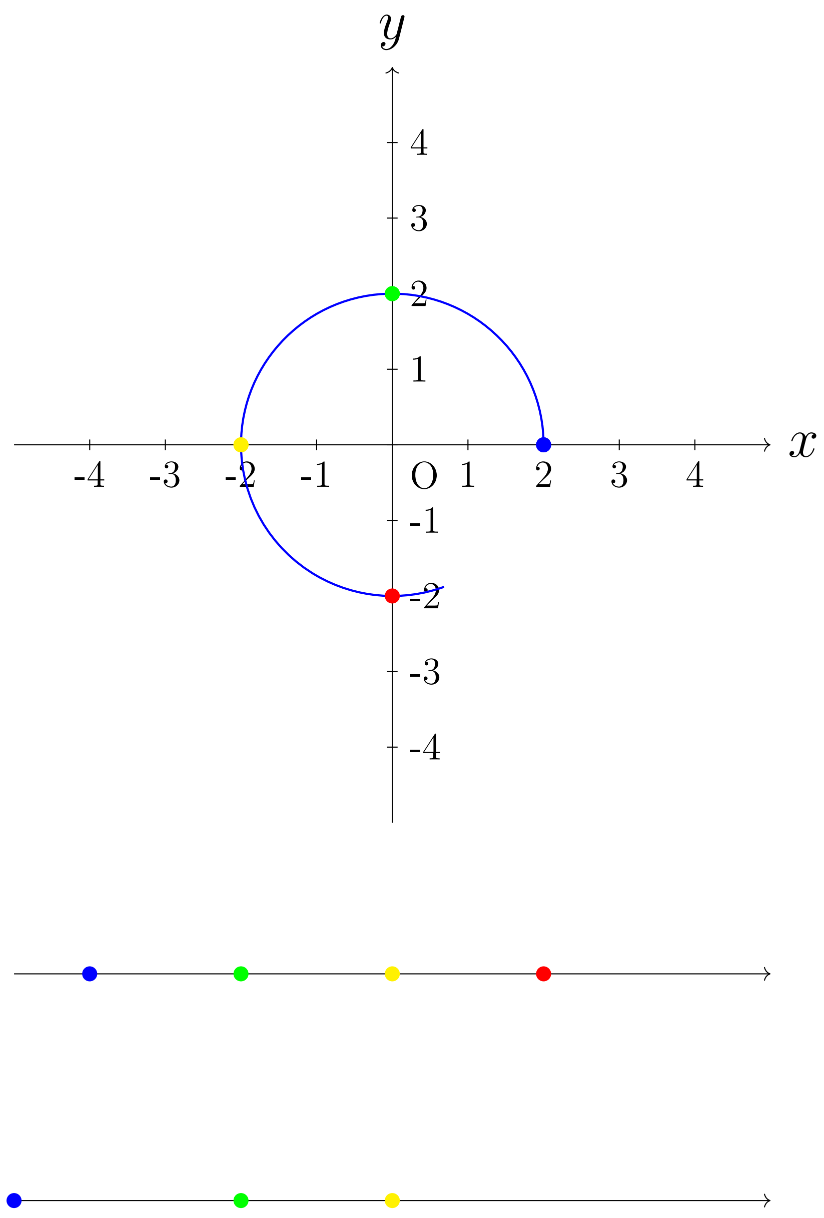

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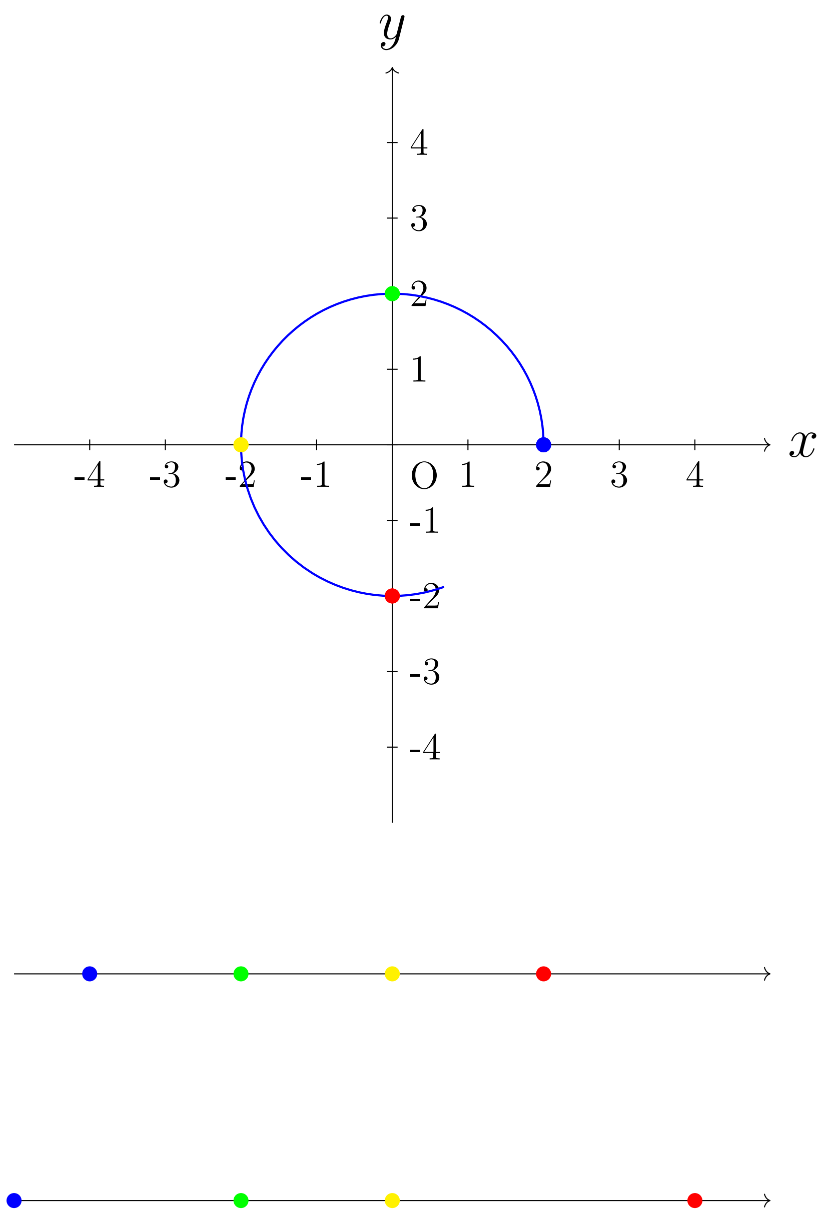

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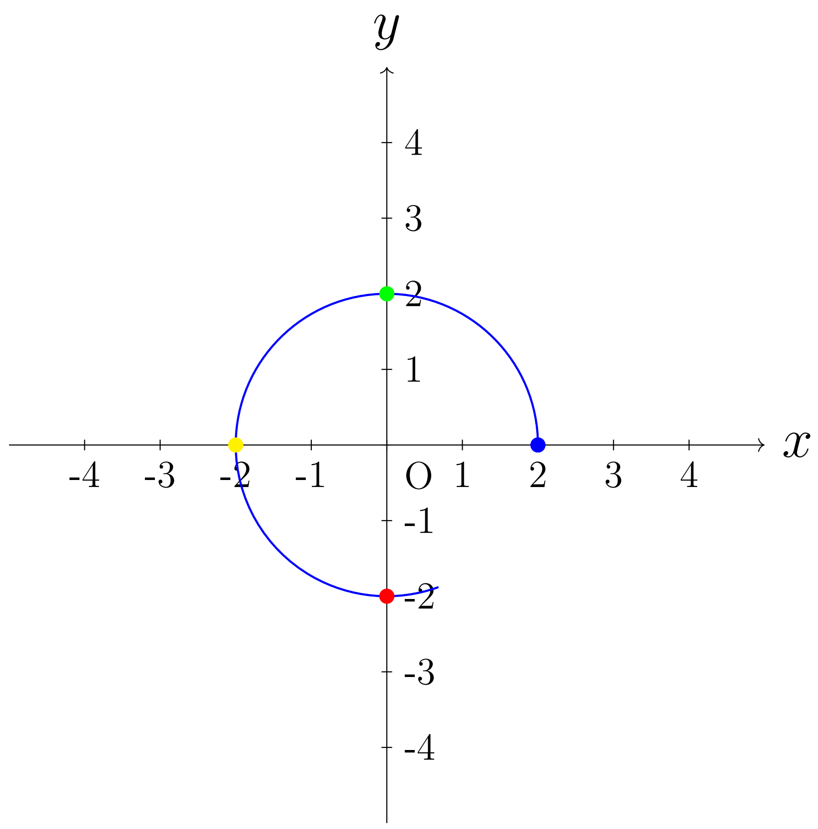

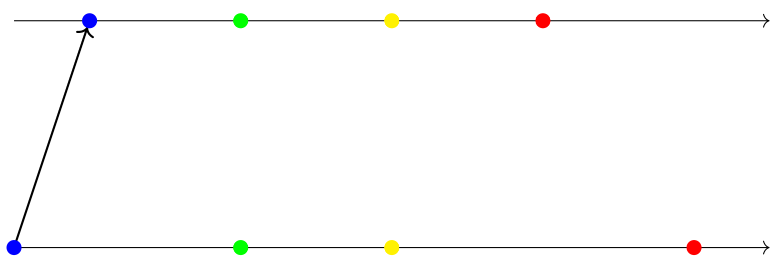

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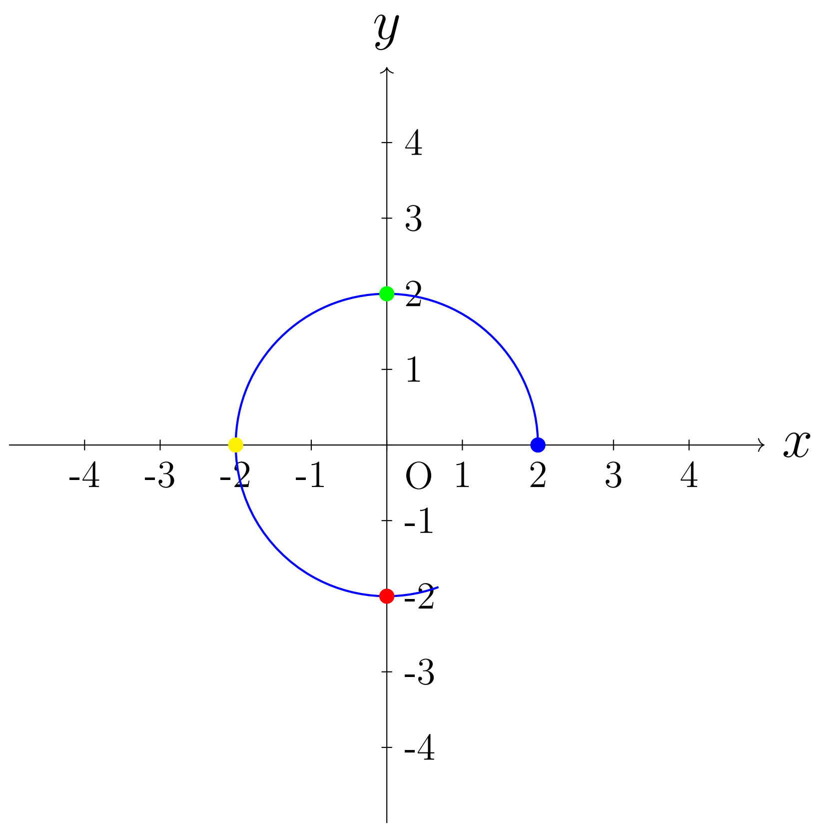

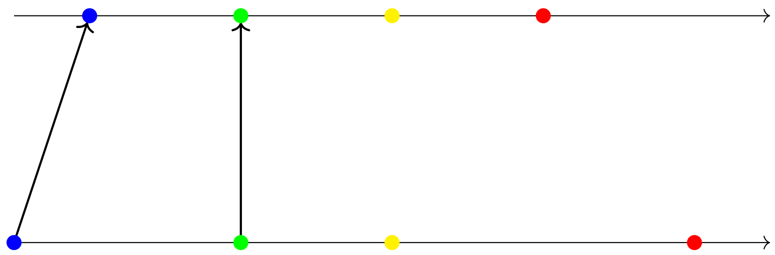

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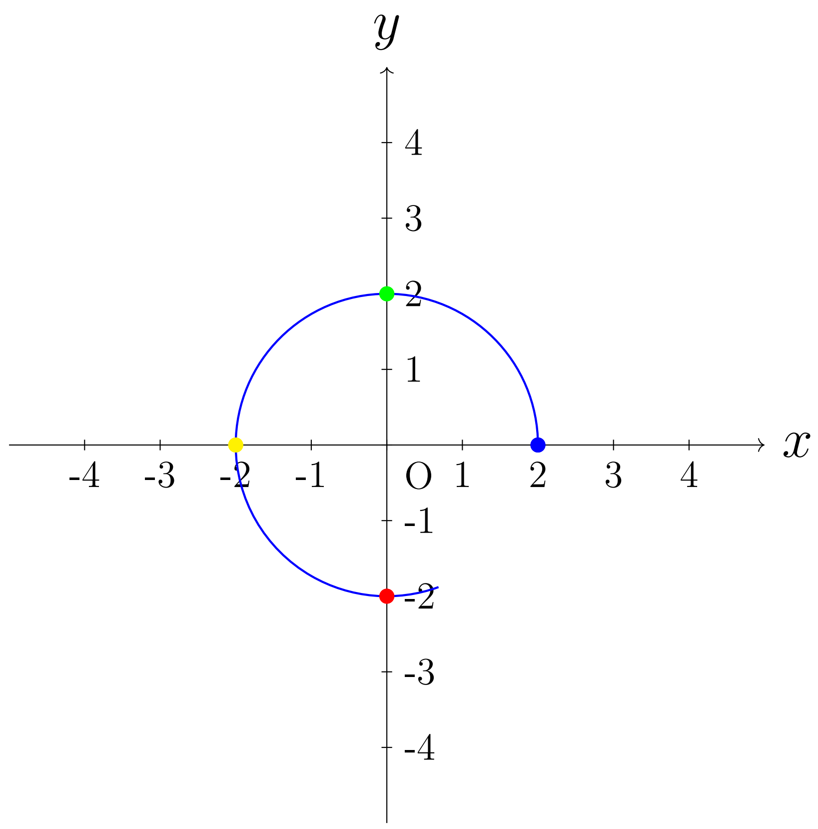

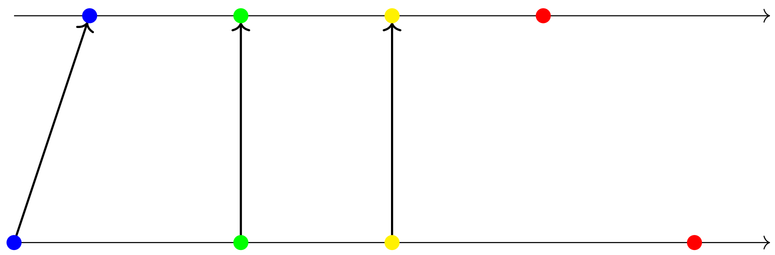

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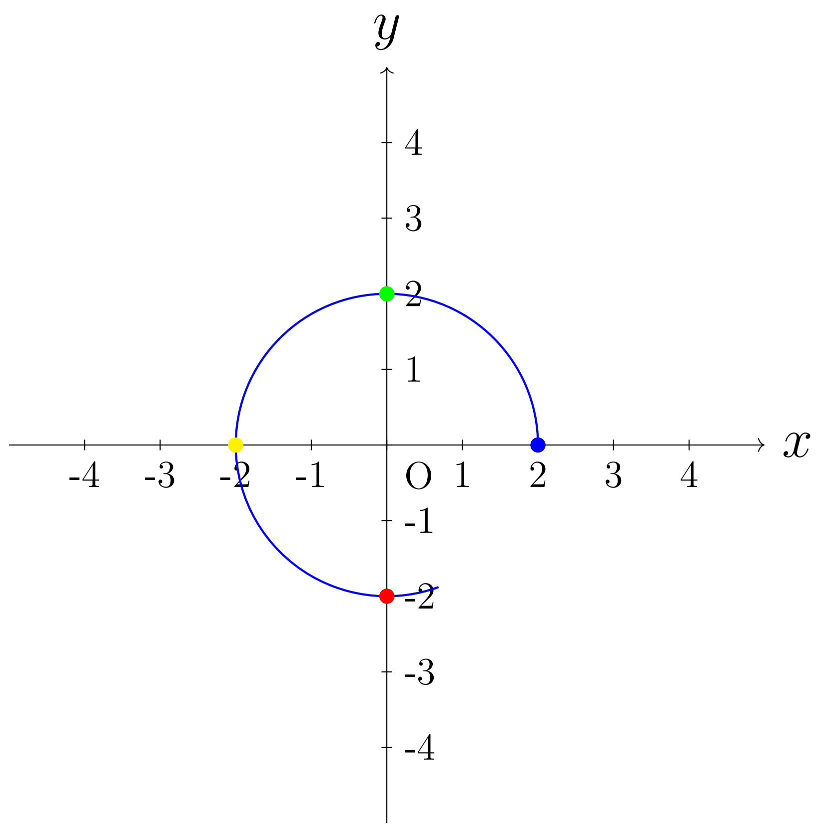

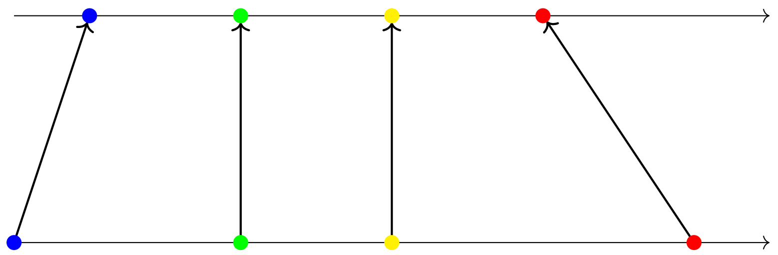

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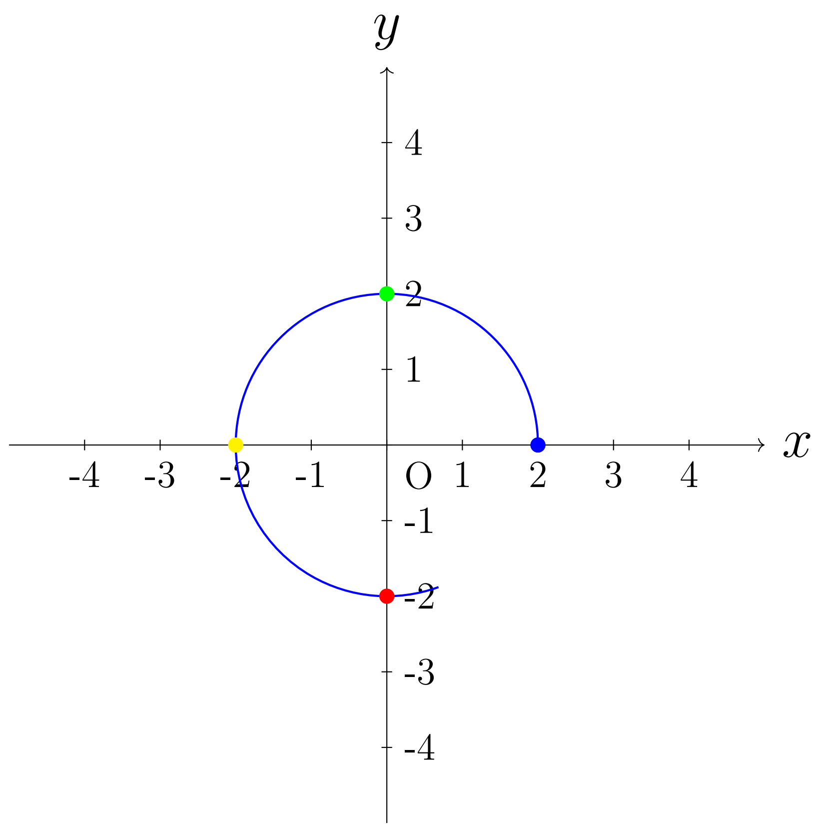

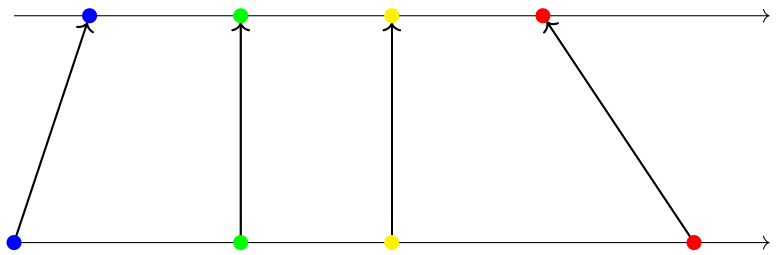

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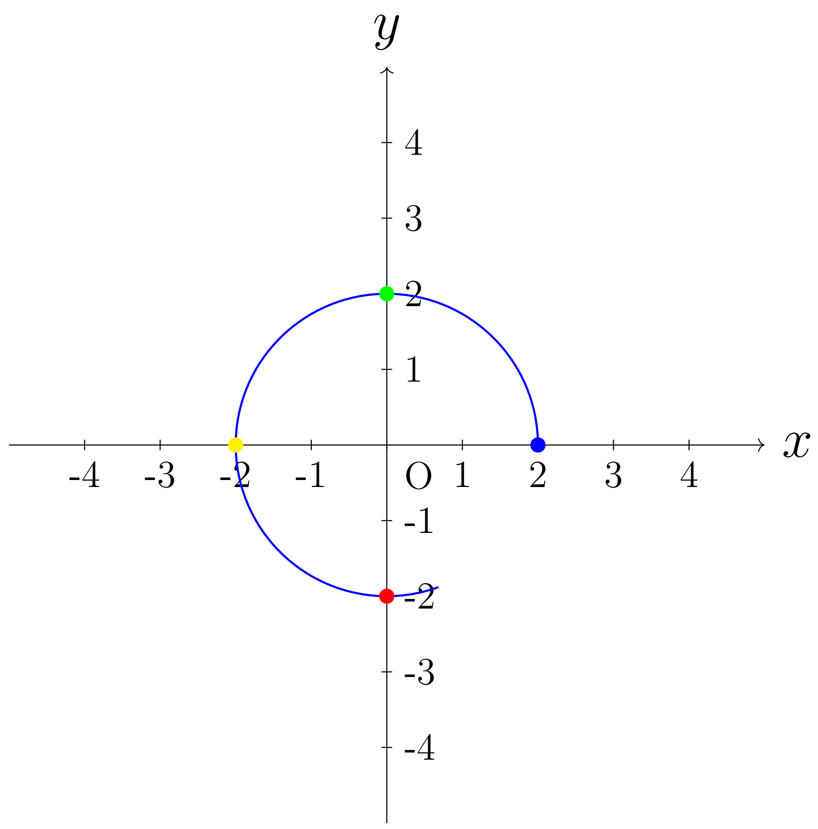

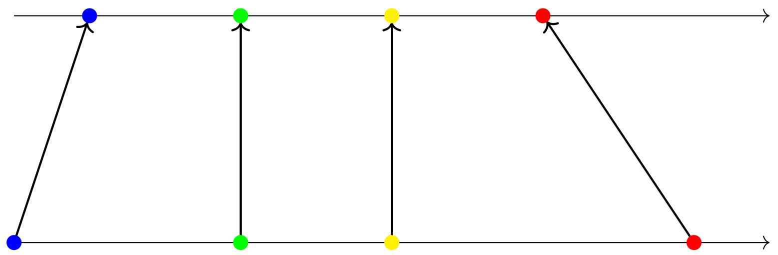

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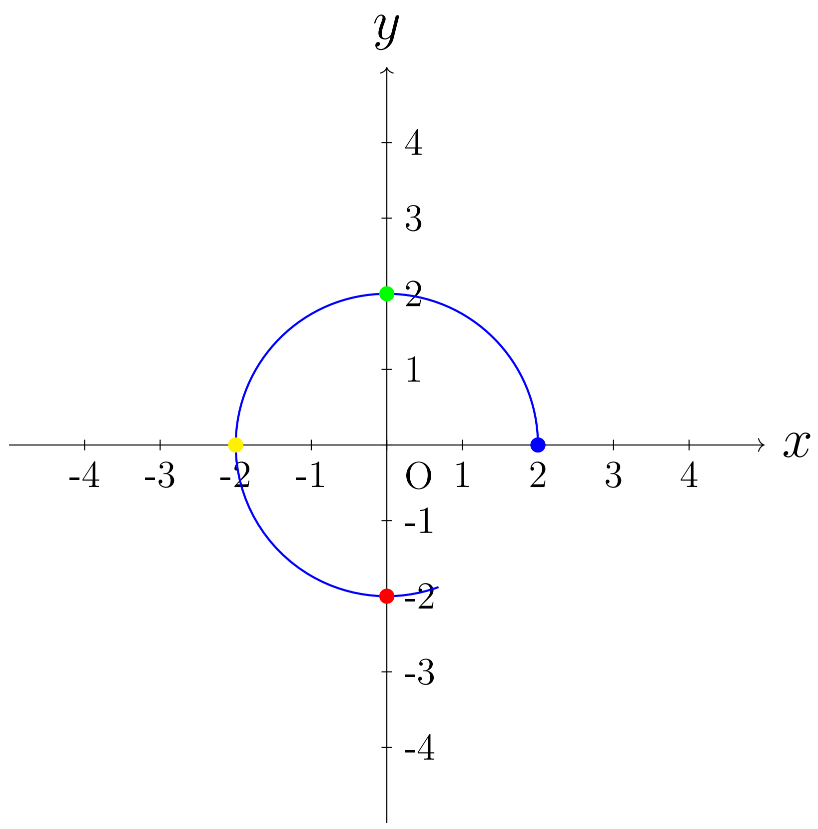

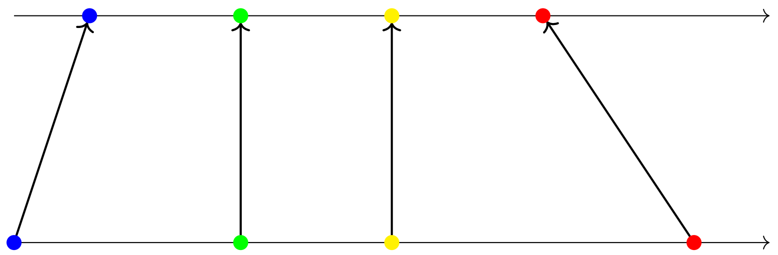

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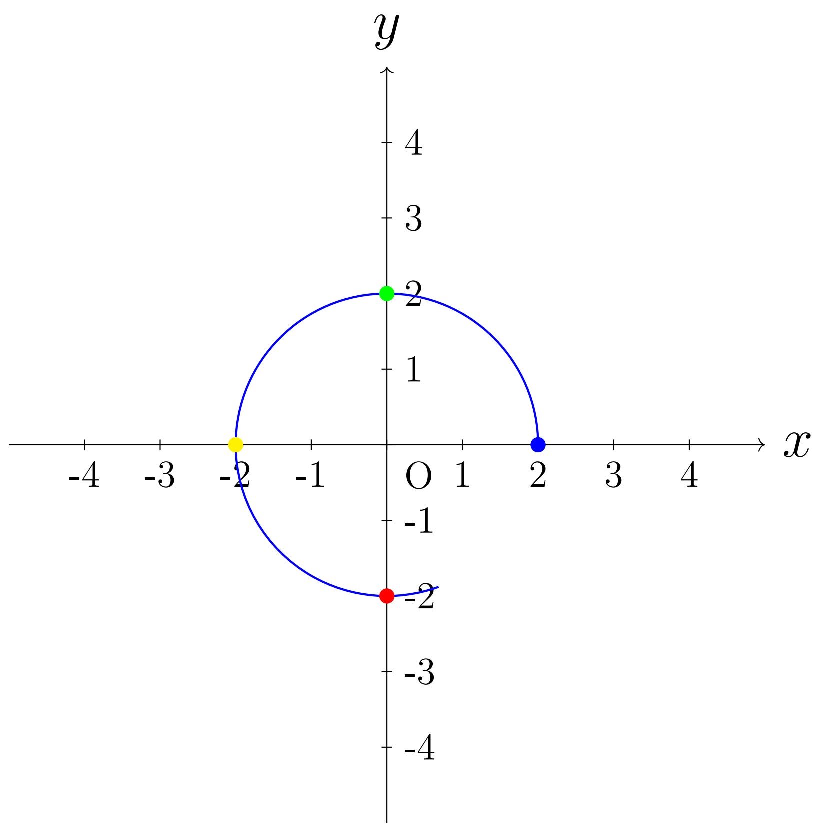

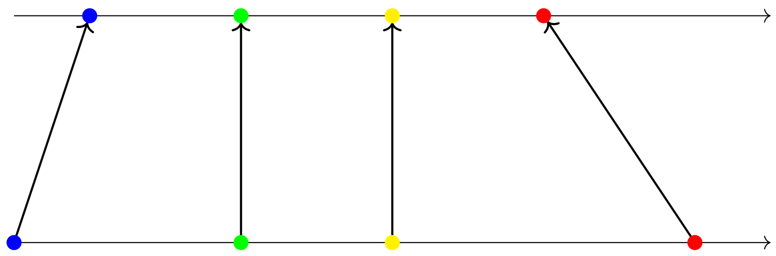

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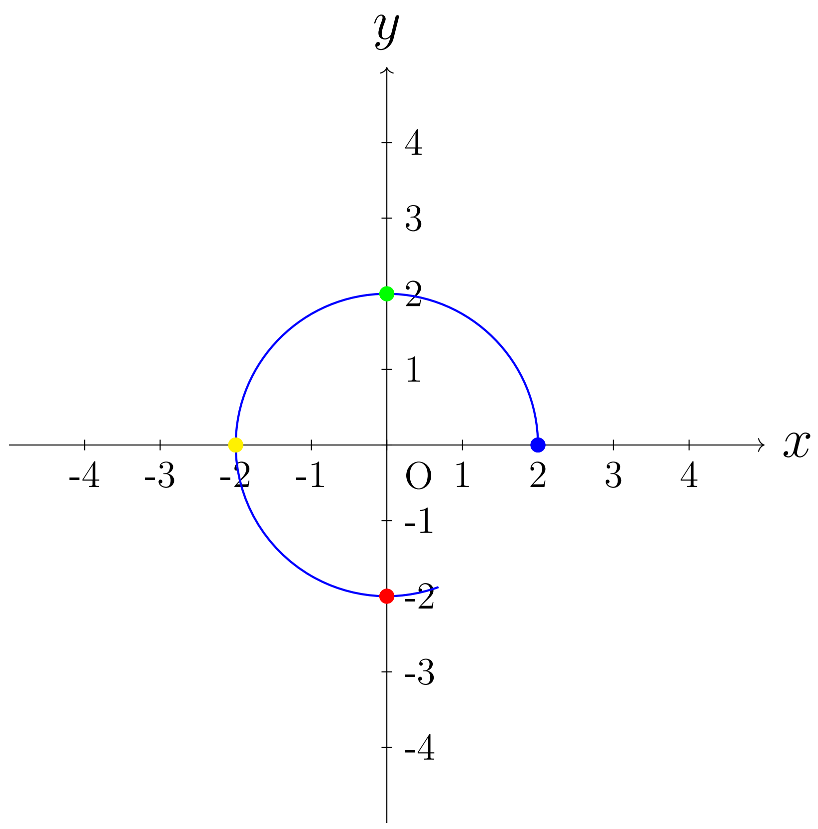

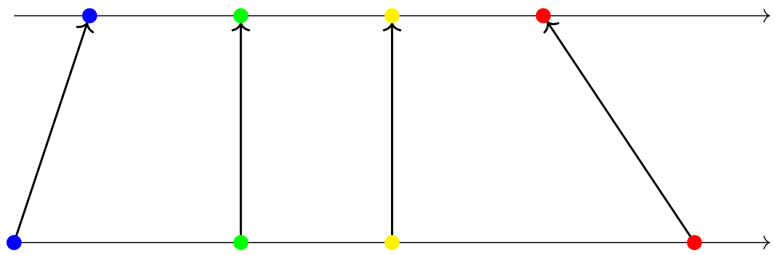

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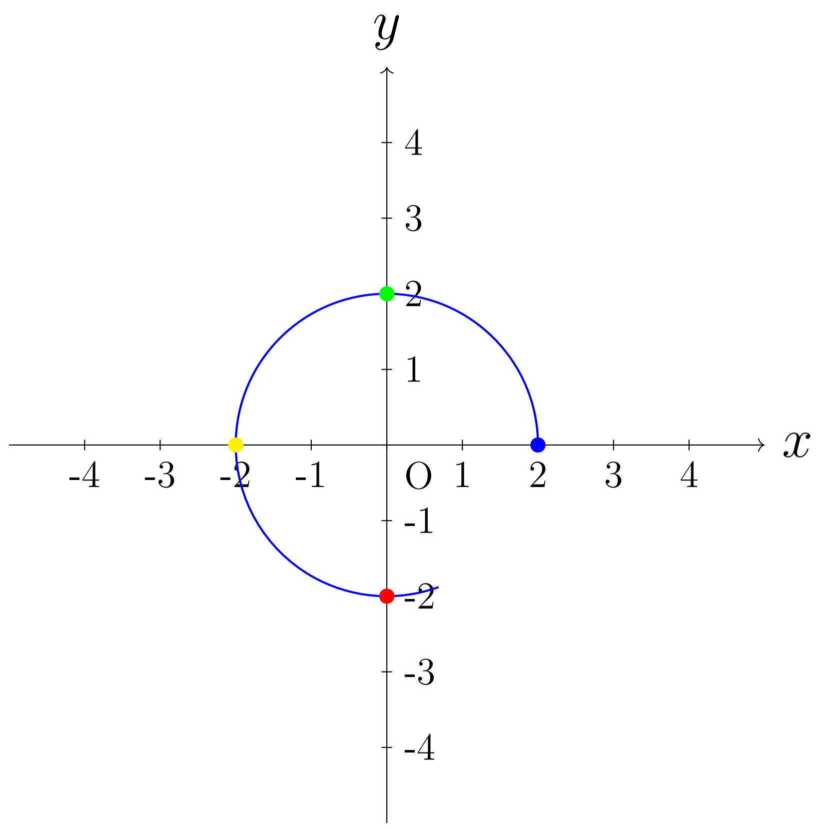

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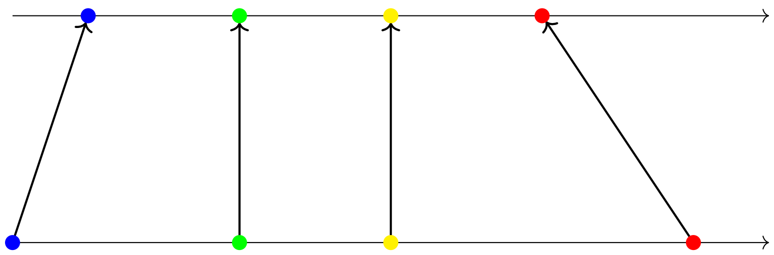

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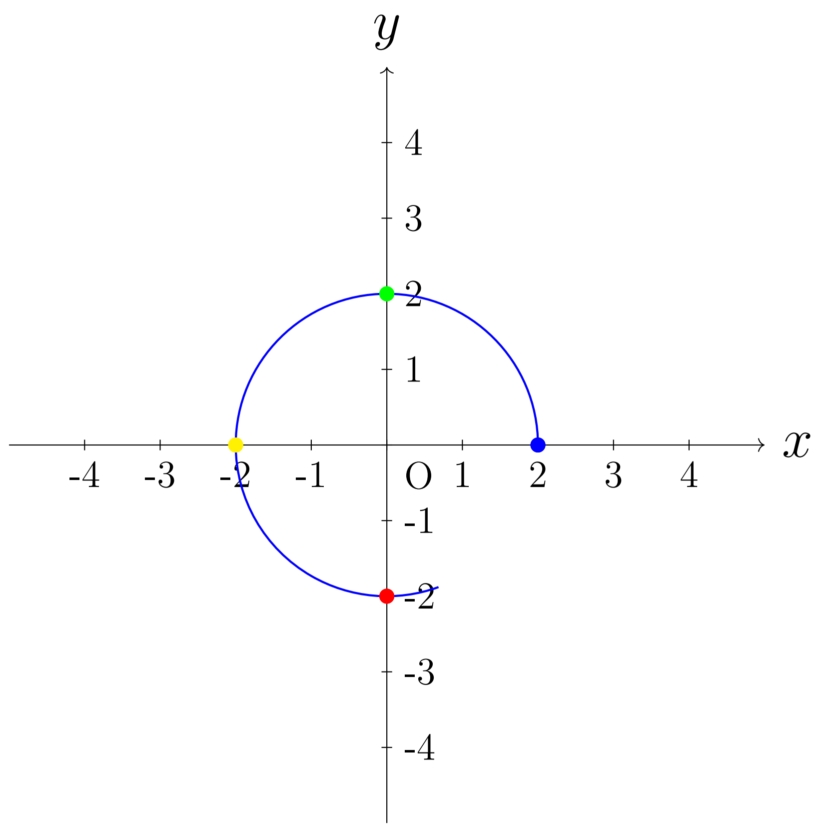

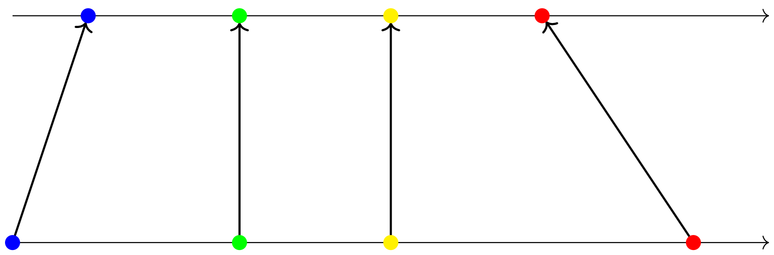

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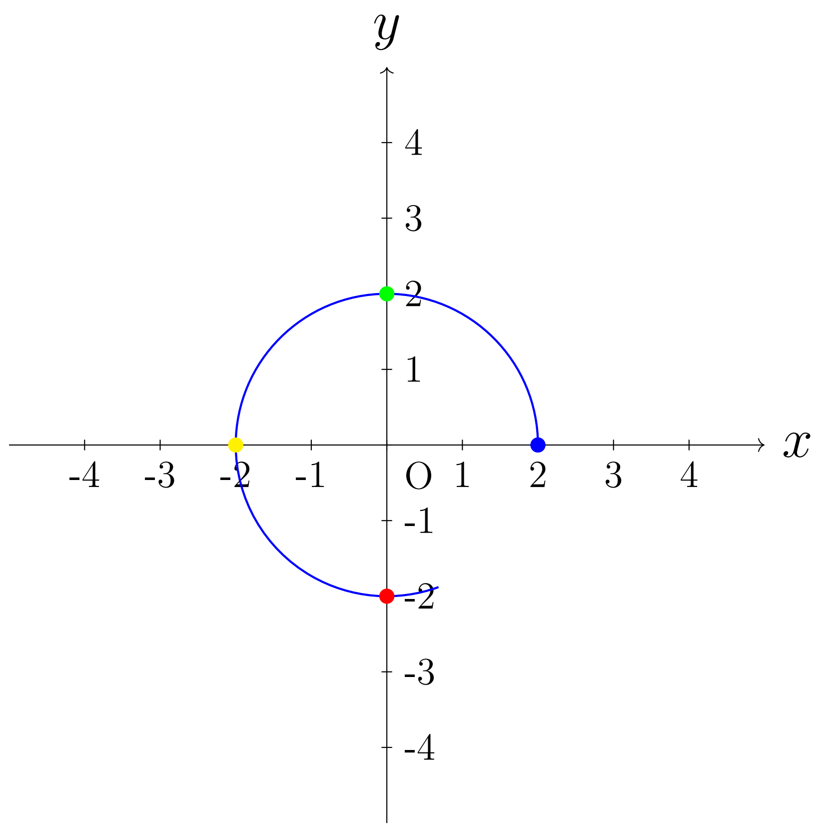

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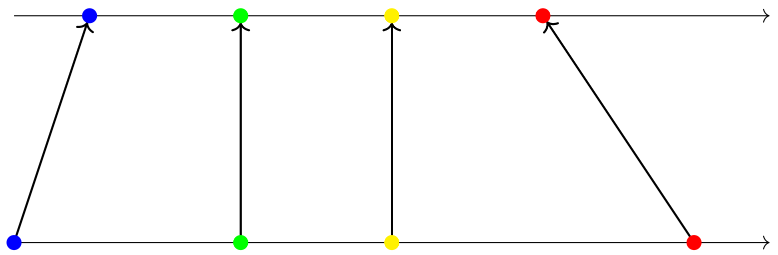

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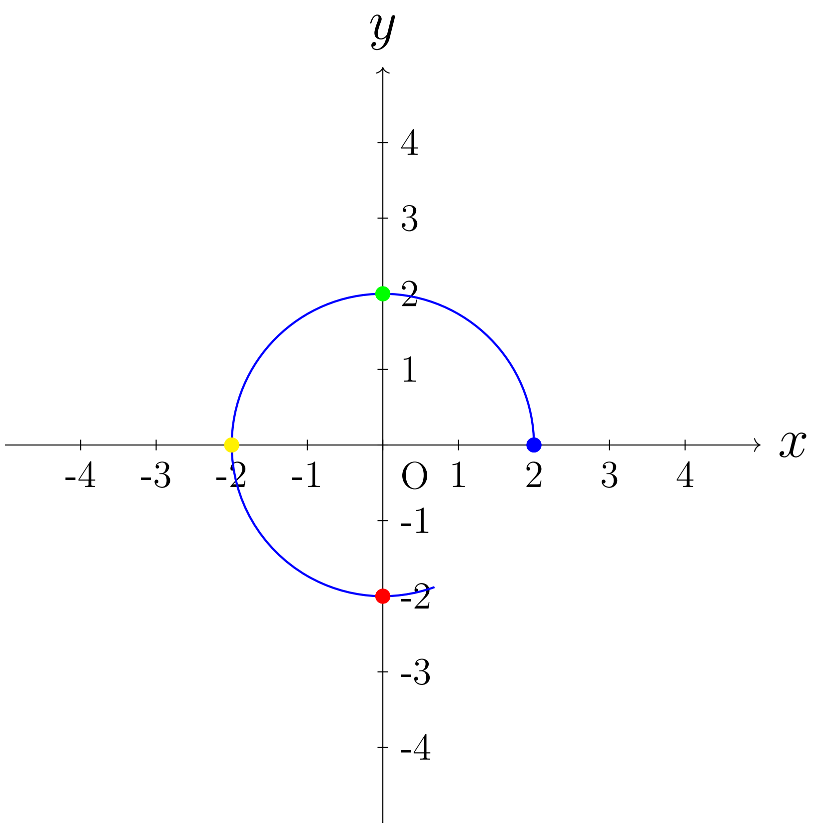

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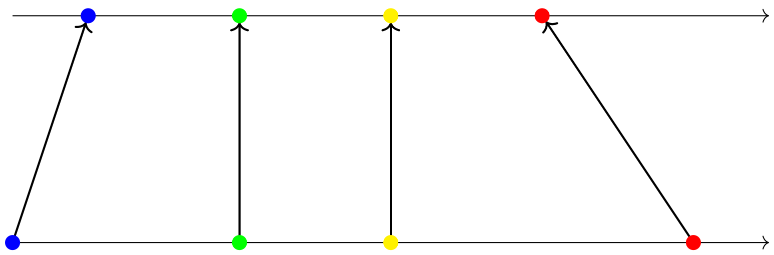

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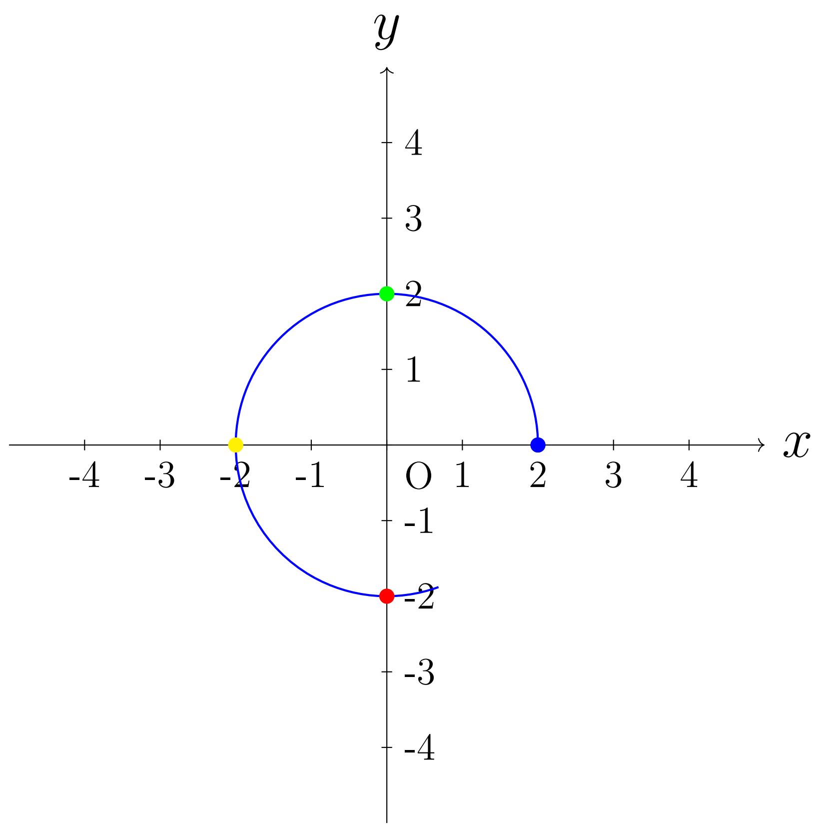

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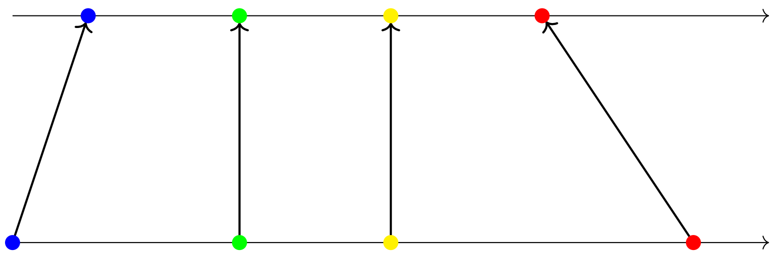

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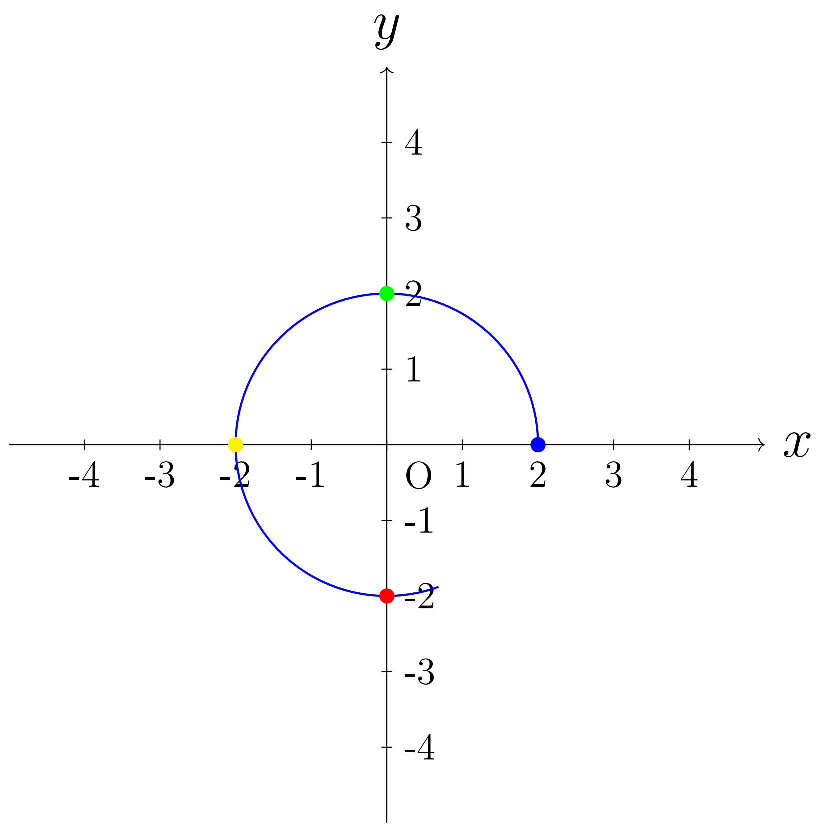
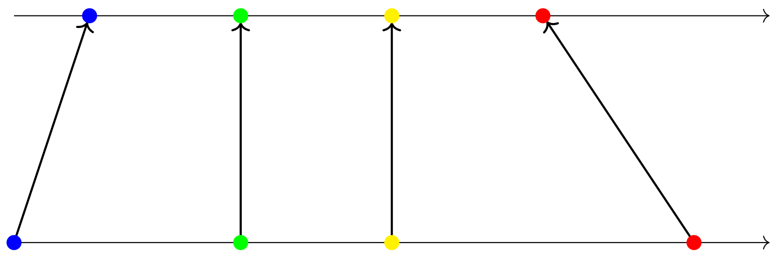
If ϕ is smooth, is ϕ^{-1} smooth? Not necessarily!



Definition.
$\gamma : (\alpha, \beta) \to \mathbb{R}^2$.
$\tilde{\gamma} : (\tilde{\alpha}, \tilde{\beta}) \to \mathbb{R}^2$.
If $\phi : (\tilde{\alpha}, \tilde{\beta}) \to (\alpha, \beta)$ is bijective, smooth,
and its inverse, ϕ^{-1} is smooth,
and $\tilde{\gamma}(t) = \gamma(\phi(t))$,
then ϕ is called a reparametrization of γ.



Explictly:

$\gamma(t) = (f_1(t), f_2(t))$
$\gamma(\phi(t)) = (f_1(\phi(t)), f_2(\phi(t)))$

Example

$\gamma : (-1, 1) \to \mathbb{R}^2$

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Example

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$\tilde{\gamma}(t) = (2t, 2t)$

$\phi : (-1/2, 1/2) \to (-1, 1)$
$\phi(t) = 2t$

Example

$\gamma : (-1, 1) \to \mathbb{R}^2$
$\gamma(t) = (t, t)$

$\tilde{\gamma} : (-1/2, 1/2) \to \mathbb{R}^2.$
$\tilde{\gamma}(t) = (2t, 2t)$

$\phi : (-1/2, 1/2) \to (-1, 1)$
$\phi(t) = 2t$
So that $\tilde{\gamma}(t)$

Example

$\gamma : (-1, 1) \to \mathbb{R}^2$
$\gamma(t) = (t, t)$

$\tilde{\gamma} : (-1/2, 1/2) \to \mathbb{R}^2.$
$\tilde{\gamma}(t) = (2t, 2t)$

$\phi : (-1/2, 1/2) \to (-1, 1)$
$\phi(t) = 2t$
So that $\tilde{\gamma}(t) = \gamma(\phi(t))$

Example

$\gamma : (-1, 1) \to \mathbb{R}^2$
$\gamma(t) = (t, t)$

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$\phi(t) = 2t$
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Example

$\gamma : (-1, 1) \to \mathbb{R}^2$
$\gamma(t) = (t, t)$

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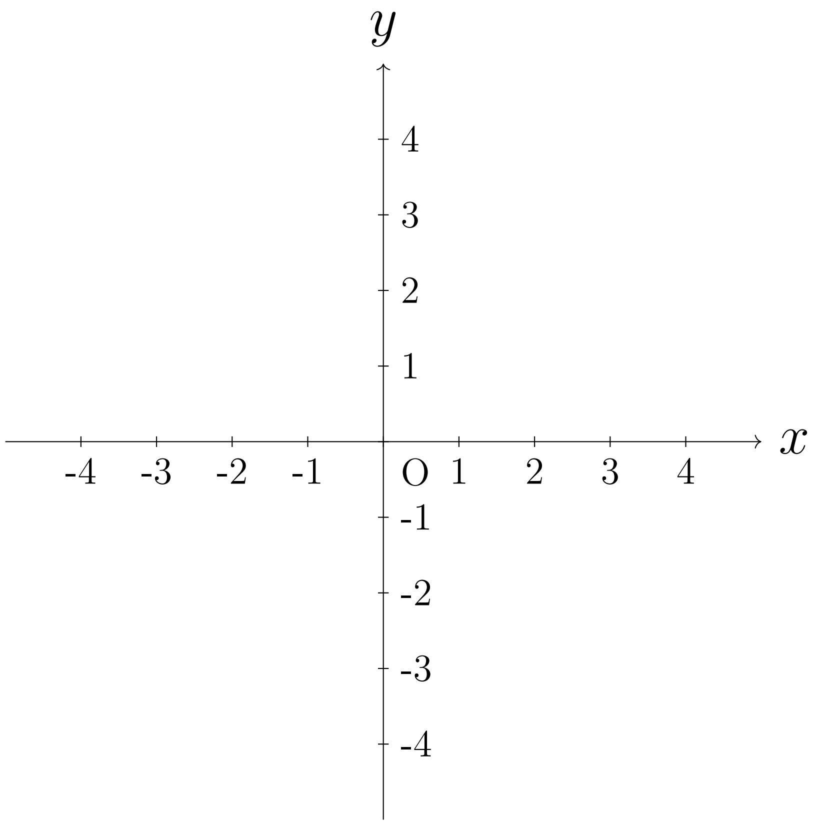
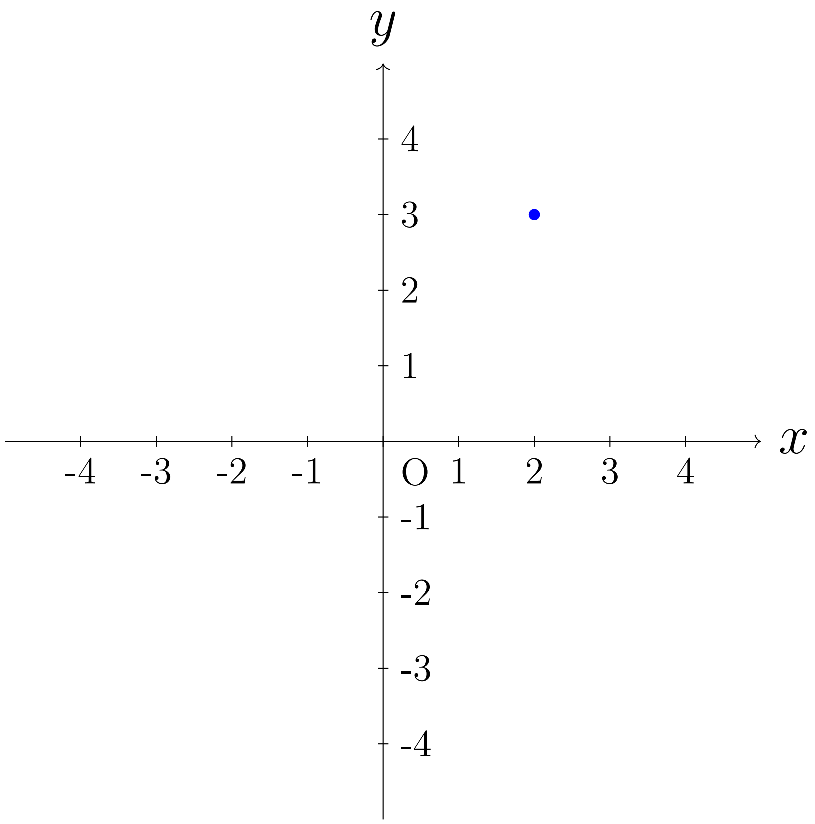
$\tilde{\gamma}(t) = (2t, 2t)$

$\phi : (-1/2, 1/2) \to (-1, 1)$
$\phi(t) = 2t$
So that $\tilde{\gamma}(t) = \gamma(\phi(t)) = \gamma(2t) = (2t, 2t)$

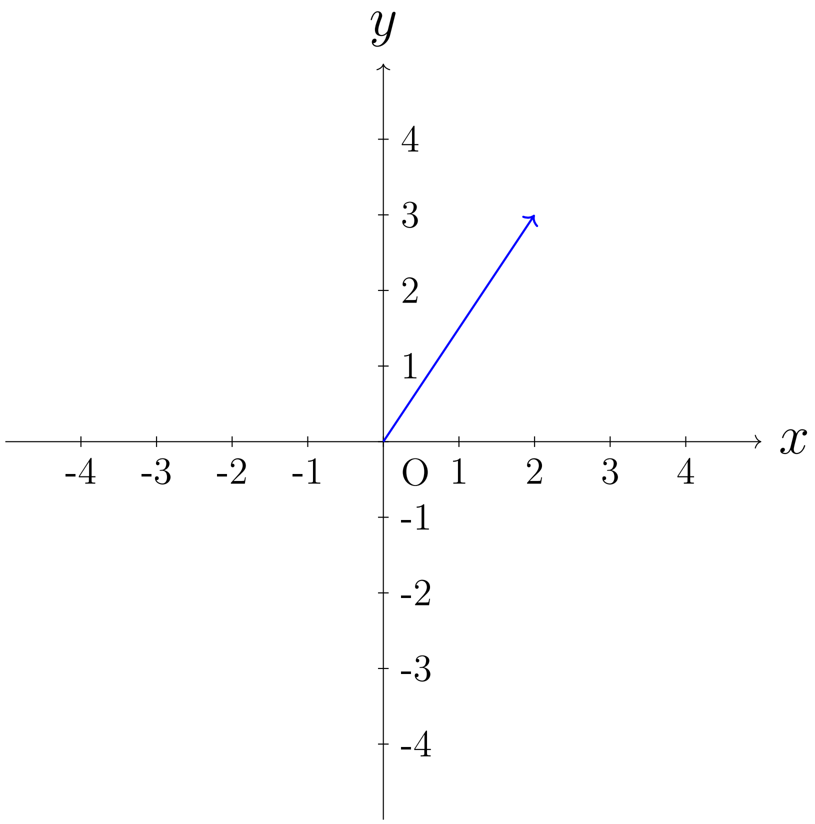
Vectors



Vectors

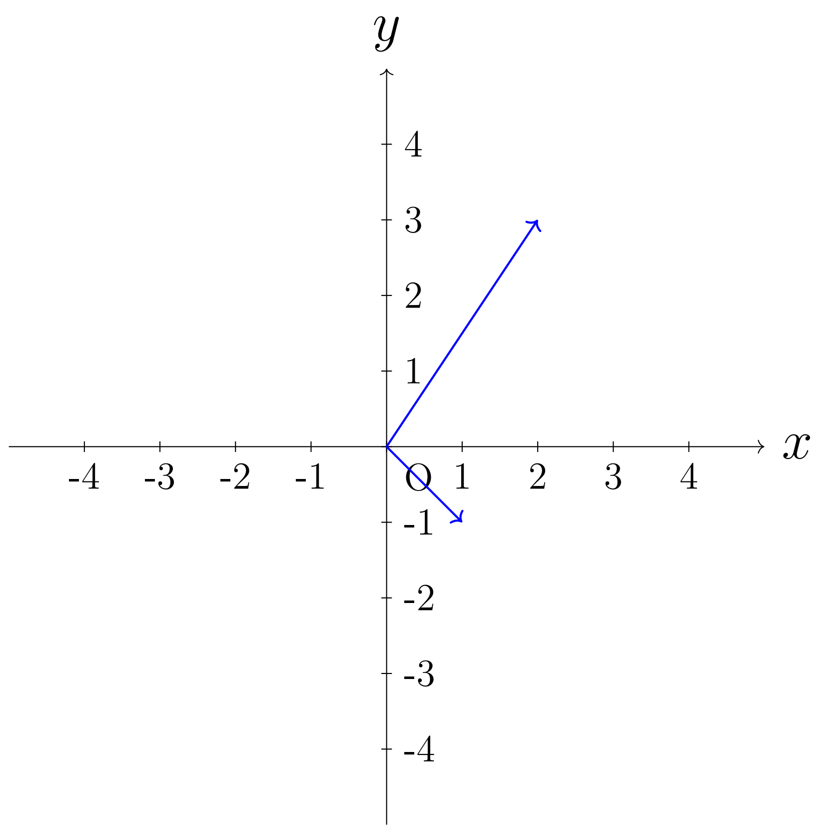


Vectors



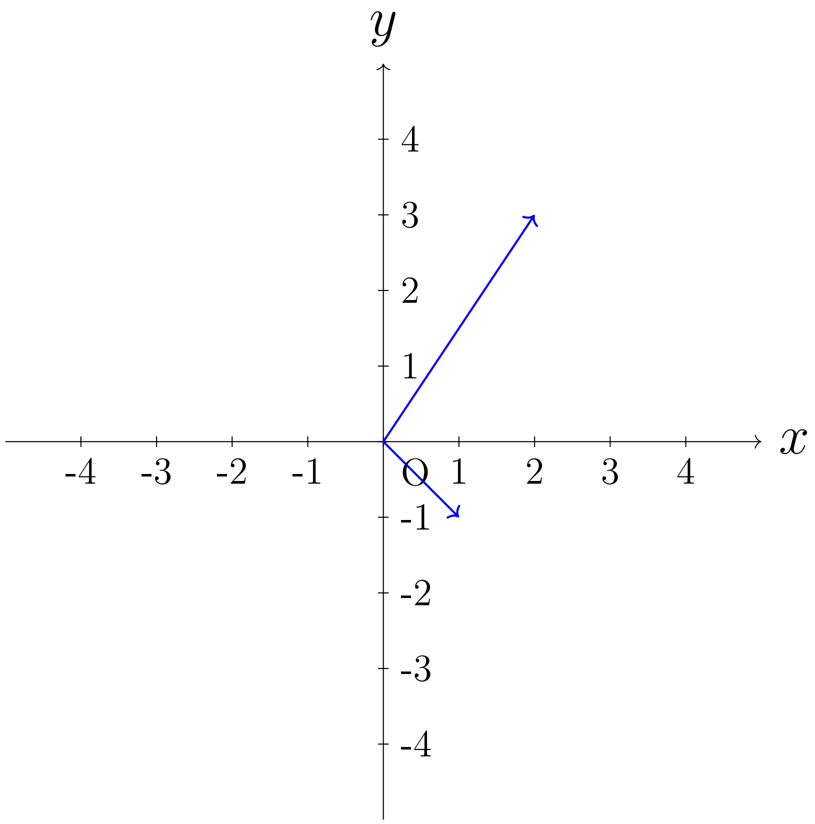
Vectors

$$v = (2, 3)$$

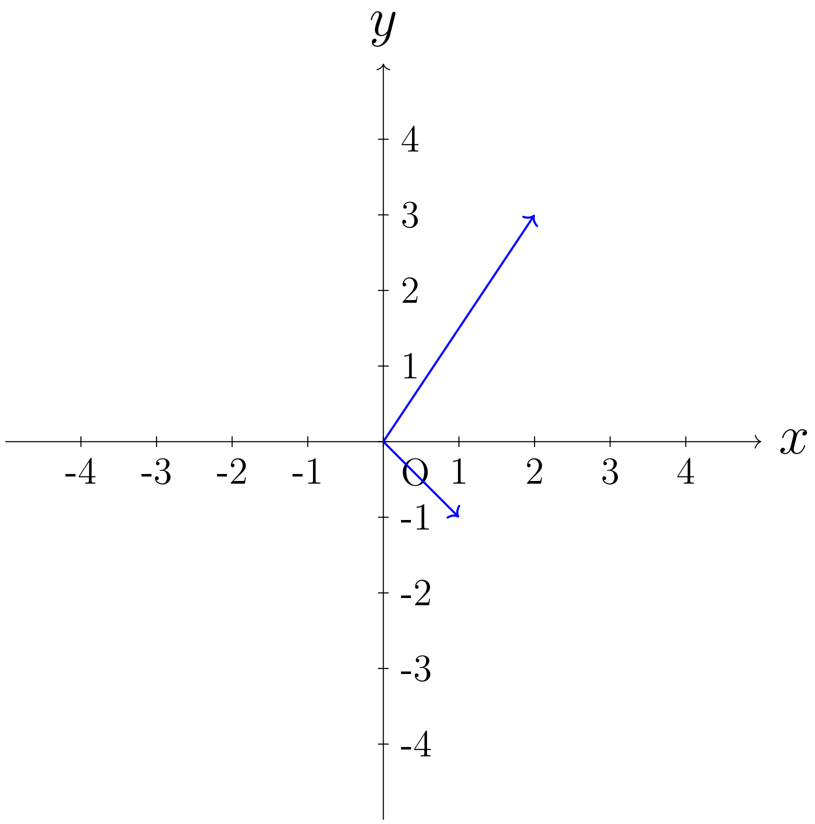


Vectors

$v = (2, 3)$
$w = (1, -1)$



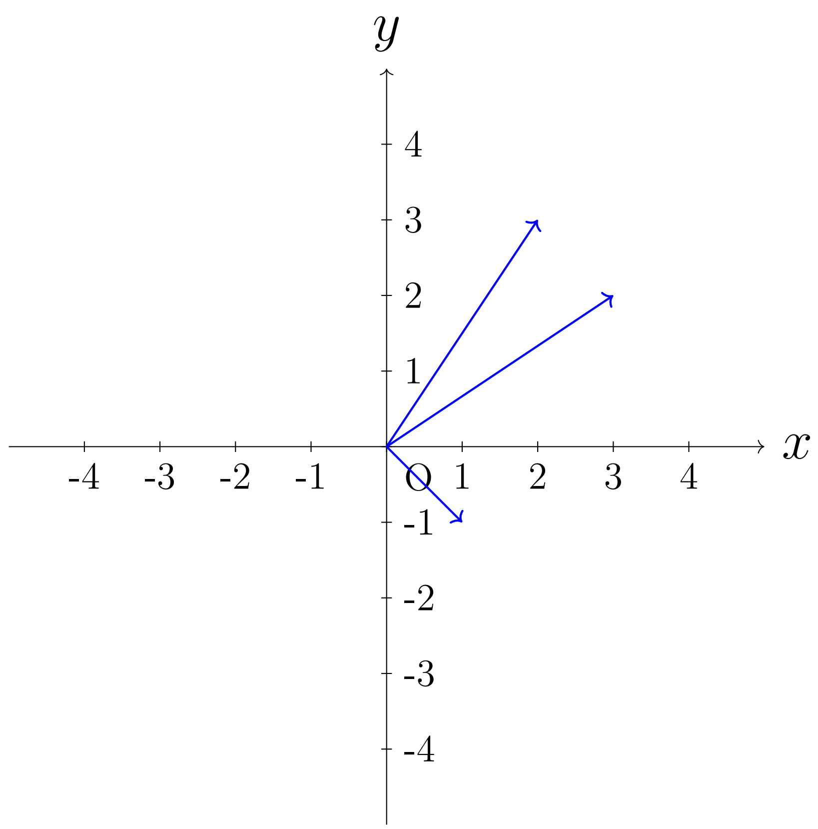
Vectors



$v = (2, 3)$
$w = (1, -1)$

Vector addition :

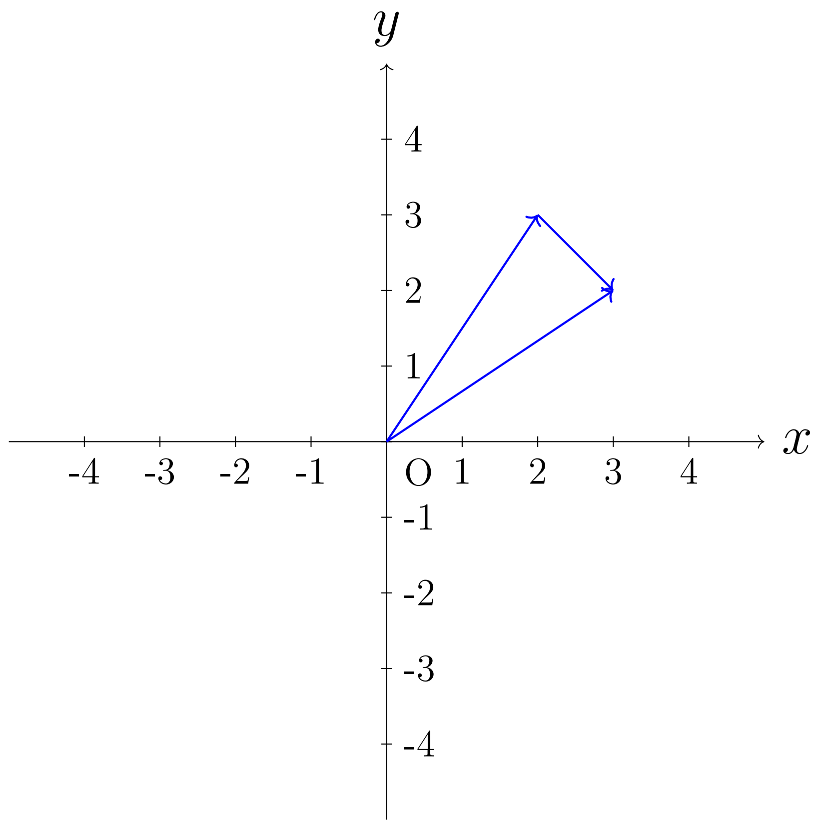
Vectors



$v = (2, 3)$
$w = (1, -1)$

Vector addition :
$v + w = (3, 2)$

Vectors

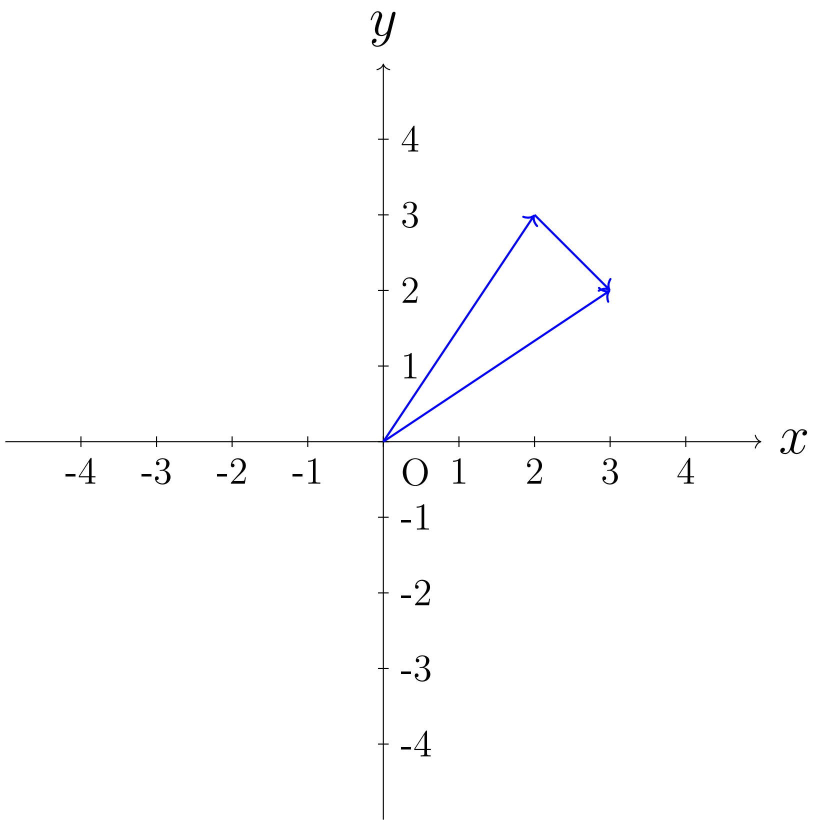


$$v = (2, 3)$$
$$w = (1, -1)$$

Vector addition :

$$v + w = (3, 2)$$

Vectors

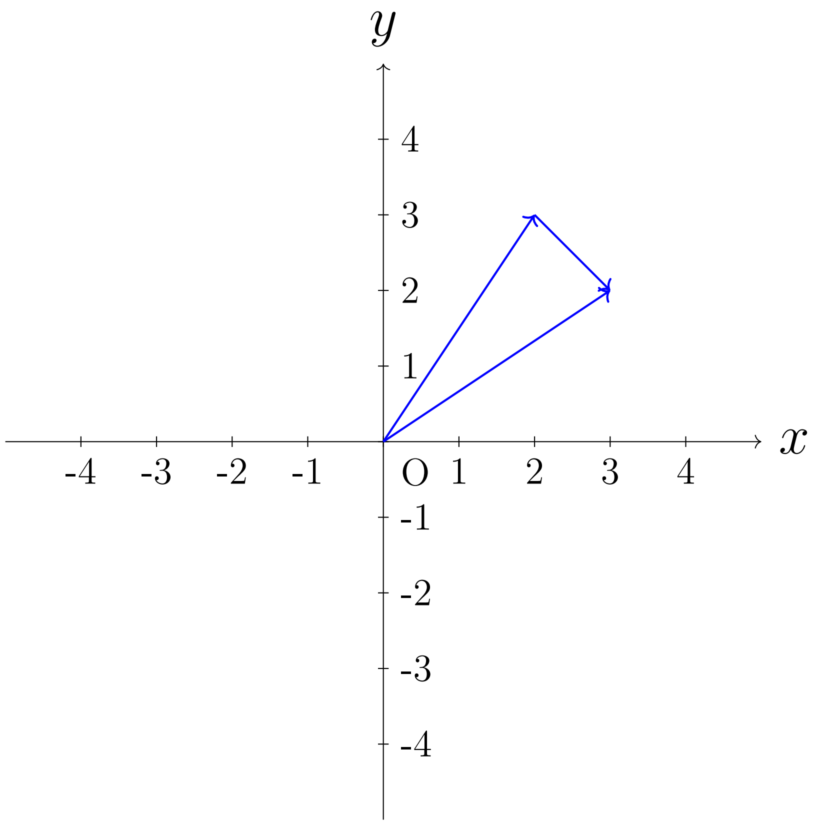


$v = (2, 3)$
$w = (1, -1)$

Vector addition :
$v + w = (3, 2)$

In general:

Vectors

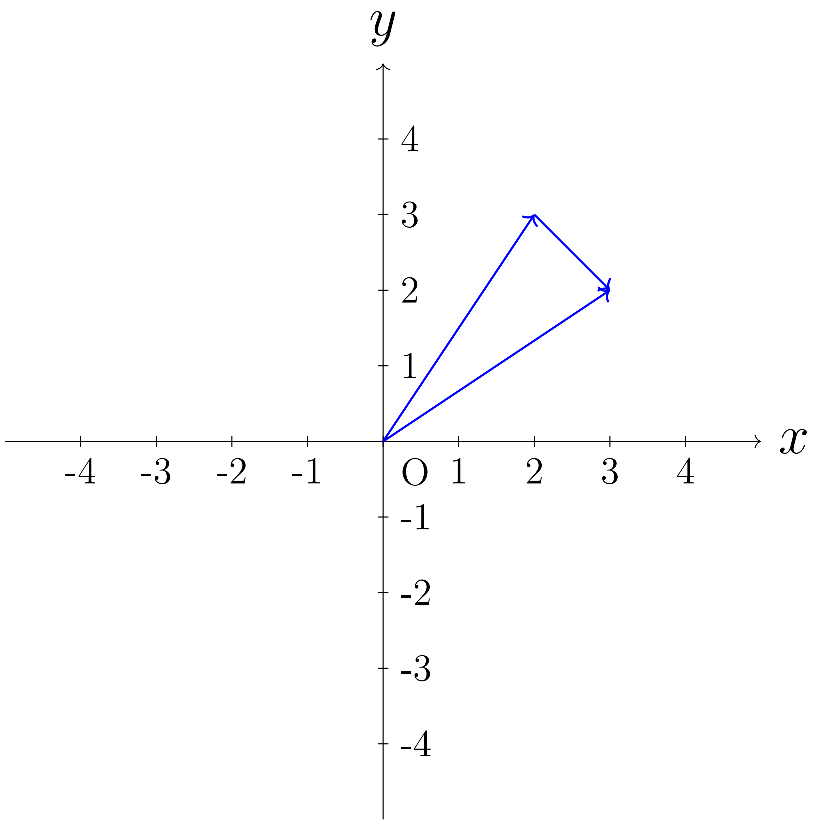


$v = (2, 3)$
$w = (1, -1)$

Vector addition :
$v + w = (3, 2)$

In general:
$(x_1, y_2) + (x_2, y_2)$

Vectors

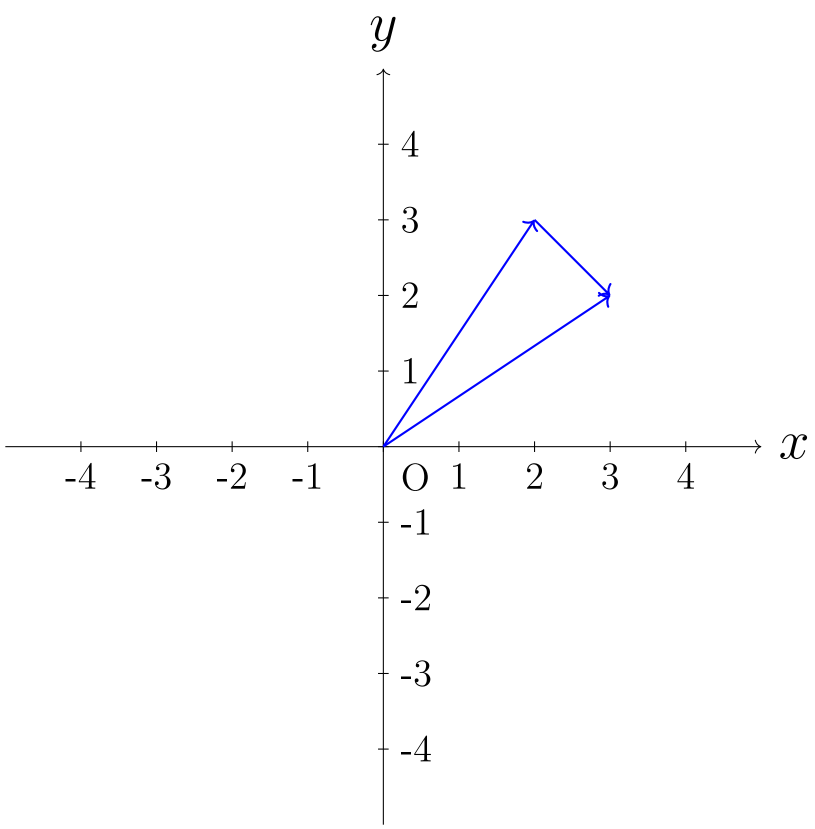


$v = (2, 3)$
$w = (1, -1)$

Vector addition :
$v + w = (3, 2)$

In general:
$(x_1, y_2) + (x_2, y_2) := (x_1 + x_2$

Vectors



$v = (2, 3)$
$w = (1, -1)$

Vector addition :
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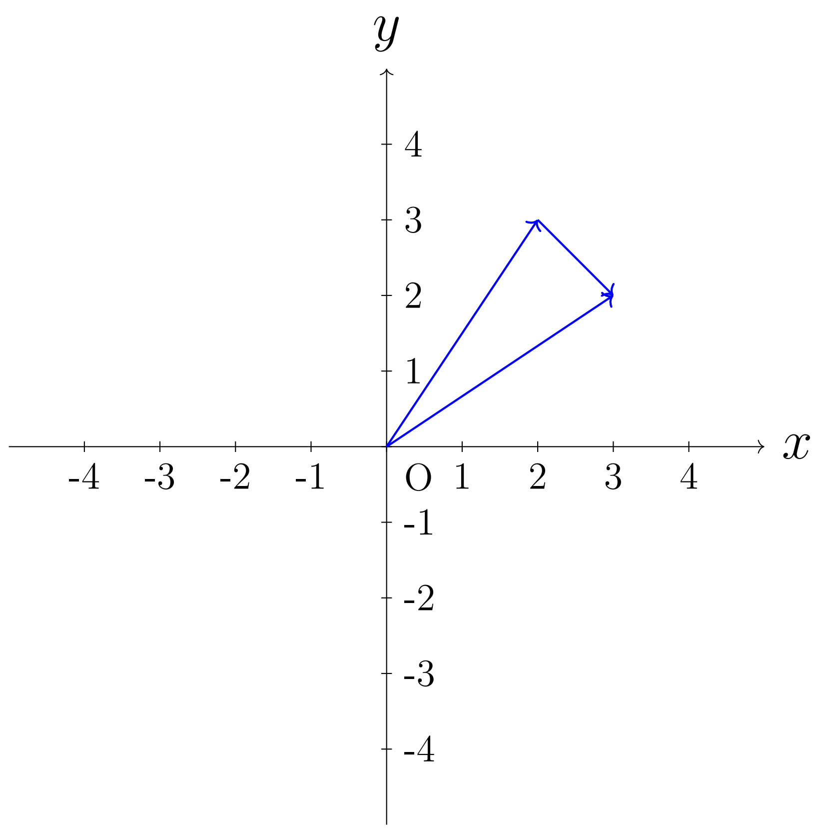
In general:
$(x_1, y_2) + (x_2, y_2) := (x_1 + x_2, y_1 + y_2)$

Vectors

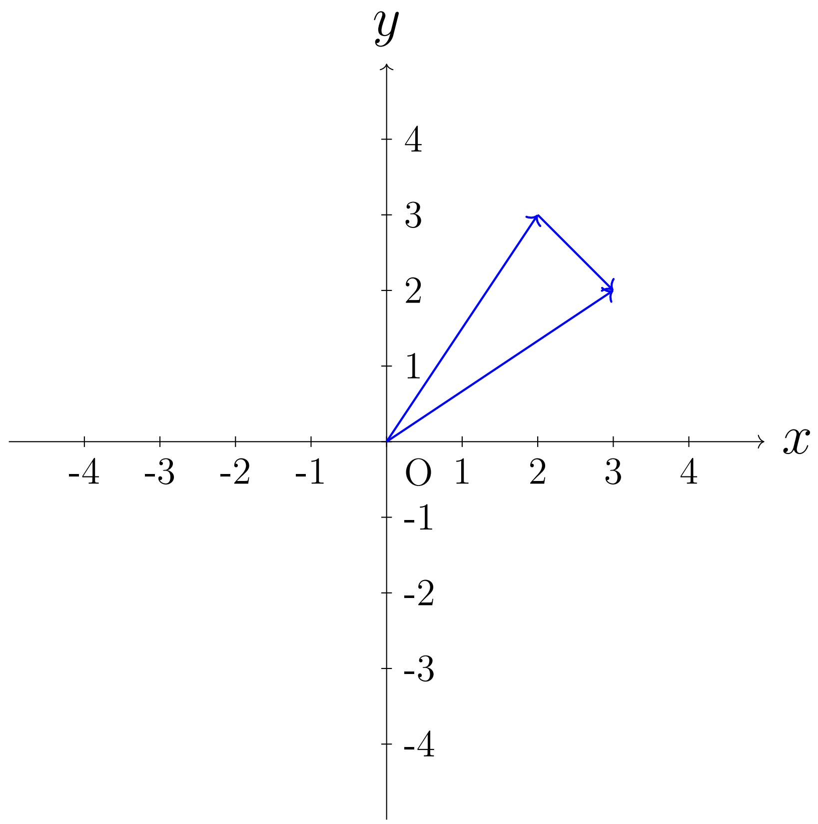


$v = (2, 3)$
$w = (1, -1)$

Vector addition :
$v + w = (3, 2)$

In general:
$(x_1, y_2) + (x_2, y_2) := (x_1 + x_2, y_1 + y_2)$
$(x_1, y_2) - (x_2, y_2)$

Vectors



$v = (2, 3)$
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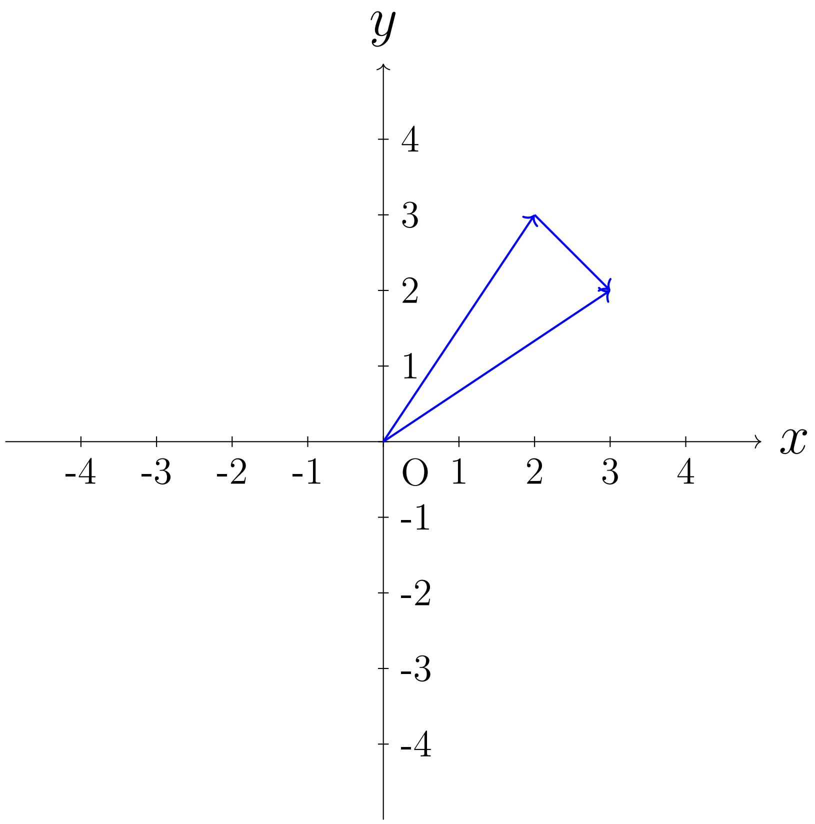
Vector addition :
$v + w = (3, 2)$

In general:
$(x_1, y_2) + (x_2, y_2) := (x_1 + x_2, y_1 + y_2)$
$(x_1, y_2) - (x_2, y_2) := (x_1 - x_2$

Vectors

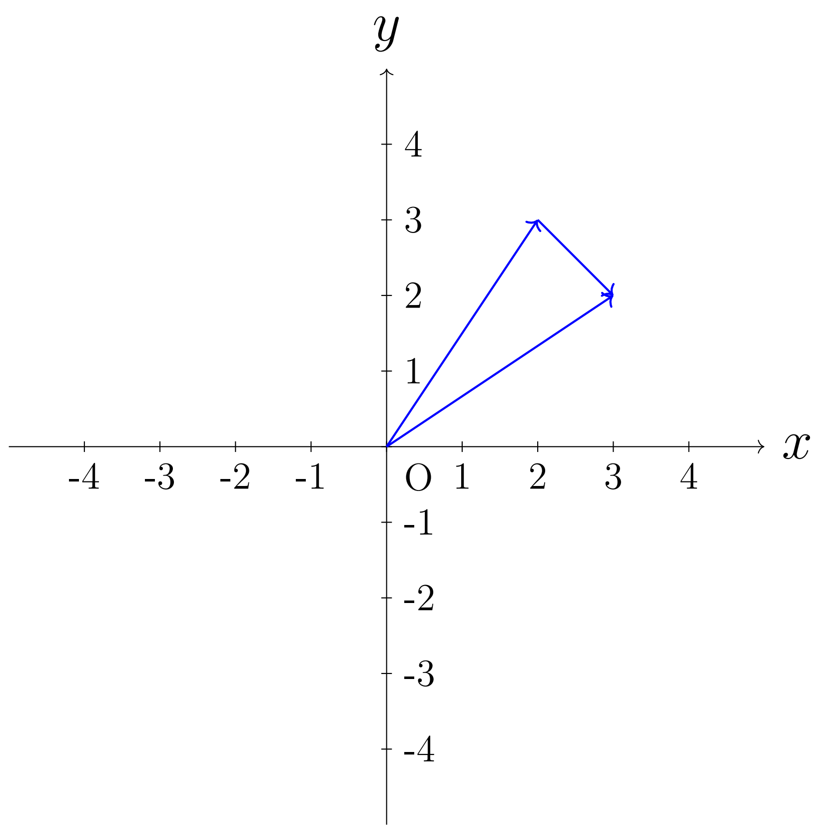


$v = (2, 3)$
$w = (1, -1)$

Vector addition :
$v + w = (3, 2)$

In general:
$(x_1, y_2) + (x_2, y_2) := (x_1 + x_2, y_1 + y_2)$
$(x_1, y_2) - (x_2, y_2) := (x_1 - x_2, y_1 - y_2)$

Vectors

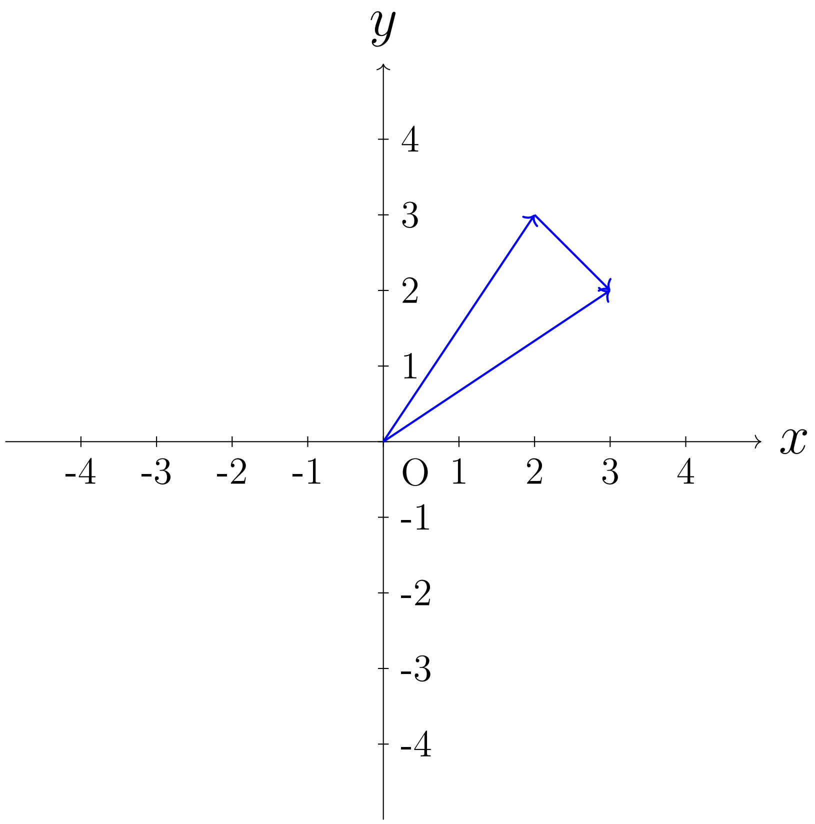


$v = (2, 3)$
$w = (1, -1)$

Vector addition and subtraction :
$v + w = (3, 2)$

In general:
$(x_1, y_2) + (x_2, y_2) := (x_1 + x_2, y_1 + y_2)$
$(x_1, y_2) - (x_2, y_2) := (x_1 - x_2, y_1 - y_2)$

Vectors



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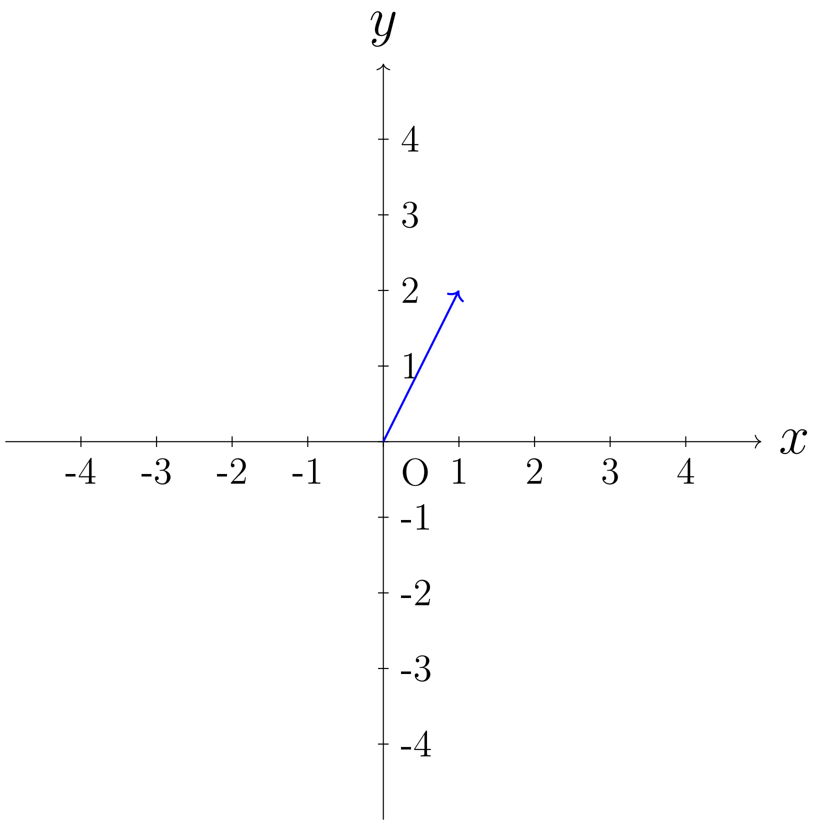
Vector addition and subtraction :
$v + w = (3, 2)$

In general:
$(x_1, y_2) + (x_2, y_2) := (x_1 + x_2, y_1 + y_2)$
$(x_1, y_2) - (x_2, y_2) := (x_1 - x_2, y_1 - y_2)$

Scalar multiplication:

Vectors



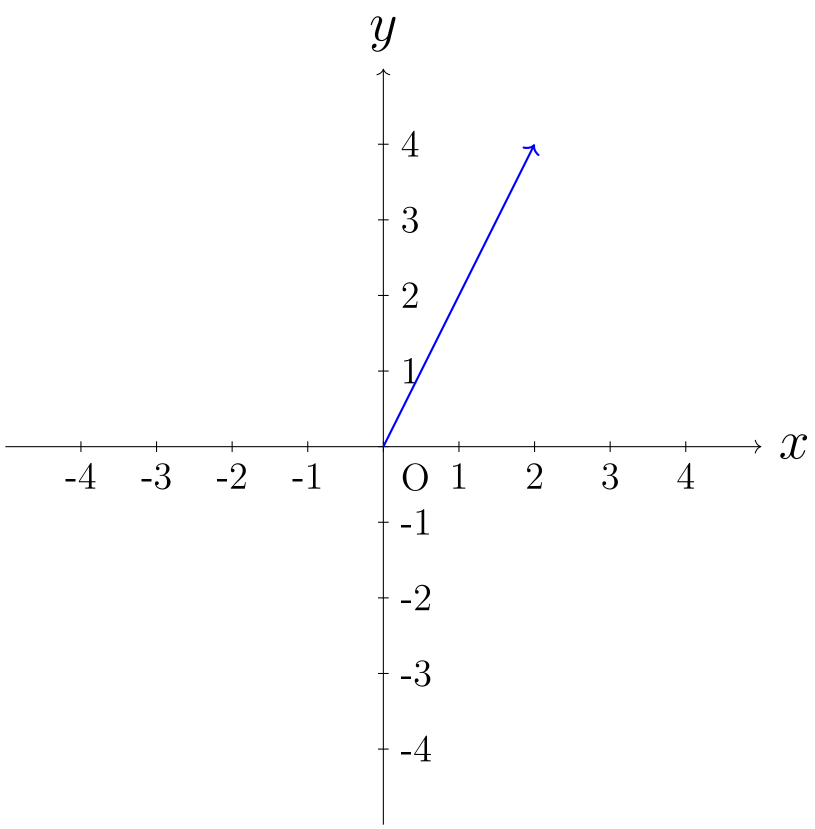
$v = (2, 3)$
$w = (1, -1)$

Vector addition and subtraction :
$v + w = (3, 2)$

In general:
$(x_1, y_2) + (x_2, y_2) := (x_1 + x_2, y_1 + y_2)$
$(x_1, y_2) - (x_2, y_2) := (x_1 - x_2, y_1 - y_2)$

Scalar multiplication:
$v := (1, 2)$

Vectors



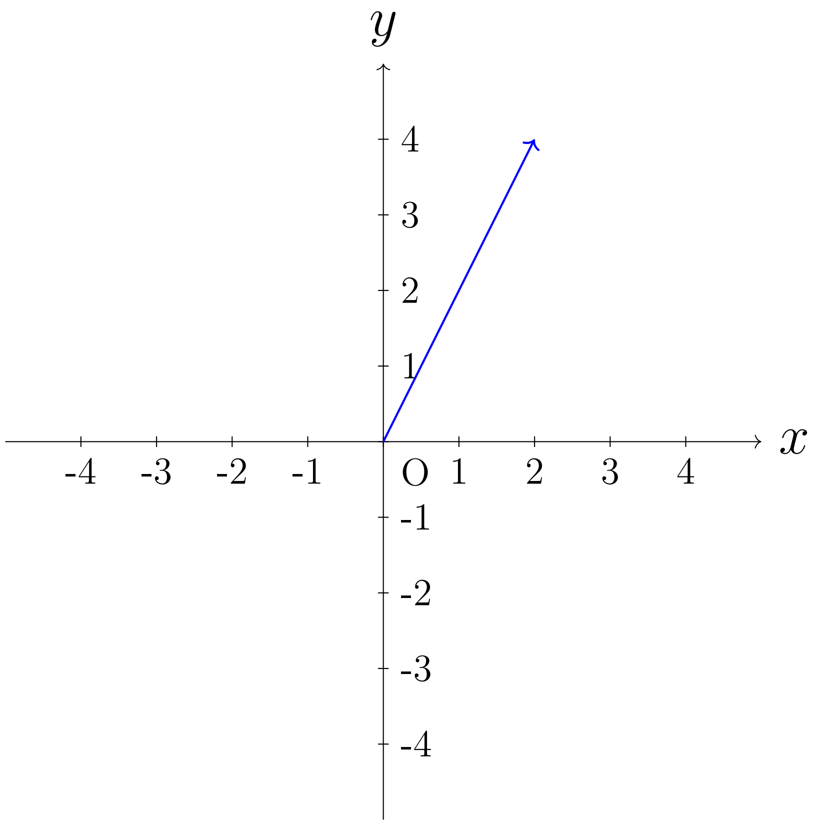
$v = (2,3)$
$w = (1,-1)$

Vector addition and subtraction :
$v + w = (3,2)$

In general:
$(x_1, y_2) + (x_2, y_2) := (x_1 + x_2, y_1 + y_2)$
$(x_1, y_2) - (x_2, y_2) := (x_1 - x_2, y_1 - y_2)$

Scalar multiplication:
$v := (1,2)$
$2v$

Vectors



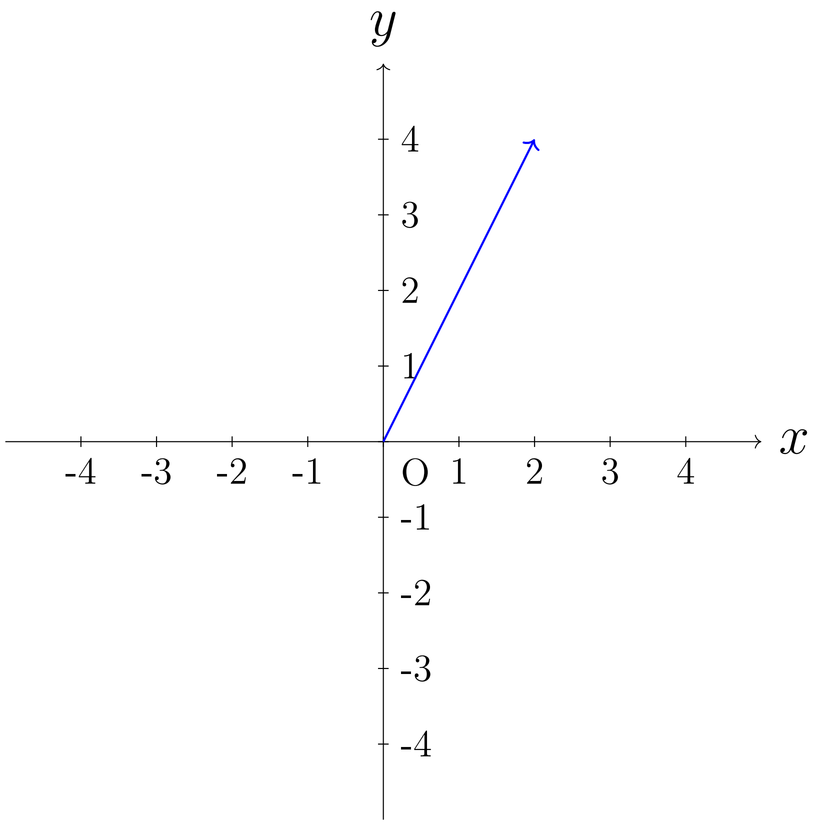
$v = (2, 3)$
$w = (1, -1)$

Vector addition and subtraction :
$v + w = (3, 2)$

In general:
$(x_1, y_2) + (x_2, y_2) := (x_1 + x_2, y_1 + y_2)$
$(x_1, y_2) - (x_2, y_2) := (x_1 - x_2, y_1 - y_2)$

Scalar multiplication:
$v := (1, 2)$
$2v = 2(1, 2)$

Vectors



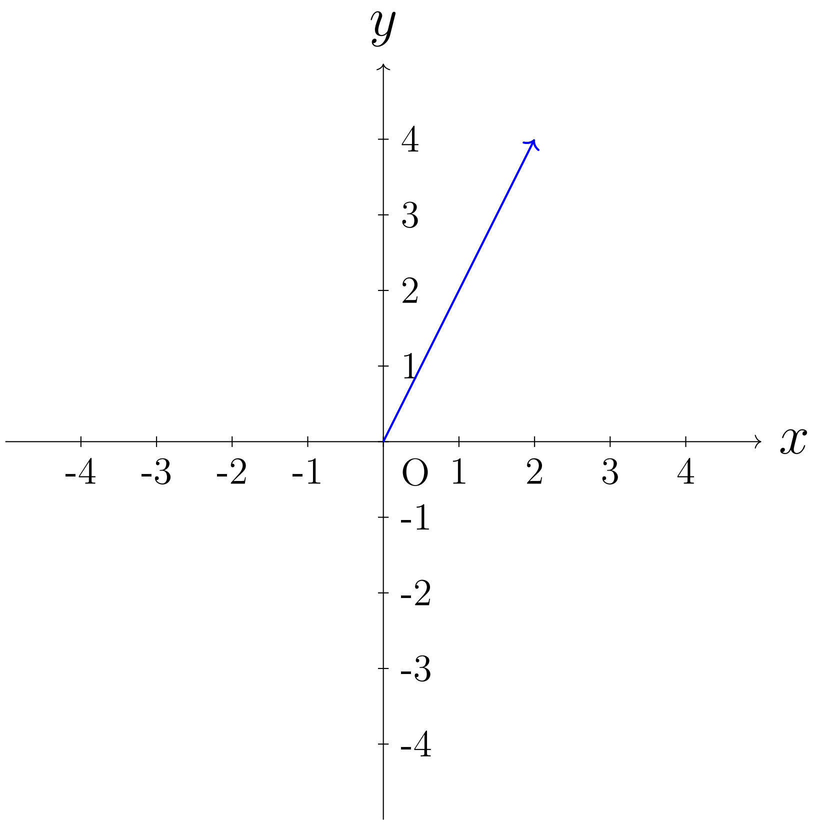
$v = (2, 3)$
$w = (1, -1)$

Vector addition and subtraction :
$v + w = (3, 2)$

In general:
$(x_1, y_2) + (x_2, y_2) := (x_1 + x_2, y_1 + y_2)$
$(x_1, y_2) - (x_2, y_2) := (x_1 - x_2, y_1 - y_2)$

Scalar multiplication:
$v := (1, 2)$
$2v = 2(1, 2) = (2, 4)$

Vectors



$v = (2, 3)$
$w = (1, -1)$

Vector addition and subtraction :
$v + w = (3, 2)$

In general:
$(x_1, y_2) + (x_2, y_2) := (x_1 + x_2, y_1 + y_2)$
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Scalar multiplication:
$v := (1, 2)$
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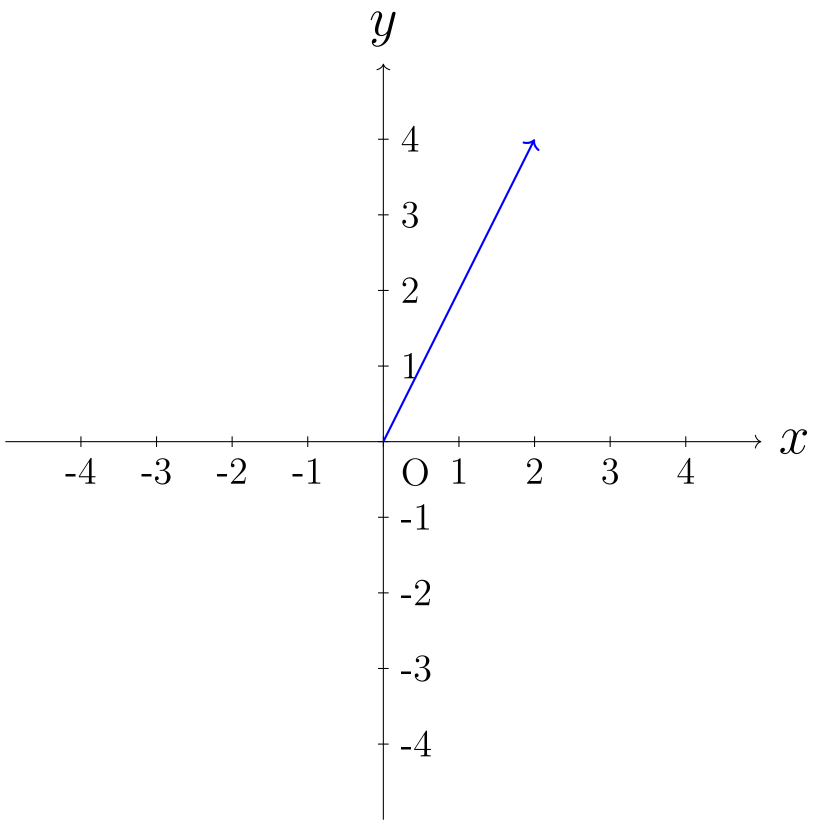
In general:

Vectors



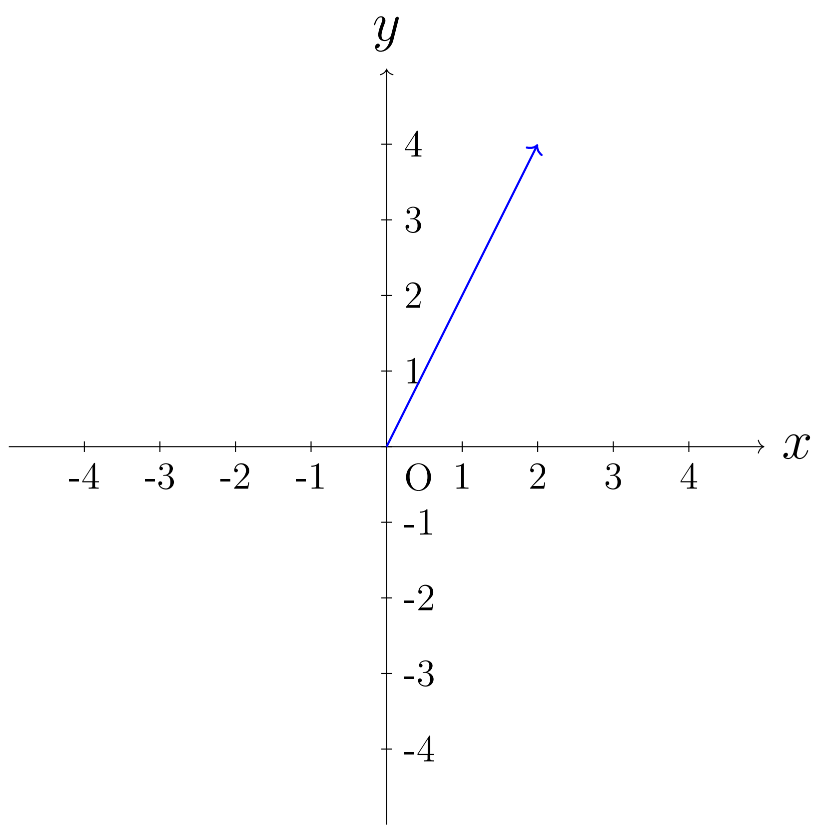
$v = (2, 3)$
$w = (1, -1)$

Vector addition and subtraction :
$v + w = (3, 2)$

In general:
$(x_1, y_2) + (x_2, y_2) := (x_1 + x_2, y_1 + y_2)$
$(x_1, y_2) - (x_2, y_2) := (x_1 - x_2, y_1 - y_2)$

Scalar multiplication:
$v := (1, 2)$
$2v = 2(1, 2) = (2, 4)$
In general:
$\lambda(x, y)$

Vectors



$v = (2, 3)$
$w = (1, -1)$

Vector addition and subtraction :
$v + w = (3, 2)$

In general:
$(x_1, y_2) + (x_2, y_2) := (x_1 + x_2, y_1 + y_2)$
$(x_1, y_2) - (x_2, y_2) := (x_1 - x_2, y_1 - y_2)$

Scalar multiplication:
$v := (1, 2)$
$2v = 2(1, 2) = (2, 4)$
In general:
$\lambda(x, y) := (\lambda x, \lambda y)$

$p := (2, 3),$

$$p := (2,3),$$
$$\mathbf{w} := (1,1),$$

$$p := (2,3),$$
$$\mathbf{w} := (1,1),$$
$$q := p + \mathbf{w} = (2,3) + (1,1) = (3,4)$$

$p := (2, 3),$
$\mathbf{w} := (1, 1),$
$q := p + \mathbf{w} = (2, 3) + (1, 1) = (3, 4)$
(displacement of p by $\mathbf{w}$).

$p := (2, 3)$,
$\mathbf{w} := (1, 1)$,
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$p := (2, 3)$ and $q = (3, 4)$,

$p := (2, 3)$,
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$q := p + \mathbf{w} = (2, 3) + (1, 1) = (3, 4)$
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$p := (2, 3)$ and $q = (3, 4)$,
$\mathbf{v} = q - p$

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(displacement of p by $\mathbf{w}$).

$p := (2, 3)$ and $q = (3, 4)$,
$\mathbf{v} = q - p$ is the displacement

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$q := p + \mathbf{w} = (2, 3) + (1, 1) = (3, 4)$
(displacement of p by $\mathbf{w}$).

$p := (2, 3)$ and $q = (3, 4)$,
$\mathbf{v} = q - p$ is the displacement that takes p to q

$p := (2, 3)$,
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$p := (2, 3)$ and $q = (3, 4)$,
$\mathbf{v} = q - p$ is the displacement that takes p to q

$\gamma : (\alpha, \beta) \to \mathbb{R}^2$

$p := (2, 3),$
$\mathbf{w} := (1, 1),$
$q := p + \mathbf{w} = (2, 3) + (1, 1) = (3, 4)$
(displacement of p by $\mathbf{w}$).

$p := (2, 3)$ and $q = (3, 4),$
$\mathbf{v} = q - p$ is the displacement that takes p to q

$\gamma : (\alpha, \beta) \to \mathbb{R}^2$ is a smooth parametrization.

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$\mathbf{w} := (1,1)$,
$q := p + \mathbf{w} = (2,3) + (1,1) = (3,4)$
(displacement of p by $\mathbf{w}$).

$p := (2,3)$ and $q = (3,4)$,
$\mathbf{v} = q - p$ is the displacement that takes p to q

$\gamma : (\alpha, \beta) \to \mathbb{R}^2$ is a smooth parametrization.
$\gamma(t)$ is the *point* at t

$p := (2,3)$,
$\mathbf{w} := (1,1)$,
$q := p + \mathbf{w} = (2,3) + (1,1) = (3,4)$
(displacement of p by $\mathbf{w}$).

$p := (2,3)$ and $q = (3,4)$,
$\mathbf{v} = q - p$ is the displacement that takes p to q

$\gamma : (\alpha, \beta) \to \mathbb{R}^2$ is a smooth parametrization.
$\gamma(t)$ is the *point* at t
$\gamma(t+h)$ is the *point* at $t+h$

$p := (2, 3)$,
$\mathbf{w} := (1, 1)$,
$q := p + \mathbf{w} = (2, 3) + (1, 1) = (3, 4)$
(displacement of p by $\mathbf{w}$).

$p := (2, 3)$ and $q = (3, 4)$,
$\mathbf{v} = q - p$ is the displacement that takes p to q

$\gamma : (\alpha, \beta) \to \mathbb{R}^2$ is a smooth parametrization.
$\gamma(t)$ is the *point* at t
$\gamma(t + h)$ is the *point* at $t + h$
$\gamma(t + h) - \gamma(t)$ is the displacement *vector* at $t + h$

$p := (2, 3)$,
$\mathbf{w} := (1, 1)$,
$q := p + \mathbf{w} = (2, 3) + (1, 1) = (3, 4)$
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$p := (2, 3)$ and $q = (3, 4)$,
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$\gamma(t)$ is the *point* at t
$\gamma(t + h)$ is the *point* at $t + h$
$\gamma(t + h) - \gamma(t)$ is the displacement *vector* at $t + h$

Definition. $\gamma : (\alpha, \beta) \to \mathbb{R}^2$

$p := (2, 3)$,
$\mathbf{w} := (1, 1)$,
$q := p + \mathbf{w} = (2, 3) + (1, 1) = (3, 4)$
(displacement of p by $\mathbf{w}$).

$p := (2, 3)$ and $q = (3, 4)$,
$\mathbf{v} = q - p$ is the displacement that takes p to q

$\gamma : (\alpha, \beta) \to \mathbb{R}^2$ is a smooth parametrization.
$\gamma(t)$ is the *point* at t
$\gamma(t + h)$ is the *point* at $t + h$
$\gamma(t + h) - \gamma(t)$ is the displacement *vector* at $t + h$

Definition. $\gamma : (\alpha, \beta) \to \mathbb{R}^2$ is a smooth parametrization.

$$\dot{\gamma}(t)$$

$p := (2, 3)$,
$\mathbf{w} := (1, 1)$,
$q := p + \mathbf{w} = (2, 3) + (1, 1) = (3, 4)$
(displacement of p by $\mathbf{w}$).

$p := (2, 3)$ and $q = (3, 4)$,
$\mathbf{v} = q - p$ is the displacement that takes p to q

$\gamma : (\alpha, \beta) \to \mathbb{R}^2$ is a smooth parametrization.
$\gamma(t)$ is the *point* at t
$\gamma(t + h)$ is the *point* at $t + h$
$\gamma(t + h) - \gamma(t)$ is the displacement *vector* at $t + h$

Definition. $\gamma : (\alpha, \beta) \to \mathbb{R}^2$ is a smooth parametrization.

$$\dot{\gamma}(t) = \lim_{h \to 0}$$

$p := (2,3)$,
$\mathbf{w} := (1,1)$,
$q := p + \mathbf{w} = (2,3) + (1,1) = (3,4)$
(displacement of p by $\mathbf{w}$).

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Definition. A smooth parametrized curve, $\gamma : (\alpha, \beta) \to \mathbb{R}^2$,

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Points on the straight line passing through p, parallel to $\mathbf{v} \neq 0$:

$$\{q \in \mathbb{R}^2 \mid q = p + k\mathbf{v}, k \in \mathbb{R}\}$$

Definition. If $\dot{\gamma}(t) \neq 0$, the line tangent to γ at t is,

$$T_\gamma(t) := \{q \in \mathbb{R}^2 \mid q = \gamma(t) + k\dot{\gamma}(t), k \in \mathbb{R}\}$$

Definition. A smooth parametrized curve, $\gamma : (\alpha, \beta) \to \mathbb{R}^2$, is called a **regular parametrized curve**

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is called the velocity vector at t and $\dot{\gamma} : (\alpha, \beta) \to \mathbb{R}^2$ is called the velocity vector field of the parametrization γ.

Points on the straight line passing through p, parallel to $\mathbf{v} \neq 0$:

$$\{q \in \mathbb{R}^2 \mid q = p + k\mathbf{v}, k \in \mathbb{R}\}$$

Definition. If $\dot{\gamma}(t) \neq 0$, the line tangent to γ at t is,

$$T_\gamma(t) := \{q \in \mathbb{R}^2 \mid q = \gamma(t) + k\dot{\gamma}(t), k \in \mathbb{R}\}$$

Definition. A smooth parametrized curve, $\gamma : (\alpha, \beta) \to \mathbb{R}^2$, is called a **regular parametrized curve** if $\dot{\gamma}(t) \neq 0$ for each $t \in (\alpha, \beta)$.

From now on, we will assume all parametrized curves to be regular

Lemma. *If* $\tilde{\gamma}(t) = \gamma(\phi(t))$ *is a reparametrization,*

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Proof.

$$\tilde{\gamma}(t) = \gamma(\phi(t))$$

$\square$

Lemma. *If* $\tilde{\gamma}(t) = \gamma(\phi(t))$ *is a reparametrization, then* $\dot{\tilde{\gamma}}(t) = \dot{\gamma}(\phi(t))\phi'(t)$

Proof.

$$\tilde{\gamma}(t) = \gamma(\phi(t))$$

$$\tilde{\gamma}(t) = (f_1(\phi(t)), f_2(\phi(t)))$$

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□

Lemma. *If* $\tilde{\gamma}(t) = \gamma(\phi(t))$ *is a reparametrization, then* $\dot{\tilde{\gamma}}(t) = \dot{\gamma}(\phi(t))\phi'(t)$

Proof.

$$\tilde{\gamma}(t) = \gamma(\phi(t))$$

$$\begin{aligned}
\tilde{\gamma}(t) &= (f_1(\phi(t)), f_2(\phi(t))) \\
\dot{\tilde{\gamma}}(t) &= (f_1'(\phi(t))\phi'(t), f_2'(\phi(t))\phi'(t)) \\
\dot{\tilde{\gamma}}(t) &= (f_1'(\phi(t)), f_2'(\phi(t)))\phi'(t) \\
\dot{\tilde{\gamma}}(t) &= \dot{\gamma}(\phi(t))\phi'(t)
\end{aligned}$$

□

Corollary. *The tangent line is invariant under a reparametrization,* $\phi(t)$.

Proof.

$$\{\gamma(t) + k\dot{\tilde{\gamma}}(t)\}$$

□

Lemma. *If* $\tilde{\gamma}(t) = \gamma(\phi(t))$ *is a reparametrization, then* $\dot{\tilde{\gamma}}(t) = \dot{\gamma}(\phi(t))\phi'(t)$

Proof.

$$\tilde{\gamma}(t) = \gamma(\phi(t))$$

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\tilde{\gamma}(t) &= (f_1(\phi(t)), f_2(\phi(t))) \\
\dot{\tilde{\gamma}}(t) &= (f_1'(\phi(t))\phi'(t), f_2'(\phi(t))\phi'(t)) \\
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\end{aligned}$$

□

Corollary. *The tangent line is invariant under a reparametrization,* $\phi(t)$*.*

Proof.

$$\{\gamma(t) + k\dot{\tilde{\gamma}}(t) \mid k \in \mathbb{R}\} = \{\gamma(t) + k\dot{\gamma}(\phi(t))\phi'(t)\}$$

□

Lemma. *If* $\tilde{\gamma}(t) = \gamma(\phi(t))$ *is a reparametrization, then* $\dot{\tilde{\gamma}}(t) = \dot{\gamma}(\phi(t))\phi'(t)$

Proof.

$$\tilde{\gamma}(t) = \gamma(\phi(t))$$

$$\tilde{\gamma}(t) = (f_1(\phi(t)), f_2(\phi(t)))$$
$$\dot{\tilde{\gamma}}(t) = (f_1'(\phi(t))\phi'(t), f_2'(\phi(t))\phi'(t))$$
$$\dot{\tilde{\gamma}}(t) = (f_1'(\phi(t)), f_2'(\phi(t)))\phi'(t)$$
$$\dot{\tilde{\gamma}}(t) = \dot{\gamma}(\phi(t))\phi'(t)$$

□

Corollary. *The tangent line is invariant under a reparametrization,* $\phi(t)$.

Proof.

$$\{\gamma(t) + k\dot{\tilde{\gamma}}(t) \mid k \in \mathbb{R}\} = \{\gamma(t) + k\dot{\gamma}(\phi(t))\phi'(t) \mid k \in \mathbb{R}\}$$

□

Lemma. *If* $\tilde{\gamma}(t) = \gamma(\phi(t))$ *is a reparametrization, then* $\dot{\tilde{\gamma}}(t) = \dot{\gamma}(\phi(t))\phi'(t)$

Proof.

$$\tilde{\gamma}(t) = \gamma(\phi(t))$$

$$\begin{aligned}
\tilde{\gamma}(t) &= (f_1(\phi(t)), f_2(\phi(t))) \\
\dot{\tilde{\gamma}}(t) &= (f_1'(\phi(t))\phi'(t), f_2'(\phi(t))\phi'(t)) \\
\dot{\tilde{\gamma}}(t) &= (f_1'(\phi(t)), f_2'(\phi(t)))\phi'(t) \\
\dot{\tilde{\gamma}}(t) &= \dot{\gamma}(\phi(t))\phi'(t)
\end{aligned}$$

□

Corollary. *The tangent line is invariant under a reparametrization,* $\phi(t)$*.*

Proof.

$$\{\gamma(t) + k\dot{\tilde{\gamma}}(t) \mid k \in \mathbb{R}\} \;=\; \{\gamma(t) + k\dot{\gamma}(\phi(t))\phi'(t) \mid k \in \mathbb{R}\}$$

□

Lemma. *If $\tilde{\gamma}(t) = \gamma(\phi(t))$ is a reparametrization, then $\dot{\tilde{\gamma}}(t) = \dot{\gamma}(\phi(t))\phi'(t)$*

Proof.

$$
\begin{aligned}
\tilde{\gamma}(t) &= \gamma(\phi(t)) \\
\tilde{\gamma}(t) &= (f_1(\phi(t)), f_2(\phi(t))) \\
\dot{\tilde{\gamma}}(t) &= (f_1'(\phi(t))\phi'(t), f_2'(\phi(t))\phi'(t)) \\
\dot{\tilde{\gamma}}(t) &= (f_1'(\phi(t)), f_2'(\phi(t)))\phi'(t) \\
\dot{\tilde{\gamma}}(t) &= \dot{\gamma}(\phi(t))\phi'(t)
\end{aligned}
$$

□

Corollary. *The tangent line is invariant under a reparametrization, $\phi(t)$, if $\phi'(t) \neq 0$*

Proof.

$$
\begin{aligned}
\{\gamma(t) + k\dot{\tilde{\gamma}}(t) \mid k \in \mathbb{R}\} &= \{\gamma(t) + k\dot{\gamma}(\phi(t))\phi'(t) \mid k \in \mathbb{R}\} \\
&= \{\gamma(t) + k\dot{\gamma}(\phi(t))\}
\end{aligned}
$$

□

Lemma. *If* $\tilde{\gamma}(t) = \gamma(\phi(t))$ *is a reparametrization, then* $\dot{\tilde{\gamma}}(t) = \dot{\gamma}(\phi(t))\phi'(t)$

Proof.

$$\begin{aligned}
\tilde{\gamma}(t) &= \gamma(\phi(t)) \\
\tilde{\gamma}(t) &= (f_1(\phi(t)), f_2(\phi(t))) \\
\dot{\tilde{\gamma}}(t) &= (f_1'(\phi(t))\phi'(t), f_2'(\phi(t))\phi'(t)) \\
\dot{\tilde{\gamma}}(t) &= (f_1'(\phi(t)), f_2'(\phi(t)))\phi'(t) \\
\dot{\tilde{\gamma}}(t) &= \dot{\gamma}(\phi(t))\phi'(t)
\end{aligned}$$

□

Corollary. *The tangent line is invariant under a reparametrization,* $\phi(t)$, *if* $\phi'(t) \neq 0$

Proof.

$$\begin{aligned}
\{\gamma(t) + k\dot{\tilde{\gamma}}(t) \mid k \in \mathbb{R}\} &= \{\gamma(t) + k\dot{\gamma}(\phi(t))\phi'(t) \mid k \in \mathbb{R}\} \\
&= \{\gamma(t) + k\dot{\gamma}(\phi(t)) \mid k \in \mathbb{R}\}
\end{aligned}$$

□

Lemma. *If* $\tilde{\gamma}(t) = \gamma(\phi(t))$ *is a reparametrization,* Note: $\tilde{\gamma}(t)$ is the same point, p, as $\gamma(\phi(t))$ *then* $\dot{\tilde{\gamma}}(t) = \dot{\gamma}(\phi(t))\phi'(t)$

Proof.

$$\begin{aligned}
\tilde{\gamma}(t) &= \gamma(\phi(t)) \\
\tilde{\gamma}(t) &= (f_1(\phi(t)), f_2(\phi(t))) \\
\dot{\tilde{\gamma}}(t) &= (f_1'(\phi(t))\phi'(t), f_2'(\phi(t))\phi'(t)) \\
\dot{\tilde{\gamma}}(t) &= (f_1'(\phi(t)), f_2'(\phi(t)))\phi'(t) \\
\dot{\tilde{\gamma}}(t) &= \dot{\gamma}(\phi(t))\phi'(t)
\end{aligned}$$

$\square$

Corollary. *The tangent line is invariant under a reparametrization,* $\phi(t)$, *if* $\phi'(t) \neq 0$

Proof.

$$\begin{aligned}
\{\gamma(t) + k\dot{\tilde{\gamma}}(t) \mid k \in \mathbb{R}\} &= \{\gamma(t) + k\dot{\gamma}(\phi(t))\phi'(t) \mid k \in \mathbb{R}\} \\
&= \{\gamma(t) + k\dot{\gamma}(\phi(t)) \mid k \in \mathbb{R}\}
\end{aligned}$$

$\square$

Lemma. *If* $\tilde{\gamma}(t) = \gamma(\phi(t))$ *is a reparametrization, then* $\dot{\tilde{\gamma}}(t) = \dot{\gamma}(\phi(t))\phi'(t)$

Note: $\tilde{\gamma}(t)$ is the same point, p, as $\gamma(\phi(t))$
When using $\tilde{\gamma}$, the point p "appears at time t"

Proof.

$$\begin{aligned}
\tilde{\gamma}(t) &= \gamma(\phi(t)) \\
\tilde{\gamma}(t) &= (f_1(\phi(t)), f_2(\phi(t))) \\
\dot{\tilde{\gamma}}(t) &= (f_1'(\phi(t))\phi'(t), f_2'(\phi(t))\phi'(t)) \\
\dot{\tilde{\gamma}}(t) &= (f_1'(\phi(t)), f_2'(\phi(t)))\phi'(t) \\
\dot{\tilde{\gamma}}(t) &= \dot{\gamma}(\phi(t))\phi'(t)
\end{aligned}$$

□

Corollary. *The tangent line is invariant under a reparametrization,* $\phi(t)$, *if* $\phi'(t) \neq 0$

Proof.

$$\begin{aligned}
\{\gamma(t) + k\dot{\tilde{\gamma}}(t) \mid k \in \mathbb{R}\} &= \{\gamma(t) + k\dot{\gamma}(\phi(t))\phi'(t) \mid k \in \mathbb{R}\} \\
&= \{\gamma(t) + k\dot{\gamma}(\phi(t)) \mid k \in \mathbb{R}\}
\end{aligned}$$

□

Lemma. *If $\tilde{\gamma}(t) = \gamma(\phi(t))$ is a reparametrization, then $\dot{\tilde{\gamma}}(t) = \dot{\gamma}(\phi(t))\phi'(t)$*

Note: $\tilde{\gamma}(t)$ is the same point, p, as $\gamma(\phi(t))$
When using $\tilde{\gamma}$, the point p "appears at time t"
When using γ, the point p "appears at time $\phi(t)$"

Proof.

$$\tilde{\gamma}(t) = \gamma(\phi(t))$$

$$\begin{aligned}
\tilde{\gamma}(t) &= (f_1(\phi(t)), f_2(\phi(t))) \\
\dot{\tilde{\gamma}}(t) &= (f_1'(\phi(t))\phi'(t), f_2'(\phi(t))\phi'(t)) \\
\dot{\tilde{\gamma}}(t) &= (f_1'(\phi(t)), f_2'(\phi(t)))\phi'(t) \\
\dot{\tilde{\gamma}}(t) &= \dot{\gamma}(\phi(t))\phi'(t)
\end{aligned}$$

$\square$

Corollary. *The tangent line is invariant under a reparametrization, $\phi(t)$, if $\phi'(t) \neq 0$*

Proof.

$$\begin{aligned}
\{\gamma(t) + k\dot{\tilde{\gamma}}(t) \mid k \in \mathbb{R}\} &= \{\gamma(t) + k\dot{\gamma}(\phi(t))\phi'(t) \mid k \in \mathbb{R}\} \\
&= \{\gamma(t) + k\dot{\gamma}(\phi(t)) \mid k \in \mathbb{R}\}
\end{aligned}$$

$\square$

Lemma. *If $\tilde{\gamma}(t) = \gamma(\phi(t))$ is a reparametrization, then $\dot{\tilde{\gamma}}(t) = \dot{\gamma}(\phi(t))\phi'(t)$*

Note: $\tilde{\gamma}(t)$ is the same point, p, as $\gamma(\phi(t))$
When using $\tilde{\gamma}$, the point p "appears at time t"
When using γ, the point p "appears at time $\phi(t)$"
So, $\dot{\tilde{\gamma}}(t)$ and $\dot{\gamma}(\phi(t))$ are velocity vectors at the same point p

Proof.

$$\tilde{\gamma}(t) = \gamma(\phi(t))$$

$$\begin{aligned}
\tilde{\gamma}(t) &= (f_1(\phi(t)), f_2(\phi(t))) \\
\dot{\tilde{\gamma}}(t) &= (f_1'(\phi(t))\phi'(t), f_2'(\phi(t))\phi'(t)) \\
\dot{\tilde{\gamma}}(t) &= (f_1'(\phi(t)), f_2'(\phi(t)))\phi'(t) \\
\dot{\tilde{\gamma}}(t) &= \dot{\gamma}(\phi(t))\phi'(t)
\end{aligned}$$

□

Corollary. *The tangent line is invariant under a reparametrization, $\phi(t)$, if $\phi'(t) \neq 0$*

Proof.

$$\begin{aligned}
\{\gamma(t) + k\dot{\tilde{\gamma}}(t) \mid k \in \mathbb{R}\} &= \{\gamma(t) + k\dot{\gamma}(\phi(t))\phi'(t) \mid k \in \mathbb{R}\} \\
&= \{\gamma(t) + k\dot{\gamma}(\phi(t)) \mid k \in \mathbb{R}\}
\end{aligned}$$

□