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Show that $\dot{\gamma}(t) = s'(t)$

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$g'(f(t))f'(t) = 1$, therefore, if $f'(t) \neq 0$ for any t,

$$\boxed{g'(f(t)) = \frac{1}{f'(t)}}$$

Taking $f(t) = s(t)$ and $g(t) = s^{-1}(t)$

$$(s^{-1})'(s(t)) = \frac{1}{s'(t)} = \frac{1}{\|\gamma'(t)\|}$$

$$\boxed{(s^{-1})'(\tilde{t}) = \frac{1}{s'(s^{-1}(\tilde{t}))} = \frac{1}{\|\dot{\gamma}(s^{-1}(\tilde{t}))\|}}$$

$\square$

Definition.

$\gamma : (\alpha, \beta) \to \mathbb{R}^2$ a regular parametrization.

$\mathbf{T}(t) = \frac{\dot\gamma(t)}{\|\dot\gamma(t)\|}$ is the *unit tangent vector* at t.

Exercise.

$\gamma : (\alpha, \beta) \to \mathbb{R}^2$ a regular parametrization.

$\tilde\gamma : (\tilde\alpha, \tilde\beta) \to \mathbb{R}^2$ a unit speed reparametrization.

Show that,

$\dot{\tilde\gamma}(\tilde t) = \mathbf{T}(s^{-1}(\tilde t))$

Solution. $\tilde\gamma(\widetilde{t}) = \gamma(s^{-1}(\tilde t))$

$$
\begin{aligned}
\dot{\tilde\gamma}(\widetilde{t}) &= \dot\gamma(s^{-1}(\tilde t))(s^{-1}(\tilde t))' \\
&= \dot\gamma(s^{-1}(\tilde t))\frac{1}{\|\dot\gamma(s^{-1}(\tilde t))\|}
\end{aligned}
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$\gamma : (\alpha, \beta) \to \mathbb{R}^2$ a regular parametrization.
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Show that,
$\dot{\tilde{\gamma}}(\tilde{t}) = \mathbf{T}(s^{-1}(\tilde{t}))$

Solution. $\tilde{\gamma}(\widetilde{t}) = \gamma(s^{-1}(\tilde{t}))$

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If $\|\dot{\gamma}(t)\|$ is a constant c, show that $t - s(t)/c$ is a constant. Here $s(t)$ is the arc-length function "measured from (fixed) t_0".

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Definition. $\gamma : (\alpha, \beta) \to \mathbb{R}^2$

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Exercise. Show that the curvature at any point of any line

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$\kappa(t) = \|\ddot{\gamma}(t)\| = 1/r$

Exercise. Show that the curvature at any point of any line is 0.

Deriving a general formula for curvature

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$$\begin{aligned}\kappa(t) &= \|\ddot{\tilde{\gamma}}(s(t))\| \\ &= \left\|\frac{\ddot{\gamma}(t) - \dot{\tilde{\gamma}}(s(t))\frac{\dot{\gamma}(t).\ddot{\gamma}(t)}{\|\dot{\gamma}(t)\|}}{\|\dot{\gamma}(t)\|^2}\right\|\end{aligned}$$

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$$\begin{aligned}
\kappa(t) &= \|\ddot{\tilde{\gamma}}(s(t))\| \\
&= \left\| \frac{\ddot{\gamma}(t) - \dot{\tilde{\gamma}}(s(t))\frac{\dot{\gamma}(t).\ddot{\gamma}(t)}{\|\dot{\gamma}(t)\|}}{\|\dot{\gamma}(t)\|^2} \right\| \\
&= \left\| \frac{\ddot{\gamma}(t) - \frac{\dot{\gamma}(t)}{\|\dot{\gamma}(t)\|}\frac{\dot{\gamma}(t).\ddot{\gamma}(t)}{\|\dot{\gamma}(t)\|}}{\|\dot{\gamma}(t)\|^2} \right\|
\end{aligned}$$