

$\mathcal{K} = \det \mathcal{W}$, Gaussian curvature

We saw that the determinant of the Weingarten map depends only on the first fundamental form

$\mathcal{K} = \det \mathcal{W}$, Gaussian curvature

This is called the Gaussian curvature

$\mathcal{K} = \det \mathcal{W}$, Gaussian curvature

This makes sense because the determinant does not depend on the basis

$\mathcal{K} = \det \mathcal{W}$, Gaussian curvature
 $\mathcal{H} = \frac{\text{trace } \mathcal{W}}{2}$, Mean Curvature

Similarly, we can define the mean curvature as the trace of the Weingarten map

$\mathcal{K} = \det \mathcal{W}$, Gaussian curvature

$\mathcal{H} = \frac{\text{trace } \mathcal{W}}{2}$, Mean Curvature

The trace also does not depend on the basis chosen

$\mathcal{K} = \det \mathcal{W}$, Gaussian curvature

$\mathcal{H} = \frac{\text{trace } \mathcal{W}}{2}$, Mean Curvature

\mathcal{W} is symmetric.

Recall, from linear algebra that symmetric matrices are diagonalizable

$\mathcal{K} = \det \mathcal{W}$, Gaussian curvature

$\mathcal{H} = \frac{\text{trace } \mathcal{W}}{2}$, Mean Curvature

\mathcal{W} is symmetric.

and the eigenvalues make up the diagonal

$\mathcal{K} = \det \mathcal{W}$, Gaussian curvature

$\mathcal{H} = \frac{\text{trace } \mathcal{W}}{2}$, Mean Curvature

\mathcal{W} is symmetric.

Denote its eigenvalues, κ_1 and κ_2 .

We will see that these eigenvalues have a special meaning for surfaces

$\mathcal{K} = \det \mathcal{W}$, Gaussian curvature

$\mathcal{H} = \frac{\text{trace } \mathcal{W}}{2}$, Mean Curvature

\mathcal{W} is symmetric.

Denote its eigenvalues, κ_1 and κ_2 .

i.e. there exist \mathbf{t}_1 and \mathbf{t}_2 *unit* vectors so that

$\mathcal{W}\mathbf{t}_1 = \kappa_1\mathbf{t}_1$ and

$\mathcal{W}\mathbf{t}_2 = \kappa_2\mathbf{t}_2$

Even their eigenvectors have a special significance

$\mathcal{K} = \det \mathcal{W}$, Gaussian curvature

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\mathcal{W} is symmetric.

Denote its eigenvalues, κ_1 and κ_2 .

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$\mathcal{W}\mathbf{t}_1 = \kappa_1\mathbf{t}_1$ and

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Assume $\kappa_1 \neq \kappa_2$, then $\mathbf{t}_1 \cdot \mathbf{t}_2 = 0$

If the eigenvalues are distinct, then the eigenvectors are orthogonal

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Assume $\kappa_1 \neq \kappa_2$, then $\mathbf{t}_1 \cdot \mathbf{t}_2 = 0$

Writing \mathcal{W} in terms of the basis \mathbf{t}_1 and \mathbf{t}_2 ,

$$\begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix}$$

It is with respect to this orthonormal basis, that the map is diagonal

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Assume $\kappa_1 \neq \kappa_2$, then $\mathbf{t}_1 \cdot \mathbf{t}_2 = 0$

Writing \mathcal{W} in terms of the basis \mathbf{t}_1 and \mathbf{t}_2 ,

$$\begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix}$$

Since the determinant remains unchanged by a change of basis

$\mathcal{K} = \det \mathcal{W}$, Gaussian curvature

$\mathcal{H} = \frac{\text{trace } \mathcal{W}}{2}$, Mean Curvature

\mathcal{W} is symmetric.

Denote its eigenvalues, κ_1 and κ_2 .

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$$\mathcal{W}\mathbf{t}_1 = \kappa_1\mathbf{t}_1 \text{ and}$$

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Assume $\kappa_1 \neq \kappa_2$, then $\mathbf{t}_1 \cdot \mathbf{t}_2 = 0$

Writing \mathcal{W} in terms of the basis \mathbf{t}_1 and \mathbf{t}_2 ,

$$\begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix}$$

$\kappa_1\kappa_2 = \text{Gaussian curvature}$

We can write the Gaussian curvature in terms of the eigenvalues

$\mathcal{K} = \det \mathcal{W}$, Gaussian curvature

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\mathcal{W} is symmetric.

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i.e. there exist \mathbf{t}_1 and \mathbf{t}_2 *unit* vectors so that

$$\mathcal{W}\mathbf{t}_1 = \kappa_1\mathbf{t}_1 \text{ and}$$

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Assume $\kappa_1 \neq \kappa_2$, then $\mathbf{t}_1 \cdot \mathbf{t}_2 = 0$

Writing \mathcal{W} in terms of the basis \mathbf{t}_1 and \mathbf{t}_2 ,

$$\begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix}$$

$\kappa_1\kappa_2 = \text{Gaussian curvature}$

$\frac{\kappa_1 + \kappa_2}{2} = \text{Mean curvature}$

And similarly for the mean curvature

$\mathcal{K} = \det \mathcal{W}$, Gaussian curvature

$\mathcal{H} = \frac{\text{trace } \mathcal{W}}{2}$, Mean Curvature

\mathcal{W} is symmetric.

Denote its eigenvalues, κ_1 and κ_2 .

i.e. there exist \mathbf{t}_1 and \mathbf{t}_2 *unit* vectors so that

$\mathcal{W}\mathbf{t}_1 = \kappa_1\mathbf{t}_1$ and

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Assume $\kappa_1 \neq \kappa_2$, then $\mathbf{t}_1 \cdot \mathbf{t}_2 = 0$

Writing \mathcal{W} in terms of the basis \mathbf{t}_1 and \mathbf{t}_2 ,

$$\begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix}$$

$\kappa_1\kappa_2 = \text{Gaussian curvature}$

$\frac{\kappa_1 + \kappa_2}{2} = \text{Mean curvature}$

Let us try and understand the geometric significance of the eigenvalue s

$\mathcal{K} = \det \mathcal{W}$, Gaussian curvature

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\mathcal{W} is symmetric.

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i.e. there exist \mathbf{t}_1 and \mathbf{t}_2 *unit* vectors so that

$\mathcal{W}\mathbf{t}_1 = \kappa_1\mathbf{t}_1$ and

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Assume $\kappa_1 \neq \kappa_2$, then $\mathbf{t}_1 \cdot \mathbf{t}_2 = 0$

Writing \mathcal{W} in terms of the basis \mathbf{t}_1 and \mathbf{t}_2 ,

$$\begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix}$$

$\kappa_1\kappa_2 = \text{Gaussian curvature}$

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As usual we begin with the study of a curve on a surface

$\mathcal{K} = \det \mathcal{W}$, Gaussian curvature

$\mathcal{H} = \frac{\text{trace } \mathcal{W}}{2}$, Mean Curvature

κ_n

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i.e. there exist \mathbf{t}_1 and \mathbf{t}_2 *unit* vectors so that

$$\mathcal{W}\mathbf{t}_1 = \kappa_1\mathbf{t}_1 \text{ and}$$

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Assume $\kappa_1 \neq \kappa_2$, then $\mathbf{t}_1 \cdot \mathbf{t}_2 = 0$

Writing \mathcal{W} in terms of the basis \mathbf{t}_1 and \mathbf{t}_2 ,

$$\begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix}$$

$\kappa_1\kappa_2 =$ Gaussian curvature

$\frac{\kappa_1+\kappa_2}{2} =$ Mean curvature

and try and understand the normal curvature

$\mathcal{K} = \det \mathcal{W}$, Gaussian curvature

$\mathcal{H} = \frac{\text{trace } \mathcal{W}}{2}$, Mean Curvature

$$\kappa_n = \mathcal{W} \dot{\gamma} . \dot{\gamma}$$

\mathcal{W} is symmetric.

Denote its eigenvalues, κ_1 and κ_2 .

i.e. there exist \mathbf{t}_1 and \mathbf{t}_2 *unit* vectors so that

$$\mathcal{W} \mathbf{t}_1 = \kappa_1 \mathbf{t}_1 \text{ and}$$

$$\mathcal{W} \mathbf{t}_2 = \kappa_2 \mathbf{t}_2$$

Assume $\kappa_1 \neq \kappa_2$, then $\mathbf{t}_1 . \mathbf{t}_2 = 0$

Writing \mathcal{W} in terms of the basis \mathbf{t}_1 and \mathbf{t}_2 ,

$$\begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix}$$

$\kappa_1 \kappa_2 =$ Gaussian curvature

$\frac{\kappa_1 + \kappa_2}{2} =$ Mean curvature

which we now know can be written using \mathcal{W}

$\mathcal{K} = \det \mathcal{W}$, Gaussian curvature

$\mathcal{H} = \frac{\text{trace } \mathcal{W}}{2}$, Mean Curvature

\mathcal{W} is symmetric.

Denote its eigenvalues, κ_1 and κ_2 .

i.e. there exist \mathbf{t}_1 and \mathbf{t}_2 *unit* vectors so that

$\mathcal{W}\mathbf{t}_1 = \kappa_1\mathbf{t}_1$ and

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Assume $\kappa_1 \neq \kappa_2$, then $\mathbf{t}_1 \cdot \mathbf{t}_2 = 0$

Writing \mathcal{W} in terms of the basis \mathbf{t}_1 and \mathbf{t}_2 ,

$$\begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix}$$

$\kappa_1\kappa_2 = \text{Gaussian curvature}$

$\frac{\kappa_1 + \kappa_2}{2} = \text{Mean curvature}$

$$\begin{aligned} \kappa_n &= \mathcal{W}\dot{\gamma} \cdot \dot{\gamma} \\ &= (\mathcal{W}(c_1\mathbf{t}_1 + c_2\mathbf{t}_2)) \cdot (c_1\mathbf{t}_1 + c_2\mathbf{t}_2) \end{aligned}$$

But now we have a new basis, \mathbf{t}_1 and \mathbf{t}_2 convenient for the Weingarten map

$\mathcal{K} = \det \mathcal{W}$, Gaussian curvature

$\mathcal{H} = \frac{\text{trace } \mathcal{W}}{2}$, Mean Curvature

\mathcal{W} is symmetric.

Denote its eigenvalues, κ_1 and κ_2 .

i.e. there exist \mathbf{t}_1 and \mathbf{t}_2 *unit* vectors so that

$$\mathcal{W}\mathbf{t}_1 = \kappa_1\mathbf{t}_1 \text{ and}$$

$$\mathcal{W}\mathbf{t}_2 = \kappa_2\mathbf{t}_2$$

Assume $\kappa_1 \neq \kappa_2$, then $\mathbf{t}_1 \cdot \mathbf{t}_2 = 0$

Writing \mathcal{W} in terms of the basis \mathbf{t}_1 and \mathbf{t}_2 ,

$$\begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix}$$

$\kappa_1\kappa_2 = \text{Gaussian curvature}$

$\frac{\kappa_1 + \kappa_2}{2} = \text{Mean curvature}$

$$\kappa_n = \mathcal{W}\dot{\gamma} \cdot \dot{\gamma}$$

$$= (\mathcal{W}(c_1\mathbf{t}_1 + c_2\mathbf{t}_2)) \cdot (c_1\mathbf{t}_1 + c_2\mathbf{t}_2)$$

$$= (c_1\mathcal{W}(\mathbf{t}_1) + c_2\mathcal{W}(\mathbf{t}_2)) \cdot (c_1\mathbf{t}_1 + c_2\mathbf{t}_2)$$

Of course, we chose this basis so that the Weingarten map has a better form

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\mathcal{W} is symmetric.

Denote its eigenvalues, κ_1 and κ_2 .

i.e. there exist \mathbf{t}_1 and \mathbf{t}_2 *unit* vectors so that

$$\mathcal{W}\mathbf{t}_1 = \kappa_1\mathbf{t}_1 \text{ and}$$

$$\mathcal{W}\mathbf{t}_2 = \kappa_2\mathbf{t}_2$$

Assume $\kappa_1 \neq \kappa_2$, then $\mathbf{t}_1 \cdot \mathbf{t}_2 = 0$

Writing \mathcal{W} in terms of the basis \mathbf{t}_1 and \mathbf{t}_2 ,

$$\begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix}$$

$\kappa_1\kappa_2 = \text{Gaussian curvature}$

$\frac{\kappa_1 + \kappa_2}{2} = \text{Mean curvature}$

$$\kappa_n = \mathcal{W}\dot{\gamma} \cdot \dot{\gamma}$$

$$= (\mathcal{W}(c_1\mathbf{t}_1 + c_2\mathbf{t}_2)) \cdot (c_1\mathbf{t}_1 + c_2\mathbf{t}_2)$$

$$= (c_1\mathcal{W}(\mathbf{t}_1) + c_2\mathcal{W}(\mathbf{t}_2)) \cdot (c_1\mathbf{t}_1 + c_2\mathbf{t}_2)$$

$$= (c_1\kappa_1\mathbf{t}_1 + c_2\kappa_2\mathbf{t}_2) \cdot (c_1\mathbf{t}_1 + c_2\mathbf{t}_2)$$

And its application can now be written in terms of κ_1 and κ_2

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Denote its eigenvalues, κ_1 and κ_2 .

i.e. there exist \mathbf{t}_1 and \mathbf{t}_2 *unit* vectors so that

$$\mathcal{W}\mathbf{t}_1 = \kappa_1\mathbf{t}_1 \text{ and}$$

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Assume $\kappa_1 \neq \kappa_2$, then $\mathbf{t}_1.\mathbf{t}_2 = 0$

Writing \mathcal{W} in terms of the basis \mathbf{t}_1 and \mathbf{t}_2 ,

$$\begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix}$$

$\kappa_1\kappa_2 =$ Gaussian curvature

$\frac{\kappa_1+\kappa_2}{2} =$ Mean curvature

$$\begin{aligned} \kappa_n &= \mathcal{W}\dot{\gamma}.\dot{\gamma} \\ &= (\mathcal{W}(c_1\mathbf{t}_1 + c_2\mathbf{t}_2)).(c_1\mathbf{t}_1 + c_2\mathbf{t}_2) \\ &= (c_1\mathcal{W}(\mathbf{t}_1) + c_2\mathcal{W}(\mathbf{t}_2)).(c_1\mathbf{t}_1 + c_2\mathbf{t}_2) \\ &= (\textcolor{red}{c}_1\textcolor{red}{\kappa}_1\textcolor{red}{t}_1 + c_2\kappa_2\mathbf{t}_2).(\textcolor{red}{c}_1\textcolor{red}{t}_1 + c_2\mathbf{t}_2) \\ &= c_1^2\kappa_1(\mathbf{t}_1.\mathbf{t}_1) + \end{aligned}$$

Now we distribute the terms

$\mathcal{K} = \det \mathcal{W}$, Gaussian curvature

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$$\mathcal{W}\mathbf{t}_1 = \kappa_1\mathbf{t}_1 \text{ and}$$

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Assume $\kappa_1 \neq \kappa_2$, then $\mathbf{t}_1.\mathbf{t}_2 = 0$

Writing \mathcal{W} in terms of the basis \mathbf{t}_1 and \mathbf{t}_2 ,

$$\begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix}$$

$\kappa_1\kappa_2 =$ Gaussian curvature

$\frac{\kappa_1+\kappa_2}{2} =$ Mean curvature

$$\kappa_n = \mathcal{W}\dot{\gamma}.\dot{\gamma}$$

$$= (\mathcal{W}(c_1\mathbf{t}_1 + c_2\mathbf{t}_2)).(c_1\mathbf{t}_1 + c_2\mathbf{t}_2)$$

$$= (c_1\mathcal{W}(\mathbf{t}_1) + c_2\mathcal{W}(\mathbf{t}_2)).(c_1\mathbf{t}_1 + c_2\mathbf{t}_2)$$

$$= (\textcolor{red}{c}_1\textcolor{red}{\kappa}_1\textcolor{red}{\mathbf{t}}_1 + c_2\kappa_2\mathbf{t}_2).(c_1\mathbf{t}_1 + \textcolor{red}{c}_2\textcolor{red}{\mathbf{t}}_2)$$

$$= c_1^2\kappa_1(\mathbf{t}_1.\mathbf{t}_1) + c_1c_2\kappa_1(\mathbf{t}_1.\mathbf{t}_2)$$

+



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Assume $\kappa_1 \neq \kappa_2$, then $\mathbf{t}_1.\mathbf{t}_2 = 0$

Writing \mathcal{W} in terms of the basis \mathbf{t}_1 and \mathbf{t}_2 ,

$$\begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix}$$

$\kappa_1\kappa_2 =$ Gaussian curvature

$\frac{\kappa_1+\kappa_2}{2} =$ Mean curvature

$$\begin{aligned} \kappa_n &= \mathcal{W}\dot{\gamma}.\dot{\gamma} \\ &= (\mathcal{W}(c_1\mathbf{t}_1 + c_2\mathbf{t}_2)).(c_1\mathbf{t}_1 + c_2\mathbf{t}_2) \\ &= (c_1\mathcal{W}(\mathbf{t}_1) + c_2\mathcal{W}(\mathbf{t}_2)).(c_1\mathbf{t}_1 + c_2\mathbf{t}_2) \\ &= (c_1\kappa_1\mathbf{t}_1 + \textcolor{red}{c_2\kappa_2\mathbf{t}_2}).(\textcolor{red}{c_1\mathbf{t}_1} + c_2\mathbf{t}_2) \\ &= c_1^2\kappa_1(\mathbf{t}_1.\mathbf{t}_1) + c_1c_2\kappa_1(\mathbf{t}_1.\mathbf{t}_2) \\ &\quad + c_2c_1\kappa_2(\mathbf{t}_2.\mathbf{t}_1) + \end{aligned}$$



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Assume $\kappa_1 \neq \kappa_2$, then $\mathbf{t}_1.\mathbf{t}_2 = 0$

Writing \mathcal{W} in terms of the basis \mathbf{t}_1 and \mathbf{t}_2 ,

$$\begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix}$$

$\kappa_1\kappa_2 =$ Gaussian curvature

$\frac{\kappa_1+\kappa_2}{2} =$ Mean curvature

$$\begin{aligned} \kappa_n &= \mathcal{W}\dot{\gamma}.\dot{\gamma} \\ &= (\mathcal{W}(c_1\mathbf{t}_1 + c_2\mathbf{t}_2)).(c_1\mathbf{t}_1 + c_2\mathbf{t}_2) \\ &= (c_1\mathcal{W}(\mathbf{t}_1) + c_2\mathcal{W}(\mathbf{t}_2)).(c_1\mathbf{t}_1 + c_2\mathbf{t}_2) \\ &= (c_1\kappa_1\mathbf{t}_1 + \textcolor{red}{c_2\kappa_2\mathbf{t}_2}).(c_1\mathbf{t}_1 + \textcolor{red}{c_2\mathbf{t}_2}) \\ &= c_1^2\kappa_1(\mathbf{t}_1.\mathbf{t}_1) + c_1c_2\kappa_1(\mathbf{t}_1.\mathbf{t}_2) \\ &\quad + c_2c_1\kappa_2(\mathbf{t}_2.\mathbf{t}_1) + c_2^2\kappa_2(\mathbf{t}_2.\mathbf{t}_2) \end{aligned}$$



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$\mathcal{W}\mathbf{t}_1 = \kappa_1\mathbf{t}_1$ and

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Assume $\kappa_1 \neq \kappa_2$, then $\mathbf{t}_1 \cdot \mathbf{t}_2 = 0$

Writing \mathcal{W} in terms of the basis \mathbf{t}_1 and \mathbf{t}_2 ,

$$\begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix}$$

$\kappa_1\kappa_2 = \text{Gaussian curvature}$

$\frac{\kappa_1 + \kappa_2}{2} = \text{Mean curvature}$

$$\begin{aligned} \kappa_n &= \mathcal{W}\dot{\gamma} \cdot \dot{\gamma} \\ &= (\mathcal{W}(c_1\mathbf{t}_1 + c_2\mathbf{t}_2)) \cdot (c_1\mathbf{t}_1 + c_2\mathbf{t}_2) \\ &= (c_1\mathcal{W}(\mathbf{t}_1) + c_2\mathcal{W}(\mathbf{t}_2)) \cdot (c_1\mathbf{t}_1 + c_2\mathbf{t}_2) \\ &= (c_1\kappa_1\mathbf{t}_1 + c_2\kappa_2\mathbf{t}_2) \cdot (c_1\mathbf{t}_1 + c_2\mathbf{t}_2) \\ &= c_1^2\kappa_1(\mathbf{t}_1 \cdot \mathbf{t}_1) + c_1c_2\kappa_1 \underbrace{(\mathbf{t}_1 \cdot \mathbf{t}_2)}_0 \\ &\quad + c_2c_1\kappa_2 \underbrace{(\mathbf{t}_2 \cdot \mathbf{t}_1)}_0 + c_2^2\kappa_2(\mathbf{t}_2 \cdot \mathbf{t}_2) \end{aligned}$$

By orthogonality, two dot products are 0

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Denote its eigenvalues, κ_1 and κ_2 .

i.e. there exist \mathbf{t}_1 and \mathbf{t}_2 *unit vectors* so that

$\mathcal{W}\mathbf{t}_1 = \kappa_1\mathbf{t}_1$ and

$\mathcal{W}\mathbf{t}_2 = \kappa_2\mathbf{t}_2$

Assume $\kappa_1 \neq \kappa_2$, then $\mathbf{t}_1 \cdot \mathbf{t}_2 = 0$

Writing \mathcal{W} in terms of the basis \mathbf{t}_1 and \mathbf{t}_2 ,

$$\begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix}$$

$\kappa_1\kappa_2 = \text{Gaussian curvature}$

$\frac{\kappa_1 + \kappa_2}{2} = \text{Mean curvature}$

$$\begin{aligned} \kappa_n &= \mathcal{W}\dot{\gamma} \cdot \dot{\gamma} \\ &= (\mathcal{W}(c_1\mathbf{t}_1 + c_2\mathbf{t}_2)) \cdot (c_1\mathbf{t}_1 + c_2\mathbf{t}_2) \\ &= (c_1\mathcal{W}(\mathbf{t}_1) + c_2\mathcal{W}(\mathbf{t}_2)) \cdot (c_1\mathbf{t}_1 + c_2\mathbf{t}_2) \\ &= (c_1\kappa_1\mathbf{t}_1 + c_2\kappa_2\mathbf{t}_2) \cdot (c_1\mathbf{t}_1 + c_2\mathbf{t}_2) \\ &= c_1^2\kappa_1 \underbrace{(\mathbf{t}_1 \cdot \mathbf{t}_1)}_1 + c_1c_2\kappa_1 \underbrace{(\mathbf{t}_1 \cdot \mathbf{t}_2)}_0 \\ &\quad + c_2c_1\kappa_2 \underbrace{(\mathbf{t}_2 \cdot \mathbf{t}_1)}_0 + c_2^2\kappa_2 \underbrace{(\mathbf{t}_2 \cdot \mathbf{t}_2)}_1 \end{aligned}$$

By orthonormality, two of them are 1

$\mathcal{K} = \det \mathcal{W}$, Gaussian curvature

$\mathcal{H} = \frac{\text{trace } \mathcal{W}}{2}$, Mean Curvature

\mathcal{W} is symmetric.

Denote its eigenvalues, κ_1 and κ_2 .

i.e. there exist \mathbf{t}_1 and \mathbf{t}_2 *unit* vectors so that

$\mathcal{W}\mathbf{t}_1 = \kappa_1\mathbf{t}_1$ and

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Assume $\kappa_1 \neq \kappa_2$, then $\mathbf{t}_1 \cdot \mathbf{t}_2 = 0$

Writing \mathcal{W} in terms of the basis \mathbf{t}_1 and \mathbf{t}_2 ,

$$\begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix}$$

$\kappa_1\kappa_2 = \text{Gaussian curvature}$

$\frac{\kappa_1 + \kappa_2}{2} = \text{Mean curvature}$

$$\kappa_n = \mathcal{W}\dot{\gamma} \cdot \dot{\gamma}$$

$$= (\mathcal{W}(c_1\mathbf{t}_1 + c_2\mathbf{t}_2)) \cdot (c_1\mathbf{t}_1 + c_2\mathbf{t}_2)$$

$$= (c_1\mathcal{W}(\mathbf{t}_1) + c_2\mathcal{W}(\mathbf{t}_2)) \cdot (c_1\mathbf{t}_1 + c_2\mathbf{t}_2)$$

$$= (c_1\kappa_1\mathbf{t}_1 + c_2\kappa_2\mathbf{t}_2) \cdot (c_1\mathbf{t}_1 + c_2\mathbf{t}_2)$$

$$= c_1^2\kappa_1 \underbrace{(\mathbf{t}_1 \cdot \mathbf{t}_1)}_1 + c_1c_2\kappa_1 \underbrace{(\mathbf{t}_1 \cdot \mathbf{t}_2)}_0$$

$$+ c_2c_1\kappa_2 \underbrace{(\mathbf{t}_2 \cdot \mathbf{t}_1)}_0 + c_2^2\kappa_2 \underbrace{(\mathbf{t}_2 \cdot \mathbf{t}_2)}_1$$

$$= c_1^2\kappa_1 + c_2^2\kappa_2$$

So finally we get the following relation

$$\kappa_n = c_1^2 \kappa_1 + c_2^2 \kappa_2$$

where, $\dot{\gamma}(t_0) = c_1 \mathbf{t}_1 + c_2 \mathbf{t}_2$

So we have the computation of the normal curvature entirely in terms of $\dot{\gamma}(t_0)$, κ_i and \mathbf{t}_i

$\kappa_n = c_1^2 \kappa_1 + c_2^2 \kappa_2$
where, $\dot{\gamma}(t_0) = c_1 \mathbf{t}_1 + c_2 \mathbf{t}_2$

We use \mathcal{W} only to obtain κ_i and \mathbf{t}_i

$$\kappa_n = c_1^2 \kappa_1 + c_2^2 \kappa_2$$

$$\text{where, } \dot{\gamma}(t_0) = c_1 \mathbf{t}_1 + c_2 \mathbf{t}_2$$

Now given any curve, to compute κ_n , all we need from the curve is its velocity vector

$\kappa_n = c_1^2 \kappa_1 + c_2^2 \kappa_2$
where, $\dot{\gamma}(t_0) = \textcolor{red}{c}_1 \mathbf{t}_1 + \textcolor{red}{c}_2 \mathbf{t}_2$

We find out the coefficients of it when written in terms of \mathbf{t}_1 and \mathbf{t}_2

$$\kappa_n = c_1^2 \kappa_1 + c_2^2 \kappa_2$$

where, $\dot{\gamma}(t_0) = c_1 \mathbf{t}_1 + c_2 \mathbf{t}_2$

And we use them along with κ_1 and κ_2 in the highlighted formula for κ_n .

$$\kappa_n = c_1^2 \kappa_1 + c_2^2 \kappa_2$$

where, $\dot{\gamma}(t_0) = c_1 \mathbf{t}_1 + c_2 \mathbf{t}_2$

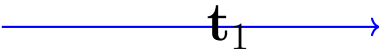
We will now use the fact that $\dot{\gamma}(t_0)$, \mathbf{t}_1 and \mathbf{t}_2 are all unit vectors

$$\kappa_n = c_1^2 \kappa_1 + c_2^2 \kappa_2$$

where, $\dot{\gamma}(t_0) = c_1 \mathbf{t}_1 + c_2 \mathbf{t}_2$

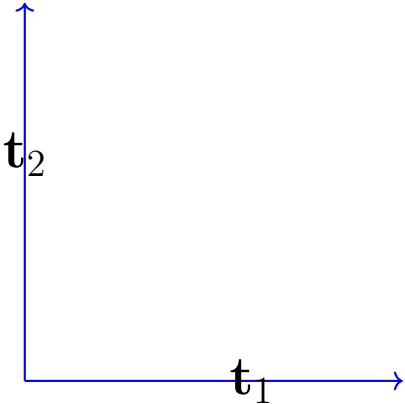
to write the coefficients in a more revealing form

$\kappa_n = c_1^2 \kappa_1 + c_2^2 \kappa_2$
where, $\dot{\gamma}(t_0) = c_1 \mathbf{t}_1 + c_2 \mathbf{t}_2$



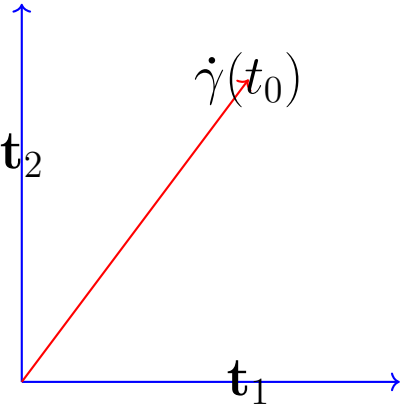
The vectors \mathbf{t}_1

$\kappa_n = c_1^2 \kappa_1 + c_2^2 \kappa_2$
where, $\dot{\gamma}(t_0) = c_1 \mathbf{t}_1 + c_2 \mathbf{t}_2$



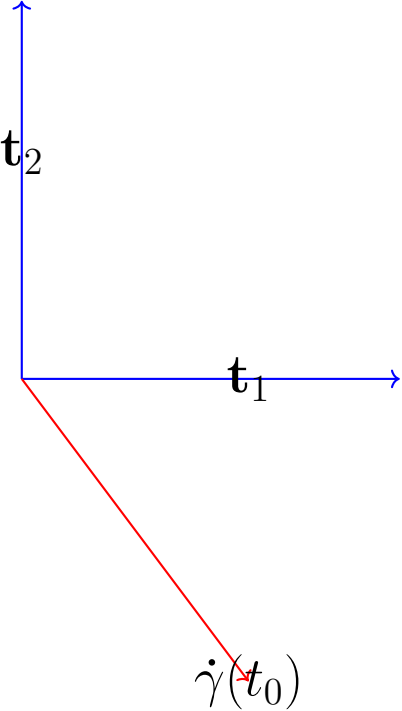
and \mathbf{t}_2 form a basis

$\kappa_n = c_1^2 \kappa_1 + c_2^2 \kappa_2$
where, $\dot{\gamma}(t_0) = c_1 \mathbf{t}_1 + c_2 \mathbf{t}_2$



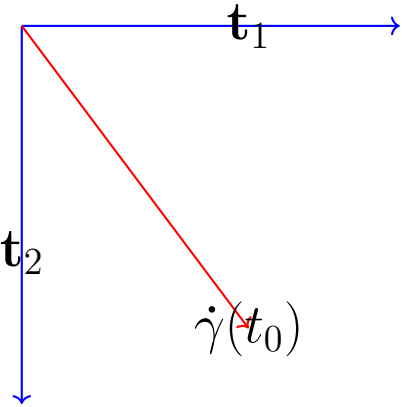
in terms of which we can write $\dot{\gamma}(t_0)$

$\kappa_n = c_1^2 \kappa_1 + c_2^2 \kappa_2$
where, $\dot{\gamma}(t_0) = c_1 \mathbf{t}_1 + c_2 \mathbf{t}_2$



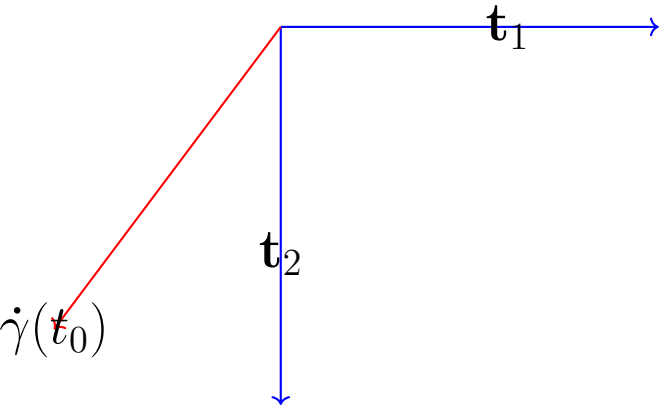
Even if $\dot{\gamma}(t_0)$ is not between the eigen-vectors

$\kappa_n = c_1^2 \kappa_1 + c_2^2 \kappa_2$
where, $\dot{\gamma}(t_0) = c_1 \mathbf{t}_1 + c_2 \mathbf{t}_2$



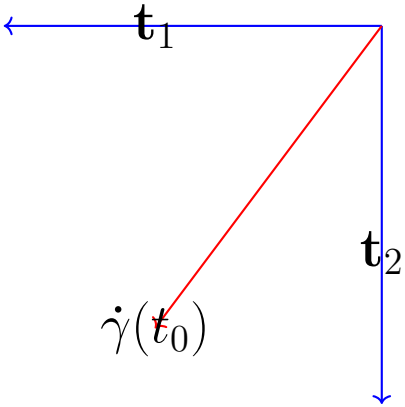
we can always replace one vector by the negative

$\kappa_n = c_1^2 \kappa_1 + c_2^2 \kappa_2$
where, $\dot{\gamma}(t_0) = c_1 \mathbf{t}_1 + c_2 \mathbf{t}_2$



No matter where $\dot{\gamma}(t_0)$ is placed,

$\kappa_n = c_1^2 \kappa_1 + c_2^2 \kappa_2$
where, $\dot{\gamma}(t_0) = c_1 \mathbf{t}_1 + c_2 \mathbf{t}_2$



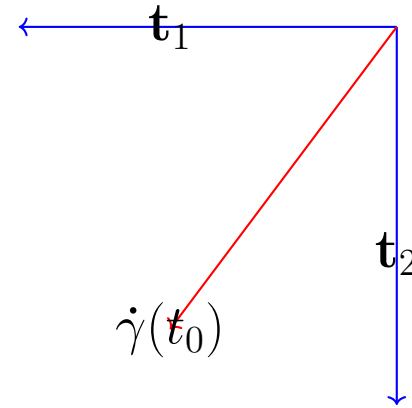
we can ensure that it is in between by replacing one or more eigenvectors by their negatives

$$\kappa_n = c_1^2 \kappa_1 + c_2^2 \kappa_2$$

where, $\dot{\gamma}(t_0) = c_1 \mathbf{t}_1 + c_2 \mathbf{t}_2$

We can always choose \mathbf{t}_1 and \mathbf{t}_2 so that,

$$\dot{\gamma}(t_0) = \cos(\theta) \mathbf{t}_1 + \sin(\theta) \mathbf{t}_2$$



Since all the vectors are unit vectors, the coefficients can be written in a better form

$$\kappa_n = c_1^2 \kappa_1 + c_2^2 \kappa_2$$

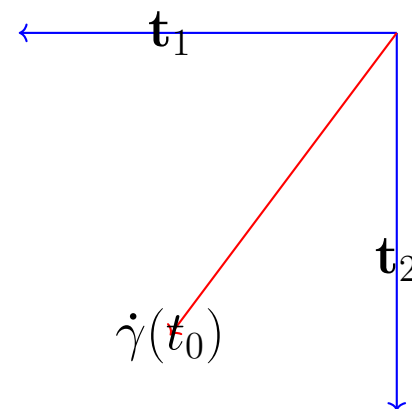
where, $\dot{\gamma}(t_0) = c_1 \mathbf{t}_1 + c_2 \mathbf{t}_2$

We can always choose \mathbf{t}_1 and \mathbf{t}_2 so that,

$$\dot{\gamma}(t_0) = \cos(\theta) \mathbf{t}_1 + \sin(\theta) \mathbf{t}_2$$

Therefore,

$$\kappa_n = \cos^2(\theta) \kappa_1 + \sin^2(\theta) \kappa_2$$



We can, therefore, express the normal curvature along $\dot{\gamma}(t_0)$ in terms of the angle it makes with \mathbf{t}_1 .

$$\kappa_n = c_1^2 \kappa_1 + c_2^2 \kappa_2$$

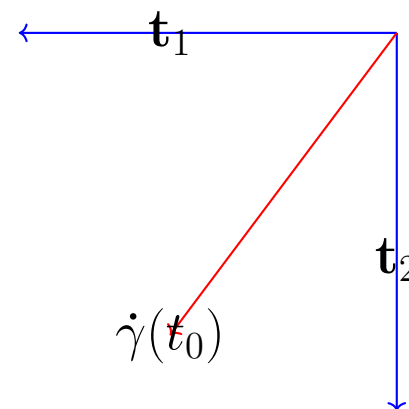
where, $\dot{\gamma}(t_0) = c_1 \mathbf{t}_1 + c_2 \mathbf{t}_2$

We can always choose \mathbf{t}_1 and \mathbf{t}_2 so that,

$$\dot{\gamma}(t_0) = \cos(\theta) \mathbf{t}_1 + \sin(\theta) \mathbf{t}_2$$

Therefore,

$$\begin{aligned} \kappa_n &= \cos^2(\theta) \kappa_1 + \sin^2(\theta) \kappa_2 \\ &= \cos^2(\theta) \kappa_1 + (1 - \cos^2(\theta)) \kappa_2 \end{aligned}$$



and exploit a standard trigonometric identity

$$\kappa_n = c_1^2 \kappa_1 + c_2^2 \kappa_2$$

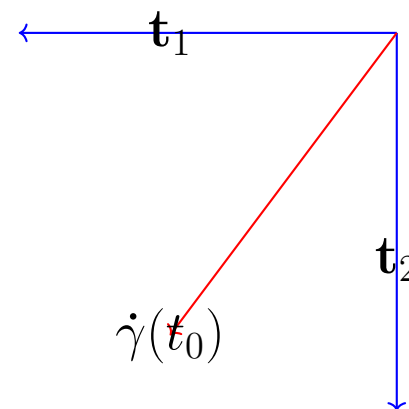
where, $\dot{\gamma}(t_0) = c_1 \mathbf{t}_1 + c_2 \mathbf{t}_2$

We can always choose \mathbf{t}_1 and \mathbf{t}_2 so that,

$$\dot{\gamma}(t_0) = \cos(\theta) \mathbf{t}_1 + \sin(\theta) \mathbf{t}_2$$

Therefore,

$$\begin{aligned} \kappa_n &= \cos^2(\theta) \kappa_1 + \sin^2(\theta) \kappa_2 \\ &= \cos^2(\theta) \kappa_1 + (1 - \cos^2(\theta)) \kappa_2 \\ &= \kappa_2 + (\kappa_1 - \kappa_2) \cos^2(\theta) \end{aligned}$$



to express it in a very revealing form

$$\kappa_n = c_1^2 \kappa_1 + c_2^2 \kappa_2$$

$$\text{where, } \dot{\gamma}(t_0) = c_1 \mathbf{t}_1 + c_2 \mathbf{t}_2$$

We can always choose \mathbf{t}_1 and \mathbf{t}_2 so that,

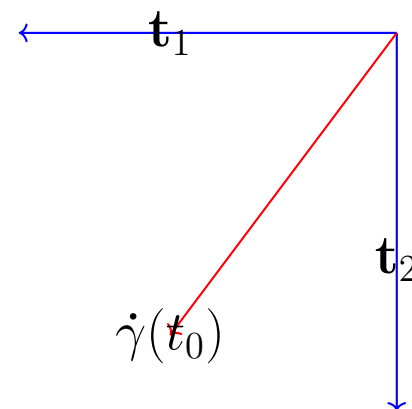
$$\dot{\gamma}(t_0) = \cos(\theta) \mathbf{t}_1 + \sin(\theta) \mathbf{t}_2$$

Therefore,

$$\begin{aligned} \kappa_n &= \cos^2(\theta) \kappa_1 + \sin^2(\theta) \kappa_2 \\ &= \cos^2(\theta) \kappa_1 + (1 - \cos^2(\theta)) \kappa_2 \\ &= \kappa_2 + (\kappa_1 - \kappa_2) \cos^2(\theta) \end{aligned}$$

Assume $\kappa_1 \leq \kappa_2$,

$$\kappa_n = \kappa_2 + \text{some negative number}$$



We may always assume that we labelled the smaller eigenvalue as the first

$$\kappa_n = c_1^2 \kappa_1 + c_2^2 \kappa_2$$

$$\text{where, } \dot{\gamma}(t_0) = c_1 \mathbf{t}_1 + c_2 \mathbf{t}_2$$

We can always choose \mathbf{t}_1 and \mathbf{t}_2 so that,

$$\dot{\gamma}(t_0) = \cos(\theta) \mathbf{t}_1 + \sin(\theta) \mathbf{t}_2$$

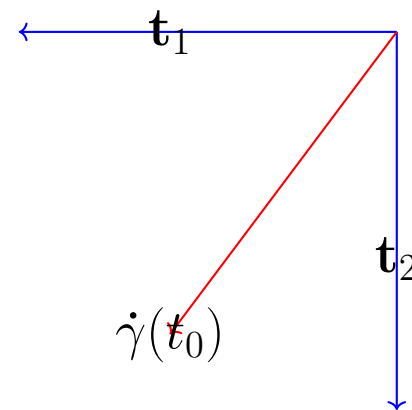
Therefore,

$$\begin{aligned} \kappa_n &= \cos^2(\theta) \kappa_1 + \sin^2(\theta) \kappa_2 \\ &= \cos^2(\theta) \kappa_1 + (1 - \cos^2(\theta)) \kappa_2 \\ &= \kappa_2 + (\kappa_1 - \kappa_2) \cos^2(\theta) \end{aligned}$$

Assume $\kappa_1 \leq \kappa_2$,

$$\kappa_n = \kappa_2 + \text{some negative number}$$

So, $\kappa_n \leq \kappa_2$.



Since we are adding a negative number or 0

$$\kappa_n = c_1^2 \kappa_1 + c_2^2 \kappa_2$$

$$\text{where, } \dot{\gamma}(t_0) = c_1 \mathbf{t}_1 + c_2 \mathbf{t}_2$$

We can always choose \mathbf{t}_1 and \mathbf{t}_2 so that,

$$\dot{\gamma}(t_0) = \cos(\theta) \mathbf{t}_1 + \sin(\theta) \mathbf{t}_2$$

Therefore,

$$\begin{aligned} \kappa_n &= \cos^2(\theta) \kappa_1 + \sin^2(\theta) \kappa_2 \\ &= \cos^2(\theta) \kappa_1 + (1 - \cos^2(\theta)) \kappa_2 \\ &= \kappa_2 + (\kappa_1 - \kappa_2) \cos^2(\theta) \end{aligned}$$

Assume $\kappa_1 \leq \kappa_2$,

$$\kappa_n = \kappa_2 + \text{some negative number}$$

So, $\kappa_n \leq \kappa_2$.

$\kappa_n = \kappa_2$ if and only if $\cos^2(\theta) = 0$
if and only if $\theta = \pi/2$

We will now check when it is equal

$$\kappa_n = c_1^2 \kappa_1 + c_2^2 \kappa_2$$

where, $\dot{\gamma}(t_0) = c_1 \mathbf{t}_1 + c_2 \mathbf{t}_2$

We can always choose \mathbf{t}_1 and \mathbf{t}_2 so that,

$$\dot{\gamma}(t_0) = \cos(\theta) \mathbf{t}_1 + \sin(\theta) \mathbf{t}_2$$

Therefore,

$$\begin{aligned} \kappa_n &= \cos^2(\theta) \kappa_1 + \sin^2(\theta) \kappa_2 \\ &= \cos^2(\theta) \kappa_1 + (1 - \cos^2(\theta)) \kappa_2 \\ &= \kappa_2 + (\kappa_1 - \kappa_2) \cos^2(\theta) \end{aligned}$$

Assume $\kappa_1 \leq \kappa_2$,

$$\kappa_n = \kappa_2 + \text{some negative number}$$

So, $\kappa_n \leq \kappa_2$.

$$\kappa_n = \kappa_2 \text{ if and only if } \cos^2(\theta) = 0$$

$$\text{if and only if } \theta = \pi/2$$

$$\text{if and only if } \dot{\gamma}(t_0) \text{ makes angle } \pi/2 \text{ with } \mathbf{t}_1$$



$$\kappa_n = c_1^2 \kappa_1 + c_2^2 \kappa_2$$

where, $\dot{\gamma}(t_0) = c_1 \mathbf{t}_1 + c_2 \mathbf{t}_2$

We can always choose \mathbf{t}_1 and \mathbf{t}_2 so that,

$$\dot{\gamma}(t_0) = \cos(\theta) \mathbf{t}_1 + \sin(\theta) \mathbf{t}_2$$

Therefore,

$$\begin{aligned} \kappa_n &= \cos^2(\theta) \kappa_1 + \sin^2(\theta) \kappa_2 \\ &= \cos^2(\theta) \kappa_1 + (1 - \cos^2(\theta)) \kappa_2 \\ &= \kappa_2 + (\kappa_1 - \kappa_2) \cos^2(\theta) \end{aligned}$$

Assume $\kappa_1 \leq \kappa_2$,

$$\kappa_n = \kappa_2 + \text{some negative number}$$

So, $\kappa_n \leq \kappa_2$.

$$\kappa_n = \kappa_2 \text{ if and only if } \cos^2(\theta) = 1$$

$$\text{if and only if } \theta = 0$$

$$\text{if and only if } \dot{\gamma}(t_0) \text{ makes angle } 0 \text{ with } \mathbf{t}_1$$

$$\text{if and only if } \dot{\gamma}(t_0) \text{ makes angle } 0 \text{ with } \mathbf{t}_2$$



$$\kappa_n = c_1^2 \kappa_1 + c_2^2 \kappa_2$$

where, $\dot{\gamma}(t_0) = c_1 \mathbf{t}_1 + c_2 \mathbf{t}_2$

We can always choose \mathbf{t}_1 and \mathbf{t}_2 so that,

$$\dot{\gamma}(t_0) = \cos(\theta) \mathbf{t}_1 + \sin(\theta) \mathbf{t}_2$$

Therefore,

$$\begin{aligned} \kappa_n &= \cos^2(\theta) \kappa_1 + \sin^2(\theta) \kappa_2 \\ &= \cos^2(\theta) \kappa_1 + (1 - \cos^2(\theta)) \kappa_2 \\ &= \kappa_2 + (\kappa_1 - \kappa_2) \cos^2(\theta) \end{aligned}$$

Assume $\kappa_1 \leq \kappa_2$,

$$\kappa_n = \kappa_2 + \text{some negative number}$$

So, $\kappa_n \leq \kappa_2$.

$$\kappa_n = \kappa_2 \text{ if and only if } \cos^2(\theta) = 0$$

$$\text{if and only if } \theta = \pi/2$$

$$\text{if and only if } \dot{\gamma}(t_0) \text{ makes angle } \pi/2 \text{ with } \mathbf{t}_1$$

$$\text{if and only if } \dot{\gamma}(t_0) \text{ makes angle } 0 \text{ with } \mathbf{t}_2$$

$$\text{i.e. } \dot{\gamma}(t_0) \text{ is aligned with } \mathbf{t}_2$$



$$\kappa_n = c_1^2 \kappa_1 + c_2^2 \kappa_2$$

$$\text{where, } \dot{\gamma}(t_0) = c_1 \mathbf{t}_1 + c_2 \mathbf{t}_2$$

We can always choose \mathbf{t}_1 and \mathbf{t}_2 so that,

$$\dot{\gamma}(t_0) = \cos(\theta) \mathbf{t}_1 + \sin(\theta) \mathbf{t}_2$$

Therefore,

$$\begin{aligned} \kappa_n &= \cos^2(\theta) \kappa_1 + \sin^2(\theta) \kappa_2 \\ &= \cos^2(\theta) \kappa_1 + (1 - \cos^2(\theta)) \kappa_2 \\ &= \kappa_2 + (\kappa_1 - \kappa_2) \cos^2(\theta) \end{aligned}$$

Assume $\kappa_1 \leq \kappa_2$,

$$\kappa_n = \kappa_2 + \text{some negative number}$$

So, $\kappa_n \leq \kappa_2$.

$$\kappa_n = \kappa_2 \text{ if and only if } \cos^2(\theta) = 1$$

$$\text{if and only if } \theta = 0$$

$$\text{if and only if } \dot{\gamma}(t_0) \text{ makes angle } 0 \text{ with } \mathbf{t}_1$$

$$\text{if and only if } \dot{\gamma}(t_0) \text{ makes angle } 0 \text{ with } \mathbf{t}_2$$

$$\text{i.e. } \dot{\gamma}(t_0) \text{ is aligned with } \mathbf{t}_2$$

Therefore,

Proposition. κ_2 is the maximum possible normal curvature of a curve at that point.

Now we see that κ_1 and κ_2 have a geometric interpretation

$$\kappa_n = c_1^2 \kappa_1 + c_2^2 \kappa_2$$

where, $\dot{\gamma}(t_0) = c_1 \mathbf{t}_1 + c_2 \mathbf{t}_2$

We can always choose \mathbf{t}_1 and \mathbf{t}_2 so that,

$$\dot{\gamma}(t_0) = \cos(\theta) \mathbf{t}_1 + \sin(\theta) \mathbf{t}_2$$

Therefore,

$$\begin{aligned} \kappa_n &= \cos^2(\theta) \kappa_1 + \sin^2(\theta) \kappa_2 \\ &= \cos^2(\theta) \kappa_1 + (1 - \cos^2(\theta)) \kappa_2 \\ &= \kappa_2 + (\kappa_1 - \kappa_2) \cos^2(\theta) \end{aligned}$$

Assume $\kappa_1 \leq \kappa_2$,

$$\kappa_n = \kappa_2 + \text{some negative number}$$

So, $\kappa_n \leq \kappa_2$.

$\kappa_n = \kappa_2$ if and only if $\cos^2(\theta) = 1$

if and only if $\theta = 0$

if and only if $\dot{\gamma}(t_0)$ makes angle 0 with \mathbf{t}_1

if and only if $\dot{\gamma}(t_0)$ makes angle 0 with \mathbf{t}_2

i.e. $\dot{\gamma}(t_0)$ is aligned with \mathbf{t}_2

Therefore,

Proposition. κ_2 is the maximum possible normal curvature of a curve at that point.

Exercise. κ_1 is the minimum possible normal curvature of a curve at that point.

This exercise can be worked out in exactly the same way

$$\kappa_n = c_1^2 \kappa_1 + c_2^2 \kappa_2$$

where, $\dot{\gamma}(t_0) = c_1 \mathbf{t}_1 + c_2 \mathbf{t}_2$

We can always choose \mathbf{t}_1 and \mathbf{t}_2 so that,

$$\dot{\gamma}(t_0) = \cos(\theta) \mathbf{t}_1 + \sin(\theta) \mathbf{t}_2$$

Therefore,

$$\begin{aligned} \kappa_n &= \cos^2(\theta) \kappa_1 + \sin^2(\theta) \kappa_2 \\ &= \cos^2(\theta) \kappa_1 + (1 - \cos^2(\theta)) \kappa_2 \\ &= \kappa_2 + (\kappa_1 - \kappa_2) \cos^2(\theta) \end{aligned}$$

Assume $\kappa_1 \leq \kappa_2$,

$$\kappa_n = \kappa_2 + \text{some negative number}$$

So, $\kappa_n \leq \kappa_2$.

$\kappa_n = \kappa_2$ if and only if $\cos^2(\theta) = 1$

if and only if $\theta = 0$

if and only if $\dot{\gamma}(t_0)$ makes angle 0 with \mathbf{t}_1

if and only if $\dot{\gamma}(t_0)$ is aligned with \mathbf{t}_1

i.e. $\dot{\gamma}(t_0)$ is aligned with \mathbf{t}_1

Therefore,

Proposition. κ_2 is the maximum possible normal curvature of a curve at that point. \mathbf{t}_1 is the direction along which the normal curvature is maximum.

Exercise. κ_1 is the minimum possible normal curvature of a curve at that point.

We can even give a geometric interpretation to t_1 and t_2

$$\kappa_n = c_1^2 \kappa_1 + c_2^2 \kappa_2$$

where, $\dot{\gamma}(t_0) = c_1 \mathbf{t}_1 + c_2 \mathbf{t}_2$

We can always choose \mathbf{t}_1 and \mathbf{t}_2 so that,

$$\dot{\gamma}(t_0) = \cos(\theta) \mathbf{t}_1 + \sin(\theta) \mathbf{t}_2$$

Therefore,

$$\begin{aligned} \kappa_n &= \cos^2(\theta) \kappa_1 + \sin^2(\theta) \kappa_2 \\ &= \cos^2(\theta) \kappa_1 + (1 - \cos^2(\theta)) \kappa_2 \\ &= \kappa_2 + (\kappa_1 - \kappa_2) \cos^2(\theta) \end{aligned}$$

Assume $\kappa_1 \leq \kappa_2$,

$$\kappa_n = \kappa_2 + \text{some negative number}$$

So, $\kappa_n \leq \kappa_2$.

$\kappa_n = \kappa_2$ if and only if $\cos^2(\theta) = 0$

if and only if $\theta = \pi/2$

if and only if $\dot{\gamma}(t_0)$ makes angle $\pi/2$ with \mathbf{t}_1

if and only if $\dot{\gamma}(t_0)$ makes angle 0 with \mathbf{t}_2

i.e. $\dot{\gamma}(t_0)$ is aligned with \mathbf{t}_2

Therefore,

Proposition. κ_2 is the maximum possible normal curvature of a curve at that point. \mathbf{t}_2 is the direction along which the normal curvature is maximum.

Exercise. κ_1 is the minimum possible normal curvature of a curve at that point. \mathbf{t}_1 is the direction along which the normal curvature is minimum.

κ_1 and κ_2 are called the **principal** curvatures

\mathbf{t}_1 and \mathbf{t}_2 are called the **principal** directions

