

# Exercise sheet 4

Curves and Surfaces, MTH201

1. Show that the curvature at any point of a line segment is always 0.
2. Find a parametrization of an ellipse, i.e.  $\{(x, y) \in \mathbb{R}^2 \mid \frac{x^2}{a} + \frac{y^2}{b} = 1\}$  and use it to compute its curvature function  $\kappa(t)$ .
3. Given *any* smooth parametrization,  $\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$ , is the curvature function  $\kappa(t)$  always smooth? Do you need to add some condition? What is it?
4. Compute the signed curvature of the circle parametrized by  $\gamma(t) = (5\cos(t), -5\sin(t))$ .
5. If  $\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$  parametrizes a curve, compute the curvature of the curve parametrized by  $\tilde{\gamma}(t) = \gamma(-t)$  in terms of the curvature of  $\gamma$ . What about the relation between the signed curvatures of  $\gamma$  and  $\tilde{\gamma}$ ?
6. Compare the signed curvatures of a curve and its reflection, i.e.  $\gamma(t)$  and  $-\gamma(t)$ .
7. By finding a unit speed parametrization of a circle of radius  $r$ , compute its curvature. Let  $\gamma(t)$  be some other \*constant\* speed parametrization of a circle of radius  $r$ , where  $v := \|\dot{\gamma}(t)\|$  is the (constant) speed, and prove that  $\|\ddot{\gamma}(t)\| = v^2/r$  (Do you recognize the significance of this?).
8. Can you draw a curve whose signed curvature in terms of a unit speed parametrization is  $\kappa_s(t) = t$ ?
9. This exercise will help you to simplify the general formula for curvature that we derived during the lecture.
  - (a) Recall the triple product identity  $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$  for any three vectors  $\mathbf{u}$ ,  $\mathbf{v}$ , and  $\mathbf{w}$ . Simplify the expression for curvature defined during the lecture so that you can use the above equation to rewrite it entirely in terms of cross products instead of dot products.
  - (b) For orthogonal vectors  $\mathbf{v}$  and  $\mathbf{w}$ , why is  $\|\mathbf{v} \times \mathbf{w}\| = \|\mathbf{v}\|\|\mathbf{w}\|$ ? Use this to show that the curvature of a curve parametrized by  $\gamma$  can be computed at the point  $\gamma(t)$  by  $\frac{\|\ddot{\gamma}(t) \times \dot{\gamma}(t)\|}{\|\dot{\gamma}(t)\|^3}$ .