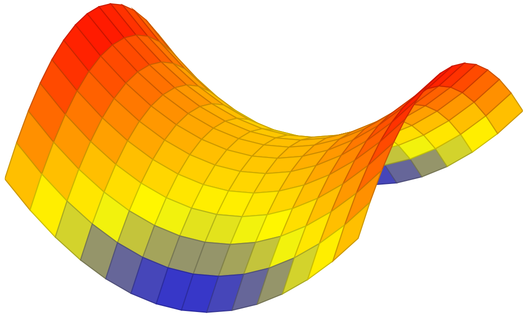
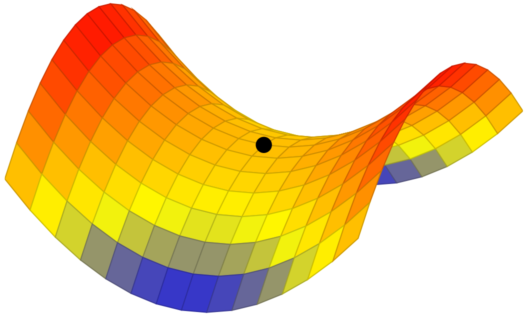
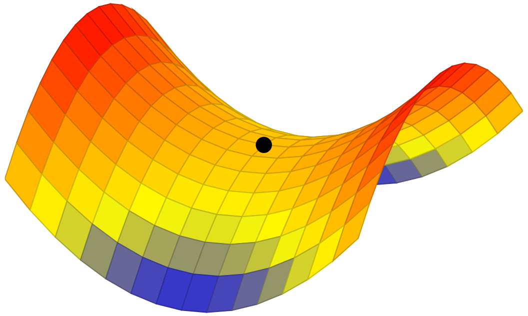


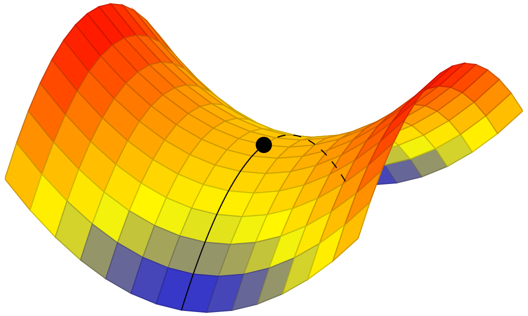
Consider this saddle shaped surface



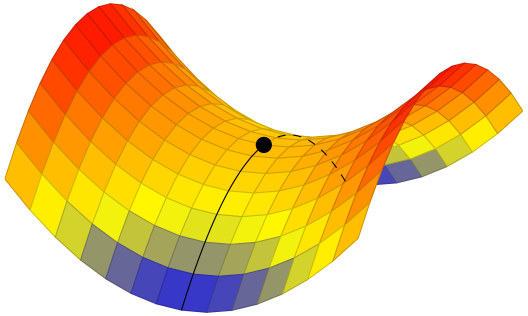
and this point on it



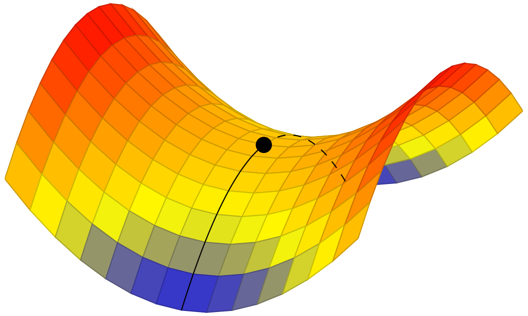
If we assume that the normal points in the upward direction



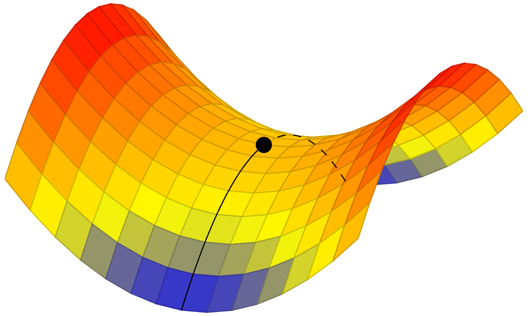
then this curve has negative normal curvature on that point



In fact, it is the smallest possible normal curveture at this point

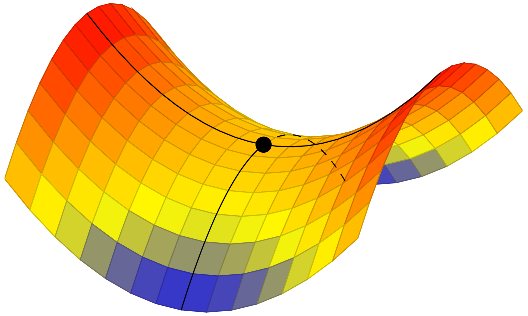


and, therefore, is one of the principal curvatures.

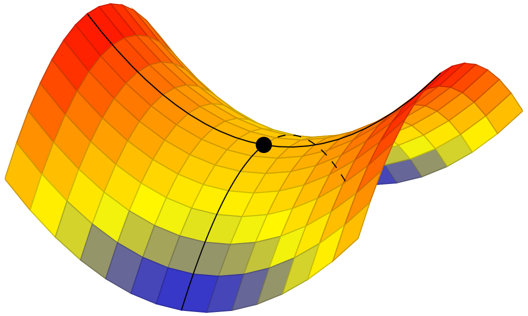


and its velocity vector is the principal direction

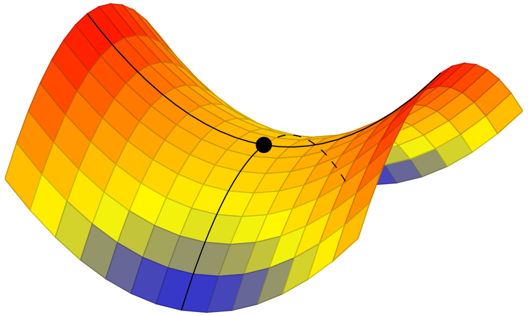
.



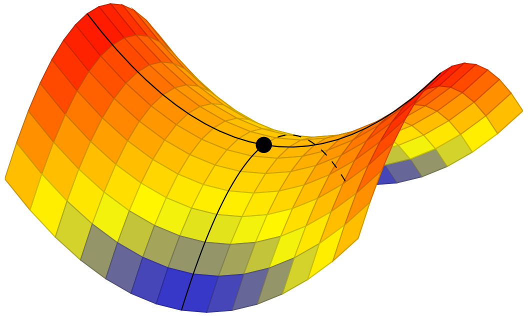
Now consider this curve. It has positive curvature at that point



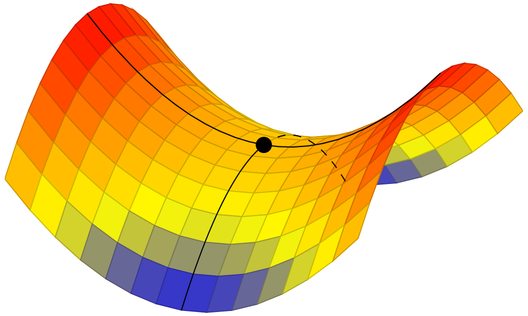
and is the maximum possible normal curvature at that point



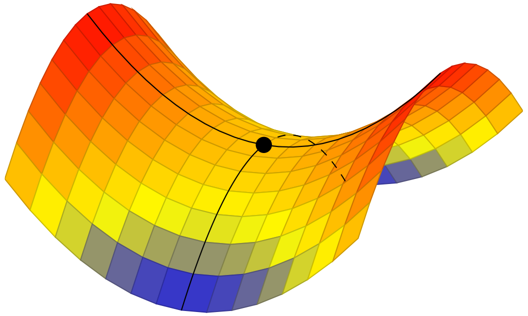
and, therefore, is the other principal curvature.



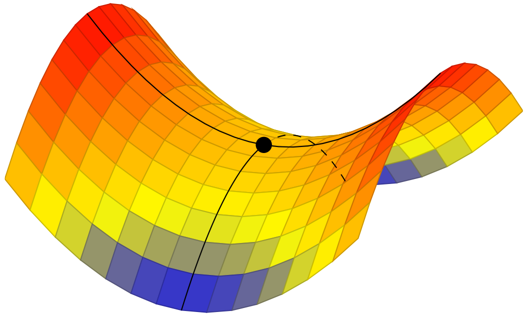
and its velocity vector is the other principal direction



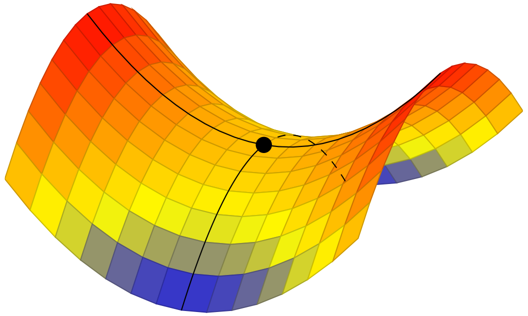
Note that both directions are perpendicular to each other



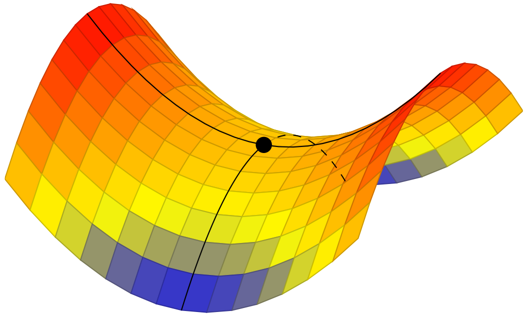
and that the Gaussian curvature is negative because the principal curvatures have different signs



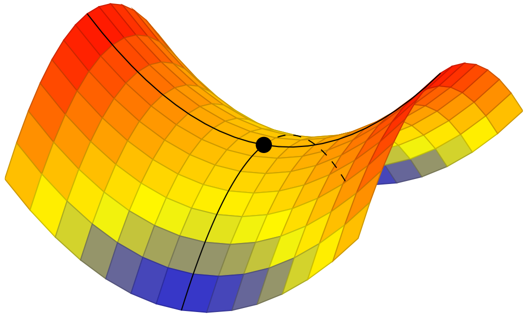
On the other hand, at the north pole of the sphere, all curves have the same normal curvature

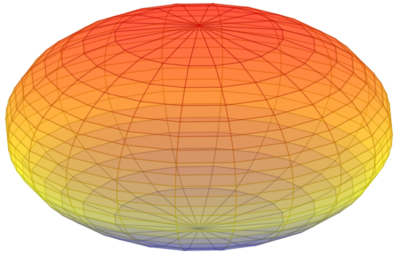


no matter which direction they point in.



So its principal curvatures are the same





and whether they are both positive or both negative, the product will be positive

$f : S_1 \to S_2$ smooth

To fully appreciate the implication of Gauss' theorem Egregium

$f : S_1 \to S_2$ smooth

let us study functions between surfaces.

$f : S_1 \to S_2$ smooth
$\gamma : (\alpha, \beta) \to S_1$

and how they change the arc-length of curves

$f : S_1 \to S_2$ smooth
$\gamma : (\alpha, \beta) \to S_1$
$f \circ \gamma : (\alpha, \beta) \to S_2$

i.e. the arc-length of the curve on S_2.

.

$f : S_1 \to S_2$ smooth
$\gamma : (\alpha, \beta) \to S_1$
$f \circ \gamma : (\alpha, \beta) \to S_2$

$$\text{arc-length of } f \circ \gamma = \int_{t_0}^{t_1} \sqrt{\frac{\mathrm{d}}{\mathrm{d}t} f(\gamma(t)) . \frac{\mathrm{d}}{\mathrm{d}t} f(\gamma(t))} \mathrm{d}t$$

Recall the formula for arc length

$f : S_1 \to S_2$ smooth
$\gamma : (\alpha, \beta) \to S_1$
$f \circ \gamma : (\alpha, \beta) \to S_2$

$$\text{arc-length of } f \circ \gamma = \int_{t_0}^{t_1} \sqrt{\frac{\mathrm{d}}{\mathrm{d}t} f(\gamma(t)) . \frac{\mathrm{d}}{\mathrm{d}t} f(\gamma(t))} \mathrm{d}t$$

Recall that $\frac{\mathrm{d}}{\mathrm{d}t} f(\gamma(t))$ at t

$f : S_1 \to S_2$ smooth
$\gamma : (\alpha, \beta) \to S_1$
$f \circ \gamma : (\alpha, \beta) \to S_2$

$$\text{arc-length of } f \circ \gamma = \int_{t_0}^{t_1} \sqrt{\frac{\mathrm{d}}{\mathrm{d}t} f(\gamma(t)) . \frac{\mathrm{d}}{\mathrm{d}t} f(\gamma(t))} \mathrm{d}t$$

depends only on the velocity of γ at t

$f : S_1 \to S_2$ smooth
$\gamma : (\alpha, \beta) \to S_1$
$f \circ \gamma : (\alpha, \beta) \to S_2$

$$\text{arc-length of } f \circ \gamma = \int_{t_0}^{t_1} \sqrt{\frac{\mathrm{d}}{\mathrm{d}t} f(\gamma(t)) . \frac{\mathrm{d}}{\mathrm{d}t} f(\gamma(t))} \mathrm{d}t$$
$$= \int_{t_0}^{t_1} \sqrt{D_{\gamma(t)}(f)\dot{\gamma}(t) . D_{\gamma(t)}(f)\dot{\gamma}(t)} \mathrm{d}t$$

and on what is denoted as $D_p(f)$

$f : S_1 \to S_2$ smooth
$\gamma : (\alpha, \beta) \to S_1$
$f \circ \gamma : (\alpha, \beta) \to S_2$

$$\begin{aligned}
\text{arc-length of } f \circ \gamma &= \int_{t_0}^{t_1} \sqrt{\frac{\mathrm{d}}{\mathrm{d}t} f(\gamma(t)) . \frac{\mathrm{d}}{\mathrm{d}t} f(\gamma(t))}\mathrm{d}t \\
&= \int_{t_0}^{t_1} \sqrt{D_{\gamma(t)}(f)\dot{\gamma}(t) . D_{\gamma(t)}(f)\dot{\gamma}(t)}\mathrm{d}t \\
&= \int_{t_0}^{t_1} \sqrt{f^*\langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle}\mathrm{d}t
\end{aligned}$$

where

$$f^*\langle \mathbf{v}, \mathbf{w} \rangle_p = \langle D_p(f)\mathbf{v} . D_p(f)\mathbf{w} \rangle_{f(p)}$$

We now define what is called the “pull back” of the first fundamental form

$f : S_1 \to S_2$ smooth
$\gamma : (\alpha, \beta) \to S_1$
$f \circ \gamma : (\alpha, \beta) \to S_2$

$$
\begin{aligned}
\text{arc-length of } f \circ \gamma &= \int_{t_0}^{t_1} \sqrt{\frac{\mathrm{d}}{\mathrm{d}t} f(\gamma(t)) . \frac{\mathrm{d}}{\mathrm{d}t} f(\gamma(t))} \mathrm{d}t \\
&= \int_{t_0}^{t_1} \sqrt{D_{\gamma(t)}(f) \dot{\gamma}(t) . D_{\gamma(t)}(f) \dot{\gamma}(t)} \mathrm{d}t \\
&= \int_{t_0}^{t_1} \sqrt{f^* \langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle} \mathrm{d}t
\end{aligned}
$$

where

$$
f^* \langle \mathbf{v}, \mathbf{w} \rangle_p = \langle D_p(f) \mathbf{v} . D_p(f) \mathbf{w} \rangle_{f(p)}
$$

and see that the arc-length can be written in terms of it

$f : S_1 \to S_2$ smooth
$\gamma : (\alpha, \beta) \to S_1$
$f \circ \gamma : (\alpha, \beta) \to S_2$

$$
\begin{aligned}
\text{arc-length of } f \circ \gamma &= \int_{t_0}^{t_1} \sqrt{\frac{\mathrm{d}}{\mathrm{d}t} f(\gamma(t)) . \frac{\mathrm{d}}{\mathrm{d}t} f(\gamma(t))} \mathrm{d}t \\
&= \int_{t_0}^{t_1} \sqrt{D_{\gamma(t)}(f)\dot{\gamma}(t) . D_{\gamma(t)}(f)\dot{\gamma}(t)} \mathrm{d}t \\
&= \int_{t_0}^{t_1} \sqrt{f^* \langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle} \mathrm{d}t
\end{aligned}
$$

where

$$
f^* \langle \mathbf{v}, \mathbf{w} \rangle_p = \langle D_p(f)\mathbf{v} . D_p(f)\mathbf{w} \rangle_{f(p)}
$$

As before, this is telling us that if we find compute $f^* \langle , \rangle_p$ point on the surface

.

$f : S_1 \to S_2$ smooth
$\gamma : (\alpha, \beta) \to S_1$
$f \circ \gamma : (\alpha, \beta) \to S_2$

$$\begin{aligned}\text{arc-length of } f \circ \gamma &= \int_{t_0}^{t_1} \sqrt{\frac{\mathrm{d}}{\mathrm{d}t} f(\gamma(t)) . \frac{\mathrm{d}}{\mathrm{d}t} f(\gamma(t))} \mathrm{d}t \\ &= \int_{t_0}^{t_1} \sqrt{D_{\gamma(t)}(f)\dot{\gamma}(t) . D_{\gamma(t)}(f)\dot{\gamma}(t)} \mathrm{d}t \\ &= \int_{t_0}^{t_1} \sqrt{f^* \langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle} \mathrm{d}t\end{aligned}$$

where

$$f^* \langle \mathbf{v}, \mathbf{w} \rangle_p = \langle D_p(f)\mathbf{v} . D_p(f)\mathbf{w} \rangle_{f(p)}$$

We only need to know the velocity vector field of any curve whose arc-length we wish to compute

.

$f : S_1 \to S_2$ smooth

$\gamma : (\alpha, \beta) \to S_1$

$f \circ \gamma : (\alpha, \beta) \to S_2$

arc-length of $f \circ \gamma =$ arc-length of γ

$$\begin{aligned}\text{arc-length of } f \circ \gamma &= \int_{t_0}^{t_1} \sqrt{\frac{\mathrm{d}}{\mathrm{d}t} f(\gamma(t)).\frac{\mathrm{d}}{\mathrm{d}t} f(\gamma(t))}\mathrm{d}t \\ &= \int_{t_0}^{t_1} \sqrt{D_{\gamma(t)}(f)\dot{\gamma}(t).D_{\gamma(t)}(f)\dot{\gamma}(t)}\mathrm{d}t \\ &= \int_{t_0}^{t_1} \sqrt{f^*\langle \dot{\gamma}(t), \dot{\gamma}(t)\rangle}\mathrm{d}t\end{aligned}$$

where

$$f^*\langle \mathbf{v}, \mathbf{w}\rangle_p = \langle D_p(f)\mathbf{v}.D_p(f)\mathbf{w}\rangle_{f(p)}$$

We now have an alternative criterion for when a function preserves the arc length of a curve

.

$f : S_1 \to S_2$ smooth
$\gamma : (\alpha, \beta) \to S_1$
$f \circ \gamma : (\alpha, \beta) \to S_2$

arc-length of $f \circ \gamma$ = arc-length of γ
if $\int_{t_0}^{t_1} f^*\langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle \mathrm{d}t = \int_{t_0}^{t_1} \langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle \mathrm{d}t$

$$\begin{aligned}
\text{arc-length of } f \circ \gamma &= \int_{t_0}^{t_1} \sqrt{\frac{\mathrm{d}}{\mathrm{d}t} f(\gamma(t)) . \frac{\mathrm{d}}{\mathrm{d}t} f(\gamma(t))} \mathrm{d}t \\
&= \int_{t_0}^{t_1} \sqrt{D_{\gamma(t)}(f)\dot{\gamma}(t) . D_{\gamma(t)}(f)\dot{\gamma}(t)} \mathrm{d}t \\
&= \int_{t_0}^{t_1} \sqrt{f^*\langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle} \mathrm{d}t
\end{aligned}$$

where

$$f^*\langle \mathbf{v}, \mathbf{w} \rangle_p = \langle D_p(f)\mathbf{v} . D_p(f)\mathbf{w} \rangle_{f(p)}$$

By definition it is when these integrals are the same

.

$f : S_1 \to S_2$ smooth
$\gamma : (\alpha, \beta) \to S_1$
$f \circ \gamma : (\alpha, \beta) \to S_2$

arc-length of $f \circ \gamma =$ arc-length of γ
if $\int_{t_0}^{t_1} f^* \langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle \mathrm{d}t = \int_{t_0}^{t_1} \langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle \mathrm{d}t$
if $f^* \langle \mathbf{v}, \mathbf{w} \rangle_p = \langle \mathbf{v}, \mathbf{w} \rangle_p$ for all $p \in S_1$.

$$\begin{aligned} \text{arc-length of } f \circ \gamma &= \int_{t_0}^{t_1} \sqrt{\frac{\mathrm{d}}{\mathrm{d}t} f(\gamma(t)). \frac{\mathrm{d}}{\mathrm{d}t} f(\gamma(t))} \mathrm{d}t \\ &= \int_{t_0}^{t_1} \sqrt{D_{\gamma(t)}(f) \dot{\gamma}(t). D_{\gamma(t)}(f) \dot{\gamma}(t)} \mathrm{d}t \\ &= \int_{t_0}^{t_1} \sqrt{f^* \langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle} \mathrm{d}t \end{aligned}$$

where

$$f^* \langle \mathbf{v}, \mathbf{w} \rangle_p = \langle D_p(f) \mathbf{v}. D_p(f) \mathbf{w} \rangle_{f(p)}$$

And that will happen when these two forms are the same

$f : S_1 \to S_2$ smooth
$\gamma : (\alpha, \beta) \to S_1$
$f \circ \gamma : (\alpha, \beta) \to S_2$

arc-length of $f \circ \gamma$ = arc-length of γ for *any* γ
if and only if $\int_{t_0}^{t_1} f^*\langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle \mathrm{d}t = \int_{t_0}^{t_1} \langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle \mathrm{d}t$
if and only if $f^*\langle \mathbf{v}, \mathbf{w} \rangle_p = \langle \mathbf{v}, \mathbf{w} \rangle_p$ for all $p \in S_1$.

$$\begin{aligned}
\text{arc-length of } f \circ \gamma &= \int_{t_0}^{t_1} \sqrt{\frac{\mathrm{d}}{\mathrm{d}t} f(\gamma(t)) . \frac{\mathrm{d}}{\mathrm{d}t} f(\gamma(t))} \mathrm{d}t \\
&= \int_{t_0}^{t_1} \sqrt{D_{\gamma(t)}(f)\dot{\gamma}(t) . D_{\gamma(t)}(f)\dot{\gamma}(t)} \mathrm{d}t \\
&= \int_{t_0}^{t_1} \sqrt{f^*\langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle} \mathrm{d}t
\end{aligned}$$

where

$$f^*\langle \mathbf{v}, \mathbf{w} \rangle_p = \langle D_p(f)\mathbf{v} . D_p(f)\mathbf{w} \rangle_{f(p)}$$

The converse is true if and only if f preserves arc-length of *any* curve

.