Exercise sheet 4

Curves and Surfaces, MTH201

1. Show that the curvature at any point of a line segment is always 0.
2. Find a parametrization of an ellipse, i.e. $\{(x, y) \in \mathbb{R}^2 \mid \frac{x^2}{a} + \frac{y^2}{b} = 1\}$ and use it compute its curvature function $\kappa(t)$.
3. Given *any* smooth parametrization, $\gamma : (\alpha, \beta) \to \mathbb{R}^2$, is the curvature function $\kappa(t)$ always smooth? Do you need to add some condition? What is it?
4. Compute the signed curvature of the circle parametrized by $\gamma(t) = (5\cos(t), -5\sin(t))$.
5. If $\gamma : (\alpha, \beta) \to \mathbb{R}^2$ parametrizes a curve, compute the curvature of the curve parametrized by $\tilde{\gamma}(t) = \gamma(-t)$ in terms of the curvature of γ. What about the relation between the signed curvatures of γ and $\tilde{\gamma}$?
6. Compare the signed curvatures of a curve and its reflection, i.e. $\gamma(t)$ and $-\gamma(t)$.
7. By finding a unit speed parametrization of a circle of radius r, compute its curvature. Let $\gamma(t)$ be some other *constant* speed parametrization of a circle of radius r, where $v := \|\dot{\gamma}(t)\|$ is the (constant) speed, and prove that $\|\ddot{\gamma}(t)\| = v^2/r$ (Do you recognize the significance of this?).
8. Can you draw a curve whose signed curvature in terms of a unit speed parametrization is $\kappa_s(t) = t$?
9. This exercise will help you to simplify the general formula for curvature that we derived during the lecture.
   (a) Recall the triple product identity $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$ for any three vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$. Simplify the expression for curvature defined during the lecture so that you can use the above equation to rewrite it entirely in terms of cross products instead of dot products.
   (b) For orthogonal vectors $\mathbf{v}$ and $\mathbf{w}$, why is $\|\mathbf{v} \times \mathbf{w}\| = \|\mathbf{v}\|\|\mathbf{w}\|$? Use this to show that the curvature of a curve parametrized by γ can be computed at the point $\gamma(t)$ by $\frac{\|\ddot{\gamma}(t) \times \dot{\gamma}(t)\|}{\|\dot{\gamma}(t)\|^3}$.