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Show that $\dot{\gamma}(t) = \underbrace{s'(t)}_{\text{"Change in distance"}} \times \mathbf{T}(t)$.

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$\gamma : (\alpha, \beta) \to \mathbb{R}^2$ a regular parametrization.
$\mathbf{T}(t) = \frac{\dot{\gamma}(t)}{\|\dot{\gamma}(t)\|}$ is the *unit tangent vector* at t.

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$\gamma : (\alpha, \beta) \to \mathbb{R}^2$ a regular parametrization.
$\tilde{\gamma} : (\tilde{\alpha}, \tilde{\beta}) \to \mathbb{R}^2$ a unit speed reparametrization.
Show that,
$\dot{\tilde{\gamma}}(\tilde{t}) = \mathbf{T}(s^{-1}(\tilde{t}))$

Recall,
If $g(f(t)) = t$, then
$g'(f(t))f'(t) = 1$, therefore, if $f'(t) \neq 0$ for any t,

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Exercise. Show that the curvature at any point of any line

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Exercise. Show that the curvature at any point of any line is 0.

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$$\begin{aligned}\kappa(t) &= \|\ddot{\tilde{\gamma}}(s(t))\| \\ &= \left\| \frac{\ddot{\gamma}(t) - \dot{\tilde{\gamma}}(s(t))\frac{\dot{\gamma}(t).\ddot{\gamma}(t)}{\|\dot{\gamma}(t)\|}}{\|\dot{\gamma}(t)\|^2} \right\|\end{aligned}$$

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$$\begin{aligned}
\kappa(t) &= \|\ddot{\tilde{\gamma}}(s(t))\| \\
&= \left\| \frac{\ddot{\gamma}(t) - \dot{\tilde{\gamma}}(s(t))\frac{\dot{\gamma}(t).\ddot{\gamma}(t)}{\|\dot{\gamma}(t)\|}}{\|\dot{\gamma}(t)\|^2} \right\| \\
&= \left\| \frac{\ddot{\gamma}(t) - \frac{\dot{\gamma}(t)}{\|\dot{\gamma}(t)\|}\frac{\dot{\gamma}(t).\ddot{\gamma}(t)}{\|\dot{\gamma}(t)\|}}{\|\dot{\gamma}(t)\|^2} \right\|
\end{aligned}$$