$\kappa_n :=$

Remember that the normal curvature of a curve is owing to a component of the acceleration

.

$$\kappa_n := \hat{\mathbf{n}}(\gamma(t)).\ddot{\gamma}(t) =$$

that keeps the curve on the surface.

$$\kappa_n := \hat{\mathbf{n}}(\gamma(t)).\ddot{\gamma}(t) =$$

We will try to understand the curvature of a surface by the normal curvature it forces on curves that lie on it.

.

$$\kappa_n := \hat{\mathbf{n}}(\gamma(t)).\ddot{\gamma}(t) = -(\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{n}}(\gamma(t))).\dot{\gamma}(t)$$

Product rule helps us shift the derivative

$$\kappa_n := \hat{\mathbf{n}}(\gamma(t)).\ddot{\gamma}(t) = -(\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{n}}(\gamma(t))).\dot{\gamma}(t)$$

Therefore, κ_n can actually can be expressed in terms of

.

$$\kappa_n := \hat{\mathbf{n}}(\gamma(t)).\ddot{\gamma}(t) = -(\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{n}}(\gamma(t))).\dot{\gamma}(t)$$

The derivative of $\hat{\mathbf{n}}$ along γ and its velocity (rather than its acceleration)

.

$$\kappa_n := \hat{\mathbf{n}}(\gamma(t)).\ddot{\gamma}(t) = -(\tfrac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{n}}(\gamma(t))).\dot{\gamma}(t)$$

Let us examine the derivative of $\hat{\mathbf{n}}$ more closely

$$\kappa_n := \hat{\mathbf{n}}(\gamma(t)).\ddot{\gamma}(t) = -(\tfrac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{n}}(\gamma(t))).\dot{\gamma}(t)$$
$$\hat{\mathbf{n}}(x, y, z) = (n_1(x, y, z), n_2(x, y, z), n_3(x, y, z))$$

Writing the coordinates of $\hat{\mathbf{n}}$ more explicitly

.

$$\kappa_n := \hat{\mathbf{n}}(\gamma(t)).\ddot{\gamma}(t) = -(\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{n}}(\gamma(t))).\dot{\gamma}(t)$$
$$\hat{\mathbf{n}}(x, y, z) = (n_1(x, y, z), n_2(x, y, z), n_3(x, y, z))$$
$$\hat{\mathbf{n}}(\gamma(t)) = (n_1(x(t), y(t), z(t)), n_2(x(t), y(t), z(t)), n_3(x(t), y(t), z(t)))$$

And same for γ and the composition of the two

$$\kappa_n := \hat{\mathbf{n}}(\gamma(t)).\ddot{\gamma}(t) = -(\tfrac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{n}}(\gamma(t))).\dot{\gamma}(t)$$
$$\hat{\mathbf{n}}(x, y, z) = (n_1(x, y, z), n_2(x, y, z), n_3(x, y, z))$$
$$\hat{\mathbf{n}}(\gamma(t)) = (n_1(x(t), y(t), z(t)), n_2(x(t), y(t), z(t)), n_3(x(t), y(t), z(t)))$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{n}}(\gamma(t)) = (n_1'(x(t), y(t), z(t)), n_2'(x(t), y(t), z(t)), n_3'(x(t), y(t), z(t)))$$

We can differentiate a vector valued function coordinate-wise

$$\kappa_n := \hat{\mathbf{n}}(\gamma(t)).\ddot{\gamma}(t) = -(\tfrac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{n}}(\gamma(t))).\dot{\gamma}(t)$$
$$\hat{\mathbf{n}}(x, y, z) = (n_1(x, y, z), n_2(x, y, z), n_3(x, y, z))$$
$$\hat{\mathbf{n}}(\gamma(t)) = (n_1(x(t), y(t), z(t)), n_2(x(t), y(t), z(t)), n_3(x(t), y(t), z(t)))$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{n}}(\gamma(t)) = (n_1'(x(t), y(t), z(t)), n_2'(x(t), y(t), z(t)), n_3'(x(t), y(t), z(t)))$$

Chain rule tells allows us to express it in terms of the partial derivatives of n_i

.

$$\kappa_n := \hat{\mathbf{n}}(\gamma(t)).\ddot{\gamma}(t) = -(\tfrac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{n}}(\gamma(t))).\dot{\gamma}(t)$$
$$\hat{\mathbf{n}}(x,y,z) = (n_1(x,y,z), n_2(x,y,z), n_3(x,y,z))$$
$$\hat{\mathbf{n}}(\gamma(t)) = (n_1(x(t),y(t),z(t)), n_2(x(t),y(t),z(t)), n_3(x(t),y(t),z(t)))$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{n}}(\gamma(t)) &= (n_1'(x(t),y(t),z(t)), n_2'(x(t),y(t),z(t)), n_3'(x(t),y(t),z(t))) \\ &= (n_{1x}x' + n_{1y}y' + n_{1z}z', \\ &\quad n_{2x}x' + n_{2y}y' + n_{2z}z', \\ &\quad n_{3x}x' + n_{3y}y' + n_{3z}z')\end{aligned}$$

and the coordinates of the γ

.

$\kappa_n := \hat{\mathbf{n}}(\gamma(t)).\ddot{\gamma}(t) = -(\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{n}}(\gamma(t))).\dot{\gamma}(t)$
$\hat{\mathbf{n}}(x, y, z) = (n_1(x, y, z), n_2(x, y, z), n_3(x, y, z))$
$\hat{\mathbf{n}}(\gamma(t)) = (n_1(x(t), y(t), z(t)), n_2(x(t), y(t), z(t)), n_3(x(t), y(t), z(t)))$

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{n}}(\gamma(t)) &= (n_1'(x(t), y(t), z(t)), n_2'(x(t), y(t), z(t)), n_3'(x(t), y(t), z(t))) \\
&= (n_{1_x}x' + n_{1_y}y' + n_{1_z}z', \\
&\quad n_{2_x}x' + n_{2_y}y' + n_{2_z}z', \\
&\quad n_{3_x}x' + n_{3_y}y' + n_{3_z}z') \\
&= \begin{pmatrix} n_{1_x} & n_{1_y} & n_{1_z} \\ n_{2_x} & n_{2_y} & n_{2_z} \\ n_{3_x} & n_{3_y} & n_{3_z} \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}
\end{aligned}
$$

This can easily be arranged as a product of two matrices

.

$$\kappa_n := \hat{\mathbf{n}}(\gamma(t)).\ddot{\gamma}(t) = -(\tfrac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{n}}(\gamma(t))).\dot{\gamma}(t)$$
$$\hat{\mathbf{n}}(x, y, z) = (n_1(x, y, z), n_2(x, y, z), n_3(x, y, z))$$
$$\hat{\mathbf{n}}(\gamma(t)) = (n_1(x(t), y(t), z(t)), n_2(x(t), y(t), z(t)), n_3(x(t), y(t), z(t)))$$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{n}}(\gamma(t)) &= (n_1'(x(t), y(t), z(t)), n_2'(x(t), y(t), z(t)), n_3'(x(t), y(t), z(t))) \\
&= (n_{1x}x' + n_{1y}y' + n_{1z}z', \\
&\quad\; n_{2x}x' + n_{2y}y' + n_{2z}z', \\
&\quad\; n_{3x}x' + n_{3y}y' + n_{3z}z') \\
&= \begin{pmatrix} n_{1x} & n_{1y} & n_{1z} \\ n_{2x} & n_{2y} & n_{2z} \\ n_{3x} & n_{3y} & n_{3z} \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} \\
&= J(\hat{\mathbf{n}})\dot{\gamma}(t)
\end{aligned}$$

The 3×3 matrix is our familiar Jacobian (but this time for 3-dimensions)

$\kappa_n := \hat{\mathbf{n}}(\gamma(t)).\ddot{\gamma}(t) = -(\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{n}}(\gamma(t))).\dot{\gamma}(t)$
$\hat{\mathbf{n}}(x, y, z) = (n_1(x, y, z), n_2(x, y, z), n_3(x, y, z))$
$\hat{\mathbf{n}}(\gamma(t)) = (n_1(x(t), y(t), z(t)), n_2(x(t), y(t), z(t)), n_3(x(t), y(t), z(t)))$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{n}}(\gamma(t)) &= (n_1'(x(t), y(t), z(t)), n_2'(x(t), y(t), z(t)), n_3'(x(t), y(t), z(t))) \\
&= (n_{1x}x' + n_{1y}y' + n_{1z}z', \\
&\quad n_{2x}x' + n_{2y}y' + n_{2z}z', \\
&\quad n_{3x}x' + n_{3y}y' + n_{3z}z') \\
&= \begin{pmatrix} n_{1x} & n_{1y} & n_{1z} \\ n_{2x} & n_{2y} & n_{2z} \\ n_{3x} & n_{3y} & n_{3z} \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} \\
&= J(\hat{\mathbf{n}})\dot{\gamma}(t)
\end{aligned}$$

$\mathcal{W} : T_p(S) \to \mathbb{R}^3$ is a linear transformation

Therefore, we can define a linear transformation

$$\kappa_n := \hat{\mathbf{n}}(\gamma(t)).\ddot{\gamma}(t) = -(\tfrac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{n}}(\gamma(t))).\dot{\gamma}(t)$$
$$\hat{\mathbf{n}}(x,y,z) = (n_1(x,y,z), n_2(x,y,z), n_3(x,y,z))$$
$$\hat{\mathbf{n}}(\gamma(t)) = (n_1(x(t),y(t),z(t)), n_2(x(t),y(t),z(t)), n_3(x(t),y(t),z(t)))$$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{n}}(\gamma(t)) &= (n_1'(x(t),y(t),z(t)), n_2'(x(t),y(t),z(t)), n_3'(x(t),y(t),z(t))) \\
&= (n_{1x}x' + n_{1y}y' + n_{1z}z', \\
&\quad n_{2x}x' + n_{2y}y' + n_{2z}z', \\
&\quad n_{3x}x' + n_{3y}y' + n_{3z}z') \\
&= \begin{pmatrix} n_{1x} & n_{1y} & n_{1z} \\ n_{2x} & n_{2y} & n_{2z} \\ n_{3x} & n_{3y} & n_{3z} \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} \\
&= J(\hat{\mathbf{n}})\dot{\gamma}(t)
\end{aligned}$$

$\mathcal{W} : T_p(S) \to \mathbb{R}^3$ is a linear transformation

The above matrix form shows that it depends only on the velocity of the curve

.

$$\kappa_n := \hat{\mathbf{n}}(\gamma(t)).\ddot{\gamma}(t) = -(\tfrac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{n}}(\gamma(t))).\dot{\gamma}(t)$$
$$\hat{\mathbf{n}}(x,y,z) = (n_1(x,y,z), n_2(x,y,z), n_3(x,y,z))$$
$$\hat{\mathbf{n}}(\gamma(t)) = (n_1(x(t),y(t),z(t)), n_2(x(t),y(t),z(t)), n_3(x(t),y(t),z(t)))$$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{n}}(\gamma(t)) &= (n_1'(x(t),y(t),z(t)), n_2'(x(t),y(t),z(t)), n_3'(x(t),y(t),z(t))) \\
&= (n_{1x}x' + n_{1y}y' + n_{1z}z', \\
&\quad n_{2x}x' + n_{2y}y' + n_{2z}z', \\
&\quad n_{3x}x' + n_{3y}y' + n_{3z}z') \\
&= \begin{pmatrix} n_{1x} & n_{1y} & n_{1z} \\ n_{2x} & n_{2y} & n_{2z} \\ n_{3x} & n_{3y} & n_{3z} \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} \\
&= J(\hat{\mathbf{n}})\dot{\gamma}(t)
\end{aligned}$$

$\mathcal{W} : T_p(S) \to \mathbb{R}^3$ is a linear transformation

Remember that a vector is defined as a velocity vector of a curve on the surface

$$\kappa_n := \hat{\mathbf{n}}(\gamma(t)).\ddot{\gamma}(t) = -(\tfrac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{n}}(\gamma(t))).\dot{\gamma}(t)$$
$$\hat{\mathbf{n}}(x, y, z) = (n_1(x, y, z), n_2(x, y, z), n_3(x, y, z))$$
$$\hat{\mathbf{n}}(\gamma(t)) = (n_1(x(t), y(t), z(t)), n_2(x(t), y(t), z(t)), n_3(x(t), y(t), z(t)))$$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{n}}(\gamma(t)) &= (n_1'(x(t), y(t), z(t)), n_2'(x(t), y(t), z(t)), n_3'(x(t), y(t), z(t))) \\
&= (n_{1x}x' + n_{1y}y' + n_{1z}z', \\
&\quad n_{2x}x' + n_{2y}y' + n_{2z}z', \\
&\quad n_{3x}x' + n_{3y}y' + n_{3z}z') \\
&= \begin{pmatrix} n_{1x} & n_{1y} & n_{1z} \\ n_{2x} & n_{2y} & n_{2z} \\ n_{3x} & n_{3y} & n_{3z} \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} \\
&= J(\hat{\mathbf{n}})\dot{\gamma}(t)
\end{aligned}$$

$\mathcal{W} : T_p(S) \to \mathbb{R}^3$ is a linear transformation

The linear transformation maps it to the velocity vector of the image of the curve under $\hat{\mathbf{n}}$

$$\kappa_n := \hat{\mathbf{n}}(\gamma(t)).\ddot{\gamma}(t) = -(\tfrac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{n}}(\gamma(t))).\dot{\gamma}(t)$$
$$\hat{\mathbf{n}}(x,y,z) = (n_1(x,y,z), n_2(x,y,z), n_3(x,y,z))$$
$$\hat{\mathbf{n}}(\gamma(t)) = (n_1(x(t),y(t),z(t)), n_2(x(t),y(t),z(t)), n_3(x(t),y(t),z(t)))$$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{n}}(\gamma(t)) &= (n_1'(x(t),y(t),z(t)), n_2'(x(t),y(t),z(t)), n_3'(x(t),y(t),z(t))) \\
&= (n_{1x}x' + n_{1y}y' + n_{1z}z', \\
&\quad n_{2x}x' + n_{2y}y' + n_{2z}z', \\
&\quad n_{3x}x' + n_{3y}y' + n_{3z}z') \\
&= \begin{pmatrix} n_{1x} & n_{1y} & n_{1z} \\ n_{2x} & n_{2y} & n_{2z} \\ n_{3x} & n_{3y} & n_{3z} \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} \\
&= J(\hat{\mathbf{n}})\dot{\gamma}(t)
\end{aligned}$$

$\mathcal{W} : T_p(S) \to \mathbb{R}^3$ is a linear transformation
$\mathcal{W}(\mathbf{v}) = -\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{n}}(\gamma(t))$, where $v = \dot{\gamma}(t)$

And the matrix form tells us that the curve may be replaced with any other curve with the same velocity vector

$$\kappa_n := \hat{\mathbf{n}}(\gamma(t)).\ddot{\gamma}(t) = -(\tfrac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{n}}(\gamma(t))).\dot{\gamma}(t)$$
$$\hat{\mathbf{n}}(x,y,z) = (n_1(x,y,z), n_2(x,y,z), n_3(x,y,z))$$
$$\hat{\mathbf{n}}(\gamma(t)) = (n_1(x(t),y(t),z(t)), n_2(x(t),y(t),z(t)), n_3(x(t),y(t),z(t)))$$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{n}}(\gamma(t)) &= (n_1'(x(t),y(t),z(t)), n_2'(x(t),y(t),z(t)), n_3'(x(t),y(t),z(t))) \\
&= (n_{1x}x' + n_{1y}y' + n_{1z}z', \\
&\quad n_{2x}x' + n_{2y}y' + n_{2z}z', \\
&\quad n_{3x}x' + n_{3y}y' + n_{3z}z') \\
&= \begin{pmatrix} n_{1x} & n_{1y} & n_{1z} \\ n_{2x} & n_{2y} & n_{2z} \\ n_{3x} & n_{3y} & n_{3z} \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} \\
&= J(\hat{\mathbf{n}})\dot{\gamma}(t)
\end{aligned}$$

$\mathcal{W} : T_p(S) \to \mathbb{R}^3$ is a linear transformation
$\mathcal{W}(\mathbf{v}) = -\tfrac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{n}}(\gamma(t))$, where $v = \dot{\gamma}(t)$
$\hat{\mathbf{n}}(\gamma(t)).\ddot{\gamma}(t) = \mathcal{W}(\dot{\gamma}(t)).\dot{\gamma}(t)$

Finally, we get a useful expression in terms of the normal curvature

$$\kappa_n := \hat{\mathbf{n}}(\gamma(t)).\ddot{\gamma}(t) = -(\tfrac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{n}}(\gamma(t))).\dot{\gamma}(t)$$
$$\hat{\mathbf{n}}(x, y, z) = (n_1(x, y, z), n_2(x, y, z), n_3(x, y, z))$$
$$\hat{\mathbf{n}}(\gamma(t)) = (n_1(x(t), y(t), z(t)), n_2(x(t), y(t), z(t)), n_3(x(t), y(t), z(t)))$$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{n}}(\gamma(t)) &= (n_1'(x(t), y(t), z(t)), n_2'(x(t), y(t), z(t)), n_3'(x(t), y(t), z(t))) \\
&= (n_{1x}x' + n_{1y}y' + n_{1z}z', \\
&\quad n_{2x}x' + n_{2y}y' + n_{2z}z', \\
&\quad n_{3x}x' + n_{3y}y' + n_{3z}z') \\
&= \begin{pmatrix} n_{1x} & n_{1y} & n_{1z} \\ n_{2x} & n_{2y} & n_{2z} \\ n_{3x} & n_{3y} & n_{3z} \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} \\
&= J(\hat{\mathbf{n}})\dot{\gamma}(t)
\end{aligned}$$

$\mathcal{W} : T_p(S) \to \mathbb{R}^3$ is a linear transformation
$\mathcal{W}(\mathbf{v}) = -\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{n}}(\gamma(t))$, where $v = \dot{\gamma}(t)$
$\hat{\mathbf{n}}(\gamma(t)).\ddot{\gamma}(t) = \mathcal{W}(\dot{\gamma}(t)).\dot{\gamma}(t)$

Notice the entire formula for the normal curvature depends only on the velocity of the curve

$$\kappa_n := \hat{\mathbf{n}}(\gamma(t)).\ddot{\gamma}(t) = -(\tfrac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{n}}(\gamma(t))).\dot{\gamma}(t)$$
$$\hat{\mathbf{n}}(x,y,z) = (n_1(x,y,z), n_2(x,y,z), n_3(x,y,z))$$
$$\hat{\mathbf{n}}(\gamma(t)) = (n_1(x(t),y(t),z(t)), n_2(x(t),y(t),z(t)), n_3(x(t),y(t),z(t)))$$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{n}}(\gamma(t)) &= (n_1'(x(t),y(t),z(t)), n_2'(x(t),y(t),z(t)), n_3'(x(t),y(t),z(t))) \\
&= (n_{1x}x' + n_{1y}y' + n_{1z}z', \\
&\quad n_{2x}x' + n_{2y}y' + n_{2z}z', \\
&\quad n_{3x}x' + n_{3y}y' + n_{3z}z') \\
&= \begin{pmatrix} n_{1x} & n_{1y} & n_{1z} \\ n_{2x} & n_{2y} & n_{2z} \\ n_{3x} & n_{3y} & n_{3z} \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} \\
&= J(\hat{\mathbf{n}})\dot{\gamma}(t)
\end{aligned}$$

$\mathcal{W} : T_p(S) \to \mathbb{R}^3$ is a linear transformation
$\mathcal{W}(\mathbf{v}) = -\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{n}}(\gamma(t))$, where $v = \dot{\gamma}(t)$
$\hat{\mathbf{n}}(\gamma(t)).\ddot{\gamma}(t) = \mathcal{W}(\dot{\gamma}(t)).\dot{\gamma}(t)$

This makes intuitive sense, since the normal curvature is owing to the surface

$$\kappa_n := \hat{\mathbf{n}}(\gamma(t)).\ddot{\gamma}(t) = -(\tfrac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{n}}(\gamma(t))).\dot{\gamma}(t)$$
$$\hat{\mathbf{n}}(x,y,z) = (n_1(x,y,z), n_2(x,y,z), n_3(x,y,z))$$
$$\hat{\mathbf{n}}(\gamma(t)) = (n_1(x(t),y(t),z(t)), n_2(x(t),y(t),z(t)), n_3(x(t),y(t),z(t)))$$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{n}}(\gamma(t)) &= (n_1'(x(t),y(t),z(t)), n_2'(x(t),y(t),z(t)), n_3'(x(t),y(t),z(t))) \\
&= (n_{1x}x' + n_{1y}y' + n_{1z}z', \\
&\quad n_{2x}x' + n_{2y}y' + n_{2z}z', \\
&\quad n_{3x}x' + n_{3y}y' + n_{3z}z') \\
&= \begin{pmatrix} n_{1x} & n_{1y} & n_{1z} \\ n_{2x} & n_{2y} & n_{2z} \\ n_{3x} & n_{3y} & n_{3z} \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} \\
&= J(\hat{\mathbf{n}})\dot{\gamma}(t)
\end{aligned}$$

$\mathcal{W} : T_p(S) \to \mathbb{R}^3$ is a linear transformation
$\mathcal{W}(\mathbf{v}) = -\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{n}}(\gamma(t))$, where $v = \dot{\gamma}(t)$
$\hat{\mathbf{n}}(\gamma(t)).\ddot{\gamma}(t) = \mathcal{W}(\dot{\gamma}(t)).\dot{\gamma}(t)$

Any other curve on the surface whose velocity vector is in the same direction will have the same κ_n

$$\hat{\mathbf{n}}(\gamma(t)).\hat{\mathbf{n}}(\gamma(t)) = 1$$

As we have done often, we will use product rule to exploit $||\hat{\mathbf{n}}|| = 1$

$$\hat{\mathbf{n}}(\gamma(t)).\hat{\mathbf{n}}(\gamma(t)) = 1$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{n}}(\gamma(t)).\hat{\mathbf{n}}(\gamma(t)) = 0$$

Product rule tells us that the derivative is perpendicular to the normal

$$\hat{\mathbf{n}}(\gamma(t)).\hat{\mathbf{n}}(\gamma(t)) = 1$$
$$\tfrac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{n}}(\gamma(t)).\hat{\mathbf{n}}(\gamma(t)) = 0$$
$$\mathcal{W}(\dot{\gamma}(t)).\hat{\mathbf{n}}(\gamma(t)) = 0$$

So the image of $\mathcal{W}$ is perpendicular to the normal

.

$$\hat{\mathbf{n}}(\gamma(t)).\hat{\mathbf{n}}(\gamma(t)) = 1$$
$$\tfrac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{n}}(\gamma(t)).\hat{\mathbf{n}}(\gamma(t)) = 0$$
$$\mathcal{W}(\dot{\gamma}(t)).\hat{\mathbf{n}}(\gamma(t)) = 0$$
$$\mathcal{W}(\dot{\gamma}(t)) = ??\sigma_x + ??\sigma_y$$

And therefore, belongs to the tangent space

.

$$\hat{\mathbf{n}}(\gamma(t)).\hat{\mathbf{n}}(\gamma(t)) = 1$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{n}}(\gamma(t)).\hat{\mathbf{n}}(\gamma(t)) = 0$$
$$\mathcal{W}(\dot{\gamma}(t)).\hat{\mathbf{n}}(\gamma(t)) = 0$$
$$\mathcal{W}(\dot{\gamma}(t)) = ??\sigma_x + ??\sigma_y$$

$$\mathcal{W}(\sigma_x)$$
$$\mathcal{W}(\sigma_y)$$

Since $\mathcal{W}$ is linear, we just need to find out what it does to the basis, σ_x and σ_y

.

$$\hat{\mathbf{n}}(\gamma(t)).\hat{\mathbf{n}}(\gamma(t)) = 1$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{n}}(\gamma(t)).\hat{\mathbf{n}}(\gamma(t)) = 0$$
$$\mathcal{W}(\dot{\gamma}(t)).\hat{\mathbf{n}}(\gamma(t)) = 0$$
$$\mathcal{W}(\dot{\gamma}(t)) = ??\sigma_x + ??\sigma_y$$

$$\mathcal{W}(\sigma_x) = -\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{n}}(\sigma(t, y_0))$$
$$\mathcal{W}(\sigma_y) = -\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{n}}(\sigma(x_0, t))$$

But the basis vectors are also velocity vectors of certain curves

$$\hat{\mathbf{n}}(\gamma(t)).\hat{\mathbf{n}}(\gamma(t)) = 1$$
$$\tfrac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{n}}(\gamma(t)).\hat{\mathbf{n}}(\gamma(t)) = 0$$
$$\mathcal{W}(\dot{\gamma}(t)).\hat{\mathbf{n}}(\gamma(t)) = 0$$
$$\mathcal{W}(\dot{\gamma}(t)) = ??\sigma_x + ??\sigma_y$$

$$\mathcal{W}(\sigma_x) = -\tfrac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{n}}(\sigma(t, y_0)) = -(\hat{\mathbf{n}} \circ \sigma)_x := -\hat{\mathbf{n}}_x$$
$$\mathcal{W}(\sigma_y) = -\tfrac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{n}}(\sigma(x_0, t)) = -\hat{\mathbf{n}}_y$$

$$-\hat{\mathbf{n}}_x = a\sigma_x + b\sigma_y$$
$$-\hat{\mathbf{n}}_y = c\sigma_x + d\sigma_y$$

These just work out to be the partial derivative of $\hat{\mathbf{n}}$ in terms of the coordinates given by the patch

$$\hat{\mathbf{n}}(\gamma(t)).\hat{\mathbf{n}}(\gamma(t)) = 1$$
$$\tfrac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{n}}(\gamma(t)).\hat{\mathbf{n}}(\gamma(t)) = 0$$
$$\mathcal{W}(\dot{\gamma}(t)).\hat{\mathbf{n}}(\gamma(t)) = 0$$
$$\mathcal{W}(\dot{\gamma}(t)) = ??\sigma_x + ??\sigma_y$$

$$\mathcal{W}(\sigma_x) = -\tfrac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{n}}(\sigma(t, y_0)) = -(\hat{\mathbf{n}} \circ \sigma)_x := -\hat{\mathbf{n}}_x$$
$$\mathcal{W}(\sigma_y) = -\tfrac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{n}}(\sigma(x_0, t)) = -\hat{\mathbf{n}}_y$$

$$-\hat{\mathbf{n}}_x = a\sigma_x + b\sigma_y$$
$$-\hat{\mathbf{n}}_y = c\sigma_x + d\sigma_y$$

This time, the basis $\sigma_x, \sigma_y, \hat{\mathbf{n}}$ are not orthogonal

$$\hat{\mathbf{n}}(\gamma(t)).\hat{\mathbf{n}}(\gamma(t)) = 1$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{n}}(\gamma(t)).\hat{\mathbf{n}}(\gamma(t)) = 0$$
$$\mathcal{W}(\dot{\gamma}(t)).\hat{\mathbf{n}}(\gamma(t)) = 0$$
$$\mathcal{W}(\dot{\gamma}(t)) = ??\sigma_x + ??\sigma_y$$

$$\mathcal{W}(\sigma_x) = -\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{n}}(\sigma(t, y_0)) = -(\hat{\mathbf{n}} \circ \sigma)_x := -\hat{\mathbf{n}}_x$$
$$\mathcal{W}(\sigma_y) = -\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{n}}(\sigma(x_0, t)) = -\hat{\mathbf{n}}_y$$

$$-\hat{\mathbf{n}}_x = a\sigma_x + b\sigma_y$$
$$-\hat{\mathbf{n}}_y = c\sigma_x + d\sigma_y$$

$\hat{\mathbf{n}}$ is orthogonal to the other too, and this can be used

.

$$\hat{\mathbf{n}}(\gamma(t)).\hat{\mathbf{n}}(\gamma(t)) = 1$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{n}}(\gamma(t)).\hat{\mathbf{n}}(\gamma(t)) = 0$$
$$\mathcal{W}(\dot{\gamma}(t)).\hat{\mathbf{n}}(\gamma(t)) = 0$$
$$\mathcal{W}(\dot{\gamma}(t)) = ??\sigma_x + ??\sigma_y$$

$$\mathcal{W}(\sigma_x) = -\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{n}}(\sigma(t, y_0)) = -(\hat{\mathbf{n}} \circ \sigma)_x := -\hat{\mathbf{n}}_x$$
$$\mathcal{W}(\sigma_y) = -\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{n}}(\sigma(x_0, t)) = -\hat{\mathbf{n}}_y$$

$$-\hat{\mathbf{n}}_x = a\sigma_x + b\sigma_y$$
$$-\hat{\mathbf{n}}_y = c\sigma_x + d\sigma_y$$

$$-\hat{\mathbf{n}}_x.\sigma_x = a\sigma_x.\sigma_x + b\sigma_y.\sigma_x$$
$$-\hat{\mathbf{n}}_y.\sigma_x = c\sigma_x.\sigma_x + d\sigma_y.\sigma_x$$
$$-\hat{\mathbf{n}}_y.\sigma_y = c\sigma_x.\sigma_y + d\sigma_y.\sigma_y$$
$$-\hat{\mathbf{n}}_x.\sigma_y = a\sigma_x.\sigma_y + b\sigma_y.\sigma_y$$

We proceed in the usual manner to figure out coefficients but this time we do not have as many 0s

$$\hat{\mathbf{n}}(\gamma(t)).\hat{\mathbf{n}}(\gamma(t)) = 1$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{n}}(\gamma(t)).\hat{\mathbf{n}}(\gamma(t)) = 0$$
$$\mathcal{W}(\dot{\gamma}(t)).\hat{\mathbf{n}}(\gamma(t)) = 0$$
$$\mathcal{W}(\dot{\gamma}(t)) = ??\sigma_x + ??\sigma_y$$

$$\mathcal{W}(\sigma_x) = -\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{n}}(\sigma(t, y_0)) = -(\hat{\mathbf{n}} \circ \sigma)_x := -\hat{\mathbf{n}}_x$$
$$\mathcal{W}(\sigma_y) = -\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{n}}(\sigma(x_0, t)) = -\hat{\mathbf{n}}_y$$

$$-\hat{\mathbf{n}}_x = a\sigma_x + b\sigma_y$$
$$-\hat{\mathbf{n}}_y = c\sigma_x + d\sigma_y$$

$$-\hat{\mathbf{n}}_x.\sigma_x = a\sigma_x.\sigma_x + b\sigma_y.\sigma_x$$
$$-\hat{\mathbf{n}}_y.\sigma_x = c\sigma_x.\sigma_x + d\sigma_y.\sigma_x$$
$$-\hat{\mathbf{n}}_y.\sigma_y = c\sigma_x.\sigma_y + d\sigma_y.\sigma_y$$
$$-\hat{\mathbf{n}}_x.\sigma_y = a\sigma_x.\sigma_y + b\sigma_y.\sigma_y$$

Nevertheless, we have found 4 relations for the 4 unknowns

.

$$\hat{\mathbf{n}}(\gamma(t)).\hat{\mathbf{n}}(\gamma(t)) = 1$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{n}}(\gamma(t)).\hat{\mathbf{n}}(\gamma(t)) = 0$$
$$\mathcal{W}(\dot{\gamma}(t)).\hat{\mathbf{n}}(\gamma(t)) = 0$$
$$\mathcal{W}(\dot{\gamma}(t)) = ??\sigma_x + ??\sigma_y$$

$$\mathcal{W}(\sigma_x) = -\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{n}}(\sigma(t, y_0)) = -(\hat{\mathbf{n}} \circ \sigma)_x := -\hat{\mathbf{n}}_x$$
$$\mathcal{W}(\sigma_y) = -\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{n}}(\sigma(x_0, t)) = -\hat{\mathbf{n}}_y$$

$$-\hat{\mathbf{n}}_x = a\sigma_x + b\sigma_y$$
$$-\hat{\mathbf{n}}_y = c\sigma_x + d\sigma_y$$

$$-\hat{\mathbf{n}}_x.\sigma_x = a\sigma_x.\sigma_x + b\sigma_y.\sigma_x$$
$$-\hat{\mathbf{n}}_y.\sigma_x = c\sigma_x.\sigma_x + d\sigma_y.\sigma_x$$
$$-\hat{\mathbf{n}}_y.\sigma_y = c\sigma_x.\sigma_y + d\sigma_y.\sigma_y$$
$$-\hat{\mathbf{n}}_x.\sigma_y = a\sigma_x.\sigma_y + b\sigma_y.\sigma_y$$

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -\hat{\mathbf{n}}_x.\sigma_x & -\hat{\mathbf{n}}_y.\sigma_x \\ -\hat{\mathbf{n}}_x.\sigma_y & -\hat{\mathbf{n}}_y.\sigma_y \end{pmatrix}$$

Writing it in matrix form

$$\hat{\mathbf{n}}(\gamma(t)).\hat{\mathbf{n}}(\gamma(t)) = 1$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{n}}(\gamma(t)).\hat{\mathbf{n}}(\gamma(t)) = 0$$
$$\mathcal{W}(\dot{\gamma}(t)).\hat{\mathbf{n}}(\gamma(t)) = 0$$
$$\mathcal{W}(\dot{\gamma}(t)) = ??\sigma_x + ??\sigma_y$$

$$\mathcal{W}(\sigma_x) = -\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{n}}(\sigma(t, y_0)) = -(\hat{\mathbf{n}} \circ \sigma)_x := -\hat{\mathbf{n}}_x$$
$$\mathcal{W}(\sigma_y) = -\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{n}}(\sigma(x_0, t)) = -\hat{\mathbf{n}}_y$$

$$-\hat{\mathbf{n}}_x = a\sigma_x + b\sigma_y$$
$$-\hat{\mathbf{n}}_y = c\sigma_x + d\sigma_y$$

$$-\hat{\mathbf{n}}_x.\sigma_x = a\sigma_x.\sigma_x + b\sigma_y.\sigma_x$$
$$-\hat{\mathbf{n}}_y.\sigma_x = c\sigma_x.\sigma_x + d\sigma_y.\sigma_x$$
$$-\hat{\mathbf{n}}_y.\sigma_y = c\sigma_x.\sigma_y + d\sigma_y.\sigma_y$$
$$-\hat{\mathbf{n}}_x.\sigma_y = a\sigma_x.\sigma_y + b\sigma_y.\sigma_y$$

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -\hat{\mathbf{n}}_x.\sigma_x & -\hat{\mathbf{n}}_y.\sigma_x \\ -\hat{\mathbf{n}}_x.\sigma_y & -\hat{\mathbf{n}}_y.\sigma_y \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} -\hat{\mathbf{n}}_x.\sigma_x & -\hat{\mathbf{n}}_y.\sigma_x \\ -\hat{\mathbf{n}}_x.\sigma_y & -\hat{\mathbf{n}}_y.\sigma_y \end{pmatrix}$$
$$=$$

We are able to express the coefficients in terms of the first fundamental form and some new terms

$$\hat{\mathbf{n}}(\gamma(t)).\hat{\mathbf{n}}(\gamma(t)) = 1$$
$$\tfrac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{n}}(\gamma(t)).\hat{\mathbf{n}}(\gamma(t)) = 0$$
$$\mathcal{W}(\dot{\gamma}(t)).\hat{\mathbf{n}}(\gamma(t)) = 0$$
$$\mathcal{W}(\dot{\gamma}(t)) = ??\sigma_x + ??\sigma_y$$

$$\mathcal{W}(\sigma_x) = -\tfrac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{n}}(\sigma(t, y_0)) = -(\hat{\mathbf{n}} \circ \sigma)_x := -\hat{\mathbf{n}}_x$$
$$\mathcal{W}(\sigma_y) = -\tfrac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{n}}(\sigma(x_0, t)) = -\hat{\mathbf{n}}_y$$

$$-\hat{\mathbf{n}}_x = a\sigma_x + b\sigma_y$$
$$-\hat{\mathbf{n}}_y = c\sigma_x + d\sigma_y$$

$$-\hat{\mathbf{n}}_x.\sigma_x = a\sigma_x.\sigma_x + b\sigma_y.\sigma_x$$
$$-\hat{\mathbf{n}}_y.\sigma_x = c\sigma_x.\sigma_x + d\sigma_y.\sigma_x$$
$$-\hat{\mathbf{n}}_y.\sigma_y = c\sigma_x.\sigma_y + d\sigma_y.\sigma_y$$
$$-\hat{\mathbf{n}}_x.\sigma_y = a\sigma_x.\sigma_y + b\sigma_y.\sigma_y$$

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -\hat{\mathbf{n}}_x.\sigma_x & -\hat{\mathbf{n}}_y.\sigma_x \\ -\hat{\mathbf{n}}_x.\sigma_y & -\hat{\mathbf{n}}_y.\sigma_y \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} -\hat{\mathbf{n}}_x.\sigma_x & -\hat{\mathbf{n}}_y.\sigma_x \\ -\hat{\mathbf{n}}_x.\sigma_y & -\hat{\mathbf{n}}_y.\sigma_y \end{pmatrix}$$
$$=$$

Product rule allows us to rewrite these new terms of (second order) partial derivatives of the patch

$$\hat{\mathbf{n}}(\gamma(t)).\hat{\mathbf{n}}(\gamma(t)) = 1$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{n}}(\gamma(t)).\hat{\mathbf{n}}(\gamma(t)) = 0$$
$$\mathcal{W}(\dot{\gamma}(t)).\hat{\mathbf{n}}(\gamma(t)) = 0$$
$$\mathcal{W}(\dot{\gamma}(t)) = ??\sigma_x + ??\sigma_y$$

$$\mathcal{W}(\sigma_x) = -\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{n}}(\sigma(t, y_0)) = -(\hat{\mathbf{n}} \circ \sigma)_x := -\hat{\mathbf{n}}_x$$
$$\mathcal{W}(\sigma_y) = -\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{n}}(\sigma(x_0, t)) = -\hat{\mathbf{n}}_y$$

$$-\hat{\mathbf{n}}_x = a\sigma_x + b\sigma_y$$
$$-\hat{\mathbf{n}}_y = c\sigma_x + d\sigma_y$$

$$-\hat{\mathbf{n}}_x.\sigma_x = a\sigma_x.\sigma_x + b\sigma_y.\sigma_x$$
$$-\hat{\mathbf{n}}_y.\sigma_x = c\sigma_x.\sigma_x + d\sigma_y.\sigma_x$$
$$-\hat{\mathbf{n}}_y.\sigma_y = c\sigma_x.\sigma_y + d\sigma_y.\sigma_y$$
$$-\hat{\mathbf{n}}_x.\sigma_y = a\sigma_x.\sigma_y + b\sigma_y.\sigma_y$$

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -\hat{\mathbf{n}}_x.\sigma_x & -\hat{\mathbf{n}}_y.\sigma_x \\ -\hat{\mathbf{n}}_x.\sigma_y & -\hat{\mathbf{n}}_y.\sigma_y \end{pmatrix}$$

$$\begin{aligned}\begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}\begin{pmatrix} -\hat{\mathbf{n}}_x.\sigma_x & -\hat{\mathbf{n}}_y.\sigma_x \\ -\hat{\mathbf{n}}_x.\sigma_y & -\hat{\mathbf{n}}_y.\sigma_y \end{pmatrix} \\ &= \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}\begin{pmatrix} \hat{\mathbf{n}}.\sigma_{xx} & \hat{\mathbf{n}}.\sigma_{yx} \\ \hat{\mathbf{n}}.\sigma_{xy} & \hat{\mathbf{n}}.\sigma_{yy} \end{pmatrix} \\ &= \end{aligned}$$

rather than differentiating $\hat{\mathbf{n}}$

.

$$\hat{\mathbf{n}}(\gamma(t)).\hat{\mathbf{n}}(\gamma(t)) = 1$$
$$\tfrac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{n}}(\gamma(t)).\hat{\mathbf{n}}(\gamma(t)) = 0$$
$$\mathcal{W}(\dot{\gamma}(t)).\hat{\mathbf{n}}(\gamma(t)) = 0$$
$$\mathcal{W}(\dot{\gamma}(t)) = ??\sigma_x + ??\sigma_y$$

$$\mathcal{W}(\sigma_x) = -\tfrac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{n}}(\sigma(t, y_0)) = -(\hat{\mathbf{n}} \circ \sigma)_x := -\hat{\mathbf{n}}_x$$
$$\mathcal{W}(\sigma_y) = -\tfrac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{n}}(\sigma(x_0, t)) = -\hat{\mathbf{n}}_y$$

$$-\hat{\mathbf{n}}_x = a\sigma_x + b\sigma_y$$
$$-\hat{\mathbf{n}}_y = c\sigma_x + d\sigma_y$$

$$-\hat{\mathbf{n}}_x.\sigma_x = a\sigma_x.\sigma_x + b\sigma_y.\sigma_x$$
$$-\hat{\mathbf{n}}_y.\sigma_x = c\sigma_x.\sigma_x + d\sigma_y.\sigma_x$$
$$-\hat{\mathbf{n}}_y.\sigma_y = c\sigma_x.\sigma_y + d\sigma_y.\sigma_y$$
$$-\hat{\mathbf{n}}_x.\sigma_y = a\sigma_x.\sigma_y + b\sigma_y.\sigma_y$$

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -\hat{\mathbf{n}}_x.\sigma_x & -\hat{\mathbf{n}}_y.\sigma_x \\ -\hat{\mathbf{n}}_x.\sigma_y & -\hat{\mathbf{n}}_y.\sigma_y \end{pmatrix}$$

$$\begin{aligned} \begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} -\hat{\mathbf{n}}_x.\sigma_x & -\hat{\mathbf{n}}_y.\sigma_x \\ -\hat{\mathbf{n}}_x.\sigma_y & -\hat{\mathbf{n}}_y.\sigma_y \end{pmatrix} \\ &= \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} \hat{\mathbf{n}}.\sigma_{xx} & \hat{\mathbf{n}}.\sigma_{yx} \\ \hat{\mathbf{n}}.\sigma_{xy} & \hat{\mathbf{n}}.\sigma_{yy} \end{pmatrix} \\ &= \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} L & M \\ M & N \end{pmatrix} \end{aligned}$$

These new terms are denoted L, M, and N and are called the second fundamental form

$$\hat{\mathbf{n}}(\gamma(t)).\hat{\mathbf{n}}(\gamma(t)) = 1$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{n}}(\gamma(t)).\hat{\mathbf{n}}(\gamma(t)) = 0$$
$$\mathcal{W}(\dot{\gamma}(t)).\hat{\mathbf{n}}(\gamma(t)) = 0$$
$$\mathcal{W}(\dot{\gamma}(t)) = ??\sigma_x + ??\sigma_y$$

$$\mathcal{W}(\sigma_x) = -\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{n}}(\sigma(t, y_0)) = -(\hat{\mathbf{n}} \circ \sigma)_x := -\hat{\mathbf{n}}_x$$
$$\mathcal{W}(\sigma_y) = -\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{n}}(\sigma(x_0, t)) = -\hat{\mathbf{n}}_y$$

$$-\hat{\mathbf{n}}_x = a\sigma_x + b\sigma_y$$
$$-\hat{\mathbf{n}}_y = c\sigma_x + d\sigma_y$$

$$-\hat{\mathbf{n}}_x.\sigma_x = a\sigma_x.\sigma_x + b\sigma_y.\sigma_x$$
$$-\hat{\mathbf{n}}_y.\sigma_x = c\sigma_x.\sigma_x + d\sigma_y.\sigma_x$$
$$-\hat{\mathbf{n}}_y.\sigma_y = c\sigma_x.\sigma_y + d\sigma_y.\sigma_y$$
$$-\hat{\mathbf{n}}_x.\sigma_y = a\sigma_x.\sigma_y + b\sigma_y.\sigma_y$$

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -\hat{\mathbf{n}}_x.\sigma_x & -\hat{\mathbf{n}}_y.\sigma_x \\ -\hat{\mathbf{n}}_x.\sigma_y & -\hat{\mathbf{n}}_y.\sigma_y \end{pmatrix}$$

$$\begin{aligned}\begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}\begin{pmatrix} -\hat{\mathbf{n}}_x.\sigma_x & -\hat{\mathbf{n}}_y.\sigma_x \\ -\hat{\mathbf{n}}_x.\sigma_y & -\hat{\mathbf{n}}_y.\sigma_y \end{pmatrix} \\ &= \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}\begin{pmatrix} \hat{\mathbf{n}}.\sigma_{xx} & \hat{\mathbf{n}}.\sigma_{yx} \\ \hat{\mathbf{n}}.\sigma_{xy} & \hat{\mathbf{n}}.\sigma_{yy} \end{pmatrix} \\ &= \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}\begin{pmatrix} L & M \\ M & N \end{pmatrix}\end{aligned}$$

Note that for any smooth function $f_{xy} = f_{yx}$

.

$$\hat{\mathbf{n}}(\gamma(t)).\hat{\mathbf{n}}(\gamma(t)) = 1$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{n}}(\gamma(t)).\hat{\mathbf{n}}(\gamma(t)) = 0$$
$$\mathcal{W}(\dot{\gamma}(t)).\hat{\mathbf{n}}(\gamma(t)) = 0$$
$$\mathcal{W}(\dot{\gamma}(t)) = ??\sigma_x + ??\sigma_y$$

$$\mathcal{W}(\sigma_x) = -\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{n}}(\sigma(t, y_0)) = -(\hat{\mathbf{n}} \circ \sigma)_x := -\hat{\mathbf{n}}_x$$
$$\mathcal{W}(\sigma_y) = -\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{n}}(\sigma(x_0, t)) = -\hat{\mathbf{n}}_y$$

$$-\hat{\mathbf{n}}_x = a\sigma_x + b\sigma_y$$
$$-\hat{\mathbf{n}}_y = c\sigma_x + d\sigma_y$$

$$-\hat{\mathbf{n}}_x.\sigma_x = a\sigma_x.\sigma_x + b\sigma_y.\sigma_x$$
$$-\hat{\mathbf{n}}_y.\sigma_x = c\sigma_x.\sigma_x + d\sigma_y.\sigma_x$$
$$-\hat{\mathbf{n}}_y.\sigma_y = c\sigma_x.\sigma_y + d\sigma_y.\sigma_y$$
$$-\hat{\mathbf{n}}_x.\sigma_y = a\sigma_x.\sigma_y + b\sigma_y.\sigma_y$$

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -\hat{\mathbf{n}}_x.\sigma_x & -\hat{\mathbf{n}}_y.\sigma_x \\ -\hat{\mathbf{n}}_x.\sigma_y & -\hat{\mathbf{n}}_y.\sigma_y \end{pmatrix}$$

$$\begin{aligned} \begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} -\hat{\mathbf{n}}_x.\sigma_x & -\hat{\mathbf{n}}_y.\sigma_x \\ -\hat{\mathbf{n}}_x.\sigma_y & -\hat{\mathbf{n}}_y.\sigma_y \end{pmatrix} \\ &= \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} \hat{\mathbf{n}}.\sigma_{xx} & \hat{\mathbf{n}}.\sigma_{yx} \\ \hat{\mathbf{n}}.\sigma_{xy} & \hat{\mathbf{n}}.\sigma_{yy} \end{pmatrix} \\ &= \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} L & M \\ M & N \end{pmatrix} \end{aligned}$$

That is why $M = \sigma_{yx} = \sigma_{xy}$

$$\hat{\mathbf{n}}(\gamma(t)).\hat{\mathbf{n}}(\gamma(t)) = 1$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{n}}(\gamma(t)).\hat{\mathbf{n}}(\gamma(t)) = 0$$
$$\mathcal{W}(\dot{\gamma}(t)).\hat{\mathbf{n}}(\gamma(t)) = 0$$
$$\mathcal{W}(\dot{\gamma}(t)) = ??\sigma_x + ??\sigma_y$$

$$\mathcal{W}(\sigma_x) = -\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{n}}(\sigma(t, y_0)) = -(\hat{\mathbf{n}} \circ \sigma)_x := -\hat{\mathbf{n}}_x$$
$$\mathcal{W}(\sigma_y) = -\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{n}}(\sigma(x_0, t)) = -\hat{\mathbf{n}}_y$$

$$-\hat{\mathbf{n}}_x = a\sigma_x + b\sigma_y$$
$$-\hat{\mathbf{n}}_y = c\sigma_x + d\sigma_y$$

$$-\hat{\mathbf{n}}_x.\sigma_x = a\sigma_x.\sigma_x + b\sigma_y.\sigma_x$$
$$-\hat{\mathbf{n}}_y.\sigma_x = c\sigma_x.\sigma_x + d\sigma_y.\sigma_x$$
$$-\hat{\mathbf{n}}_y.\sigma_y = c\sigma_x.\sigma_y + d\sigma_y.\sigma_y$$
$$-\hat{\mathbf{n}}_x.\sigma_y = a\sigma_x.\sigma_y + b\sigma_y.\sigma_y$$

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -\hat{\mathbf{n}}_x.\sigma_x & -\hat{\mathbf{n}}_y.\sigma_x \\ -\hat{\mathbf{n}}_x.\sigma_y & -\hat{\mathbf{n}}_y.\sigma_y \end{pmatrix}$$

$$\begin{aligned}\begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}\begin{pmatrix} -\hat{\mathbf{n}}_x.\sigma_x & -\hat{\mathbf{n}}_y.\sigma_x \\ -\hat{\mathbf{n}}_x.\sigma_y & -\hat{\mathbf{n}}_y.\sigma_y \end{pmatrix} \\ &= \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}\begin{pmatrix} \hat{\mathbf{n}}.\sigma_{xx} & \hat{\mathbf{n}}.\sigma_{yx} \\ \hat{\mathbf{n}}.\sigma_{xy} & \hat{\mathbf{n}}.\sigma_{yy} \end{pmatrix} \\ &= \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}\begin{pmatrix} L & M \\ M & N \end{pmatrix}\end{aligned}$$

We now introduce the counterpart in surfaces to Frenet-Serret in curves

$$\hat{\mathbf{n}}(\gamma(t)).\hat{\mathbf{n}}(\gamma(t)) = 1$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{n}}(\gamma(t)).\hat{\mathbf{n}}(\gamma(t)) = 0$$
$$\mathcal{W}(\dot{\gamma}(t)).\hat{\mathbf{n}}(\gamma(t)) = 0$$
$$\mathcal{W}(\dot{\gamma}(t)) = ??\sigma_x + ??\sigma_y$$

$$\mathcal{W}(\sigma_x) = -\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{n}}(\sigma(t, y_0)) = -(\hat{\mathbf{n}} \circ \sigma)_x := -\hat{\mathbf{n}}_x$$
$$\mathcal{W}(\sigma_y) = -\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{n}}(\sigma(x_0, t)) = -\hat{\mathbf{n}}_y$$

$$-\hat{\mathbf{n}}_x = a\sigma_x + b\sigma_y$$
$$-\hat{\mathbf{n}}_y = c\sigma_x + d\sigma_y$$

$$-\hat{\mathbf{n}}_x.\sigma_x = a\sigma_x.\sigma_x + b\sigma_y.\sigma_x$$
$$-\hat{\mathbf{n}}_y.\sigma_x = c\sigma_x.\sigma_x + d\sigma_y.\sigma_x$$
$$-\hat{\mathbf{n}}_y.\sigma_y = c\sigma_x.\sigma_y + d\sigma_y.\sigma_y$$
$$-\hat{\mathbf{n}}_x.\sigma_y = a\sigma_x.\sigma_y + b\sigma_y.\sigma_y$$

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -\hat{\mathbf{n}}_x.\sigma_x & -\hat{\mathbf{n}}_y.\sigma_x \\ -\hat{\mathbf{n}}_x.\sigma_y & -\hat{\mathbf{n}}_y.\sigma_y \end{pmatrix}$$

$$\begin{aligned}\begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}\begin{pmatrix} -\hat{\mathbf{n}}_x.\sigma_x & -\hat{\mathbf{n}}_y.\sigma_x \\ -\hat{\mathbf{n}}_x.\sigma_y & -\hat{\mathbf{n}}_y.\sigma_y \end{pmatrix} \\ &= \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}\begin{pmatrix} \hat{\mathbf{n}}.\sigma_{xx} & \hat{\mathbf{n}}.\sigma_{yx} \\ \hat{\mathbf{n}}.\sigma_{xy} & \hat{\mathbf{n}}.\sigma_{yy} \end{pmatrix} \\ &= \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}\begin{pmatrix} L & M \\ M & N \end{pmatrix}\end{aligned}$$

Frenet-Serret told us how to express the derivatives of convenient basis vector fields in terms of the same basis

.

$$\hat{\mathbf{n}}(\gamma(t)).\hat{\mathbf{n}}(\gamma(t)) = 1$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{n}}(\gamma(t)).\hat{\mathbf{n}}(\gamma(t)) = 0$$
$$\mathcal{W}(\dot{\gamma}(t)).\hat{\mathbf{n}}(\gamma(t)) = 0$$
$$\mathcal{W}(\dot{\gamma}(t)) = ??\sigma_x + ??\sigma_y$$

$$\mathcal{W}(\sigma_x) = -\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{n}}(\sigma(t, y_0)) = -(\hat{\mathbf{n}} \circ \sigma)_x := -\hat{\mathbf{n}}_x$$
$$\mathcal{W}(\sigma_y) = -\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{n}}(\sigma(x_0, t)) = -\hat{\mathbf{n}}_y$$

$$-\hat{\mathbf{n}}_x = a\sigma_x + b\sigma_y$$
$$-\hat{\mathbf{n}}_y = c\sigma_x + d\sigma_y$$

$$-\hat{\mathbf{n}}_x.\sigma_x = a\sigma_x.\sigma_x + b\sigma_y.\sigma_x$$
$$-\hat{\mathbf{n}}_y.\sigma_x = c\sigma_x.\sigma_x + d\sigma_y.\sigma_x$$
$$-\hat{\mathbf{n}}_y.\sigma_y = c\sigma_x.\sigma_y + d\sigma_y.\sigma_y$$
$$-\hat{\mathbf{n}}_x.\sigma_y = a\sigma_x.\sigma_y + b\sigma_y.\sigma_y$$

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -\hat{\mathbf{n}}_x.\sigma_x & -\hat{\mathbf{n}}_y.\sigma_x \\ -\hat{\mathbf{n}}_x.\sigma_y & -\hat{\mathbf{n}}_y.\sigma_y \end{pmatrix}$$

$$\begin{aligned} \begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} -\hat{\mathbf{n}}_x.\sigma_x & -\hat{\mathbf{n}}_y.\sigma_x \\ -\hat{\mathbf{n}}_x.\sigma_y & -\hat{\mathbf{n}}_y.\sigma_y \end{pmatrix} \\ &= \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} \hat{\mathbf{n}}.\sigma_{xx} & \hat{\mathbf{n}}.\sigma_{yx} \\ \hat{\mathbf{n}}.\sigma_{xy} & \hat{\mathbf{n}}.\sigma_{yy} \end{pmatrix} \\ &= \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} L & M \\ M & N \end{pmatrix} \end{aligned}$$

$$\sigma_{xx} = ??\sigma_x + ??\sigma_y + ??\hat{\mathbf{n}}$$
$$\sigma_{xy} = ??\sigma_x + ??\sigma_y + ??\hat{\mathbf{n}}$$
$$\sigma_{yy} = ??\sigma_x + ??\sigma_y + ??\hat{\mathbf{n}}$$

Here we find the derivatives of the basis vectors $\{\sigma_x, \sigma_y, \hat{\mathbf{n}}\}$ in terms of $\sigma_x, \sigma_y, \hat{\mathbf{n}}$

$$\hat{\mathbf{n}}(\gamma(t)).\hat{\mathbf{n}}(\gamma(t)) = 1$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{n}}(\gamma(t)).\hat{\mathbf{n}}(\gamma(t)) = 0$$
$$\mathcal{W}(\dot{\gamma}(t)).\hat{\mathbf{n}}(\gamma(t)) = 0$$
$$\mathcal{W}(\dot{\gamma}(t)) = ??\sigma_x + ??\sigma_y$$

$$\mathcal{W}(\sigma_x) = -\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{n}}(\sigma(t, y_0)) = -(\hat{\mathbf{n}} \circ \sigma)_x := -\hat{\mathbf{n}}_x$$
$$\mathcal{W}(\sigma_y) = -\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{n}}(\sigma(x_0, t)) = -\hat{\mathbf{n}}_y$$

$$-\hat{\mathbf{n}}_x = a\sigma_x + b\sigma_y$$
$$-\hat{\mathbf{n}}_y = c\sigma_x + d\sigma_y$$

$$-\hat{\mathbf{n}}_x.\sigma_x = a\sigma_x.\sigma_x + b\sigma_y.\sigma_x$$
$$-\hat{\mathbf{n}}_y.\sigma_x = c\sigma_x.\sigma_x + d\sigma_y.\sigma_x$$
$$-\hat{\mathbf{n}}_y.\sigma_y = c\sigma_x.\sigma_y + d\sigma_y.\sigma_y$$
$$-\hat{\mathbf{n}}_x.\sigma_y = a\sigma_x.\sigma_y + b\sigma_y.\sigma_y$$

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -\hat{\mathbf{n}}_x.\sigma_x & -\hat{\mathbf{n}}_y.\sigma_x \\ -\hat{\mathbf{n}}_x.\sigma_y & -\hat{\mathbf{n}}_y.\sigma_y \end{pmatrix}$$

$$\begin{aligned} \begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}\begin{pmatrix} -\hat{\mathbf{n}}_x.\sigma_x & -\hat{\mathbf{n}}_y.\sigma_x \\ -\hat{\mathbf{n}}_x.\sigma_y & -\hat{\mathbf{n}}_y.\sigma_y \end{pmatrix} \\ &= \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}\begin{pmatrix} \hat{\mathbf{n}}.\sigma_{xx} & \hat{\mathbf{n}}.\sigma_{yx} \\ \hat{\mathbf{n}}.\sigma_{xy} & \hat{\mathbf{n}}.\sigma_{yy} \end{pmatrix} \\ &= \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}\begin{pmatrix} L & M \\ M & N \end{pmatrix} \end{aligned}$$

$$\sigma_{xx} = ??\sigma_x + ??\sigma_y + ??\hat{\mathbf{n}}$$
$$\sigma_{xy} = ??\sigma_x + ??\sigma_y + ??\hat{\mathbf{n}}$$
$$\sigma_{yy} = ??\sigma_x + ??\sigma_y + ??\hat{\mathbf{n}}$$

The derivatives of $\hat{\mathbf{n}}$ were already found.

$$\hat{\mathbf{n}}(\gamma(t)).\hat{\mathbf{n}}(\gamma(t)) = 1$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{n}}(\gamma(t)).\hat{\mathbf{n}}(\gamma(t)) = 0$$
$$\mathcal{W}(\dot{\gamma}(t)).\hat{\mathbf{n}}(\gamma(t)) = 0$$
$$\mathcal{W}(\dot{\gamma}(t)) = ??\sigma_x + ??\sigma_y$$

$$\mathcal{W}(\sigma_x) = -\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{n}}(\sigma(t, y_0)) = -(\hat{\mathbf{n}} \circ \sigma)_x := -\hat{\mathbf{n}}_x$$
$$\mathcal{W}(\sigma_y) = -\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{n}}(\sigma(x_0, t)) = -\hat{\mathbf{n}}_y$$

$$-\hat{\mathbf{n}}_x = a\sigma_x + b\sigma_y$$
$$-\hat{\mathbf{n}}_y = c\sigma_x + d\sigma_y$$

$$-\hat{\mathbf{n}}_x.\sigma_x = a\sigma_x.\sigma_x + b\sigma_y.\sigma_x$$
$$-\hat{\mathbf{n}}_y.\sigma_x = c\sigma_x.\sigma_x + d\sigma_y.\sigma_x$$
$$-\hat{\mathbf{n}}_y.\sigma_y = c\sigma_x.\sigma_y + d\sigma_y.\sigma_y$$
$$-\hat{\mathbf{n}}_x.\sigma_y = a\sigma_x.\sigma_y + b\sigma_y.\sigma_y$$

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -\hat{\mathbf{n}}_x.\sigma_x & -\hat{\mathbf{n}}_y.\sigma_x \\ -\hat{\mathbf{n}}_x.\sigma_y & -\hat{\mathbf{n}}_y.\sigma_y \end{pmatrix}$$

$$\begin{aligned}\begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}\begin{pmatrix} -\hat{\mathbf{n}}_x.\sigma_x & -\hat{\mathbf{n}}_y.\sigma_x \\ -\hat{\mathbf{n}}_x.\sigma_y & -\hat{\mathbf{n}}_y.\sigma_y \end{pmatrix} \\ &= \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}\begin{pmatrix} \hat{\mathbf{n}}.\sigma_{xx} & \hat{\mathbf{n}}.\sigma_{yx} \\ \hat{\mathbf{n}}.\sigma_{xy} & \hat{\mathbf{n}}.\sigma_{yy} \end{pmatrix} \\ &= \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}\begin{pmatrix} L & M \\ M & N \end{pmatrix}\end{aligned}$$

$$\sigma_{xx} = ??\sigma_x + ??\sigma_y + ??\hat{\mathbf{n}}$$
$$\sigma_{xy} = ??\sigma_x + ??\sigma_y + ??\hat{\mathbf{n}}$$
$$\sigma_{yy} = ??\sigma_x + ??\sigma_y + ??\hat{\mathbf{n}}$$

The a, b, c, d that we just found out were precisely those coefficients.

$$\hat{\mathbf{n}}(\gamma(t)).\hat{\mathbf{n}}(\gamma(t)) = 1$$
$$\tfrac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{n}}(\gamma(t)).\hat{\mathbf{n}}(\gamma(t)) = 0$$
$$\mathcal{W}(\dot{\gamma}(t)).\hat{\mathbf{n}}(\gamma(t)) = 0$$
$$\mathcal{W}(\dot{\gamma}(t)) = ??\sigma_x + ??\sigma_y$$

$$\mathcal{W}(\sigma_x) = -\tfrac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{n}}(\sigma(t, y_0)) = -(\hat{\mathbf{n}} \circ \sigma)_x := -\hat{\mathbf{n}}_x$$
$$\mathcal{W}(\sigma_y) = -\tfrac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{n}}(\sigma(x_0, t)) = -\hat{\mathbf{n}}_y$$

$$-\hat{\mathbf{n}}_x = a\sigma_x + b\sigma_y$$
$$-\hat{\mathbf{n}}_y = c\sigma_x + d\sigma_y$$

$$-\hat{\mathbf{n}}_x.\sigma_x = a\sigma_x.\sigma_x + b\sigma_y.\sigma_x$$
$$-\hat{\mathbf{n}}_y.\sigma_x = c\sigma_x.\sigma_x + d\sigma_y.\sigma_x$$
$$-\hat{\mathbf{n}}_y.\sigma_y = c\sigma_x.\sigma_y + d\sigma_y.\sigma_y$$
$$-\hat{\mathbf{n}}_x.\sigma_y = a\sigma_x.\sigma_y + b\sigma_y.\sigma_y$$

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -\hat{\mathbf{n}}_x.\sigma_x & -\hat{\mathbf{n}}_y.\sigma_x \\ -\hat{\mathbf{n}}_x.\sigma_y & -\hat{\mathbf{n}}_y.\sigma_y \end{pmatrix}$$

$$\begin{aligned}\begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}\begin{pmatrix} -\hat{\mathbf{n}}_x.\sigma_x & -\hat{\mathbf{n}}_y.\sigma_x \\ -\hat{\mathbf{n}}_x.\sigma_y & -\hat{\mathbf{n}}_y.\sigma_y \end{pmatrix} \\ &= \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}\begin{pmatrix} \hat{\mathbf{n}}.\sigma_{xx} & \hat{\mathbf{n}}.\sigma_{yx} \\ \hat{\mathbf{n}}.\sigma_{xy} & \hat{\mathbf{n}}.\sigma_{yy} \end{pmatrix} \\ &= \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}\begin{pmatrix} L & M \\ M & N \end{pmatrix}\end{aligned}$$

$$\sigma_{xx} = ??\sigma_x + ??\sigma_y + L\hat{\mathbf{n}}$$
$$\sigma_{xy} = ??\sigma_x + ??\sigma_y + M\hat{\mathbf{n}}$$
$$\sigma_{yy} = ??\sigma_x + ??\sigma_y + N\hat{\mathbf{n}}$$

We now know some coefficients

$$\hat{\mathbf{n}}(\gamma(t)).\hat{\mathbf{n}}(\gamma(t)) = 1$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{n}}(\gamma(t)).\hat{\mathbf{n}}(\gamma(t)) = 0$$
$$\mathcal{W}(\dot{\gamma}(t)).\hat{\mathbf{n}}(\gamma(t)) = 0$$
$$\mathcal{W}(\dot{\gamma}(t)) = ??\sigma_x + ??\sigma_y$$

$$\mathcal{W}(\sigma_x) = -\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{n}}(\sigma(t, y_0)) = -(\hat{\mathbf{n}} \circ \sigma)_x := -\hat{\mathbf{n}}_x$$
$$\mathcal{W}(\sigma_y) = -\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{n}}(\sigma(x_0, t)) = -\hat{\mathbf{n}}_y$$

$$-\hat{\mathbf{n}}_x = a\sigma_x + b\sigma_y$$
$$-\hat{\mathbf{n}}_y = c\sigma_x + d\sigma_y$$

$$-\hat{\mathbf{n}}_x.\sigma_x = a\sigma_x.\sigma_x + b\sigma_y.\sigma_x$$
$$-\hat{\mathbf{n}}_y.\sigma_x = c\sigma_x.\sigma_x + d\sigma_y.\sigma_x$$
$$-\hat{\mathbf{n}}_y.\sigma_y = c\sigma_x.\sigma_y + d\sigma_y.\sigma_y$$
$$-\hat{\mathbf{n}}_x.\sigma_y = a\sigma_x.\sigma_y + b\sigma_y.\sigma_y$$

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -\hat{\mathbf{n}}_x.\sigma_x & -\hat{\mathbf{n}}_y.\sigma_x \\ -\hat{\mathbf{n}}_x.\sigma_y & -\hat{\mathbf{n}}_y.\sigma_y \end{pmatrix}$$

$$\begin{aligned}\begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}\begin{pmatrix} -\hat{\mathbf{n}}_x.\sigma_x & -\hat{\mathbf{n}}_y.\sigma_x \\ -\hat{\mathbf{n}}_x.\sigma_y & -\hat{\mathbf{n}}_y.\sigma_y \end{pmatrix} \\ &= \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}\begin{pmatrix} \hat{\mathbf{n}}.\sigma_{xx} & \hat{\mathbf{n}}.\sigma_{yx} \\ \hat{\mathbf{n}}.\sigma_{xy} & \hat{\mathbf{n}}.\sigma_{yy} \end{pmatrix} \\ &= \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}\begin{pmatrix} L & M \\ M & N \end{pmatrix}\end{aligned}$$

$$\sigma_{xx} = \Gamma^1_{11}\sigma_x + \Gamma^2_{11}\sigma_y + L\hat{\mathbf{n}}$$
$$\sigma_{xy} = \Gamma^1_{12}\sigma_x + \Gamma^2_{12}\sigma_y + M\hat{\mathbf{n}}$$
$$\sigma_{yy} = \Gamma^1_{22}\sigma_x + \Gamma^2_{22}\sigma_y + N\hat{\mathbf{n}}$$

The others are called the "Christoffel symbols"

$$
\begin{aligned}
\sigma_{xx} &= \Gamma^1_{11}\sigma_x + \Gamma^2_{11}\sigma_y + L\hat{\mathbf{n}} \\
\sigma_{xy} &= \Gamma^1_{12}\sigma_x + \Gamma^2_{12}\sigma_y + M\hat{\mathbf{n}} \\
\sigma_{yy} &= \Gamma^1_{22}\sigma_x + \Gamma^2_{22}\sigma_y + N\hat{\mathbf{n}}
\end{aligned}
$$

Can the Christoffel symbols be expressed in terms of known quantities?

$$\sigma_{xx} = \Gamma^1_{11}\sigma_x + \Gamma^2_{11}\sigma_y + L\hat{\mathbf{n}}$$
$$\sigma_{xy} = \Gamma^1_{12}\sigma_x + \Gamma^2_{12}\sigma_y + M\hat{\mathbf{n}}$$
$$\sigma_{yy} = \Gamma^1_{22}\sigma_x + \Gamma^2_{22}\sigma_y + N\hat{\mathbf{n}}$$

There are 6 of them so do they appear in 6 relations involving known quantities?

$$\begin{aligned}
\sigma_{xx} &= \Gamma^1_{11}\sigma_x + \Gamma^2_{11}\sigma_y + L\hat{\mathbf{n}} \\
\sigma_{xy} &= \Gamma^1_{12}\sigma_x + \Gamma^2_{12}\sigma_y + M\hat{\mathbf{n}} \\
\sigma_{yy} &= \Gamma^1_{22}\sigma_x + \Gamma^2_{22}\sigma_y + N\hat{\mathbf{n}}
\end{aligned}$$

$$\begin{aligned}
\sigma_{xx}.\sigma_x &= \Gamma^1_{11}\sigma_x.\sigma_x + \Gamma^2_{11}\sigma_y.\sigma_x + L\hat{\mathbf{n}}.\sigma_x \\
&= \Gamma^1_{11}E + \Gamma^2_{11}F
\end{aligned}$$

As usual we do that by taking the dot product with some of the basis vectors

$$\sigma_{xx} = \Gamma_{11}^{1}\sigma_x + \Gamma_{11}^{2}\sigma_y + L\hat{\mathbf{n}}$$
$$\sigma_{xy} = \Gamma_{12}^{1}\sigma_x + \Gamma_{12}^{2}\sigma_y + M\hat{\mathbf{n}}$$
$$\sigma_{yy} = \Gamma_{22}^{1}\sigma_x + \Gamma_{22}^{2}\sigma_y + N\hat{\mathbf{n}}$$

$$\begin{aligned}\sigma_{xx}.\sigma_x &= \Gamma_{11}^{1}\sigma_x.\sigma_x + \Gamma_{11}^{2}\sigma_y.\sigma_x + L\hat{\mathbf{n}}.\sigma_x \\ &= \Gamma_{11}^{1}E + \Gamma_{11}^{2}F\end{aligned}$$

$$E_x = (\sigma_x.\sigma_x)_x = 2\sigma_{xx}.\sigma_x$$

However, product rule tells us about $\sigma_{xx}.\sigma_x$

$$\sigma_{xx} = \Gamma^1_{11}\sigma_x + \Gamma^2_{11}\sigma_y + L\hat{\mathbf{n}}$$
$$\sigma_{xy} = \Gamma^1_{12}\sigma_x + \Gamma^2_{12}\sigma_y + M\hat{\mathbf{n}}$$
$$\sigma_{yy} = \Gamma^1_{22}\sigma_x + \Gamma^2_{22}\sigma_y + N\hat{\mathbf{n}}$$

$$\begin{aligned}\sigma_{xx}.\sigma_x &= \Gamma^1_{11}\sigma_x.\sigma_x + \Gamma^2_{11}\sigma_y.\sigma_x + L\hat{\mathbf{n}}.\sigma_x \\ &= \Gamma^1_{11}E + \Gamma^2_{11}F\end{aligned}$$

$E_x = (\sigma_x.\sigma_x)_x = 2\sigma_{xx}.\sigma_x$

Therefore,

$$E_x/2 = \Gamma^1_{11}E + \Gamma^2_{11}F$$

So we obtain a relation entirely in terms of E, F, G (and their derivatives)

$$\begin{aligned}
\sigma_{xx} &= \Gamma^1_{11}\sigma_x + \Gamma^2_{11}\sigma_y + L\hat{\mathbf{n}} \\
\sigma_{xy} &= \Gamma^1_{12}\sigma_x + \Gamma^2_{12}\sigma_y + M\hat{\mathbf{n}} \\
\sigma_{yy} &= \Gamma^1_{22}\sigma_x + \Gamma^2_{22}\sigma_y + N\hat{\mathbf{n}}
\end{aligned}$$

$$\begin{aligned}
\sigma_{xx}.\sigma_x &= \Gamma^1_{11}\sigma_x.\sigma_x + \Gamma^2_{11}\sigma_y.\sigma_x + L\hat{\mathbf{n}}.\sigma_x \\
&= \Gamma^1_{11}E + \Gamma^2_{11}F
\end{aligned}$$

$$E_x = (\sigma_x.\sigma_x)_x = 2\sigma_{xx}.\sigma_x$$

Therefore,

$$E_x/2 = \Gamma^1_{11}E + \Gamma^2_{11}F$$

It may seem we got lucky here because of the double derivative was with respect to the same variable

$$\sigma_{xx} = \Gamma^1_{11}\sigma_x + \Gamma^2_{11}\sigma_y + L\hat{\mathbf{n}}$$
$$\sigma_{xy} = \Gamma^1_{12}\sigma_x + \Gamma^2_{12}\sigma_y + M\hat{\mathbf{n}}$$
$$\sigma_{yy} = \Gamma^1_{22}\sigma_x + \Gamma^2_{22}\sigma_y + N\hat{\mathbf{n}}$$

$$\begin{aligned}\sigma_{xx}.\sigma_y &= \Gamma^1_{11}\sigma_x.\sigma_y + \Gamma^2_{11}\sigma_y.\sigma_y + L\hat{\mathbf{n}}.\sigma_x \\ &= \Gamma^1_{11}F + \Gamma^2_{11}G\end{aligned}$$

$$\begin{aligned}\sigma_{xx}.\sigma_x &= \Gamma^1_{11}\sigma_x.\sigma_x + \Gamma^2_{11}\sigma_y.\sigma_x + L\hat{\mathbf{n}}.\sigma_x \\ &= \Gamma^1_{11}E + \Gamma^2_{11}F\end{aligned}$$

$$F_x = (\sigma_x.\sigma_y)_x = \sigma_{xx}.\sigma_y +$$

$$E_x = (\sigma_x.\sigma_x)_x = 2\sigma_{xx}.\sigma_x$$

Therefore,

$$E_x/2 = \Gamma^1_{11}E + \Gamma^2_{11}F$$

But, we can do the same thing, this time, exploiting the fact that mixed partial derivatives are equal

.

$$\sigma_{xx} = \Gamma^1_{11}\sigma_x + \Gamma^2_{11}\sigma_y + L\hat{\mathbf{n}}$$
$$\sigma_{xy} = \Gamma^1_{12}\sigma_x + \Gamma^2_{12}\sigma_y + M\hat{\mathbf{n}}$$
$$\sigma_{yy} = \Gamma^1_{22}\sigma_x + \Gamma^2_{22}\sigma_y + N\hat{\mathbf{n}}$$

$$\begin{aligned}\sigma_{xx}.\sigma_x &= \Gamma^1_{11}\sigma_x.\sigma_x + \Gamma^2_{11}\sigma_y.\sigma_x + L\hat{\mathbf{n}}.\sigma_x \\ &= \Gamma^1_{11}E + \Gamma^2_{11}F\end{aligned}$$

$$E_x = (\sigma_x.\sigma_x)_x = 2\sigma_{xx}.\sigma_x$$

Therefore,

$$E_x/2 = \Gamma^1_{11}E + \Gamma^2_{11}F$$

$$\begin{aligned}\sigma_{xx}.\sigma_y &= \Gamma^1_{11}\sigma_x.\sigma_y + \Gamma^2_{11}\sigma_y.\sigma_y + L\hat{\mathbf{n}}.\sigma_x \\ &= \Gamma^1_{11}F + \Gamma^2_{11}G\end{aligned}$$

$$F_x = (\sigma_x.\sigma_y)_x = \sigma_{xx}.\sigma_y +$$

Which gives a relation involving what we want to find out

$$\sigma_{xx} = \Gamma^1_{11}\sigma_x + \Gamma^2_{11}\sigma_y + L\hat{\mathbf{n}}$$
$$\sigma_{xy} = \Gamma^1_{12}\sigma_x + \Gamma^2_{12}\sigma_y + M\hat{\mathbf{n}}$$
$$\sigma_{yy} = \Gamma^1_{22}\sigma_x + \Gamma^2_{22}\sigma_y + N\hat{\mathbf{n}}$$

$$\begin{aligned}\sigma_{xx}.\sigma_x &= \Gamma^1_{11}\sigma_x.\sigma_x + \Gamma^2_{11}\sigma_y.\sigma_x + L\hat{\mathbf{n}}.\sigma_x \\ &= \Gamma^1_{11}E + \Gamma^2_{11}F\end{aligned}$$

$$E_x = (\sigma_x.\sigma_x)_x = 2\sigma_{xx}.\sigma_x$$

Therefore,

$$E_x/2 = \Gamma^1_{11}E + \Gamma^2_{11}F$$

$$\begin{aligned}\sigma_{xx}.\sigma_y &= \Gamma^1_{11}\sigma_x.\sigma_y + \Gamma^2_{11}\sigma_y.\sigma_y + L\hat{\mathbf{n}}.\sigma_x \\ &= \Gamma^1_{11}F + \Gamma^2_{11}G\end{aligned}$$

$$F_x = (\sigma_x.\sigma_y)_x = \sigma_{xx}.\sigma_y + \sigma_x \cdot \sigma_{yx}$$

but also another term

$$\sigma_{xx} = \Gamma^1_{11}\sigma_x + \Gamma^2_{11}\sigma_y + L\hat{\mathbf{n}}$$
$$\sigma_{xy} = \Gamma^1_{12}\sigma_x + \Gamma^2_{12}\sigma_y + M\hat{\mathbf{n}}$$
$$\sigma_{yy} = \Gamma^1_{22}\sigma_x + \Gamma^2_{22}\sigma_y + N\hat{\mathbf{n}}$$

$$\begin{aligned}\sigma_{xx}.\sigma_y &= \Gamma^1_{11}\sigma_x.\sigma_y + \Gamma^2_{11}\sigma_y.\sigma_y + L\hat{\mathbf{n}}.\sigma_x \\ &= \Gamma^1_{11}F + \Gamma^2_{11}G\end{aligned}$$

$$\begin{aligned}\sigma_{xx}.\sigma_x &= \Gamma^1_{11}\sigma_x.\sigma_x + \Gamma^2_{11}\sigma_y.\sigma_x + L\hat{\mathbf{n}}.\sigma_x \\ &= \Gamma^1_{11}E + \Gamma^2_{11}F\end{aligned}$$

$$F_x = (\sigma_x.\sigma_y)_x = \sigma_{xx}.\sigma_y + \sigma_x \cdot \sigma_{yx}$$

$$E_x = (\sigma_x.\sigma_x)_x = 2\sigma_{xx}.\sigma_x$$

Therefore,

$$E_x/2 = \Gamma^1_{11}E + \Gamma^2_{11}F$$

There is no point making it appear in product rule applied to $(\sigma_x.\sigma_y)_x$

$$\sigma_{xx} = \Gamma^1_{11}\sigma_x + \Gamma^2_{11}\sigma_y + L\hat{\mathbf{n}}$$
$$\sigma_{xy} = \Gamma^1_{12}\sigma_x + \Gamma^2_{12}\sigma_y + M\hat{\mathbf{n}}$$
$$\sigma_{yy} = \Gamma^1_{22}\sigma_x + \Gamma^2_{22}\sigma_y + N\hat{\mathbf{n}}$$

$$\begin{aligned}\sigma_{xx}.\sigma_y &= \Gamma^1_{11}\sigma_x.\sigma_y + \Gamma^2_{11}\sigma_y.\sigma_y + L\hat{\mathbf{n}}.\sigma_x \\ &= \Gamma^1_{11}F + \Gamma^2_{11}G\end{aligned}$$

$$\begin{aligned}\sigma_{xx}.\sigma_x &= \Gamma^1_{11}\sigma_x.\sigma_x + \Gamma^2_{11}\sigma_y.\sigma_x + L\hat{\mathbf{n}}.\sigma_x \\ &= \Gamma^1_{11}E + \Gamma^2_{11}F\end{aligned}$$

$$F_x = (\sigma_x.\sigma_y)_x = \sigma_{xx}.\sigma_y + \sigma_x \cdot \sigma_{yx}$$

$$E_x = (\sigma_x.\sigma_x)_x = 2\sigma_{xx}.\sigma_x$$

Therefore,

$$E_x/2 = \Gamma^1_{11}E + \Gamma^2_{11}F$$

The above equation already extracts everything we can from it

$$\sigma_{xx} = \Gamma^1_{11}\sigma_x + \Gamma^2_{11}\sigma_y + L\hat{\mathbf{n}}$$
$$\sigma_{xy} = \Gamma^1_{12}\sigma_x + \Gamma^2_{12}\sigma_y + M\hat{\mathbf{n}}$$
$$\sigma_{yy} = \Gamma^1_{22}\sigma_x + \Gamma^2_{22}\sigma_y + N\hat{\mathbf{n}}$$

$$\begin{aligned}\sigma_{xx}.\sigma_x &= \Gamma^1_{11}\sigma_x.\sigma_x + \Gamma^2_{11}\sigma_y.\sigma_x + L\hat{\mathbf{n}}.\sigma_x \\ &= \Gamma^1_{11}E + \Gamma^2_{11}F\end{aligned}$$

$$E_x = (\sigma_x.\sigma_x)_x = 2\sigma_{xx}.\sigma_x$$

Therefore,

$$E_x/2 = \Gamma^1_{11}E + \Gamma^2_{11}F$$

$$\begin{aligned}\sigma_{xx}.\sigma_y &= \Gamma^1_{11}\sigma_x.\sigma_y + \Gamma^2_{11}\sigma_y.\sigma_y + L\hat{\mathbf{n}}.\sigma_x \\ &= \Gamma^1_{11}F + \Gamma^2_{11}G\end{aligned}$$

$$F_x = (\sigma_x.\sigma_y)_x = \sigma_{xx}.\sigma_y + \sigma_x \cdot \sigma_{yx}$$
$$E_y = (\sigma_x.\sigma_x)_y = 2\sigma_x.\sigma_{xx}$$

But $\sigma_x \cdot \sigma_{yx} = \sigma_x \cdot \sigma_{xy}$ because mixed partial derivatives are equal

$$\sigma_{xx} = \Gamma^1_{11}\sigma_x + \Gamma^2_{11}\sigma_y + L\hat{\mathbf{n}}$$
$$\sigma_{xy} = \Gamma^1_{12}\sigma_x + \Gamma^2_{12}\sigma_y + M\hat{\mathbf{n}}$$
$$\sigma_{yy} = \Gamma^1_{22}\sigma_x + \Gamma^2_{22}\sigma_y + N\hat{\mathbf{n}}$$

$$\begin{aligned}\sigma_{xx}.\sigma_x &= \Gamma^1_{11}\sigma_x.\sigma_x + \Gamma^2_{11}\sigma_y.\sigma_x + L\hat{\mathbf{n}}.\sigma_x \\ &= \Gamma^1_{11}E + \Gamma^2_{11}F\end{aligned}$$

$$E_x = (\sigma_x.\sigma_x)_x = 2\sigma_{xx}.\sigma_x$$

Therefore,

$$E_x/2 = \Gamma^1_{11}E + \Gamma^2_{11}F$$

$$\begin{aligned}\sigma_{xx}.\sigma_y &= \Gamma^1_{11}\sigma_x.\sigma_y + \Gamma^2_{11}\sigma_y.\sigma_y + L\hat{\mathbf{n}}.\sigma_x \\ &= \Gamma^1_{11}F + \Gamma^2_{11}G\end{aligned}$$

$$F_x = (\sigma_x.\sigma_y)_x = \sigma_{xx}.\sigma_y + \sigma_x \cdot \sigma_{yx}$$
$$E_y = (\sigma_x.\sigma_x)_y = 2\sigma_x.\sigma_{xx}$$

$$F_x - E_y/2 = \Gamma^1_{11}F + \Gamma^2_{11}G$$

So we finally obtain the relation

$$\sigma_{xx} = \Gamma^1_{11}\sigma_x + \Gamma^2_{11}\sigma_y + L\hat{\mathbf{n}}$$
$$\sigma_{xy} = \Gamma^1_{12}\sigma_x + \Gamma^2_{12}\sigma_y + M\hat{\mathbf{n}}$$
$$\sigma_{yy} = \Gamma^1_{22}\sigma_x + \Gamma^2_{22}\sigma_y + N\hat{\mathbf{n}}$$

$$\begin{aligned}\sigma_{xx}.\sigma_x &= \Gamma^1_{11}\sigma_x.\sigma_x + \Gamma^2_{11}\sigma_y.\sigma_x + L\hat{\mathbf{n}}.\sigma_x\\ &= \Gamma^1_{11}E + \Gamma^2_{11}F\end{aligned}$$

$$E_x = (\sigma_x.\sigma_x)_x = 2\sigma_{xx}.\sigma_x$$

Therefore,

$$E_x/2 = \Gamma^1_{11}E + \Gamma^2_{11}F$$

$$\begin{aligned}\sigma_{xx}.\sigma_y &= \Gamma^1_{11}\sigma_x.\sigma_y + \Gamma^2_{11}\sigma_y.\sigma_y + L\hat{\mathbf{n}}.\sigma_x\\ &= \Gamma^1_{11}F + \Gamma^2_{11}G\end{aligned}$$

$$F_x = (\sigma_x.\sigma_y)_x = \sigma_{xx}.\sigma_y + \sigma_x \cdot \sigma_{yx}$$
$$E_y = (\sigma_x.\sigma_x)_y = 2\sigma_x.\sigma_{xx}$$

$$F_x - E_y/2 = \Gamma^1_{11}F + \Gamma^2_{11}G$$

Proposition. *The Christoffel symbols depend only on the first fundamental form.*

Similarly, we can obtain all the relations and solve

$$\sigma_{xx} = \Gamma^1_{11}\sigma_x + \Gamma^2_{11}\sigma_y + L\hat{\mathbf{n}}$$
$$\sigma_{xy} = \Gamma^1_{12}\sigma_x + \Gamma^2_{12}\sigma_y + M\hat{\mathbf{n}}$$
$$\sigma_{yy} = \Gamma^1_{22}\sigma_x + \Gamma^2_{22}\sigma_y + N\hat{\mathbf{n}}$$

$$\begin{aligned}\sigma_{xx}.\sigma_x &= \Gamma^1_{11}\sigma_x.\sigma_x + \Gamma^2_{11}\sigma_y.\sigma_x + L\hat{\mathbf{n}}.\sigma_x \\ &= \Gamma^1_{11}E + \Gamma^2_{11}F\end{aligned}$$

$$E_x = (\sigma_x.\sigma_x)_x = 2\sigma_{xx}.\sigma_x$$

Therefore,

$$E_x/2 = \Gamma^1_{11}E + \Gamma^2_{11}F$$

$$\begin{aligned}\sigma_{xx}.\sigma_y &= \Gamma^1_{11}\sigma_x.\sigma_y + \Gamma^2_{11}\sigma_y.\sigma_y + L\hat{\mathbf{n}}.\sigma_x \\ &= \Gamma^1_{11}F + \Gamma^2_{11}G\end{aligned}$$

$$F_x = (\sigma_x.\sigma_y)_x = \sigma_{xx}.\sigma_y + \sigma_x \cdot \sigma_{yx}$$
$$E_y = (\sigma_x.\sigma_x)_y = 2\sigma_x.\sigma_{xx}$$

$$F_x - E_y/2 = \Gamma^1_{11}F + \Gamma^2_{11}G$$

Proposition. *The Christoffel symbols depend only on the first fundamental form.*

The derivatives of any (tangent) vector field to a surface, can be resolved in two components

$$\begin{aligned}\sigma_{xx} &= \Gamma^1_{11}\sigma_x + \Gamma^2_{11}\sigma_y + L\hat{\mathbf{n}}\\ \sigma_{xy} &= \Gamma^1_{12}\sigma_x + \Gamma^2_{12}\sigma_y + M\hat{\mathbf{n}}\\ \sigma_{yy} &= \Gamma^1_{22}\sigma_x + \Gamma^2_{22}\sigma_y + N\hat{\mathbf{n}}\end{aligned}$$

$$\begin{aligned}\sigma_{xx}.\sigma_x &= \Gamma^1_{11}\sigma_x.\sigma_x + \Gamma^2_{11}\sigma_y.\sigma_x + L\hat{\mathbf{n}}.\sigma_x\\ &= \Gamma^1_{11}E + \Gamma^2_{11}F\end{aligned}$$

$$E_x = (\sigma_x.\sigma_x)_x = 2\sigma_{xx}.\sigma_x$$

Therefore,

$$E_x/2 = \Gamma^1_{11}E + \Gamma^2_{11}F$$

$$\begin{aligned}\sigma_{xx}.\sigma_y &= \Gamma^1_{11}\sigma_x.\sigma_y + \Gamma^2_{11}\sigma_y.\sigma_y + L\hat{\mathbf{n}}.\sigma_x\\ &= \Gamma^1_{11}F + \Gamma^2_{11}G\end{aligned}$$

$$\begin{aligned}F_x &= (\sigma_x.\sigma_y)_x = \sigma_{xx}.\sigma_y + \sigma_x \cdot \sigma_{yx}\\ E_y &= (\sigma_x.\sigma_x)_y = 2\sigma_x.\sigma_{xx}\end{aligned}$$

$$F_x - E_y/2 = \Gamma^1_{11}F + \Gamma^2_{11}G$$

Proposition. *The Christoffel symbols depend only on the first fundamental form.*

The component in the direction of the normal, which will depend only on the second fundamental form

$$\sigma_{xx} = \Gamma^1_{11}\sigma_x + \Gamma^2_{11}\sigma_y + L\hat{\mathbf{n}}$$
$$\sigma_{xy} = \Gamma^1_{12}\sigma_x + \Gamma^2_{12}\sigma_y + M\hat{\mathbf{n}}$$
$$\sigma_{yy} = \Gamma^1_{22}\sigma_x + \Gamma^2_{22}\sigma_y + N\hat{\mathbf{n}}$$

$$\begin{aligned}\sigma_{xx}.\sigma_x &= \Gamma^1_{11}\sigma_x.\sigma_x + \Gamma^2_{11}\sigma_y.\sigma_x + L\hat{\mathbf{n}}.\sigma_x \\ &= \Gamma^1_{11}E + \Gamma^2_{11}F\end{aligned}$$

$$E_x = (\sigma_x.\sigma_x)_x = 2\sigma_{xx}.\sigma_x$$

Therefore,

$$E_x/2 = \Gamma^1_{11}E + \Gamma^2_{11}F$$

$$\begin{aligned}\sigma_{xx}.\sigma_y &= \Gamma^1_{11}\sigma_x.\sigma_y + \Gamma^2_{11}\sigma_y.\sigma_y + L\hat{\mathbf{n}}.\sigma_x \\ &= \Gamma^1_{11}F + \Gamma^2_{11}G\end{aligned}$$

$$F_x = (\sigma_x.\sigma_y)_x = \sigma_{xx}.\sigma_y + \sigma_x \cdot \sigma_{yx}$$
$$E_y = (\sigma_x.\sigma_x)_y = 2\sigma_x.\sigma_{xx}$$

$$F_x - E_y/2 = \Gamma^1_{11}F + \Gamma^2_{11}G$$

Proposition. *The Christoffel symbols depend only on the first fundamental form.*

The other component is tangent to the surface and will depend only on the first fundamental form.

$$\sigma_{xx} = \Gamma^1_{11}\sigma_x + \Gamma^2_{11}\sigma_y + L\hat{\mathbf{n}}$$
$$\sigma_{xy} = \Gamma^1_{12}\sigma_x + \Gamma^2_{12}\sigma_y + M\hat{\mathbf{n}}$$
$$\sigma_{yy} = \Gamma^1_{22}\sigma_x + \Gamma^2_{22}\sigma_y + N\hat{\mathbf{n}}$$

So,

$$\sigma_{xyy} = \text{an expression involving second derivatives of } \sigma_x, \sigma_y, \hat{\mathbf{n}}$$

Now that we can all first derivatives in terms of the basis, we can repeatedly do it for others

$$\sigma_{xx} = \Gamma^1_{11}\sigma_x + \Gamma^2_{11}\sigma_y + L\hat{\mathbf{n}}$$
$$\sigma_{xy} = \Gamma^1_{12}\sigma_x + \Gamma^2_{12}\sigma_y + M\hat{\mathbf{n}}$$
$$\sigma_{yy} = \Gamma^1_{22}\sigma_x + \Gamma^2_{22}\sigma_y + N\hat{\mathbf{n}}$$

So,

$$\begin{aligned} \sigma_{xyy} &= \text{an expression involving second derivatives of } \sigma_x, \sigma_y, \hat{\mathbf{n}} \\ &= \text{an expression in terms of } \sigma_x, \sigma_y, \hat{\mathbf{n}} \end{aligned}$$

$\blacksquare$

$\sigma_{xx} = \Gamma^1_{11}\sigma_x + \Gamma^2_{11}\sigma_y + L\hat{\mathbf{n}}$
$\sigma_{xy} = \Gamma^1_{12}\sigma_x + \Gamma^2_{12}\sigma_y + M\hat{\mathbf{n}}$
$\sigma_{yy} = \Gamma^1_{22}\sigma_x + \Gamma^2_{22}\sigma_y + N\hat{\mathbf{n}}$

So,

$$\begin{aligned}\sigma_{xyy} &= \text{an expression involving second derivatives of } \sigma_x, \sigma_y, \hat{\mathbf{n}} \\ &= \text{an expression in terms of } \sigma_x, \sigma_y, \hat{\mathbf{n}}\end{aligned}$$

But $\sigma_{xyy} = \sigma_{yyx}$

Similarly,

$$\sigma_{yyx} = \text{some other expression involving second derivatives of } \sigma_x, \sigma_y, \hat{\mathbf{n}}$$

$\blacksquare$

$$\sigma_{xx} = \Gamma_{11}^1 \sigma_x + \Gamma_{11}^2 \sigma_y + L\hat{\mathbf{n}}$$
$$\sigma_{xy} = \Gamma_{12}^1 \sigma_x + \Gamma_{12}^2 \sigma_y + M\hat{\mathbf{n}}$$
$$\sigma_{yy} = \Gamma_{22}^1 \sigma_x + \Gamma_{22}^2 \sigma_y + N\hat{\mathbf{n}}$$

So,

$$\begin{aligned} \sigma_{xyy} &= \text{an expression involving second derivatives of } \sigma_x, \sigma_y, \hat{\mathbf{n}} \\ &= \text{an expression in terms of } \sigma_x, \sigma_y, \hat{\mathbf{n}} \end{aligned}$$

But $\sigma_{xyy} = \sigma_{yyx}$

Similarly,

$$\begin{aligned} \sigma_{yyx} &= \text{some other expression involving second derivatives of } \sigma_x, \sigma_y, \hat{\mathbf{n}} \\ &= \text{some other expression in terms of } \sigma_x, \sigma_y, \hat{\mathbf{n}} \end{aligned}$$

$\blacksquare$

$$\sigma_{xx} = \Gamma^1_{11}\sigma_x + \Gamma^2_{11}\sigma_y + L\hat{\mathbf{n}}$$
$$\sigma_{xy} = \Gamma^1_{12}\sigma_x + \Gamma^2_{12}\sigma_y + M\hat{\mathbf{n}}$$
$$\sigma_{yy} = \Gamma^1_{22}\sigma_x + \Gamma^2_{22}\sigma_y + N\hat{\mathbf{n}}$$

So,

$$\begin{aligned}\sigma_{xyy} &= \text{an expression involving second derivatives of } \sigma_x, \sigma_y, \hat{\mathbf{n}} \\ &= \text{an expression in terms of } \sigma_x, \sigma_y, \hat{\mathbf{n}}\end{aligned}$$

But $\sigma_{xyy} = \sigma_{yyx}$

Similarly,

$$\begin{aligned}\sigma_{yyx} &= \text{some other expression involving second derivatives of } \sigma_x, \sigma_y, \hat{\mathbf{n}} \\ &= \text{some other expression in terms of } \sigma_x, \sigma_y, \hat{\mathbf{n}}\end{aligned}$$

Equating coeffients,

Theorem (Gauss' Theorem Egregium)**.**

$$\det\begin{pmatrix} L & M \\ M & N \end{pmatrix}$$

can be expressed entirely in terms of the Christoffel symbols (i.e. first fundamental form)

$\blacksquare$