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Proof.

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$(\mathbf{e}_i.\mathbf{e}_j)' = 0$ $\square$

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By the theory of differential equations, always has a solution, unique if we fix,
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$(\mathbf{e}_i.\mathbf{e}_j)' = 0$ (exercise!) $\square$

Theorem. *Given,*

1. *functions* $\tilde{\kappa} : (\alpha, \beta) \to \mathbb{R}$, $\tilde{\kappa}(t) > 0$ *for all* t, *and* $\tilde{\tau} : (\alpha, \beta) \to \mathbb{R}$
2. $p \in \mathbb{R}^3$, $t_0 \in (\alpha, \beta)$,
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there is a unit speed parametrization, $\gamma : (\alpha, \beta) \to \mathbb{R}^3$, *so that,*

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Proof.

$$\begin{aligned}
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Proof.

$$\dot{\mathbf{e}}_1(t) = 0\mathbf{e}_1(t) + \kappa(t)\mathbf{e}_2(t) + 0\mathbf{e}_3(t)$$
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Can find $\gamma(t)$, so that, $\mathbf{T}(t) = \dot{\gamma}(t) = \mathbf{e}_1(t)$ and $\gamma(t_0) = p$ (exercise!! Integration (anti-derivative)!)

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Theorem. *Given,*

1. *functions* $\tilde{\kappa} : (\alpha, \beta) \to \mathbb{R}$, $\tilde{\kappa}(t) > 0$ *for all* t, *and* $\tilde{\tau} : (\alpha, \beta) \to \mathbb{R}$
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$\mathbf{e}_3(t)$ unit $\square$

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$\mathbf{e}_3(t)$ unit and orthogonal $\square$

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$\mathbf{e}_3(t)$ unit and orthogonal to $\mathbf{e}_1(t)$ and $\mathbf{e}_2(t)$. only two choices, and negatives of each other. So one is $\mathbf{e}_1(t) \times \mathbf{e}_2(t)$, other is $-\mathbf{e}_1(t) \times \mathbf{e}_2(t)$. $\square$

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$$\begin{aligned}
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products...

$\square$