$\mathcal{K} = \det\, \mathcal{W}$, Gaussian curvature

We saw that the determinant of the Weingarten map depends only on the first fundamental form

$\mathcal{K} = \det \mathcal{W}$, Gaussian curvature

This is called the Gaussian curvature

$\mathcal{K} = \det\, \mathcal{W}$, Gaussian curvature

This makes sense because the determinant does not depend on the basis

$\mathcal{K} = \det \mathcal{W}$, Gaussian curvature

$\mathcal{H} = \frac{\text{trace } \mathcal{W}}{2}$, Mean Curvature

Similarly, we can define the mean curvature as the trace of the Weingarten map

$\mathcal{K} = \det \mathcal{W}$, Gaussian curvature
$\mathcal{H} = \frac{\text{trace } \mathcal{W}}{2}$, Mean Curvature

The trace also does not depend on the basis chosen

$\mathcal{K} = \det \mathcal{W}$, Gaussian curvature
$\mathcal{H} = \frac{\text{trace } \mathcal{W}}{2}$, Mean Curvature

$\mathcal{W}$ is symmetric.

Recall, from linear algebra that symmetric matrices are diagonalizable

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$\mathcal{K} = \det \mathcal{W}$, Gaussian curvature
$\mathcal{H} = \frac{\text{trace } \mathcal{W}}{2}$, Mean Curvature

$\mathcal{W}$ is symmetric.

and the eigenvalues make up the diagonal

$\mathcal{K} = \det \mathcal{W}$, Gaussian curvature
$\mathcal{H} = \frac{\text{trace } \mathcal{W}}{2}$, Mean Curvature

$\mathcal{W}$ is symmetric.
Denote its eigenvalues, κ_1 and κ_2.

We will see that these eigenvalues have a special meaning for surfaces

$\mathcal{K} = \det \mathcal{W}$, Gaussian curvature
$\mathcal{H} = \frac{\text{trace } \mathcal{W}}{2}$, Mean Curvature

$\mathcal{W}$ is symmetric.
Denote its eigenvalues, κ_1 and κ_2.
i.e. there exist $\mathbf{t}_1$ and $\mathbf{t}_2$ *unit* vectors so that
$\mathcal{W}\mathbf{t}_1 = \kappa_1\mathbf{t}_1$ and
$\mathcal{W}\mathbf{t}_2 = \kappa_2\mathbf{t}_2$

Even their eigenvectors have a special significance

$\mathcal{K} = \det \mathcal{W}$, Gaussian curvature
$\mathcal{H} = \frac{\text{trace } \mathcal{W}}{2}$, Mean Curvature

$\mathcal{W}$ is symmetric.
Denote its eigenvalues, κ_1 and κ_2.
i.e. there exist $\mathbf{t}_1$ and $\mathbf{t}_2$ *unit* vectors so that
$\mathcal{W}\mathbf{t}_1 = \kappa_1 \mathbf{t}_1$ and
$\mathcal{W}\mathbf{t}_2 = \kappa_2 \mathbf{t}_2$
Assume $\kappa_1 \neq \kappa_2$, then $\mathbf{t}_1 . \mathbf{t}_2 = 0$

If the eigenvalues are distinct, then the eigenvectors are orthogonal

$\mathcal{K} = \det \mathcal{W}$, Gaussian curvature
$\mathcal{H} = \frac{\text{trace } \mathcal{W}}{2}$, Mean Curvature

$\mathcal{W}$ is symmetric.
Denote its eigenvalues, κ_1 and κ_2.
i.e. there exist $\mathbf{t}_1$ and $\mathbf{t}_2$ *unit* vectors so that
$\mathcal{W}\mathbf{t}_1 = \kappa_1 \mathbf{t}_1$ and
$\mathcal{W}\mathbf{t}_2 = \kappa_2 \mathbf{t}_2$
Assume $\kappa_1 \neq \kappa_2$, then $\mathbf{t}_1.\mathbf{t}_2 = 0$

Writing $\mathcal{W}$ in terms of the basis $\mathbf{t}_1$ and $\mathbf{t}_2$,

$$\begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix}$$

It is with respect to this orthonormal basis, that the map is diagonal

$\mathcal{K} = \det \mathcal{W}$, Gaussian curvature
$\mathcal{H} = \frac{\text{trace } \mathcal{W}}{2}$, Mean Curvature

$\mathcal{W}$ is symmetric.
Denote its eigenvalues, κ_1 and κ_2.
i.e. there exist $\mathbf{t}_1$ and $\mathbf{t}_2$ *unit* vectors so that
$\mathcal{W}\mathbf{t}_1 = \kappa_1 \mathbf{t}_1$ and
$\mathcal{W}\mathbf{t}_2 = \kappa_2 \mathbf{t}_2$
Assume $\kappa_1 \neq \kappa_2$, then $\mathbf{t}_1.\mathbf{t}_2 = 0$

Writing $\mathcal{W}$ in terms of the basis $\mathbf{t}_1$ and $\mathbf{t}_2$,

$$\begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix}$$

Since the determinant remains unchanged by a change of basis

.

$\mathcal{K} = \det \mathcal{W}$, Gaussian curvature
$\mathcal{H} = \frac{\text{trace } \mathcal{W}}{2}$, Mean Curvature

$\mathcal{W}$ is symmetric.
Denote its eigenvalues, κ_1 and κ_2.
i.e. there exist $\mathbf{t}_1$ and $\mathbf{t}_2$ *unit* vectors so that
$\mathcal{W}\mathbf{t}_1 = \kappa_1\mathbf{t}_1$ and
$\mathcal{W}\mathbf{t}_2 = \kappa_2\mathbf{t}_2$
Assume $\kappa_1 \neq \kappa_2$, then $\mathbf{t}_1.\mathbf{t}_2 = 0$

Writing $\mathcal{W}$ in terms of the basis $\mathbf{t}_1$ and $\mathbf{t}_2$,

$$\begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix}$$

$\kappa_1\kappa_2 =$ Gaussian curvature

We can write the Gaussian curvature in terms of the eigenvalues

.

$\mathcal{K} = \det \mathcal{W}$, Gaussian curvature
$\mathcal{H} = \frac{\text{trace } \mathcal{W}}{2}$, Mean Curvature

$\mathcal{W}$ is symmetric.
Denote its eigenvalues, κ_1 and κ_2.
i.e. there exist $\mathbf{t}_1$ and $\mathbf{t}_2$ *unit* vectors so that
$\mathcal{W}\mathbf{t}_1 = \kappa_1 \mathbf{t}_1$ and
$\mathcal{W}\mathbf{t}_2 = \kappa_2 \mathbf{t}_2$
Assume $\kappa_1 \neq \kappa_2$, then $\mathbf{t}_1 . \mathbf{t}_2 = 0$

Writing $\mathcal{W}$ in terms of the basis $\mathbf{t}_1$ and $\mathbf{t}_2$,

$$\begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix}$$

$\kappa_1 \kappa_2 =$ Gaussian curvature
$\frac{\kappa_1 + \kappa_2}{2} =$ Mean curvature

And similarly for the mean curvature

$\mathcal{K} = \det \mathcal{W}$, Gaussian curvature
$\mathcal{H} = \frac{\text{trace } \mathcal{W}}{2}$, Mean Curvature

$\mathcal{W}$ is symmetric.
Denote its eigenvalues, κ_1 and κ_2.
i.e. there exist $\mathbf{t}_1$ and $\mathbf{t}_2$ *unit* vectors so that
$\mathcal{W}\mathbf{t}_1 = \kappa_1 \mathbf{t}_1$ and
$\mathcal{W}\mathbf{t}_2 = \kappa_2 \mathbf{t}_2$
Assume $\kappa_1 \neq \kappa_2$, then $\mathbf{t}_1.\mathbf{t}_2 = 0$

Writing $\mathcal{W}$ in terms of the basis $\mathbf{t}_1$ and $\mathbf{t}_2$,

$$\begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix}$$

$\kappa_1 \kappa_2 =$ Gaussian curvature
$\frac{\kappa_1 + \kappa_2}{2} =$ Mean curvature

Let us try and understand the geometric significance of the eigenvalue s

$\mathcal{K} = \det \mathcal{W}$, Gaussian curvature
$\mathcal{H} = \frac{\text{trace } \mathcal{W}}{2}$, Mean Curvature

$\mathcal{W}$ is symmetric.
Denote its eigenvalues, κ_1 and κ_2.
i.e. there exist $\mathbf{t}_1$ and $\mathbf{t}_2$ *unit* vectors so that
$\mathcal{W}\mathbf{t}_1 = \kappa_1 \mathbf{t}_1$ and
$\mathcal{W}\mathbf{t}_2 = \kappa_2 \mathbf{t}_2$
Assume $\kappa_1 \neq \kappa_2$, then $\mathbf{t}_1.\mathbf{t}_2 = 0$

Writing $\mathcal{W}$ in terms of the basis $\mathbf{t}_1$ and $\mathbf{t}_2$,

$$\begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix}$$

$\kappa_1 \kappa_2 =$ Gaussian curvature
$\frac{\kappa_1 + \kappa_2}{2} =$ Mean curvature

As usual we begin with the study of a curve on a surface

$\mathcal{K} = \det \mathcal{W}$, Gaussian curvature
$\mathcal{H} = \frac{\text{trace } \mathcal{W}}{2}$, Mean Curvature

κ_n

$\mathcal{W}$ is symmetric.
Denote its eigenvalues, κ_1 and κ_2.
i.e. there exist $\mathbf{t}_1$ and $\mathbf{t}_2$ *unit* vectors so that
$\mathcal{W}\mathbf{t}_1 = \kappa_1 \mathbf{t}_1$ and
$\mathcal{W}\mathbf{t}_2 = \kappa_2 \mathbf{t}_2$
Assume $\kappa_1 \neq \kappa_2$, then $\mathbf{t}_1.\mathbf{t}_2 = 0$

Writing $\mathcal{W}$ in terms of the basis $\mathbf{t}_1$ and $\mathbf{t}_2$,

$$\begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix}$$

$\kappa_1 \kappa_2 =$ Gaussian curvature
$\frac{\kappa_1 + \kappa_2}{2} =$ Mean curvature

and try and understand the normal curvature

$\mathcal{K} = \det \mathcal{W}$, Gaussian curvature
$\mathcal{H} = \frac{\text{trace } \mathcal{W}}{2}$, Mean Curvature

$$\kappa_n = \mathcal{W}\dot{\gamma}.\dot{\gamma}$$

$\mathcal{W}$ is symmetric.
Denote its eigenvalues, κ_1 and κ_2.
i.e. there exist $\mathbf{t}_1$ and $\mathbf{t}_2$ *unit* vectors so that
$\mathcal{W}\mathbf{t}_1 = \kappa_1 \mathbf{t}_1$ and
$\mathcal{W}\mathbf{t}_2 = \kappa_2 \mathbf{t}_2$
Assume $\kappa_1 \neq \kappa_2$, then $\mathbf{t}_1.\mathbf{t}_2 = 0$

Writing $\mathcal{W}$ in terms of the basis $\mathbf{t}_1$ and $\mathbf{t}_2$,

$$\begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix}$$

$\kappa_1 \kappa_2 =$ Gaussian curvature
$\frac{\kappa_1 + \kappa_2}{2} =$ Mean curvature

which we now know can be written using $\mathcal{W}$

$\mathcal{K} = \det \mathcal{W}$, Gaussian curvature
$\mathcal{H} = \frac{\text{trace } \mathcal{W}}{2}$, Mean Curvature

$\mathcal{W}$ is symmetric.
Denote its eigenvalues, κ_1 and κ_2.
i.e. there exist $\mathbf{t}_1$ and $\mathbf{t}_2$ *unit* vectors so that
$\mathcal{W}\mathbf{t}_1 = \kappa_1 \mathbf{t}_1$ and
$\mathcal{W}\mathbf{t}_2 = \kappa_2 \mathbf{t}_2$
Assume $\kappa_1 \neq \kappa_2$, then $\mathbf{t}_1 . \mathbf{t}_2 = 0$

Writing $\mathcal{W}$ in terms of the basis $\mathbf{t}_1$ and $\mathbf{t}_2$,

$$\begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix}$$

$\kappa_1 \kappa_2 =$ Gaussian curvature
$\frac{\kappa_1 + \kappa_2}{2} =$ Mean curvature

$$\begin{aligned} \kappa_n &= \mathcal{W}\dot{\gamma}.\dot{\gamma} \\ &= (\mathcal{W}(c_1\mathbf{t}_1 + c_2\mathbf{t}_2)).(c_1\mathbf{t}_1 + c_2\mathbf{t}_2) \end{aligned}$$

But now we have a new basis, $\mathbf{t}_1$ and $\mathbf{t}_2$ convenient for the Weingarten map

$\mathcal{K} = \det \mathcal{W}$, Gaussian curvature
$\mathcal{H} = \frac{\text{trace } \mathcal{W}}{2}$, Mean Curvature

$$\begin{aligned}\kappa_n &= \mathcal{W}\dot{\gamma}.\dot{\gamma} \\ &= (\mathcal{W}(c_1\mathbf{t}_1 + c_2\mathbf{t}_2)).(c_1\mathbf{t}_1 + c_2\mathbf{t}_2) \\ &= (c_1\mathcal{W}(\mathbf{t}_1) + c_2\mathcal{W}(\mathbf{t}_2)).(c_1\mathbf{t}_1 + c_2\mathbf{t}_2))\end{aligned}$$

$\mathcal{W}$ is symmetric.
Denote its eigenvalues, κ_1 and κ_2.
i.e. there exist $\mathbf{t}_1$ and $\mathbf{t}_2$ *unit* vectors so that
$\mathcal{W}\mathbf{t}_1 = \kappa_1\mathbf{t}_1$ and
$\mathcal{W}\mathbf{t}_2 = \kappa_2\mathbf{t}_2$
Assume $\kappa_1 \neq \kappa_2$, then $\mathbf{t}_1.\mathbf{t}_2 = 0$

Writing $\mathcal{W}$ in terms of the basis $\mathbf{t}_1$ and $\mathbf{t}_2$,

$$\begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix}$$

$\kappa_1\kappa_2 =$ Gaussian curvature
$\frac{\kappa_1+\kappa_2}{2} =$ Mean curvature

Of course, we chose this basis so that the Weingarten map has a better form

$\mathcal{K} = \det\, \mathcal{W}$, Gaussian curvature
$\mathcal{H} = \frac{\text{trace}\, \mathcal{W}}{2}$, Mean Curvature

$\mathcal{W}$ is symmetric.
Denote its eigenvalues, κ_1 and κ_2.
i.e. there exist $\mathbf{t}_1$ and $\mathbf{t}_2$ *unit* vectors so that
$\mathcal{W}\mathbf{t}_1 = \kappa_1 \mathbf{t}_1$ and
$\mathcal{W}\mathbf{t}_2 = \kappa_2 \mathbf{t}_2$
Assume $\kappa_1 \neq \kappa_2$, then $\mathbf{t}_1 . \mathbf{t}_2 = 0$

Writing $\mathcal{W}$ in terms of the basis $\mathbf{t}_1$ and $\mathbf{t}_2$,

$$\begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix}$$

$\kappa_1 \kappa_2 =$ Gaussian curvature
$\frac{\kappa_1 + \kappa_2}{2} =$ Mean curvature

$$\begin{aligned} \kappa_n &= \mathcal{W}\dot{\gamma}.\dot{\gamma} \\ &= (\mathcal{W}(c_1\mathbf{t}_1 + c_2\mathbf{t}_2)).(c_1\mathbf{t}_1 + c_2\mathbf{t}_2) \\ &= (c_1\mathcal{W}(\mathbf{t}_1) + c_2\mathcal{W}(\mathbf{t}_2)).(c_1\mathbf{t}_1 + c_2\mathbf{t}_2)) \\ &= (c_1\kappa_1\mathbf{t}_1 + c_2\kappa_2\mathbf{t}_2).(c_1\mathbf{t}_1 + c_2\mathbf{t}_2)) \end{aligned}$$

And its application can now be written in terms of κ_1 and κ_2

$\mathcal{K} = \det \mathcal{W}$, Gaussian curvature
$\mathcal{H} = \frac{\text{trace } \mathcal{W}}{2}$, Mean Curvature

$\mathcal{W}$ is symmetric.
Denote its eigenvalues, κ_1 and κ_2.
i.e. there exist $\mathbf{t}_1$ and $\mathbf{t}_2$ *unit* vectors so that
$\mathcal{W}\mathbf{t}_1 = \kappa_1 \mathbf{t}_1$ and
$\mathcal{W}\mathbf{t}_2 = \kappa_2 \mathbf{t}_2$
Assume $\kappa_1 \neq \kappa_2$, then $\mathbf{t}_1.\mathbf{t}_2 = 0$

Writing $\mathcal{W}$ in terms of the basis $\mathbf{t}_1$ and $\mathbf{t}_2$,

$$\begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix}$$

$\kappa_1 \kappa_2 =$ Gaussian curvature
$\frac{\kappa_1 + \kappa_2}{2} =$ Mean curvature

$$\begin{aligned}
\kappa_n &= \mathcal{W}\dot{\gamma}.\dot{\gamma} \\
&= (\mathcal{W}(c_1\mathbf{t}_1 + c_2\mathbf{t}_2)).(c_1\mathbf{t}_1 + c_2\mathbf{t}_2) \\
&= (c_1\mathcal{W}(\mathbf{t}_1) + c_2\mathcal{W}(\mathbf{t}_2)).(c_1\mathbf{t}_1 + c_2\mathbf{t}_2)) \\
&= (c_1\kappa_1\mathbf{t}_1 + c_2\kappa_2\mathbf{t}_2).(c_1\mathbf{t}_1 + c_2\mathbf{t}_2)) \\
&= c_1^2\kappa_1(\mathbf{t}_1.\mathbf{t}_1) +
\end{aligned}$$

Now we distribute the terms

$\mathcal{K} = \det \mathcal{W}$, Gaussian curvature
$\mathcal{H} = \frac{\text{trace } \mathcal{W}}{2}$, Mean Curvature

$\mathcal{W}$ is symmetric.
Denote its eigenvalues, κ_1 and κ_2.
i.e. there exist $\mathbf{t}_1$ and $\mathbf{t}_2$ *unit* vectors so that
$\mathcal{W}\mathbf{t}_1 = \kappa_1 \mathbf{t}_1$ and
$\mathcal{W}\mathbf{t}_2 = \kappa_2 \mathbf{t}_2$
Assume $\kappa_1 \neq \kappa_2$, then $\mathbf{t}_1.\mathbf{t}_2 = 0$

Writing $\mathcal{W}$ in terms of the basis $\mathbf{t}_1$ and $\mathbf{t}_2$,

$$\begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix}$$

$\kappa_1 \kappa_2 =$ Gaussian curvature
$\frac{\kappa_1 + \kappa_2}{2} =$ Mean curvature

$$\begin{aligned} \kappa_n &= \mathcal{W}\dot{\gamma}.\dot{\gamma} \\ &= (\mathcal{W}(c_1\mathbf{t}_1 + c_2\mathbf{t}_2)).(c_1\mathbf{t}_1 + c_2\mathbf{t}_2) \\ &= (c_1\mathcal{W}(\mathbf{t}_1) + c_2\mathcal{W}(\mathbf{t}_2)).(c_1\mathbf{t}_1 + c_2\mathbf{t}_2)) \\ &= (c_1\kappa_1\mathbf{t}_1 + c_2\kappa_2\mathbf{t}_2).(c_1\mathbf{t}_1 + c_2\mathbf{t}_2)) \\ &= c_1^2\kappa_1(\mathbf{t}_1.\mathbf{t}_1) + c_1c_2\kappa_1(\mathbf{t}_1.\mathbf{t}_2) \\ &\quad + \end{aligned}$$

$\mathcal{K} = \det \mathcal{W}$, Gaussian curvature
$\mathcal{H} = \frac{\text{trace } \mathcal{W}}{2}$, Mean Curvature

$\mathcal{W}$ is symmetric.
Denote its eigenvalues, κ_1 and κ_2.
i.e. there exist $\mathbf{t}_1$ and $\mathbf{t}_2$ *unit* vectors so that
$\mathcal{W}\mathbf{t}_1 = \kappa_1 \mathbf{t}_1$ and
$\mathcal{W}\mathbf{t}_2 = \kappa_2 \mathbf{t}_2$
Assume $\kappa_1 \neq \kappa_2$, then $\mathbf{t}_1.\mathbf{t}_2 = 0$

Writing $\mathcal{W}$ in terms of the basis $\mathbf{t}_1$ and $\mathbf{t}_2$,

$$\begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix}$$

$\kappa_1\kappa_2 =$ Gaussian curvature
$\frac{\kappa_1+\kappa_2}{2} =$ Mean curvature

$$\begin{aligned}
\kappa_n &= \mathcal{W}\dot{\gamma}.\dot{\gamma} \\
&= (\mathcal{W}(c_1\mathbf{t}_1 + c_2\mathbf{t}_2)).(c_1\mathbf{t}_1 + c_2\mathbf{t}_2) \\
&= (c_1\mathcal{W}(\mathbf{t}_1) + c_2\mathcal{W}(\mathbf{t}_2)).(c_1\mathbf{t}_1 + c_2\mathbf{t}_2)) \\
&= (c_1\kappa_1\mathbf{t}_1 + c_2\kappa_2\mathbf{t}_2).(c_1\mathbf{t}_1 + c_2\mathbf{t}_2)) \\
&= c_1^2\kappa_1(\mathbf{t}_1.\mathbf{t}_1) + c_1c_2\kappa_1(\mathbf{t}_1.\mathbf{t}_2) \\
&\quad + c_2c_1\kappa_2(\mathbf{t}_2.\mathbf{t}_1) +
\end{aligned}$$

$\mathcal{K} = \det \mathcal{W}$, Gaussian curvature
$\mathcal{H} = \frac{\text{trace } \mathcal{W}}{2}$, Mean Curvature

$\mathcal{W}$ is symmetric.
Denote its eigenvalues, κ_1 and κ_2.
i.e. there exist $\mathbf{t}_1$ and $\mathbf{t}_2$ *unit* vectors so that
$\mathcal{W}\mathbf{t}_1 = \kappa_1 \mathbf{t}_1$ and
$\mathcal{W}\mathbf{t}_2 = \kappa_2 \mathbf{t}_2$
Assume $\kappa_1 \neq \kappa_2$, then $\mathbf{t}_1.\mathbf{t}_2 = 0$

Writing $\mathcal{W}$ in terms of the basis $\mathbf{t}_1$ and $\mathbf{t}_2$,

$$\begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix}$$

$\kappa_1\kappa_2 =$ Gaussian curvature
$\frac{\kappa_1+\kappa_2}{2} =$ Mean curvature

$$\begin{aligned}
\kappa_n &= \mathcal{W}\dot{\gamma}.\dot{\gamma} \\
&= (\mathcal{W}(c_1\mathbf{t}_1 + c_2\mathbf{t}_2)).(c_1\mathbf{t}_1 + c_2\mathbf{t}_2) \\
&= (c_1\mathcal{W}(\mathbf{t}_1) + c_2\mathcal{W}(\mathbf{t}_2)).(c_1\mathbf{t}_1 + c_2\mathbf{t}_2)) \\
&= (c_1\kappa_1\mathbf{t}_1 + c_2\kappa_2\mathbf{t}_2).(c_1\mathbf{t}_1 + c_2\mathbf{t}_2)) \\
&= c_1^2\kappa_1(\mathbf{t}_1.\mathbf{t}_1) + c_1c_2\kappa_1(\mathbf{t}_1.\mathbf{t}_2) \\
&\quad + c_2c_1\kappa_2(\mathbf{t}_2.\mathbf{t}_1) + c_2^2\kappa_2(\mathbf{t}_2.\mathbf{t}_2)
\end{aligned}$$

$\mathcal{K} = \det \mathcal{W}$, Gaussian curvature
$\mathcal{H} = \frac{\text{trace } \mathcal{W}}{2}$, Mean Curvature

$\mathcal{W}$ is symmetric.
Denote its eigenvalues, κ_1 and κ_2.
i.e. there exist $\mathbf{t}_1$ and $\mathbf{t}_2$ *unit* vectors so that
$\mathcal{W}\mathbf{t}_1 = \kappa_1 \mathbf{t}_1$ and
$\mathcal{W}\mathbf{t}_2 = \kappa_2 \mathbf{t}_2$
Assume $\kappa_1 \neq \kappa_2$, then $\mathbf{t}_1.\mathbf{t}_2 = 0$

Writing $\mathcal{W}$ in terms of the basis $\mathbf{t}_1$ and $\mathbf{t}_2$,

$$\begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix}$$

$\kappa_1 \kappa_2 =$ Gaussian curvature
$\frac{\kappa_1 + \kappa_2}{2} =$ Mean curvature

$$\begin{aligned}
\kappa_n &= \mathcal{W}\dot{\gamma}.\dot{\gamma} \\
&= (\mathcal{W}(c_1\mathbf{t}_1 + c_2\mathbf{t}_2)).(c_1\mathbf{t}_1 + c_2\mathbf{t}_2) \\
&= (c_1\mathcal{W}(\mathbf{t}_1) + c_2\mathcal{W}(\mathbf{t}_2)).(c_1\mathbf{t}_1 + c_2\mathbf{t}_2)) \\
&= (c_1\kappa_1\mathbf{t}_1 + c_2\kappa_2\mathbf{t}_2).(c_1\mathbf{t}_1 + c_2\mathbf{t}_2)) \\
&= c_1^2\kappa_1(\mathbf{t}_1.\mathbf{t}_1) + c_1c_2\kappa_1 \underbrace{(\mathbf{t}_1.\mathbf{t}_2)}_{0} \\
&\quad + c_2c_1\kappa_2 \underbrace{(\mathbf{t}_2.\mathbf{t}_1)}_{0} + c_2^2\kappa_2(\mathbf{t}_2.\mathbf{t}_2)
\end{aligned}$$

By orthogonality, two dot products are 0

$\mathcal{K} = \det \mathcal{W}$, Gaussian curvature
$\mathcal{H} = \frac{\text{trace } \mathcal{W}}{2}$, Mean Curvature

$\mathcal{W}$ is symmetric.
Denote its eigenvalues, κ_1 and κ_2.
i.e. there exist $\mathbf{t}_1$ and $\mathbf{t}_2$ *unit* vectors so that
$\mathcal{W}\mathbf{t}_1 = \kappa_1 \mathbf{t}_1$ and
$\mathcal{W}\mathbf{t}_2 = \kappa_2 \mathbf{t}_2$
Assume $\kappa_1 \neq \kappa_2$, then $\mathbf{t}_1.\mathbf{t}_2 = 0$

Writing $\mathcal{W}$ in terms of the basis $\mathbf{t}_1$ and $\mathbf{t}_2$,

$$\begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix}$$

$\kappa_1\kappa_2 =$ Gaussian curvature
$\frac{\kappa_1+\kappa_2}{2} =$ Mean curvature

$$\begin{aligned}
\kappa_n &= \mathcal{W}\dot{\gamma}.\dot{\gamma} \\
&= (\mathcal{W}(c_1\mathbf{t}_1 + c_2\mathbf{t}_2)).(c_1\mathbf{t}_1 + c_2\mathbf{t}_2) \\
&= (c_1\mathcal{W}(\mathbf{t}_1) + c_2\mathcal{W}(\mathbf{t}_2)).(c_1\mathbf{t}_1 + c_2\mathbf{t}_2)) \\
&= (c_1\kappa_1\mathbf{t}_1 + c_2\kappa_2\mathbf{t}_2).(c_1\mathbf{t}_1 + c_2\mathbf{t}_2)) \\
&= c_1^2\kappa_1 \underbrace{(\mathbf{t}_1.\mathbf{t}_1)}_{1} + c_1c_2\kappa_1 \underbrace{(\mathbf{t}_1.\mathbf{t}_2)}_{0} \\
&\quad + c_2c_1\kappa_2 \underbrace{(\mathbf{t}_2.\mathbf{t}_1)}_{0} + c_2^2\kappa_2 \underbrace{(\mathbf{t}_2.\mathbf{t}_2)}_{1}
\end{aligned}$$

By orthonormality, two of them are 1

$\mathcal{K} = \det \mathcal{W}$, Gaussian curvature
$\mathcal{H} = \frac{\text{trace } \mathcal{W}}{2}$, Mean Curvature

$\mathcal{W}$ is symmetric.
Denote its eigenvalues, κ_1 and κ_2.
i.e. there exist $\mathbf{t}_1$ and $\mathbf{t}_2$ *unit* vectors so that
$\mathcal{W}\mathbf{t}_1 = \kappa_1 \mathbf{t}_1$ and
$\mathcal{W}\mathbf{t}_2 = \kappa_2 \mathbf{t}_2$
Assume $\kappa_1 \neq \kappa_2$, then $\mathbf{t}_1.\mathbf{t}_2 = 0$

Writing $\mathcal{W}$ in terms of the basis $\mathbf{t}_1$ and $\mathbf{t}_2$,

$$\begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix}$$

$\kappa_1\kappa_2 =$ Gaussian curvature
$\frac{\kappa_1+\kappa_2}{2} =$ Mean curvature

$$\begin{aligned}
\kappa_n &= \mathcal{W}\dot{\gamma}.\dot{\gamma} \\
&= (\mathcal{W}(c_1\mathbf{t}_1 + c_2\mathbf{t}_2)).(c_1\mathbf{t}_1 + c_2\mathbf{t}_2) \\
&= (c_1\mathcal{W}(\mathbf{t}_1) + c_2\mathcal{W}(\mathbf{t}_2)).(c_1\mathbf{t}_1 + c_2\mathbf{t}_2)) \\
&= (c_1\kappa_1\mathbf{t}_1 + c_2\kappa_2\mathbf{t}_2).(c_1\mathbf{t}_1 + c_2\mathbf{t}_2)) \\
&= c_1^2\kappa_1 \underbrace{(\mathbf{t}_1.\mathbf{t}_1)}_{1} + c_1c_2\kappa_1 \underbrace{(\mathbf{t}_1.\mathbf{t}_2)}_{0} \\
&\quad + c_2c_1\kappa_2 \underbrace{(\mathbf{t}_2.\mathbf{t}_1)}_{0} + c_2^2\kappa_2 \underbrace{(\mathbf{t}_2.\mathbf{t}_2)}_{1} \\
&= c_1^2\kappa_1 + c_2^2\kappa_2
\end{aligned}$$

So finally we get the following relation

$\kappa_n = c_1^2 \kappa_1 + c_2^2 \kappa_2$
where, $\dot{\gamma}(t_0) = c_1 \mathbf{t}_1 + c_2 \mathbf{t}_2$

So we have the computation of the normal curvature entirely in terms of $\dot{\gamma}(t_0)$, κ_i and $\mathbf{t}_i$

$\kappa_n = c_1^2 \kappa_1 + c_2^2 \kappa_2$
where, $\dot{\gamma}(t_0) = c_1 \mathbf{t}_1 + c_2 \mathbf{t}_2$

We use $\mathcal{W}$ only to obtain κ_i and $\mathbf{t}_i$

.

$\kappa_n = c_1^2\kappa_1 + c_2^2\kappa_2$

where, $\dot{\gamma}(t_0) = c_1\mathbf{t}_1 + c_2\mathbf{t}_2$

Now given any curve, to compute κ_n, all we need from the curve is its velocity vector

.

$\kappa_n = c_1^2 \kappa_1 + c_2^2 \kappa_2$

where, $\dot{\gamma}(t_0) = c_1 \mathbf{t}_1 + c_2 \mathbf{t}_2$

We find out the coefficients of it when written in terms of $\mathbf{t}_1$ and $\mathbf{t}_2$

.

$\kappa_n = c_1^2\kappa_1 + c_2^2\kappa_2$

where, $\dot{\gamma}(t_0) = c_1\mathbf{t}_1 + c_2\mathbf{t}_2$

And we use them along with κ_1 and κ_2 in the highlighted formula for κ_n.

$$\kappa_n = c_1^2 \kappa_1 + c_2^2 \kappa_2$$
where, $\dot{\gamma}(t_0) = c_1 \mathbf{t}_1 + c_2 \mathbf{t}_2$

We will now use the fact that $\dot{\gamma}(t_0)$, $\mathbf{t}_1$ and $\mathbf{t}_2$ are all unit vectors

$$\kappa_n = c_1^2\kappa_1 + c_2^2\kappa_2$$

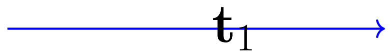
where, $\dot{\gamma}(t_0) = c_1\mathbf{t}_1 + c_2\mathbf{t}_2$

to write the coefficients in a more revealing form

.

$$\kappa_n = c_1^2\kappa_1 + c_2^2\kappa_2$$

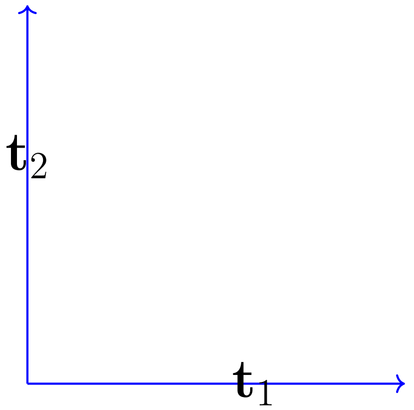
where, $\dot{\gamma}(t_0) = c_1\mathbf{t}_1 + c_2\mathbf{t}_2$



The vectors $\mathbf{t}_1$

$\kappa_n = c_1^2 \kappa_1 + c_2^2 \kappa_2$

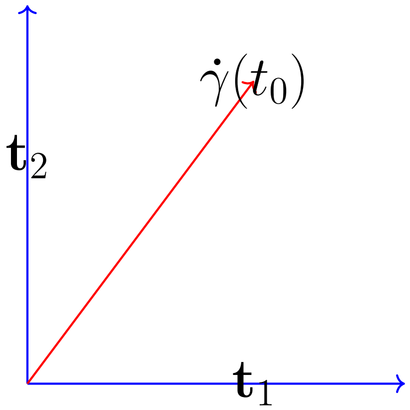
where, $\dot{\gamma}(t_0) = c_1 \mathbf{t}_1 + c_2 \mathbf{t}_2$



and $\mathbf{t}_2$ form a basis

$$\kappa_n = c_1^2\kappa_1 + c_2^2\kappa_2$$

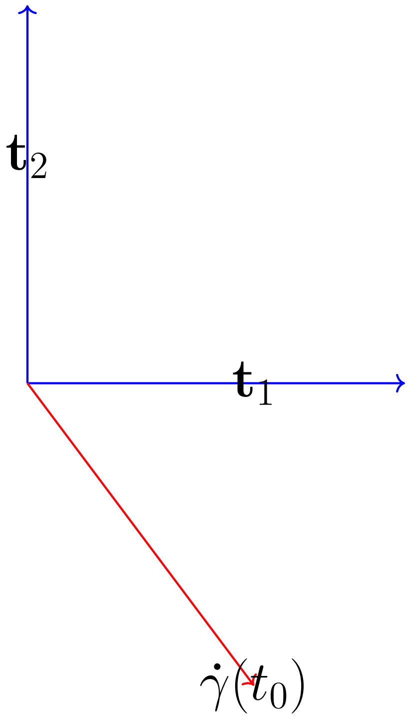
where, $\dot{\gamma}(t_0) = c_1\mathbf{t}_1 + c_2\mathbf{t}_2$



in terms of which we can write $\dot{\gamma}(t_0)$

$$\kappa_n = c_1^2\kappa_1 + c_2^2\kappa_2$$

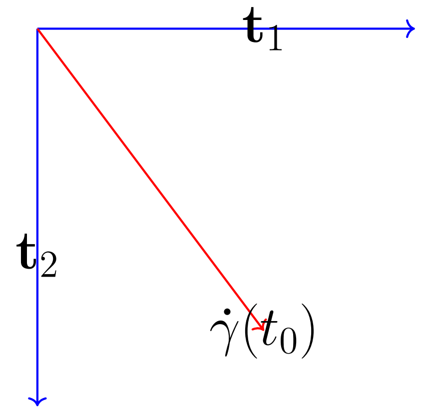
where, $\dot{\gamma}(t_0) = c_1\mathbf{t}_1 + c_2\mathbf{t}_2$



Even if $\dot{\gamma}(t_0)$ is not between the eigen-vectors

$\kappa_n = c_1^2\kappa_1 + c_2^2\kappa_2$

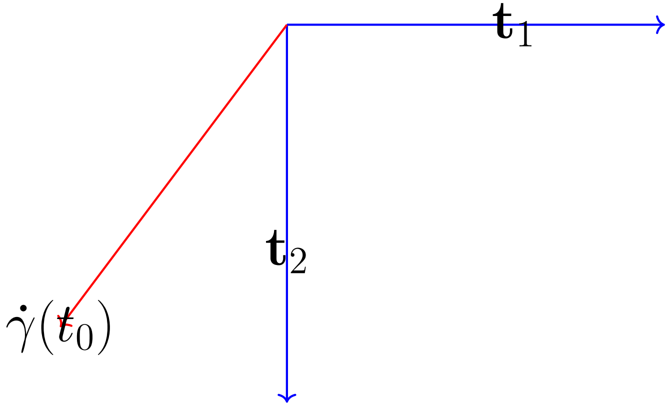
where, $\dot{\gamma}(t_0) = c_1\mathbf{t}_1 + c_2\mathbf{t}_2$



we can always replace one vector by the negative

$\kappa_n = c_1^2\kappa_1 + c_2^2\kappa_2$

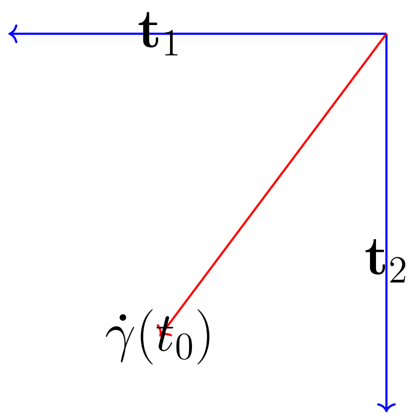
where, $\dot{\gamma}(t_0) = c_1\mathbf{t}_1 + c_2\mathbf{t}_2$



No matter where $\dot{\gamma}(t_0)$ is placed,

$\kappa_n = c_1^2 \kappa_1 + c_2^2 \kappa_2$

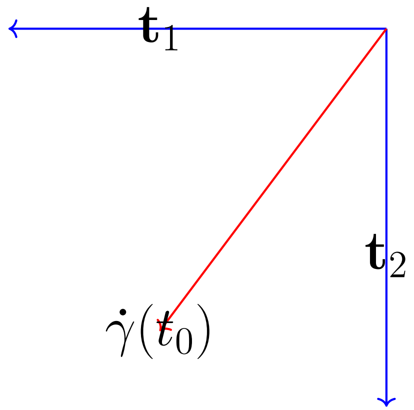
where, $\dot{\gamma}(t_0) = c_1 \mathbf{t}_1 + c_2 \mathbf{t}_2$



we can ensure that it is in between by replacing one or more eigenvectors by their negatives

$\kappa_n = c_1^2\kappa_1 + c_2^2\kappa_2$
where, $\dot{\gamma}(t_0) = c_1\mathbf{t}_1 + c_2\mathbf{t}_2$

We can always choose $\mathbf{t}_1$ and $\mathbf{t}_2$ so that,
$\dot{\gamma}(t_0) = \cos(\theta)\mathbf{t}_1 + \sin(\theta)\mathbf{t}_2$



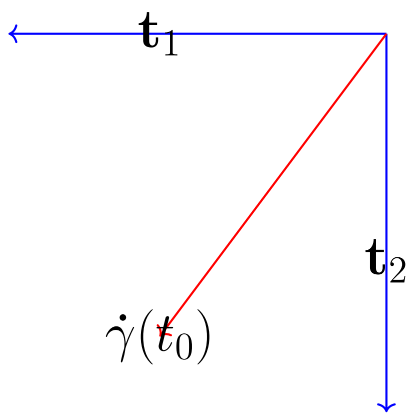
Since all the vectors are unit vectors, the coefficients can be written in a better form

$\kappa_n = c_1^2\kappa_1 + c_2^2\kappa_2$
where, $\dot{\gamma}(t_0) = c_1\mathbf{t}_1 + c_2\mathbf{t}_2$

We can always choose $\mathbf{t}_1$ and $\mathbf{t}_2$ so that,
$\dot{\gamma}(t_0) = \cos(\theta)\mathbf{t}_1 + \sin(\theta)\mathbf{t}_2$

Therefore,

$$\kappa_n = \cos^2(\theta)\kappa_1 + \sin^2(\theta)\kappa_2$$



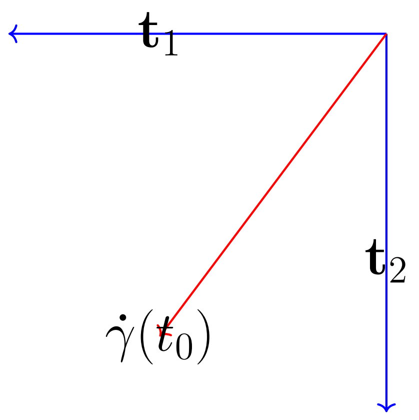
We can, therefore, express the normal curvature along $\dot{\gamma}(t_0)$ in terms of the angle it makes with $\mathbf{t}_1$.

$\kappa_n = c_1^2 \kappa_1 + c_2^2 \kappa_2$
where, $\dot{\gamma}(t_0) = c_1 \mathbf{t}_1 + c_2 \mathbf{t}_2$

We can always choose $\mathbf{t}_1$ and $\mathbf{t}_2$ so that,
$\dot{\gamma}(t_0) = \cos(\theta)\mathbf{t}_1 + \sin(\theta)\mathbf{t}_2$

Therefore,

$$\begin{aligned}\kappa_n &= \cos^2(\theta)\kappa_1 + \sin^2(\theta)\kappa_2 \\ &= \cos^2(\theta)\kappa_1 + (1 - \cos^2(\theta))\kappa_2\end{aligned}$$



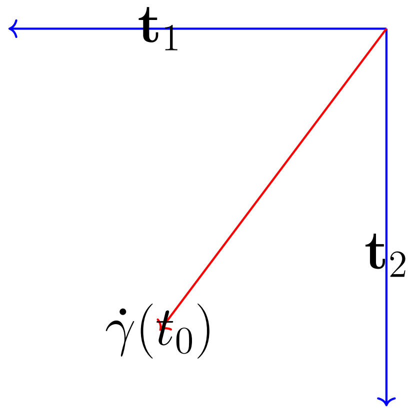
and exploit a standard trigonometric identity

$\kappa_n = c_1^2\kappa_1 + c_2^2\kappa_2$
where, $\dot{\gamma}(t_0) = c_1\mathbf{t}_1 + c_2\mathbf{t}_2$

We can always choose $\mathbf{t}_1$ and $\mathbf{t}_2$ so that,
$\dot{\gamma}(t_0) = \cos(\theta)\mathbf{t}_1 + \sin(\theta)\mathbf{t}_2$

Therefore,

$$\begin{aligned}
\kappa_n &= \cos^2(\theta)\kappa_1 + \sin^2(\theta)\kappa_2 \\
&= \cos^2(\theta)\kappa_1 + (1 - \cos^2(\theta))\kappa_2 \\
&= \kappa_2 + (\kappa_1 - \kappa_2)\cos^2(\theta)
\end{aligned}$$



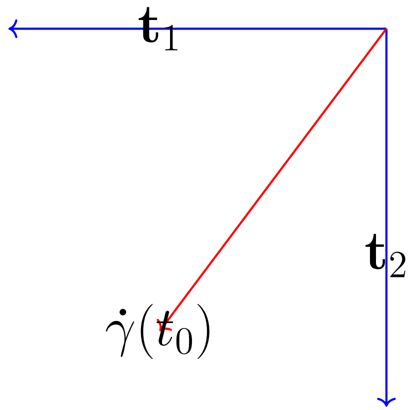
to express it in a very revealing form

$\kappa_n = c_1^2\kappa_1 + c_2^2\kappa_2$
where, $\dot{\gamma}(t_0) = c_1\mathbf{t}_1 + c_2\mathbf{t}_2$

We can always choose $\mathbf{t}_1$ and $\mathbf{t}_2$ so that,
$\dot{\gamma}(t_0) = \cos(\theta)\mathbf{t}_1 + \sin(\theta)\mathbf{t}_2$

Therefore,

$$\begin{aligned}\kappa_n &= \cos^2(\theta)\kappa_1 + \sin^2(\theta)\kappa_2 \\ &= \cos^2(\theta)\kappa_1 + (1-\cos^2(\theta))\kappa_2 \\ &= \kappa_2 + (\kappa_1 - \kappa_2)\cos^2(\theta)\end{aligned}$$



Assume $\kappa_1 \leq \kappa_2$,

$$\kappa_n = \kappa_2 + \text{some negative number}$$

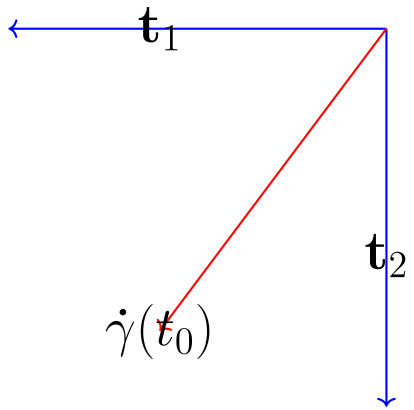
We may always assume that we labelled the smaller eigenvalue as the first

$\kappa_n = c_1^2 \kappa_1 + c_2^2 \kappa_2$
where, $\dot{\gamma}(t_0) = c_1 \mathbf{t}_1 + c_2 \mathbf{t}_2$

We can always choose $\mathbf{t}_1$ and $\mathbf{t}_2$ so that,
$\dot{\gamma}(t_0) = \cos(\theta)\mathbf{t}_1 + \sin(\theta)\mathbf{t}_2$

Therefore,

$$\begin{aligned}\kappa_n &= \cos^2(\theta)\kappa_1 + \sin^2(\theta)\kappa_2 \\ &= \cos^2(\theta)\kappa_1 + (1 - \cos^2(\theta))\kappa_2 \\ &= \kappa_2 + (\kappa_1 - \kappa_2)\cos^2(\theta)\end{aligned}$$



Assume $\kappa_1 \leq \kappa_2$,

$$\kappa_n = \kappa_2 + \text{some negative number}$$

So, $\kappa_n \leq \kappa_2$.

Since we are adding a negative number or 0

$\kappa_n = c_1^2\kappa_1 + c_2^2\kappa_2$
where, $\dot{\gamma}(t_0) = c_1\mathbf{t}_1 + c_2\mathbf{t}_2$

We can always choose $\mathbf{t}_1$ and $\mathbf{t}_2$ so that,
$\dot{\gamma}(t_0) = \cos(\theta)\mathbf{t}_1 + \sin(\theta)\mathbf{t}_2$

Therefore,

$$\begin{aligned}\kappa_n &= \cos^2(\theta)\kappa_1 + \sin^2(\theta)\kappa_2 \\ &= \cos^2(\theta)\kappa_1 + (1-\cos^2(\theta))\kappa_2 \\ &= \kappa_2 + (\kappa_1 - \kappa_2)\cos^2(\theta)\end{aligned}$$

Assume $\kappa_1 \leq \kappa_2$,

$$\kappa_n = \kappa_2 + \text{some negative number}$$

So, $\kappa_n \leq \kappa_2$.

$\kappa_n = \kappa_2$ if and only if $\cos^2(\theta) = 0$
if and only if $\theta = \pi/2$

We will now check when it is equal

$\kappa_n = c_1^2\kappa_1 + c_2^2\kappa_2$
where, $\dot{\gamma}(t_0) = c_1\mathbf{t}_1 + c_2\mathbf{t}_2$

We can always choose $\mathbf{t}_1$ and $\mathbf{t}_2$ so that,
$\dot{\gamma}(t_0) = \cos(\theta)\mathbf{t}_1 + \sin(\theta)\mathbf{t}_2$

Therefore,

$$\begin{aligned}\kappa_n &= \cos^2(\theta)\kappa_1 + \sin^2(\theta)\kappa_2 \\ &= \cos^2(\theta)\kappa_1 + (1 - \cos^2(\theta))\kappa_2 \\ &= \kappa_2 + (\kappa_1 - \kappa_2)\cos^2(\theta)\end{aligned}$$

Assume $\kappa_1 \leq \kappa_2$,

$$\kappa_n = \kappa_2 + \text{some negative number}$$

So, $\kappa_n \leq \kappa_2$.

$\kappa_n = \kappa_2$ if and only if $\cos^2(\theta) = 0$
if and only if $\theta = \pi/2$
if and only if $\dot{\gamma}(t_0)$ makes angle $\pi/2$ with $\mathbf{t}_1$

$\blacksquare$

$\kappa_n = c_1^2\kappa_1 + c_2^2\kappa_2$
where, $\dot{\gamma}(t_0) = c_1\mathbf{t}_1 + c_2\mathbf{t}_2$

We can always choose $\mathbf{t}_1$ and $\mathbf{t}_2$ so that,
$\dot{\gamma}(t_0) = \cos(\theta)\mathbf{t}_1 + \sin(\theta)\mathbf{t}_2$

Therefore,

$$\begin{aligned}\kappa_n &= \cos^2(\theta)\kappa_1 + \sin^2(\theta)\kappa_2 \\ &= \cos^2(\theta)\kappa_1 + (1-\cos^2(\theta))\kappa_2 \\ &= \kappa_2 + (\kappa_1 - \kappa_2)\cos^2(\theta)\end{aligned}$$

Assume $\kappa_1 \leq \kappa_2$,

$$\kappa_n = \kappa_2 + \text{some negative number}$$

So, $\kappa_n \leq \kappa_2$.

$\kappa_n = \kappa_2$ if and only if $\cos^2(\theta) = 0$
if and only if $\theta = \pi/2$
if and only if $\dot{\gamma}(t_0)$ makes angle $\pi/2$ with $\mathbf{t}_1$
if and only if $\dot{\gamma}(t_0)$ makes angle 0 with $\mathbf{t}_2$

$\blacksquare$

$\kappa_n = c_1^2\kappa_1 + c_2^2\kappa_2$
where, $\dot{\gamma}(t_0) = c_1\mathbf{t}_1 + c_2\mathbf{t}_2$

We can always choose $\mathbf{t}_1$ and $\mathbf{t}_2$ so that,
$\dot{\gamma}(t_0) = \cos(\theta)\mathbf{t}_1 + \sin(\theta)\mathbf{t}_2$

Therefore,

$$\begin{aligned}\kappa_n &= \cos^2(\theta)\kappa_1 + \sin^2(\theta)\kappa_2\\ &= \cos^2(\theta)\kappa_1 + (1-\cos^2(\theta))\kappa_2\\ &= \kappa_2 + (\kappa_1 - \kappa_2)\cos^2(\theta)\end{aligned}$$

Assume $\kappa_1 \leq \kappa_2$,

$$\kappa_n = \kappa_2 + \text{some negative number}$$

So, $\kappa_n \leq \kappa_2$.

$\kappa_n = \kappa_2$ if and only if $\cos^2(\theta) = 0$
if and only if $\theta = \pi/2$
if and only if $\dot{\gamma}(t_0)$ makes angle $\pi/2$ with $\mathbf{t}_1$
if and only if $\dot{\gamma}(t_0)$ makes angle 0 with $\mathbf{t}_2$
i.e. $\dot{\gamma}(t_0)$ is aligned with $\mathbf{t}_2$

$\blacksquare$

$\kappa_n = c_1^2\kappa_1 + c_2^2\kappa_2$
where, $\dot{\gamma}(t_0) = c_1\mathbf{t}_1 + c_2\mathbf{t}_2$

We can always choose $\mathbf{t}_1$ and $\mathbf{t}_2$ so that,
$\dot{\gamma}(t_0) = \cos(\theta)\mathbf{t}_1 + \sin(\theta)\mathbf{t}_2$

Therefore,

$$\begin{aligned}\kappa_n &= \cos^2(\theta)\kappa_1 + \sin^2(\theta)\kappa_2 \\ &= \cos^2(\theta)\kappa_1 + (1-\cos^2(\theta))\kappa_2 \\ &= \kappa_2 + (\kappa_1 - \kappa_2)\cos^2(\theta)\end{aligned}$$

Assume $\kappa_1 \leq \kappa_2$,

$$\kappa_n = \kappa_2 + \text{some negative number}$$

So, $\kappa_n \leq \kappa_2$.

$\kappa_n = \kappa_2$ if and only if $\cos^2(\theta) = 0$
if and only if $\theta = \pi/2$
if and only if $\dot{\gamma}(t_0)$ makes angle $\pi/2$ with $\mathbf{t}_1$
if and only if $\dot{\gamma}(t_0)$ makes angle 0 with $\mathbf{t}_2$
i.e. $\dot{\gamma}(t_0)$ is aligned with $\mathbf{t}_2$

Therefore,

Proposition. κ_2 *is the maximum possible normal curvature of a curve at that point.*

Now we see that κ_1 and κ_2 have a geometric interpretation

$\kappa_n = c_1^2\kappa_1 + c_2^2\kappa_2$
where, $\dot{\gamma}(t_0) = c_1\mathbf{t}_1 + c_2\mathbf{t}_2$

We can always choose $\mathbf{t}_1$ and $\mathbf{t}_2$ so that,
$\dot{\gamma}(t_0) = \cos(\theta)\mathbf{t}_1 + \sin(\theta)\mathbf{t}_2$

Therefore,

$$\begin{aligned}\kappa_n &= \cos^2(\theta)\kappa_1 + \sin^2(\theta)\kappa_2 \\ &= \cos^2(\theta)\kappa_1 + (1-\cos^2(\theta))\kappa_2 \\ &= \kappa_2 + (\kappa_1 - \kappa_2)\cos^2(\theta)\end{aligned}$$

Assume $\kappa_1 \leq \kappa_2$,

$$\kappa_n = \kappa_2 + \text{some negative number}$$

So, $\kappa_n \leq \kappa_2$.

$\kappa_n = \kappa_2$ if and only if $\cos^2(\theta) = 0$
if and only if $\theta = \pi/2$
if and only if $\dot{\gamma}(t_0)$ makes angle $\pi/2$ with $\mathbf{t}_1$
if and only if $\dot{\gamma}(t_0)$ makes angle 0 with $\mathbf{t}_2$
i.e. $\dot{\gamma}(t_0)$ is aligned with $\mathbf{t}_2$

Therefore,

Proposition. *κ_2 is the maximum possible normal curvature of a curve at that point.*

Exercise. κ_1 is the minimum possible normal curvature of a curve at that point.

This exercise can be worked out in exactly the same way

$\kappa_n = c_1^2\kappa_1 + c_2^2\kappa_2$
where, $\dot{\gamma}(t_0) = c_1\mathbf{t}_1 + c_2\mathbf{t}_2$

We can always choose $\mathbf{t}_1$ and $\mathbf{t}_2$ so that,
$\dot{\gamma}(t_0) = \cos(\theta)\mathbf{t}_1 + \sin(\theta)\mathbf{t}_2$

Therefore,

$$\begin{aligned}\kappa_n &= \cos^2(\theta)\kappa_1 + \sin^2(\theta)\kappa_2\\ &= \cos^2(\theta)\kappa_1 + (1-\cos^2(\theta))\kappa_2\\ &= \kappa_2 + (\kappa_1 - \kappa_2)\cos^2(\theta)\end{aligned}$$

Assume $\kappa_1 \leq \kappa_2$,

$$\kappa_n = \kappa_2 + \text{some negative number}$$

So, $\kappa_n \leq \kappa_2$.

$\kappa_n = \kappa_2$ if and only if $\cos^2(\theta) = 0$
if and only if $\theta = \pi/2$
if and only if $\dot{\gamma}(t_0)$ makes angle $\pi/2$ with $\mathbf{t}_1$
if and only if $\dot{\gamma}(t_0)$ makes angle 0 with $\mathbf{t}_2$
i.e. $\dot{\gamma}(t_0)$ is aligned with $\mathbf{t}_2$

Therefore,

Proposition. *κ_2 is the maximum possible normal curvature of a curve at that point. $\mathbf{t}_2$ is the direction along which the normal curvature is maximum.*

Exercise. κ_1 is the minimum possible normal curvature of a curve at that point.

We can even give a geometric intepretation to t_1 and t_2

$\kappa_n = c_1^2\kappa_1 + c_2^2\kappa_2$
where, $\dot{\gamma}(t_0) = c_1\mathbf{t}_1 + c_2\mathbf{t}_2$

We can always choose $\mathbf{t}_1$ and $\mathbf{t}_2$ so that,
$\dot{\gamma}(t_0) = \cos(\theta)\mathbf{t}_1 + \sin(\theta)\mathbf{t}_2$

Therefore,

$$\begin{aligned}\kappa_n &= \cos^2(\theta)\kappa_1 + \sin^2(\theta)\kappa_2 \\ &= \cos^2(\theta)\kappa_1 + (1-\cos^2(\theta))\kappa_2 \\ &= \kappa_2 + (\kappa_1 - \kappa_2)\cos^2(\theta)\end{aligned}$$

Assume $\kappa_1 \leq \kappa_2$,

$$\kappa_n = \kappa_2 + \text{some negative number}$$

So, $\kappa_n \leq \kappa_2$.

$\kappa_n = \kappa_2$ if and only if $\cos^2(\theta) = 0$
if and only if $\theta = \pi/2$
if and only if $\dot{\gamma}(t_0)$ makes angle $\pi/2$ with $\mathbf{t}_1$
if and only if $\dot{\gamma}(t_0)$ makes angle 0 with $\mathbf{t}_2$
i.e. $\dot{\gamma}(t_0)$ is aligned with $\mathbf{t}_2$

Therefore,

Proposition. *κ_2 is the maximum possible normal curvature of a curve at that point. $\mathbf{t}_2$ is the direction along which the normal curvature is maximum.*

Exercise. κ_1 is the minimum possible normal curvature of a curve at that point. $\mathbf{t}_1$ is the direction along which the normal curvature is minimum.

κ_1 and κ_2 are called the **principal** curvatures
$\mathbf{t}_1$ and $\mathbf{t}_2$ are called the **principal** directions

$\blacksquare$