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$g'(f(t))f'(t) = 1$, therefore, if $f'(t) \neq 0$ for any t,

$$\boxed{g'(f(t)) = \frac{1}{f'(t)}}$$

Taking $f(t) = s(t)$ and $g(t) = s^{-1}(t)$

$$(s^{-1})'(s(t)) = \frac{1}{s'(t)} = \frac{1}{\|\gamma'(t)\|}$$

$$\boxed{(s^{-1})'(\tilde{t}) = \frac{1}{s'(s^{-1}(\tilde{t}))} = \frac{1}{\|\dot{\gamma}(s^{-1}(\tilde{t}))\|}}$$

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Definition.
$\gamma : (\alpha, \beta) \to \mathbb{R}^2$ a regular parametrization.
$\mathbf{T}(t) = \frac{\dot{\gamma}(t)}{\|\dot{\gamma}(t)\|}$ is the *unit tangent vector* at t.

Exercise.
$\gamma : (\alpha, \beta) \to \mathbb{R}^2$ a regular parametrization.
$\tilde{\gamma} : (\tilde{\alpha}, \tilde{\beta}) \to \mathbb{R}^2$ a unit speed reparametrization.
Show that,
$\dot{\tilde{\gamma}}(\tilde{t}) = \mathbf{T}(s^{-1}(\tilde{t}))$

Solution. $\tilde{\gamma}(\widetilde{t}) = \gamma(s^{-1}(\tilde{t}))$

$$\begin{aligned} \dot{\tilde{\gamma}}(\widetilde{t}) &= \dot{\gamma}(s^{-1}(\tilde{t}))(s^{-1}(\tilde{t}))' \\ &= \dot{\gamma}(s^{-1}(\tilde{t}))\frac{1}{\|\dot{\gamma}(s^{-1}(\tilde{t}))\|} \end{aligned}$$

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Exercise. Show that the curvature at any point of any line

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Exercise. Show that the curvature at any point of any line is 0.

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$$\begin{aligned} \kappa(t) &= \|\ddot{\tilde{\gamma}}(s(t))\| \\ &= \left\| \frac{\ddot{\gamma}(t) - \dot{\tilde{\gamma}}(s(t))\frac{\dot{\gamma}(t).\ddot{\gamma}(t)}{\|\dot{\gamma}(t)\|}}{\|\dot{\gamma}(t)\|^2} \right\| \end{aligned}$$

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$$\begin{aligned}
\kappa(t) &= \|\ddot{\tilde{\gamma}}(s(t))\| \\
&= \left\| \frac{\ddot{\gamma}(t) - \dot{\tilde{\gamma}}(s(t))\frac{\dot{\gamma}(t).\ddot{\gamma}(t)}{\|\dot{\gamma}(t)\|}}{\|\dot{\gamma}(t)\|^2} \right\| \\
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\end{aligned}$$