Exercise sheet 6

Curves and Surfaces, MTH201

Let $\gamma : (\alpha, \beta) \to \mathbb{R}^2$ be a regular *unit speed* parametrization of **space** curves. As with plane curves, we can define $\mathbf{T}(t) := \dot{\gamma}(t)$.

For a plane curve, there are only two unit vectors normal to $\mathbf{T}(t)$ for a given t, but now there are infinitely many. This is why we choose the one which is in the direction of the acceleration:

1. Assume that the (ordinary, *not* signed) curvature, $\kappa(t) \neq 0$ for all t. Show that $\mathbf{N}(t) := \frac{\ddot{\gamma}(t)}{\kappa(t)}$ is a non-zero unit vector orthogonal to $\mathbf{T}(t)$.

2. Prove that $\mathbf{N}(t)$ is smoothly varying.

3. Define $\mathbf{B}(t) := \mathbf{T}(t) \times \mathbf{N}(t)$. Show that $\mathbf{B}(t)$ is smoothly varying and that $\mathbf{B}(t)$ is perpendicular to both $\mathbf{T}(t)$ and $\mathbf{N}(t)$.

4. By the previous part, $\{\mathbf{T}(t), \mathbf{N}(t), \mathbf{B}(t)\}$ form an orthonormal basis. So,
$$\dot{\mathbf{T}}(t) = x_T(t)\mathbf{T}(t) + y_T(t)\mathbf{N}(t) + z_T(t)\mathbf{B}(t)$$
for some $x_T(t), y_T(t), z_T(t)$. What are these coefficients, $x_T(t), y_T(t), z_T(t)$? This should be straightforward.

5. Similarly,
$$\dot{\mathbf{N}}(t) = x_N(t)\mathbf{T}(t) + y_N(t)\mathbf{N}(t) + z_N(t)\mathbf{B}(t)$$
for some $x_N(t), y_N(t), z_N(t)$. What are these coefficients, $x_N(t)$ and $y_N(t)$? $z_N(t)$ will need to be done later.

6. Similarly,
$$\dot{\mathbf{B}}(t) = x_B(t)\mathbf{T}(t) + y_B(t)\mathbf{N}(t) + z_B(t)\mathbf{B}(t)$$
for some $x_B(t), y_B(t), z_B(t)$. Show that $x_B(t) = 0$ and $z_B(t) = 0$. In other words $\dot{\mathbf{B}}(t)$ is always a scalar multiple of $\mathbf{N}(t)$. Let us denote the *negative* of this coefficient by $\tau(t)$. The negative sign is only a convention and simplifies some notation later. In other words, $\tau(t) = -y_B(t)$ and is called the "torsion" at t.

7. Now prove that the missing coefficient $c(t)$ in part 5. is $-\tau(t)$.