Remember that replacement rules characterize the operators

A very useful observation is that we only need the replacement rules and Modus Ponens

The other inference rules can be derived using only these

Notation: $p \vdash_* q$ if q follows from p using only Modus-Ponens and replacement rules

But first let us use this notation for a proof that involves only Modus Ponens and replacement rules (no inference ru

Notation: $p \vdash_* q$ if q follows from p using only Modus-Ponens and replacement rules

Theorem 1 $p \vdash q$ *iff* $p \vdash_* q$

Using this notation we can state our theorem as above

Notation: $p \vdash_* q$ if q follows from p using only Modus-Ponens and replacement rules

Theorem 1 $p \vdash q$ *iff* $p \vdash_* q$

Proof.

The proof will involve showing that each of the inference rules can be inferred from the replacement rules and Modu

Notation: $p \vdash_* q$ if q follows from p using only Modus-Ponens and replacement rules

Theorem 1 $p \vdash q$ *iff* $p \vdash_* q$

Proof.

Lemma 2 $p \rightarrow q,\ \neg q\ \vdash_* \neg p$

We will take one example and leave the rest as exercises

Notation: $p \vdash_* q$ if q follows from p using only Modus-Ponens and replacement rules

Theorem 1 $p \vdash q$ *iff* $p \vdash_* q$

Proof.

Lemma 2 $p \to q, \neg q \ \vdash_* \neg p$

From now on I will shift to the following notation for proofs

Notation: $p \vdash_* q$ if q follows from p using only Modus-Ponens and replacement rules

Theorem 1$p \vdash q$ *iff* $p \vdash_* q$

Proof.

Lemma 2 $p \to q$, $\neg q \ \vdash_* \neg p$

First we state the two hypotheses

Notation: $p \vdash_* q$ if q follows from p using only Modus-Ponens and replacement rules

Theorem 1$p \vdash q$ *iff* $p \vdash_* q$

Proof.

Lemma 2 $p \to q, \neg q \ \vdash_* \neg p$

$$
\begin{array}{r|l}
0 & p \to q \text{ (Given)}
\end{array}
$$

which is this

Notation: $p \vdash_* q$ if q follows from p using only Modus-Ponens and replacement rules

Theorem 1$p \vdash q$ *iff* $p \vdash_* q$

Proof.

Lemma 2 $p \to q,\ \neg q\ \vdash_* \neg p$

0	$p \to q$ (Given)
1	$\neg q$ (Given)

and this

Notation: $p \vdash_* q$ if q follows from p using only Modus-Ponens and replacement rules

Theorem 1$p \vdash q$ *iff* $p \vdash_* q$

Proof.

Lemma 2 $p \rightarrow q$, $\neg q \ \vdash_* \neg p$

0	$p \rightarrow q$ (Given)
1	$\neg q$ (Given)
2	$\neg\neg q \rightarrow \neg\neg p$ (Double negation)

Notice here that the replacement rule is applied to sub expressions.

Notation: $p \vdash_* q$ if q follows from p using only Modus-Ponens and replacement rules

Theorem 1 $p \vdash q$ *iff* $p \vdash_* q$

Proof.

Lemma 2 $p \to q, \neg q \ \vdash_* \neg p$

0	$p \to q$ (Given)
1	$\neg q$ (Given)
2	$\neg\neg q \to \neg\neg p$ (Double negation)
3	$\neg\neg q \to \neg\neg p \to (\neg p \to \neg q)$ (FL3)

Now this is a perfect application of axiom FL3

Notation: $p \vdash_* q$ if q follows from p using only Modus-Ponens and replacement rules

Theorem 1$p \vdash q$ *iff* $p \vdash_* q$

Proof.

Lemma 2 $p \to q, \neg q \ \vdash_* \neg p$

0	$p \to q$ (Given)
1	$\neg q$ (Given)
2	$\neg\neg q \to \neg\neg p$ (Double negation)
3	$\neg\neg q \to \neg\neg p \to (\neg p \to \neg q)$ (FL3)
4	$\neg p \to \neg q$ (MP on 2 & 3)

Applying Modus Ponens on it gives an expression

Notation: $p \vdash_* q$ if q follows from p using only Modus-Ponens and replacement rules

Theorem 1$p \vdash q$ *iff* $p \vdash_* q$

Proof.

Lemma 2 $p \to q,\ \neg q\ \ \vdash_* \neg p$

0	$p \to q$ (Given)
1	$\neg q$ (Given)
2	$\neg\neg q \to \neg\neg p$ (Double negation)
3	$\neg\neg q \to \neg\neg p \to (\neg p \to \neg q)$ (FL3)
4	$\neg p \to \neg q$ (MP on 2 & 3)
5	$\neg q$ (MP on 1 & 4)

$\square$

on which we can apply Modus Ponens again to get the conclusion

Notation: $p \vdash_* q$ if q follows from p using only Modus-Ponens and replacement rules

Theorem 1 $p \vdash q$ *iff* $p \vdash_* q$

Proof.

Lemma 2 $p \to q, \neg q \ \vdash_* \neg p$

0	$p \to q$ (Given)
1	$\neg q$ (Given)
2	$\neg\neg q \to \neg\neg p$ (Double negation)
3	$\neg\neg q \to \neg\neg p \to (\neg p \to \neg q)$ (FL3)
4	$\neg p \to \neg q$ (MP on 2 & 3)
5	$\neg q$ (MP on 1 & 4)

$\square$

The other inference rules can also be proved similarly

Notation: $p \vdash_* q$ if q follows from p using only Modus-Ponens and replacement rules

Theorem 1$p \vdash q$ *iff* $p \vdash_* q$

Proof.

Lemma 2 $p \rightarrow q,\ \neg q\ \vdash_* \neg p$

0	$p \rightarrow q$ (Given)
1	$\neg q$ (Given)
2	$\neg\neg q \rightarrow \neg\neg p$ (Double negation)
3	$\neg\neg q \rightarrow \neg\neg p \rightarrow (\neg p \rightarrow \neg q)$ (FL3)
4	$\neg p \rightarrow \neg q$ (MP on 2 & 3)
5	$\neg q$ (MP on 1 & 4)

$\square$

We now prove a lemma that will be useful later.

Notation: $p \vdash_* q$ if q follows from p using only Modus-Ponens and replacement rules

Theorem 1 $p \vdash q$ *iff* $p \vdash_* q$

Proof.

Lemma 2 $p \to q, \neg q \ \vdash_* \neg p$

0	$p \to q$ (Given)
1	$\neg q$ (Given)
2	$\neg\neg q \to \neg\neg p$ (Double negation)
3	$\neg\neg q \to \neg\neg p \to (\neg p \to \neg q)$ (FL3)
4	$\neg p \to \neg q$ (MP on 2 & 3)
5	$\neg q$ (MP on 1 & 4)

$\square$

It essentially proves that if a statement is true,

Notation: $p \vdash_* q$ if q follows from p using only Modus-Ponens and replacement rules

Theorem 1 $p \vdash q$ *iff* $p \vdash_* q$

Proof.

Lemma 2 $p \to q,\ \neg q\ \vdash_* \neg p$

0	$p \to q$ (Given)
1	$\neg q$ (Given)
2	$\neg\neg q \to \neg\neg p$ (Double negation)
3	$\neg\neg q \to \neg\neg p \to (\neg p \to \neg q)$ (FL3)
4	$\neg p \to \neg q$ (MP on 2 & 3)
5	$\neg q$ (MP on 1 & 4)

$\square$

it remains true even if you add some hypothesis

Notation: $p \vdash_* q$ if q follows from p using only Modus-Ponens and replacement rules

Theorem 1$p \vdash q$ *iff* $p \vdash_* q$

Proof.

Lemma 2 $p \to q, \neg q \ \vdash_* \neg p$

0	$p \to q$ (Given)
1	$\neg q$ (Given)
2	$\neg\neg q \to \neg\neg p$ (Double negation)
3	$\neg\neg q \to \neg\neg p \to (\neg p \to \neg q)$ (FL3)
4	$\neg p \to \neg q$ (MP on 2 & 3)
5	$\neg q$ (MP on 1 & 4)

$\square$

In terms of semantics, if q is true, then $p \to q$ is also true

Notation: $p \vdash_* q$ if q follows from p using only Modus-Ponens and replacement rules

Theorem 1 $p \vdash q$ *iff* $p \vdash_* q$

Proof.

Lemma 2 $p \to q, \neg q \ \vdash_* \neg p$

0	$p \to q$ (Given)
1	$\neg q$ (Given)
2	$\neg\neg q \to \neg\neg p$ (Double negation)
3	$\neg\neg q \to \neg\neg p \to (\neg p \to \neg q)$ (FL3)
4	$\neg p \to \neg q$ (MP on 2 & 3)
5	$\neg q$ (MP on 1 & 4)

$\square$

because if the right hand side of $\to$ is true, then the implication is true

Notation: $p \vdash_* q$ if q follows from p using only Modus-Ponens and replacement rules

Lemma 4 $\vdash_* q$

Theorem 1 $p \vdash q$ *iff* $p \vdash_* q$

Proof.

Lemma 2 $p \to q, \neg q \ \vdash_* \neg p$

0	$p \to q$ (Given)
1	$\neg q$ (Given)
2	$\neg\neg q \to \neg\neg p$ (Double negation)
3	$\neg\neg q \to \neg\neg p \to (\neg p \to \neg q)$ (FL3)
4	$\neg p \to \neg q$ (MP on 2 & 3)
5	$\neg q$ (MP on 1 & 4)

$\square$

If we have a proof of q

Notation: $p \vdash_* q$ if q follows from p using only Modus-Ponens and replacement rules

Lemma 4 $\vdash_* q$ *implies* $\vdash_* p \to q$

Theorem 1 $p \vdash q$ *iff* $p \vdash_* q$

Proof.

Lemma 2 $p \to q, \neg q \ \vdash_* \neg p$

0	$p \to q$ (Given)
1	$\neg q$ (Given)
2	$\neg\neg q \to \neg\neg p$ (Double negation)
3	$\neg\neg q \to \neg\neg p \to (\neg p \to \neg q)$ (FL3)
4	$\neg p \to \neg q$ (MP on 2 & 3)
5	$\neg q$ (MP on 1 & 4)

$\square$

we can modify it to a proof of $p \to q$

Notation: $p \vdash_* q$ if q follows from p using only Modus-Ponens and replacement rules

Theorem 1 $p \vdash q$ *iff* $p \vdash_* q$

Proof.

Lemma 2 $p \to q$, $\neg q \ \vdash_* \neg p$

0	$p \to q$ (Given)
1	$\neg q$ (Given)
2	$\neg\neg q \to \neg\neg p$ (Double negation)
3	$\neg\neg q \to \neg\neg p \to (\neg p \to \neg q)$ (FL3)
4	$\neg p \to \neg q$ (MP on 2 & 3)
5	$\neg q$ (MP on 1 & 4)

□

Lemma 4 $\vdash_* q$ *implies* $\vdash_* p \to q$

Proof.

Notation: $p \vdash_* q$ if q follows from p using only Modus-Ponens and replacement rules

Theorem 1 $p \vdash q$ *iff* $p \vdash_* q$

Proof.

Lemma 2 $p \rightarrow q$, $\neg q \ \vdash_* \neg p$

0	$p \rightarrow q$ (Given)
1	$\neg q$ (Given)
2	$\neg\neg q \rightarrow \neg\neg p$ (Double negation)
3	$\neg\neg q \rightarrow \neg\neg p \rightarrow (\neg p \rightarrow \neg q)$ (FL3)
4	$\neg p \rightarrow \neg q$ (MP on 2 & 3)
5	$\neg q$ (MP on 1 & 4)

□

Lemma 4 $\vdash_* q$ *implies* $\vdash_* p \rightarrow q$

Proof.

Notation: $p \vdash_* q$ if q follows from p using only Modus-Ponens and replacement rules

Theorem 1 $p \vdash q$ *iff* $p \vdash_* q$

Proof.

Lemma 2 $p \to q$, $\neg q \ \vdash_* \neg p$

0	$p \to q$ (Given)
1	$\neg q$ (Given)
2	$\neg\neg q \to \neg\neg p$ (Double negation)
3	$\neg\neg q \to \neg\neg p \to (\neg p \to \neg q)$ (FL3)
4	$\neg p \to \neg q$ (MP on 2 & 3)
5	$\neg q$ (MP on 1 & 4)

$\square$

Lemma 4 $\vdash_* q$ *implies* $\vdash_* p \to q$

Proof.

$n-1$	$\dots$ $(\dots)$

A proof of q would have many steps

Notation: $p \vdash_* q$ if q follows from p using only Modus-Ponens and replacement rules

Theorem 1 $p \vdash q$ *iff* $p \vdash_* q$

Proof.

Lemma 2 $p \to q$, $\neg q \ \vdash_* \neg p$

$$\begin{array}{r|l} 0 & p \to q \text{ (Given)} \\ 1 & \neg q \text{ (Given)} \\ 2 & \neg\neg q \to \neg\neg p \text{ (Double negation)} \\ 3 & \neg\neg q \to \neg\neg p \to (\neg p \to \neg q) \text{ (FL3)} \\ 4 & \neg p \to \neg q \text{ (MP on 2 \& 3)} \\ 5 & \neg q \text{ (MP on 1 \& 4)} \end{array}$$

□

Lemma 4 $\vdash_* q$ *implies* $\vdash_* p \to q$

Proof.

$$\begin{array}{r|l} n-1 & \ldots\ (\ldots) \\ n & q\ (\ldots) \end{array}$$

of which the last line would be q

Notation: $p \vdash_* q$ if q follows from p using only Modus-Ponens and replacement rules

Theorem 1 $p \vdash q$ *iff* $p \vdash_* q$

Proof.

Lemma 2 $p \to q$, $\neg q \vdash_* \neg p$

0	$p \to q$ (Given)
1	$\neg q$ (Given)
2	$\neg\neg q \to \neg\neg p$ (Double negation)
3	$\neg\neg q \to \neg\neg p \to (\neg p \to \neg q)$ (FL3)
4	$\neg p \to \neg q$ (MP on 2 & 3)
5	$\neg q$ (MP on 1 & 4)

□

Lemma 4 $\vdash_* q$ *implies* $\vdash_* p \to q$

Proof.

$n-1$	$\ldots (\ldots)$
n	$q (\ldots)$
$n+1$	$q \to (p \to q)$ (FL1)

Now we simply use FL1

Notation: $p \vdash_* q$ if q follows from p using only Modus-Ponens and replacement rules

Theorem 1 $p \vdash q$ *iff* $p \vdash_* q$

Proof.

Lemma 2 $p \to q, \neg q \ \vdash_* \neg p$

0	$p \to q$ (Given)
1	$\neg q$ (Given)
2	$\neg\neg q \to \neg\neg p$ (Double negation)
3	$\neg\neg q \to \neg\neg p \to (\neg p \to \neg q)$ (FL3)
4	$\neg p \to \neg q$ (MP on 2 & 3)
5	$\neg q$ (MP on 1 & 4)

□

Lemma 4 $\vdash_* q$ *implies* $\vdash_* p \to q$

Proof.

$n-1$	$\ldots$ $(\ldots)$
n	q $(\ldots)$
$n+1$	$q \to (p \to q)$ (FL1)
$n+2$	$p \to q$ (MP on n & $n+1$)

to give what we need using Modus Ponens

Notation: $p \vdash_* q$ if q follows from p using only Modus-Ponens and replacement rules

Theorem 1$p \vdash q$ *iff* $p \vdash_* q$

Proof.

Lemma 2 $p \to q$, $\neg q \ \vdash_* \neg p$

0	$p \to q$ (Given)
1	$\neg q$ (Given)
2	$\neg\neg q \to \neg\neg p$ (Double negation)
3	$\neg\neg q \to \neg\neg p \to (\neg p \to \neg q)$ (FL3)
4	$\neg p \to \neg q$ (MP on 2 & 3)
5	$\neg q$ (MP on 1 & 4)

Lemma 4 $\vdash_* q$ *implies* $\vdash_* p \to q$

Proof.

$n-1$	$\ldots\ (\ldots)$
n	$q\ (\ldots)$
$n+1$	$q \to (p \to q)$ (FL1)
$n+2$	$p \to q$ (MP on n & $n+1$)

So, assuming a proof of q,

□

Lemma 4 $\vdash_* q$ *implies* $\vdash_* p \to q$

Proof.

$$\begin{array}{r|l} n-1 & \ldots\ (\ldots) \\ n & q\ (\ldots) \\ n+1 & q \to (p \to q)\ \text{(FL1)} \\ n+2 & p \to q\ (\text{MP on } n \ \&\ n+1) \end{array}$$

So, assuming a proof of q, we can extend it to a proof of $p \to q$ by FL1 and Modus Ponens $\square$

Theorem 3 (Direct proof)

Lemma 4 $\vdash_* q$ *implies* $\vdash_* p \rightarrow q$

Proof.

$$\begin{array}{r|l} n-1 & \ldots\ (\ldots) \\ n & q\ (\ldots) \\ n+1 & q \rightarrow (p \rightarrow q)\ \text{(FL1)} \\ n+2 & p \rightarrow q\ (\text{MP on } n\ \&\ n+1) \end{array}$$

So, assuming a proof of q, we can extend it to a proof of $p \rightarrow q$ by FL1 and Modus Ponens $\square$

We are now going to prove something that we use without thinking

Theorem 3 (Direct proof)

Lemma 4 $\vdash_* q$ *implies* $\vdash_* p \to q$

Proof.

$$\begin{array}{r|l} n-1 & \ldots\ (\ldots) \\ n & q\ (\ldots) \\ n+1 & q \to (p \to q)\ \text{(FL1)} \\ n+2 & p \to q\ (\text{MP on } n \ \&\ n+1) \end{array}$$

So, assuming a proof of q, we can extend it to a proof of $p \to q$ by FL1 and Modus Ponens □

because it seems so obvious.

Theorem 3 (Direct proof) $p \vdash_* q$

Lemma 4 $\vdash_* q$ *implies* $\vdash_* p \to q$

Proof.

$$\begin{array}{r|l} n-1 & \ldots\ (\ldots) \\ n & q\ (\ldots) \\ n+1 & q \to (p \to q)\ \text{(FL1)} \\ n+2 & p \to q\ (\text{MP on } n \ \&\ n+1) \end{array}$$

So, assuming a proof of q, we can extend it to a proof of $p \to q$ by FL1 and Modus Ponens □

If we can deduce q from p

Theorem 3 (Direct proof) $p \vdash_* q$ *implies* $\vdash_* p \to q$

Lemma 4 $\vdash_* q$ *implies* $\vdash_* p \to q$

Proof.

$$\begin{array}{r|l} n-1 & \ldots\ (\ldots) \\ n & q\ (\ldots) \\ n+1 & q \to (p \to q)\ \text{(FL1)} \\ n+2 & p \to q\ \text{(MP on } n \text{ \& } n+1\text{)} \end{array}$$

So, assuming a proof of q, we can extend it to a proof of $p \to q$ by FL1 and Modus Ponens □

Then we have essentially proved that $p \to q$

Theorem 3 (Direct proof) $p \vdash_* q$ *implies* $\vdash_* p \rightarrow q$

Lemma 4 $\vdash_* q$ *implies* $\vdash_* p \rightarrow q$

Proof.

$$\begin{array}{r|l} n-1 & \ldots\ (\ldots) \\ n & q\ (\ldots) \\ n+1 & q \rightarrow (p \rightarrow q)\ \text{(FL1)} \\ n+2 & p \rightarrow q\ \text{(MP on } n \text{ \& } n+1\text{)} \end{array}$$

So, assuming a proof of q, we can extend it to a proof of $p \rightarrow q$ by FL1 and Modus Ponens □

Observe that while a proof of q given p sounds like $p \rightarrow q$, the two are very different

Theorem 3 (Direct proof) $p \vdash_* q$ *implies* $\vdash_* p \to q$

Lemma 4 $\vdash_* q$ *implies* $\vdash_* p \to q$

Proof.

$$\begin{array}{r|l} n-1 & \ldots\ (\ldots) \\ n & q\ (\ldots) \\ n+1 & q \to (p \to q)\ \text{(FL1)} \\ n+2 & p \to q\ \text{(MP on } n \text{ \& } n+1\text{)} \end{array}$$

So, assuming a proof of q, we can extend it to a proof of $p \to q$ by FL1 and Modus Ponens □

A proof is a sequence of statements following some rules

Theorem 3 (Direct proof) $p \vdash_* q$ *implies* $\vdash_* p \to q$

Lemma 4 $\vdash_* q$ *implies* $\vdash_* p \to q$

Proof.

$$
\begin{array}{r|l}
n-1 & \ldots\ (\ldots) \\
n & q\ (\ldots) \\
n+1 & q \to (p \to q)\ \text{(FL1)} \\
n+2 & p \to q\ \text{(MP on } n \text{ \& } n+1\text{)}
\end{array}
$$

So, assuming a proof of q, we can extend it to a proof of $p \to q$ by FL1 and Modus Ponens □

On the other hand, $\implies$ is an operator within our language of Propositional Logic

Theorem 3 (Direct proof) $p \vdash_* q$ *implies* $\vdash_* p \to q$

Lemma 4 $\vdash_* q$ *implies* $\vdash_* p \to q$

Proof.

$$\begin{array}{r|l} n-1 & \ldots\ (\ldots) \\ n & q\ (\ldots) \\ n+1 & q \to (p \to q)\ \text{(FL1)} \\ n+2 & p \to q\ \text{(MP on } n \text{ \& } n+1\text{)} \end{array}$$

So, assuming a proof of q, we can extend it to a proof of $p \to q$ by FL1 and Modus Ponens □

which can be a statement in a proof

Theorem 3 (Direct proof) $p \vdash_* q$ *implies* $\vdash_* p \to q$

Lemma 4 $\vdash_* q$ *implies* $\vdash_* p \to q$

Proof.

$$\begin{array}{r|l} n-1 & \dots \ (\dots) \\ n & q \ (\dots) \\ n+1 & q \to (p \to q) \text{ (FL1)} \\ n+2 & p \to q \text{ (MP on } n \text{ \& } n+1\text{)} \end{array}$$

So, assuming a proof of q, we can extend it to a proof of $p \to q$ by FL1 and Modus Ponens □

But it is not yet clear why a proof that q *follows* from p

Theorem 3 (Direct proof) $p \vdash_* q$ *implies* $\vdash_* p \to q$

Lemma 4 $\vdash_* q$ *implies* $\vdash_* p \to q$

Proof.

$$\begin{array}{r|l} n-1 & \ldots\ (\ldots) \\ n & q\ (\ldots) \\ n+1 & q \to (p \to q)\ \text{(FL1)} \\ n+2 & p \to q\ (\text{MP on } n\ \&\ n+1) \end{array}$$

So, assuming a proof of q, we can extend it to a proof of $p \to q$ by FL1 and Modus Ponens $\square$

also means that q is *implied* by p

Theorem 3 (Direct proof) $p \vdash_* q$ *implies* $\vdash_* p \to q$

Lemma 4 $\vdash_* q$ *implies* $\vdash_* p \to q$

Proof.

$$\begin{array}{r|l} n-1 & \ldots\ (\ldots) \\ n & q\ (\ldots) \\ n+1 & q \to (p \to q)\ \text{(FL1)} \\ n+2 & p \to q\ \text{(MP on } n \text{ \& } n+1\text{)} \end{array}$$

So, assuming a proof of q, we can extend it to a proof of $p \to q$ by FL1 and Modus Ponens □

But it ought to be true if our idea of a formal proof makes sense

Theorem 3 (Direct proof) $p \vdash_* q$ *implies* $\vdash_* p \to q$

Lemma 4 $\vdash_* q$ *implies* $\vdash_* p \to q$

Proof.

$$\begin{array}{r|l} n-1 & \ldots\ (\ldots) \\ n & q\ (\ldots) \\ n+1 & q \to (p \to q)\ \text{(FL1)} \\ n+2 & p \to q\ \text{(MP on } n \text{ \& } n+1\text{)} \end{array}$$

So, assuming a proof of q, we can extend it to a proof of $p \to q$ by FL1 and Modus Ponens $\square$

That it does follow from the axioms and rules

Theorem 3 (Direct proof) $p \vdash_* q$ *implies* $\vdash_* p \to q$

Lemma 4 $\vdash_* q$ *implies* $\vdash_* p \to q$

Proof.

$$\begin{array}{r|l} n-1 & \ldots\ (\ldots) \\ n & q\ (\ldots) \\ n+1 & q \to (p \to q)\ \text{(FL1)} \\ n+2 & p \to q\ \text{(MP on } n \text{ \& } n+1\text{)} \end{array}$$

So, assuming a proof of q, we can extend it to a proof of $p \to q$ by FL1 and Modus Ponens □

adds further weight to our axioms and rules.

Theorem 3 (Direct proof) $p \vdash_* q$ *implies* $\vdash_* p \to q$

Proof Assume true for proofs of length $\leq k$

Lemma 4 $\vdash_* q$ *implies* $\vdash_* p \to q$

Proof.

$$\begin{array}{r|l} n-1 & \ldots\ (\ldots) \\ n & q\ (\ldots) \\ n+1 & q \to (p \to q)\ \text{(FL1)} \\ n+2 & p \to q\ \text{(MP on } n \text{ \& } n+1\text{)} \end{array}$$

So, assuming a proof of q, we can extend it to a proof of $p \to q$ by FL1 and Modus Ponens □

We will use induction.

Theorem 3 (Direct proof) $p \vdash_* q$ *implies* $\vdash_* p \to q$

Proof Assume true for proofs of length $\leq k$

Lemma 4 $\vdash_* q$ *implies* $\vdash_* p \to q$

Proof.

$$\begin{array}{r|l} n-1 & \ldots\ (\ldots) \\ n & q\ (\ldots) \\ n+1 & q \to (p \to q)\ \text{(FL1)} \\ n+2 & p \to q\ (\text{MP on } n\ \&\ n+1) \end{array}$$

So, assuming a proof of q, we can extend it to a proof of $p \to q$ by FL1 and Modus Ponens □

Or rather we will use a very specific version of induction

Theorem 3 (Direct proof) $p \vdash_* q$ *implies* $\vdash_* p \to q$

Proof Assume true for proofs of length $\leq k$

Lemma 4 $\vdash_* q$ *implies* $\vdash_* p \to q$

Proof.

$$
\begin{array}{r|l}
n-1 & \ldots\ (\ldots) \\
n & q\ (\ldots) \\
n+1 & q \to (p \to q)\ \text{(FL1)} \\
n+2 & p \to q\ \text{(MP on } n \text{ \& } n+1\text{)}
\end{array}
$$

So, assuming a proof of q, we can extend it to a proof of $p \to q$ by FL1 and Modus Ponens □

because one can rightfully argue that induction needs justification

Theorem 3 (Direct proof) $p \vdash_* q$ *implies* $\vdash_* p \to q$

Proof Assume true for proofs of length $\leq k$

Lemma 4 $\vdash_* q$ *implies* $\vdash_* p \to q$

Proof.

$$\begin{array}{r|l} n-1 & \ldots\ (\ldots) \\ n & q\ (\ldots) \\ n+1 & q \to (p \to q)\ \text{(FL1)} \\ n+2 & p \to q\ (\text{MP on } n\ \&\ n+1) \end{array}$$

So, assuming a proof of q, we can extend it to a proof of $p \to q$ by FL1 and Modus Ponens □

However, all we will prove is that we can reduce the proof for proof with k steps

Theorem 3 (Direct proof) $p \vdash_* q$ *implies* $\vdash_* p \to q$

Proof Assume true for proofs of length $\leq k$

Lemma 4 $\vdash_* q$ *implies* $\vdash_* p \to q$

Proof.

$$\begin{array}{r|l} n-1 & \ldots\ (\ldots) \\ n & q\ (\ldots) \\ n+1 & q \to (p \to q)\ \text{(FL1)} \\ n+2 & p \to q\ (\text{MP on } n\ \&\ n+1) \end{array}$$

So, assuming a proof of q, we can extend it to a proof of $p \to q$ by FL1 and Modus Ponens □

to those with $k-1$ steps

Theorem 3 (Direct proof) $p \vdash_* q$ *implies* $\vdash_* p \to q$

Proof Assume true for proofs of length $\leq k$

Lemma 4 $\vdash_* q$ *implies* $\vdash_* p \to q$

Proof.

$$\begin{array}{r|l} n-1 & \ldots\ (\ldots) \\ n & q\ (\ldots) \\ n+1 & q \to (p \to q)\ \text{(FL1)} \\ n+2 & p \to q\ (\text{MP on } n\ \&\ n+1) \end{array}$$

So, assuming a proof of q, we can extend it to a proof of $p \to q$ by FL1 and Modus Ponens □

Right now, do not worry about the justification of induction

Theorem 3 (Direct proof) $p \vdash_* q$ *implies* $\vdash_* p \to q$

Proof Assume true for proofs of length $\leq k$

Lemma 4 $\vdash_* q$ *implies* $\vdash_* p \to q$

Proof.

$$\begin{array}{r|l} n-1 & \dots\ (\dots) \\ n & q\ (\dots) \\ n+1 & q \to (p \to q)\ \text{(FL1)} \\ n+2 & p \to q\ (\text{MP on } n\ \&\ n+1) \end{array}$$

So, assuming a proof of q, we can extend it to a proof of $p \to q$ by FL1 and Modus Ponens □

and focus on how it can be used to prove what we want

Theorem 3 (Direct proof) $p \vdash_* q$ *implies* $\vdash_* p \to q$

Proof Assume true for proofs of length $\leq k$

Case 1: Concluded q using Modus Ponens

Lemma 4 $\vdash_* q$ *implies* $\vdash_* p \to q$

Proof.

$$\begin{array}{r|l} n-1 & \ldots\ (\ldots) \\ n & q\ (\ldots) \\ n+1 & q \to (p \to q)\ \text{(FL1)} \\ n+2 & p \to q\ (\text{MP on } n\ \&\ n+1) \end{array}$$

So, assuming a proof of q, we can extend it to a proof of $p \to q$ by FL1 and Modus Ponens □

Let us first consider the case where in the proof of $p \vdash q$,

Theorem 3 (Direct proof) $p \vdash_* q$ *implies* $\vdash_* p \to q$

Proof Assume true for proofs of length $\leq k$

Case 1: Concluded q using Modus Ponens

Lemma 4 $\vdash_* q$ *implies* $\vdash_* p \to q$

Proof.

$$\begin{array}{r|l} n-1 & \ldots \ (\ldots) \\ n & q \ (\ldots) \\ n+1 & q \to (p \to q) \text{ (FL1)} \\ n+2 & p \to q \text{ (MP on } n \text{ \& } n+1\text{)} \end{array}$$

So, assuming a proof of q, we can extend it to a proof of $p \to q$ by FL1 and Modus Ponens $\square$

the last statement, i.e. q, follows from Modus Ponens

Theorem 3 (Direct proof) $p \vdash_* q$ *implies* $\vdash_* p \to q$

Proof Assume true for proofs of length $\leq k$

Case 1: Concluded q using Modus Ponens

Lemma 4 $\vdash_* q$ *implies* $\vdash_* p \to q$

Proof.

$$\begin{array}{r|l} n-1 & \ldots\ (\ldots) \\ n & q\ (\ldots) \\ n+1 & q \to (p \to q)\ \text{(FL1)} \\ n+2 & p \to q\ (\text{MP on } n \ \&\ n+1) \end{array}$$

So, assuming a proof of q, we can extend it to a proof of $p \to q$ by FL1 and Modus Ponens □

(rather than a replacement rule)

Theorem 3 (Direct proof) $p \vdash_* q$ *implies* $\vdash_* p \to q$

Proof Assume true for proofs of length $\leq k$

Case 1: Concluded q using Modus Ponens

Lemma 4 $\vdash_* q$ *implies* $\vdash_* p \to q$

Proof.

$$
\begin{array}{r|l}
n-1 & \ldots\ (\ldots) \\
n & q\ (\ldots) \\
n+1 & q \to (p \to q)\ \text{(FL1)} \\
n+2 & p \to q\ \text{(MP on } n \text{ \& } n+1\text{)}
\end{array}
$$

So, assuming a proof of q, we can extend it to a proof of $p \to q$ by FL1 and Modus Ponens □

Theorem 3 (Direct proof) $p \vdash_* q$ *implies* $\vdash_* p \to q$

Proof Assume true for proofs of length $\leq k$

Case 1: Concluded q using Modus Ponens

0	p (Given)

Lemma 4 $\vdash_* q$ *implies* $\vdash_* p \to q$

Proof.

$n-1$	$\ldots$ $(\ldots)$
n	q $(\ldots)$
$n+1$	$q \to (p \to q)$ (FL1)
$n+2$	$p \to q$ (MP on n & $n+1$)

So, assuming a proof of q, we can extend it to a proof of $p \to q$ by FL1 and Modus Ponens □

So we have some proof of q beginning with the given p

Theorem 3 (Direct proof) $p \vdash_* q$ *implies* $\vdash_* p \to q$

Proof Assume true for proofs of length $\leq k$

Case 1: Concluded q using Modus Ponens

$$\begin{array}{r|l} 0 & p \text{ (Given)} \\ \dots & \dots \ (\dots) \\ & \\ \dots & \dots \ (\dots) \\ & \\ \dots & \dots \ (\dots) \end{array}$$

Lemma 4 $\vdash_* q$ *implies* $\vdash_* p \to q$

Proof.

$$\begin{array}{r|l} n-1 & \dots \ (\dots) \\ n & q \ (\dots) \\ n+1 & q \to (p \to q) \text{ (FL1)} \\ n+2 & p \to q \text{ (MP on } n \text{ \& } n+1) \end{array}$$

So, assuming a proof of q, we can extend it to a proof of $p \to q$ by FL1 and Modus Ponens $\square$

Now we have some other intermediate steps

Theorem 3 (Direct proof) $p \vdash_* q$ *implies* $\vdash_* p \to q$

Proof Assume true for proofs of length $\leq k$

Case 1: Concluded q using Modus Ponens

$$\begin{array}{r|l} 0 & p \text{ (Given)} \\ \dots & \dots\ (\dots) \\ & \\ \dots & \dots\ (\dots) \\ & \\ \dots & \dots\ (\dots) \\ k & q \end{array}$$

Lemma 4 $\vdash_* q$ *implies* $\vdash_* p \to q$

Proof.

$$\begin{array}{r|l} n-1 & \dots\ (\dots) \\ n & q\ (\dots) \\ n+1 & q \to (p \to q) \text{ (FL1)} \\ n+2 & p \to q \text{ (MP on } n \text{ \& } n+1\text{)} \end{array}$$

So, assuming a proof of q, we can extend it to a proof of $p \to q$ by FL1 and Modus Ponens $\square$

Eventually proving q

Theorem 3 (Direct proof) $p \vdash_* q$ *implies* $\vdash_* p \to q$

Proof Assume true for proofs of length $\leq k$

Case 1: Concluded q using Modus Ponens

$$\begin{array}{r|l} 0 & p \text{ (Given)} \\ \ldots & \ldots (\ldots) \\ & \\ \ldots & \ldots (\ldots) \\ & \\ \ldots & \ldots (\ldots) \\ k & q \end{array}$$

Lemma 4 $\vdash_* q$ *implies* $\vdash_* p \to q$

Proof.

$$\begin{array}{r|l} n-1 & \ldots (\ldots) \\ n & q (\ldots) \\ n+1 & q \to (p \to q) \text{ (FL1)} \\ n+2 & p \to q \text{ (MP on } n \text{ \& } n+1) \end{array}$$

So, assuming a proof of q, we can extend it to a proof of $p \to q$ by FL1 and Modus Ponens □

Since we are assuming in this case that it follows from Modus Ponens

Theorem 3 (Direct proof) $p \vdash_* q$ *implies* $\vdash_* p \to q$

Proof Assume true for proofs of length $\leq k$

Case 1: Concluded q using Modus Ponens

$$\begin{array}{r|l} 0 & p \text{ (Given)} \\ \ldots & \ldots\ (\ldots) \\ & \\ \ldots & \ldots\ (\ldots) \\ & \\ \ldots & \ldots\ (\ldots) \\ k & q \end{array}$$

Lemma 4 $\vdash_* q$ *implies* $\vdash_* p \to q$

Proof.

$$\begin{array}{r|l} n-1 & \ldots\ (\ldots) \\ n & q\ (\ldots) \\ n+1 & q \to (p \to q) \text{ (FL1)} \\ n+2 & p \to q \text{ (MP on } n \text{ \& } n+1) \end{array}$$

So, assuming a proof of q, we can extend it to a proof of $p \to q$ by FL1 and Modus Ponens □

there will be some intermediate steps that Modus Ponens is used on

Theorem 3 (Direct proof) $p \vdash_* q$ *implies* $\vdash_* p \to q$

Proof Assume true for proofs of length $\leq k$

Case 1: Concluded q using Modus Ponens

$$\begin{array}{r|l} 0 & p \text{ (Given)} \\ \dots & \dots\ (\dots) \end{array}$$

$$\begin{array}{r|l} \dots & \dots\ (\dots) \\ n & r \to q\ (\dots) \\ \dots & \dots\ (\dots) \\ k & q \end{array}$$

Lemma 4 $\vdash_* q$ *implies* $\vdash_* p \to q$

Proof.

$$\begin{array}{r|l} n-1 & \dots\ (\dots) \\ n & q\ (\dots) \\ n+1 & q \to (p \to q) \text{ (FL1)} \\ n+2 & p \to q \text{ (MP on } n \text{ \& } n+1) \end{array}$$

So, assuming a proof of q, we can extend it to a proof of $p \to q$ by FL1 and Modus Ponens □

One of those steps, say the nth, will be of the form $r \to q$ for some q

Theorem 3 (Direct proof) $p \vdash_* q$ *implies* $\vdash_* p \to q$

Proof Assume true for proofs of length $\leq k$

Case 1: Concluded q using Modus Ponens

$$\begin{array}{r|l}
0 & p \text{ (Given)} \\
\ldots & \ldots\ (\ldots) \\
m & r\ (\ldots) \\
\ldots & \ldots\ (\ldots) \\
n & r \to q\ (\ldots) \\
\ldots & \ldots\ (\ldots) \\
k & q
\end{array}$$

Lemma 4 $\vdash_* q$ *implies* $\vdash_* p \to q$

Proof.

$$\begin{array}{r|l}
n-1 & \ldots\ (\ldots) \\
n & q\ (\ldots) \\
n+1 & q \to (p \to q) \text{ (FL1)} \\
n+2 & p \to q \text{ (MP on } n \text{ \& } n+1\text{)}
\end{array}$$

So, assuming a proof of q, we can extend it to a proof of $p \to q$ by FL1 and Modus Ponens □

while the mth statement would be r

Theorem 3 (Direct proof) $p \vdash_* q$ *implies* $\vdash_* p \rightarrow q$

Proof Assume true for proofs of length $\leq k$

Case 1: Concluded q using Modus Ponens

0	p (Given)
$\ldots$	$\ldots$ $(\ldots)$
m	r $(\ldots)$
$\ldots$	$\ldots$ $(\ldots)$
n	$r \rightarrow q$ $(\ldots)$
$\ldots$	$\ldots$ $(\ldots)$
k	q (MP on m & n)

Lemma 4 $\vdash_* q$ *implies* $\vdash_* p \rightarrow q$

Proof.

$n-1$	$\ldots$ $(\ldots)$
n	q $(\ldots)$
$n+1$	$q \rightarrow (p \rightarrow q)$ (FL1)
$n+2$	$p \rightarrow q$ (MP on n & $n+1$)

So, assuming a proof of q, we can extend it to a proof of $p \rightarrow q$ by FL1 and Modus Ponens □

so that Modus Ponens can be applied to it.

Theorem 3 (Direct proof) $p \vdash_* q$ *implies* $\vdash_* p \to q$

Proof Assume true for proofs of length $\leq k$

Case 1: Concluded q using Modus Ponens

$$\begin{array}{r|l} 0 & p \text{ (Given)} \\ \dots & \dots \ (\dots) \\ m & r \ (\dots) \\ \dots & \dots \ (\dots) \\ n & r \to q \ (\dots) \\ \dots & \dots \ (\dots) \\ k & q \text{ (MP on } m \text{ \& } n) \end{array}$$

Lemma 4 $\vdash_* q$ *implies* $\vdash_* p \to q$

Proof.

$$\begin{array}{r|l} n-1 & \dots \ (\dots) \\ n & q \ (\dots) \\ n+1 & q \to (p \to q) \text{ (FL1)} \\ n+2 & p \to q \text{ (MP on } n \text{ \& } n+1) \end{array}$$

So, assuming a proof of q, we can extend it to a proof of $p \to q$ by FL1 and Modus Ponens □

Notice that the steps from p to r is fewer than k steps

Theorem 3 (Direct proof) $p \vdash_* q$ *implies* $\vdash_* p \to q$

Proof Assume true for proofs of length $\leq k$

Case 1: Concluded q using Modus Ponens

$$\begin{array}{r|l}
0 & p \text{ (Given)} \\
\dots & \dots\ (\dots) \\
m & r\ (\dots) \\
\dots & \dots\ (\dots) \\
n & r \to q\ (\dots) \\
\dots & \dots\ (\dots) \\
k & q \text{ (MP on } m \text{ \& } n) \\
k+1 & p \to r \text{ (Induction)}
\end{array}$$

Lemma 4 $\vdash_* q$ *implies* $\vdash_* p \to q$

Proof.

$$\begin{array}{r|l}
n-1 & \dots\ (\dots) \\
n & q\ (\dots) \\
n+1 & q \to (p \to q) \text{ (FL1)} \\
n+2 & p \to q \text{ (MP on } n \text{ \& } n+1)
\end{array}$$

So, assuming a proof of q, we can extend it to a proof of $p \to q$ by FL1 and Modus Ponens $\square$

So we can deduce this by induction

Theorem 3 (Direct proof) $p \vdash_* q$ *implies* $\vdash_* p \to q$

Proof Assume true for proofs of length $\le k$

Case 1: Concluded q using Modus Ponens

$$\begin{array}{r|l} 0 & p \text{ (Given)} \\ \dots & \dots\ (\dots) \\ m & r\ (\dots) \\ \dots & \dots\ (\dots) \\ n & r \to q\ (\dots) \\ \dots & \dots\ (\dots) \\ k & q \text{ (MP on } m \ \&\ n) \\ k+1 & p \to r \text{ (Induction)} \end{array}$$

Lemma 4 $\vdash_* q$ *implies* $\vdash_* p \to q$

Proof.

$$\begin{array}{r|l} n-1 & \dots\ (\dots) \\ n & q\ (\dots) \\ n+1 & q \to (p \to q) \text{ (FL1)} \\ n+2 & p \to q \text{ (MP on } n \ \&\ n+1) \end{array}$$

So, assuming a proof of q, we can extend it to a proof of $p \to q$ by FL1 and Modus Ponens □

Similarly notice that the steps from p to $r \to q$ is fewer than k steps

Theorem 3 (Direct proof) $p \vdash_* q$ *implies* $\vdash_* p \to q$

Proof Assume true for proofs of length $\leq k$

Case 1: Concluded q using Modus Ponens

0	p (Given)
$\dots$	$\dots$ $(\dots)$
m	r $(\dots)$
$\dots$	$\dots$ $(\dots)$
n	$r \to q$ $(\dots)$
$\dots$	$\dots$ $(\dots)$
k	q (MP on m & n)
$k+1$	$p \to r$ (Induction)
$k+2$	$p \to (r \to q)$ (Induction)

Lemma 4 $\vdash_* q$ *implies* $\vdash_* p \to q$

Proof.

$n-1$	$\dots$ $(\dots)$
n	q $(\dots)$
$n+1$	$q \to (p \to q)$ (FL1)
$n+2$	$p \to q$ (MP on n & $n+1$)

So, assuming a proof of q, we can extend it to a proof of $p \to q$ by FL1 and Modus Ponens □

So we have can also infer this

Theorem 3 (Direct proof) $p \vdash_* q$ *implies* $\vdash_* p \to q$

Proof Assume true for proofs of length $\leq k$

Case 1: Concluded q using Modus Ponens

$$\begin{array}{r|l}
0 & p \text{ (Given)} \\
\dots & \dots \ (\dots) \\
m & r \ (\dots) \\
\dots & \dots \ (\dots) \\
n & r \to q \ (\dots) \\
\dots & \dots \ (\dots) \\
k & q \text{ (MP on } m \text{ \& } n) \\
k+1 & p \to r \text{ (Induction)} \\
k+2 & p \to (r \to q) \text{ (Induction)} \\
k+3 & p \to (r \to q) \to (p \to r \to (p \to q)) \text{ (FL2)}
\end{array}$$

Lemma 4 $\vdash_* q$ *implies* $\vdash_* p \to q$

Proof.

$$\begin{array}{r|l}
n-1 & \dots \ (\dots) \\
n & q \ (\dots) \\
n+1 & q \to (p \to q) \text{ (FL1)} \\
n+2 & p \to q \text{ (MP on } n \text{ \& } n+1)
\end{array}$$

So, assuming a proof of q, we can extend it to a proof of $p \to q$ by FL1 and Modus Ponens □

Now we are ready to use FL3

Theorem 3 (Direct proof) $p \vdash_* q$ *implies* $\vdash_* p \to q$

Proof Assume true for proofs of length $\leq k$

Case 1: Concluded q using Modus Ponens

$$
\begin{array}{r|l}
0 & p \text{ (Given)} \\
\ldots & \ldots\ (\ldots) \\
m & r\ (\ldots) \\
\ldots & \ldots\ (\ldots) \\
n & r \to q\ (\ldots) \\
\ldots & \ldots\ (\ldots) \\
k & q \text{ (MP on } m \text{ \& } n) \\
k+1 & p \to r \text{ (Induction)} \\
k+2 & p \to (r \to q) \text{ (Induction)} \\
k+3 & p \to (r \to q) \to (p \to r \to (p \to q)) \text{ (FL2)} \\
k+4 & p \to r \to (p \to q) \text{ (MP on } k+1, k+3)
\end{array}
$$

Lemma 4 $\vdash_* q$ *implies* $\vdash_* p \to q$

Proof.

$$
\begin{array}{r|l}
n-1 & \ldots\ (\ldots) \\
n & q\ (\ldots) \\
n+1 & q \to (p \to q) \text{ (FL1)} \\
n+2 & p \to q \text{ (MP on } n \text{ \& } n+1)
\end{array}
$$

So, assuming a proof of q, we can extend it to a proof of $p \to q$ by FL1 and Modus Ponens $\square$

Notice that FL3 is designed to allow us to use Modus Ponens

Theorem 3 (Direct proof) $p \vdash_* q$ *implies* $\vdash_* p \to q$

Proof Assume true for proofs of length $\leq k$

Case 1: Concluded q using Modus Ponens

$$
\begin{array}{r|l}
0 & p \text{ (Given)} \\
\dots & \dots \ (\dots) \\
m & r \ (\dots) \\
\dots & \dots \ (\dots) \\
n & r \to q \ (\dots) \\
\dots & \dots \ (\dots) \\
k & q \text{ (MP on } m \text{ \& } n\text{)} \\
k+1 & p \to r \text{ (Induction)} \\
k+2 & p \to (r \to q) \text{ (Induction)} \\
k+3 & p \to (r \to q) \to (p \to r \to (p \to q)) \text{ (FL2)} \\
k+4 & p \to r \to (p \to q) \text{ (MP on } k+1, k+3\text{)}
\end{array}
$$

Lemma 4 $\vdash_* q$ *implies* $\vdash_* p \to q$

Proof.

$$
\begin{array}{r|l}
n-1 & \dots \ (\dots) \\
n & q \ (\dots) \\
n+1 & q \to (p \to q) \text{ (FL1)} \\
n+2 & p \to q \text{ (MP on } n \text{ \& } n+1\text{)}
\end{array}
$$

So, assuming a proof of q, we can extend it to a proof of $p \to q$ by FL1 and Modus Ponens $\square$

to give us $p \to r \to (p \to q)$

Theorem 3 (Direct proof) $p \vdash_* q$ *implies* $\vdash_* p \to q$

Proof Assume true for proofs of length $\leq k$

Case 1: Concluded q using Modus Ponens

$$
\begin{array}{r|l}
0 & p \text{ (Given)} \\
\dots & \dots (\dots) \\
m & r (\dots) \\
\dots & \dots (\dots) \\
n & r \to q (\dots) \\
\dots & \dots (\dots) \\
k & q \text{ (MP on } m \text{ \& } n) \\
k+1 & p \to r \text{ (Induction)} \\
k+2 & p \to (r \to q) \text{ (Induction)} \\
k+3 & p \to (r \to q) \to (p \to r \to (p \to q)) \text{ (FL2)} \\
k+4 & p \to r \to (p \to q) \text{ (MP on } k+1, k+3) \\
k+5 & p \to q \text{ (MP on } k+1, k+4)
\end{array}
$$

Lemma 4 $\vdash_* q$ *implies* $\vdash_* p \to q$

Proof.

$$
\begin{array}{r|l}
n-1 & \dots (\dots) \\
n & q (\dots) \\
n+1 & q \to (p \to q) \text{ (FL1)} \\
n+2 & p \to q \text{ (MP on } n \text{ \& } n+1)
\end{array}
$$

So, assuming a proof of q, we can extend it to a proof of $p \to q$ by FL1 and Modus Ponens □

Which in turn is precisely what Modus Ponens needs to give us what we seek