# Exercise sheet 5

Curves and Surfaces, MTH201

## Additional exercises

**NOTE:** These exercises repeat many of the concepts / exrecises covered earlier and are meant for you to identify gaps in your understanding. They are not exhaustive and the mid-semester examination will not be restricted to these questions.

Let $S \subset \mathbb{R}^3$ be a part of a surface and $\sigma : U \to S$ be a regular surface patch.

1. For each of the surface patches below, identify the surface that they (partially) cover:

   (a) $\sigma : \mathbb{R}^2 \to \mathbb{R}^3$, $\sigma(x, y) = (x, y, 0)$.

   (b) $\sigma : \mathbb{R}^2 \to \mathbb{R}^3$, $\sigma(x, y) = (x, y, x + y)$.

   (c) $\sigma : \mathbb{R}^2 \to \mathbb{R}^3$, $\sigma(x, y) = (\cos(x), \sin(x), y)$.

   (d) $\sigma : \mathbb{R}^2 \to \mathbb{R}^3$, $\sigma(x, y) = (x, y, \sqrt{r^2 - x^2 - y^2})$.

   (e) $\sigma : \mathbb{R}^2 \to \mathbb{R}^3$, $\sigma(x, y) = (x, y, \sqrt{r^2 - x^2 + y^2})$.

2. If $U$ is an open subset in $\mathbb{R}^2$ and $f : U \to \mathbb{R}$ is a smooth function, then show that $\sigma(x, y) := (x, y, f(x, y))$ is a regular surface patch.

3. Consider a $\gamma : (a, b) \to S \subset \mathbb{R}^3$ parametrizing a curve that lies on the part of the surface covered by the surface patch. In other words, for each $t$, $\gamma(t)$ must, be in the image of $\sigma$, i.e. there is some $x(t)$, and $y(t)$ in $U$, so that $\gamma(t) = \sigma(x(t), y(t))$. Assuming that $x(t)$ and $y(t)$ are smooth,

   (a) Consider the part of the surface covered by $\sigma : \mathbb{R}^2 \to \mathbb{R}^3$, $\sigma(x, y) = (\cos(x), \sin(x), y)$ and consider the curve $\gamma(t) = (0, 0, t)$. Note that it lies on the surface. Write it in the form, $\gamma(t) = \sigma(x(t), y(t))$ by finding suitable functions $x(t)$ and $y(t)$. Do the same for the curve $\gamma_2(t) = (\cos(t), -\sin(t), 0)$ which also lies on the surface.

   (b) Show that

   $$\dot{\gamma}(t_0) = x'(t_0)\sigma_x(x(t_0), y(t_0)) + y'(t_0)\sigma_y(x(t_0), y(t_0))$$

4. Show that $\sigma_x(x_0, y_0)$ and $\sigma_y(x_0, y_0)$ are each velocity vectors of curves that lie on the surface. Why are they linearly independent?

5. Why do the previous two exercises show that $\sigma_x(x_0, y_0)$ and $\sigma_y(x_0, y_0)$ are a basis for the tangent vectors?

6. Compute $\hat{n}(p)$ for any point $p$ on a plane. Show that it is constant.

7. Compute $\hat{n}(p)$ for any point $p$ on a sphere.

8. Consider a point $p$ on the part of the surface covered by a surface patch. Therefore, it is of the form $p = \sigma(x_0, y_0)$ for some $x_0$ an $y_0$. Consider $\hat{n}(p) = \sigma_x(x_0, y_0) \times \sigma_y(x_0, y_0)$ which is a vector in $\mathbb{R}^3$ based at $p$.

   (a) Is it a tangent vector? Why or why not?

   (b) Why is its dot product with $\sigma_x(x_0, y_0)$ and $\sigma_y(x_0, y_0)$ zero?

   (c) Why is its dot product with *any* tangent vector (of the surface at $p$) zero?

9. Consider a smooth function from the surface to $\mathbb{R}$, $f : S \to \mathbb{R}$. Show that the rate of change along any parametrization $\gamma$, i.e. $\frac{\mathrm{d}}{\mathrm{d}t}|_{t=t_0} f(\gamma(t)))$ depends on the partial derivatives of $f$ at the point $\gamma(t_0)$ and the velocity of $\gamma$ at $t_0$. (*Hint:* This is just a way of interpreting chain rule)

10. Consider $\hat{\hat{n}}(x, y) = \hat{n}(\sigma(x, y))$. Note that if $p = \sigma(x, y)$, then $\hat{\hat{n}}(x, y) = \hat{n}(p)$, i.e. $\hat{\hat{n}}$ is simply $\hat{n}$ written in terms of the coordinates provided by $\sigma$. Note also that if $\gamma(t) = \sigma(x(t), y(t))$, then $\hat{\hat{n}}(x(t), y(t)) = \hat{n}(\gamma(t))$.

    (a) Show that the rate of change of $\hat{n}$ along a parametrization $\gamma$ of a curve on the surface (i.e. $\frac{\mathrm{d}}{\mathrm{d}t}|_{t=t_0} \hat{n}(\gamma(t))))$ depends only on $\hat{n}$ at the point $\gamma(t_0)$ and the velocity of $\gamma$ at $t_0$. (*Hint:* Apply the previous exercise to each coordinate of the function $\hat{\hat{n}}$. Why is it important to use $\hat{\hat{n}}$ and not $\hat{n}$?)

11. From now on, we will assume that $\gamma$ is a unit speed parametrization. Show that $\ddot{\gamma}(t_0).\hat{n}(\gamma(t_0)) = -\dot{\gamma}(t_0).\frac{\mathrm{d}}{\mathrm{d}t}|_{t=t_0} \hat{n}(\gamma(t))$. Along with the previous exercise, this shows that the component of the acceleration in the direction of the normal, (denoted $\kappa_n(t_0)$) depends only on the normal to the surface and the direction of a unit speed parametrization.

12. Consider the other component of the acceleration, $\ddot{\gamma}(t_0) - \ddot{\gamma}(t_0).\hat{n}(\gamma(t_0))$. Why is $\ddot{\gamma}(t_0)$ perpendicular to $\hat{n}(\gamma(t_0))$? Why is $\ddot{\gamma}(t_0)$ parallel to $\mathbf{T}(t_0) \times \hat{n}(\gamma(t_0))$? Let the magnitude be denoted by $\kappa_g(t_0)$

13. Show that $\kappa^2(t_0) = \kappa_n^2(t_0) + \kappa_g^2(t_0)$.