Chain rule for mult-variable functions

f

Chain rule for mult-variable functions

$f : \mathbb{R}^2$

Chain rule for mult-variable functions

$f : \mathbb{R}^2 \to \mathbb{R}$

Chain rule for mult-variable functions

$f : \mathbb{R}^2 \to \mathbb{R}$
γ

Chain rule for mult-variable functions

$f : \mathbb{R}^2 \to \mathbb{R}$
$\gamma : (\alpha, \beta)$

Chain rule for mult-variable functions

$f : \mathbb{R}^2 \to \mathbb{R}$
$\gamma : (\alpha, \beta) \to \mathbb{R}^2$

Chain rule for mult-variable functions

$f : \mathbb{R}^2 \to \mathbb{R}$
$\gamma : (\alpha, \beta) \to \mathbb{R}^2$
$\gamma(t)$

Chain rule for mult-variable functions

$f : \mathbb{R}^2 \to \mathbb{R}$
$\gamma : (\alpha, \beta) \to \mathbb{R}^2$
$\gamma(t) = (x(t), y(t))$

Chain rule for mult-variable functions

$f : \mathbb{R}^2 \to \mathbb{R}$
$\gamma : (\alpha, \beta) \to \mathbb{R}^2$
$\gamma(t) = (x(t), y(t))$
$f \circ \gamma$

Chain rule for mult-variable functions

$f : \mathbb{R}^2 \to \mathbb{R}$
$\gamma : (\alpha, \beta) \to \mathbb{R}^2$
$\gamma(t) = (x(t), y(t))$
$f \circ \gamma : (\alpha, \beta)$

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$f \circ \gamma : (\alpha, \beta) \to \mathbb{R}$

Chain rule for mult-variable functions

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$f \circ \gamma : (\alpha, \beta) \to \mathbb{R}$

$(f \circ \gamma)'(t_0)$

Chain rule for mult-variable functions

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$\gamma : (\alpha, \beta) \to \mathbb{R}^2$
$\gamma(t) = (x(t), y(t))$
$f \circ \gamma : (\alpha, \beta) \to \mathbb{R}$

$$(f \circ \gamma)'(t_0) = f_x(x(t_0), y(t_0))x'(t_0) + f_y(x(t_0), y(t_0))y'(t_0)$$

Chain rule for mult-variable functions

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$(f \circ \gamma)'(t_0) = f_x(x(t_0), y(t_0))x'(t_0) + f_y(x(t_0), y(t_0))y'(t_0) = (f_x(x, y), f_y(x, y)).\dot{\gamma}(t_0)$

Chain rule for mult-variable functions

$f : \mathbb{R}^2 \to \mathbb{R}$
$\gamma : (\alpha, \beta) \to \mathbb{R}^2$
$\gamma(t) = (x(t), y(t))$
$f \circ \gamma : (\alpha, \beta) \to \mathbb{R}$

$(f \circ \gamma)'(t_0) = f_x(x(t_0), y(t_0))x'(t_0) + f_y(x(t_0), y(t_0))y'(t_0) = \nabla(f)(x(t_0), y(t_0)).\dot{\gamma}(t_0)$,
where $\nabla(f)(x, y) = (f_x(x, y).f_y(x, y))$,

Chain rule for mult-variable functions

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$\boxed{f_{\mathbf{v}}(x(t_0), y(t_0)) := (f \circ \gamma)'(t_0) = \nabla(f)(x(t_0), y(t_0)).\mathbf{v}}$,
where $\nabla(f)(x, y) = (f_x(x, y).f_y(x, y))$,
$\mathbf{v} = \dot{\gamma}(t_0)$,

Chain rule for mult-variable functions

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$f \circ \gamma : (\alpha, \beta) \to \mathbb{R}$

$\boxed{f_{\mathbf{v}}(x(t_0), y(t_0)) := (f \circ \gamma)'(t_0) = \nabla(f)(p).\mathbf{v}}$,
where $\nabla(f)(x, y) = (f_x(x, y).f_y(x, y))$,
$\mathbf{v} = \dot{\gamma}(t_0)$,
and $p = (x(t_0), y(t_0))$

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Chain rule for mult-variable functions

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$f : \mathbb{R}^2$

Chain rule for mult-variable functions

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$f : \mathbb{R}^2 \to \mathbb{R}$

Chain rule for mult-variable functions

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$f : \mathbb{R}^2 \to \mathbb{R}$
γ

Chain rule for mult-variable functions

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$f : \mathbb{R}^2 \to \mathbb{R}$
$\gamma : \mathbb{R}^2$

Chain rule for mult-variable functions

$f : \mathbb{R}^2 \to \mathbb{R}$
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$f : \mathbb{R}^2 \to \mathbb{R}$
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Chain rule for mult-variable functions

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$f : \mathbb{R}^2 \to \mathbb{R}$
$\gamma : \mathbb{R}^2 \to \mathbb{R}^2$
$\gamma(u, v)$

Chain rule for mult-variable functions

$f : \mathbb{R}^2 \to \mathbb{R}$
$\gamma : (\alpha, \beta) \to \mathbb{R}^2$
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$\gamma(u, v) = (x(u, v), y(u, v))$

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$f : \mathbb{R}^2 \to \mathbb{R}$
$\gamma : \mathbb{R}^2 \to \mathbb{R}^2$
$\gamma(u, v) = (x(u, v), y(u, v))$
$f \circ \gamma : \mathbb{R}^2$

Chain rule for mult-variable functions

$f : \mathbb{R}^2 \to \mathbb{R}$
$\gamma : (\alpha, \beta) \to \mathbb{R}^2$
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$f : \mathbb{R}^2 \to \mathbb{R}$
$\gamma : \mathbb{R}^2 \to \mathbb{R}^2$
$\gamma(u, v) = (x(u, v), y(u, v))$
$f \circ \gamma : \mathbb{R}^2 \to \mathbb{R}$

Chain rule for mult-variable functions

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$f : \mathbb{R}^2 \to \mathbb{R}$
$\gamma : \mathbb{R}^2 \to \mathbb{R}^2$
$\gamma(u, v) = (x(u, v), y(u, v))$
$f \circ \gamma : \mathbb{R}^2 \to \mathbb{R}$

$(f \circ \gamma)_u(u_0, v_0)$

Chain rule for mult-variable functions

$f : \mathbb{R}^2 \to \mathbb{R}$
$\gamma : (\alpha, \beta) \to \mathbb{R}^2$
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$\gamma : \mathbb{R}^2 \to \mathbb{R}^2$
$\gamma(u, v) = (x(u, v), y(u, v))$
$f \circ \gamma : \mathbb{R}^2 \to \mathbb{R}$

$(f \circ \gamma)_u(u_0, v_0)$
$= f_x(x(u_0, v_0), y(u_0, v_0))x_u(u_0, v_0)$
$+f_y(x(u_0, v_0), y(u_0, v_0))y_u(u_0, v_0)$

Curves on surfaces

Definition. $\mathbf{v} \in \mathbb{R}^3$ is a tangent vector of the surface S at a point p, if there is a γ

Chain rule for mult-variable functions

$f : \mathbb{R}^2 \to \mathbb{R}$
$\gamma : (\alpha, \beta) \to \mathbb{R}^2$
$\gamma(t) = (x(t), y(t))$
$f \circ \gamma : (\alpha, \beta) \to \mathbb{R}$

$\boxed{f_{\mathbf{v}}(x(t_0), y(t_0)) := (f \circ \gamma)'(t_0) = \nabla(f)(p).\mathbf{v}}$,
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and $p = (x(t_0), y(t_0))$

$f : \mathbb{R}^2 \to \mathbb{R}$
$\gamma : \mathbb{R}^2 \to \mathbb{R}^2$
$\gamma(u, v) = (x(u, v), y(u, v))$
$f \circ \gamma : \mathbb{R}^2 \to \mathbb{R}$

$(f \circ \gamma)_u(u_0, v_0)$
$= f_x(x(u_0, v_0), y(u_0, v_0))x_u(u_0, v_0)$
$+f_y(x(u_0, v_0), y(u_0, v_0))y_u(u_0, v_0)$

Curves on surfaces

Definition. $\mathbf{v} \in \mathbb{R}^3$ is a tangent vector of the surface S at a point p, if there is a $\gamma : (\alpha, \beta)$

Chain rule for mult-variable functions

$f : \mathbb{R}^2 \to \mathbb{R}$
$\gamma : (\alpha, \beta) \to \mathbb{R}^2$
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$\boxed{f_{\mathbf{v}}(x(t_0), y(t_0)) := (f \circ \gamma)'(t_0) = \nabla(f)(p).\mathbf{v}}$,
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$f : \mathbb{R}^2 \to \mathbb{R}$
$\gamma : \mathbb{R}^2 \to \mathbb{R}^2$
$\gamma(u, v) = (x(u, v), y(u, v))$
$f \circ \gamma : \mathbb{R}^2 \to \mathbb{R}$

$(f \circ \gamma)_u(u_0, v_0)$
$= f_x(x(u_0, v_0), y(u_0, v_0))x_u(u_0, v_0)$
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Curves on surfaces

Definition. $\mathbf{v} \in \mathbb{R}^3$ is a tangent vector of the surface S at a point p, if there is a $\gamma : (\alpha, \beta) \to S$

Chain rule for mult-variable functions

$f : \mathbb{R}^2 \to \mathbb{R}$
$\gamma : (\alpha, \beta) \to \mathbb{R}^2$
$\gamma(t) = (x(t), y(t))$
$f \circ \gamma : (\alpha, \beta) \to \mathbb{R}$

$\boxed{f_{\mathbf{v}}(x(t_0), y(t_0)) := (f \circ \gamma)'(t_0) = \nabla(f)(p).\mathbf{v}}$,
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and $p = (x(t_0), y(t_0))$

$f : \mathbb{R}^2 \to \mathbb{R}$
$\gamma : \mathbb{R}^2 \to \mathbb{R}^2$
$\gamma(u, v) = (x(u, v), y(u, v))$
$f \circ \gamma : \mathbb{R}^2 \to \mathbb{R}$

$(f \circ \gamma)_u(u_0, v_0)$
$= f_x(x(u_0, v_0), y(u_0, v_0))x_u(u_0, v_0)$
$+f_y(x(u_0, v_0), y(u_0, v_0))y_u(u_0, v_0)$

Curves on surfaces

Definition. $\mathbf{v} \in \mathbb{R}^3$ is a tangent vector of the surface S at a point p, if there is a $\gamma : (\alpha, \beta) \to S \subset \mathbb{R}^3$ so that $p = \gamma(t)$ and

Chain rule for mult-variable functions

$f : \mathbb{R}^2 \to \mathbb{R}$
$\gamma : (\alpha, \beta) \to \mathbb{R}^2$
$\gamma(t) = (x(t), y(t))$
$f \circ \gamma : (\alpha, \beta) \to \mathbb{R}$

$\boxed{f_{\mathbf{v}}(x(t_0), y(t_0)) := (f \circ \gamma)'(t_0) = \nabla(f)(p).\mathbf{v}}$,
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$\mathbf{v} = \dot{\gamma}(t_0)$,
and $p = (x(t_0), y(t_0))$

$f : \mathbb{R}^2 \to \mathbb{R}$
$\gamma : \mathbb{R}^2 \to \mathbb{R}^2$
$\gamma(u, v) = (x(u, v), y(u, v))$
$f \circ \gamma : \mathbb{R}^2 \to \mathbb{R}$

$(f \circ \gamma)_u(u_0, v_0)$
$= f_x(x(u_0, v_0), y(u_0, v_0))x_u(u_0, v_0)$
$+f_y(x(u_0, v_0), y(u_0, v_0))y_u(u_0, v_0)$

Curves on surfaces

Definition. $\mathbf{v} \in \mathbb{R}^3$ is a tangent vector of the surface S at a point p, if there is a $\gamma : (\alpha, \beta) \to S \subset \mathbb{R}^3$ so that $p = \gamma(t)$ and $\mathbf{v} = \dot{\gamma}(t)$

$\gamma : (\alpha, \beta) \to S \subset \mathbb{R}^3$ is a curve.

Chain rule for mult-variable functions

$f : \mathbb{R}^2 \to \mathbb{R}$
$\gamma : (\alpha, \beta) \to \mathbb{R}^2$
$\gamma(t) = (x(t), y(t))$
$f \circ \gamma : (\alpha, \beta) \to \mathbb{R}$

$\boxed{f_{\mathbf{v}}(x(t_0), y(t_0)) := (f \circ \gamma)'(t_0) = \nabla(f)(p).\mathbf{v}}$,
where $\nabla(f)(x, y) = (f_x(x, y).f_y(x, y))$,
$\mathbf{v} = \dot{\gamma}(t_0)$,
and $p = (x(t_0), y(t_0))$

$f : \mathbb{R}^2 \to \mathbb{R}$
$\gamma : \mathbb{R}^2 \to \mathbb{R}^2$
$\gamma(u, v) = (x(u, v), y(u, v))$
$f \circ \gamma : \mathbb{R}^2 \to \mathbb{R}$

$(f \circ \gamma)_u(u_0, v_0)$
$= f_x(x(u_0, v_0), y(u_0, v_0))x_u(u_0, v_0)$
$+ f_y(x(u_0, v_0), y(u_0, v_0))y_u(u_0, v_0)$

Curves on surfaces

Definition. $\mathbf{v} \in \mathbb{R}^3$ is a tangent vector of the surface S at a point p, if there is a $\gamma : (\alpha, \beta) \to S \subset \mathbb{R}^3$ so that $p = \gamma(t)$ and $\mathbf{v} = \dot{\gamma}(t)$

$\gamma : (\alpha, \beta) \to S \subset \mathbb{R}^3$ is a curve.
$\sigma : U \to S$ a surface patch.

Chain rule for mult-variable functions

$f : \mathbb{R}^2 \to \mathbb{R}$
$\gamma : (\alpha, \beta) \to \mathbb{R}^2$
$\gamma(t) = (x(t), y(t))$
$f \circ \gamma : (\alpha, \beta) \to \mathbb{R}$

$\boxed{f_{\mathbf{v}}(x(t_0), y(t_0)) := (f \circ \gamma)'(t_0) = \nabla(f)(p).\mathbf{v}}$,
where $\nabla(f)(x, y) = (f_x(x, y).f_y(x, y))$,
$\mathbf{v} = \dot{\gamma}(t_0)$,
and $p = (x(t_0), y(t_0))$

$f : \mathbb{R}^2 \to \mathbb{R}$
$\gamma : \mathbb{R}^2 \to \mathbb{R}^2$
$\gamma(u, v) = (x(u, v), y(u, v))$
$f \circ \gamma : \mathbb{R}^2 \to \mathbb{R}$

$(f \circ \gamma)_u(u_0, v_0)$
$= f_x(x(u_0, v_0), y(u_0, v_0))x_u(u_0, v_0)$
$+ f_y(x(u_0, v_0), y(u_0, v_0))y_u(u_0, v_0)$

Curves on surfaces

Definition. $\mathbf{v} \in \mathbb{R}^3$ is a tangent vector of the surface S at a point p, if there is a $\gamma : (\alpha, \beta) \to S \subset \mathbb{R}^3$ so that $p = \gamma(t)$ and $\mathbf{v} = \dot{\gamma}(t)$

$\gamma : (\alpha, \beta) \to S \subset \mathbb{R}^3$ is a curve.
$\sigma : U \to S$ a surface patch.
So, $\gamma(t) = \sigma(x(t), y(t))$

Chain rule for mult-variable functions

$f : \mathbb{R}^2 \to \mathbb{R}$
$\gamma : (\alpha, \beta) \to \mathbb{R}^2$
$\gamma(t) = (x(t), y(t))$
$f \circ \gamma : (\alpha, \beta) \to \mathbb{R}$

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Exercise. Show that σ is regular at p if and only if the tangent vectors at p form a two dimensional subspace of $\mathbb{R}^3$.

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Given $f(x, y, z) = t$, can we "solve for" z in terms of x and y?
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Equivalently, can we find,

Applications of the inverse function theorem

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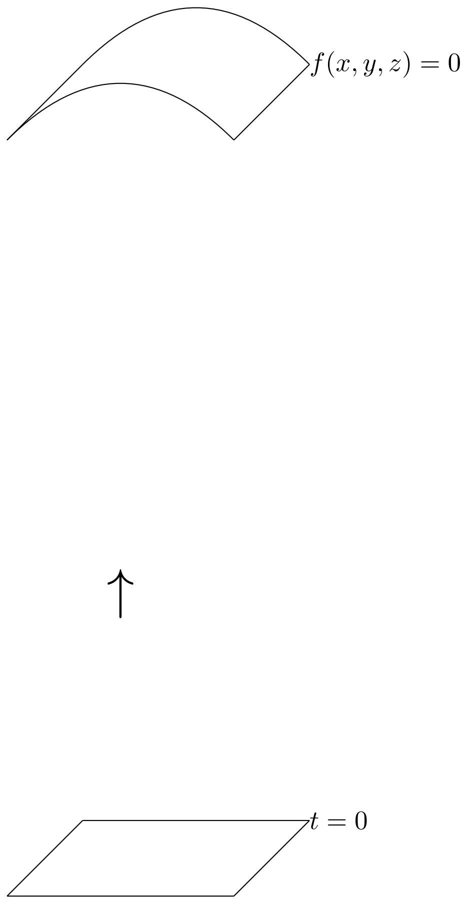
Equivalently, can we find,
$F(x, y, t) := (x, y, \theta(x, y))$ so that $f(x, y, \theta(x, y)) = t$

Applications of the inverse function theorem



Given $f(x, y, z) = 0$, can we "solve for" z in terms of x and y?
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Making it stronger works!

Given $f(x, y, z) = t$, can we "solve for" z in terms of x and y?
Can we write $z = \theta(x, y)$ so that $f(x, y, \theta(x, y)) = t$?

Equivalently, can we find,
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Applications of the inverse function theorem

Given $f(x,y,z)=0$, can we "solve for" z in terms of x and y?
Can we write $z=\theta(x,y)$ so that $f(x,y,\theta(x,y))=0$?

Making it stronger works!

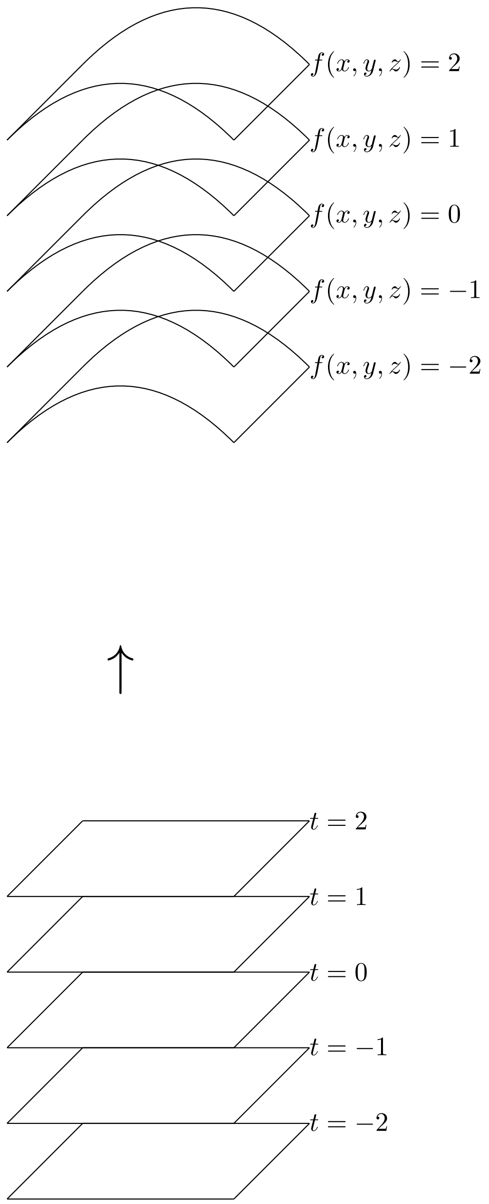
Given $f(x,y,z)=t$, can we "solve for" z in terms of x and y?
Can we write $z=\theta(x,y)$ so that $f(x,y,\theta(x,y))=t$?

Equivalently, can we find,
$F(x,y,t):=(x,y,\theta(x,y))$ so that $f(x,y,\theta(x,y))=t$

Equivalently, can we define an inverse of

$$G(x,y,z):=(x,y,f(x,y,z))$$

so that $G(F(x,y,z))=(x,y,z)$



$F_x = (1, 0, f_x(x, y, z))$
$F_y = (0, 1, f_y(x, y, z))$
$F_z = (0, 0, f_z(x, y, z))$
linearly independent if and only if $f_z(x, y, z) \neq 0$

Can "solve for z" if and only if $f_z(x, y, z) \neq 0$

$F_x = (1, 0, f_x(x, y, z))$
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Can "solve for z" if and only if $f_z(x, y, z) \neq 0$
Can "solve for y" if and only if $f_y(x, y, z) \neq 0$

$F_x = (1, 0, f_x(x, y, z))$
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linearly independent if and only if $f_z(x, y, z) \neq 0$

Can "solve for z" if and only if $f_z(x, y, z) \neq 0$
Can "solve for y" if and only if $f_y(x, y, z) \neq 0$
Can "solve for x" if and only if $f_x(x, y, z) \neq 0$

$F_x = (1, 0, f_x(x, y, z))$
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Can "solve for z" if and only if $f_z(x, y, z) \neq 0$
Can "solve for y" if and only if $f_y(x, y, z) \neq 0$
Can "solve for x" if and only if $f_x(x, y, z) \neq 0$

Can solve for either x, y, or z if and only if $\nabla(f)(x, y, z) \neq 0$

Understanding the cross products

Understanding the cross products

$\mathbf{T}(t) \times \mathbf{N}(t) = \mathbf{B}(t)$

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$\mathbf{T}(t) \times \mathbf{N}(t) = \mathbf{B}(t)$
$\mathbf{B}(t) \times \mathbf{T}(t) = \mathbf{N}(t)$

$$\mathbf{T}(t) = (\cos(\theta(t)), \sin(\theta(t)))$$

$$\mathbf{T}(t) = (\cos(\theta(t)), \sin(\theta(t)))$$
$$\dot{\mathbf{T}}(t)$$

$$\mathbf{T}(t) = (\cos(\theta(t)), \sin(\theta(t)))$$
$$\dot{\mathbf{T}}(t) = \theta'(t)(-\sin(\theta(t)), \cos(\theta(t)))$$

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$$\kappa_s(t) = \theta'(t)$$

If $\kappa_s(t) = \kappa$,

$$\mathbf{T}(t) = (\cos(\theta(t)), \sin(\theta(t)))$$
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If $\kappa_s(t) = \kappa$,

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If $\kappa_s(t) = \kappa$,
$$\theta'(t) = \kappa$$
$$\theta(t) = \kappa t$$

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$$\kappa_s(t) = \theta'(t)$$

If $\kappa_s(t) = \kappa$,
$$\theta'(t) = \kappa$$
$$\theta(t) = \kappa t + \theta_0$$

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$$\gamma'(t) = \mathbf{T}(t) = (\cos(at + \theta_0), \sin(at + \theta_0))$$

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$$\gamma(t) = (\tfrac{1}{\kappa}\sin(\kappa t + \theta_0) + p_1, -\tfrac{1}{\kappa}\cos(\kappa t + \theta_0) + p_2)$$

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$$\gamma(t) = (\tfrac{1}{\kappa}\sin(\kappa t + \theta_0), -\tfrac{1}{\kappa}\cos(\kappa t + \theta_0)) + p$$

$$\mathbf{T}(t) = (\cos(\theta(t)), \sin(\theta(t)))$$
$$\dot{\mathbf{T}}(t) = \theta'(t)(-\sin(\theta(t)), \cos(\theta(t))) = \theta'(t)\mathbf{N}(t)$$
$$\kappa_s(t) = \theta'(t)$$

If $\kappa_s(t) = \kappa$,
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$$\gamma(t) = (\tfrac{1}{\kappa}\sin(\kappa t + \theta_0), -\tfrac{1}{\kappa}\cos(\kappa t + \theta_0)) + p$$
where $p = (p_1, p_2)$

Exercise. If $\gamma : (\alpha, \beta) \to \mathbb{R}^2$ is a unit speed parametrization with constant curvature 0, then $\gamma(t)$ lies on a line for all t.

Solution. $\ddot{\gamma}(t) = 0$ $\square$

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Solution. $\ddot{\gamma}(t) = 0$
$\dot{\gamma}(t) = \mathbf{v}$ for some $\mathbf{v} \in \mathbb{R}$
$\gamma(t) = t\mathbf{v} + p$ $\square$

$$\mathbf{N}(t).\mathbf{T}(t) = 0$$

$$\mathbf{N}(t).\mathbf{T}(t) = 0$$
$$(\mathbf{N}(t).\mathbf{T}(t))' = 0$$

$$\mathbf{N}(t).\mathbf{T}(t) = 0$$
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$$\dot{\mathbf{N}}(t).\mathbf{T}(t) + \mathbf{N}(t).\dot{\mathbf{T}}(t) = 0$$

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$$\dot{\mathbf{N}}(t).\mathbf{T}(t) + \mathbf{N}(t).\dot{\mathbf{T}}(t) = 0$$
$$\dot{\mathbf{N}}(t).\mathbf{T}(t) + \kappa_s(t)\mathbf{N}(t).\dot{\mathbf{N}}(t) = 0$$

$$
\begin{aligned}
&\mathbf{N}(t).\mathbf{T}(t) = 0\\
&(\mathbf{N}(t).\mathbf{T}(t))' = 0\\
&\dot{\mathbf{N}}(t).\mathbf{T}(t) + \mathbf{N}(t).\dot{\mathbf{T}}(t) = 0\\
&\dot{\mathbf{N}}(t).\mathbf{T}(t) + \kappa_s(t)\mathbf{N}(t).\dot{\mathbf{N}}(t) = 0\\
&\dot{\mathbf{N}}(t).\mathbf{T}(t) + \kappa_s(t) = 0
\end{aligned}
$$

$$
\begin{aligned}
&\mathbf{N}(t).\mathbf{T}(t) = 0 \\
&(\mathbf{N}(t).\mathbf{T}(t))' = 0 \\
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&\dot{\mathbf{N}}(t).\mathbf{T}(t) + \kappa_s(t)\mathbf{N}(t).\dot{\mathbf{N}}(t) = 0 \\
&\dot{\mathbf{N}}(t).\mathbf{T}(t) + \kappa_s(t) = 0 \\
&\dot{\mathbf{N}}(t).\mathbf{T}(t) = -\kappa_s(t)
\end{aligned}
$$

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$$\dot{\mathbf{N}}(t).\mathbf{T}(t) + \mathbf{N}(t).\dot{\mathbf{T}}(t) = 0$$
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$$\dot{\mathbf{N}}(t).\mathbf{T}(t) = -\kappa_s(t)$$

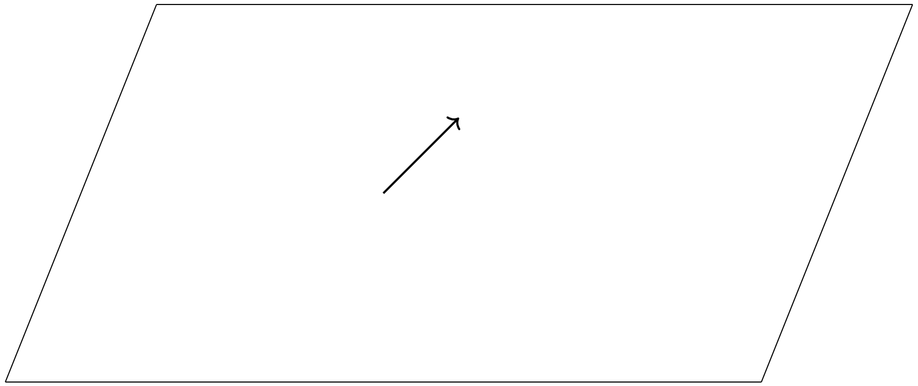
$$\mathbf{N}(t).\mathbf{N}(t) = 0$$
$$\dot{\mathbf{N}}(t).\mathbf{N}(t) = 0$$

So,

$\mathbf{N}(t).\mathbf{T}(t) = 0$
$(\mathbf{N}(t).\mathbf{T}(t))' = 0$
$\dot{\mathbf{N}}(t).\mathbf{T}(t) + \mathbf{N}(t).\dot{\mathbf{T}}(t) = 0$
$\dot{\mathbf{N}}(t).\mathbf{T}(t) + \kappa_s(t)\mathbf{N}(t).\dot{\mathbf{N}}(t) = 0$
$\dot{\mathbf{N}}(t).\mathbf{T}(t) + \kappa_s(t) = 0$
$\dot{\mathbf{N}}(t).\mathbf{T}(t) = -\kappa_s(t)$

$\mathbf{N}(t).\mathbf{N}(t) = 0$
$\dot{\mathbf{N}}(t).\mathbf{N}(t) = 0$

So,
$\dot{\mathbf{N}}(t) = -\kappa(t)\mathbf{T}(t)$



$$P = \{(x, y, z) \in \mathbb{R}^3 \mid ax + by + cz = d\}$$

$$
\begin{aligned}
P &= \{(x, y, z) \in \mathbb{R}^3 \mid ax + by + cz = d\} \\
&= \{(x, y, z) \in \mathbb{R}^3 \mid (a, b, c).(x, y, z) = d\}
\end{aligned}
$$

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\begin{aligned}
P &= \{(x, y, z) \in \mathbb{R}^3 \mid ax + by + cz = d\} \\
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\end{aligned}
$$

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\begin{aligned}
P &= \{(x,y,z)\in\mathbb{R}^3 \mid ax+by+cz=d\}\\
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\end{aligned}
$$

$\gamma :$

$$
\begin{aligned}
P &= \{(x, y, z) \in \mathbb{R}^3 \mid ax + by + cz = d\} \\
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\end{aligned}
$$

$$\gamma : (\alpha, \beta)$$

$$
\begin{aligned}
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&= \{(x, y, z) \in \mathbb{R}^3 \mid (a, b, c).(x, y, z) = d\} \\
&= \{\mathbf{v} \in \mathbb{R}^3 \mid \mathbf{n}.\mathbf{v} = d\}
\end{aligned}
$$

$$\gamma : (\alpha, \beta) \to \mathbb{R}^3$$

$$
\begin{aligned}
P &= \{(x,y,z) \in \mathbb{R}^3 \mid ax + by + cz = d\} \\
&= \{(x,y,z) \in \mathbb{R}^3 \mid (a,b,c).(x,y,z) = d\} \\
&= \{\mathbf{v} \in \mathbb{R}^3 \mid \mathbf{n}.\mathbf{v} = d\}
\end{aligned}
$$

$\gamma : (\alpha, \beta) \to \mathbb{R}^3$ parametrizes a curve

$$
\begin{aligned}
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&= \{\mathbf{v} \in \mathbb{R}^3 \mid \mathbf{n}.\mathbf{v} = d\}
\end{aligned}
$$

$\gamma : (\alpha, \beta) \to \mathbb{R}^3$ parametrizes a curve that lies on the plane, P,

$$\begin{aligned} P &= \{(x, y, z) \in \mathbb{R}^3 \mid ax + by + cz = d\} \\ &= \{(x, y, z) \in \mathbb{R}^3 \mid (a, b, c).(x, y, z) = d\} \\ &= \{\mathbf{v} \in \mathbb{R}^3 \mid \mathbf{n}.\mathbf{v} = d\} \end{aligned}$$

$\gamma : (\alpha, \beta) \to \mathbb{R}^3$ parametrizes a curve that lies on the plane, P, if and only if

$$\mathbf{n}.\gamma(t) = d$$

$$\begin{aligned} P &= \{(x,y,z) \in \mathbb{R}^3 \mid ax+by+cz=d\} \\ &= \{(x,y,z) \in \mathbb{R}^3 \mid (a,b,c).(x,y,z)=d\} \\ &= \{\mathbf{v} \in \mathbb{R}^3 \mid \mathbf{n}.\mathbf{v}=d\} \end{aligned}$$

$\gamma : (\alpha, \beta) \to \mathbb{R}^3$ parametrizes a curve that lies on the plane, P, if and only if

$$\mathbf{n}.\gamma(t) = d$$

for all t

$$\begin{aligned} P &= \{(x,y,z) \in \mathbb{R}^3 \mid ax + by + cz = d\} \\ &= \{(x,y,z) \in \mathbb{R}^3 \mid (a,b,c).(x,y,z) = d\} \\ &= \{\mathbf{v} \in \mathbb{R}^3 \mid \mathbf{n}.\mathbf{v} = d\} \end{aligned}$$

$\gamma : (\alpha, \beta) \to \mathbb{R}^3$ parametrizes a curve that lies on the plane, P, if and only if

$$\mathbf{n}.\gamma(t) = d$$

for all $t \in (\alpha, \beta)$.

Alternatively,

$$\begin{aligned} P &= \{(x,y,z) \in \mathbb{R}^3 \mid ax + by + cz = d\} \\ &= \{(x,y,z) \in \mathbb{R}^3 \mid (a,b,c).(x,y,z) = d\} \\ &= \{\mathbf{v} \in \mathbb{R}^3 \mid \mathbf{n}.\mathbf{v} = d\} \end{aligned}$$

$\gamma : (\alpha, \beta) \to \mathbb{R}^3$ parametrizes a curve that lies on the plane, P, if and only if

$$\mathbf{n}.\gamma(t) = d$$

for all $t \in (\alpha, \beta)$.