

Dicussion before the lecture

Reparametrization example by “shifting the phase”

$$\dot{\tilde{\gamma}}(t) = \dot{\gamma}(\phi(t))\phi'(t)$$

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$$\gamma(t) := (\cos(t), \sin(t))$$

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$$\dot{\tilde{\gamma}}(t) = \dot{\gamma}(\phi(t))\phi'(t)$$

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$$\tilde{\gamma}(t) := (\cos(t + \pi/2), \sin(t + \pi/2))$$

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$$\tilde{\gamma}(t) := \gamma(\phi(t))$$

Regular parametrization

Definition.

The point $\gamma(t)$

Regular parametrization

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The point $\gamma(t)$ of γ :

Regular parametrization

Definition.

The point $\gamma(t)$ of $\gamma : (\alpha, \beta) \rightarrow$

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The point $\gamma(t)$ of $\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$

Regular parametrization

Definition.

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Example. $\gamma(t) = (t^3, t^3)$ is singular at 0 because $\dot{\gamma}(t) = (3t^2, 3t^2)$.

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Proof.

$$\psi(\phi(t)) = t$$

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$\psi(\phi(t)) = t$ for *each* t

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Proof.

$\psi(\phi(t)) = t$ for *each* t

$\psi'(\phi(t))\phi'(t) = 1$ for *each* t

So, $\psi'(\phi(t)) \neq 0$ and $\phi'(t) \neq 0$ for *each* t

So, $\psi'(t) \neq 0$ and $\phi'(t) \neq 0$ for *each* t because ϕ is bijective. \square

Regular parametrization

Proposition. *A reparametrization of a regular parametrization*

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The point $\gamma(t)$ of $\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$ is regular point if $\dot{\gamma}(t) \neq 0$ and singular if $\dot{\gamma}(t) = 0$. The parametrization γ is called regular if $\dot{\gamma}(t) \neq 0$ for *every* $t \in (\alpha, \beta)$.

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The point $\gamma(t)$ of $\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$ is regular point if *Proof.*

$\dot{\gamma}(t) \neq 0$ and singular if $\dot{\gamma}(t) = 0$. The parametrization $\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$
 γ is called regular if $\dot{\gamma}(t) \neq 0$ for every $t \in (\alpha, \beta)$. $\tilde{\gamma} : (\tilde{\alpha}, \tilde{\beta}) \rightarrow \mathbb{R}^2$

□

Example. $\gamma(t) = (t^3, t^3)$ is singular at 0 because $\dot{\gamma}(t) = (3t^2, 3t^2)$.

Lemma.

If $\phi : (\tilde{\alpha}, \tilde{\beta}) \rightarrow (\alpha, \beta)$ is invertible with inverse $\psi : (\alpha, \beta) \rightarrow (\tilde{\alpha}, \tilde{\beta})$, then $\phi'(t) \neq 0$ for all $t \in (\tilde{\alpha}, \tilde{\beta})$ and $\psi'(t) \neq 0$ for all $t \in (\alpha, \beta)$.

Proof.

$\psi(\phi(t)) = t$ for each t

$\psi'(\phi(t))\phi'(t) = 1$ for each t

So, $\psi'(\phi(t)) \neq 0$ and $\phi'(t) \neq 0$ for each t

So, $\psi'(t) \neq 0$ and $\phi'(t) \neq 0$ for each t because ϕ is bijective. □

Regular parametrization

Definition.

The point $\gamma(t)$ of $\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$ is regular point if $\dot{\gamma}(t) \neq 0$ and singular if $\dot{\gamma}(t) = 0$. The parametrization γ is called regular if $\dot{\gamma}(t) \neq 0$ for every $t \in (\alpha, \beta)$.

Example. $\gamma(t) = (t^3, t^3)$ is singular at 0 because $\dot{\gamma}(t) = (3t^2, 3t^2)$.

Lemma.

If $\phi : (\tilde{\alpha}, \tilde{\beta}) \rightarrow (\alpha, \beta)$ is invertible with inverse $\psi : (\alpha, \beta) \rightarrow (\tilde{\alpha}, \tilde{\beta})$, then $\phi'(t) \neq 0$ for all $t \in (\tilde{\alpha}, \tilde{\beta})$ and $\psi'(t) \neq 0$ for all $t \in (\alpha, \beta)$.

Proof.

$\psi(\phi(t)) = t$ for each t
 $\psi'(\phi(t))\phi'(t) = 1$ for each t
So, $\psi'(\phi(t)) \neq 0$ and $\phi'(t) \neq 0$ for each t
So, $\psi'(t) \neq 0$ and $\phi'(t) \neq 0$ for each t because ϕ is bijective. □

Proposition. A reparametrization of a regular parametrization is regular.

Proof.

$$\begin{aligned} \gamma &: (\alpha, \beta) \rightarrow \mathbb{R}^2 \\ \tilde{\gamma} &: (\tilde{\alpha}, \tilde{\beta}) \rightarrow \mathbb{R}^2 \\ \tilde{\gamma}(\tilde{t}) \end{aligned}$$

□

Regular parametrization

Definition.

The point $\gamma(t)$ of $\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$ is regular point if $\dot{\gamma}(t) \neq 0$ and singular if $\dot{\gamma}(t) = 0$. The parametrization γ is called regular if $\dot{\gamma}(t) \neq 0$ for every $t \in (\alpha, \beta)$.

Example. $\gamma(t) = (t^3, t^3)$ is singular at 0 because $\dot{\gamma}(t) = (3t^2, 3t^2)$.

Lemma.

If $\phi : (\tilde{\alpha}, \tilde{\beta}) \rightarrow (\alpha, \beta)$ is invertible with inverse $\psi : (\alpha, \beta) \rightarrow (\tilde{\alpha}, \tilde{\beta})$, then $\phi'(t) \neq 0$ for all $t \in (\tilde{\alpha}, \tilde{\beta})$ and $\psi'(t) \neq 0$ for all $t \in (\alpha, \beta)$.

Proof.

$\psi(\phi(t)) = t$ for each t
 $\psi'(\phi(t))\phi'(t) = 1$ for each t
So, $\psi'(\phi(t)) \neq 0$ and $\phi'(t) \neq 0$ for each t
So, $\psi'(t) \neq 0$ and $\phi'(t) \neq 0$ for each t because ϕ is bijective. □

Proposition. A reparametrization of a regular parametrization is regular.

Proof.

$$\begin{aligned} \gamma &: (\alpha, \beta) \rightarrow \mathbb{R}^2 \\ \tilde{\gamma} &: (\tilde{\alpha}, \tilde{\beta}) \rightarrow \mathbb{R}^2 \\ \tilde{\gamma}(\tilde{t}) &= \gamma(\phi(\tilde{t})) \end{aligned}$$

□

Regular parametrization

Definition.

The point $\gamma(t)$ of $\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$ is regular point if $\dot{\gamma}(t) \neq 0$ and singular if $\dot{\gamma}(t) = 0$. The parametrization γ is called regular if $\dot{\gamma}(t) \neq 0$ for every $t \in (\alpha, \beta)$.

Example. $\gamma(t) = (t^3, t^3)$ is singular at 0 because $\dot{\gamma}(t) = (3t^2, 3t^2)$.

Lemma.

If $\phi : (\tilde{\alpha}, \tilde{\beta}) \rightarrow (\alpha, \beta)$ is invertible with inverse $\psi : (\alpha, \beta) \rightarrow (\tilde{\alpha}, \tilde{\beta})$, then $\phi'(t) \neq 0$ for all $t \in (\tilde{\alpha}, \tilde{\beta})$ and $\psi'(t) \neq 0$ for all $t \in (\alpha, \beta)$.

Proof.

$\psi(\phi(t)) = t$ for each t
 $\psi'(\phi(t))\phi'(t) = 1$ for each t
So, $\psi'(\phi(t)) \neq 0$ and $\phi'(t) \neq 0$ for each t
So, $\psi'(t) \neq 0$ and $\phi'(t) \neq 0$ for each t because ϕ is bijective. □

Proposition. A reparametrization of a regular parametrization is regular.

Proof.

$$\begin{aligned} \gamma &: (\alpha, \beta) \rightarrow \mathbb{R}^2 \\ \tilde{\gamma} &: (\tilde{\alpha}, \tilde{\beta}) \rightarrow \mathbb{R}^2 \\ \tilde{\gamma}(\tilde{t}) &= \gamma(\phi(\tilde{t})) \\ \tilde{\gamma}'(\tilde{t}) &= \gamma'(\phi(\tilde{t})) \end{aligned}$$

□

Regular parametrization

Definition.

The point $\gamma(t)$ of $\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$ is regular point if $\dot{\gamma}(t) \neq 0$ and singular if $\dot{\gamma}(t) = 0$. The parametrization γ is called regular if $\dot{\gamma}(t) \neq 0$ for every $t \in (\alpha, \beta)$.

Example. $\gamma(t) = (t^3, t^3)$ is singular at 0 because $\dot{\gamma}(t) = (3t^2, 3t^2)$.

Lemma.

If $\phi : (\tilde{\alpha}, \tilde{\beta}) \rightarrow (\alpha, \beta)$ is invertible with inverse $\psi : (\alpha, \beta) \rightarrow (\tilde{\alpha}, \tilde{\beta})$, then $\phi'(t) \neq 0$ for all $t \in (\tilde{\alpha}, \tilde{\beta})$ and $\psi'(t) \neq 0$ for all $t \in (\alpha, \beta)$.

Proof.

$\psi(\phi(t)) = t$ for each t
 $\psi'(\phi(t))\phi'(t) = 1$ for each t
So, $\psi'(\phi(t)) \neq 0$ and $\phi'(t) \neq 0$ for each t
So, $\psi'(t) \neq 0$ and $\phi'(t) \neq 0$ for each t because ϕ is bijective. □

Proposition. A reparametrization of a regular parametrization is regular.

Proof.

$$\begin{aligned} \gamma &: (\alpha, \beta) \rightarrow \mathbb{R}^2 \\ \tilde{\gamma} &: (\tilde{\alpha}, \tilde{\beta}) \rightarrow \mathbb{R}^2 \\ \tilde{\gamma}(\tilde{t}) &= \gamma(\phi(\tilde{t})) \\ \tilde{\gamma}'(\tilde{t}) &= \gamma'(\phi(\tilde{t}))\phi'(\tilde{t}) \end{aligned}$$

□

Regular parametrization

Definition.

The point $\gamma(t)$ of $\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$ is regular point if $\dot{\gamma}(t) \neq 0$ and singular if $\dot{\gamma}(t) = 0$. The parametrization γ is called regular if $\dot{\gamma}(t) \neq 0$ for every $t \in (\alpha, \beta)$.

Example. $\gamma(t) = (t^3, t^3)$ is singular at 0 because $\dot{\gamma}(t) = (3t^2, 3t^2)$.

Lemma.

If $\phi : (\tilde{\alpha}, \tilde{\beta}) \rightarrow (\alpha, \beta)$ is invertible with inverse $\psi : (\alpha, \beta) \rightarrow (\tilde{\alpha}, \tilde{\beta})$, then $\phi'(t) \neq 0$ for all $t \in (\tilde{\alpha}, \tilde{\beta})$ and $\psi'(t) \neq 0$ for all $t \in (\alpha, \beta)$.

Proof.

$\psi(\phi(t)) = t$ for each t
 $\psi'(\phi(t))\phi'(t) = 1$ for each t
So, $\psi'(\phi(t)) \neq 0$ and $\phi'(t) \neq 0$ for each t
So, $\psi'(t) \neq 0$ and $\phi'(t) \neq 0$ for each t because ϕ is bijective. □

Proposition. A reparametrization of a regular parametrization is regular.

Proof.

$$\begin{aligned} \gamma &: (\alpha, \beta) \rightarrow \mathbb{R}^2 \\ \tilde{\gamma} &: (\tilde{\alpha}, \tilde{\beta}) \rightarrow \mathbb{R}^2 \\ \tilde{\gamma}(\tilde{t}) &= \gamma(\phi(\tilde{t})) \\ \tilde{\gamma}'(\tilde{t}) &= \gamma'(\phi(\tilde{t}))\phi'(\tilde{t}) \\ \text{But, } \gamma'(t) &\neq 0 \end{aligned}$$

□

Regular parametrization

Definition.

The point $\gamma(t)$ of $\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$ is regular point if $\dot{\gamma}(t) \neq 0$ and singular if $\dot{\gamma}(t) = 0$. The parametrization γ is called regular if $\dot{\gamma}(t) \neq 0$ for every $t \in (\alpha, \beta)$.

Example. $\gamma(t) = (t^3, t^3)$ is singular at 0 because $\dot{\gamma}(t) = (3t^2, 3t^2)$.

Lemma.

If $\phi : (\tilde{\alpha}, \tilde{\beta}) \rightarrow (\alpha, \beta)$ is invertible with inverse $\psi : (\alpha, \beta) \rightarrow (\tilde{\alpha}, \tilde{\beta})$, then $\phi'(t) \neq 0$ for all $t \in (\tilde{\alpha}, \tilde{\beta})$ and $\psi'(t) \neq 0$ for all $t \in (\alpha, \beta)$.

Proof.

$$\psi(\phi(t)) = t \text{ for each } t$$

$$\psi'(\phi(t))\phi'(t) = 1 \text{ for each } t$$

So, $\psi'(\phi(t)) \neq 0$ and $\phi'(t) \neq 0$ for each t

So, $\psi'(t) \neq 0$ and $\phi'(t) \neq 0$ for each t because ϕ is bijective. □

Proposition. A reparametrization of a regular parametrization is regular.

Proof.

$$\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$$

$$\tilde{\gamma} : (\tilde{\alpha}, \tilde{\beta}) \rightarrow \mathbb{R}^2$$

$$\tilde{\gamma}(\tilde{t}) = \gamma(\phi(\tilde{t}))$$

$$\tilde{\gamma}'(\tilde{t}) = \gamma'(\phi(\tilde{t}))\phi'(\tilde{t})$$

But, $\gamma'(t) \neq 0$ for all t , □

Regular parametrization

Definition.

The point $\gamma(t)$ of $\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$ is regular point if $\dot{\gamma}(t) \neq 0$ and singular if $\dot{\gamma}(t) = 0$. The parametrization γ is called regular if $\dot{\gamma}(t) \neq 0$ for every $t \in (\alpha, \beta)$.

Example. $\gamma(t) = (t^3, t^3)$ is singular at 0 because $\dot{\gamma}(t) = (3t^2, 3t^2)$.

Lemma.

If $\phi : (\tilde{\alpha}, \tilde{\beta}) \rightarrow (\alpha, \beta)$ is invertible with inverse $\psi : (\alpha, \beta) \rightarrow (\tilde{\alpha}, \tilde{\beta})$, then $\phi'(t) \neq 0$ for all $t \in (\tilde{\alpha}, \tilde{\beta})$ and $\psi'(t) \neq 0$ for all $t \in (\alpha, \beta)$.

Proof.

$\psi(\phi(t)) = t$ for each t
 $\psi'(\phi(t))\phi'(t) = 1$ for each t
So, $\psi'(\phi(t)) \neq 0$ and $\phi'(t) \neq 0$ for each t
So, $\psi'(t) \neq 0$ and $\phi'(t) \neq 0$ for each t because ϕ is bijective. □

Proposition. A reparametrization of a regular parametrization is regular.

Proof.

$\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$
 $\tilde{\gamma} : (\tilde{\alpha}, \tilde{\beta}) \rightarrow \mathbb{R}^2$
 $\tilde{\gamma}(\tilde{t}) = \gamma(\phi(\tilde{t}))$
 $\tilde{\gamma}'(\tilde{t}) = \gamma'(\phi(\tilde{t}))\phi'(\tilde{t})$
But, $\gamma'(t) \neq 0$ for all t , therefore, even for $\phi(\tilde{t})$ □

Regular parametrization

Definition.

The point $\gamma(t)$ of $\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$ is regular point if $\dot{\gamma}(t) \neq 0$ and singular if $\dot{\gamma}(t) = 0$. The parametrization γ is called regular if $\dot{\gamma}(t) \neq 0$ for every $t \in (\alpha, \beta)$.

Example. $\gamma(t) = (t^3, t^3)$ is singular at 0 because $\dot{\gamma}(t) = (3t^2, 3t^2)$.

Lemma.

If $\phi : (\tilde{\alpha}, \tilde{\beta}) \rightarrow (\alpha, \beta)$ is invertible with inverse $\psi : (\alpha, \beta) \rightarrow (\tilde{\alpha}, \tilde{\beta})$, then $\phi'(t) \neq 0$ for all $t \in (\tilde{\alpha}, \tilde{\beta})$ and $\psi'(t) \neq 0$ for all $t \in (\alpha, \beta)$.

Proof.

$\psi(\phi(t)) = t$ for each t
 $\psi'(\phi(t))\phi'(t) = 1$ for each t
So, $\psi'(\phi(t)) \neq 0$ and $\phi'(t) \neq 0$ for each t
So, $\psi'(t) \neq 0$ and $\phi'(t) \neq 0$ for each t because ϕ is bijective. □

Proposition. A reparametrization of a regular parametrization is regular.

Proof.

$\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$
 $\tilde{\gamma} : (\tilde{\alpha}, \tilde{\beta}) \rightarrow \mathbb{R}^2$
 $\tilde{\gamma}(\tilde{t}) = \gamma(\phi(\tilde{t}))$
 $\tilde{\gamma}'(\tilde{t}) = \gamma'(\phi(\tilde{t}))\phi'(\tilde{t})$
But, $\gamma'(t) \neq 0$ for all t , therefore, even for $\phi(\tilde{t})$ and, $\phi'(\tilde{t}) \neq 0$ □

Regular parametrization

Definition.

The point $\gamma(t)$ of $\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$ is regular point if $\dot{\gamma}(t) \neq 0$ and singular if $\dot{\gamma}(t) = 0$. The parametrization γ is called regular if $\dot{\gamma}(t) \neq 0$ for every $t \in (\alpha, \beta)$.

Example. $\gamma(t) = (t^3, t^3)$ is singular at 0 because $\dot{\gamma}(t) = (3t^2, 3t^2)$.

Lemma.

If $\phi : (\tilde{\alpha}, \tilde{\beta}) \rightarrow (\alpha, \beta)$ is invertible with inverse $\psi : (\alpha, \beta) \rightarrow (\tilde{\alpha}, \tilde{\beta})$, then $\phi'(t) \neq 0$ for all $t \in (\tilde{\alpha}, \tilde{\beta})$ and $\psi'(t) \neq 0$ for all $t \in (\alpha, \beta)$.

Proof.

$\psi(\phi(t)) = t$ for each t
 $\psi'(\phi(t))\phi'(t) = 1$ for each t
So, $\psi'(\phi(t)) \neq 0$ and $\phi'(t) \neq 0$ for each t
So, $\psi'(t) \neq 0$ and $\phi'(t) \neq 0$ for each t because ϕ is bijective. □

Proposition. A reparametrization of a regular parametrization is regular.

Proof.

$\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$
 $\tilde{\gamma} : (\tilde{\alpha}, \tilde{\beta}) \rightarrow \mathbb{R}^2$
 $\tilde{\gamma}(\tilde{t}) = \gamma(\phi(\tilde{t}))$
 $\tilde{\gamma}'(\tilde{t}) = \gamma'(\phi(\tilde{t}))\phi'(\tilde{t})$
But, $\gamma'(t) \neq 0$ for all t , therefore, even for $\phi(\tilde{t})$
and, $\phi'(\tilde{t}) \neq 0$ for all \tilde{t} □

Regular parametrization

Definition.

The point $\gamma(t)$ of $\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$ is regular point if $\dot{\gamma}(t) \neq 0$ and singular if $\dot{\gamma}(t) = 0$. The parametrization γ is called regular if $\dot{\gamma}(t) \neq 0$ for every $t \in (\alpha, \beta)$.

Example. $\gamma(t) = (t^3, t^3)$ is singular at 0 because $\dot{\gamma}(t) = (3t^2, 3t^2)$.

Lemma.

If $\phi : (\tilde{\alpha}, \tilde{\beta}) \rightarrow (\alpha, \beta)$ is invertible with inverse $\psi : (\alpha, \beta) \rightarrow (\tilde{\alpha}, \tilde{\beta})$, then $\phi'(t) \neq 0$ for all $t \in (\tilde{\alpha}, \tilde{\beta})$ and $\psi'(t) \neq 0$ for all $t \in (\alpha, \beta)$.

Proof.

$\psi(\phi(t)) = t$ for each t
 $\psi'(\phi(t))\phi'(t) = 1$ for each t
So, $\psi'(\phi(t)) \neq 0$ and $\phi'(t) \neq 0$ for each t
So, $\psi'(t) \neq 0$ and $\phi'(t) \neq 0$ for each t because ϕ is bijective. □

Proposition. A reparametrization of a regular parametrization is regular.

Proof.

$\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$
 $\tilde{\gamma} : (\tilde{\alpha}, \tilde{\beta}) \rightarrow \mathbb{R}^2$
 $\tilde{\gamma}(\tilde{t}) = \gamma(\phi(\tilde{t}))$
 $\tilde{\gamma}'(\tilde{t}) = \gamma'(\phi(\tilde{t}))\phi'(\tilde{t})$
But, $\gamma'(t) \neq 0$ for all t , therefore, even for $\phi(\tilde{t})$
and, $\phi'(\tilde{t}) \neq 0$ for all \tilde{t}

□

Inner product:

Inner product:

$$v = (2, 3)$$

Inner product:

$$v = (2, 3)$$

$$w = (2, 1)$$

Inner product:

$$v = (2, 3)$$

$$w = (2, 1)$$

$$v \cdot w$$

Inner product:

$$v = (2, 3)$$

$$w = (2, 1)$$

$$v.w := (2, 3).(2, 1)$$

Inner product:

$$v = (2, 3)$$

$$w = (2, 1)$$

$$v.w := (2, 3).(2, 1) = 2 \times 2 + 3 \times 1$$

Inner product:

$$v = (2, 3)$$

$$w = (2, 1)$$

$$v.w := (2, 3).(2, 1) = 2 \times 2 + 3 \times 1 = 7$$

Inner product:

$$v = (2, 3)$$

$$w = (2, 1)$$

$$v.w := (2, 3).(2, 1) = 2 \times 2 + 3 \times 1 = 7$$

In general:

Inner product:

$$v = (2, 3)$$

$$w = (2, 1)$$

$$v.w := (2, 3).(2, 1) = 2 \times 2 + 3 \times 1 = 7$$

In general:

$$(x_1, y_1).(x_2, y_2) := x_1x_2 + y_1y_2$$

Inner product:

$$v = (2, 3)$$

$$w = (2, 1)$$

$$v.w := (2, 3).(2, 1) = 2 \times 2 + 3 \times 1 = 7$$

In general:

$$(x_1, y_1).(x_2, y_2) := x_1x_2 + y_1y_2$$

Exercise. For $\mathbf{v} : (\alpha, \beta) \rightarrow \mathbf{R}^2$

Inner product:

$$v = (2, 3)$$

$$w = (2, 1)$$

$$v.w := (2, 3).(2, 1) = 2 \times 2 + 3 \times 1 = 7$$

In general:

$$(x_1, y_1).(x_2, y_2) := x_1x_2 + y_1y_2$$

Exercise. For $\mathbf{v} : (\alpha, \beta) \rightarrow \mathbf{R}^2$ and $\mathbf{w} : (\alpha, \beta) \rightarrow \mathbf{R}^2$,

Inner product:

$$v = (2, 3)$$

$$w = (2, 1)$$

$$v.w := (2, 3).(2, 1) = 2 \times 2 + 3 \times 1 = 7$$

In general:

$$(x_1, y_1).(x_2, y_2) := x_1x_2 + y_1y_2$$

Exercise. For $\mathbf{v} : (\alpha, \beta) \rightarrow \mathbf{R}^2$ and $\mathbf{w} : (\alpha, \beta) \rightarrow \mathbf{R}^2$,
show that $(\mathbf{v}(t).\mathbf{w}(t))' = \mathbf{v}'(t).\mathbf{w}(t) + \mathbf{v}(t).\mathbf{w}'(t)$

Inner product:

$$v = (2, 3)$$

$$w = (2, 1)$$

$$v.w := (2, 3).(2, 1) = 2 \times 2 + 3 \times 1 = 7$$

In general:

$$(x_1, y_1).(x_2, y_2) := x_1x_2 + y_1y_2$$

Exercise. For $\mathbf{v} : (\alpha, \beta) \rightarrow \mathbf{R}^2$ and $\mathbf{w} : (\alpha, \beta) \rightarrow \mathbf{R}^2$,
show that $(\mathbf{v}(t).\mathbf{w}(t))' = \mathbf{v}'(t).\mathbf{w}(t) + \mathbf{v}(t).\mathbf{w}'(t)$

$$v = (x, y)$$

Inner product:

$$v = (2, 3)$$

$$w = (2, 1)$$

$$v.w := (2, 3).(2, 1) = 2 \times 2 + 3 \times 1 = 7$$

In general:

$$(x_1, y_1).(x_2, y_2) := x_1x_2 + y_1y_2$$

Exercise. For $\mathbf{v} : (\alpha, \beta) \rightarrow \mathbf{R}^2$ and $\mathbf{w} : (\alpha, \beta) \rightarrow \mathbf{R}^2$,
show that $(\mathbf{v}(t).\mathbf{w}(t))' = \mathbf{v}'(t).\mathbf{w}(t) + \mathbf{v}(t).\mathbf{w}'(t)$

$$v = (x, y)$$

$$v.v$$

Inner product:

$$v = (2, 3)$$

$$w = (2, 1)$$

$$v.w := (2, 3).(2, 1) = 2 \times 2 + 3 \times 1 = 7$$

In general:

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Exercise. For $\mathbf{v} : (\alpha, \beta) \rightarrow \mathbf{R}^2$ and $\mathbf{w} : (\alpha, \beta) \rightarrow \mathbf{R}^2$,
show that $(\mathbf{v}(t).\mathbf{w}(t))' = \mathbf{v}'(t).\mathbf{w}(t) + \mathbf{v}(t).\mathbf{w}'(t)$

$$v = (x, y)$$

$$v.v = (x, y).(x, y) = x^2 + y^2$$

Inner product:

$$v = (2, 3)$$

$$w = (2, 1)$$

$$v.w := (2, 3).(2, 1) = 2 \times 2 + 3 \times 1 = 7$$

In general:

$$(x_1, y_1).(x_2, y_2) := x_1x_2 + y_1y_2$$

Exercise. For $\mathbf{v} : (\alpha, \beta) \rightarrow \mathbf{R}^2$ and $\mathbf{w} : (\alpha, \beta) \rightarrow \mathbf{R}^2$,
show that $(\mathbf{v}(t).\mathbf{w}(t))' = \mathbf{v}'(t).\mathbf{w}(t) + \mathbf{v}(t).\mathbf{w}'(t)$

$$v = (x, y)$$

$$v.v = (x, y).(x, y) = x^2 + y^2$$

$$||(x, y)|| = \sqrt{x^2 + y^2}$$

Inner product:

$$v = (2, 3)$$

$$w = (2, 1)$$

$$v.w := (2, 3).(2, 1) = 2 \times 2 + 3 \times 1 = 7$$

In general:

$$(x_1, y_1).(x_2, y_2) := x_1x_2 + y_1y_2$$

Exercise. For $\mathbf{v} : (\alpha, \beta) \rightarrow \mathbf{R}^2$ and $\mathbf{w} : (\alpha, \beta) \rightarrow \mathbf{R}^2$,
show that $(\mathbf{v}(t).\mathbf{w}(t))' = \mathbf{v}'(t).\mathbf{w}(t) + \mathbf{v}(t).\mathbf{w}'(t)$

$$v = (x, y)$$

$$v.v = (x, y).(x, y) = x^2 + y^2$$

$$||(x, y)|| = \sqrt{x^2 + y^2}$$

$$v = (x, y)$$

Inner product:

$$v = (2, 3)$$

$$w = (2, 1)$$

$$v.w := (2, 3).(2, 1) = 2 \times 2 + 3 \times 1 = 7$$

In general:

$$(x_1, y_1).(x_2, y_2) := x_1x_2 + y_1y_2$$

Exercise. For $\mathbf{v} : (\alpha, \beta) \rightarrow \mathbf{R}^2$ and $\mathbf{w} : (\alpha, \beta) \rightarrow \mathbf{R}^2$,
show that $(\mathbf{v}(t).\mathbf{w}(t))' = \mathbf{v}'(t).\mathbf{w}(t) + \mathbf{v}(t).\mathbf{w}'(t)$

$$v = (x, y)$$

$$v.v = (x, y).(x, y) = x^2 + y^2$$

$$||(x, y)|| = \sqrt{x^2 + y^2}$$

$$v = (x, y)$$

$$v.w =$$

Inner product:

$$v = (2, 3)$$

$$w = (2, 1)$$

$$v.w := (2, 3).(2, 1) = 2 \times 2 + 3 \times 1 = 7$$

In general:

$$(x_1, y_1).(x_2, y_2) := x_1x_2 + y_1y_2$$

Exercise. For $\mathbf{v} : (\alpha, \beta) \rightarrow \mathbf{R}^2$ and $\mathbf{w} : (\alpha, \beta) \rightarrow \mathbf{R}^2$,
show that $(\mathbf{v}(t).\mathbf{w}(t))' = \mathbf{v}'(t).\mathbf{w}(t) + \mathbf{v}(t).\mathbf{w}'(t)$

$$v = (x, y)$$

$$v.v = (x, y).(x, y) = x^2 + y^2$$

$$||(x, y)|| = \sqrt{x^2 + y^2}$$

$$v = (x, y)$$

$$v.w = ||v||||w|| \cos(\theta) \text{ where,}$$

θ is the angle between v and w

Inner product:

$$v = (2, 3)$$

$$w = (2, 1)$$

$$v.w := (2, 3).(2, 1) = 2 \times 2 + 3 \times 1 = 7$$

In general:

$$(x_1, y_1).(x_2, y_2) := x_1x_2 + y_1y_2$$

Exercise. For $\mathbf{v} : (\alpha, \beta) \rightarrow \mathbf{R}^2$ and $\mathbf{w} : (\alpha, \beta) \rightarrow \mathbf{R}^2$, show that $(\mathbf{v}(t).\mathbf{w}(t))' = \mathbf{v}'(t).\mathbf{w}(t) + \mathbf{v}(t).\mathbf{w}'(t)$

$$v = (x, y)$$

$$v.v = (x, y).(x, y) = x^2 + y^2$$

$$||(x, y)|| = \sqrt{x^2 + y^2}$$

$$v = (x, y)$$

$$v.w = ||v||||w|| \cos(\theta) \text{ where,}$$

θ is the angle between v and w

Definition. $\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$ is a unit speed parametrization

Inner product:

$$v = (2, 3)$$

$$w = (2, 1)$$

$$v.w := (2, 3).(2, 1) = 2 \times 2 + 3 \times 1 = 7$$

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