The importance of orthonormal basis

Orthonormal vectors in general

If we have three vectors, $\mathbf{e_1}$, $\mathbf{e_2}$, and $\mathbf{e_3}$ in $\mathbb{R}^3$, they become especially useful if,

1. They are perpendicular to each other, i.e. $\mathbf{e}_1.\mathbf{e}_2 = 0$, $\mathbf{e}_2.\mathbf{e}_3 = 0$, and $\mathbf{e}_1.\mathbf{e}_3 = 0$.
2. They have unit magnitude, which is equivalent to $\mathbf{e}_1.\mathbf{e}_1 = 1$, $\mathbf{e}_2.\mathbf{e}_2 = 1$, and $\mathbf{e}_3.\mathbf{e}_3 = 1$.

In that case, they form a basis, which means that any vector can be represented in a unique way as a linear combination of $\mathbf{e_1}$, $\mathbf{e_2}$, and $\mathbf{e_3}$. In other words, there are three special scalars λ_1, λ_2, and λ_3 (and no others that will serve the same purpose), so that $v = \lambda_1\mathbf{e}_1 + \lambda_2\mathbf{e}_2 + \lambda_3\mathbf{e}_3$.

However, they form not just a basis, but an orthonormal basis, and this allows us to figure out the coefficients, λ_1, λ_2, and λ_3, more easily. To see how, observe, what happens if we take the dot product of $v = \lambda_1\mathbf{e}_1 + \lambda_2\mathbf{e}_2 + \lambda_3\mathbf{e}_3$ with, say, $\mathbf{e_2}$,

$$v.\mathbf{e}_2 = \lambda_1(\mathbf{e}_1.\mathbf{e}_2) + \lambda_2(\mathbf{e}_2.\mathbf{e}_2) + \lambda_3(\mathbf{e}_3.\mathbf{e}_2)$$

Condition 1. above gives the 0s and condition 2, gives the 1 on the right hand side of,

$$v.\mathbf{e}_2 = \lambda_1 0 + \lambda_2 1 + \lambda_3 0$$

and so ,

$$v.\mathbf{e}_2 = \lambda_2$$

So it tells us that $\lambda_2 = v.\mathbf{e}_2$. In the same way, we can show that $\lambda_1 = v.\mathbf{e}_1$, and $\lambda_3 = v.\mathbf{e}_3$.

To summarize, *Given an orthonormal basis* $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$*, any vector,* v*, can be written as a linear combination of this basis and the coefficient of of* $\mathbf{e}_i$ *is simply* $\mathbf{v}.\mathbf{e}_i$

Applying this to curves

Now in our case of curves, the vectors that we are interested in happen to be the tangent vector, acceleration vector, normal etc. and they are all based at a point $\gamma(t)$ of a curve parametrized by γ, so they are functions of t. It would be nice to have a useful orthonormal basis for each point $\gamma(t)$.

A paricularly useful orthonormal basis for each point $\gamma(t)$ is $\{\mathbf{T}(t), \mathbf{N}(t), \mathbf{B}(t)\}$. Once we fix this, for any given t we should be able to write any vector at $\gamma(t)$ in terms of this basis. This should include the derivatives $\dot{\mathbf{T}}(t)$, $\dot{\mathbf{N}}(t)$, and $\dot{\mathbf{B}}(t)$. Since the basis is orthonormal we can simply use the dot product to figure out the coefficient. So, for instance, the coefficient of $\dot{\mathbf{N}}(t)$ in terms of $\mathbf{T}(t)$ is simply $\dot{\mathbf{N}}(t).\mathbf{T}(t)$.

Product rule tells us more

It is a useful habit to see how product rule helps us to rewrite the coefficient,

$$\dot{\mathbf{N}}(t).\mathbf{T}(t) = (\mathbf{N}(t).\mathbf{T}(t))' - \mathbf{N}(t).\dot{\mathbf{T}}(t)$$

Since the vectors are orthonormal, the first derivative on the right hand side is 0, and therefore,

$$\dot{\mathbf{N}}(t).\mathbf{T}(t) = -\mathbf{N}(t).\dot{\mathbf{T}}(t) = -\dot{\mathbf{T}}(t).\mathbf{N}(t)$$

Note that $\dot{\mathbf{T}}(t).\mathbf{N}(t)$ is simply the coefficient of $\dot{\mathbf{T}}(t)$ in terms of $\mathbf{N}(t)$. So product rule helps us to write a coefficient of $\dot{\mathbf{N}}(t)$ in terms of another known coefficient. This game works for any coefficient of $\dot{\mathbf{T}}(t)$, $\dot{\mathbf{N}}(t)$, and $\dot{\mathbf{B}}(t)$: *product rule tells us that half the coefficients are negatives of others* and this is really the main observation that simplifies the understanding of the coefficients of $\dot{\mathbf{T}}(t)$, $\dot{\mathbf{N}}(t)$, and $\dot{\mathbf{B}}(t)$ when written in terms of $\{\mathbf{T}(t), \mathbf{N}(t), \mathbf{B}(t)\}$.

We still have to figure out half the coefficients but this usually follows from the definition. For instance, $\mathbf{N}(t)$ is *defined* so that $\dot{\mathbf{T}}(t) = \kappa(t)\mathbf{N}(t)$, or to make the other coefficients explicit, $\dot{\mathbf{T}}(t) = 0\mathbf{T}(t) + \kappa(t)\mathbf{N}(t) + 0\mathbf{B}(t)$. So three coefficients known in one stroke! Combined with the product rule as applied above, we now know some coefficients of $\dot{\mathbf{N}}(t)$ and $\dot{\mathbf{B}}(t)$ too. For plane curves, in fact, that is all that there is to know, but for curves in space, there remain two missing coefficients that are negatives of each other, so one of them is abstracted out as a new quantity called the torsion, and the other is simply a negative of it.