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Prove that  $s_{\beta}(t) - s_{\alpha}(t)$  is a constant.

**Theorem** (First Fundamental theorem of calculus).

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*Proof.*

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by the First Fundamental Theorem of Calculus,  $s'(t) = \|\dot{\gamma}(t)\|$   $\square$

**Corollary.** *The arc length function*



**Theorem** (First Fundamental theorem of calculus).  
 $f : [\alpha, \beta] \rightarrow \mathbb{R}$  continuous

$$F(t) := \int_{t_0}^t f(u) du$$

then,  $F'(t) = f(t)$

**Corollary.**  $\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$  is a smooth and regular parametrization.

and  $s(t)$  its arc-length function beginning at  $t_0$  then,  
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**Corollary.** The arc length function  $s(t)$

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by the First Fundamental Theorem of Calculus,  $s'(t) =$

$$||\dot{\gamma}(t)|| \quad \square$$

**Corollary.** The arc length function  $s(t)$  is smooth.

*Proof.* □

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If  $g(f(t)) = t$

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$$(s^{-1})'(\tilde{t}) = \frac{1}{s'(s^{-1}(\tilde{t}))} = \frac{1}{||\gamma'(s^{-1}(\tilde{t}))||}$$

$\gamma :$

$$(s^{-1})'(\tilde{t}) = \frac{1}{s'(s^{-1}(\tilde{t}))} = \frac{1}{||\gamma'(s^{-1}(\tilde{t}))||}$$

$$\gamma : (\alpha, \beta) \rightarrow$$

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$$\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$$

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$$\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$$

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$$\tilde{\gamma}(\tilde{t}) = \gamma(\phi(\tilde{t}))$$

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$$\text{If } \phi(t) = s^{-1}(t),$$

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$$(s^{-1})'(\tilde{t}) = \frac{1}{s'(s^{-1}(\tilde{t}))} = \frac{1}{||\gamma'(s^{-1}(\tilde{t}))||}$$

Proved that,

**Theorem.**

$\gamma :$

$$\begin{aligned} \gamma &: (\alpha, \beta) \rightarrow \mathbb{R}^2 \\ \tilde{\gamma} &: (\tilde{\alpha}, \tilde{\beta}) \rightarrow \mathbb{R}^2 \\ \tilde{\gamma}(\tilde{t}) &= \gamma(\phi(\tilde{t})) \\ \tilde{\gamma}'(\tilde{t}) &= \gamma'(\phi(\tilde{t}))\phi'(\tilde{t}) \end{aligned}$$

If  $\phi(t) = s^{-1}(t)$ ,

$$\tilde{\gamma}'(\tilde{t}) = \gamma'(s^{-1}(\tilde{t}))(s^{-1})'(\tilde{t}) = \gamma'(s^{-1}(\tilde{t}))\frac{1}{||\gamma'(s^{-1}(\tilde{t}))||}$$

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$$\gamma : (\alpha, \beta) \rightarrow$$

$$\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$$

$$\tilde{\gamma} : (\tilde{\alpha}, \tilde{\beta}) \rightarrow \mathbb{R}^2$$

$$\tilde{\gamma}(\tilde{t}) = \gamma(\phi(\tilde{t}))$$

$$\tilde{\gamma}'(\tilde{t}) = \gamma'(\phi(\tilde{t}))\phi'(\tilde{t})$$

$$\text{If } \phi(t) = s^{-1}(t),$$

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**Theorem.**

$\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$  a regular smooth parametrization

$$\begin{aligned} \gamma &: (\alpha, \beta) \rightarrow \mathbb{R}^2 \\ \tilde{\gamma} &: (\tilde{\alpha}, \tilde{\beta}) \rightarrow \mathbb{R}^2 \\ \tilde{\gamma}(\tilde{t}) &= \gamma(\phi(\tilde{t})) \\ \tilde{\gamma}'(\tilde{t}) &= \gamma'(\phi(\tilde{t}))\phi'(\tilde{t}) \end{aligned}$$

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$\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$  a regular smooth parametrization  $s(t)$ , the arc-length from  $t_0$  to  $t$  (where  $t, t_0 \in (\alpha, \beta)$ )

$$\begin{aligned} \gamma &: (\alpha, \beta) \rightarrow \mathbb{R}^2 \\ \tilde{\gamma} &: (\tilde{\alpha}, \tilde{\beta}) \rightarrow \mathbb{R}^2 \\ \tilde{\gamma}(\tilde{t}) &= \gamma(\phi(\tilde{t})) \\ \tilde{\gamma}'(\tilde{t}) &= \gamma'(\phi(\tilde{t}))\phi'(\tilde{t}) \end{aligned}$$

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 $\tilde{\gamma}(t) = \gamma(s^{-1}(t))$ , then  
 $\tilde{\gamma}$  is a unit speed re-parametrization.

$$\begin{aligned} \gamma &: (\alpha, \beta) \rightarrow \mathbb{R}^2 \\ \tilde{\gamma} &: (\tilde{\alpha}, \tilde{\beta}) \rightarrow \mathbb{R}^2 \\ \tilde{\gamma}(\tilde{t}) &= \gamma(\phi(\tilde{t})) \\ \tilde{\gamma}'(\tilde{t}) &= \gamma'(\phi(\tilde{t}))\phi'(\tilde{t}) \end{aligned}$$

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