The same point under different parametrizations.

Since there remains confusion about viewing the same point under different parametrizations, let me clarify it with an example. The concept is very simple.

If we consider the parametrization,

$$\gamma : (-5, 5) \to \mathbb{R}^2$$

$$\gamma(t) = (t, t)$$

it traces a straight line segment. The point $(1, 1)$, for instance, lies on this line segment because it is the image under γ of 1, which belongs to the domain γ.

Consider a reparametrization tracing the same straight line segment,

$$\tilde{\gamma} : (-5, 5) \to \mathbb{R}^2$$

$$\gamma(t) = (-t, -t)$$

Of course, $\tilde{\gamma}(1) = (-1, -1)$ is *not* the same point as $\gamma(1) = (1, 1)$! Different parametrizations take the same parameter t to different points otherwise they would define the same function! That is why, when comparing the same point under different parametrizations, we never plug in the same t. We need something to relate the parameters that are taken to the same point, and that is precisely the role of the reparametrization map, which we often denote by ϕ (or by s if it is a unit speed one)!

In this example,

$$(1, 1) = \gamma(1) = \tilde{\gamma}(-1)$$

In fact, for any a,

$$(a, a) = \gamma(a) = \tilde{\gamma}(-a)$$

So, taking $\phi : \langle -5, 5) \to (-5, 5)$ to be $\phi(\tilde{t}) = -\tilde{t}$,

$$\tilde{\gamma}(\tilde{t}) = \gamma(\phi(\tilde{t}))$$

In other words, if a point $p = \tilde{\gamma}(b)$ for some b (in the domain of $\tilde{\gamma}$), then the *same* point $p = \gamma(\phi(b))$.

Equivalently, if a point $p = \gamma(c)$ for some c (in the domain of γ), then the *same* point $p = \gamma(\phi^{-1}(c))$.

Note that differentiating both sides of,

$$\tilde{\gamma}(\tilde{t}) = \gamma(\phi(\tilde{t}))$$

gives, by chain rule,

$$\dot{\tilde{\gamma}}(\tilde{t}) = \dot{\gamma}(\phi(\tilde{t}))\phi'(\tilde{t})$$

This is comparing velocity vectors of the same point p. and so, on the right hand side, which is using γ (and not $\tilde{\gamma}$), we are applying $\dot{\gamma}$ to $\phi(\tilde{t})$ and not simply to $\tilde{t}$).