Exercise sheet 1

Curves and Surfaces, MTH201

1. Find a parametrization $\gamma(t)$ for a line segment joining two given points $(x_1.y_1)$ and $(x_2.y_2)$. Find $\dot{\gamma}(t)$.

2. What does the parametrization trace out $\gamma(t) = (\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2})$?

3. Show that the parametrization $\gamma(t) := (t^2-1, t(t^2-1))$ is not injective, i.e. there are two *distinct* real numbers t_1 and t_2 so that $\gamma(t_1) = \gamma(t_2)$. Can you deduce the shape[1] of this curve? Can you express the set of points defined by this curve as the zero set[2] of some function $f(x, y)$?

4. For $\mathbf{v} : (\alpha, \beta) \to \mathbf{R}^2$ and $\mathbf{w} : (\alpha, \beta) \to \mathbf{R}^2$, show that $(\mathbf{v}(t).\mathbf{w}(t))' = \mathbf{v}'(t).\mathbf{w}(t) + \mathbf{v}(t).\mathbf{w}'(t)$.

5. If $\mathbf{n} : (\alpha, \beta) \to \mathbf{R}^2$ is such that $||\mathbf{n}(t)||$ is constant, then prove that $\dot{\mathbf{n}}(t)$ is either 0 or perpendicular to $\mathbf{n}(t)$.

6. if we denote,
$$s_\alpha(t) := \int_{t_\alpha}^{t} ||\dot{\gamma}(u)||\mathrm{d}u$$
$$s_\beta(t) := \int_{t_\beta}^{t} ||\dot{\gamma}(u)||\mathrm{d}u$$
prove that $s_\beta(t) - s_\alpha(t)$ is a constant **(assume that $t_\alpha < t_\beta$)**.

7. If $\gamma : (\alpha, \beta) \to \mathbb{R}^2$ is a smooth **and regular** parametrization, then show that $||\dot{\gamma}(t)|| : (\alpha, \beta) \to \mathbb{R}$ is smooth.

8. For the parametrization $\gamma : (-\pi/2, \pi/2) \to \mathbb{R}^2$ given by $\gamma(t) = (5\cos(t), 5\sin(t))$,
   (a) Find the arc-length function $s(t)$ (starting at, say, 0)
   (b) Find a reparametrization map ϕ so that $\gamma(\phi(t))$ is a unit-speed parametrization.

[1] Just a rough drawing showing where the curve intersects the axes and where it self-intersects etc.

[2] The zero set of a function $f : \mathbb{R}^2 \to \mathbb{R}$ is $\{(x, y) \mid f(x, y) = 0\}$, i.e. the set of points (x, y) in the plane for which $f(x, y) = 0$