Signed vs unsigned curvature

First, we will note what is true for both plane curves and space curves. $\|\ddot{\gamma}(t)\|$ is the curvature of a curve defined by a unit speed parametrization $\gamma : (\alpha, \beta) \to \mathbb{R}^n$, where $n = 2, 3$. This has to be 0 or positive because it is the norm of a vector. Also, the acceleration vector $\ddot{\gamma}(t)$, for any curve is always perpendicular to the unit tangent vector. These two are true for plane *and* space curves.

For plane curves we have a nice opportunity. Consider any unit vector $\mathbf{v}$ in the plane, based at a point p. There are exactly two vectors perpendicular to $\mathbf{v}$: one obtained by rotating $\mathbf{v}$ around p in the counterclockwise direction, and one obtained by rotating it in the clockwise direction. Furthermore, both these vector are negatives of each other, so if we denote one of them as $\mathbf{w}$, the other is $-\mathbf{w}$.

Since the acceleration vector at t is always perpendicular to the unit tangent vector at t, it can have two possible directions which can be represented by unit vectors obtained by rotating $\mathbf{T}(t)$ counterclockwise or clockwise (around the point $\gamma(t)$). Since they are negatives of each other, all that remains is to somehow decide on the main one. The convention is to give importance to the one that is obtained by rotating the unit tangent vector counterclockwise, and denote that by $\mathbf{N}_s(t)$. Now, since the acceleration vector points in the direction of $\mathbf{N}_s(t)$ or its negative, $-\mathbf{N}_s(t)$, it will always be a scalar multiple (denoted $\kappa_s(t)$) of $\mathbf{N}_s(t)$. Note that unlike the ordinary curvature, $\kappa(t)$, this scalar multiple, $\kappa_s(t)$, will have a negative sign if the acceleration vector points in the direction of $-\mathbf{N}_s(t)$ rather than $\mathbf{N}_s(t)$. However, $|\kappa_s(t)| = \kappa(t)$.

To summarize, the curvature is merely the maginitude of the acceleration. Since the acceleration is always orthogonal to the unit tangent vector, for plane curves and *only* for plane curves, we *almost* know its direction: we know it is one of two possibilities. The signed curvature, by enhancing the curvature with a sign, tells us which of these two are possible. This is why, for plane curves, and *only* for plane curves, the signed curvature tells us the magnitude and direction of the the acceleration vector.

For space curves, we cannot define a signed curvature. In space, there are infinitely many directions (unit vectors) orthogonal to a unit tangent vector (or any fixed vector, for that matter). It now makes no sense to say "rotate the unit tangent vector" in the counterclockwise direction because we have to rotate about some axis but now there are infinitely many (in the case of plane curves, there was a natural axis of rotation: the one perpendicular to the plane in which the curve lies). So for space curves, we only talk about the ordinary curvature, and while that tells us the magnitude of the acceleration, it does not single out the direction.

However, as we saw in one of our lectures, $\mathbf{N}_s(t)$ was also important because together with $\mathbf{T}_s(t)$ it formed a very convenient basis that helped us to figure out why constant curvature curves lie one a circle etc. Much of the power of this (moving) basis came from the fact that the derivatives of them could be written in terms of these basis again and the signed curvature.

It would be nice if a similar basis exists for space curves too. Remember that just as in plane curves, it would be more powerful if their derivatives could be written in terms of the basis. We already have $\mathbf{T}(t)$, and this time the natural normal $\mathbf{N}(t)$ is simply the direction of the accelaration. Since space is 3-dimensional, we also need one more vector which should be orthogonal to these two, but all this is discussed in the 6th question from exercise set 5.