Observe that both arguments are the same in the first fundamental form in the integrals for arc length

To prove the converse, i.e. that arc-length preserving functions preserve the first fundamental form,

Polar identity

we will need to highlight an algebraic propery satisfied by the first fundamental form

Polar identity

The idea is simple:

Polar identity

the distributivity of the first fundamental form can be used to relate terms

Polar identity

$$\langle v+w, v+w\rangle = \langle v,v\rangle + 2\langle v,w\rangle + \langle w,w\rangle$$

whose arguments are different, with those whose arguments are the same.

Polar identity

$\langle v+w, v+w\rangle = \langle v, v\rangle + 2\langle v, w\rangle + \langle w, w\rangle$

Notice that since $\langle v, w\rangle = \langle w, v\rangle$, we have only one term with different arguments

Polar identity

$$\langle v+w, v+w\rangle = \langle v, v\rangle + 2\langle v, w\rangle + \langle w, w\rangle$$

$$\langle v, w\rangle = \frac{\langle v+w, v+w\rangle - \langle v, v\rangle - \langle w, w\rangle}{2}$$

allowing us to express it entirely in terms of those that have both arguments the same

.

Polar identity

$$\langle v+w, v+w\rangle = \langle v, v\rangle + 2\langle v, w\rangle + \langle w, w\rangle$$

$$\langle v, w\rangle = \frac{\langle v+w, v+w\rangle - \langle v, v\rangle - \langle w, w\rangle}{2}$$

Notice that we only used two properties of the first fundamental form:

Polar identity

$$\langle v+w, v+w\rangle = \langle v, v\rangle + 2\langle v, w\rangle + \langle w, w\rangle$$

$$\langle v, w\rangle = \frac{\langle v+w, v+w\rangle - \langle v, v\rangle - \langle w, w\rangle}{2}$$

distributivity and symmetry

Polar identity

$$\langle v+w, v+w\rangle = \langle v, v\rangle + 2\langle v, w\rangle + \langle w, w\rangle$$

$$\langle v, w\rangle = \frac{\langle v+w, v+w\rangle - \langle v, v\rangle - \langle w, w\rangle}{2}$$

Similarly,

$$f^*\langle v+w, v+w\rangle = f^*\langle v, v\rangle + 2f^*\langle v, w\rangle + f^*\langle w, w\rangle$$

Luckily, these are the same properties the pull back also satisfies (easy exercise!)

.

Polar identity

$$\langle v+w, v+w\rangle = \langle v, v\rangle + 2\langle v, w\rangle + \langle w, w\rangle$$

$$\langle v, w\rangle = \frac{\langle v+w, v+w\rangle - \langle v, v\rangle - \langle w, w\rangle}{2}$$

Similarly,

$$f^*\langle v+w, v+w\rangle = f^*\langle v, v\rangle + 2f^*\langle v, w\rangle + f^*\langle w, w\rangle$$

$$f^*\langle v, w\rangle = \frac{f^*\langle v+w, v+w\rangle - f^*\langle v, v\rangle - f^*\langle w, w\rangle}{2}$$

and once again we can express the mixed term in terms of the one which has both terms equal

Polar identity

$$\langle v+w, v+w\rangle = \langle v,v\rangle + 2\langle v,w\rangle + \langle w,w\rangle$$

$$\langle v,w\rangle = \frac{\langle v+w,v+w\rangle - \langle v,v\rangle - \langle w,w\rangle}{2}$$

Similarly,

$$f^*\langle v+w, v+w\rangle = f^*\langle v,v\rangle + 2f^*\langle v,w\rangle + f^*\langle w,w\rangle$$

$$f^*\langle v,w\rangle = \frac{f^*\langle v+w,v+w\rangle - f^*\langle v,v\rangle - f^*\langle w,w\rangle}{2}$$

If, $f^*\langle v,v\rangle = \langle v,v\rangle$ for all v,

All this proves that if the two forms are equal when tested on pairs of same vectors

Polar identity

$$\langle v+w, v+w\rangle = \langle v,v\rangle + 2\langle v,w\rangle + \langle w,w\rangle$$

$$\langle v,w\rangle = \frac{\langle v+w,v+w\rangle - \langle v,v\rangle - \langle w,w\rangle}{2}$$

Similarly,

$$f^*\langle v+w, v+w\rangle = f^*\langle v,v\rangle + 2f^*\langle v,w\rangle + f^*\langle w,w\rangle$$

$$f^*\langle v,w\rangle = \frac{f^*\langle v+w,v+w\rangle - f^*\langle v,v\rangle - f^*\langle w,w\rangle}{2}$$

If, $f^*\langle v,v\rangle = \langle v,v\rangle$ for all v, then $f^*\langle v,w\rangle = \langle v,w\rangle$ for all v, w

they will be equal even when applied to pairs where the vectors are different

Polar identity

$$\langle v+w, v+w\rangle = \langle v, v\rangle + 2\langle v, w\rangle + \langle w, w\rangle$$

$$\langle v, w\rangle = \frac{\langle v+w, v+w\rangle - \langle v, v\rangle - \langle w, w\rangle}{2}$$

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$$f^*\langle v+w, v+w\rangle = f^*\langle v, v\rangle + 2f^*\langle v, w\rangle + f^*\langle w, w\rangle$$

$$f^*\langle v, w\rangle = \frac{f^*\langle v+w, v+w\rangle - f^*\langle v, v\rangle - f^*\langle w, w\rangle}{2}$$

If, $f^*\langle v, v\rangle = \langle v, v\rangle$ for all v, then $f^*\langle v, w\rangle = \langle v, w\rangle$ for all v, w

$\blacksquare$

Polar identity

$\langle v+w, v+w\rangle = \langle v,v\rangle + 2\langle v,w\rangle + \langle w,w\rangle$

$\langle v,w\rangle = \frac{\langle v+w,v+w\rangle - \langle v,v\rangle - \langle w,w\rangle}{2}$

Similarly,

$f^*\langle v+w, v+w\rangle = f^*\langle v,v\rangle + 2f^*\langle v,w\rangle + f^*\langle w,w\rangle$

$f^*\langle v,w\rangle = \frac{f^*\langle v+w,v+w\rangle - f^*\langle v,v\rangle - f^*\langle w,w\rangle}{2}$

If, $f^*\langle v,v\rangle = \langle v,v\rangle$ for all v, then $f^*\langle v,w\rangle = \langle v,w\rangle$ for all v, w

If $f^*\langle \mathbf{w}_1, \mathbf{w}_2\rangle_p \neq \langle \mathbf{w}_1, \mathbf{w}_2\rangle_p$ then $f^*\langle \mathbf{v}, \mathbf{v}\rangle_p \neq \langle \mathbf{v}, \mathbf{v}\rangle_p$ for some $\mathbf{v}$

So if the first fundamental forms are different, the integrands must be different

Polar identity

$\langle v+w, v+w\rangle = \langle v,v\rangle + 2\langle v,w\rangle + \langle w,w\rangle$

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Assume, $f^*\langle \mathbf{v}, \mathbf{v}\rangle_p > \langle \mathbf{v}, \mathbf{v}\rangle_p$

We can assume this, because if it is strictly smaller, we can proceed similarly

Polar identity

$\langle v+w, v+w\rangle = \langle v, v\rangle + 2\langle v, w\rangle + \langle w, w\rangle$

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Assume, $f^*\langle \mathbf{v}, \mathbf{v}\rangle_p > \langle \mathbf{v}, \mathbf{v}\rangle_p$

then, $f^*\langle \mathbf{v}, \mathbf{v}\rangle_p - \langle \mathbf{v}, \mathbf{v}\rangle_p > 0$ for some p

and instead concludes that this difference is strictly negative

Polar identity

$\langle v+w, v+w\rangle = \langle v,v\rangle + 2\langle v,w\rangle + \langle w,w\rangle$

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Similarly,

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Assume, $f^*\langle \mathbf{v}, \mathbf{v}\rangle_p > \langle \mathbf{v}, \mathbf{v}\rangle_p$

then, $f^*\langle \mathbf{v}, \mathbf{v}\rangle_p - \langle \mathbf{v}, \mathbf{v}\rangle_p > 0$ for some p

Let $p = \gamma(t_0)$ and $\mathbf{v} = \dot{\gamma}(t_0)$ for some γ

Remember that tangent vectors are defined as velocity vectors of some curve on the surface

Polar identity

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Assume, $f^*\langle \mathbf{v}, \mathbf{v}\rangle_p > \langle \mathbf{v}, \mathbf{v}\rangle_p$

then, $f^*\langle \mathbf{v}, \mathbf{v}\rangle_p - \langle \mathbf{v}, \mathbf{v}\rangle_p > 0$ for some p

Let $p = \gamma(t_0)$ and $\mathbf{v} = \dot{\gamma}(t_0)$ for some γ

So, $f^*\langle \dot{\gamma}(t_0), \dot{\gamma}(t_0)\rangle_{\gamma(t_0)} - \langle \dot{\gamma}(t_0), \dot{\gamma}(t_0)\rangle_{\gamma(t_0)} > 0$ for some t_0

so we can rephrase this in terms of a function from an interval to $\mathbb{R}$ being strictly positive at some point

Polar identity
$\langle v+w, v+w\rangle = \langle v, v\rangle + 2\langle v, w\rangle + \langle w, w\rangle$
$\langle v, w\rangle = \frac{\langle v+w,v+w\rangle - \langle v,v\rangle - \langle w,w\rangle}{2}$

Similarly,
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Assume, $f^*\langle \mathbf{v}, \mathbf{v}\rangle_p > \langle \mathbf{v}, \mathbf{v}\rangle_p$
then, $f^*\langle \mathbf{v}, \mathbf{v}\rangle_p - \langle \mathbf{v}, \mathbf{v}\rangle_p > 0$ for some p
Let $p = \gamma(t_0)$ and $\mathbf{v} = \dot{\gamma}(t_0)$ for some γ

So, $f^*\langle \dot{\gamma}(t_0), \dot{\gamma}(t_0)\rangle_{\gamma(t_0)} - \langle \dot{\gamma}(t_0), \dot{\gamma}(t_0)\rangle_{\gamma(t_0)} > 0$ for some t_0

By continuity,
$f^*\langle \dot{\gamma}(t), \dot{\gamma}(t)\rangle_p - \langle \dot{\gamma}(t), \dot{\gamma}(t)\rangle_p > 0$ in some interval $[t_1, t_2]$

Continuity will never allow only one point to be strictly positive

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Polar identity
$\langle v+w, v+w\rangle = \langle v, v\rangle + 2\langle v, w\rangle + \langle w, w\rangle$
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Similarly,
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then, $f^*\langle \mathbf{v}, \mathbf{v}\rangle_p - \langle \mathbf{v}, \mathbf{v}\rangle_p > 0$ for some p
Let $p = \gamma(t_0)$ and $\mathbf{v} = \dot{\gamma}(t_0)$ for some γ

So, $f^*\langle \dot{\gamma}(t_0), \dot{\gamma}(t_0)\rangle_{\gamma(t_0)} - \langle \dot{\gamma}(t_0), \dot{\gamma}(t_0)\rangle_{\gamma(t_0)} > 0$ for some t_0

By continuity,
$f^*\langle \dot{\gamma}(t), \dot{\gamma}(t)\rangle_p - \langle \dot{\gamma}(t), \dot{\gamma}(t)\rangle_p > 0$ in some interval $[t_1, t_2]$

some interval around it must also be strictly positive even if the interval is very small

Polar identity
$\langle v+w, v+w\rangle = \langle v,v\rangle + 2\langle v,w\rangle + \langle w,w\rangle$
$\langle v,w\rangle = \frac{\langle v+w,v+w\rangle - \langle v,v\rangle - \langle w,w\rangle}{2}$

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Assume, $f^*\langle \mathbf{v}, \mathbf{v}\rangle_p > \langle \mathbf{v}, \mathbf{v}\rangle_p$
then, $f^*\langle \mathbf{v}, \mathbf{v}\rangle_p - \langle \mathbf{v}, \mathbf{v}\rangle_p > 0$ for some p
Let $p = \gamma(t_0)$ and $\mathbf{v} = \dot{\gamma}(t_0)$ for some γ

So, $f^*\langle \dot{\gamma}(t_0), \dot{\gamma}(t_0)\rangle_{\gamma(t_0)} - \langle \dot{\gamma}(t_0), \dot{\gamma}(t_0)\rangle_{\gamma(t_0)} > 0$ for some t_0

By continuity,
$f^*\langle \dot{\gamma}(t), \dot{\gamma}(t)\rangle_p - \langle \dot{\gamma}(t), \dot{\gamma}(t)\rangle_p > 0$ in some interval $[t_1, t_2]$
So, $\int_{t_1}^{t_2} f^*\langle \dot{\gamma}(t), \dot{\gamma}(t)\rangle_p - \langle \dot{\gamma}(t), \dot{\gamma}(t)\rangle_p \mathrm{d}t > 0$

At least in that interval the integral is forced to be positive

Polar identity

$\langle v+w, v+w\rangle = \langle v,v\rangle + 2\langle v,w\rangle + \langle w,w\rangle$

$\langle v,w\rangle = \frac{\langle v+w,v+w\rangle - \langle v,v\rangle - \langle w,w\rangle}{2}$

Similarly,

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If, $f^*\langle v,v\rangle = \langle v,v\rangle$ for all v, then $f^*\langle v,w\rangle = \langle v,w\rangle$ for all v, w

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Assume, $f^*\langle \mathbf{v}, \mathbf{v}\rangle_p > \langle \mathbf{v}, \mathbf{v}\rangle_p$

then, $f^*\langle \mathbf{v}, \mathbf{v}\rangle_p - \langle \mathbf{v}, \mathbf{v}\rangle_p > 0$ for some p

Let $p = \gamma(t_0)$ and $\mathbf{v} = \dot{\gamma}(t_0)$ for some γ

So, $f^*\langle \dot{\gamma}(t_0), \dot{\gamma}(t_0)\rangle_{\gamma(t_0)} - \langle \dot{\gamma}(t_0), \dot{\gamma}(t_0)\rangle_{\gamma(t_0)} > 0$ for some t_0

By continuity,

$f^*\langle \dot{\gamma}(t), \dot{\gamma}(t)\rangle_p - \langle \dot{\gamma}(t), \dot{\gamma}(t)\rangle_p > 0$ in some interval $[t_1, t_2]$

So, $\int_{t_1}^{t_2} f^*\langle \dot{\gamma}(t), \dot{\gamma}(t)\rangle_p - \langle \dot{\gamma}(t), \dot{\gamma}(t)\rangle_p \mathrm{d}t > 0$

$\int_{t_1}^{t_2} f^*\langle \dot{\gamma}(t), \dot{\gamma}(t)\rangle_p \mathrm{d}t - \int_{t_1}^{t_2}\langle \dot{\gamma}(t), \dot{\gamma}(t)\rangle_p \mathrm{d}t > 0$

and the difference of integrals is positive

Polar identity
$\langle v+w, v+w\rangle = \langle v, v\rangle + 2\langle v, w\rangle + \langle w, w\rangle$
$\langle v, w\rangle = \frac{\langle v+w, v+w\rangle - \langle v,v\rangle - \langle w,w\rangle}{2}$

Similarly,
$f^*\langle v+w, v+w\rangle = f^*\langle v, v\rangle + 2f^*\langle v, w\rangle + f^*\langle w, w\rangle$
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Assume, $f^*\langle \mathbf{v}, \mathbf{v}\rangle_p > \langle \mathbf{v}, \mathbf{v}\rangle_p$
then, $f^*\langle \mathbf{v}, \mathbf{v}\rangle_p - \langle \mathbf{v}, \mathbf{v}\rangle_p > 0$ for some p
Let $p = \gamma(t_0)$ and $\mathbf{v} = \dot{\gamma}(t_0)$ for some γ

So, $f^*\langle \dot{\gamma}(t_0), \dot{\gamma}(t_0)\rangle_{\gamma(t_0)} - \langle \dot{\gamma}(t_0), \dot{\gamma}(t_0)\rangle_{\gamma(t_0)} > 0$ for some t_0

By continuity,
$f^*\langle \dot{\gamma}(t), \dot{\gamma}(t)\rangle_p - \langle \dot{\gamma}(t), \dot{\gamma}(t)\rangle_p > 0$ in some interval $[t_1, t_2]$

So, $\int_{t_1}^{t_2} f^*\langle \dot{\gamma}(t), \dot{\gamma}(t)\rangle_p - \langle \dot{\gamma}(t), \dot{\gamma}(t)\rangle_p \mathrm{d}t > 0$
$\int_{t_1}^{t_2} f^*\langle \dot{\gamma}(t), \dot{\gamma}(t)\rangle_p \mathrm{d}t - \int_{t_1}^{t_2} \langle \dot{\gamma}(t), \dot{\gamma}(t)\rangle_p \mathrm{d}t > 0$
$\int_{t_1}^{t_2} f^*\langle \dot{\gamma}(t), \dot{\gamma}(t)\rangle_p \mathrm{d}t > \int_{t_1}^{t_2} \langle \dot{\gamma}(t), \dot{\gamma}(t)\rangle_p \mathrm{d}t$

and the two integrals are forced to be different

Polar identity

$\langle v+w, v+w\rangle = \langle v, v\rangle + 2\langle v, w\rangle + \langle w, w\rangle$

$\langle v, w\rangle = \frac{\langle v+w, v+w\rangle - \langle v, v\rangle - \langle w, w\rangle}{2}$

Similarly,

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Assume, $f^*\langle \mathbf{v}, \mathbf{v}\rangle_p > \langle \mathbf{v}, \mathbf{v}\rangle_p$

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Let $p = \gamma(t_0)$ and $\mathbf{v} = \dot{\gamma}(t_0)$ for some γ

So, $f^*\langle \dot{\gamma}(t_0), \dot{\gamma}(t_0)\rangle_{\gamma(t_0)} - \langle \dot{\gamma}(t_0), \dot{\gamma}(t_0)\rangle_{\gamma(t_0)} > 0$ for some t_0

By continuity,

$f^*\langle \dot{\gamma}(t), \dot{\gamma}(t)\rangle_p - \langle \dot{\gamma}(t), \dot{\gamma}(t)\rangle_p > 0$ in some interval $[t_1, t_2]$

So, $\int_{t_1}^{t_2} f^*\langle \dot{\gamma}(t), \dot{\gamma}(t)\rangle_p - \langle \dot{\gamma}(t), \dot{\gamma}(t)\rangle_p \mathrm{d}t > 0$

$\int_{t_1}^{t_2} f^*\langle \dot{\gamma}(t), \dot{\gamma}(t)\rangle_p \mathrm{d}t - \int_{t_1}^{t_2}\langle \dot{\gamma}(t), \dot{\gamma}(t)\rangle_p \mathrm{d}t > 0$

$\int_{t_1}^{t_2} f^*\langle \dot{\gamma}(t), \dot{\gamma}(t)\rangle_p \mathrm{d}t > \int_{t_1}^{t_2}\langle \dot{\gamma}(t), \dot{\gamma}(t)\rangle_p \mathrm{d}t$

$\blacksquare$

Polar identity

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Assume, $f^*\langle \mathbf{v}, \mathbf{v}\rangle_p > \langle \mathbf{v}, \mathbf{v}\rangle_p$
then, $f^*\langle \mathbf{v}, \mathbf{v}\rangle_p - \langle \mathbf{v}, \mathbf{v}\rangle_p > 0$ for some p
Let $p = \gamma(t_0)$ and $\mathbf{v} = \dot\gamma(t_0)$ for some γ

So, $f^*\langle \dot\gamma(t_0), \dot\gamma(t_0)\rangle_{\gamma(t_0)} - \langle \dot\gamma(t_0), \dot\gamma(t_0)\rangle_{\gamma(t_0)} > 0$ for some t_0

By continuity,
$f^*\langle \dot\gamma(t), \dot\gamma(t)\rangle_p - \langle \dot\gamma(t), \dot\gamma(t)\rangle_p > 0$ in some interval $[t_1, t_2]$

So, $\int_{t_1}^{t_2} f^*\langle \dot\gamma(t), \dot\gamma(t)\rangle_p - \langle \dot\gamma(t), \dot\gamma(t)\rangle_p \mathrm{d}t > 0$
$\int_{t_1}^{t_2} f^*\langle \dot\gamma(t), \dot\gamma(t)\rangle_p \mathrm{d}t - \int_{t_1}^{t_2} \langle \dot\gamma(t), \dot\gamma(t)\rangle_p \mathrm{d}t > 0$
$\int_{t_1}^{t_2} f^*\langle \dot\gamma(t), \dot\gamma(t)\rangle_p \mathrm{d}t > \int_{t_1}^{t_2} \langle \dot\gamma(t), \dot\gamma(t)\rangle_p \mathrm{d}t$
arc length of $f \circ \gamma$ *from* t_1 *to* $t_2 \neq$ arc-length of γ *from* t_1 *to* t_2

And we have, therefore, proved the converse

Polar identity

$\langle v+w, v+w\rangle = \langle v,v\rangle + 2\langle v,w\rangle + \langle w,w\rangle$

$\langle v,w\rangle = \frac{\langle v+w,v+w\rangle - \langle v,v\rangle - \langle w,w\rangle}{2}$

Similarly,

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Let $p = \gamma(t_0)$ and $\mathbf{v} = \dot\gamma(t_0)$ for some γ

So, $f^*\langle \dot\gamma(t_0), \dot\gamma(t_0)\rangle_{\gamma(t_0)} - \langle \dot\gamma(t_0), \dot\gamma(t_0)\rangle_{\gamma(t_0)} > 0$ for some t_0

By continuity,

$f^*\langle \dot\gamma(t), \dot\gamma(t)\rangle_p - \langle \dot\gamma(t), \dot\gamma(t)\rangle_p > 0$ in some interval $[t_1, t_2]$

So, $\int_{t_1}^{t_2} f^*\langle \dot\gamma(t), \dot\gamma(t)\rangle_p - \langle \dot\gamma(t), \dot\gamma(t)\rangle_p \mathrm{d}t > 0$

$\int_{t_1}^{t_2} f^*\langle \dot\gamma(t), \dot\gamma(t)\rangle_p \mathrm{d}t - \int_{t_1}^{t_2}\langle \dot\gamma(t), \dot\gamma(t)\rangle_p \mathrm{d}t > 0$

$\int_{t_1}^{t_2} f^*\langle \dot\gamma(t), \dot\gamma(t)\rangle_p \mathrm{d}t > \int_{t_1}^{t_2}\langle \dot\gamma(t), \dot\gamma(t)\rangle_p \mathrm{d}t$

arc length of $f \circ \gamma$ *from* t_1 *to* $t_2 \neq$ arc-length of γ *from* t_1 *to* t_2

If $\langle , \rangle$ and $f^*\langle , \rangle$ fail to be equal for even one point,

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arc length of $f \circ \gamma$ *from* t_1 *to* $t_2 \neq$ arc-length of γ *from* t_1 *to* t_2

then f must fail to preserve the arc-length of some curve

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arc length of $f \circ \gamma$ *from* t_1 *to* $t_2 \neq$ arc-length of γ *from* t_1 *to* t_2

Alternatively, if a function preserves the arc-lengths of all curves,

.

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So, $f^*\langle \dot\gamma(t_0), \dot\gamma(t_0)\rangle_{\gamma(t_0)} - \langle \dot\gamma(t_0), \dot\gamma(t_0)\rangle_{\gamma(t_0)} > 0$ for some t_0

By continuity,

$f^*\langle \dot\gamma(t), \dot\gamma(t)\rangle_p - \langle \dot\gamma(t), \dot\gamma(t)\rangle_p > 0$ in some interval $[t_1, t_2]$

So, $\int_{t_1}^{t_2} f^*\langle \dot\gamma(t), \dot\gamma(t)\rangle_p - \langle \dot\gamma(t), \dot\gamma(t)\rangle_p \mathrm{d}t > 0$

$\int_{t_1}^{t_2} f^*\langle \dot\gamma(t), \dot\gamma(t)\rangle_p \mathrm{d}t - \int_{t_1}^{t_2} \langle \dot\gamma(t), \dot\gamma(t)\rangle_p \mathrm{d}t > 0$

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arc length of $f \circ \gamma$ *from* t_1 *to* $t_2 \neq$ arc-length of γ *from* t_1 *to* t_2

$\mathbf{v}_1 = \dot\gamma_1(t)$

$\mathbf{v}_2 = \dot\gamma_2(t)$

$D_p(f)\mathbf{v}_1 = \frac{\mathrm{d}}{\mathrm{d}t}(f(\gamma_1(t)))$

$D_p(f)\mathbf{v}_2 = \frac{\mathrm{d}}{\mathrm{d}t}(f(\gamma_2(t)))$

then it also preserves the first fundamental form

Covariant derivative

$\gamma : (\alpha, \beta) \to S$

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Definition. $\mathbf{v}$ is parallel along γ if $\nabla_\gamma \mathbf{v}(t) = 0$

γ is a geodesic if and only if $\nabla_\gamma \dot{\gamma} = 0$

In terms of a surface patch:
$\gamma(t) = \sigma(x(t), y(t))$
$\mathbf{v}(t) = \alpha(t)\sigma_x(x(t), y(t)) + \beta(t)\sigma_y(x(t), y(t))$

$$\dot{\mathbf{v}}(t) = (\alpha(t)\sigma_x(x(t), y(t)) + \beta(t)\sigma_y(x(t), y(t)))'$$

$$
\begin{aligned}
\dot{\mathbf{v}}(t) &= (\alpha(t)\sigma_x(x(t), y(t)) + \beta(t)\sigma_y(x(t), y(t)))' \\
&= \alpha'(t)\sigma_x(x(t), y(t)) + \alpha(t)(\sigma_x(x(t), y(t)))' + \beta'(t)\sigma_y(x(t), y(t)) + \beta(t)(\sigma_y(x(t), y(t)))'
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&= \alpha'(t)\sigma_x(x(t), y(t)) + \alpha(t)(x'(t)\sigma_{xx}(x(t), y(t)) + y'(t)\sigma_{xy}(x(t), y(t))) \\
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Proposition. $\nabla_\gamma \mathbf{v}$ *depends only on the first fundamental form of the surface*

$$
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Corollary. *The geodesic curvature of a curve on a surface depends only on the first fundamental form.*

$$
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Proposition. $\nabla_\gamma \mathbf{v}$ *depends only on the first fundamental form of the surface*

Corollary. *The geodesic curvature of a curve on a surface depends only on the first fundamental form.*

Proposition. $\mathbf{v}(t)$ *is a parallel vector field along* γ *if and only if* $\alpha(t)$ *and* $\beta(t)$ *satisfy the following first order differential equation:*

$$\ldots$$

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Corollary. *Any vector* $\mathbf{v}_0$ *at* $\gamma(t_0)$ *can be extended to exactly one tangent vector field* $\mathbf{v}(t)$ *along* γ.