Curvature of a curve on a surface

Recall:

Consider an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$.

It will be important to be absolutely clear about this

Curvature of a curve on a surface

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Consider an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$.

$v = \alpha_1 \mathbf{e}_1 + \alpha_2 \mathbf{e}_2 + \alpha_3 \mathbf{e}_3$

So we can write any vector as linear combination of them

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$\alpha_i = v.\mathbf{e}_i$

And the coefficients can be recovered by a dot product

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$\alpha_i = v.\mathbf{e}_i$

If v is perpendicular to $\mathbf{e}_1$ and $\mathbf{e}_2$

$v.\mathbf{e}_1 = 0$

$v.\mathbf{e}_2 = 0$

$\blacksquare$

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$\alpha_1 = v.\mathbf{e}_1 = 0$

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So, $v = \alpha_3\mathbf{e}_3$

A vector perpendicular to two of the basis vectors will have to be in the direction of the third

Curvature of a curve on a surface

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So, $v = \alpha_3 \mathbf{e}_3$

$\gamma : (\alpha, \beta) \to S \subset \mathbb{R}^3$

We continue our study of the consequences of a curve being on a surfaces

.

Curvature of a curve on a surface

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$\alpha_1 = v.\mathbf{e}_1 = 0$
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So, $v = \alpha_3\mathbf{e}_3$

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Once again, we can think of a curve using the usual coordinates

.

Curvature of a curve on a surface

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So, $v = \alpha_3\mathbf{e}_3$

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or using the coordinates provided by the surface patch.

Curvature of a curve on a surface

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let us now study acceleration and curvature of a curve on a surface

.

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$\alpha_1 = v.\mathbf{e}_1 = 0$

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So, $v = \alpha_3\mathbf{e}_3$

$\gamma : (\alpha, \beta) \to S \subset \mathbb{R}^3$ unit speed

$\ddot{\gamma}(t).\mathbf{T}(t) = 0$

If the parametrization is unit speed, then $\ddot{\gamma}(t)$ is cetainly perpendicular to $\mathbf{T}(t)$.

.

Curvature of a curve on a surface

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So, $v = \alpha_3\mathbf{e}_3$

$\gamma : (\alpha, \beta) \to S \subset \mathbb{R}^3$ unit speed
$\ddot{\gamma}(t).\mathbf{T}(t) = 0$

$\sigma : U \to S$ a regular surface patch.

As usual, we study the surface with a surface patch

.

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$\sigma : U \to S$ a regular surface patch.
$\hat{\mathbf{n}}(p) := \frac{\sigma_x(p) \times \sigma_y(p)}{\|\sigma_x(p) \times \sigma_y(p)\|}$

A regular surface has a natural normal.

Curvature of a curve on a surface

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Exercise. Prove that $\hat{\mathbf{n}}(\gamma(t))$ is perpendicular to $\mathbf{T}(t)$.

Try to solve this exercise (Hint: $\mathbf{T}(t)$ is in the span of?)

Curvature of a curve on a surface

Recall:
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So, $v = \alpha_3\mathbf{e}_3$

$\gamma : (\alpha, \beta) \to S \subset \mathbb{R}^3$ unit speed
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The following discussion is motivated by this intuitive obervation:

Curvature of a curve on a surface

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$\gamma : (\alpha, \beta) \to S \subset \mathbb{R}^3$ unit speed
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Exercise. Prove that $\hat{\mathbf{n}}(\gamma(t))$ is perpendicular to $\mathbf{T}(t)$.

The curve lies on a surface because

.

Curvature of a curve on a surface

$\ddot{\gamma}(t).\hat{\mathbf{n}}(\gamma(t))$

Recall:
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$\alpha_1 = v.\mathbf{e}_1 = 0$
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So, $v = \alpha_3\mathbf{e}_3$

$\gamma : (\alpha, \beta) \to S \subset \mathbb{R}^3$ unit speed
$\ddot{\gamma}(t).\mathbf{T}(t) = 0$

$\sigma : U \to S$ a regular surface patch.
$\hat{\mathbf{n}}(p) := \frac{\sigma_x(p)\times\sigma_y(p)}{\|\sigma_x(p)\times\sigma_y(p)\|}$

Exercise. Prove that $\hat{\mathbf{n}}(\gamma(t))$ is perpendicular to $\mathbf{T}(t)$.

some component of the acceleration keeps the curve on the surface

Curvature of a curve on a surface

$\ddot{\gamma}(t).\hat{\mathbf{n}}(\gamma(t))$

Recall:
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$\gamma : (\alpha, \beta) \to S \subset \mathbb{R}^3$ unit speed
$\ddot{\gamma}(t).\mathbf{T}(t) = 0$

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Exercise. Prove that $\hat{\mathbf{n}}(\gamma(t))$ is perpendicular to $\mathbf{T}(t)$.

This is obviously normal to the surface

Curvature of a curve on a surface

$\kappa_n(t) := \ddot{\gamma}(t).\hat{\mathbf{n}}(\gamma(t))$ (normal curvature)

Recall:
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So, $v = \alpha_3\mathbf{e}_3$

$\gamma : (\alpha, \beta) \to S \subset \mathbb{R}^3$ unit speed
$\ddot{\gamma}(t).\mathbf{T}(t) = 0$

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Exercise. Prove that $\hat{\mathbf{n}}(\gamma(t))$ is perpendicular to $\mathbf{T}(t)$.

The magnitude of this components is called the normal curvature

Curvature of a curve on a surface

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Exercise. Prove that $\hat{\mathbf{n}}(\gamma(t))$ is perpendicular to $\mathbf{T}(t)$.

$\kappa_n(t) := \ddot{\gamma}(t).\hat{\mathbf{n}}(\gamma(t))$ (normal curvature)

$\ddot{\gamma}(t) - (\ddot{\gamma}(t).\hat{\mathbf{n}}(\gamma(t)))\hat{\mathbf{n}}(\gamma(t)) =??$

If we subtract this component, what direction does the rest point in?

Curvature of a curve on a surface

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So, $v = \alpha_3\mathbf{e}_3$

$\gamma : (\alpha, \beta) \to S \subset \mathbb{R}^3$ unit speed
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Exercise. Prove that $\hat{\mathbf{n}}(\gamma(t))$ is perpendicular to $\mathbf{T}(t)$.

$\kappa_n(t) := \ddot{\gamma}(t).\hat{\mathbf{n}}(\gamma(t))$ (normal curvature)

$\ddot{\gamma}(t) - (\ddot{\gamma}(t).\hat{\mathbf{n}}(\gamma(t)))\hat{\mathbf{n}}(\gamma(t)) =??$

As we have seen before, we can find that if we find two orthonormal vectors

Curvature of a curve on a surface

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$\gamma : (\alpha, \beta) \to S \subset \mathbb{R}^3$ unit speed
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Exercise. Prove that $\hat{\mathbf{n}}(\gamma(t))$ is perpendicular to $\mathbf{T}(t)$.

$\kappa_n(t) := \ddot{\gamma}(t).\hat{\mathbf{n}}(\gamma(t))$ (normal curvature)

$\ddot{\gamma}(t) - (\ddot{\gamma}(t).\hat{\mathbf{n}}(\gamma(t)))\hat{\mathbf{n}}(\gamma(t)) =??$

that both are perpendicular to

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Exercise. Prove that $\hat{\mathbf{n}}(\gamma(t))$ is perpendicular to $\mathbf{T}(t)$.

$\kappa_n(t) := \ddot{\gamma}(t).\hat{\mathbf{n}}(\gamma(t))$ (normal curvature)

$\ddot{\gamma}(t) - (\ddot{\gamma}(t).\hat{\mathbf{n}}(\gamma(t)))\hat{\mathbf{n}}(\gamma(t)) =??$

and then take the cross product

Curvature of a curve on a surface

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So, $v = \alpha_3\mathbf{e}_3$

$\ddot{\gamma}(t) - (\ddot{\gamma}(t).\hat{\mathbf{n}}(\gamma(t)))\hat{\mathbf{n}}(\gamma(t)) =??$

$\gamma : (\alpha, \beta) \to S \subset \mathbb{R}^3$ unit speed
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$\hat{\mathbf{n}}(p) := \frac{\sigma_x(p)\times\sigma_y(p)}{\|\sigma_x(p)\times\sigma_y(p)\|}$

Exercise. Prove that $\hat{\mathbf{n}}(\gamma(t))$ is perpendicular to $\mathbf{T}(t)$.

The acceleration is already perpendicular to the unit tangent

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If v is perpendicular to $\mathbf{e}_1$ and $\mathbf{e}_2$
$\alpha_1 = v.\mathbf{e}_1 = 0$
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So, $v = \alpha_3\mathbf{e}_3$

$\gamma : (\alpha, \beta) \to S \subset \mathbb{R}^3$ unit speed
$\ddot{\gamma}(t).\mathbf{T}(t) = 0$

$\sigma : U \to S$ a regular surface patch.
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Exercise. Prove that $\hat{\mathbf{n}}(\gamma(t))$ is perpendicular to $\mathbf{T}(t)$.

$\kappa_n(t) := \ddot{\gamma}(t).\hat{\mathbf{n}}(\gamma(t))$ (normal curvature)

$\ddot{\gamma}(t) - (\ddot{\gamma}(t).\hat{\mathbf{n}}(\gamma(t)))\hat{\mathbf{n}}(\gamma(t)) =??$

So is the component in the direction of the normal

Curvature of a curve on a surface

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$\alpha_1 = v.\mathbf{e}_1 = 0$
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So, $v = \alpha_3\mathbf{e}_3$

$\ddot{\gamma}(t) - (\ddot{\gamma}(t).\hat{\mathbf{n}}(\gamma(t)))\hat{\mathbf{n}}(\gamma(t)) = ??$

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Exercise. Prove that $\hat{\mathbf{n}}(\gamma(t))$ is perpendicular to $\mathbf{T}(t)$.

Therefore, so is the difference

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So, $v = \alpha_3\mathbf{e}_3$

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Exercise. Prove that $\hat{\mathbf{n}}(\gamma(t))$ is perpendicular to $\mathbf{T}(t)$.

because they both lie in the plane perpendicular to $\mathbf{T}$

$\kappa_n(t) := \ddot{\gamma}(t).\hat{\mathbf{n}}(\gamma(t))$ (normal curvature)

$\ddot{\gamma}(t) - (\ddot{\gamma}(t).\hat{\mathbf{n}}(\gamma(t)))\hat{\mathbf{n}}(\gamma(t)) = ??$

Curvature of a curve on a surface

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Exercise. Prove that $\hat{\mathbf{n}}(\gamma(t))$ is perpendicular to $\mathbf{T}(t)$.

$\kappa_n(t) := \ddot{\gamma}(t).\hat{\mathbf{n}}(\gamma(t))$ (normal curvature)

$\ddot{\gamma}(t) - (\ddot{\gamma}(t).\hat{\mathbf{n}}(\gamma(t)))\hat{\mathbf{n}}(\gamma(t)) =??$

Also, since we have removed the component of the normal, it must be perpendicular to it

Curvature of a curve on a surface

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> $\alpha_1 = v.\mathbf{e}_1 = 0$
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> So, $v = \alpha_3\mathbf{e}_3$

$\gamma : (\alpha, \beta) \to S \subset \mathbb{R}^3$ unit speed
$\ddot{\gamma}(t).\mathbf{T}(t) = 0$

$\sigma : U \to S$ a regular surface patch.
$\hat{\mathbf{n}}(p) := \frac{\sigma_x(p)\times\sigma_y(p)}{\|\sigma_x(p)\times\sigma_y(p)\|}$

Exercise. Prove that $\hat{\mathbf{n}}(\gamma(t))$ is perpendicular to $\mathbf{T}(t)$.

$\kappa_n(t) := \ddot{\gamma}(t).\hat{\mathbf{n}}(\gamma(t))$ (normal curvature)

$\ddot{\gamma}(t) - (\ddot{\gamma}(t).\hat{\mathbf{n}}(\gamma(t)))\hat{\mathbf{n}}(\gamma(t)) =??$

So we know that it will be perpendicular to $\hat{\mathbf{n}}$ and $\mathbf{T}$

Curvature of a curve on a surface

Recall:
Consider an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$.
$v = \alpha_1 \mathbf{e}_1 + \alpha_2 \mathbf{e}_2 + \alpha_3 \mathbf{e}_3$
$\alpha_i = v.\mathbf{e}_i$
If v is perpendicular to $\mathbf{e}_1$ and $\mathbf{e}_2$
$\alpha_1 = v.\mathbf{e}_1 = 0$
$\alpha_2 = v.\mathbf{e}_2 = 0$
So, $v = \alpha_3 \mathbf{e}_3$

$\gamma : (\alpha, \beta) \to S \subset \mathbb{R}^3$ unit speed
$\ddot{\gamma}(t).\mathbf{T}(t) = 0$

$\sigma : U \to S$ a regular surface patch.
$\hat{\mathbf{n}}(p) := \frac{\sigma_x(p) \times \sigma_y(p)}{\|\sigma_x(p) \times \sigma_y(p)\|}$

Exercise. Prove that $\hat{\mathbf{n}}(\gamma(t))$ is perpendicular to $\mathbf{T}(t)$.

$\kappa_n(t) := \ddot{\gamma}(t).\hat{\mathbf{n}}(\gamma(t))$ (normal curvature)

$$\ddot{\gamma}(t) - (\ddot{\gamma}(t).\hat{\mathbf{n}}(\gamma(t)))\hat{\mathbf{n}}(\gamma(t)) = \kappa_g(t)(\hat{\mathbf{n}}(\gamma(t)) \times \mathbf{T}(t))$$

And is therefore some scalar multiple of $\hat{\mathbf{n}} \times \mathbf{T}$

Curvature of a curve on a surface

> *Recall:*
> Consider an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$.
> $v = \alpha_1 \mathbf{e}_1 + \alpha_2 \mathbf{e}_2 + \alpha_3 \mathbf{e}_3$
> $\alpha_i = v.\mathbf{e}_i$
> If v is perpendicular to $\mathbf{e}_1$ and $\mathbf{e}_2$
> $\alpha_1 = v.\mathbf{e}_1 = 0$
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> So, $v = \alpha_3 \mathbf{e}_3$

$\gamma : (\alpha, \beta) \to S \subset \mathbb{R}^3$ unit speed
$\ddot{\gamma}(t).\mathbf{T}(t) = 0$

$\sigma : U \to S$ a regular surface patch.
$\hat{\mathbf{n}}(p) := \frac{\sigma_x(p) \times \sigma_y(p)}{\|\sigma_x(p) \times \sigma_y(p)\|}$

Exercise. Prove that $\hat{\mathbf{n}}(\gamma(t))$ is perpendicular to $\mathbf{T}(t)$.

$\blacksquare$

$\kappa_n(t) := \ddot{\gamma}(t).\hat{\mathbf{n}}(\gamma(t))$ (normal curvature)

$\ddot{\gamma}(t) - (\ddot{\gamma}(t).\hat{\mathbf{n}}(\gamma(t)))\hat{\mathbf{n}}(\gamma(t)) = \kappa_g(t)(\hat{\mathbf{n}}(\gamma(t)) \times \mathbf{T}(t))$

$\kappa_g(t)$ is the geodesic curvature

Curvature of a curve on a surface

> *Recall:*
> Consider an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$.
> $v = \alpha_1 \mathbf{e}_1 + \alpha_2 \mathbf{e}_2 + \alpha_3 \mathbf{e}_3$
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> $\alpha_1 = v.\mathbf{e}_1 = 0$
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> So, $v = \alpha_3 \mathbf{e}_3$

$\gamma : (\alpha, \beta) \to S \subset \mathbb{R}^3$ unit speed
$\ddot{\gamma}(t).\mathbf{T}(t) = 0$

$\sigma : U \to S$ a regular surface patch.
$\hat{\mathbf{n}}(p) := \frac{\sigma_x(p) \times \sigma_y(p)}{\|\sigma_x(p) \times \sigma_y(p)\|}$

Exercise. Prove that $\hat{\mathbf{n}}(\gamma(t))$ is perpendicular to $\mathbf{T}(t)$.

$\kappa_n(t) := \ddot{\gamma}(t).\hat{\mathbf{n}}(\gamma(t))$ (normal curvature)

$\ddot{\gamma}(t) - (\ddot{\gamma}(t).\hat{\mathbf{n}}(\gamma(t)))\hat{\mathbf{n}}(\gamma(t)) = \kappa_g(t)(\hat{\mathbf{n}}(\gamma(t)) \times \mathbf{T}(t))$

$\kappa_g(t)$ is the geodesic curvature

$$\kappa_g(t) = \ddot{\gamma}(t).(\hat{\mathbf{n}}(\gamma(t)) \times \mathbf{T}(t))$$

Taking the dot product with $\hat{\mathbf{n}}(\gamma(t)) \times \mathbf{T}(t)$ on both sides of the above equation

Curvature of a curve on a surface

Recall:
Consider an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$.
$v = \alpha_1\mathbf{e}_1 + \alpha_2\mathbf{e}_2 + \alpha_3\mathbf{e}_3$
$\alpha_i = v.\mathbf{e}_i$
If v is perpendicular to $\mathbf{e}_1$ and $\mathbf{e}_2$
$\alpha_1 = v.\mathbf{e}_1 = 0$
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So, $v = \alpha_3\mathbf{e}_3$

$\gamma : (\alpha, \beta) \to S \subset \mathbb{R}^3$ unit speed
$\ddot{\gamma}(t).\mathbf{T}(t) = 0$

$\sigma : U \to S$ a regular surface patch.
$\hat{\mathbf{n}}(p) := \frac{\sigma_x(p)\times\sigma_y(p)}{\|\sigma_x(p)\times\sigma_y(p)\|}$

Exercise. Prove that $\hat{\mathbf{n}}(\gamma(t))$ is perpendicular to $\mathbf{T}(t)$.

$\kappa_n(t) := \ddot{\gamma}(t).\hat{\mathbf{n}}(\gamma(t))$ (normal curvature)

$\ddot{\gamma}(t) - (\ddot{\gamma}(t).\hat{\mathbf{n}}(\gamma(t)))\hat{\mathbf{n}}(\gamma(t)) = \kappa_g(t)(\hat{\mathbf{n}}(\gamma(t)) \times \mathbf{T}(t))$

$\kappa_g(t)$ is the geodesic curvature

$$\kappa_g(t) = \ddot{\gamma}(t).(\hat{\mathbf{n}}(\gamma(t)) \times \mathbf{T}(t))$$

$$\ddot{\gamma}(t) = \kappa_g(t)(\hat{\mathbf{n}}(\gamma(t)) \times \mathbf{T}(t)) + (\ddot{\gamma}(t).\hat{\mathbf{n}}(\gamma(t)))\hat{\mathbf{n}}(\gamma(t))$$

Rearranging

Curvature of a curve on a surface

> *Recall:*
> Consider an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$.
> $v = \alpha_1 \mathbf{e}_1 + \alpha_2 \mathbf{e}_2 + \alpha_3 \mathbf{e}_3$
> $\alpha_i = v.\mathbf{e}_i$
> If v is perpendicular to $\mathbf{e}_1$ and $\mathbf{e}_2$
> $\alpha_1 = v.\mathbf{e}_1 = 0$
> $\alpha_2 = v.\mathbf{e}_2 = 0$
> So, $v = \alpha_3 \mathbf{e}_3$

$\gamma : (\alpha, \beta) \to S \subset \mathbb{R}^3$ unit speed

$\ddot{\gamma}(t).\mathbf{T}(t) = 0$

$\sigma : U \to S$ a regular surface patch.

$\hat{\mathbf{n}}(p) := \frac{\sigma_x(p) \times \sigma_y(p)}{\|\sigma_x(p) \times \sigma_y(p)\|}$

Exercise. Prove that $\hat{\mathbf{n}}(\gamma(t))$ is perpendicular to $\mathbf{T}(t)$.

$\kappa_n(t) := \ddot{\gamma}(t).\hat{\mathbf{n}}(\gamma(t))$ (normal curvature)

$$\ddot{\gamma}(t) - (\ddot{\gamma}(t).\hat{\mathbf{n}}(\gamma(t)))\hat{\mathbf{n}}(\gamma(t)) = \kappa_g(t)(\hat{\mathbf{n}}(\gamma(t)) \times \mathbf{T}(t))$$

$\kappa_g(t)$ is the geodesic curvature

$$\kappa_g(t) = \ddot{\gamma}(t).(\hat{\mathbf{n}}(\gamma(t)) \times \mathbf{T}(t))$$

$$\begin{aligned}\ddot{\gamma}(t) &= \kappa_g(t)(\hat{\mathbf{n}}(\gamma(t)) \times \mathbf{T}(t)) + (\ddot{\gamma}(t).\hat{\mathbf{n}}(\gamma(t)))\hat{\mathbf{n}}(\gamma(t)) \\ &= \kappa_g(t)(\hat{\mathbf{n}}(\gamma(t)) \times \mathbf{T}(t)) + \kappa_n(t)\hat{\mathbf{n}}(\gamma(t))\end{aligned}$$

Plugging in the formula for normal curvature as defined above

Curvature of a curve on a surface

> *Recall:*
> Consider an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$.
> $v = \alpha_1 \mathbf{e}_1 + \alpha_2 \mathbf{e}_2 + \alpha_3 \mathbf{e}_3$
> $\alpha_i = v.\mathbf{e}_i$
> If v is perpendicular to $\mathbf{e}_1$ and $\mathbf{e}_2$
> $\alpha_1 = v.\mathbf{e}_1 = 0$
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> So, $v = \alpha_3 \mathbf{e}_3$

$\gamma : (\alpha, \beta) \to S \subset \mathbb{R}^3$ unit speed
$\ddot{\gamma}(t).\mathbf{T}(t) = 0$

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Exercise. Prove that $\hat{\mathbf{n}}(\gamma(t))$ is perpendicular to $\mathbf{T}(t)$.

$\kappa_n(t) := \ddot{\gamma}(t).\hat{\mathbf{n}}(\gamma(t))$ (normal curvature)

$$\ddot{\gamma}(t) - (\ddot{\gamma}(t).\hat{\mathbf{n}}(\gamma(t)))\hat{\mathbf{n}}(\gamma(t)) = \kappa_g(t)(\hat{\mathbf{n}}(\gamma(t)) \times \mathbf{T}(t))$$

$\kappa_g(t)$ is the geodesic curvature

$$\kappa_g(t) = \ddot{\gamma}(t).(\hat{\mathbf{n}}(\gamma(t)) \times \mathbf{T}(t))$$

$$\begin{aligned}\ddot{\gamma}(t) &= \kappa_g(t)(\hat{\mathbf{n}}(\gamma(t)) \times \mathbf{T}(t)) + (\ddot{\gamma}(t).\hat{\mathbf{n}}(\gamma(t)))\hat{\mathbf{n}}(\gamma(t)) \\ &= \kappa_g(t)(\hat{\mathbf{n}}(\gamma(t)) \times \mathbf{T}(t)) + \kappa_n(t)\hat{\mathbf{n}}(\gamma(t))\end{aligned}$$

$$\|\ddot{\gamma}(t)\|^2 = \kappa_g^2(t) + \kappa_n^2(t)$$

Taking dot product with itself (on both sides)

Curvature of a curve on a surface

> *Recall:*
> Consider an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$.
> $v = \alpha_1\mathbf{e}_1 + \alpha_2\mathbf{e}_2 + \alpha_3\mathbf{e}_3$
> $\alpha_i = v.\mathbf{e}_i$
> If v is perpendicular to $\mathbf{e}_1$ and $\mathbf{e}_2$
> $\alpha_1 = v.\mathbf{e}_1 = 0$
> $\alpha_2 = v.\mathbf{e}_2 = 0$
> So, $v = \alpha_3\mathbf{e}_3$

$\gamma : (\alpha, \beta) \to S \subset \mathbb{R}^3$ unit speed
$\ddot{\gamma}(t).\mathbf{T}(t) = 0$

$\sigma : U \to S$ a regular surface patch.
$\hat{\mathbf{n}}(p) := \frac{\sigma_x(p)\times\sigma_y(p)}{\|\sigma_x(p)\times\sigma_y(p)\|}$

Exercise. Prove that $\hat{\mathbf{n}}(\gamma(t))$ is perpendicular to $\mathbf{T}(t)$.

$\kappa_n(t) := \ddot{\gamma}(t).\hat{\mathbf{n}}(\gamma(t))$ (normal curvature)

$\ddot{\gamma}(t) - (\ddot{\gamma}(t).\hat{\mathbf{n}}(\gamma(t)))\hat{\mathbf{n}}(\gamma(t)) = \kappa_g(t)(\hat{\mathbf{n}}(\gamma(t)) \times \mathbf{T}(t))$

$\kappa_g(t)$ is the geodesic curvature

$$\kappa_g(t) = \ddot{\gamma}(t).(\hat{\mathbf{n}}(\gamma(t)) \times \mathbf{T}(t))$$

$$\begin{aligned}\ddot{\gamma}(t) &= \kappa_g(t)(\hat{\mathbf{n}}(\gamma(t)) \times \mathbf{T}(t)) + (\ddot{\gamma}(t).\hat{\mathbf{n}}(\gamma(t)))\hat{\mathbf{n}}(\gamma(t)) \\ &= \kappa_g(t)(\hat{\mathbf{n}}(\gamma(t)) \times \mathbf{T}(t)) + \kappa_n(t)\hat{\mathbf{n}}(\gamma(t))\end{aligned}$$

$$\begin{aligned}\|\ddot{\gamma}(t)\|^2 &= \kappa_g^2(t) + \kappa_n^2(t) \\ \kappa^2(t) &= \kappa_g^2(t) + \kappa_n^2(t)\end{aligned}$$

But the left hand side is the square of the usual curvature!

Curvature of a curve on a surface

> *Recall:*
> Consider an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$.
> $v = \alpha_1\mathbf{e}_1 + \alpha_2\mathbf{e}_2 + \alpha_3\mathbf{e}_3$
> $\alpha_i = v.\mathbf{e}_i$
> If v is perpendicular to $\mathbf{e}_1$ and $\mathbf{e}_2$
> $\alpha_1 = v.\mathbf{e}_1 = 0$
> $\alpha_2 = v.\mathbf{e}_2 = 0$
> So, $v = \alpha_3\mathbf{e}_3$

$\gamma : (\alpha, \beta) \to S \subset \mathbb{R}^3$ unit speed
$\ddot{\gamma}(t).\mathbf{T}(t) = 0$

$\sigma : U \to S$ a regular surface patch.
$\hat{\mathbf{n}}(p) := \frac{\sigma_x(p)\times\sigma_y(p)}{\|\sigma_x(p)\times\sigma_y(p)\|}$

Exercise. Prove that $\hat{\mathbf{n}}(\gamma(t))$ is perpendicular to $\mathbf{T}(t)$.

$\kappa_n(t) := \ddot{\gamma}(t).\hat{\mathbf{n}}(\gamma(t))$ (normal curvature)

$\ddot{\gamma}(t) - (\ddot{\gamma}(t).\hat{\mathbf{n}}(\gamma(t)))\hat{\mathbf{n}}(\gamma(t)) = \kappa_g(t)(\hat{\mathbf{n}}(\gamma(t)) \times \mathbf{T}(t))$

$\kappa_g(t)$ is the geodesic curvature

$$\kappa_g(t) = \ddot{\gamma}(t).(\hat{\mathbf{n}}(\gamma(t)) \times \mathbf{T}(t))$$

$$\begin{aligned}\ddot{\gamma}(t) &= \kappa_g(t)(\hat{\mathbf{n}}(\gamma(t)) \times \mathbf{T}(t)) + (\ddot{\gamma}(t).\hat{\mathbf{n}}(\gamma(t)))\hat{\mathbf{n}}(\gamma(t))\\ &= \kappa_g(t)(\hat{\mathbf{n}}(\gamma(t)) \times \mathbf{T}(t)) + \kappa_n(t)\hat{\mathbf{n}}(\gamma(t))\end{aligned}$$

$$\begin{aligned}\|\ddot{\gamma}(t)\|^2 &= \kappa_g^2(t) + \kappa_n^2(t)\\ \kappa^2(t) &= \kappa_g^2(t) + \kappa_n^2(t)\end{aligned}$$

Remember that when we viewed the curve simply as a space curve,

Curvature of a curve on a surface

Recall:
Consider an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$.
$v = \alpha_1\mathbf{e}_1 + \alpha_2\mathbf{e}_2 + \alpha_3\mathbf{e}_3$
$\alpha_i = v.\mathbf{e}_i$
If v is perpendicular to $\mathbf{e}_1$ and $\mathbf{e}_2$
$\alpha_1 = v.\mathbf{e}_1 = 0$
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So, $v = \alpha_3\mathbf{e}_3$

$\gamma : (\alpha, \beta) \to S \subset \mathbb{R}^3$ unit speed
$\ddot{\gamma}(t).\mathbf{T}(t) = 0$

$\sigma : U \to S$ a regular surface patch.
$\hat{\mathbf{n}}(p) := \frac{\sigma_x(p)\times\sigma_y(p)}{\|\sigma_x(p)\times\sigma_y(p)\|}$

Exercise. Prove that $\hat{\mathbf{n}}(\gamma(t))$ is perpendicular to $\mathbf{T}(t)$.

$\mathbf{T}(t)$, $\mathbf{N}(t)$, and $B(t) := \mathbf{T}(t) \times \mathbf{N}(t)$ proved to be a useful basis

$\kappa_n(t) := \ddot{\gamma}(t).\hat{\mathbf{n}}(\gamma(t))$ (normal curvature)

$$\ddot{\gamma}(t) - (\ddot{\gamma}(t).\hat{\mathbf{n}}(\gamma(t)))\hat{\mathbf{n}}(\gamma(t)) = \kappa_g(t)(\hat{\mathbf{n}}(\gamma(t)) \times \mathbf{T}(t))$$

$\kappa_g(t)$ is the geodesic curvature

$$\kappa_g(t) = \ddot{\gamma}(t).(\hat{\mathbf{n}}(\gamma(t)) \times \mathbf{T}(t))$$

$$\begin{aligned}\ddot{\gamma}(t) &= \kappa_g(t)(\hat{\mathbf{n}}(\gamma(t)) \times \mathbf{T}(t)) + (\ddot{\gamma}(t).\hat{\mathbf{n}}(\gamma(t)))\hat{\mathbf{n}}(\gamma(t)) \\ &= \kappa_g(t)(\hat{\mathbf{n}}(\gamma(t)) \times \mathbf{T}(t)) + \kappa_n(t)\hat{\mathbf{n}}(\gamma(t))\end{aligned}$$

$$\begin{aligned}\|\ddot{\gamma}(t)\|^2 &= \kappa_g^2(t) + \kappa_n^2(t) \\ \kappa^2(t) &= \kappa_g^2(t) + \kappa_n^2(t)\end{aligned}$$

Curvature of a curve on a surface

Recall:
Consider an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$.
$v = \alpha_1\mathbf{e}_1 + \alpha_2\mathbf{e}_2 + \alpha_3\mathbf{e}_3$
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If v is perpendicular to $\mathbf{e}_1$ and $\mathbf{e}_2$
$\alpha_1 = v.\mathbf{e}_1 = 0$
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So, $v = \alpha_3\mathbf{e}_3$

$\gamma : (\alpha, \beta) \to S \subset \mathbb{R}^3$ unit speed
$\ddot{\gamma}(t).\mathbf{T}(t) = 0$

$\sigma : U \to S$ a regular surface patch.
$\hat{\mathbf{n}}(p) := \frac{\sigma_x(p)\times\sigma_y(p)}{\|\sigma_x(p)\times\sigma_y(p)\|}$

Exercise. Prove that $\hat{\mathbf{n}}(\gamma(t))$ is perpendicular to $\mathbf{T}(t)$.

$\kappa_n(t) := \ddot{\gamma}(t).\hat{\mathbf{n}}(\gamma(t))$ (normal curvature)

$\ddot{\gamma}(t) - (\ddot{\gamma}(t).\hat{\mathbf{n}}(\gamma(t)))\hat{\mathbf{n}}(\gamma(t)) = \kappa_g(t)(\hat{\mathbf{n}}(\gamma(t)) \times \mathbf{T}(t))$

$\kappa_g(t)$ is the geodesic curvature

$$\kappa_g(t) = \ddot{\gamma}(t).(\hat{\mathbf{n}}(\gamma(t)) \times \mathbf{T}(t))$$

$$\begin{aligned}\ddot{\gamma}(t) &= \kappa_g(t)(\hat{\mathbf{n}}(\gamma(t)) \times \mathbf{T}(t)) + (\ddot{\gamma}(t).\hat{\mathbf{n}}(\gamma(t)))\hat{\mathbf{n}}(\gamma(t)) \\ &= \kappa_g(t)(\hat{\mathbf{n}}(\gamma(t)) \times \mathbf{T}(t)) + \kappa_n(t)\hat{\mathbf{n}}(\gamma(t))\end{aligned}$$

$$\begin{aligned}\|\ddot{\gamma}(t)\|^2 &= \kappa_g^2(t) + \kappa_n^2(t) \\ \kappa^2(t) &= \kappa_g^2(t) + \kappa_n^2(t)\end{aligned}$$

Now, viewing the curve as a curve on the surface

Curvature of a curve on a surface

> *Recall:*
> Consider an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$.
> $v = \alpha_1\mathbf{e}_1 + \alpha_2\mathbf{e}_2 + \alpha_3\mathbf{e}_3$
> $\alpha_i = v.\mathbf{e}_i$
> If v is perpendicular to $\mathbf{e}_1$ and $\mathbf{e}_2$
> $\alpha_1 = v.\mathbf{e}_1 = 0$
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> So, $v = \alpha_3\mathbf{e}_3$

$\gamma : (\alpha, \beta) \to S \subset \mathbb{R}^3$ unit speed
$\ddot{\gamma}(t).\mathbf{T}(t) = 0$

$\sigma : U \to S$ a regular surface patch.
$\hat{\mathbf{n}}(p) := \frac{\sigma_x(p)\times\sigma_y(p)}{\|\sigma_x(p)\times\sigma_y(p)\|}$

Exercise. Prove that $\hat{\mathbf{n}}(\gamma(t))$ is perpendicular to $\mathbf{T}(t)$.

$\kappa_n(t) := \ddot{\gamma}(t).\hat{\mathbf{n}}(\gamma(t))$ (normal curvature)

$$\ddot{\gamma}(t) - (\ddot{\gamma}(t).\hat{\mathbf{n}}(\gamma(t)))\hat{\mathbf{n}}(\gamma(t)) = \kappa_g(t)(\hat{\mathbf{n}}(\gamma(t)) \times \mathbf{T}(t))$$

$\kappa_g(t)$ is the geodesic curvature

$$\kappa_g(t) = \ddot{\gamma}(t).(\hat{\mathbf{n}}(\gamma(t)) \times \mathbf{T}(t))$$

$$\begin{aligned}\ddot{\gamma}(t) &= \kappa_g(t)(\hat{\mathbf{n}}(\gamma(t)) \times \mathbf{T}(t)) + (\ddot{\gamma}(t).\hat{\mathbf{n}}(\gamma(t)))\hat{\mathbf{n}}(\gamma(t)) \\ &= \kappa_g(t)(\hat{\mathbf{n}}(\gamma(t)) \times \mathbf{T}(t)) + \kappa_n(t)\hat{\mathbf{n}}(\gamma(t))\end{aligned}$$

$$\begin{aligned}\|\ddot{\gamma}(t)\|^2 &= \kappa_g^2(t) + \kappa_n^2(t) \\ \kappa^2(t) &= \kappa_g^2(t) + \kappa_n^2(t)\end{aligned}$$

$\mathbf{T}(t)$, $\hat{\mathbf{n}}(t)$, and $\mathbf{T}(t) \times \hat{\mathbf{n}}(t)$ proves to be a useful basis

Curvature of a curve on a surface

> *Recall:*
> Consider an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$.
> $v = \alpha_1\mathbf{e}_1 + \alpha_2\mathbf{e}_2 + \alpha_3\mathbf{e}_3$
> $\alpha_i = v.\mathbf{e}_i$
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> $\alpha_1 = v.\mathbf{e}_1 = 0$
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> So, $v = \alpha_3\mathbf{e}_3$

$\gamma : (\alpha, \beta) \to S \subset \mathbb{R}^3$ unit speed
$\ddot{\gamma}(t).\mathbf{T}(t) = 0$

$\sigma : U \to S$ a regular surface patch.
$\hat{\mathbf{n}}(p) := \frac{\sigma_x(p)\times\sigma_y(p)}{\|\sigma_x(p)\times\sigma_y(p)\|}$

Exercise. Prove that $\hat{\mathbf{n}}(\gamma(t))$ is perpendicular to $\mathbf{T}(t)$.

$\kappa_n(t) := \ddot{\gamma}(t).\hat{\mathbf{n}}(\gamma(t))$ (normal curvature)

$\ddot{\gamma}(t) - (\ddot{\gamma}(t).\hat{\mathbf{n}}(\gamma(t)))\hat{\mathbf{n}}(\gamma(t)) = \kappa_g(t)(\hat{\mathbf{n}}(\gamma(t)) \times \mathbf{T}(t))$

$\kappa_g(t)$ is the geodesic curvature

$$\kappa_g(t) = \ddot{\gamma}(t).(\hat{\mathbf{n}}(\gamma(t)) \times \mathbf{T}(t))$$

$$\begin{aligned}\ddot{\gamma}(t) &= \kappa_g(t)(\hat{\mathbf{n}}(\gamma(t)) \times \mathbf{T}(t)) + (\ddot{\gamma}(t).\hat{\mathbf{n}}(\gamma(t)))\hat{\mathbf{n}}(\gamma(t)) \\ &= \kappa_g(t)(\hat{\mathbf{n}}(\gamma(t)) \times \mathbf{T}(t)) + \kappa_n(t)\hat{\mathbf{n}}(\gamma(t))\end{aligned}$$

$$\begin{aligned}\|\ddot{\gamma}(t)\|^2 &= \kappa_g^2(t) + \kappa_n^2(t) \\ \kappa^2(t) &= \kappa_g^2(t) + \kappa_n^2(t)\end{aligned}$$

Exercise. Prove that $\mathbf{N}(t) = \hat{\mathbf{n}}(t)$ if and only if $\kappa_g(t) = 0$

However, $\mathbf{N}$ and $\hat{\mathbf{n}}$ need not coincide, as this simple exercise shows!

Curvature of a curve on a surface

Recall:
Consider an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$.
$v = \alpha_1\mathbf{e}_1 + \alpha_2\mathbf{e}_2 + \alpha_3\mathbf{e}_3$
$\alpha_i = v.\mathbf{e}_i$
If v is perpendicular to $\mathbf{e}_1$ and $\mathbf{e}_2$
$\alpha_1 = v.\mathbf{e}_1 = 0$
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So, $v = \alpha_3\mathbf{e}_3$

$\gamma : (\alpha, \beta) \to S \subset \mathbb{R}^3$ unit speed
$\ddot{\gamma}(t).\mathbf{T}(t) = 0$

$\sigma : U \to S$ a regular surface patch.
$\hat{\mathbf{n}}(p) := \frac{\sigma_x(p)\times\sigma_y(p)}{\|\sigma_x(p)\times\sigma_y(p)\|}$

Exercise. Prove that $\hat{\mathbf{n}}(\gamma(t))$ is perpendicular to $\mathbf{T}(t)$.

$\kappa_n(t) := \ddot{\gamma}(t).\hat{\mathbf{n}}(\gamma(t))$ (normal curvature)

$$\ddot{\gamma}(t) - (\ddot{\gamma}(t).\hat{\mathbf{n}}(\gamma(t)))\hat{\mathbf{n}}(\gamma(t)) = \kappa_g(t)(\hat{\mathbf{n}}(\gamma(t)) \times \mathbf{T}(t))$$

$\kappa_g(t)$ is the geodesic curvature

$$\kappa_g(t) = \ddot{\gamma}(t).(\hat{\mathbf{n}}(\gamma(t)) \times \mathbf{T}(t))$$

$$\begin{aligned}\ddot{\gamma}(t) &= \kappa_g(t)(\hat{\mathbf{n}}(\gamma(t)) \times \mathbf{T}(t)) + (\ddot{\gamma}(t).\hat{\mathbf{n}}(\gamma(t)))\hat{\mathbf{n}}(\gamma(t))\\ &= \kappa_g(t)(\hat{\mathbf{n}}(\gamma(t)) \times \mathbf{T}(t)) + \kappa_n(t)\hat{\mathbf{n}}(\gamma(t))\end{aligned}$$

$$\begin{aligned}\|\ddot{\gamma}(t)\|^2 &= \kappa_g^2(t) + \kappa_n^2(t)\\ \kappa^2(t) &= \kappa_g^2(t) + \kappa_n^2(t)\end{aligned}$$

Exercise. Prove that $\mathbf{N}(t) = \hat{\mathbf{n}}(t)$ if and only if $\kappa_g(t) = 0$

Definition. A parametrization γ of a curve on a surface is called a geodesic if $\kappa_g(t) = 0$ for all t.

Such curves are called geodesics

First fundamental form

$\gamma : (\alpha, \beta) \to S$

We will study the arc length of a curve on a surface

First fundamental form

$\gamma : (\alpha, \beta) \to S \subset \mathbb{R}^3$

Again, since the surface is in $\mathbb{R}^3$ the curve is also in $\mathbb{R}^3$

First fundamental form

$\gamma : (\alpha, \beta) \to S \subset \mathbb{R}^3$
$\sigma : U \to S$ a regular surface patch.

We will now write everything in terms of a surface patch

First fundamental form

$\gamma : (\alpha, \beta) \to S \subset \mathbb{R}^3$
$\sigma : U \to S$ a regular surface patch.
$\gamma(t) = \sigma(x(t), y(t))$

Remember that $x(t)$ and $y(t)$ are the coordinates provided by σ

First fundamental form

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$\sigma : U \to S$ a regular surface patch.
$\gamma(t) = \sigma(x(t), y(t))$

$$\dot{\gamma}(t) = x'(t)\sigma_x(x(t), y(t)) + y'(t)\sigma_y(x(t), y(t))$$

As we have seen, chain rule expresses the velocity vector in terms of σ_x and σ_y

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$$\dot{\gamma}(t) = x'(t)\sigma_x(x(t), y(t)) + y'(t)\sigma_y(x(t), y(t))$$

$$\begin{aligned}\dot{\gamma}(t).\dot{\gamma}(t) &= x'^2(t)(\sigma_x(x(t), y(t)).\sigma_x(x(t), y(t)) \\ &+ 2x(t)'y'(t)(\sigma_x(x(t), y(t)).\sigma_y(x(t), y(t)) \\ &+ y'^2(t)(\sigma_y(x(t), y(t)).\sigma_y(x(t), y(t))\end{aligned}$$

So we can do the same for the dot with itself, to know its norm

.

First fundamental form

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$$\begin{aligned}\dot{\gamma}(t).\dot{\gamma}(t) &= x'^2(t)(\sigma_x(x(t), y(t)).\sigma_x(x(t), y(t)) \\ &+ 2x(t)'y'(t)(\sigma_x(x(t), y(t)).\sigma_y(x(t), y(t)) \\ &+ y'^2(t)(\sigma_y(x(t), y(t)).\sigma_y(x(t), y(t))\end{aligned}$$

Checking this formula should be a straightforward exercise

.

First fundamental form

$\gamma : (\alpha, \beta) \to S \subset \mathbb{R}^3$
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$\gamma(t) = \sigma(x(t), y(t))$

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$$\begin{aligned}\dot{\gamma}(t).\dot{\gamma}(t) &= x'^2(t)(\sigma_x(x(t), y(t)).\sigma_x(x(t), y(t)) \\ &+ 2x(t)'y'(t)(\sigma_x(x(t), y(t)).\sigma_y(x(t), y(t)) \\ &+ y'^2(t)(\sigma_y(x(t), y(t)).\sigma_y(x(t), y(t))\end{aligned}$$

$$\begin{aligned}\dot{\gamma}(t).\dot{\gamma}(t) &= x'^2(t)E(x(t), y(t)) \\ &+ 2x'(t)y'(t)F(x(t), y(t)) \\ &+ y'^2(t)G(x(t), y(t))\end{aligned}$$

where,
$E(x, y) := \sigma_x(x, y).\sigma_x(x, y)$
$F(x, y) := \sigma_x(x, y).\sigma_y(x, y)$
$G(x, y) := \sigma_y(x, y).\sigma_y(x, y)$

Let us abstract out the terms that refer to only the patch

.

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$$\dot{\gamma}(t) = x'(t)\sigma_x(x(t), y(t)) + y'(t)\sigma_y(x(t), y(t))$$

$$\begin{aligned}\dot{\gamma}(t).\dot{\gamma}(t) &= x'^2(t)(\sigma_x(x(t), y(t)).\sigma_x(x(t), y(t)) \\ &+ 2x(t)'y'(t)(\sigma_x(x(t), y(t)).\sigma_y(x(t), y(t)) \\ &+ y'^2(t)(\sigma_y(x(t), y(t)).\sigma_y(x(t), y(t))\end{aligned}$$

$$\begin{aligned}\dot{\gamma}(t).\dot{\gamma}(t) &= x'^2(t)E(x(t), y(t)) \\ &+ 2x'(t)y'(t)F(x(t), y(t)) \\ &+ y'^2(t)G(x(t), y(t))\end{aligned}$$

where,
$E(x, y) := \sigma_x(x, y).\sigma_x(x, y)$
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Calling them E, F, and G

.

First fundamental form

$\gamma : (\alpha, \beta) \to S \subset \mathbb{R}^3$
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$$\dot{\gamma}(t) = x'(t)\sigma_x(x(t), y(t)) + y'(t)\sigma_y(x(t), y(t))$$

$$\begin{aligned}\dot{\gamma}(t).\dot{\gamma}(t) &= x'^2(t)(\sigma_x(x(t), y(t)).\sigma_x(x(t), y(t)) \\ &+ 2x(t)'y'(t)(\sigma_x(x(t), y(t)).\sigma_y(x(t), y(t)) \\ &+ y'^2(t)(\sigma_y(x(t), y(t)).\sigma_y(x(t), y(t))\end{aligned}$$

$$\begin{aligned}\dot{\gamma}(t).\dot{\gamma}(t) &= x'^2(t)E(x(t), y(t)) \\ &+ 2x'(t)y'(t)F(x(t), y(t)) \\ &+ y'^2(t)G(x(t), y(t))\end{aligned}$$

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$G(x, y) := \sigma_y(x, y).\sigma_y(x, y)$

Note that E, F, and G are functions with domain U (i.e. domain of σ)

First fundamental form

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$\gamma(t) = \sigma(x(t), y(t))$

$$\dot{\gamma}(t) = x'(t)\sigma_x(x(t), y(t)) + y'(t)\sigma_y(x(t), y(t))$$

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$$\begin{aligned}\dot{\gamma}(t).\dot{\gamma}(t) &= x'^2(t)E(x(t), y(t)) \\ &+ 2x'(t)y'(t)F(x(t), y(t)) \\ &+ y'^2(t)G(x(t), y(t))\end{aligned}$$

where,
$E(x, y) := \sigma_x(x, y).\sigma_x(x, y)$
$F(x, y) := \sigma_x(x, y).\sigma_y(x, y)$
$G(x, y) := \sigma_y(x, y).\sigma_y(x, y)$

Observe that E, F, and G do not depend on the curve

First fundamental form

$\gamma : (\alpha, \beta) \to S \subset \mathbb{R}^3$
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$\gamma(t) = \sigma(x(t), y(t))$

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$$\begin{aligned}\dot{\gamma}(t).\dot{\gamma}(t) &= x'^2(t)(\sigma_x(x(t), y(t)).\sigma_x(x(t), y(t)) \\ &+ 2x(t)'y'(t)(\sigma_x(x(t), y(t)).\sigma_y(x(t), y(t)) \\ &+ y'^2(t)(\sigma_y(x(t), y(t)).\sigma_y(x(t), y(t))\end{aligned}$$

$$\begin{aligned}\dot{\gamma}(t).\dot{\gamma}(t) &= x'^2(t)E(x(t), y(t)) \\ &+ 2x'(t)y'(t)F(x(t), y(t)) \\ &+ y'^2(t)G(x(t), y(t))\end{aligned}$$

where,
$E(x, y) := \sigma_x(x, y).\sigma_x(x, y)$
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$G(x, y) := \sigma_y(x, y).\sigma_y(x, y)$

They may be computed for a surface patch

First fundamental form

$\gamma : (\alpha, \beta) \to S \subset \mathbb{R}^3$
$\sigma : U \to S$ a regular surface patch.
$\gamma(t) = \sigma(x(t), y(t))$

$$\dot{\gamma}(t) = x'(t)\sigma_x(x(t), y(t)) + y'(t)\sigma_y(x(t), y(t))$$

$$\begin{aligned}\dot{\gamma}(t).\dot{\gamma}(t) &= x'^2(t)(\sigma_x(x(t), y(t)).\sigma_x(x(t), y(t)) \\ &+ 2x(t)'y'(t)(\sigma_x(x(t), y(t)).\sigma_y(x(t), y(t)) \\ &+ y'^2(t)(\sigma_y(x(t), y(t)).\sigma_y(x(t), y(t))\end{aligned}$$

$$\begin{aligned}\dot{\gamma}(t).\dot{\gamma}(t) &= x'^2(t)E(x(t), y(t)) \\ &+ 2x'(t)y'(t)F(x(t), y(t)) \\ &+ y'^2(t)G(x(t), y(t))\end{aligned}$$

where,
$E(x, y) := \sigma_x(x, y).\sigma_x(x, y)$
$F(x, y) := \sigma_x(x, y).\sigma_y(x, y)$
$G(x, y) := \sigma_y(x, y).\sigma_y(x, y)$

and used for any curve we may want to study on that patch

First fundamental form

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$\sigma : U \to S$ a regular surface patch.
$\gamma(t) = \sigma(x(t), y(t))$

$$\dot{\gamma}(t) = x'(t)\sigma_x(x(t), y(t)) + y'(t)\sigma_y(x(t), y(t))$$

$$\begin{aligned}\dot{\gamma}(t).\dot{\gamma}(t) &= x'^2(t)(\sigma_x(x(t), y(t)).\sigma_x(x(t), y(t)) \\ &+ 2x(t)'y'(t)(\sigma_x(x(t), y(t)).\sigma_y(x(t), y(t)) \\ &+ y'^2(t)(\sigma_y(x(t), y(t)).\sigma_y(x(t), y(t))\end{aligned}$$

$$\begin{aligned}\dot{\gamma}(t).\dot{\gamma}(t) &= x'^2(t)E(x(t), y(t)) \\ &+ 2x'(t)y'(t)F(x(t), y(t)) \\ &+ y'^2(t)G(x(t), y(t))\end{aligned}$$

where,
$E(x, y) := \sigma_x(x, y).\sigma_x(x, y)$
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$G(x, y) := \sigma_y(x, y).\sigma_y(x, y)$

just like we computed σ_x, σ_y and used it for the velocity of any curve

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$$\begin{aligned}\dot{\gamma}(t).\dot{\gamma}(t) &= x'^2(t)E(x(t), y(t)) \\ &+ 2x'(t)y'(t)F(x(t), y(t)) \\ &+ y'^2(t)G(x(t), y(t))\end{aligned}$$

where,
$E(x, y) := \sigma_x(x, y).\sigma_x(x, y)$
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$G(x, y) := \sigma_y(x, y).\sigma_y(x, y)$

So the norm can be also be written in terms of E, F, and G,

First fundamental form

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$$\dot{\gamma}(t) = x'(t)\sigma_x(x(t), y(t)) + y'(t)\sigma_y(x(t), y(t))$$

$$\begin{aligned}\dot{\gamma}(t).\dot{\gamma}(t) &= x'^2(t)(\sigma_x(x(t), y(t)).\sigma_x(x(t), y(t)) \\ &+ 2x(t)'y'(t)(\sigma_x(x(t), y(t)).\sigma_y(x(t), y(t)) \\ &+ y'^2(t)(\sigma_y(x(t), y(t)).\sigma_y(x(t), y(t))\end{aligned}$$

$$\begin{aligned}\dot{\gamma}(t).\dot{\gamma}(t) &= x'^2(t)E(x(t), y(t)) \\ &+ 2x'(t)y'(t)F(x(t), y(t)) \\ &+ y'^2(t)G(x(t), y(t)) \\ \|\dot{\gamma}(t).\dot{\gamma}(t)\| &= \sqrt{x'^2(t)E + 2x'(t)y'(t)F + y'^2(t)G}\end{aligned}$$

$\blacksquare$

First fundamental form

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$$\dot{\gamma}(t) = x'(t)\sigma_x(x(t), y(t)) + y'(t)\sigma_y(x(t), y(t))$$

$$\begin{aligned}\dot{\gamma}(t).\dot{\gamma}(t) &= x'^2(t)(\sigma_x(x(t), y(t)).\sigma_x(x(t), y(t))\\ &+ 2x(t)'y'(t)(\sigma_x(x(t), y(t)).\sigma_y(x(t), y(t))\\ &+ y'^2(t)(\sigma_y(x(t), y(t)).\sigma_y(x(t), y(t))\end{aligned}$$

$$s(t) = \int_{t_0}^{t} \|\dot{\gamma}(t)\| \mathrm{d}t$$

$$\begin{aligned}\dot{\gamma}(t).\dot{\gamma}(t) &= x'^2(t)E(x(t), y(t))\\ &+ 2x'(t)y'(t)F(x(t), y(t))\\ &+ y'^2(t)G(x(t), y(t))\\ \|\dot{\gamma}(t).\dot{\gamma}(t)\| &= \sqrt{x'^2(t)E + 2x'(t)y'(t)F + y'^2(t)G}\end{aligned}$$

and, therefore, the arc length

First fundamental form

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where,
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$$\dot{\gamma}(t) = x'(t)\sigma_x(x(t), y(t)) + y'(t)\sigma_y(x(t), y(t))$$

$$\begin{aligned}\dot{\gamma}(t).\dot{\gamma}(t) &= x'^2(t)(\sigma_x(x(t), y(t)).\sigma_x(x(t), y(t)) \\ &+ 2x(t)'y'(t)(\sigma_x(x(t), y(t)).\sigma_y(x(t), y(t)) \\ &+ y'^2(t)(\sigma_y(x(t), y(t)).\sigma_y(x(t), y(t))\end{aligned}$$

$$\begin{aligned}s(t) &= \int_{t_0}^{t} \|\dot{\gamma}(t)\| \mathrm{d}t \\ &= \int_{t_0}^{t} \sqrt{x'^2(t)E + 2x'(t)y'(t)F + y'^2(t)G} \mathrm{d}t\end{aligned}$$

$$\begin{aligned}\dot{\gamma}(t).\dot{\gamma}(t) &= x'^2(t)E(x(t), y(t)) \\ &+ 2x'(t)y'(t)F(x(t), y(t)) \\ &+ y'^2(t)G(x(t), y(t)) \\ \|\dot{\gamma}(t).\dot{\gamma}(t)\| &= \sqrt{x'^2(t)E + 2x'(t)y'(t)F + y'^2(t)G}\end{aligned}$$

can also be expressed in terms of E, F, and G

First fundamental form

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$\sigma : U \to S$ a regular surface patch.
$\gamma(t) = \sigma(x(t), y(t))$

where,

$$E(x,y) := \sigma_x(x,y).\sigma_x(x,y)$$
$$F(x,y) := \sigma_x(x,y).\sigma_y(x,y)$$
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$$\dot{\gamma}(t) = x'(t)\sigma_x(x(t), y(t)) + y'(t)\sigma_y(x(t), y(t))$$

$$\begin{aligned}\dot{\gamma}(t).\dot{\gamma}(t) &= x'^2(t)(\sigma_x(x(t), y(t)).\sigma_x(x(t), y(t)) \\ &+ 2x(t)'y'(t)(\sigma_x(x(t), y(t)).\sigma_y(x(t), y(t)) \\ &+ y'^2(t)(\sigma_y(x(t), y(t)).\sigma_y(x(t), y(t))\end{aligned}$$

$$\begin{aligned}\dot{\gamma}(t).\dot{\gamma}(t) &= x'^2(t)E(x(t), y(t)) \\ &+ 2x'(t)y'(t)F(x(t), y(t)) \\ &+ y'^2(t)G(x(t), y(t)) \\ \|\dot{\gamma}(t).\dot{\gamma}(t)\| &= \sqrt{x'^2(t)E + 2x'(t)y'(t)F + y'^2(t)G}\end{aligned}$$

$$\begin{aligned}s(t) &= \int_{t_0}^{t} \|\dot{\gamma}(t)\| \mathrm{d}t \\ &= \int_{t_0}^{t} \sqrt{x'^2(t)E + 2x'(t)y'(t)F + y'^2(t)G}\mathrm{d}t\end{aligned}$$

To summarize, we compute E, F, and G for each point of the surface patch and keep it aside

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$E(x,y) := \sigma_x(x,y).\sigma_x(x,y)$
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$$\dot\gamma(t) = x'(t)\sigma_x(x(t),y(t)) + y'(t)\sigma_y(x(t),y(t))$$

$$\begin{aligned}\dot\gamma(t).\dot\gamma(t) &= x'^2(t)(\sigma_x(x(t),y(t)).\sigma_x(x(t),y(t)) \\ &+ 2x(t)'y'(t)(\sigma_x(x(t),y(t)).\sigma_y(x(t),y(t)) \\ &+ y'^2(t)(\sigma_y(x(t),y(t)).\sigma_y(x(t),y(t))\end{aligned}$$

$$\begin{aligned}s(t) &= \int_{t_0}^{t} \|\dot\gamma(t)\| \mathrm{d}t \\ &= \int_{t_0}^{t} \sqrt{x'^2(t){\color{red}E} + 2x'(t)y'(t){\color{red}F} + y'^2(t){\color{red}G}}\mathrm{d}t\end{aligned}$$

$$\begin{aligned}\dot\gamma(t).\dot\gamma(t) &= x'^2(t)E(x(t),y(t)) \\ &+ 2x'(t)y'(t)F(x(t),y(t)) \\ &+ y'^2(t)G(x(t),y(t)) \\ \|\dot\gamma(t).\dot\gamma(t)\| &= \sqrt{x'^2(t)E + 2x'(t)y'(t)F + y'^2(t)G}\end{aligned}$$

Given a curve, we take express it in terms of the surface patch

.

First fundamental form

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$$\dot\gamma(t) = x'(t)\sigma_x(x(t), y(t)) + y'(t)\sigma_y(x(t), y(t))$$

$$\begin{aligned}\dot\gamma(t).\dot\gamma(t) &= x'^2(t)(\sigma_x(x(t), y(t)).\sigma_x(x(t), y(t)) \\ &+ 2x(t)'y'(t)(\sigma_x(x(t), y(t)).\sigma_y(x(t), y(t)) \\ &+ y'^2(t)(\sigma_y(x(t), y(t)).\sigma_y(x(t), y(t))\end{aligned}$$

$$\begin{aligned}s(t) &= \int_{t_0}^{t} \|\dot\gamma(t)\| \mathrm{d}t \\ &= \int_{t_0}^{t} \sqrt{x'^2(t)E + 2x'(t)y'(t)F + y'^2(t)G} \mathrm{d}t\end{aligned}$$

$$\begin{aligned}\dot\gamma(t).\dot\gamma(t) &= x'^2(t)E(x(t), y(t)) \\ &+ 2x'(t)y'(t)F(x(t), y(t)) \\ &+ y'^2(t)G(x(t), y(t)) \\ \|\dot\gamma(t).\dot\gamma(t)\| &= \sqrt{x'^2(t)E + 2x'(t)y'(t)F + y'^2(t)G}\end{aligned}$$

, i.e. find out its $(x(t), y(t))$

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$E(x,y) := \sigma_x(x,y).\sigma_x(x,y)$
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$G(x,y) := \sigma_y(x,y).\sigma_y(x,y)$

$$\dot{\gamma}(t) = x'(t)\sigma_x(x(t), y(t)) + y'(t)\sigma_y(x(t), y(t))$$

$$\begin{aligned}\dot{\gamma}(t).\dot{\gamma}(t) &= x'^2(t)(\sigma_x(x(t), y(t)).\sigma_x(x(t), y(t))\\ &+ 2x(t)'y'(t)(\sigma_x(x(t), y(t)).\sigma_y(x(t), y(t))\\ &+ y'^2(t)(\sigma_y(x(t), y(t)).\sigma_y(x(t), y(t))\end{aligned}$$

$$\begin{aligned}s(t) &= \int_{t_0}^{t} \|\dot{\gamma}(t)\| \mathrm{d}t\\ &= \int_{t_0}^{t} \sqrt{x'^2(t)E + 2x'(t)y'(t)F + y'^2(t)G}\mathrm{d}t\end{aligned}$$

$$\begin{aligned}\dot{\gamma}(t).\dot{\gamma}(t) &= x'^2(t)E(x(t), y(t))\\ &+ 2x'(t)y'(t)F(x(t), y(t))\\ &+ y'^2(t)G(x(t), y(t))\\ \|\dot{\gamma}(t).\dot{\gamma}(t)\| &= \sqrt{x'^2(t)E + 2x'(t)y'(t)F + y'^2(t)G}\end{aligned}$$

use that to find out its $x'(t)$ and $y'(t)$ and plug it into the above formula.

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where,
$E(x,y) := \sigma_x(x,y).\sigma_x(x,y)$
$F(x,y) := \sigma_x(x,y).\sigma_y(x,y)$
$G(x,y) := \sigma_y(x,y).\sigma_y(x,y)$

$$\dot{\gamma}(t) = x'(t)\sigma_x(x(t),y(t)) + y'(t)\sigma_y(x(t),y(t))$$

$$\begin{aligned}\dot{\gamma}(t).\dot{\gamma}(t) &= x'^2(t)(\sigma_x(x(t),y(t)).\sigma_x(x(t),y(t)) \\ &+ 2x(t)'y'(t)(\sigma_x(x(t),y(t)).\sigma_y(x(t),y(t)) \\ &+ y'^2(t)(\sigma_y(x(t),y(t)).\sigma_y(x(t),y(t))\end{aligned}$$

$$\begin{aligned}s(t) &= \int_{t_0}^{t} \|\dot{\gamma}(t)\| \mathrm{d}t \\ &= \int_{t_0}^{t} \sqrt{x'^2(t)E + 2x'(t)y'(t)F + y'^2(t)G}\,\mathrm{d}t\end{aligned}$$

$$\begin{aligned}\dot{\gamma}(t).\dot{\gamma}(t) &= x'^2(t)E(x(t),y(t)) \\ &+ 2x'(t)y'(t)F(x(t),y(t)) \\ &+ y'^2(t)G(x(t),y(t)) \\ \|\dot{\gamma}(t).\dot{\gamma}(t)\| &= \sqrt{x'^2(t)E + 2x'(t)y'(t)F + y'^2(t)G}\end{aligned}$$

Observe,

$$\dot{\gamma}(t).\dot{\gamma}(t) = \begin{pmatrix} x'(t) & y'(t) \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix}$$

Later, it will prove useful to know that E, F, and G can be arranged in a matrix

.