Exercise sheet 1

1. Find a parametrization $\gamma(t)$ for a line segment joining two given points $(x_1.y_1)$ and $(x_2.y_2)$. Find $\dot{\gamma}(t)$.

2. What does the parametrization trace out $\gamma(t) = (\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2})$?

3. Show that the parametrization $\gamma(t) := (t^2-1, t(t^2-1))$ is not injective, i.e. there are two *distinct* real numbers t_1 and t_2 so that $\gamma(t_1) = \gamma(t_2)$. Can you deduce the shape[1] of this curve? Can you express the set of points defined by this curve as the zero set[2] of some function $f(x, y)$?

4. Remember that $\mathbb{R}^2$ can be given the structure of a vector space by defining, $(x_1, y_1) + (x_2, y_2) := (x_1 + x_2, y_1 + y_2)$ (vector addition) and $c(x, y) := (cx, cy)$ (scalar multiplication) for some real number c. Let $\mathbf{v}_1 : (\alpha, \beta) \to \mathbb{R}^2$ and $\mathbf{v}_2 : (\alpha, \beta) \to \mathbb{R}^2$ be smooth "vector valued" functions.

   (a) $(\mathbf{v}_1(t) + \mathbf{v}_2(t))' = \mathbf{v}_1'(t) + \mathbf{v}_2'(t)$

   (b) $(\mathbf{v}_1(t) - \mathbf{v}_2(t))' = \mathbf{v}_1'(t) + \mathbf{v}_2'(t)$

   (c) $(\mathbf{v}_1(t)\mathbf{v}_2(t))' = \mathbf{v}_1'(t) + \mathbf{v}_2'(t)$

   (d) $(\mathbf{v}_1(t)/\mathbf{v}_2(t))' = \mathbf{v}_1'(t) + \mathbf{v}_2'(t)$

   (e) $\mathbf{v}(\phi(t))' = \mathbf{v}'(\phi(t))\phi'(t)$, where $\phi : (\alpha', \beta') \to (\alpha, \beta)$ is a smooth function.

   (f) During the lecture we defined $\mathbf{v}'(t)$, where $\mathbf{v}(t) = (f(t), g(t))$ to be $(f'(t), g'(t))$. Show that,

   $$\mathbf{v}'(t) = \lim_{h \to 0} 1/h(\mathbf{v}(t+h) - \mathbf{v}(t))$$

   Remember that the subtraction above is vector subtraction and the multiplication by $1/h$ is scalar multiplication.

[1] Just a rough drawing showing where the curve intersects the axes and where it self-intersects etc.

[2] The zero set of a function $f : \mathbb{R}^2 \to \mathbb{R}$ is $\{(x, y) \mid f(x, y) = 0\}$, i.e. the set of points (x, y) in the plane for which $f(x, y) = 0$