Hints / Solutions to Exercise sheet 1

Curves and Surfaces, MTH201

Question 1: Find a parametrization $\gamma(t)$ for a line segment joining two given points (x_1, y_1) and (x_2, y_2). Find $\dot{\gamma}(t)$.

Solution 1: The line points in the direction, $\mathbf{v} = (x_2, y_2) - (x_1, y_1) = (x_2 - x_1, y_2 - y_1)$. Every other point on the line is a translate of (x_1, y_2) by scalar multiples of $\mathbf{v}$, i.e. $\gamma(t) = (x_1, y_1) + t\mathbf{v}$. When $t = 0$, we get the point (x_1, x_2) and when $t = 1$, we get the translate of (x_1, y_1) by $\mathbf{v}$, so we restrict the domain to $(0, 1)$ to get $\gamma : (0, 1) \to \mathbb{R}^2$.

Question 2: What does the parametrization trace out $\gamma(t) = (\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2})$?

Solution 2: Observe that $(\frac{2t}{1+t^2})^2 + (\frac{1-t^2}{1+t^2})^2 = 1$, and, therefore, the image of the parametrization is a subset of $\{(x, y) \mid x^2 + y^2 = 1\}$, which is a circle. The parametrization is continuous so it will trace out an arc of the circle. The arc will be determined by the domain.

This shows that the circle can also be parametrized by a parametrization which uses only polynomials, more precisely "rational functions", i.e. functions that can be represented by quotients of polynomials.

Question 3: Show that the parametrization $\gamma(t) := (t^2 - 1, t(t^2 - 1))$ is not injective

Solution 3: $\gamma(1) = (0, 0) = \gamma(-1)$

Question 3: Express the curve traced out by the parametrization $\gamma(t) := (t^2 - 1, t(t^2 - 1))$ as the zero set of some function.

Solution 3: We will try to "eliminate the variable" t, which in this case is doable (but not always!)

$x = t^2 - 1$ and $y = t(t^2 - 1)$, so $y/x = t$ as long as $t \neq \pm 1$ (because we cannot divide by 0!). Now, plugging in $t = y/x$ in $x = t^2 - 1$, we get $x = (y/x)^2 - 1$, and so $x^3 = y^2 - x^2$. We know this holds for all t except ± 1, but for $t = \pm 1$ we get the point $(0, 0)$. We can easily check that this also satisfies $x^3 = y^2 - x^2$.