

Exercise sheet 5

1. Define $\mathbb{RP}^n = (\mathbb{R}^{n+1} \setminus 0) / \sim$ where $(x_0, x_1, \dots, x_n) \sim (y_0, y_1, \dots, y_n)$ if and only if there is an $\alpha \in \mathbb{R}$ such that $(x_0, x_1, \dots, x_n) = \alpha(y_0, y_1, \dots, y_n)$. Show that it is a CW-complex for each n and compute its homology.
2. Define $\mathbb{CP}^n = (\mathbb{C}^{n+1} \setminus 0) / \sim$ where $(x_0, x_1, \dots, x_n) \sim (y_0, y_1, \dots, y_n)$ if and only if there is an $\alpha \in \mathbb{C}$ such that $(x_0, x_1, \dots, x_n) = \alpha(y_0, y_1, \dots, y_n)$. Show that it is a CW-complex for each n and compute its homology.
3. Exercises 7 and 8 from section 2.2 of Hatcher's Algebraic Topology (page 155).
4. Prove that $H_n(X_n)$ is always free, where X_n denotes the n skeleton of a CW-complex X .
5. Compute $H_k(S^n; G)$ for any abelian group, G .
6. Prove that if $f : S^n \rightarrow S^n$ has degree d , then the induced map $f : H_k(S^n; G) \rightarrow H_k(S^n; G)$ is multiplication by d .
7. Let X is a topological space such that $H_k(X)$ is always finitely generated, then prove that if F is a field, then the $\chi(X) = \sum_i (-1)^i \dim H_i(X; F)$.
8. Prove the following properties of Tor_1 :
 - (a) $Tor_1(A, B) = Tor_1(B, A)$
 - (b) $Tor(A, G) = 0$ if G is free
 - (c) If $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is a short exact sequence then, the following sequence is exact:
$$0 \rightarrow Tor_1(A, G) \rightarrow Tor_1(B, G) \rightarrow Tor_1(C, G) \rightarrow A \otimes G \rightarrow B \otimes G \rightarrow C \otimes G \rightarrow 0$$