# Exercise sheet 5

Curves and Surfaces, MTH201

## Additional exercises

**NOTE:** These exercises repeat many of the concepts / exrecises covered earlier and are meant for you to identify gaps in your understanding. They are not exhaustive and the mid-semester examination will not be restricted to these questions.

Let $S \subset \mathbb{R}^3$ be a part of a surface and $\sigma : U \to S$ be a regular surface patch.

1. For each of the surface patches below, identify the surface that they (partially) cover:
   (a) $\sigma : \mathbb{R}^2 \to \mathbb{R}^3$, $\sigma(x, y) = (x, y, 0)$.
   (b) $\sigma : \mathbb{R}^2 \to \mathbb{R}^3$, $\sigma(x, y) = (x, y, x + y)$.
   (c) $\sigma : \mathbb{R}^2 \to \mathbb{R}^3$, $\sigma(x, y) = (\cos(x), \sin(x), y)$.
   (d) $\sigma : \mathbb{R}^2 \to \mathbb{R}^3$, $\sigma(x, y) = (x, y, \sqrt{r^2 - x^2 - y^2})$.
   (e) $\sigma : \mathbb{R}^2 \to \mathbb{R}^3$, $\sigma(x, y) = (x, y, \sqrt{r^2 - x^2 + y^2})$.

2. Consider a $\gamma : (a, b) \to S \subset \mathbb{R}^3$ parametrizing a curve that lies on the part of the surface covered by the surface patch. In other words, for each $t$, $\gamma(t)$ must, be in the image of $\sigma$, i.e. there is some $x(t)$, and $y(t)$ in $U$, so that $\gamma(t) = \sigma(x(t), y(t))$. Assuming that $x(t)$ and $y(t)$ are smooth,
   (a) Consider the part of the surface covered by $\sigma : \mathbb{R}^2 \to \mathbb{R}^3$, $\sigma(x, y) = (\cos(x), \sin(x), y)$ and consider the curve $\gamma(t) = (0, 0, t)$. Note that it lies on the surface. Write it in the form, $\gamma(t) = \sigma(x(t), y(t))$ by finding suitable functions $x(t)$ and $y(t)$. Do the same for the curve $\gamma_2(t) = (\cos(t), -\sin(t), 0)$ which also lies on the surface.
   (b) Show that
   $$\dot{\gamma}(t_0) = x'(t_0)\sigma_x(x(t_0), y(t_0)) + y'(t_0)\sigma_y(x(t_0), y(t_0))$$

3. Show that $\sigma_x(x_0, y_0)$ and $\sigma_y(x_0, y_0)$ are each velocity vectors of curves that lie on the surface. Why are they linearly independent?

4. Why do the previous two exercises show that $\sigma_x(x_0, y_0)$ and $\sigma_y(x_0, y_0)$ are a basis for the tangent vectors?

5. Consider a point $p$ on the part of the surface covered by a surface patch. Therefore, it is of the form $p = \sigma(x_0, y_0)$ for some $x_0$ an $y_0$. Consider $\hat{n}(p) = \sigma_x(x_0, y_0) \times \sigma_y(x_0, y_0)$. Why is its dot product with $\sigma_x(x_0, y_0)$ and $\sigma_y(x_0, y_0)$ zero? Why is its dot product with *any* tangent vector (of the surface at $p$) zero?

6. Compute $\hat{n}(p)$ for any point $p$ on a sphere.