

Reparametrization

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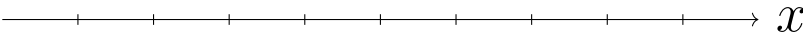
If ϕ is smooth, is ϕ^{-1} smooth?

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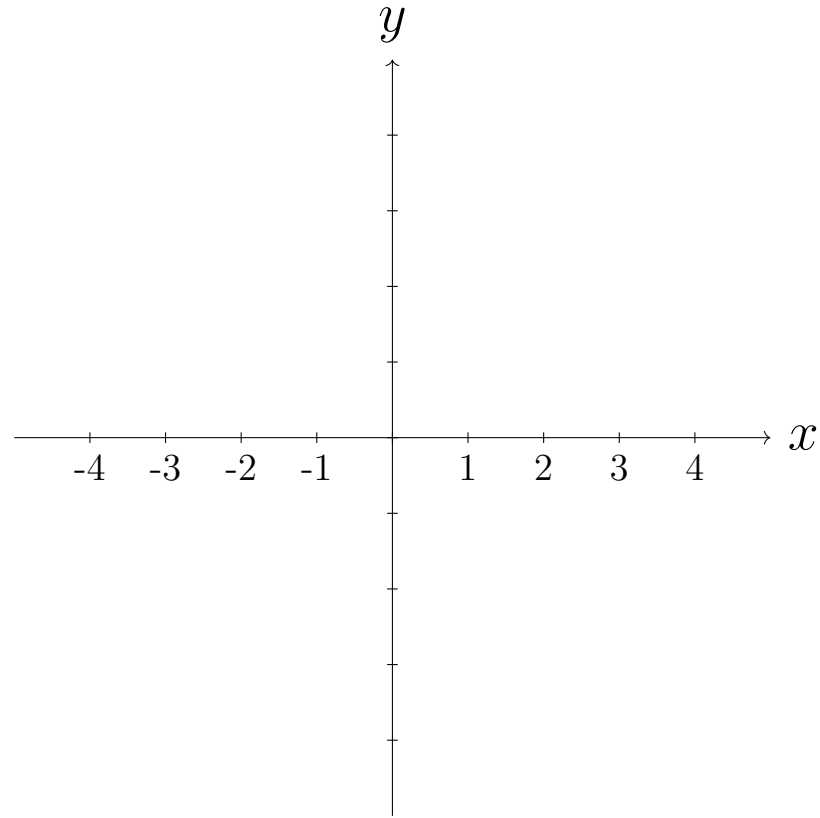
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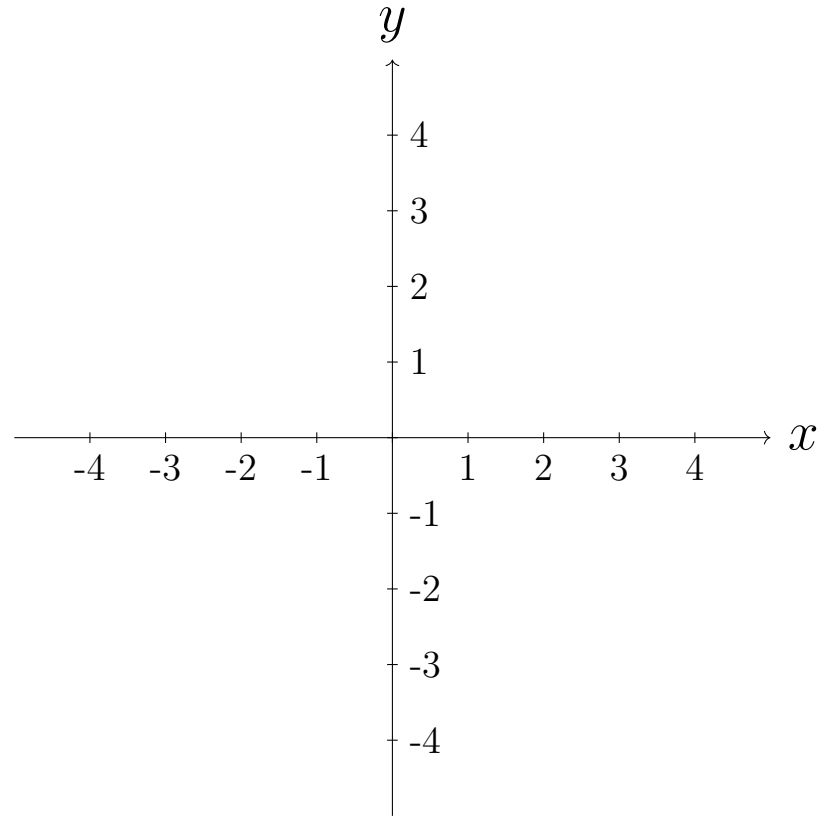
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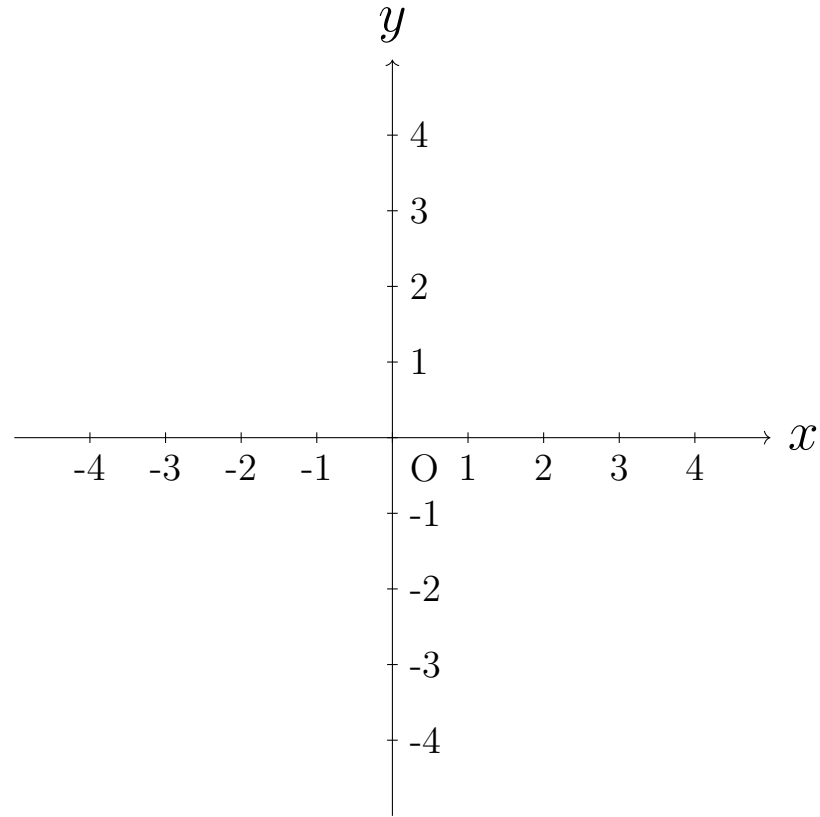
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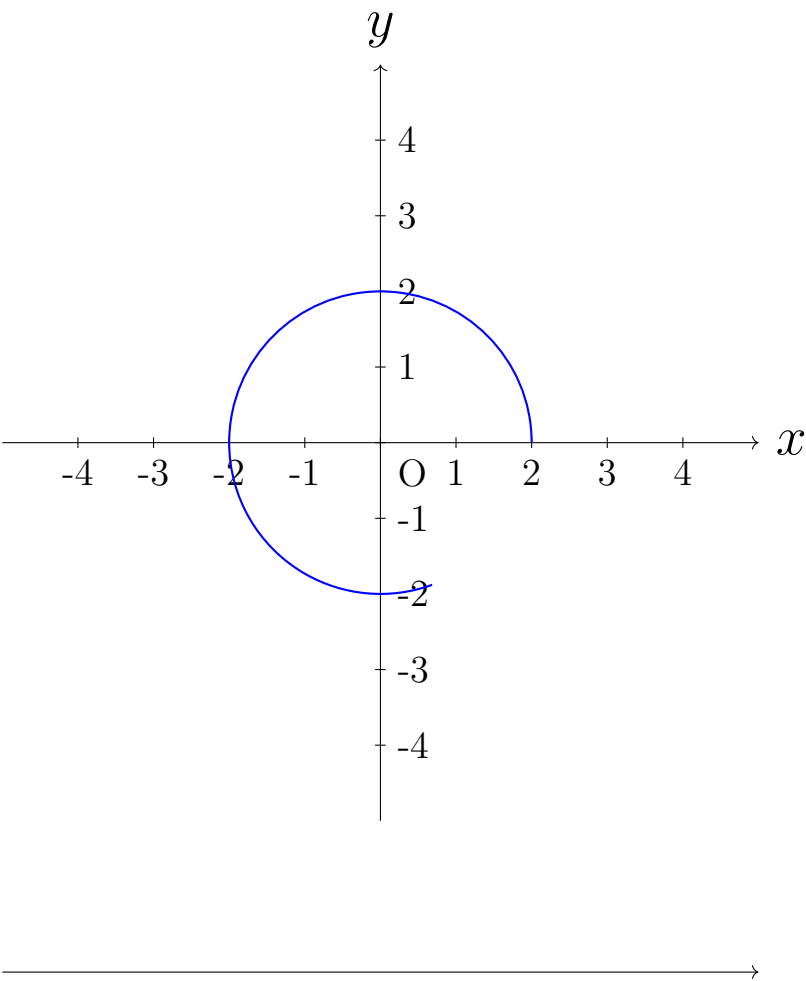
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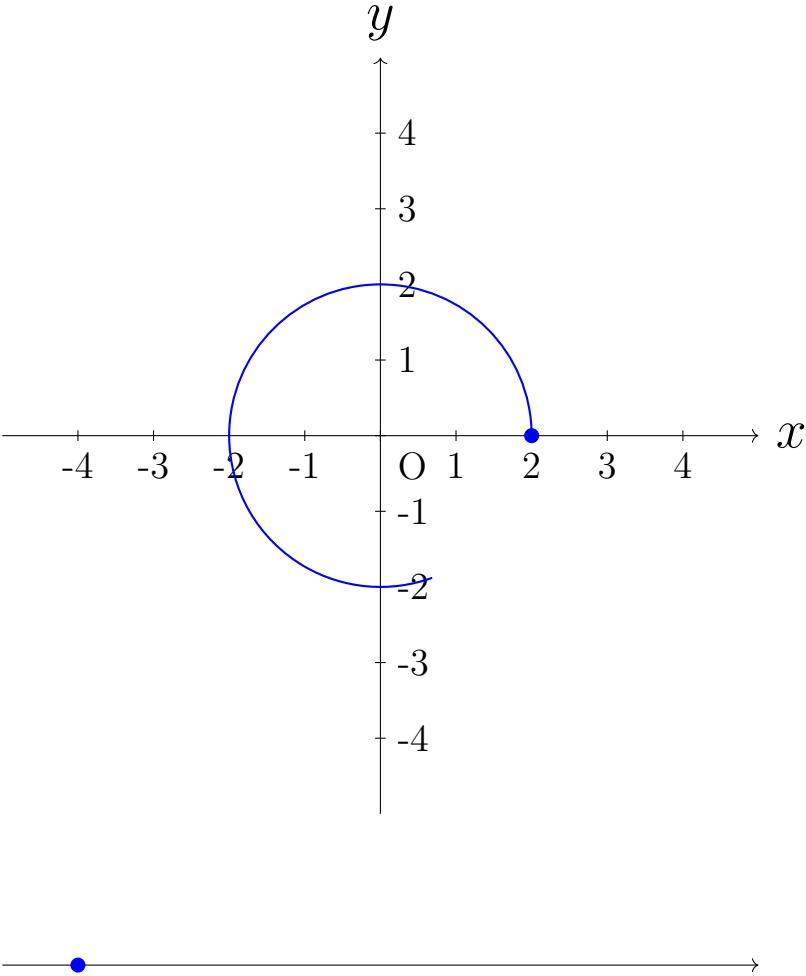
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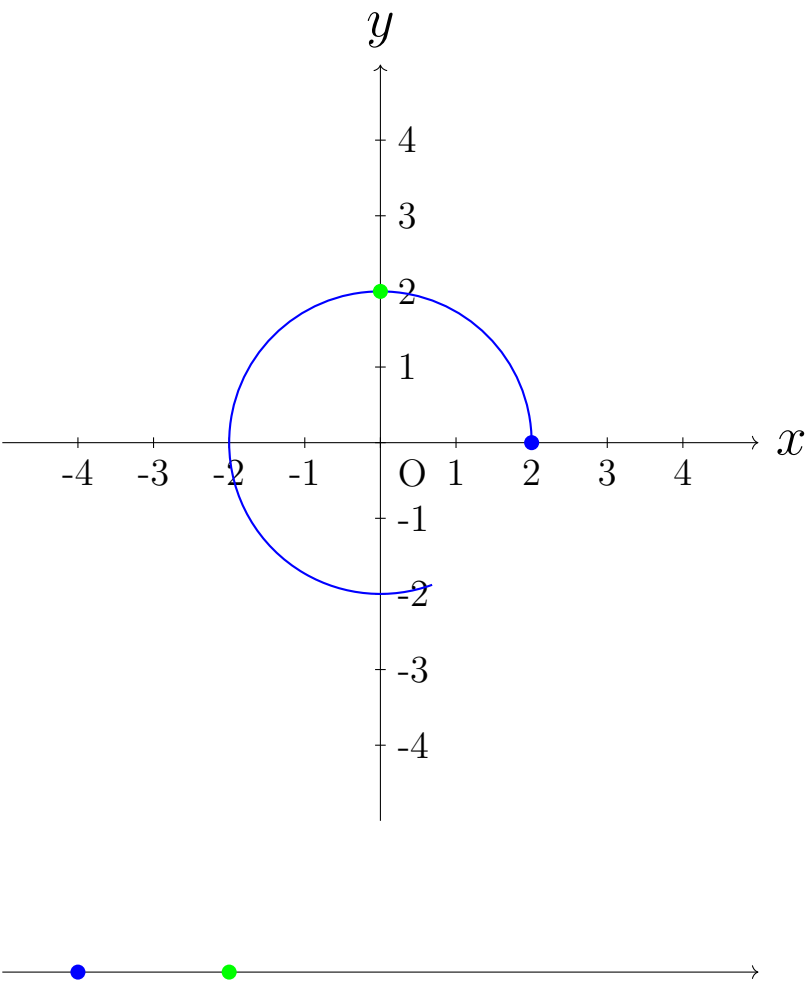
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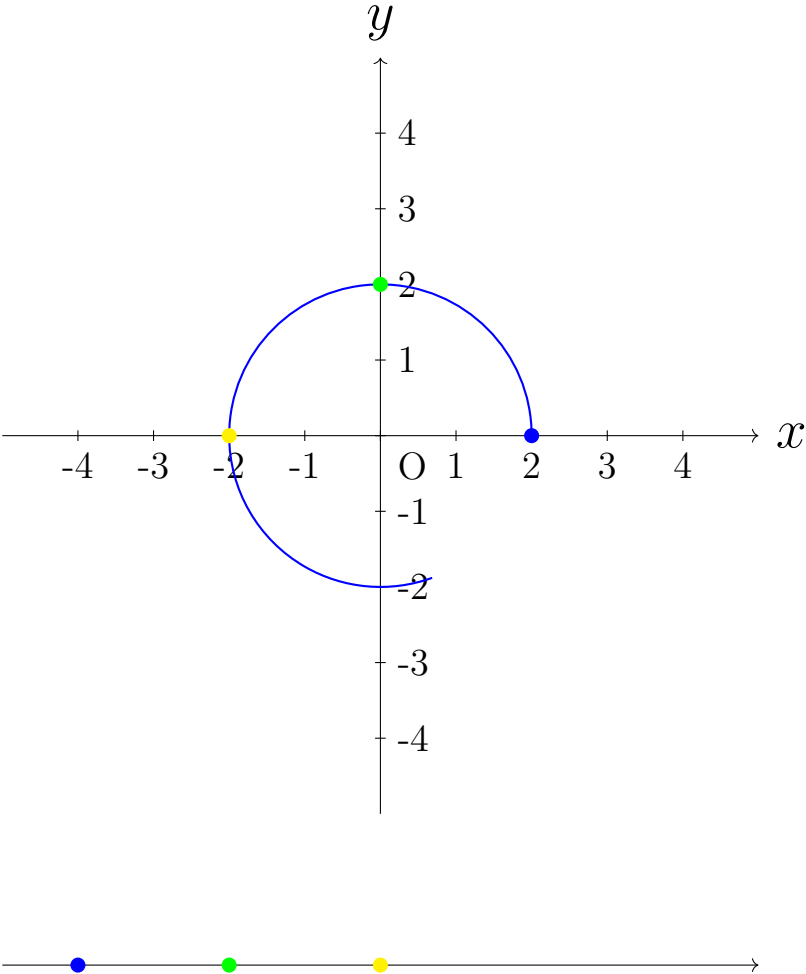
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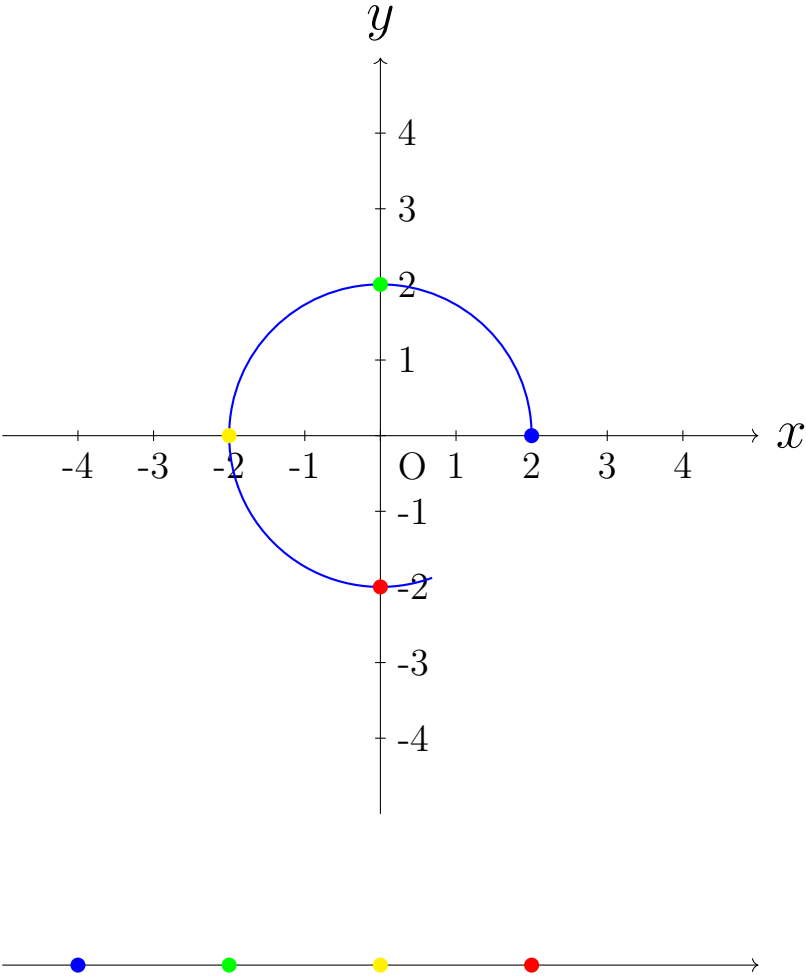
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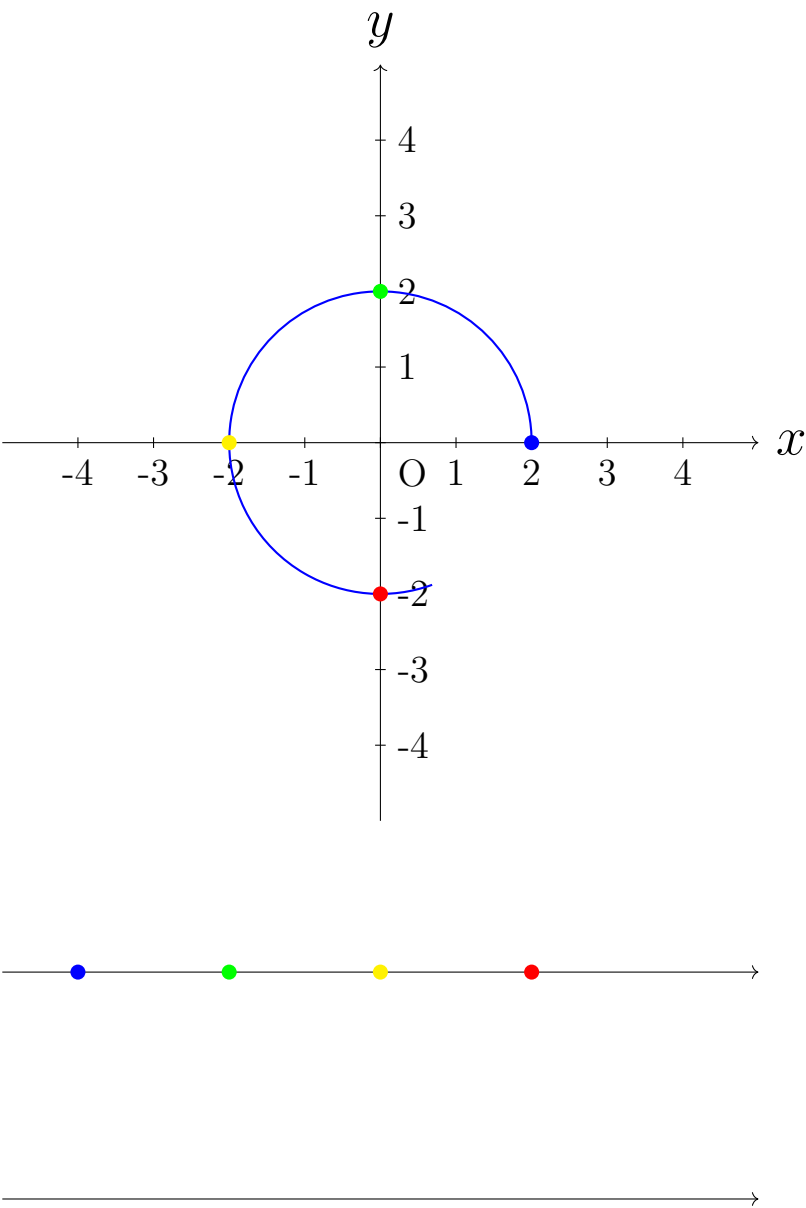
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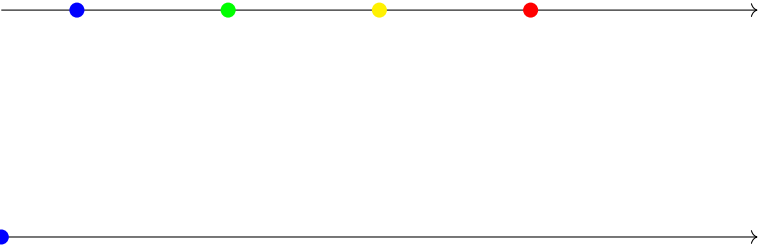
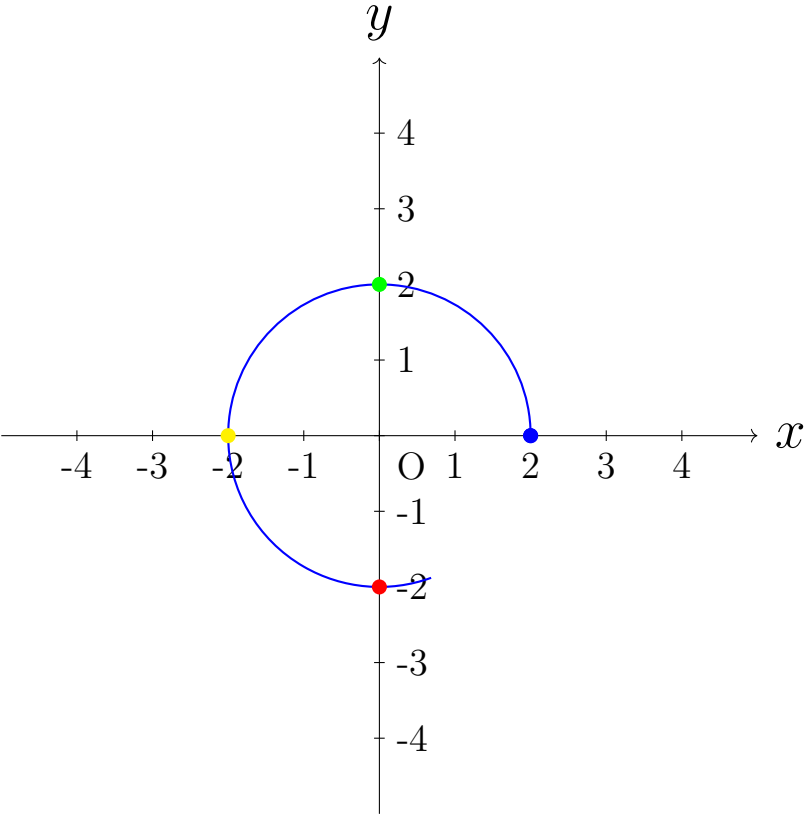
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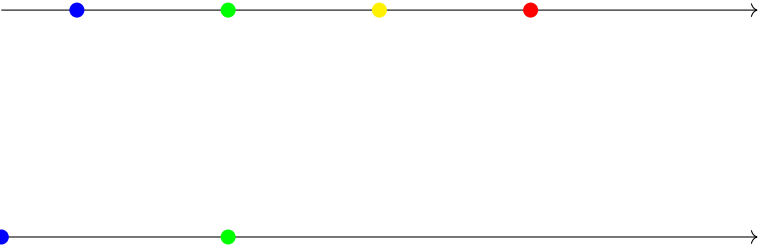
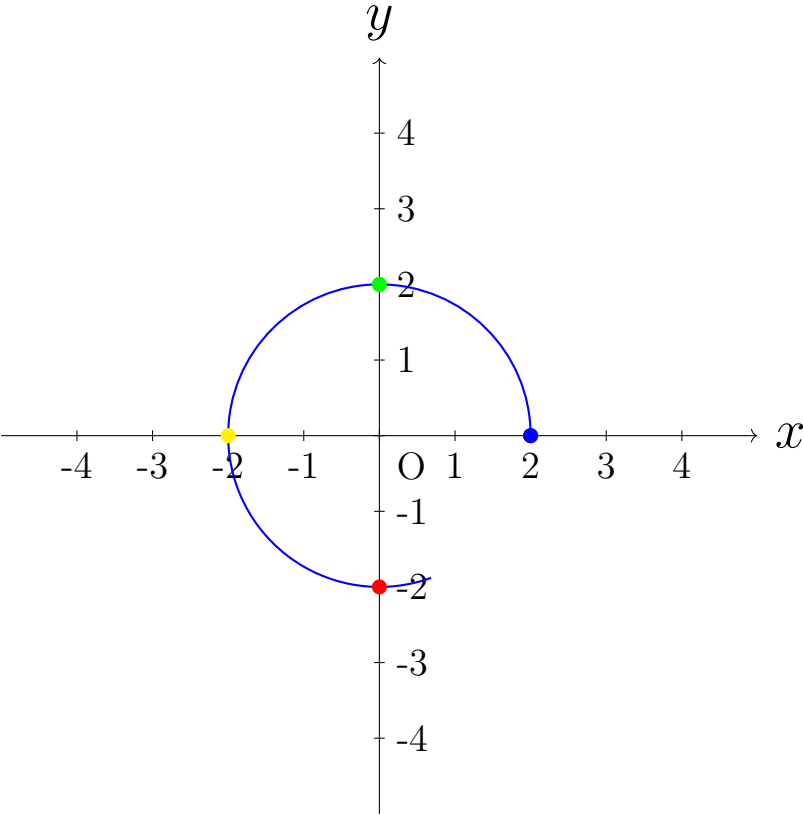
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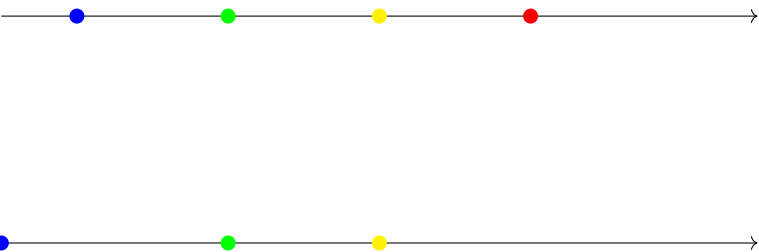
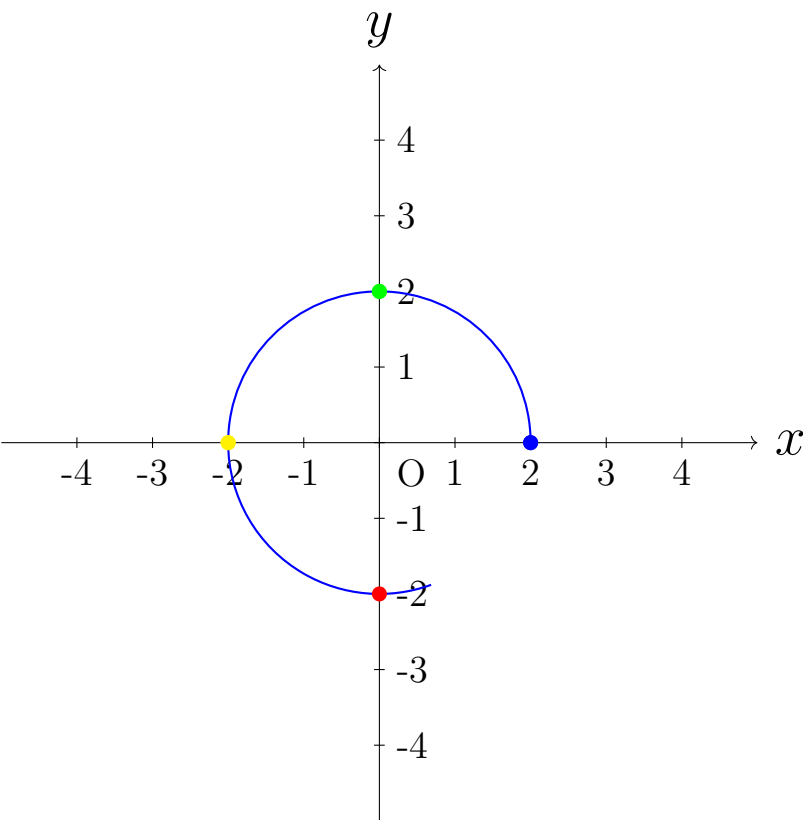
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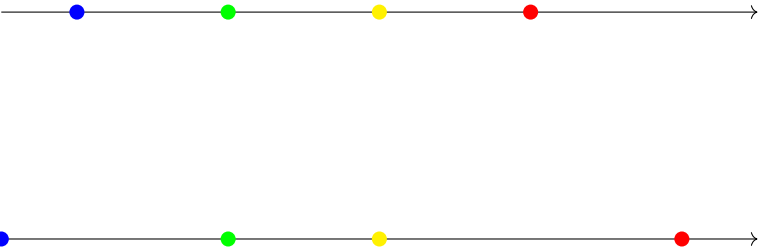
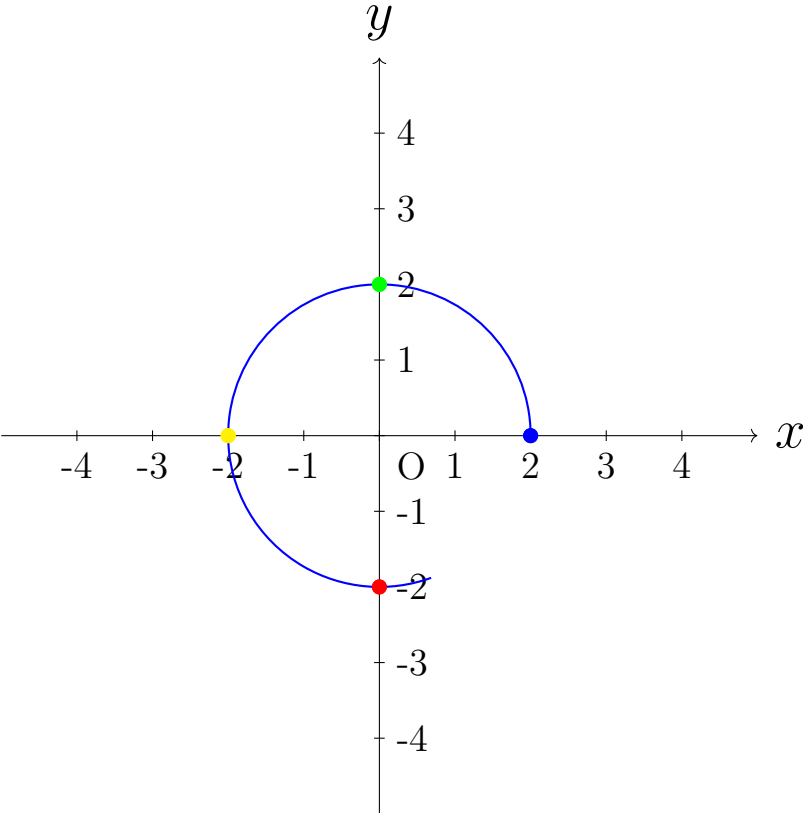
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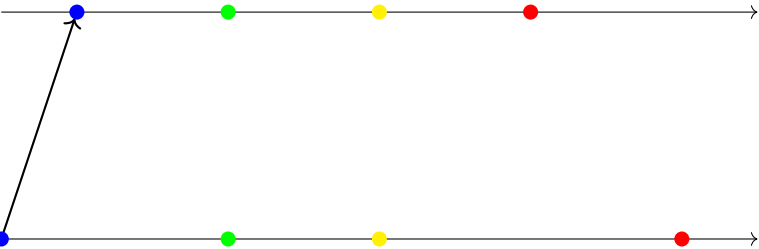
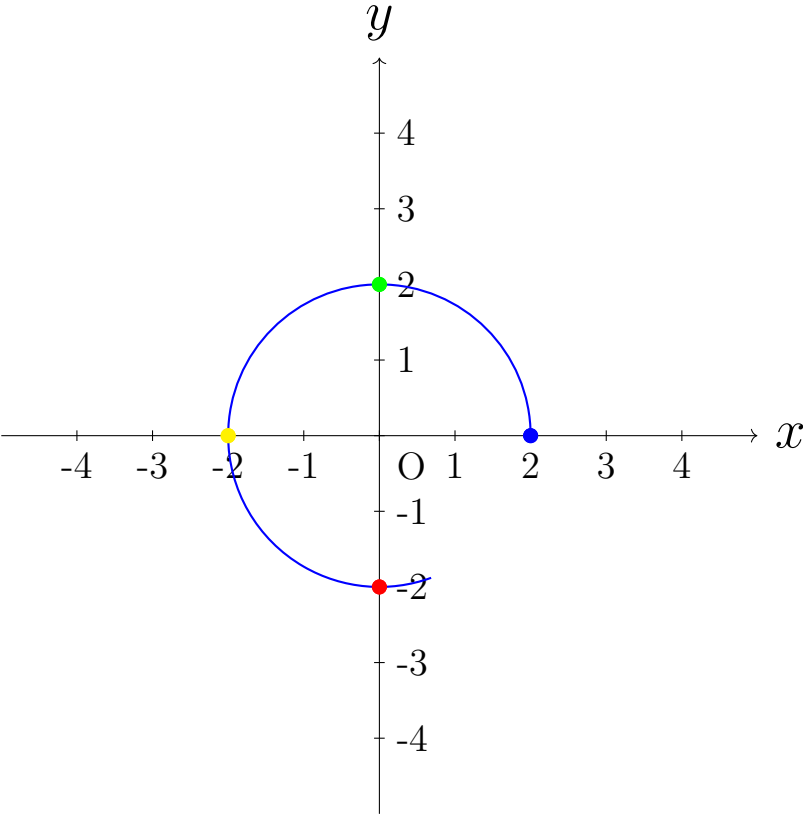
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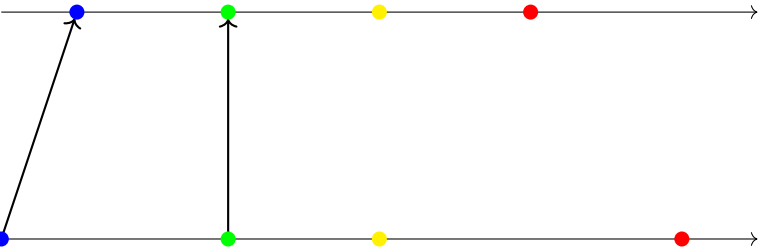
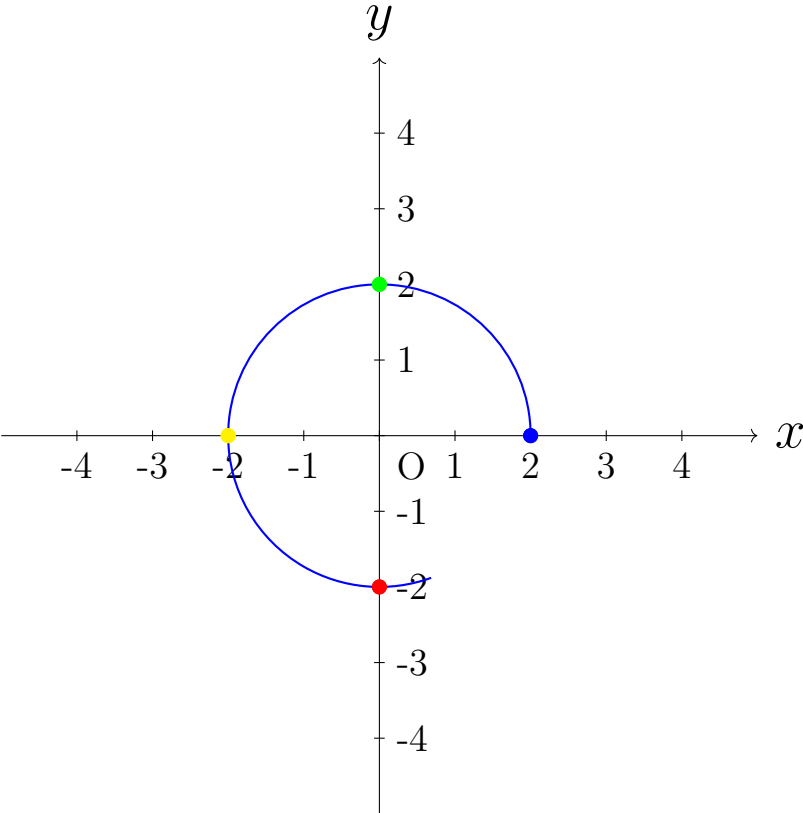
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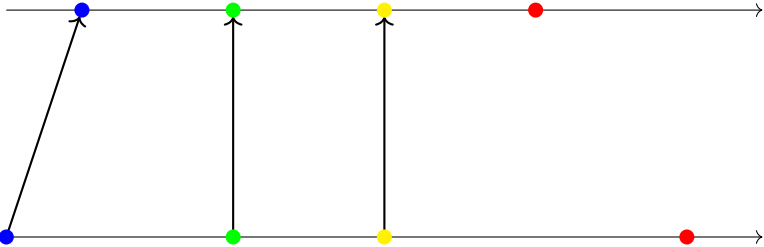
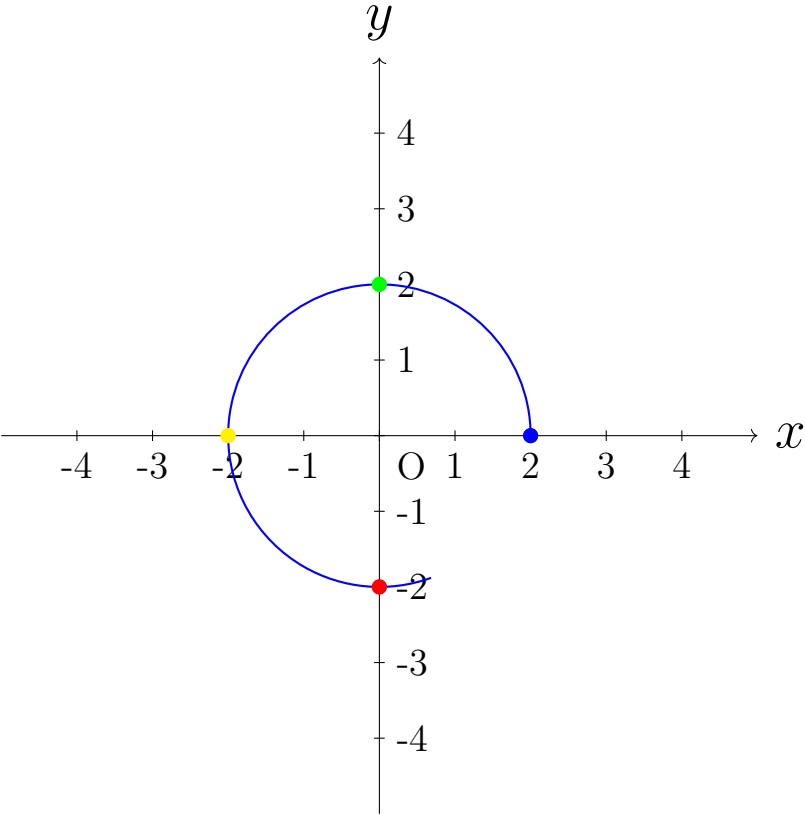
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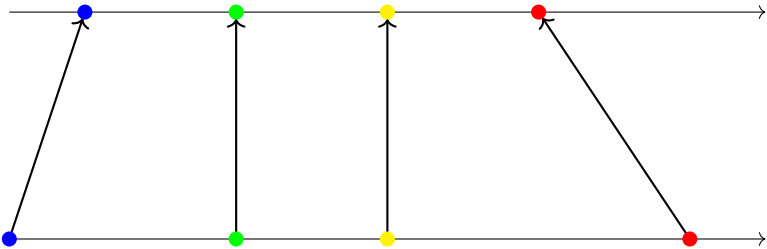
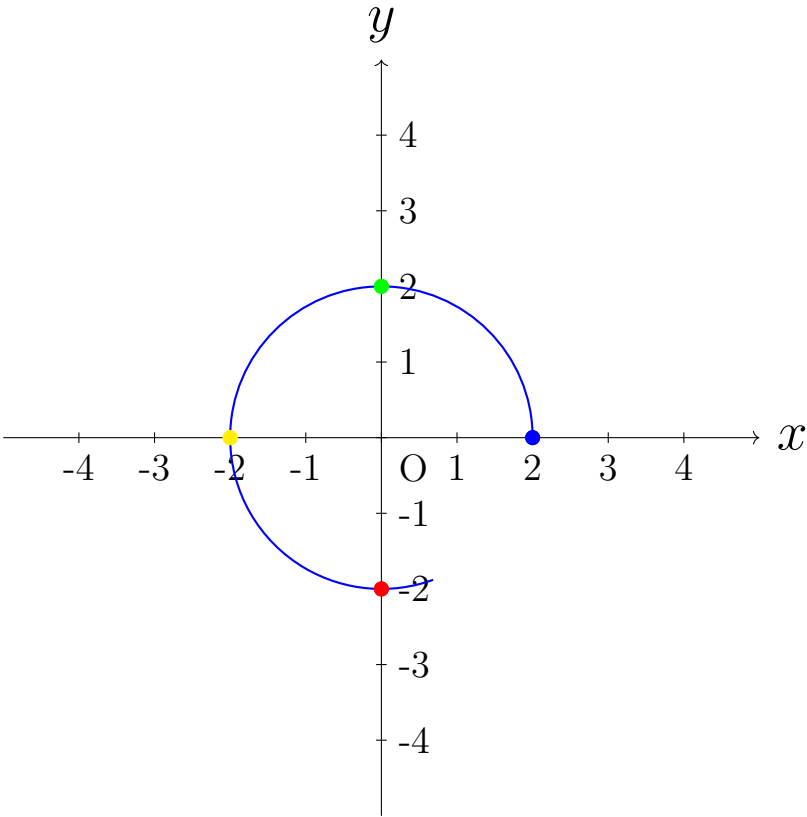
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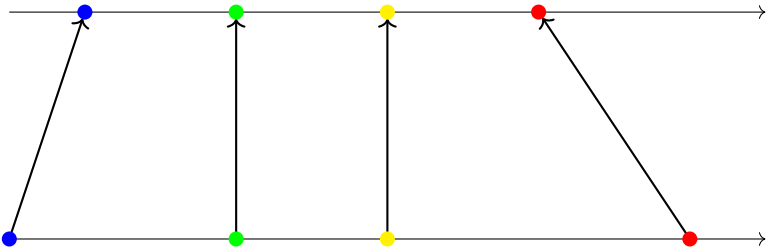
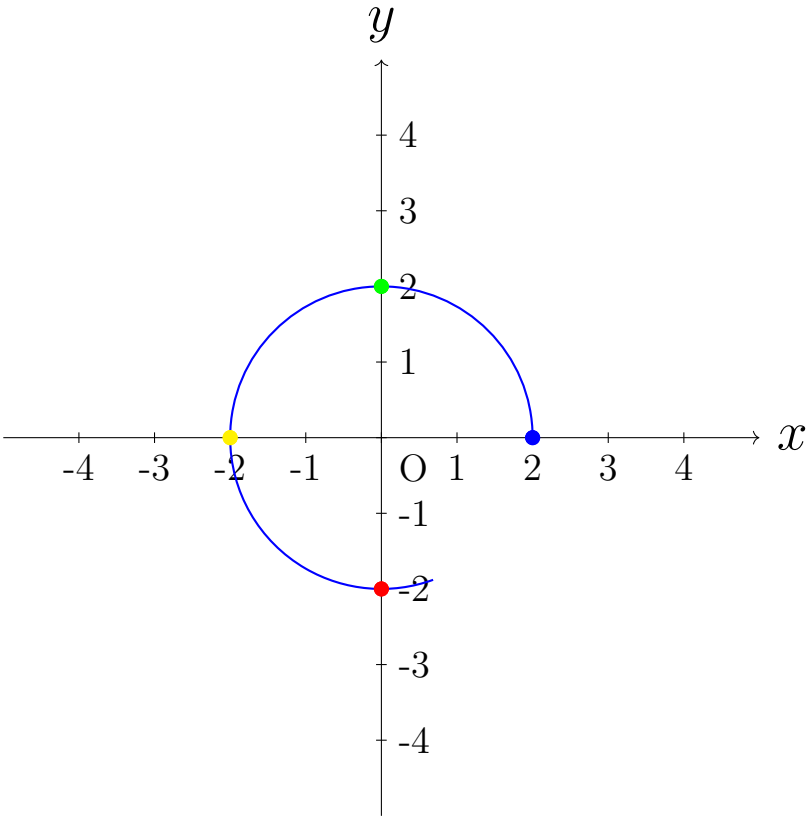
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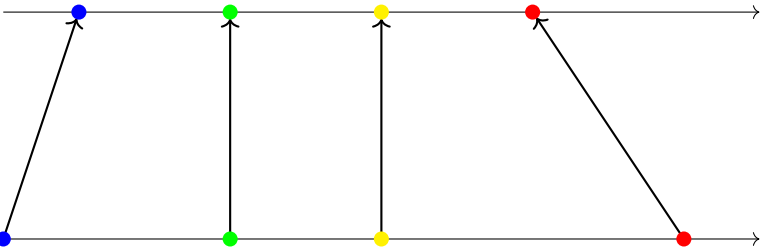
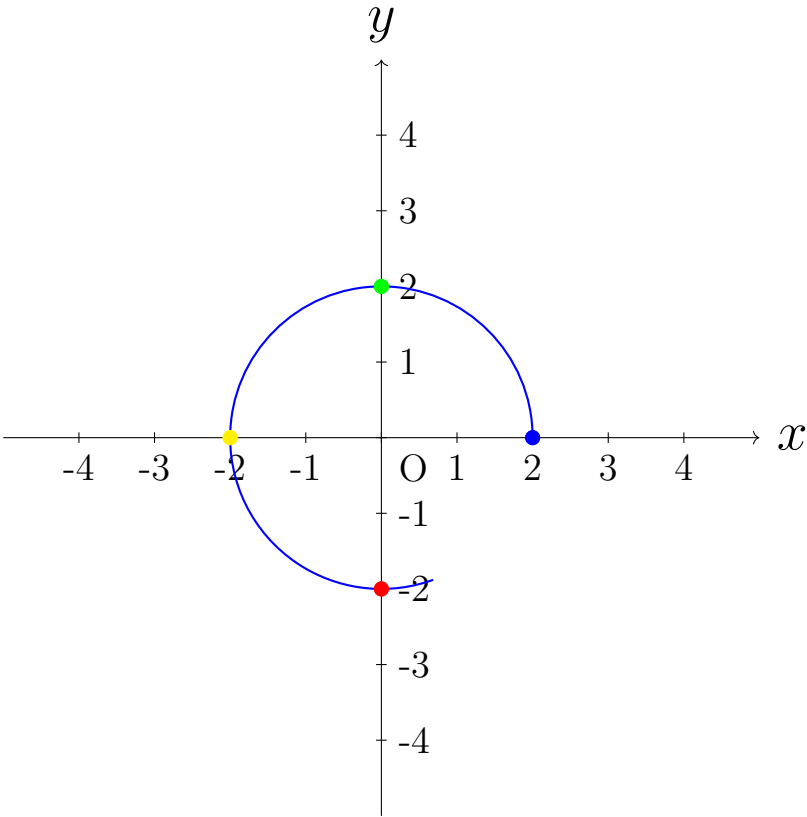
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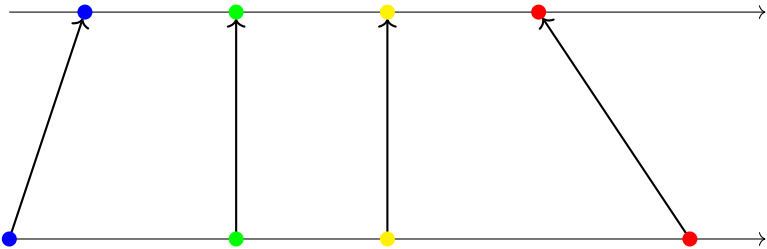
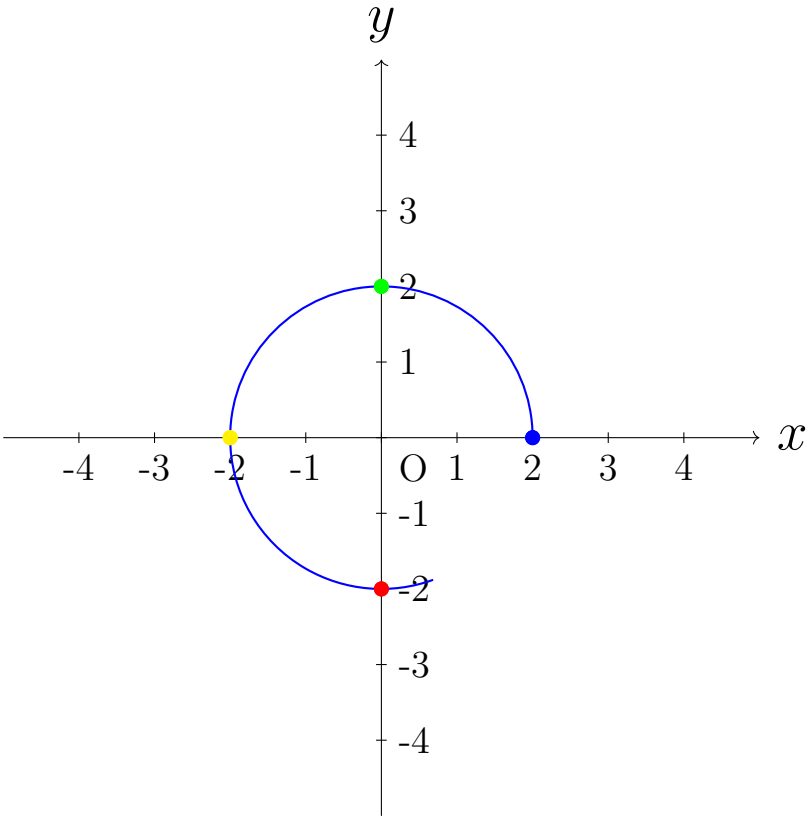
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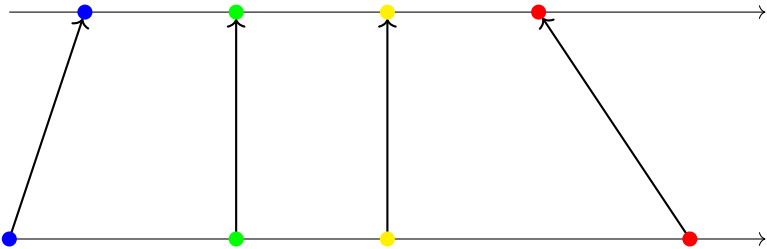
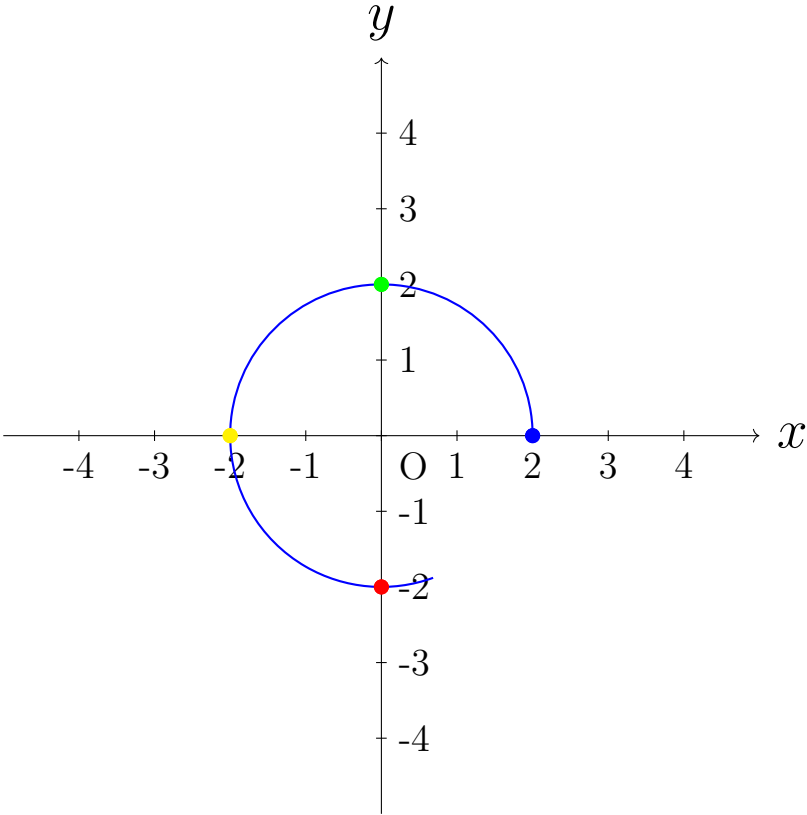
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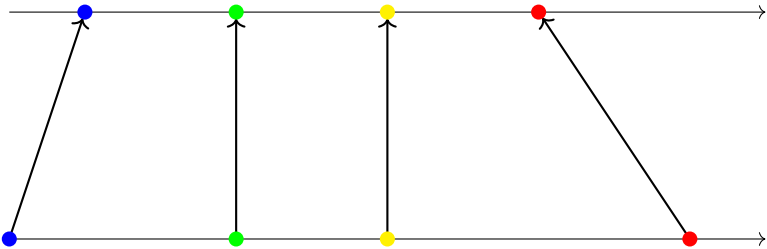
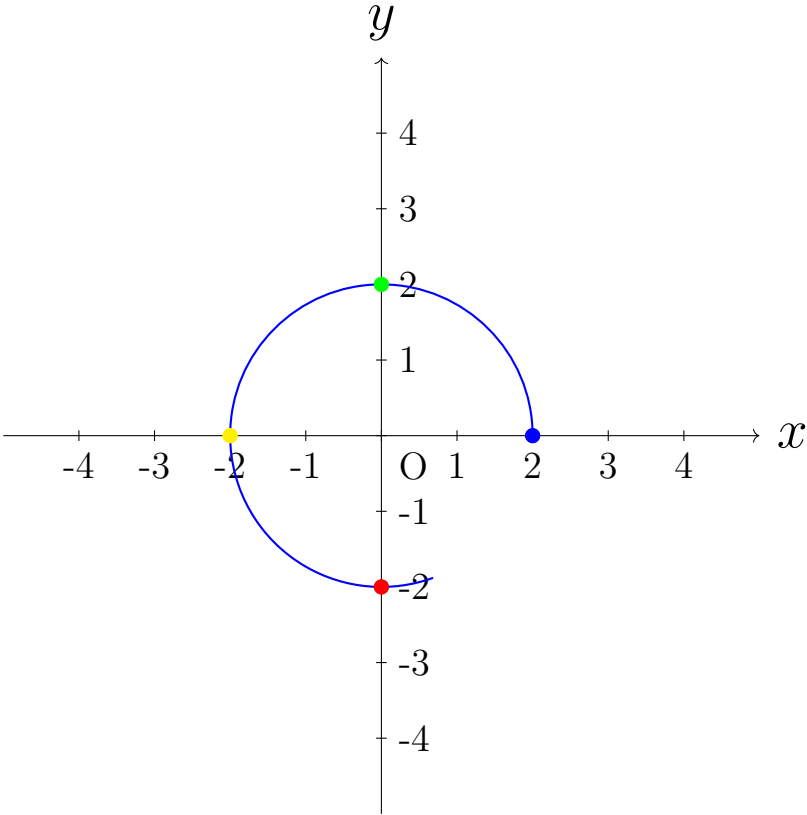
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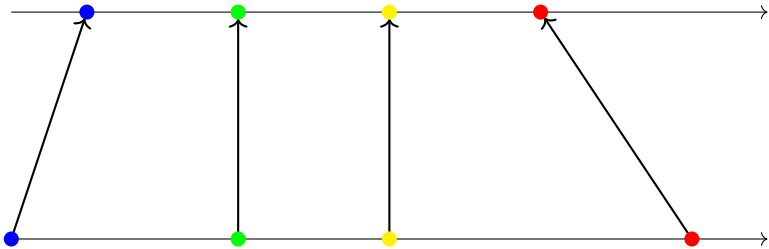
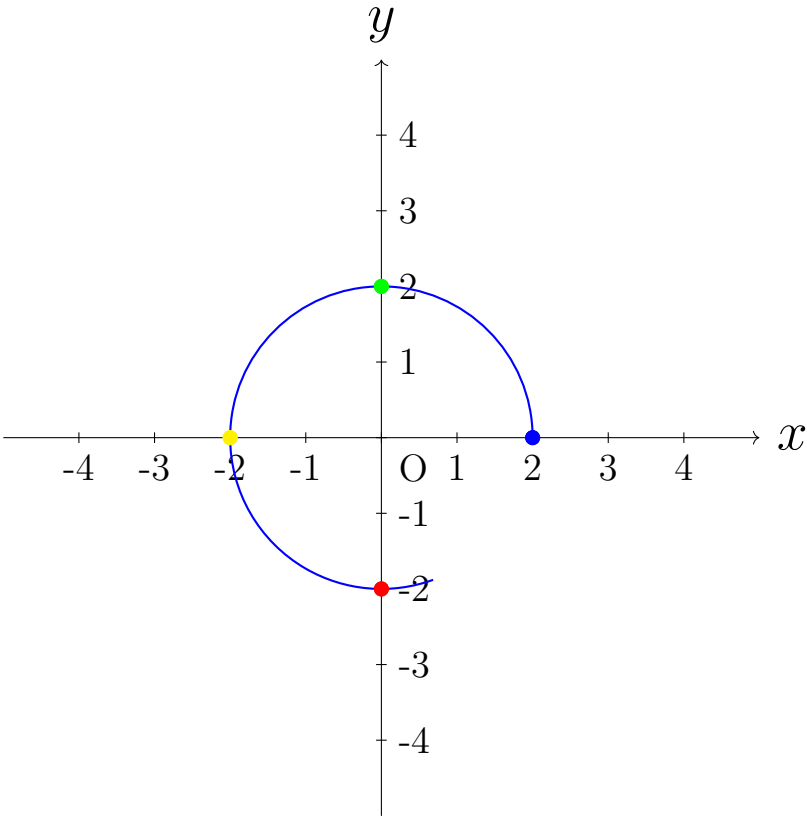
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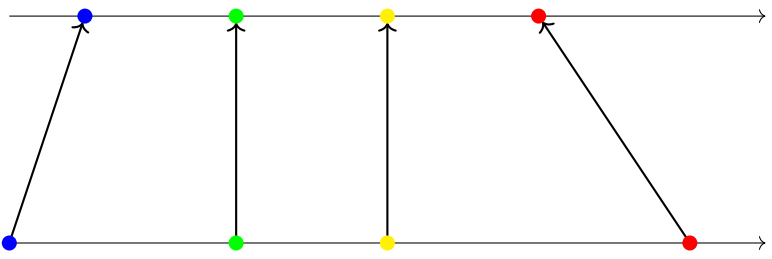
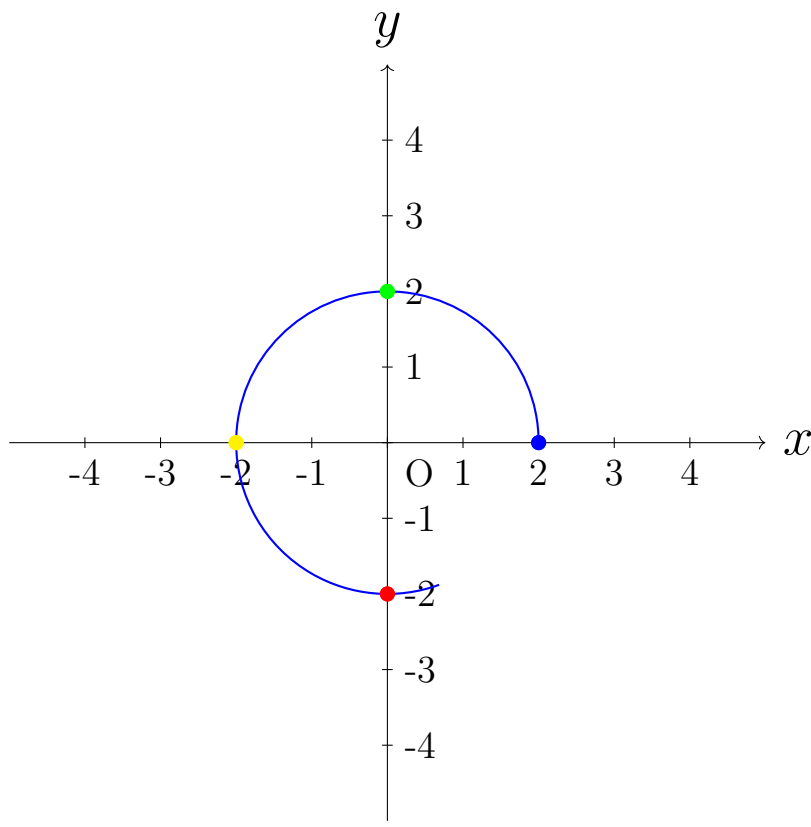
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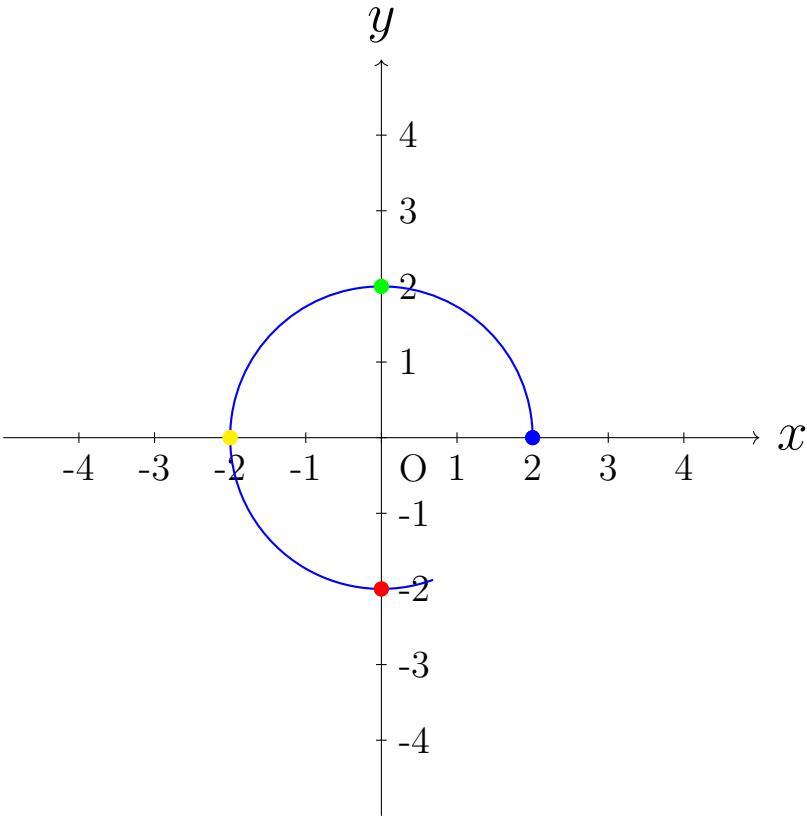
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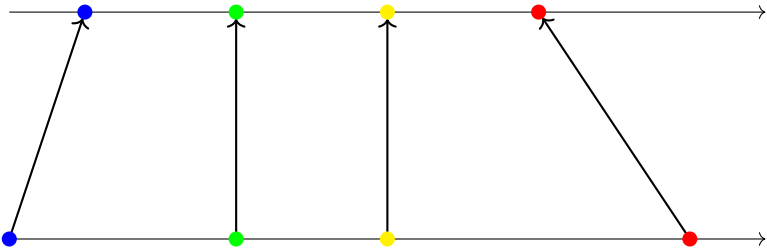
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Explictly:



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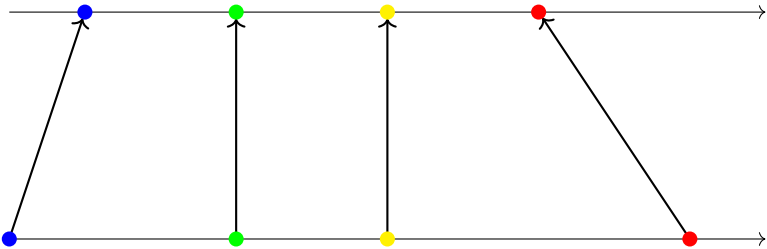
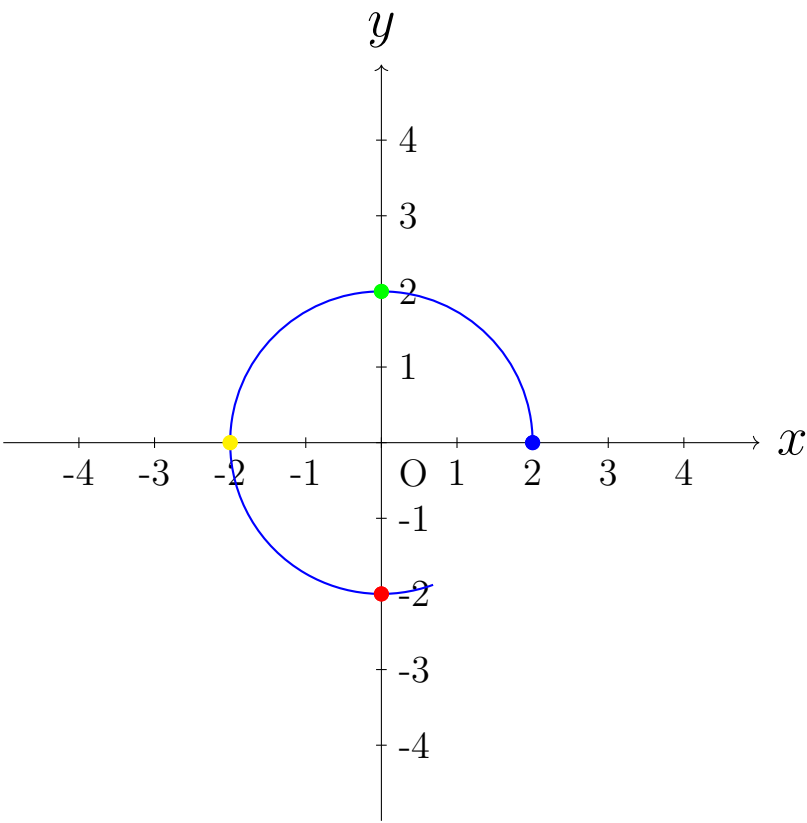
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$\gamma(t) = (f_1(t), f_2(t))$



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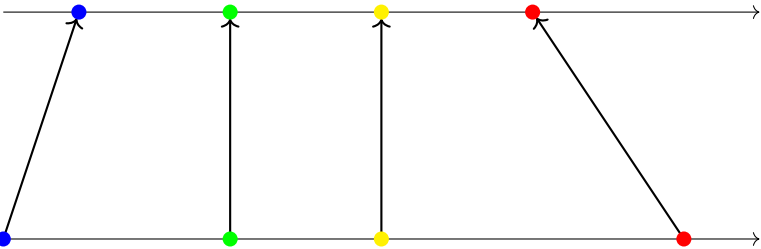
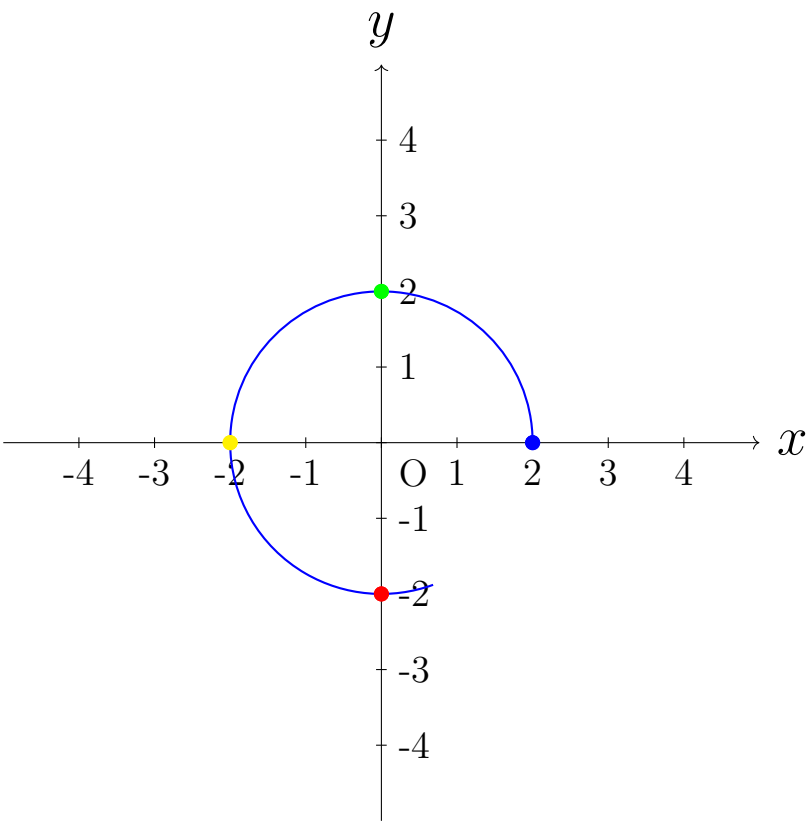
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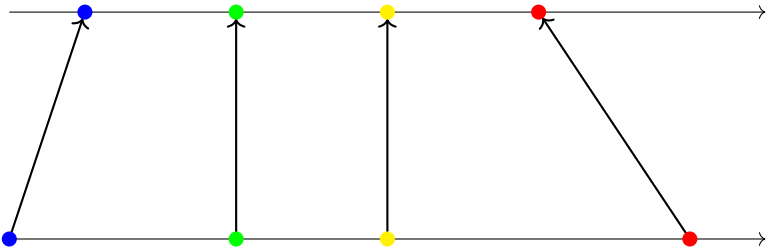
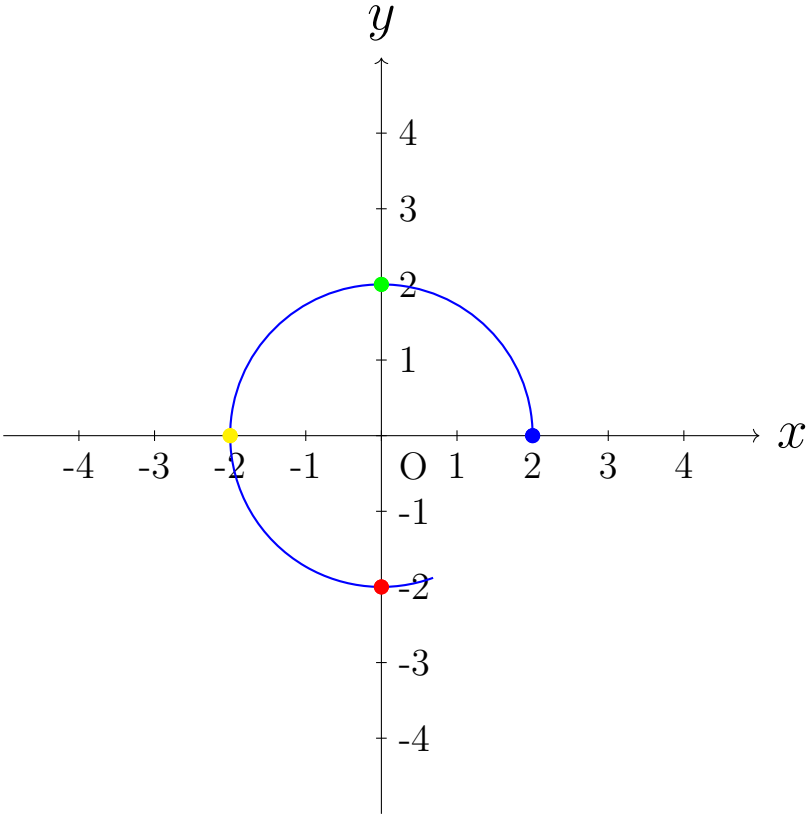
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$$\phi : (-1/2, 1/2) \rightarrow (-1, 1)$$

$$\phi(t) = 2t$$

Example

$$\gamma : (-1, 1) \rightarrow \mathbb{R}^2$$

$$\gamma(t) = (t, t)$$

$$\tilde{\gamma} : (-1/2, 1/2) \rightarrow \mathbb{R}^2.$$

$$\tilde{\gamma}(t) = (2t, 2t)$$

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So that $\tilde{\gamma}(t)$

Example

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Example

$$\gamma : (-1, 1) \rightarrow \mathbb{R}^2$$

$$\gamma(t) = (t, t)$$

$$\tilde{\gamma} : (-1/2, 1/2) \rightarrow \mathbb{R}^2.$$

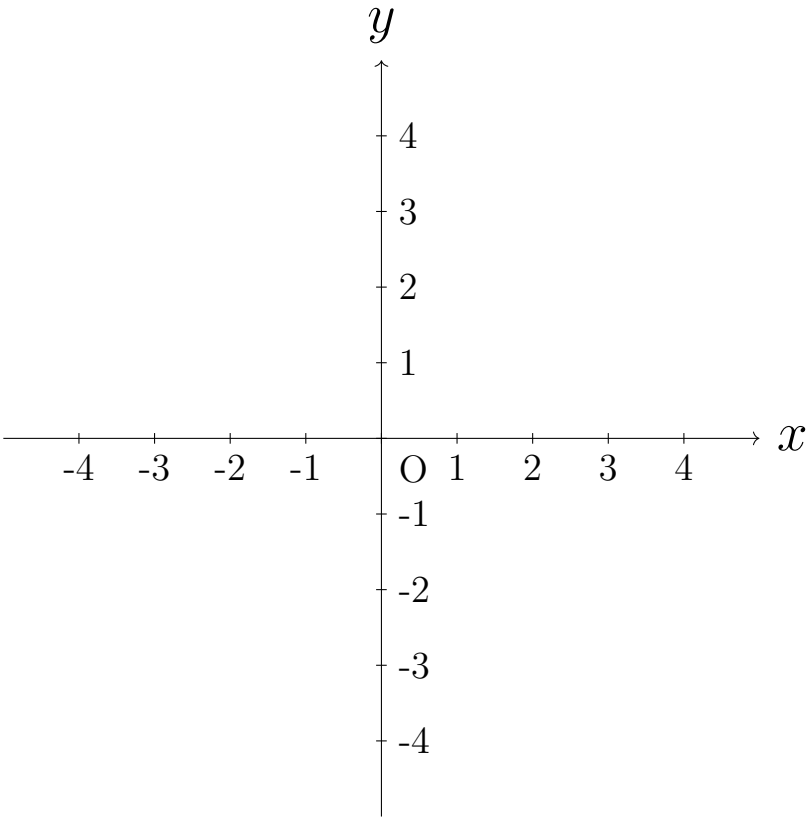
$$\tilde{\gamma}(t) = (2t, 2t)$$

$$\phi : (-1/2, 1/2) \rightarrow (-1, 1)$$

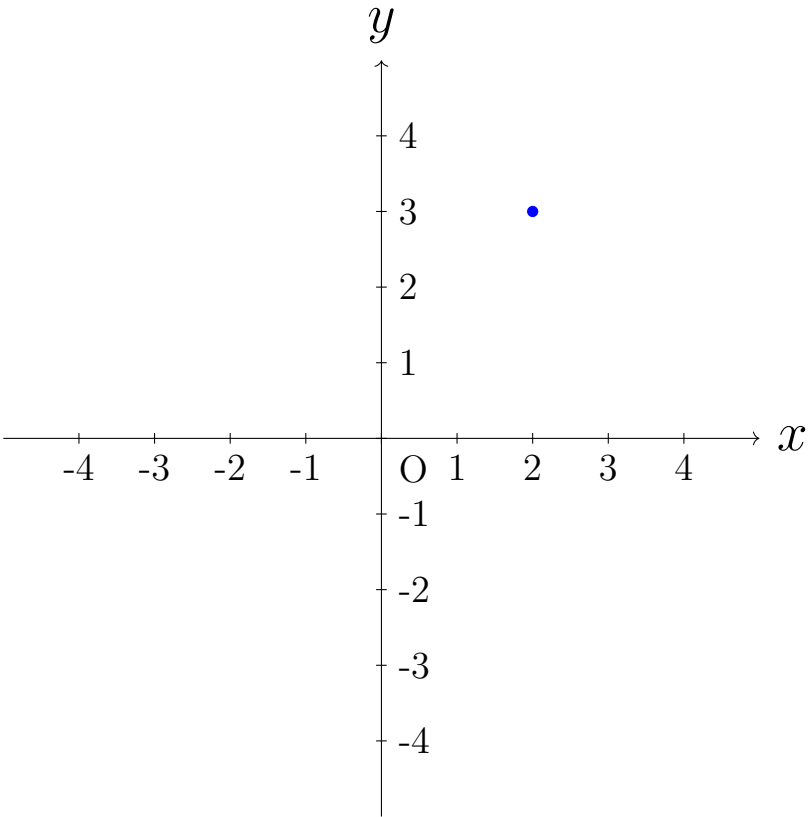
$$\phi(t) = 2t$$

$$\text{So that } \tilde{\gamma}(t) = \gamma(\phi(t)) = \gamma(2t) = (2t, 2t)$$

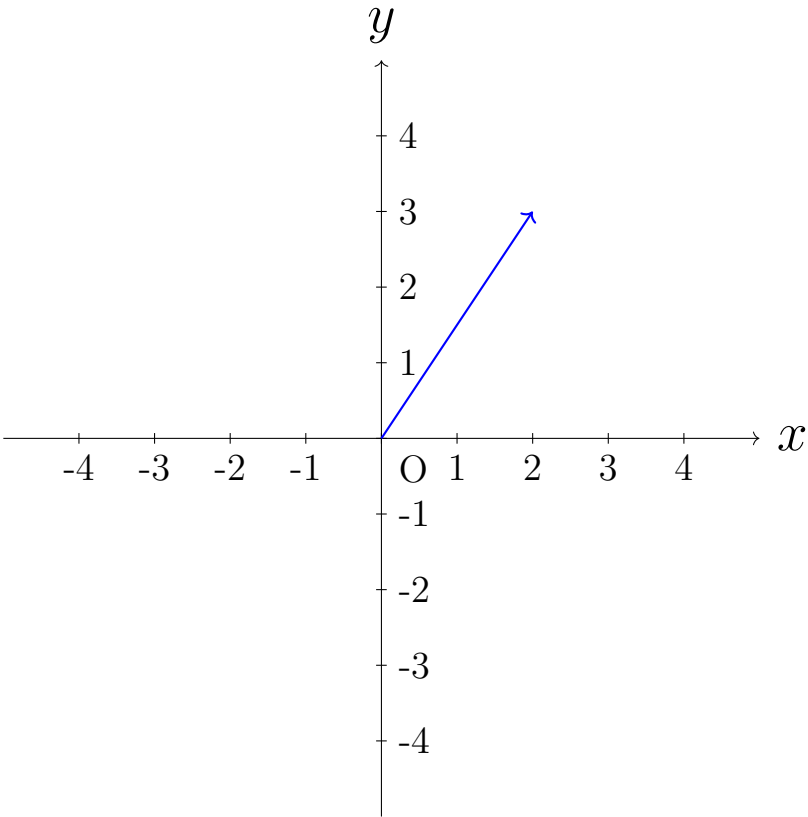
Vectors



Vectors

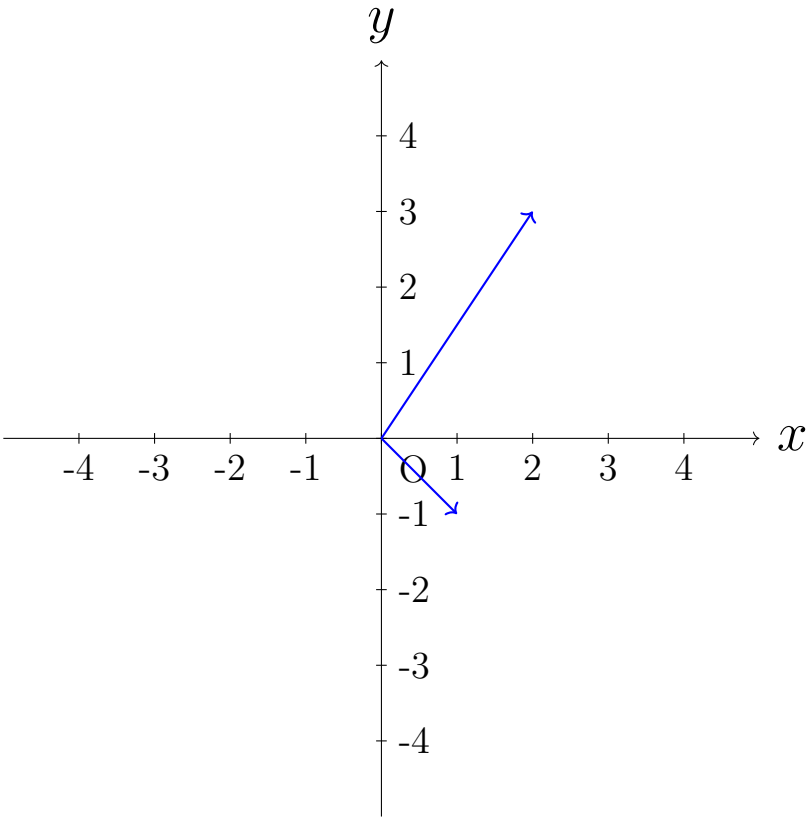


Vectors



Vectors

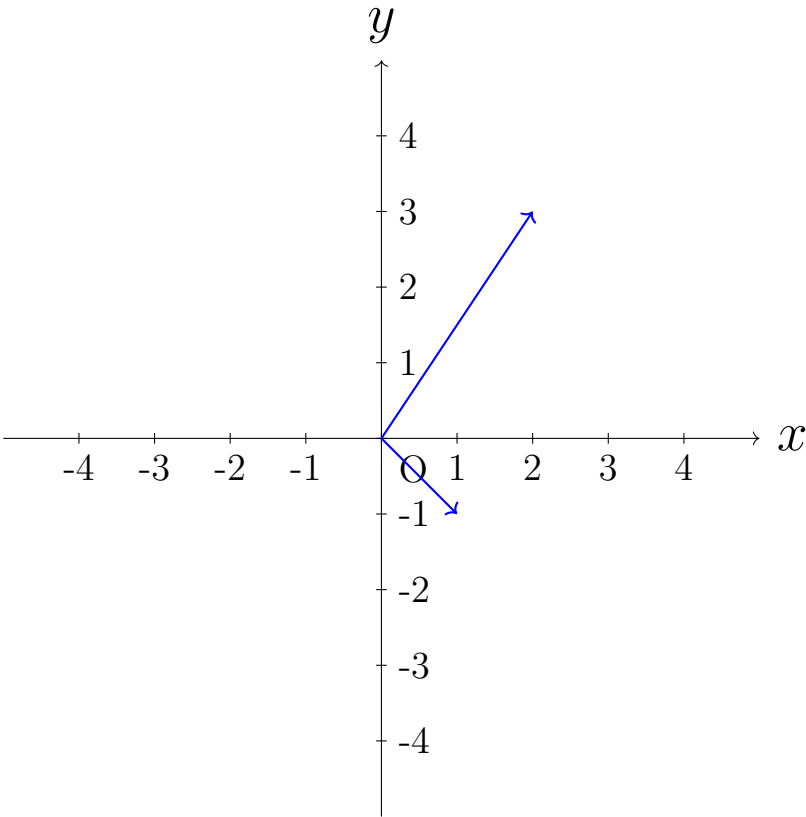
$$v = (2, 3)$$



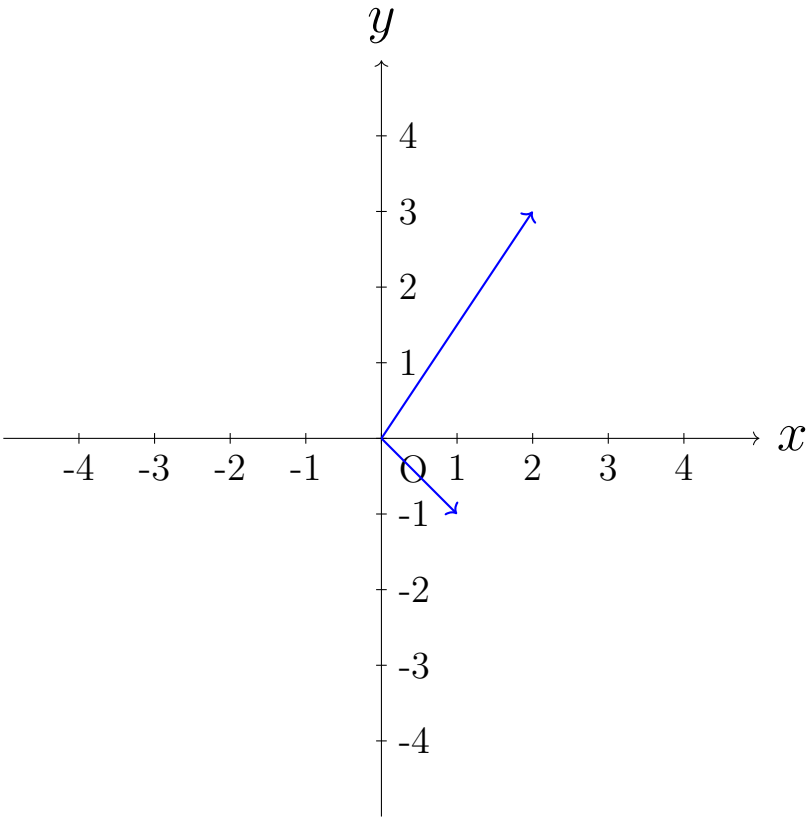
Vectors

$$v = (2, 3)$$

$$w = (1, -1)$$



Vectors

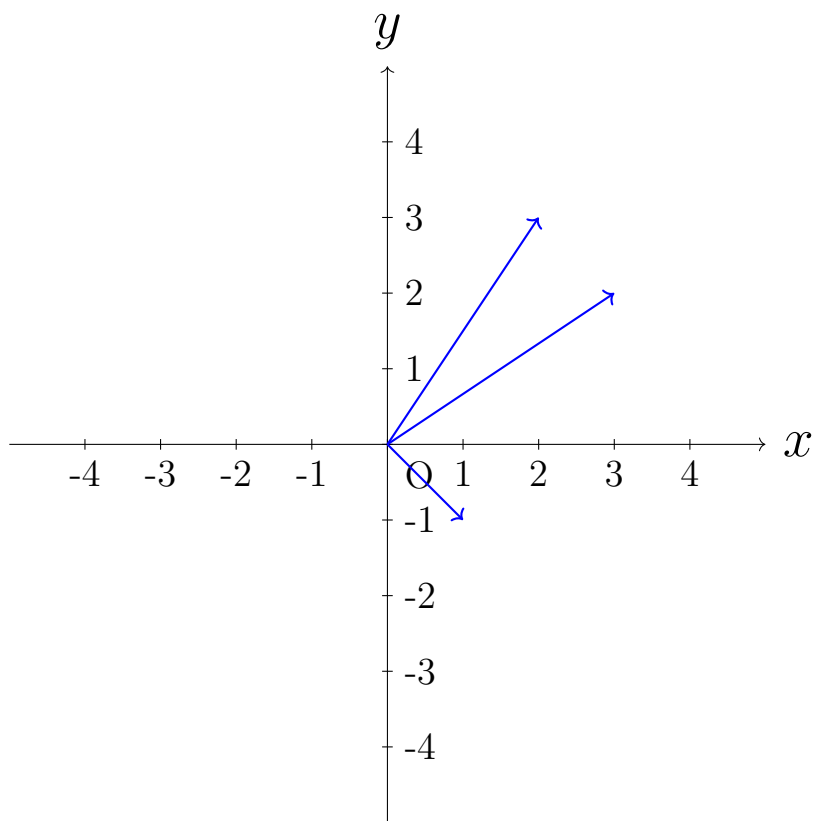


$$v = (2, 3)$$

$$w = (1, -1)$$

Vector addition :

Vectors



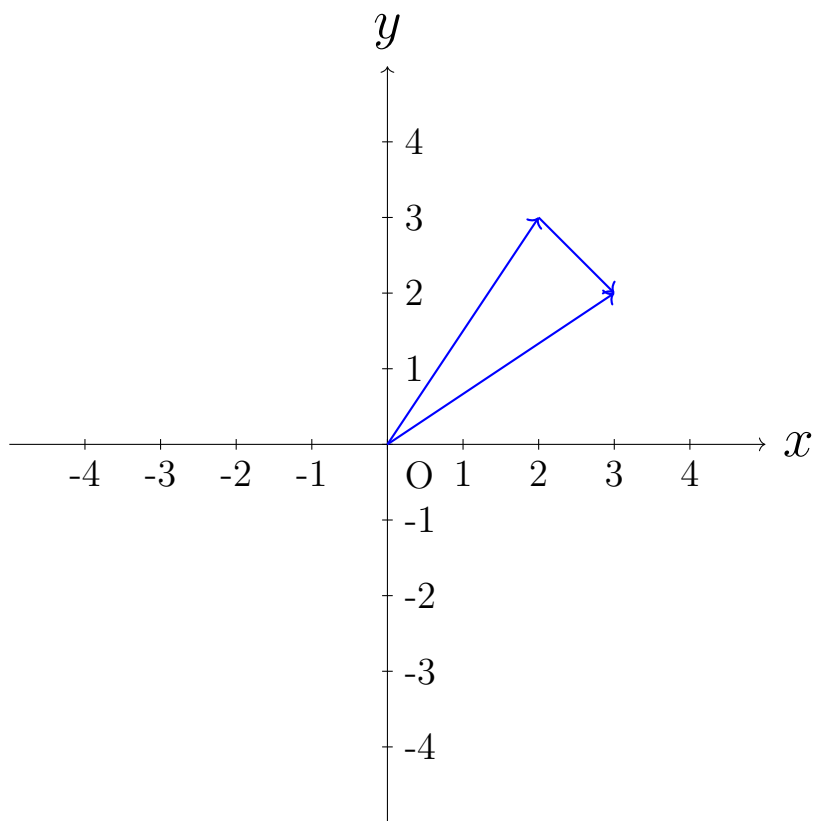
$$v = (2, 3)$$

$$w = (1, -1)$$

Vector addition :

$$v + w = (3, 2)$$

Vectors



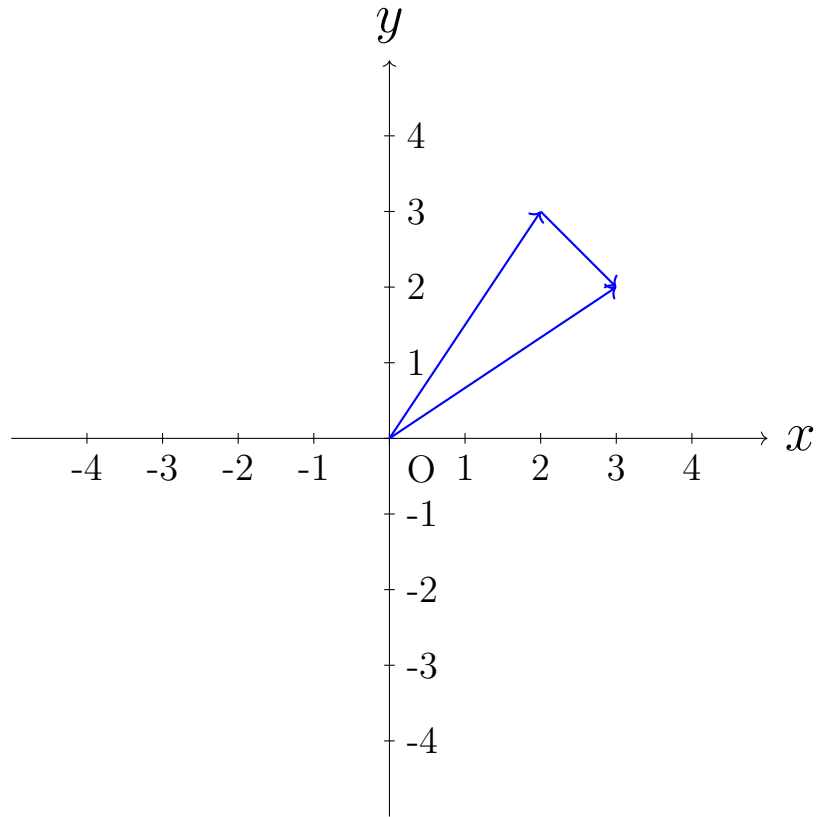
$$v = (2, 3)$$

$$w = (1, -1)$$

Vector addition :

$$v + w = (3, 2)$$

Vectors



$$v = (2, 3)$$

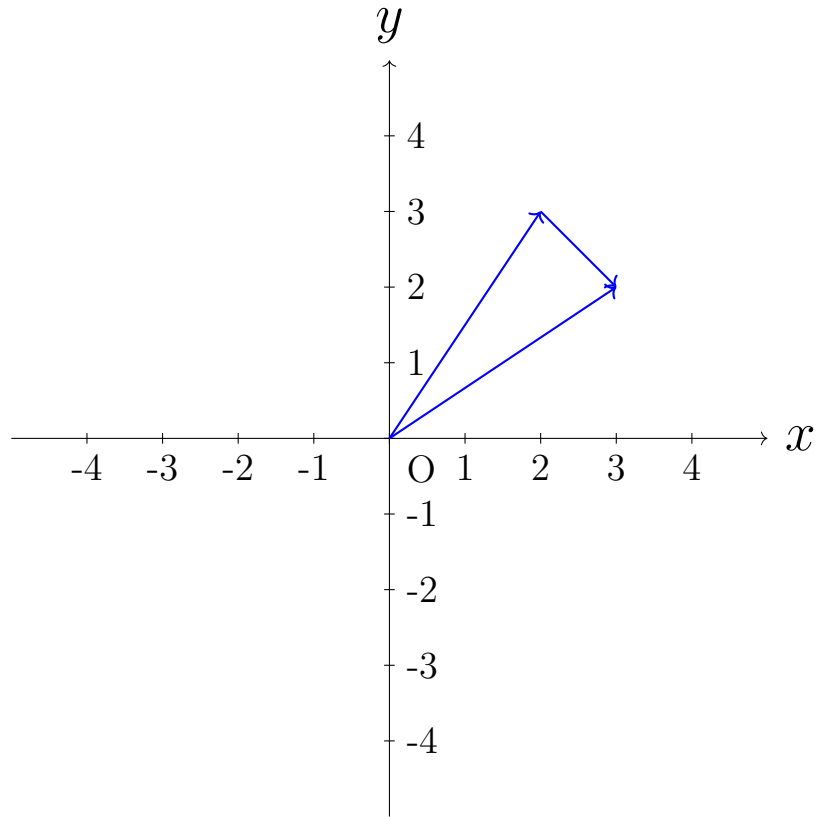
$$w = (1, -1)$$

Vector addition :

$$v + w = (3, 2)$$

In general:

Vectors



$$v = (2, 3)$$

$$w = (1, -1)$$

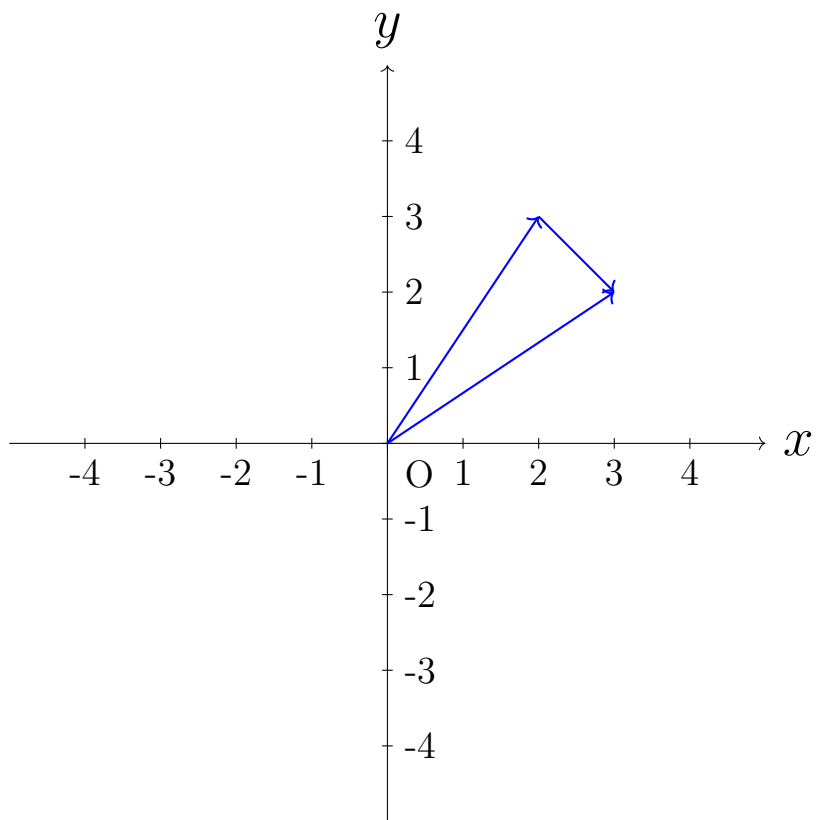
Vector addition :

$$v + w = (3, 2)$$

In general:

$$(x_1, y_1) + (x_2, y_2)$$

Vectors



$$v = (2, 3)$$

$$w = (1, -1)$$

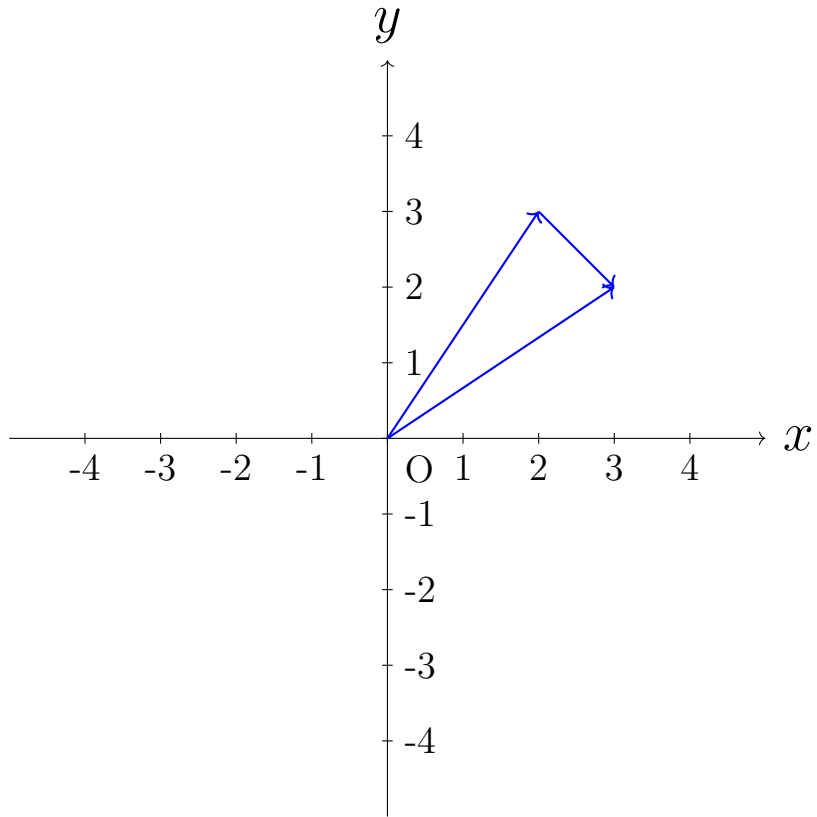
Vector addition :

$$v + w = (3, 2)$$

In general:

$$(x_1, y_1) + (x_2, y_2) := (x_1 + x_2, y_1 + y_2)$$

Vectors



$$v = (2, 3)$$

$$w = (1, -1)$$

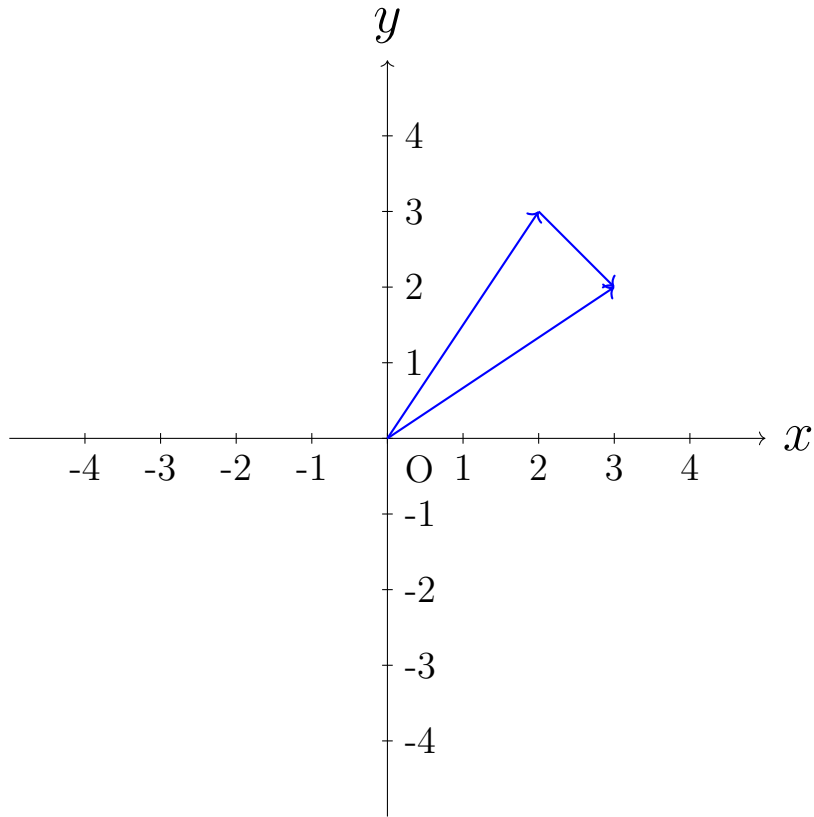
Vector addition :

$$v + w = (3, 2)$$

In general:

$$(x_1, y_1) + (x_2, y_2) := (x_1 + x_2, y_1 + y_2)$$

Vectors



$$v = (2, 3)$$

$$w = (1, -1)$$

Vector addition :

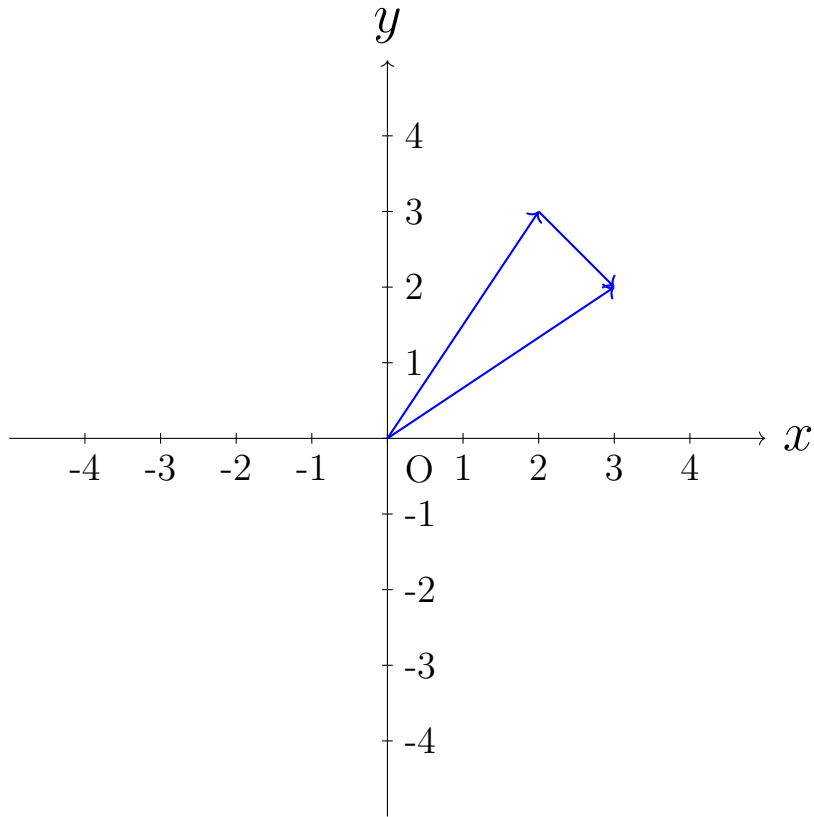
$$v + w = (3, 2)$$

In general:

$$(x_1, y_1) + (x_2, y_2) := (x_1 + x_2, y_1 + y_2)$$

$$(x_1, y_1) - (x_2, y_2)$$

Vectors



$$v = (2, 3)$$

$$w = (1, -1)$$

Vector addition :

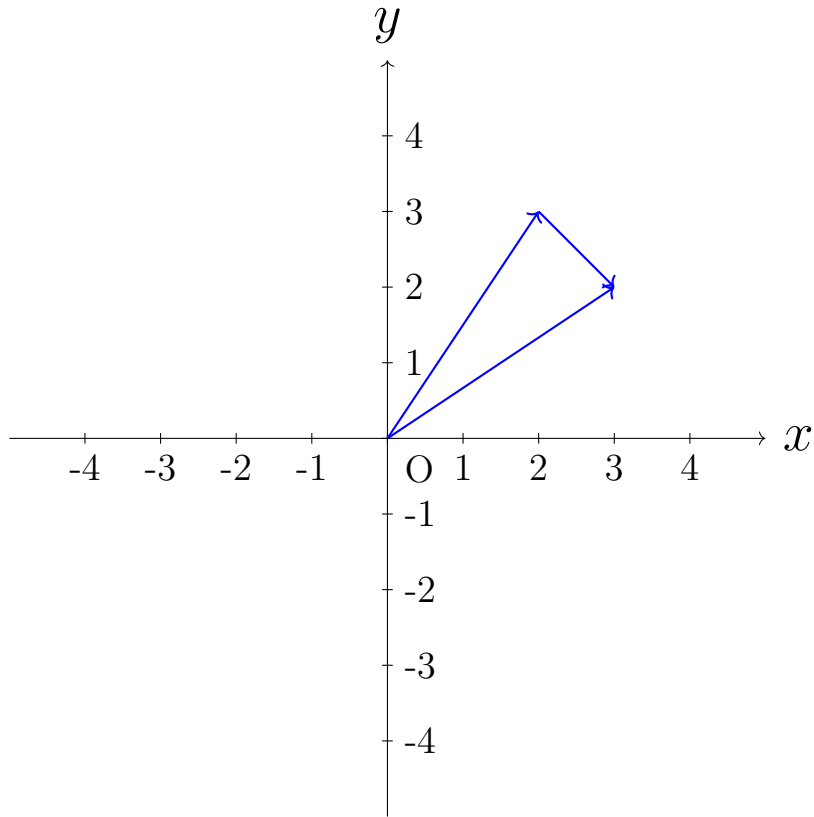
$$v + w = (3, 2)$$

In general:

$$(x_1, y_1) + (x_2, y_2) := (x_1 + x_2, y_1 + y_2)$$

$$(x_1, y_1) - (x_2, y_2) := (x_1 - x_2, y_1 - y_2)$$

Vectors



$$v = (2, 3)$$

$$w = (1, -1)$$

Vector addition :

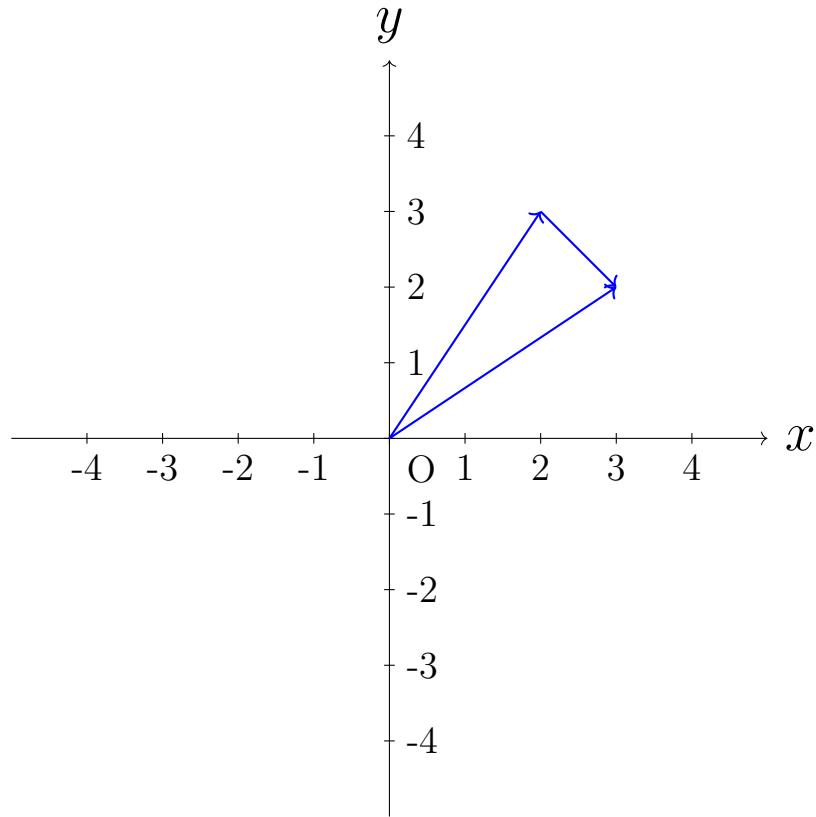
$$v + w = (3, 2)$$

In general:

$$(x_1, y_1) + (x_2, y_2) := (x_1 + x_2, y_1 + y_2)$$

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Vectors



$$v = (2, 3)$$

$$w = (1, -1)$$

Vector addition and subtraction :

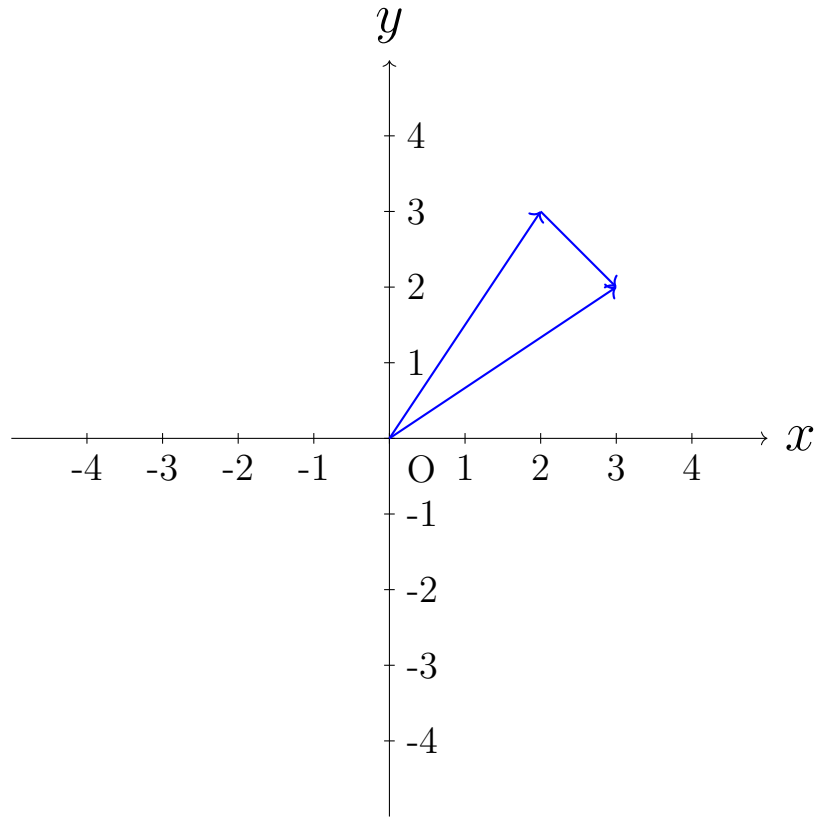
$$v + w = (3, 2)$$

In general:

$$(x_1, y_1) + (x_2, y_2) := (x_1 + x_2, y_1 + y_2)$$

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Vectors



$$v = (2, 3)$$

$$w = (1, -1)$$

Vector addition and subtraction :

$$v + w = (3, 2)$$

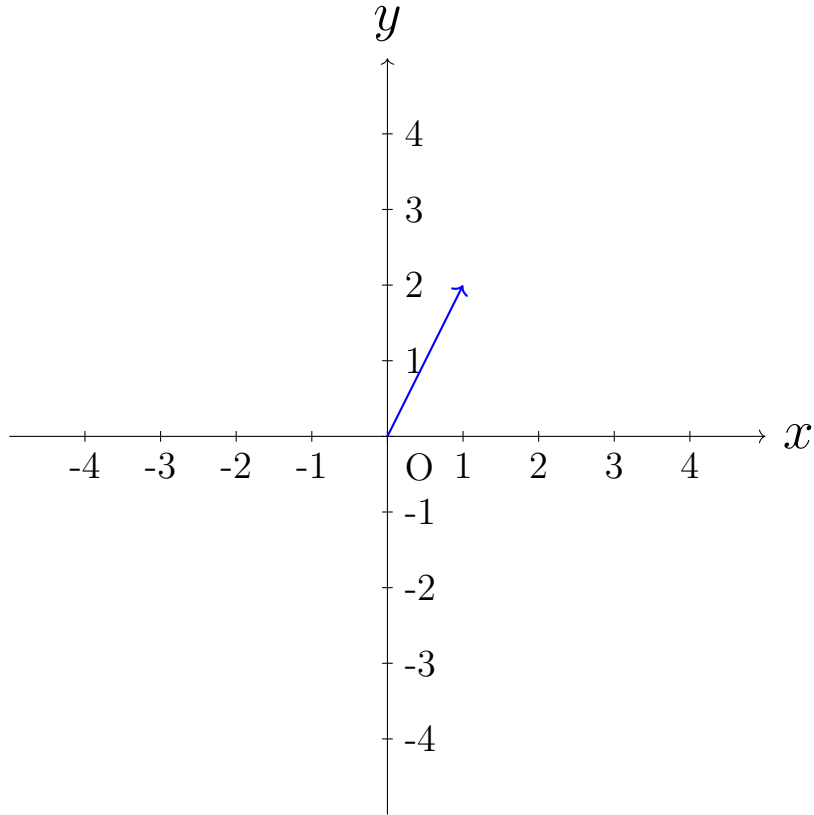
In general:

$$(x_1, y_1) + (x_2, y_2) := (x_1 + x_2, y_1 + y_2)$$

$$(x_1, y_1) - (x_2, y_2) := (x_1 - x_2, y_1 - y_2)$$

Scalar multiplication:

Vectors



$$v = (2, 3)$$

$$w = (1, -1)$$

Vector addition and subtraction :

$$v + w = (3, 2)$$

In general:

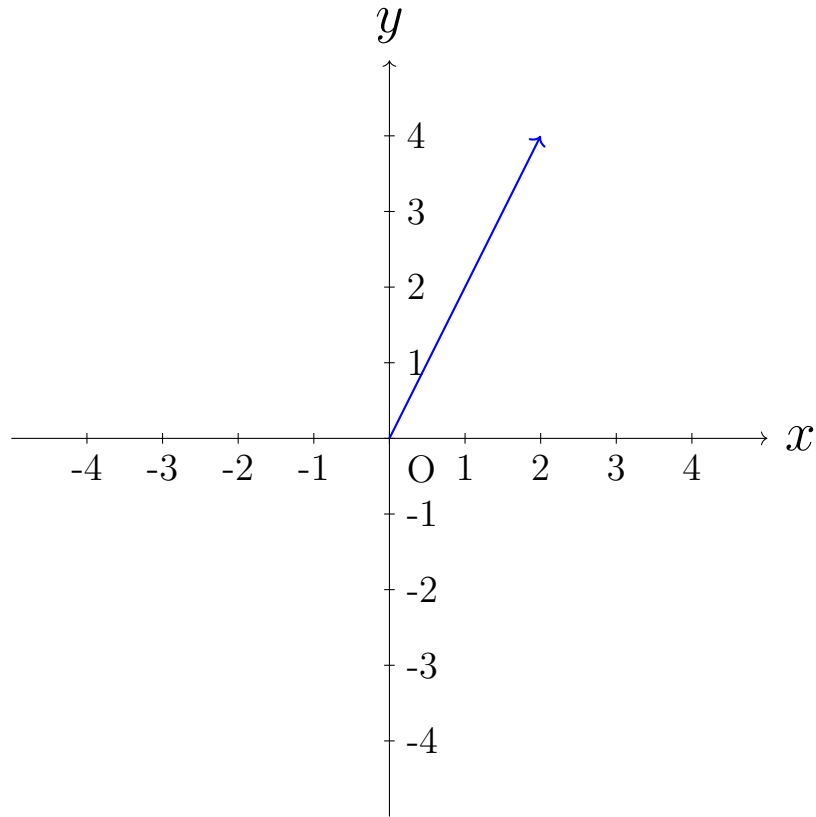
$$(x_1, y_1) + (x_2, y_2) := (x_1 + x_2, y_1 + y_2)$$

$$(x_1, y_1) - (x_2, y_2) := (x_1 - x_2, y_1 - y_2)$$

Scalar multiplication:

$$v := (1, 2)$$

Vectors



$$v = (2, 3)$$

$$w = (1, -1)$$

Vector addition and subtraction :

$$v + w = (3, 2)$$

In general:

$$(x_1, y_1) + (x_2, y_2) := (x_1 + x_2, y_1 + y_2)$$

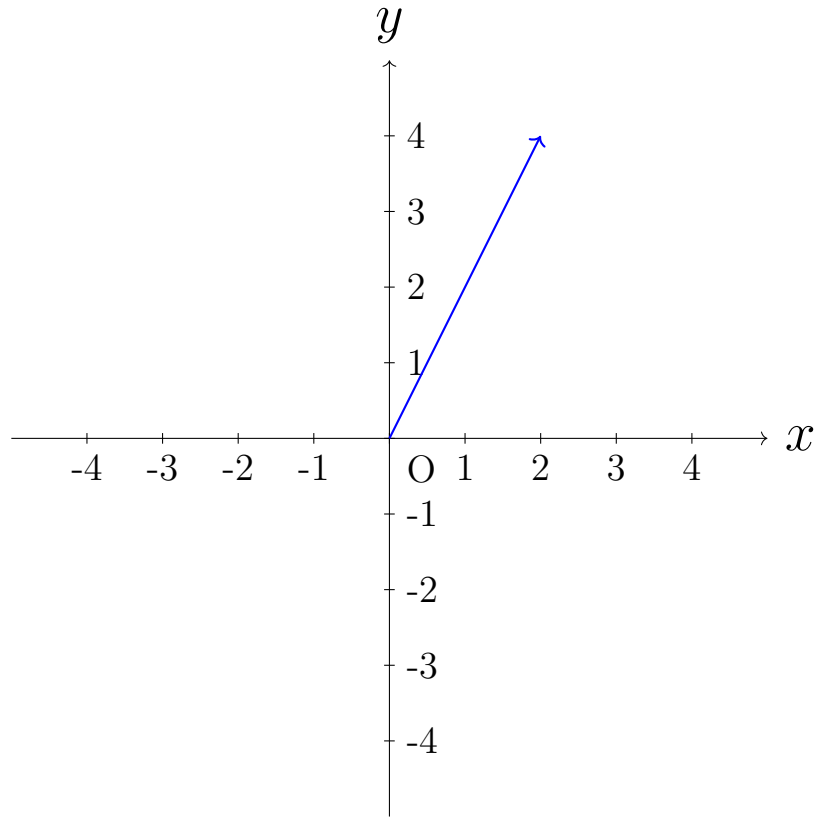
$$(x_1, y_1) - (x_2, y_2) := (x_1 - x_2, y_1 - y_2)$$

Scalar multiplication:

$$v := (1, 2)$$

$$2v$$

Vectors



$$v = (2, 3)$$

$$w = (1, -1)$$

Vector addition and subtraction :

$$v + w = (3, 2)$$

In general:

$$(x_1, y_1) + (x_2, y_2) := (x_1 + x_2, y_1 + y_2)$$

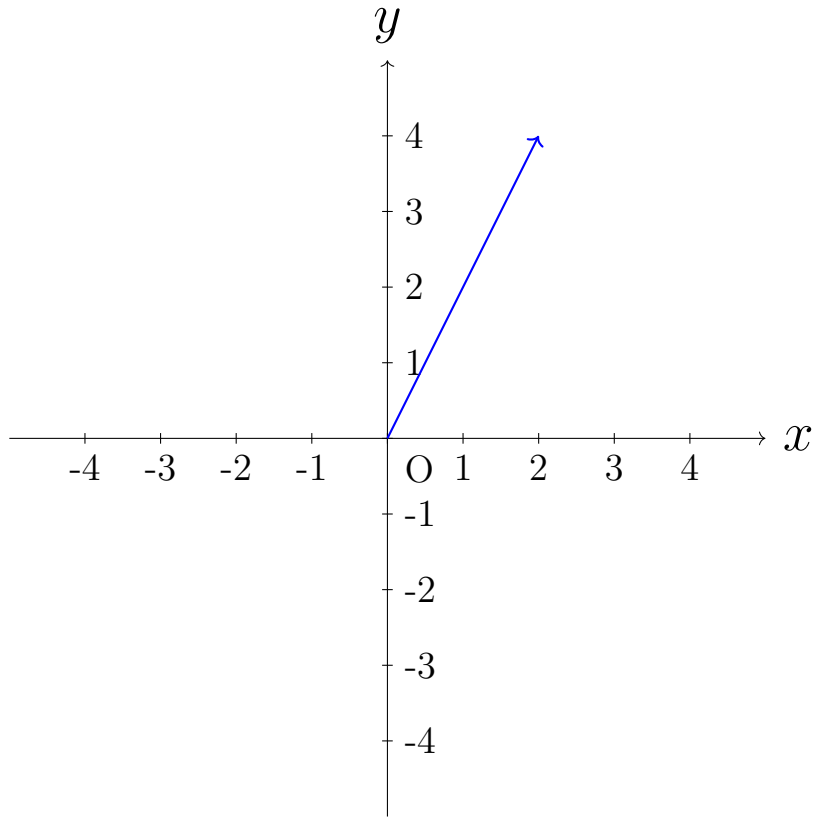
$$(x_1, y_1) - (x_2, y_2) := (x_1 - x_2, y_1 - y_2)$$

Scalar multiplication:

$$v := (1, 2)$$

$$2v = 2(1, 2)$$

Vectors



$$v = (2, 3)$$

$$w = (1, -1)$$

Vector addition and subtraction :

$$v + w = (3, 2)$$

In general:

$$(x_1, y_1) + (x_2, y_2) := (x_1 + x_2, y_1 + y_2)$$

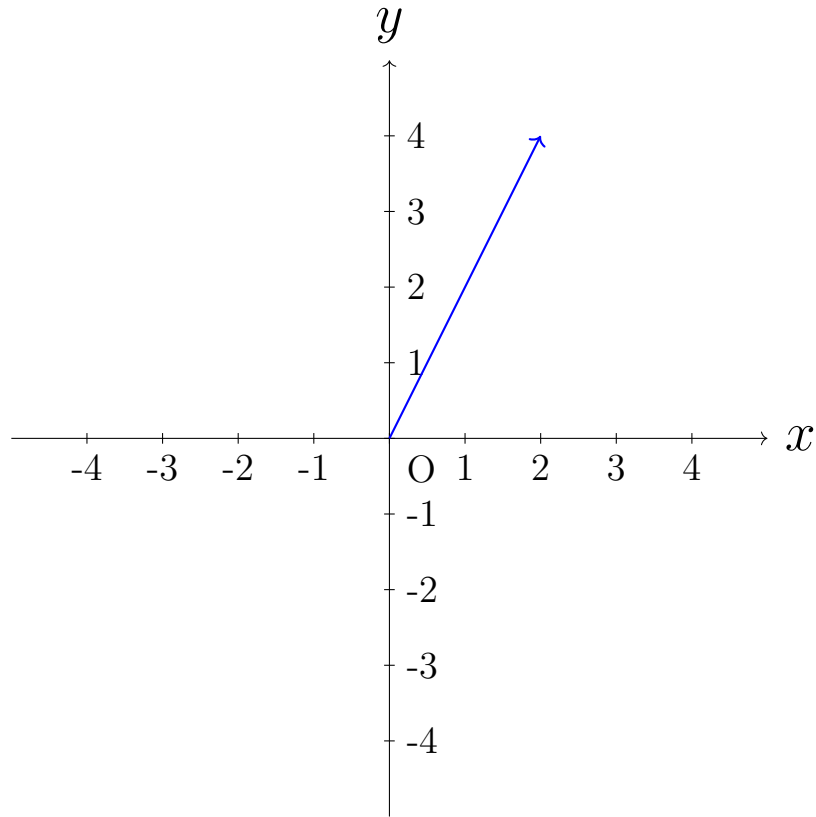
$$(x_1, y_1) - (x_2, y_2) := (x_1 - x_2, y_1 - y_2)$$

Scalar multiplication:

$$v := (1, 2)$$

$$2v = 2(1, 2) = (2, 4)$$

Vectors



$$v = (2, 3)$$

$$w = (1, -1)$$

Vector addition and subtraction :

$$v + w = (3, 2)$$

In general:

$$(x_1, y_1) + (x_2, y_2) := (x_1 + x_2, y_1 + y_2)$$

$$(x_1, y_1) - (x_2, y_2) := (x_1 - x_2, y_1 - y_2)$$

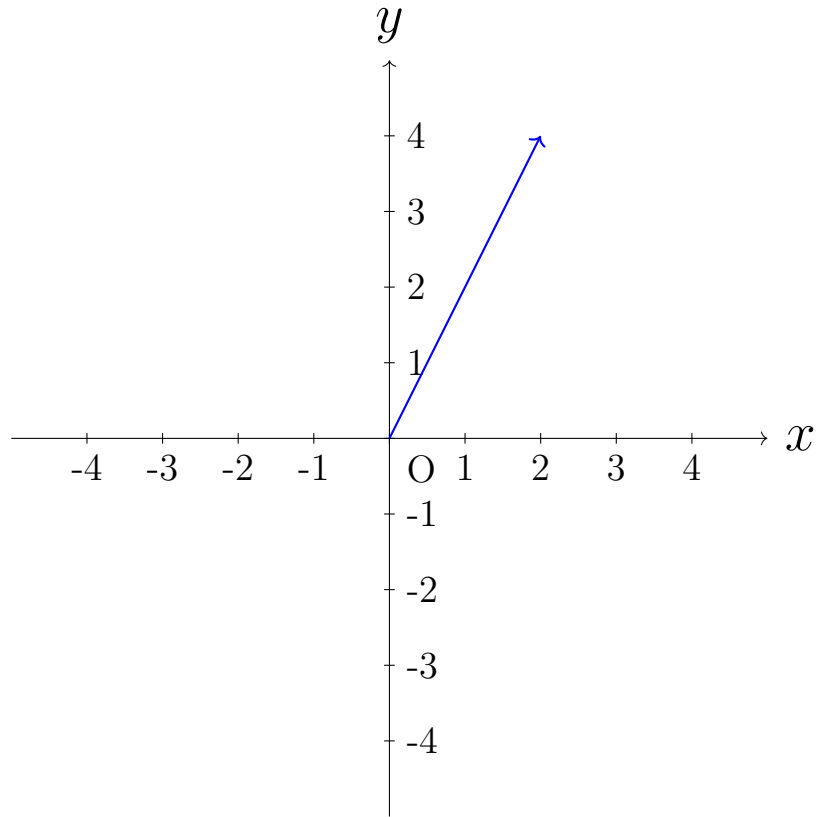
Scalar multiplication:

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In general:

Vectors



$$v = (2, 3)$$

$$w = (1, -1)$$

Vector addition and subtraction :

$$v + w = (3, 2)$$

In general:

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$$(x_1, y_1) - (x_2, y_2) := (x_1 - x_2, y_1 - y_2)$$

Scalar multiplication:

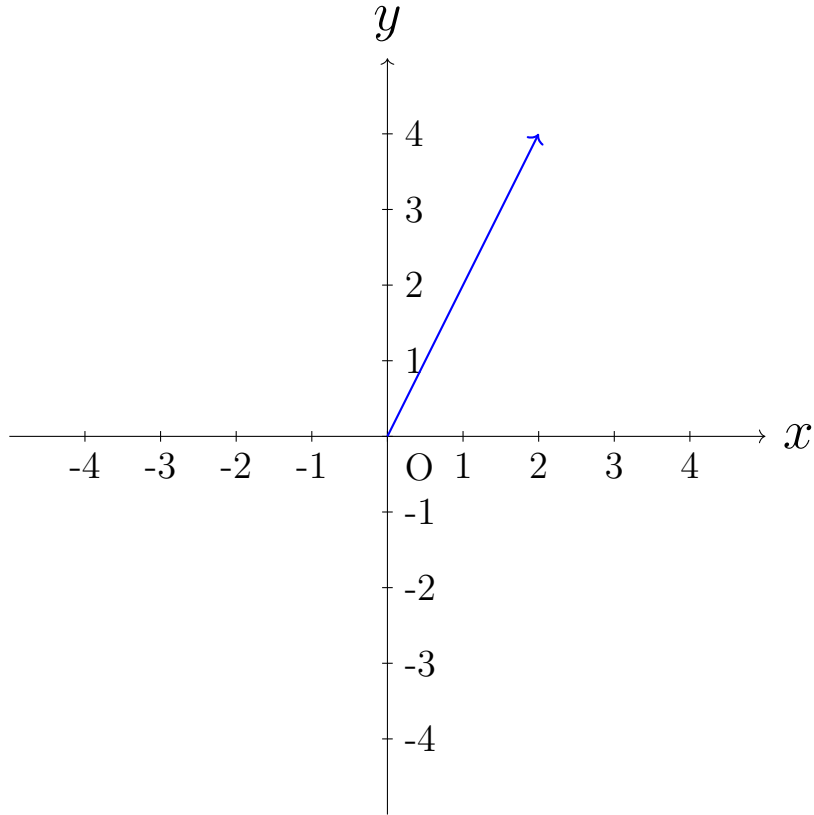
$$v := (1, 2)$$

$$2v = 2(1, 2) = (2, 4)$$

In general:

$$\lambda(x, y)$$

Vectors



$$v = (2, 3)$$

$$w = (1, -1)$$

Vector addition and subtraction :

$$v + w = (3, 2)$$

In general:

$$(x_1, y_1) + (x_2, y_2) := (x_1 + x_2, y_1 + y_2)$$

$$(x_1, y_1) - (x_2, y_2) := (x_1 - x_2, y_1 - y_2)$$

Scalar multiplication:

$$v := (1, 2)$$

$$2v = 2(1, 2) = (2, 4)$$

In general:

$$\lambda(x, y) := (\lambda x, \lambda y)$$

$$p := (2, 3),$$

$$p := (2, 3),$$

$$\mathbf{w} := (1, 1),$$

$$p := (2, 3),$$

$$\mathbf{w} := (1, 1),$$

$$q := p + \mathbf{w} = (2, 3) + (1, 1) = (3, 4)$$

$$p := (2, 3),$$

$$\mathbf{w} := (1, 1),$$

$$q := p + \mathbf{w} = (2, 3) + (1, 1) = (3, 4)$$

(displacement of p by \mathbf{w}).

$$p := (2, 3),$$

$$\mathbf{w} := (1, 1),$$

$$q := p + \mathbf{w} = (2, 3) + (1, 1) = (3, 4)$$

(displacement of p by \mathbf{w}).

$$p := (2, 3) \text{ and } q = (3, 4),$$

$$p := (2, 3),$$

$$\mathbf{w} := (1, 1),$$

$$q := p + \mathbf{w} = (2, 3) + (1, 1) = (3, 4)$$

(displacement of p by \mathbf{w}).

$$p := (2, 3) \text{ and } q = (3, 4),$$

$$\mathbf{v} = q - p$$

$p := (2, 3)$,
 $\mathbf{w} := (1, 1)$,
 $q := p + \mathbf{w} = (2, 3) + (1, 1) = (3, 4)$
(displacement of p by \mathbf{w}).

$p := (2, 3)$ and $q = (3, 4)$,
 $\mathbf{v} = q - p$ is the displacement

$$p := (2, 3),$$

$$\mathbf{w} := (1, 1),$$

$$q := p + \mathbf{w} = (2, 3) + (1, 1) = (3, 4)$$

(displacement of p by \mathbf{w}).

$$p := (2, 3) \text{ and } q = (3, 4),$$

$\mathbf{v} = q - p$ is the displacement that takes p to q

$$p := (2, 3),$$

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(displacement of p by \mathbf{w}).

$$p := (2, 3) \text{ and } q = (3, 4),$$

$\mathbf{v} = q - p$ is the displacement that takes p to q

$$\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$$

$$p := (2, 3),$$

$$\mathbf{w} := (1, 1),$$

$$q := p + \mathbf{w} = (2, 3) + (1, 1) = (3, 4)$$

(displacement of p by \mathbf{w}).

$$p := (2, 3) \text{ and } q = (3, 4),$$

$\mathbf{v} = q - p$ is the displacement that takes p to q

$\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$ is a smooth parametrization.

$p := (2, 3),$
 $\mathbf{w} := (1, 1),$
 $q := p + \mathbf{w} = (2, 3) + (1, 1) = (3, 4)$
(displacement of p by \mathbf{w}).

$p := (2, 3)$ and $q = (3, 4),$
 $\mathbf{v} = q - p$ is the displacement that takes p to q

$\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$ is a smooth parametrization.
 $\gamma(t)$ is the *point* at t

$$\begin{aligned}
p &:= (2, 3), \\
\mathbf{w} &:= (1, 1), \\
q &:= p + \mathbf{w} = (2, 3) + (1, 1) = (3, 4) \\
&\text{(displacement of } p \text{ by } \mathbf{w}\text{).}
\end{aligned}$$

$$\begin{aligned}
p &:= (2, 3) \text{ and } q = (3, 4), \\
\mathbf{v} &= q - p \text{ is the displacement that takes } p \text{ to } q
\end{aligned}$$

$$\begin{aligned}
\gamma : (\alpha, \beta) &\rightarrow \mathbb{R}^2 \text{ is a smooth parametrization.} \\
\gamma(t) &\text{ is the } \textit{point} \text{ at } t \\
\gamma(t + h) &\text{ is the } \textit{point} \text{ at } t + h
\end{aligned}$$

$$p := (2, 3),$$

$$\mathbf{w} := (1, 1),$$

$$q := p + \mathbf{w} = (2, 3) + (1, 1) = (3, 4)$$

(displacement of p by \mathbf{w}).

$$p := (2, 3) \text{ and } q = (3, 4),$$

$\mathbf{v} = q - p$ is the displacement that takes p to q

$\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$ is a smooth parametrization.

$\gamma(t)$ is the *point* at t

$\gamma(t + h)$ is the *point* at $t + h$

$\gamma(t + h) - \gamma(t)$ is the displacement *vector* at $t + h$

$$p := (2, 3),$$

$$\mathbf{w} := (1, 1),$$

$$q := p + \mathbf{w} = (2, 3) + (1, 1) = (3, 4)$$

(displacement of p by \mathbf{w}).

$$p := (2, 3) \text{ and } q = (3, 4),$$

$\mathbf{v} = q - p$ is the displacement that takes p to q

$\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$ is a smooth parametrization.

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Definition. $\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$

$p := (2, 3)$,
 $\mathbf{w} := (1, 1)$,
 $q := p + \mathbf{w} = (2, 3) + (1, 1) = (3, 4)$
(displacement of p by \mathbf{w}).

$p := (2, 3)$ and $q = (3, 4)$,
 $\mathbf{v} = q - p$ is the displacement that takes p to q

$\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$ is a smooth parametrization.
 $\gamma(t)$ is the *point* at t
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 $\gamma(t + h) - \gamma(t)$ is the displacement *vector* at $t + h$

Definition. $\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$ is a smooth parametrization.

$$\dot{\gamma}(t)$$

$$p := (2, 3),$$

$$\mathbf{w} := (1, 1),$$

$$q := p + \mathbf{w} = (2, 3) + (1, 1) = (3, 4)$$

(displacement of p by \mathbf{w}).

$$p := (2, 3) \text{ and } q = (3, 4),$$

$\mathbf{v} = q - p$ is the displacement that takes p to q

$\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$ is a smooth parametrization.

$\gamma(t)$ is the *point* at t

$\gamma(t + h)$ is the *point* at $t + h$

$\gamma(t + h) - \gamma(t)$ is the displacement *vector* at $t + h$

Definition. $\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$ is a smooth parametrization.

$$\dot{\gamma}(t) = \lim_{h \rightarrow 0}$$

$$p := (2, 3),$$

$$\mathbf{w} := (1, 1),$$

$$q := p + \mathbf{w} = (2, 3) + (1, 1) = (3, 4)$$

(displacement of p by \mathbf{w}).

$$p := (2, 3) \text{ and } q = (3, 4),$$

$\mathbf{v} = q - p$ is the displacement that takes p to q

$\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$ is a smooth parametrization.

$\gamma(t)$ is the *point* at t

$\gamma(t + h)$ is the *point* at $t + h$

$\gamma(t + h) - \gamma(t)$ is the displacement *vector* at $t + h$

Definition. $\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$ is a smooth parametrization.

$$\dot{\gamma}(t) = \lim_{h \rightarrow 0} (1/h)(\gamma(t + h) - \gamma(t))$$

$p := (2, 3)$,
 $\mathbf{w} := (1, 1)$,
 $q := p + \mathbf{w} = (2, 3) + (1, 1) = (3, 4)$
(displacement of p by \mathbf{w}).

$p := (2, 3)$ and $q = (3, 4)$,
 $\mathbf{v} = q - p$ is the displacement that takes p to q

$\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$ is a smooth parametrization.

$\gamma(t)$ is the *point* at t

$\gamma(t + h)$ is the *point* at $t + h$

$\gamma(t + h) - \gamma(t)$ is the displacement *vector* at $t + h$

Definition. $\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$ is a smooth parametrization.

$$\dot{\gamma}(t) = \lim_{h \rightarrow 0} (1/h)(\gamma(t + h) - \gamma(t))$$

is called the velocity vector at t and $\dot{\gamma} : (\alpha, \beta) \rightarrow \mathbb{R}^2$ is called the velocity vector field of the parametrization γ .

$p := (2, 3)$,
 $\mathbf{w} := (1, 1)$,
 $q := p + \mathbf{w} = (2, 3) + (1, 1) = (3, 4)$
(displacement of p by \mathbf{w}).

Points on the straight line passing through p ,

$p := (2, 3)$ and $q = (3, 4)$,
 $\mathbf{v} = q - p$ is the displacement that takes p to q

$\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$ is a smooth parametrization.

$\gamma(t)$ is the *point* at t

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$\gamma(t + h) - \gamma(t)$ is the displacement *vector* at $t + h$

Definition. $\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$ is a smooth parametrization.

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$p := (2, 3)$,
 $\mathbf{w} := (1, 1)$,
 $q := p + \mathbf{w} = (2, 3) + (1, 1) = (3, 4)$
 (displacement of p by \mathbf{w}).

Points on the straight line passing through p , parallel to \mathbf{v}

$p := (2, 3)$ and $q = (3, 4)$,
 $\mathbf{v} = q - p$ is the displacement that takes p to q

$\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$ is a smooth parametrization.

$\gamma(t)$ is the *point* at t

$\gamma(t + h)$ is the *point* at $t + h$

$\gamma(t + h) - \gamma(t)$ is the displacement *vector* at $t + h$

Definition. $\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$ is a smooth parametrization.

$$\dot{\gamma}(t) = \lim_{h \rightarrow 0} (1/h)(\gamma(t + h) - \gamma(t))$$

is called the velocity vector at t and $\dot{\gamma} : (\alpha, \beta) \rightarrow \mathbb{R}^2$ is called the velocity vector field of the parametrization γ .

$p := (2, 3)$,
 $\mathbf{w} := (1, 1)$,
 $q := p + \mathbf{w} = (2, 3) + (1, 1) = (3, 4)$
(displacement of p by \mathbf{w}).

Points on the straight line passing through p , parallel to $\mathbf{v} \neq 0$:

$p := (2, 3)$ and $q = (3, 4)$,
 $\mathbf{v} = q - p$ is the displacement that takes p to q

$\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$ is a smooth parametrization.

$\gamma(t)$ is the *point* at t

$\gamma(t + h)$ is the *point* at $t + h$

$\gamma(t + h) - \gamma(t)$ is the displacement *vector* at $t + h$

Definition. $\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$ is a smooth parametrization.

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Definition. A smooth parametrized curve, $\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$,

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Definition. A smooth parametrized curve, $\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$, is called a **regular parametrized curve**

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Points on the straight line passing through p , parallel to
 $\mathbf{v} \neq 0$:

$$\{q \in \mathbb{R}^2 \mid q = p + k\mathbf{v}, k \in \mathbb{R}\}$$

Definition. If $\dot{\gamma}(t) \neq 0$, the line tangent to γ at t is,

$$T_\gamma(t) := \{q \in \mathbb{R}^2 \mid q = \gamma(t) + k\dot{\gamma}(t), k \in \mathbb{R}\}$$

Definition. A smooth parametrized curve, $\gamma : (\alpha, \beta) \rightarrow \mathbb{R}^2$, is called a **regular parametrized curve** if $\dot{\gamma}(t) \neq 0$ for each $t \in (\alpha, \beta)$.

From now on, we will assume all parametrized curves to be regular

Lemma. *If $\tilde{\gamma}(t) = \gamma(\phi(t))$ is a reparametrization,*

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Lemma. *If $\tilde{\gamma}(t) = \gamma(\phi(t))$ is a reparametrization, then $\dot{\tilde{\gamma}}(t) = \dot{\gamma}(\phi(t))\phi'(t)$*

Proof.

$$\tilde{\gamma}(t) = \gamma(\phi(t))$$



Lemma. *If $\tilde{\gamma}(t) = \gamma(\phi(t))$ is a reparametrization, then $\dot{\tilde{\gamma}}(t) = \dot{\gamma}(\phi(t))\phi'(t)$*

Proof.

$$\tilde{\gamma}(t) = \gamma(\phi(t))$$

$$\tilde{\gamma}(t) = (f_1(\phi(t)), f_2(\phi(t)))$$



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$$\dot{\tilde{\gamma}}(t) = (f_1'(\phi(t)), f_2'(\phi(t)))\phi'(t)$$

$$\dot{\tilde{\gamma}}(t) = \dot{\gamma}(\phi(t))\phi'(t)$$

□

Corollary. *The tangent line is invariant under a reparametrization, $\phi(t)$.*

Proof.

$$\{\gamma(t) + k\dot{\tilde{\gamma}}(t)\}$$

□

Lemma. *If $\tilde{\gamma}(t) = \gamma(\phi(t))$ is a reparametrization, then $\dot{\tilde{\gamma}}(t) = \dot{\gamma}(\phi(t))\phi'(t)$*

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Corollary. *The tangent line is invariant under a reparametrization, $\phi(t)$.*

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$$\{\gamma(t) + k\dot{\tilde{\gamma}}(t) \mid k \in \mathbb{R}\} = \{\gamma(t) + k\dot{\gamma}(\phi(t))\phi'(t)\}$$

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□

Corollary. *The tangent line is invariant under a reparametrization, $\phi(t)$, if $\phi'(t) \neq 0$*

Proof.

$$\begin{aligned}\{\gamma(t) + k\dot{\tilde{\gamma}}(t) \mid k \in \mathbb{R}\} &= \{\gamma(t) + k\dot{\gamma}(\phi(t))\phi'(t) \mid k \in \mathbb{R}\} \\ &= \{\gamma(t) + k\dot{\gamma}(\phi(t))\}\end{aligned}$$

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Lemma. *If $\tilde{\gamma}(t) = \gamma(\phi(t))$ is a reparametrization, then $\dot{\tilde{\gamma}}(t) = \dot{\gamma}(\phi(t))\phi'(t)$*

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$$\begin{aligned} \{\gamma(t) + k\dot{\tilde{\gamma}}(t) \mid k \in \mathbb{R}\} &= \{\gamma(t) + k\dot{\gamma}(\phi(t))\phi'(t) \mid k \in \mathbb{R}\} \\ &= \{\gamma(t) + k\dot{\gamma}(\phi(t)) \mid k \in \mathbb{R}\} \end{aligned}$$

□

Lemma. If $\tilde{\gamma}(t) = \gamma(\phi(t))$ is a reparametrization, Note: $\tilde{\gamma}(t)$ is the same point, p , as $\gamma(\phi(t))$ then $\dot{\tilde{\gamma}}(t) = \dot{\gamma}(\phi(t))\phi'(t)$

Proof.

$$\tilde{\gamma}(t) = \gamma(\phi(t))$$

$$\tilde{\gamma}(t) = (f_1(\phi(t)), f_2(\phi(t)))$$

$$\dot{\tilde{\gamma}}(t) = (f'_1(\phi(t))\phi'(t), f'_2(\phi(t))\phi'(t))$$

$$\dot{\tilde{\gamma}}(t) = (f'_1(\phi(t)), f'_2(\phi(t)))\phi'(t)$$

$$\dot{\tilde{\gamma}}(t) = \dot{\gamma}(\phi(t))\phi'(t)$$

□

Corollary. The tangent line is invariant under a reparametrization, $\phi(t)$, if $\phi'(t) \neq 0$

Proof.

$$\begin{aligned} \{\gamma(t) + k\dot{\tilde{\gamma}}(t) \mid k \in \mathbb{R}\} &= \{\gamma(t) + k\dot{\gamma}(\phi(t))\phi'(t) \mid k \in \mathbb{R}\} \\ &= \{\gamma(t) + k\dot{\gamma}(\phi(t)) \mid k \in \mathbb{R}\} \end{aligned}$$

□

Lemma. *If $\tilde{\gamma}(t) = \gamma(\phi(t))$ is a reparametrization,*

then $\dot{\tilde{\gamma}}(t) = \dot{\gamma}(\phi(t))\phi'(t)$

Note: $\tilde{\gamma}(t)$ is the same point, p , as $\gamma(\phi(t))$

When using $\tilde{\gamma}$, the point p “appears at time t ”

Proof.

$$\begin{aligned}\tilde{\gamma}(t) &= \gamma(\phi(t)) \\ \tilde{\gamma}(t) &= (f_1(\phi(t)), f_2(\phi(t))) \\ \dot{\tilde{\gamma}}(t) &= (f'_1(\phi(t))\phi'(t), f'_2(\phi(t))\phi'(t)) \\ \dot{\tilde{\gamma}}(t) &= (f'_1(\phi(t)), f'_2(\phi(t)))\phi'(t) \\ \dot{\tilde{\gamma}}(t) &= \dot{\gamma}(\phi(t))\phi'(t)\end{aligned}$$

□

Corollary. *The tangent line is invariant under a reparametrization, $\phi(t)$, if $\phi'(t) \neq 0$*

Proof.

$$\begin{aligned}\{\gamma(t) + k\dot{\tilde{\gamma}}(t) \mid k \in \mathbb{R}\} &= \{\gamma(t) + k\dot{\gamma}(\phi(t))\phi'(t) \mid k \in \mathbb{R}\} \\ &= \{\gamma(t) + k\dot{\gamma}(\phi(t)) \mid k \in \mathbb{R}\}\end{aligned}$$

□

Lemma. *If $\tilde{\gamma}(t) = \gamma(\phi(t))$ is a reparametrization, then $\dot{\tilde{\gamma}}(t) = \dot{\gamma}(\phi(t))\phi'(t)$*

Proof.

Note: $\tilde{\gamma}(t)$ is the same point, p , as $\gamma(\phi(t))$

When using $\tilde{\gamma}$, the point p “appears at time t ”

When using γ , the point p “appears at time $\phi(t)$ ”

$$\begin{aligned}\tilde{\gamma}(t) &= \gamma(\phi(t)) \\ \tilde{\gamma}(t) &= (f_1(\phi(t)), f_2(\phi(t))) \\ \dot{\tilde{\gamma}}(t) &= (f'_1(\phi(t))\phi'(t), f'_2(\phi(t))\phi'(t)) \\ \dot{\tilde{\gamma}}(t) &= (f'_1(\phi(t)), f'_2(\phi(t)))\phi'(t) \\ \dot{\tilde{\gamma}}(t) &= \dot{\gamma}(\phi(t))\phi'(t)\end{aligned}$$

□

Corollary. *The tangent line is invariant under a reparametrization, $\phi(t)$, if $\phi'(t) \neq 0$*

Proof.

$$\begin{aligned}\{\gamma(t) + k\dot{\tilde{\gamma}}(t) \mid k \in \mathbb{R}\} &= \{\gamma(t) + k\dot{\gamma}(\phi(t))\phi'(t) \mid k \in \mathbb{R}\} \\ &= \{\gamma(t) + k\dot{\gamma}(\phi(t)) \mid k \in \mathbb{R}\}\end{aligned}$$

□

Lemma. *If $\tilde{\gamma}(t) = \gamma(\phi(t))$ is a reparametrization, then $\dot{\tilde{\gamma}}(t) = \dot{\gamma}(\phi(t))\phi'(t)$*

Proof.

$$\begin{aligned}\tilde{\gamma}(t) &= \gamma(\phi(t)) \\ \tilde{\gamma}(t) &= (f_1(\phi(t)), f_2(\phi(t))) \\ \dot{\tilde{\gamma}}(t) &= (f'_1(\phi(t))\phi'(t), f'_2(\phi(t))\phi'(t)) \\ \dot{\tilde{\gamma}}(t) &= (f'_1(\phi(t)), f'_2(\phi(t)))\phi'(t) \\ \dot{\tilde{\gamma}}(t) &= \dot{\gamma}(\phi(t))\phi'(t)\end{aligned}$$

Note: $\tilde{\gamma}(t)$ is the same point, p , as $\gamma(\phi(t))$

When using $\tilde{\gamma}$, the point p “appears at time t ”

When using γ , the point p “appears at time $\phi(t)$ ”

So, $\dot{\tilde{\gamma}}(t)$ and $\dot{\gamma}(\phi(t))$ are velocity vectors at the same point p

Corollary. *The tangent line is invariant under a reparametrization, $\phi(t)$, if $\phi'(t) \neq 0$*

Proof.

$$\begin{aligned}\{\gamma(t) + k\dot{\tilde{\gamma}}(t) \mid k \in \mathbb{R}\} &= \{\gamma(t) + k\dot{\gamma}(\phi(t))\phi'(t) \mid k \in \mathbb{R}\} \\ &= \{\gamma(t) + k\dot{\gamma}(\phi(t)) \mid k \in \mathbb{R}\}\end{aligned}$$