Exercise sheet 8

Curves and Surfaces, MTH201

1. Let $f : S_1 \to S_2$ denote a smooth function that between surfaces that is 1-1, onto, its inverse is smooth, and so that $f^*\langle \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{v}, \mathbf{w} \rangle$ (called a local isometry) and let $\sigma_1 : U \to S_1$ denote a surface patch on S_1. Let $\sigma_2 = f \circ \sigma_1$
   (a) Show that $(\sigma_2)_x = D_p(f)(\sigma_1)_x$ and $(\sigma_2)_y = D_p(f)(\sigma_1)_y$
   (b) Show that if $(\sigma_1)_x \times (\sigma_1)_y \neq 0$, then $(\sigma_2)_x \times (\sigma_2)_y \neq 0$
   (c) We can then treat σ_2 as a surface patch for S_2. Show that if E_1, F_1, G_1 denote the entries of the matrix of the first fundamental form with respect to σ_1 and E_2, F_2, G_2 denote the entries of the matrix of the first fundamental form with respect to σ_2, then $E_1 = E_2$, $F_1 = F_2$, and $G_1 = G_3$.
   (d) Why does f map geodesics to geodesics?
2. Prove that the geodesic curvature of a curve in a plane (treated as a surface in $\mathbb{R}^3$) is equal to the plane curvature.
3. Compute the normal curvature of any curve on the sphere. Can you interpret the answer physically? Using this, prove that curves on the sphere that have constant geodesic curvature are circles.