

# Dicussion before the lecture

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*Proof.*

$$\psi(\phi(t)) = t$$

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So,  $\psi'(t) \neq 0$  and  $\phi'(t) \neq 0$  for *each*  $t$  because  $\phi$  is bijective.  $\square$

# Regular parametrization

**Proposition.** *A reparametrization of a regular parametrization*

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**Proposition.** *A reparametrization of a regular parametrization is regular.*

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*If  $\phi : (\tilde{\alpha}, \tilde{\beta}) \rightarrow (\alpha, \beta)$  is invertible with inverse  $\psi : (\alpha, \beta) \rightarrow (\tilde{\alpha}, \tilde{\beta})$ , then  $\phi'(t) \neq 0$  for all  $t \in (\tilde{\alpha}, \tilde{\beta})$  and  $\psi'(t) \neq 0$  for all  $t \in (\alpha, \beta)$ .*

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**Proposition.** *A reparametrization of a regular parametrization is regular.*

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$\dot{\gamma}(t) \neq 0$  and singular if  $\dot{\gamma}(t) = 0$ . The parametrization  $\gamma : (\alpha, \beta) \rightarrow$   
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**Proposition.** *A reparametrization of a regular parametrization is regular.*

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**Proposition.** *A reparametrization of a regular parametrization is regular.*

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$$v \cdot w$$

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**Exercise.** For  $\mathbf{v} : (\alpha, \beta) \rightarrow \mathbf{R}^2$  and  $\mathbf{w} : (\alpha, \beta) \rightarrow \mathbf{R}^2$ ,  
show that  $(\mathbf{v}(t).\mathbf{w}(t))' = \mathbf{v}'(t).\mathbf{w}(t) + \mathbf{v}(t).\mathbf{w}'(t)$

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$$\ddot{\tilde{\gamma}}(t) = \ddot{\gamma}(\phi(t))\phi'(t)\phi'(t) + \dot{\gamma}(\phi(t))\phi''(t)$$

(may change the direction of acceleration)

**If the direction of velocity does not change  
with  $t$**

$$\dot{\gamma}(t) = f(t)\mathbf{v} = (f(t)v_1, f(t)v_2)$$

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**If the direction of velocity does not change  
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$$\begin{aligned}\dot{\gamma}(t) &= f(t)\mathbf{v} = (f(t)v_1, f(t)v_2) \\ \ddot{\gamma}(t) &= (f'(t)v_1, f'(t)v_2) = f'(t)\mathbf{v}\end{aligned}$$

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$$\ddot{\gamma}(t) = (f'(t)v_1, f'(t)v_2) = f'(t)\mathbf{v}$$

Direction of acceleration also unchanged