Exercise sheet 6

Curves and Surfaces, MTH201

1. If $\mathbf{v}_1(t), \mathbf{v}_2(t)$, and $\mathbf{v}_3(t)$, unit vector fields which are *not necessarily* $\mathbf{T}(t), \mathbf{N}(t), \mathbf{B}(t)$, but nevertheless satisfy the same equations:

$$\begin{aligned}
\dot{\mathbf{v}}_1 &= \kappa(t)\mathbf{v}_2(t) \\
\dot{\mathbf{v}}_2 &= -\kappa(t)\mathbf{v}_1(t) + \tau(t)\mathbf{v}_3(t) \\
\dot{\mathbf{v}}_3 &= -\tau(t)\mathbf{v}_2(t)
\end{aligned}$$

   This exercise will show that if $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are equal to $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$, respectively, for one t_0, then they are equal for all t, but in a slightly less straightforward manner than you may expect to work out.

   (a) Show that $f(t) = \mathbf{v}_1(t).\mathbf{T}(t) + \mathbf{v}_2(t).\mathbf{N}(t) + \mathbf{v}_3(t).\mathbf{B}(t)$ is constant. Note that we are not saying that the individual terms themselves are constant, so this exercise is not saying, for instance, that the the dot products, $\mathbf{v}_1(t).\mathbf{T}(t)$, $\mathbf{v}_2(t).\mathbf{N}(t)$, or $\mathbf{v}_3(t).\mathbf{B}(t)$, are individually constant; it is only when we take their sum that we can ensure the resulting sum is constant. Luckily, this will be sufficient to prove the next part.

   (b) Show that if $\mathbf{v}_1(t_0) = \mathbf{T}(t_0)$, $\mathbf{v}_2(t_0) = \mathbf{N}(t_0)$, or $\mathbf{v}_3(t_0) = \mathbf{B}(t_0)$, for some t_0, then the equalities hold for all t. (*Hint:* What is an equivalent condition, in terms of dot products, for two vectors being equal? Remember, these are unit vectors! Note that the previous part was only useful because we had equality of the corresponding vectors at t_0.)

2. For any curve parametrized by a unit speed parametrization $\gamma : (\alpha, \beta) \to \mathbb{R}^3$, let $\delta(t) = \gamma(t) + \mathbf{v}$ for some vector $\mathbf{v}$ (i.e. each point of the curve is being translated by the same $\mathbf{v}$). Prove that,

   (a) δ is also unit speed

   (b) The curvature, torsion, unit tangent, normal, and binormal of δ is the same as the curvature and torsion of γ at t.

3. Consider a $3 \times 3-$ matrix A with the property that $A\mathbf{v}.A\mathbf{w} = \mathbf{v}.\mathbf{w}$ for any two vectors, $\mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ (recall, that such matrices are called orthogonal; if you transform two vectors by the same orthgonal transformation, the dot product will remain the same as before the transformation, so the norm of). For any curve parametrized by a unit speed parametrization

$\gamma : (\alpha, \beta) \to \mathbb{R}^3$, let $\delta(t) = A\gamma(t)$ (i.e. each point of the curve is being transformed by the same matrix A). Prove that,

(a) δ is also unit speed

(b) The curvature and torsion of δ is the same as the curvature and torsion of γ at t.

(c) The curvature and torsion of δ is the same as the curvature and torsion of γ at t.

(d) If $\mathbf{T}_1(t)$, $\mathbf{N}_1(t)$ and $\mathbf{B}_1(t)$ denote the unit tangent, unit normal, and unit binormal of γ at t, while $\mathbf{T}_2(t)$, $\mathbf{N}_2(t)$ and $\mathbf{B}_2(t)$ denote the unit tangent, unit normal of δ at t, then prove that,

$$\mathbf{T}_2(t) = A(\mathbf{T}_1(t))$$

$$\mathbf{N}_2(t) = A(\mathbf{N}_1(t))$$

$$\mathbf{B}_2(t) = A(\mathbf{B}_1(t))$$

4. If $\sigma : U \to \mathbb{R}^3$ is a surface patch, and $p \in \mathbb{R}^3$ lies on its image, then prove that any linear combination of the vectors $\sigma_x(p)$ and $\sigma_y(p)$ is a tangent vector, i.e. show that it is the velocity vector of some curve on the surface.

5. Prove that if a surface is regular, then the space of tangent vectors is 2-dimensional.

6. Let $f(x, y) = x\sin(x + 2y)$. Find f_x, f_y, f_{xy}, and f_{yx}.

7. Use the chain rule of partial derivatives to show that if $f : \mathbb{R} \to \mathbb{R}$ is a smooth function such that $y = f(x)$ satisfies $F(x, y) = 0$, where $F : \mathbb{R}^2 \to \mathbb{R}$ is also a smooth function, then $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{F_x}{F_y}$. This is called implicit differentiation.

8. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function and let (x_0, y_0) be a point in $\mathbb{R}^2$ and $\mathbf{v} = (v_1, v_2)$ a vector. Define $F = f(x_0 + v_1 t, y_0 + v_2 t)$. Use the chain rule for partial derivatives to show that $F'(0) = f_x(x_0, y_0)v_1 + f_y(x_0, y_0)v_2 = (f_x(x_0, y_0), f_y(x_0, y_0))\cdot\mathbf{v}$. Note that $F'(0) = \lim_{t\to 0} \dfrac{f(x_0 + tv_1, y_0 + tv_2) - f(x_0, y_0)}{t}$; it is called the directional derivative of f in the direction of $\mathbf{v}$ and is denoted by $f_{\mathbf{v}}$. This exercise shows that one can compute the directional derivative of f in the direction of any vector $\mathbf{v}$ if one knows the partial derivatives of f.

9. Which of the following surface patches are regular?

(a) $\sigma : \mathbb{R}^2 \to \mathbb{R}^3$, $\sigma(x, y) = (x, y, x + y)$

(b) $\sigma : \mathbb{R}^2 \to \mathbb{R}^3$, $\sigma(x, y) = (x, x^2, y^3)$

10. For an open subset U of $\mathbb{R}^2$, and a smooth map $f : U \to \mathbb{R}^3$, show that around any point in the set $S := \{(x, y, f(x, y)) \mid (x, y) \in \mathbb{R}^2\}$, we can find a smooth and regular surface patch.