Inverting the arc-length function

The arc-length function is defined as,

$$s(t) := \int_{t_0}^{t} \|\dot{\gamma}(u)\| \mathrm{d}u$$

We used the inverse of $s(t)$ to reparametrize to a unit speed parametrization. However, we can only do that if it is itself smooth, is indeed invertible, and its inverse is smooth. That $s(t)$ is smooth was already demonstrated in the lecture and in an exercise.

To prove that is invertible, we need to check that it is one-one, i.e. if x and y are different, then so is $s(x)$ and $s(y)$. Any function with a positive derivative will be one-one, and we know that $s'(t) = \|\gamma(t)\|$, which is positive because γ is regular.

So we know that s is invertible. Now the question is, is the inverse smooth? First, observe that if the inverse is smooth, then we know that $s'(t)$ must be strictly positive for all t. There is a famous and important theorem, "Inverse Function Theorem", that among other things, guarantees the converse, i.e. if the derivative of the invertible function is always positive, then its inverse is smooth.

Actually, it says more:

If $f : (\alpha, \beta) \to (\alpha', \beta')$ is a smooth function, and $f'(t) > 0$ for some t, then f is invertible in a small open interval containing t, and its inverse is smooth in that interval (and, therefore, at $f(t)$)

So the theorem guarentees two things: invertibility, but only guarantees invertibility in a vicinity of the point at which the derivative is non-zero. But, luckily, we know that it is invertible in the entire domain by other means. The theorem also guarantees that the inverse will be smooth at $f(t)$. This is the part that we need.