

Remember that replacement rules characterize the operators

A very useful observation is that we only need the replacement rules and Modus Ponens

The other inference rules can be derived using only these

Notation: $p \vdash_* q$ if q follows from p using only
Modus-Ponens and replacement rules

But first let us use this notation for a proof that involves only Modus Ponens and replacement rules (no inference ru

Notation: $p \vdash_* q$ if q follows from p using only
Modus-Ponens and replacement rules

Theorem $1p \vdash q$ iff $p \vdash_* q$

Using this notation we can state our theorem as above

Notation: $p \vdash_* q$ if q follows from p using only
Modus-Ponens and replacement rules

Theorem $p \vdash q$ iff $p \vdash_* q$

Proof.

The proof will involve showing that each of the inference rules can be inferred from the replacement rules and Modu

Notation: $p \vdash_* q$ if q follows from p using only
Modus-Ponens and replacement rules

Theorem 1 $p \vdash q$ iff $p \vdash_* q$

Proof.

Lemma 2 $p \rightarrow q, \neg q \vdash_* \neg p$

We will take one example and leave the rest as exercises

Notation: $p \vdash_* q$ if q follows from p using only
Modus-Ponens and replacement rules

Theorem 1 $p \vdash q$ iff $p \vdash_* q$

Proof.

Lemma 2 $p \rightarrow q, \neg q \vdash_* \neg p$

From now on I will shift to the following notation for proofs

Notation: $p \vdash_* q$ if q follows from p using only
Modus-Ponens and replacement rules

Theorem 1 $p \vdash q$ iff $p \vdash_* q$

Proof.

Lemma 2 $p \rightarrow q, \neg q \vdash_* \neg p$

First we state the two hypotheses

Notation: $p \vdash_* q$ if q follows from p using only
Modus-Ponens and replacement rules

Theorem 1 $p \vdash q$ iff $p \vdash_* q$

Proof.

Lemma 2 $p \rightarrow q, \neg q \vdash_* \neg p$

0 | $p \rightarrow q$ (Given)

which is this

Notation: $p \vdash_* q$ if q follows from p using only
Modus-Ponens and replacement rules

Theorem 1 $p \vdash q$ iff $p \vdash_* q$

Proof.

Lemma 2 $p \rightarrow q, \neg q \vdash_* \neg p$

0		$p \rightarrow q$ (Given)
1		$\neg q$ (Given)

and this

Notation: $p \vdash_* q$ if q follows from p using only
Modus-Ponens and replacement rules

Theorem 1 $p \vdash q$ iff $p \vdash_* q$

Proof.

Lemma 2 $p \rightarrow q, \neg q \vdash_* \neg p$

0	$p \rightarrow q$ (Given)
1	$\neg q$ (Given)
2	$\neg\neg q \rightarrow \neg\neg p$ (Double negation)

Notice here that the replacement rule is applied to sub expressions.

Notation: $p \vdash_* q$ if q follows from p using only
Modus-Ponens and replacement rules

Theorem 1 $p \vdash q$ iff $p \vdash_* q$

Proof.

Lemma 2 $p \rightarrow q, \neg q \vdash_* \neg p$

0	$p \rightarrow q$ (Given)
1	$\neg q$ (Given)
2	$\neg\neg q \rightarrow \neg\neg p$ (Double negation)
3	$\neg\neg q \rightarrow \neg\neg p \rightarrow (\neg p \rightarrow \neg q)$ (FL3)

Now this is a perfect application of axiom FL3

Notation: $p \vdash_* q$ if q follows from p using only

Modus-Ponens and replacement rules

Theorem 1 $p \vdash q$ iff $p \vdash_* q$

Proof.

Lemma 2 $p \rightarrow q, \neg q \vdash_* \neg p$

0	$p \rightarrow q$ (Given)
1	$\neg q$ (Given)
2	$\neg\neg q \rightarrow \neg\neg p$ (Double negation)
3	$\neg\neg q \rightarrow \neg\neg p \rightarrow (\neg p \rightarrow \neg q)$ (FL3)
4	$\neg p \rightarrow \neg q$ (MP on 2 & 3)

Applying Modus Ponens on it gives an expression

Notation: $p \vdash_* q$ if q follows from p using only

Modus-Ponens and replacement rules

Theorem 1 $p \vdash q$ iff $p \vdash_* q$

Proof.

Lemma 2 $p \rightarrow q, \neg q \vdash_* \neg p$

0	$p \rightarrow q$ (Given)
1	$\neg q$ (Given)
2	$\neg\neg q \rightarrow \neg\neg p$ (Double negation)
3	$\neg\neg q \rightarrow \neg\neg p \rightarrow (\neg p \rightarrow \neg q)$ (FL3)
4	$\neg p \rightarrow \neg q$ (MP on 2 & 3)
5	$\neg q$ (MP on 1 & 4)

□

on which we can apply Modus Ponens again to get the conclusion

Notation: $p \vdash_* q$ if q follows from p using only
Modus-Ponens and replacement rules

Theorem 1 $p \vdash q$ iff $p \vdash_* q$

Proof.

Lemma 2 $p \rightarrow q, \neg q \vdash_* \neg p$

0	$p \rightarrow q$ (Given)
1	$\neg q$ (Given)
2	$\neg\neg q \rightarrow \neg\neg p$ (Double negation)
3	$\neg\neg q \rightarrow \neg\neg p \rightarrow (\neg p \rightarrow \neg q)$ (FL3)
4	$\neg p \rightarrow \neg q$ (MP on 2 & 3)
5	$\neg q$ (MP on 1 & 4)

□

The other inference rules can also be proved similarly

Notation: $p \vdash_* q$ if q follows from p using only

Modus-Ponens and replacement rules

Theorem 1 $p \vdash q$ iff $p \vdash_* q$

Proof.

Lemma 2 $p \rightarrow q, \neg q \vdash_* \neg p$

0	$p \rightarrow q$ (Given)
1	$\neg q$ (Given)
2	$\neg\neg q \rightarrow \neg\neg p$ (Double negation)
3	$\neg\neg q \rightarrow \neg\neg p \rightarrow (\neg p \rightarrow \neg q)$ (FL3)
4	$\neg p \rightarrow \neg q$ (MP on 2 & 3)
5	$\neg q$ (MP on 1 & 4)

□

We now prove a lemma that will be useful later.

Notation: $p \vdash_* q$ if q follows from p using only
Modus-Ponens and replacement rules

Theorem 1 $p \vdash q$ iff $p \vdash_* q$

Proof.

Lemma 2 $p \rightarrow q, \neg q \vdash_* \neg p$

0	$p \rightarrow q$ (Given)
1	$\neg q$ (Given)
2	$\neg\neg q \rightarrow \neg\neg p$ (Double negation)
3	$\neg\neg q \rightarrow \neg\neg p \rightarrow (\neg p \rightarrow \neg q)$ (FL3)
4	$\neg p \rightarrow \neg q$ (MP on 2 & 3)
5	$\neg q$ (MP on 1 & 4)

□

It essentially proves that if a statement is true,

Notation: $p \vdash_* q$ if q follows from p using only
Modus-Ponens and replacement rules

Theorem 1 $p \vdash q$ iff $p \vdash_* q$

Proof.

Lemma 2 $p \rightarrow q, \neg q \vdash_* \neg p$

0	$p \rightarrow q$ (Given)
1	$\neg q$ (Given)
2	$\neg\neg q \rightarrow \neg\neg p$ (Double negation)
3	$\neg\neg q \rightarrow \neg\neg p \rightarrow (\neg p \rightarrow \neg q)$ (FL3)
4	$\neg p \rightarrow \neg q$ (MP on 2 & 3)
5	$\neg q$ (MP on 1 & 4)

□

it remains true even if you add some hypothesis

Notation: $p \vdash_* q$ if q follows from p using only
Modus-Ponens and replacement rules

Theorem 1 $p \vdash q$ iff $p \vdash_* q$

Proof.

Lemma 2 $p \rightarrow q, \neg q \vdash_* \neg p$

0	$p \rightarrow q$ (Given)
1	$\neg q$ (Given)
2	$\neg\neg q \rightarrow \neg\neg p$ (Double negation)
3	$\neg\neg q \rightarrow \neg\neg p \rightarrow (\neg p \rightarrow \neg q)$ (FL3)
4	$\neg p \rightarrow \neg q$ (MP on 2 & 3)
5	$\neg q$ (MP on 1 & 4)

□

In terms of semantics, if q is true, then $p \rightarrow q$ is also true

Notation: $p \vdash_* q$ if q follows from p using only

Modus-Ponens and replacement rules

Theorem 1 $p \vdash q$ iff $p \vdash_* q$

Proof.

Lemma 2 $p \rightarrow q, \neg q \vdash_* \neg p$

0	$p \rightarrow q$ (Given)
1	$\neg q$ (Given)
2	$\neg\neg q \rightarrow \neg\neg p$ (Double negation)
3	$\neg\neg q \rightarrow \neg\neg p \rightarrow (\neg p \rightarrow \neg q)$ (FL3)
4	$\neg p \rightarrow \neg q$ (MP on 2 & 3)
5	$\neg q$ (MP on 1 & 4)

□

because if the right hand side of \rightarrow is true, then the implication is true

Notation: $p \vdash_* q$ if q follows from p using only

Modus-Ponens and replacement rules

Lemma 4 $\vdash_* q$

Theorem 1 $p \vdash q$ iff $p \vdash_* q$

Proof.

Lemma 2 $p \rightarrow q, \neg q \vdash_* \neg p$

0	$p \rightarrow q$ (Given)
1	$\neg q$ (Given)
2	$\neg\neg q \rightarrow \neg\neg p$ (Double negation)
3	$\neg\neg q \rightarrow \neg\neg p \rightarrow (\neg p \rightarrow \neg q)$ (FL3)
4	$\neg p \rightarrow \neg q$ (MP on 2 & 3)
5	$\neg q$ (MP on 1 & 4)

□

If we have a proof of q

Notation: $p \vdash_* q$ if q follows from p using only

Modus-Ponens and replacement rules

Lemma 4 $\vdash_* q$ *implies* $\vdash_* p \rightarrow q$

Theorem 1 $p \vdash q$ *iff* $p \vdash_* q$

Proof.

Lemma 2 $p \rightarrow q, \neg q \vdash_* \neg p$

0	$p \rightarrow q$ (Given)
1	$\neg q$ (Given)
2	$\neg\neg q \rightarrow \neg\neg p$ (Double negation)
3	$\neg\neg q \rightarrow \neg\neg p \rightarrow (\neg p \rightarrow \neg q)$ (FL3)
4	$\neg p \rightarrow \neg q$ (MP on 2 & 3)
5	$\neg q$ (MP on 1 & 4)

□

we can modify it to a proof of $p \rightarrow q$

Notation: $p \vdash_* q$ if q follows from p using only
Modus-Ponens and replacement rules

Theorem 1 $p \vdash q$ iff $p \vdash_* q$

Proof.

Lemma 2 $p \rightarrow q, \neg q \vdash_* \neg p$

0	$p \rightarrow q$ (Given)
1	$\neg q$ (Given)
2	$\neg\neg q \rightarrow \neg\neg p$ (Double negation)
3	$\neg\neg q \rightarrow \neg\neg p \rightarrow (\neg p \rightarrow \neg q)$ (FL3)
4	$\neg p \rightarrow \neg q$ (MP on 2 & 3)
5	$\neg q$ (MP on 1 & 4)

□

Lemma 4 $\vdash_* q$ implies $\vdash_* p \rightarrow q$

Proof.

Notation: $p \vdash_* q$ if q follows from p using only
Modus-Ponens and replacement rules

Theorem 1 $p \vdash q$ iff $p \vdash_* q$

Proof.

Lemma 2 $p \rightarrow q, \neg q \vdash_* \neg p$

0	$p \rightarrow q$ (Given)
1	$\neg q$ (Given)
2	$\neg\neg q \rightarrow \neg\neg p$ (Double negation)
3	$\neg\neg q \rightarrow \neg\neg p \rightarrow (\neg p \rightarrow \neg q)$ (FL3)
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Lemma 4 $\vdash_* q$ implies $\vdash_* p \rightarrow q$

Proof.

Notation: $p \vdash_* q$ if q follows from p using only
Modus-Ponens and replacement rules

Theorem 1 $p \vdash q$ iff $p \vdash_* q$

Proof.

Lemma 2 $p \rightarrow q, \neg q \vdash_* \neg p$

0	$p \rightarrow q$ (Given)
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5	$\neg q$ (MP on 1 & 4)

□

Lemma 4 $\vdash_* q$ implies $\vdash_* p \rightarrow q$

Proof.

$n - 1 \mid \dots (\dots)$

A proof of q would have many steps

Notation: $p \vdash_* q$ if q follows from p using only Modus-Ponens and replacement rules

Theorem 1 $p \vdash q$ iff $p \vdash_* q$

Proof.

Lemma 2 $p \rightarrow q, \neg q \vdash_* \neg p$

0	$p \rightarrow q$ (Given)
1	$\neg q$ (Given)
2	$\neg \neg q \rightarrow \neg \neg p$ (Double negation)
3	$\neg \neg q \rightarrow \neg \neg p \rightarrow (\neg p \rightarrow \neg q)$ (FL3)
4	$\neg p \rightarrow \neg q$ (MP on 2 & 3)
5	$\neg q$ (MP on 1 & 4)

□

Lemma 4 $\vdash_* q$ implies $\vdash_* p \rightarrow q$

Proof.

$n - 1$	$\dots (\dots)$
n	$q (\dots)$

of which the last line would be q

Notation: $p \vdash_* q$ if q follows from p using only
Modus-Ponens and replacement rules

Theorem 1 $p \vdash q$ iff $p \vdash_* q$

Proof.

Lemma 2 $p \rightarrow q, \neg q \vdash_* \neg p$

0	$p \rightarrow q$ (Given)
1	$\neg q$ (Given)
2	$\neg\neg q \rightarrow \neg\neg p$ (Double negation)
3	$\neg\neg q \rightarrow \neg\neg p \rightarrow (\neg p \rightarrow \neg q)$ (FL3)
4	$\neg p \rightarrow \neg q$ (MP on 2 & 3)
5	$\neg q$ (MP on 1 & 4)

Lemma 4 $\vdash_* q$ implies $\vdash_* p \rightarrow q$

Proof.

$n - 1$	$\dots (\dots)$
n	$q (\dots)$
$n + 1$	$q \rightarrow (p \rightarrow q)$ (FL1)

□

Now we simply use FL1

Notation: $p \vdash_* q$ if q follows from p using only
Modus-Ponens and replacement rules

Theorem 1 $p \vdash q$ iff $p \vdash_* q$

Proof.

Lemma 2 $p \rightarrow q, \neg q \vdash_* \neg p$

0	$p \rightarrow q$ (Given)
1	$\neg q$ (Given)
2	$\neg\neg q \rightarrow \neg\neg p$ (Double negation)
3	$\neg\neg q \rightarrow \neg\neg p \rightarrow (\neg p \rightarrow \neg q)$ (FL3)
4	$\neg p \rightarrow \neg q$ (MP on 2 & 3)
5	$\neg q$ (MP on 1 & 4)

Lemma 4 $\vdash_* q$ implies $\vdash_* p \rightarrow q$

Proof.

$n - 1$	$\dots (\dots)$
n	$q (\dots)$
$n + 1$	$q \rightarrow (p \rightarrow q)$ (FL1)
$n + 2$	$p \rightarrow q$ (MP on n & $n + 1$)

□

to give what we need using Modus Ponens

Notation: $p \vdash_* q$ if q follows from p using only
Modus-Ponens and replacement rules

Theorem 1 $p \vdash q$ iff $p \vdash_* q$

Proof.

Lemma 2 $p \rightarrow q, \neg q \vdash_* \neg p$

0	$p \rightarrow q$ (Given)
1	$\neg q$ (Given)
2	$\neg\neg q \rightarrow \neg\neg p$ (Double negation)
3	$\neg\neg q \rightarrow \neg\neg p \rightarrow (\neg p \rightarrow \neg q)$ (FL3)
4	$\neg p \rightarrow \neg q$ (MP on 2 & 3)
5	$\neg q$ (MP on 1 & 4)

Lemma 4 $\vdash_* q$ implies $\vdash_* p \rightarrow q$

Proof.

$n - 1$	$\dots (\dots)$
n	$q (\dots)$
$n + 1$	$q \rightarrow (p \rightarrow q)$ (FL1)
$n + 2$	$p \rightarrow q$ (MP on n & $n + 1$)

So, assuming a proof of q ,

□

Lemma 4 $\vdash_* q$ *implies* $\vdash_* p \rightarrow q$

Proof.

$n - 1$	\vdots	(\dots)
n	q	(\dots)
$n + 1$	$q \rightarrow (p \rightarrow q)$ (FL1)	
$n + 2$	$p \rightarrow q$ (MP on n & $n + 1$)	

So, assuming a proof of q , we can extend it to a proof of $p \rightarrow q$ by FL1 and Modus Ponens \square

Proof.

$n - 1$	$\dots (\dots)$
n	$q (\dots)$
$n + 1$	$q \rightarrow (p \rightarrow q)$ (FL1)
$n + 2$	$p \rightarrow q$ (MP on n & $n + 1$)

So, assuming a proof of q , we can extend it to a proof of $p \rightarrow q$ by FL1 and Modus Ponens \square

Theorem 3 (Direct proof)

Lemma 4 $\vdash_* q$ *implies* $\vdash_* p \rightarrow q$

Proof.

$n - 1$	$\dots (\dots)$
n	$q (\dots)$
$n + 1$	$q \rightarrow (p \rightarrow q)$ (FL1)
$n + 2$	$p \rightarrow q$ (MP on n & $n + 1$)

So, assuming a proof of q , we can extend it to a proof of $p \rightarrow q$ by FL1 and Modus Ponens \square

because it seems so obvious.

Theorem 3 (Direct proof) $p \vdash_* q$

Lemma 4 $\vdash_* q$ *implies* $\vdash_* p \rightarrow q$

Proof.

$n - 1$	$\dots (\dots)$
n	$q (\dots)$
$n + 1$	$q \rightarrow (p \rightarrow q)$ (FL1)
$n + 2$	$p \rightarrow q$ (MP on n & $n + 1$)

So, assuming a proof of q , we can extend it to a proof of $p \rightarrow q$ by FL1 and Modus Ponens \square

If we can deduce q from p

Theorem 3 (Direct proof) $p \vdash_* q$ *implies* $\vdash_* p \rightarrow q$

Lemma 4 $\vdash_* q$ *implies* $\vdash_* p \rightarrow q$

Proof.

$n - 1$	$\dots (\dots)$
n	$q (\dots)$
$n + 1$	$q \rightarrow (p \rightarrow q)$ (FL1)
$n + 2$	$p \rightarrow q$ (MP on n & $n + 1$)

So, assuming a proof of q , we can extend it to a proof of $p \rightarrow q$ by FL1 and Modus Ponens \square

Then we have essentially proved that $p \rightarrow q$

Theorem 3 (Direct proof) $p \vdash_* q$ *implies* $\vdash_* p \rightarrow q$

Lemma 4 $\vdash_* q$ *implies* $\vdash_* p \rightarrow q$

Proof.

$n - 1$	$\dots (\dots)$
n	$q (\dots)$
$n + 1$	$q \rightarrow (p \rightarrow q)$ (FL1)
$n + 2$	$p \rightarrow q$ (MP on n & $n + 1$)

So, assuming a proof of q , we can extend it to a proof of $p \rightarrow q$ by FL1 and Modus Ponens \square

Observe that while a proof of q given p sounds like $p \rightarrow q$, the two are very different

Theorem 3 (Direct proof) $p \vdash_* q$ *implies* $\vdash_* p \rightarrow q$

Lemma 4 $\vdash_* q$ *implies* $\vdash_* p \rightarrow q$

Proof.

$n - 1$	$\dots (\dots)$
n	$q (\dots)$
$n + 1$	$q \rightarrow (p \rightarrow q)$ (FL1)
$n + 2$	$p \rightarrow q$ (MP on n & $n + 1$)

So, assuming a proof of q , we can extend it to a proof of $p \rightarrow q$ by FL1 and Modus Ponens \square

A proof is a sequence of statements following some rules

Theorem 3 (Direct proof) $p \vdash_* q$ *implies* $\vdash_* p \rightarrow q$

Lemma 4 $\vdash_* q$ *implies* $\vdash_* p \rightarrow q$

Proof.

$n - 1$	$\dots (\dots)$
n	$q (\dots)$
$n + 1$	$q \rightarrow (p \rightarrow q)$ (FL1)
$n + 2$	$p \rightarrow q$ (MP on n & $n + 1$)

So, assuming a proof of q , we can extend it to a proof of $p \rightarrow q$ by FL1 and Modus Ponens \square

On the other hand, \implies is an operator within our language of Propositional Logic

Theorem 3 (Direct proof) $p \vdash_* q$ *implies* $\vdash_* p \rightarrow q$

Lemma 4 $\vdash_* q$ *implies* $\vdash_* p \rightarrow q$

Proof.

$n - 1$	$\dots (\dots)$
n	$q (\dots)$
$n + 1$	$q \rightarrow (p \rightarrow q)$ (FL1)
$n + 2$	$p \rightarrow q$ (MP on n & $n + 1$)

So, assuming a proof of q , we can extend it to a proof of $p \rightarrow q$ by FL1 and Modus Ponens \square

which can be a statement in a proof

Theorem 3 (Direct proof) $p \vdash_* q$ *implies* $\vdash_* p \rightarrow q$

Lemma 4 $\vdash_* q$ *implies* $\vdash_* p \rightarrow q$

Proof.

$n - 1$	$\dots (\dots)$
n	$q (\dots)$
$n + 1$	$q \rightarrow (p \rightarrow q)$ (FL1)
$n + 2$	$p \rightarrow q$ (MP on n & $n + 1$)

So, assuming a proof of q , we can extend it to a proof of $p \rightarrow q$ by FL1 and Modus Ponens \square

But it is not yet clear why a proof that q *follows* from p

Theorem 3 (Direct proof) $p \vdash_* q$ *implies* $\vdash_* p \rightarrow q$

Lemma 4 $\vdash_* q$ *implies* $\vdash_* p \rightarrow q$

Proof.

$n - 1$	$\dots (\dots)$
n	$q (\dots)$
$n + 1$	$q \rightarrow (p \rightarrow q)$ (FL1)
$n + 2$	$p \rightarrow q$ (MP on n & $n + 1$)

So, assuming a proof of q , we can extend it to a proof of $p \rightarrow q$ by FL1 and Modus Ponens \square

also means that q is *implied* by p

Theorem 3 (Direct proof) $p \vdash_* q$ *implies* $\vdash_* p \rightarrow q$

Lemma 4 $\vdash_* q$ *implies* $\vdash_* p \rightarrow q$

Proof.

$n - 1$	$\dots (\dots)$
n	$q (\dots)$
$n + 1$	$q \rightarrow (p \rightarrow q)$ (FL1)
$n + 2$	$p \rightarrow q$ (MP on n & $n + 1$)

So, assuming a proof of q , we can extend it to a proof of $p \rightarrow q$ by FL1 and Modus Ponens \square

But it ought to be true if our idea of a formal proof makes sense

Theorem 3 (Direct proof) $p \vdash_* q$ *implies* $\vdash_* p \rightarrow q$

Lemma 4 $\vdash_* q$ *implies* $\vdash_* p \rightarrow q$

Proof.

$n - 1$	$\dots (\dots)$
n	$q (\dots)$
$n + 1$	$q \rightarrow (p \rightarrow q)$ (FL1)
$n + 2$	$p \rightarrow q$ (MP on n & $n + 1$)

So, assuming a proof of q , we can extend it to a proof of $p \rightarrow q$ by FL1 and Modus Ponens \square

That it does follow from the axioms and rules

Theorem 3 (Direct proof) $p \vdash_* q$ *implies* $\vdash_* p \rightarrow q$

Lemma 4 $\vdash_* q$ *implies* $\vdash_* p \rightarrow q$

Proof.

$n - 1$	$\dots (\dots)$
n	$q (\dots)$
$n + 1$	$q \rightarrow (p \rightarrow q)$ (FL1)
$n + 2$	$p \rightarrow q$ (MP on n & $n + 1$)

So, assuming a proof of q , we can extend it to a proof of $p \rightarrow q$ by FL1 and Modus Ponens \square

adds further weight to our axioms and rules.

Theorem 3 (Direct proof) $p \vdash_* q$ *implies* $\vdash_* p \rightarrow q$

Lemma 4 $\vdash_* q$ *implies* $\vdash_* p \rightarrow q$

Proof Assume true for proofs of length $\leq k$

Proof.

$n - 1$	\vdots	(\dots)
n	q	(\dots)
$n + 1$	$q \rightarrow (p \rightarrow q)$	(FL1)
$n + 2$	$p \rightarrow q$	(MP on n & $n + 1$)

So, assuming a proof of q , we can extend it to a proof of $p \rightarrow q$ by FL1 and Modus Ponens \square

We will use induction.

Theorem 3 (Direct proof) $p \vdash_* q$ *implies* $\vdash_* p \rightarrow q$

Lemma 4 $\vdash_* q$ *implies* $\vdash_* p \rightarrow q$

Proof Assume true for proofs of length $\leq k$

Proof.

$n - 1$	$\dots (\dots)$
n	$q (\dots)$
$n + 1$	$q \rightarrow (p \rightarrow q)$ (FL1)
$n + 2$	$p \rightarrow q$ (MP on n & $n + 1$)

So, assuming a proof of q , we can extend it to a proof of $p \rightarrow q$ by FL1 and Modus Ponens \square

Or rather we will use a very specific version of induction

Theorem 3 (Direct proof) $p \vdash_* q$ *implies* $\vdash_* p \rightarrow q$

Lemma 4 $\vdash_* q$ *implies* $\vdash_* p \rightarrow q$

Proof Assume true for proofs of length $\leq k$

Proof.

$n - 1$	\vdots	\vdots
n	q	(\dots)
$n + 1$	$q \rightarrow (p \rightarrow q)$	(FL1)
$n + 2$	$p \rightarrow q$	(MP on n & $n + 1$)

So, assuming a proof of q , we can extend it to a proof of $p \rightarrow q$ by FL1 and Modus Ponens \square

because one can rightfully argue that induction needs justification

Theorem 3 (Direct proof) $p \vdash_* q$ *implies* $\vdash_* p \rightarrow q$

Proof Assume true for proofs of length $\leq k$

Lemma 4 $\vdash_* q$ *implies* $\vdash_* p \rightarrow q$

Proof.

$n - 1$	\vdots	(\dots)
n	q	(\dots)
$n + 1$	$q \rightarrow (p \rightarrow q)$	(FL1)
$n + 2$	$p \rightarrow q$	(MP on n & $n + 1$)

So, assuming a proof of q , we can extend it to a proof of $p \rightarrow q$ by FL1 and Modus Ponens \square

However, all we will prove is that we can reduce the proof for proof with k steps

Theorem 3 (Direct proof) $p \vdash_* q$ *implies* $\vdash_* p \rightarrow q$

Lemma 4 $\vdash_* q$ *implies* $\vdash_* p \rightarrow q$

Proof Assume true for proofs of length $\leq k$

Proof.

$n - 1$	\vdots	\vdots
n	q	(\dots)
$n + 1$	$q \rightarrow (p \rightarrow q)$	(FL1)
$n + 2$	$p \rightarrow q$	(MP on n & $n + 1$)

So, assuming a proof of q , we can extend it to a proof of $p \rightarrow q$ by FL1 and Modus Ponens \square

to those with $k - 1$ steps

Theorem 3 (Direct proof) $p \vdash_* q$ *implies* $\vdash_* p \rightarrow q$

Proof Assume true for proofs of length $\leq k$

Lemma 4 $\vdash_* q$ *implies* $\vdash_* p \rightarrow q$

Proof.

$n - 1$	$\dots (\dots)$
n	$q (\dots)$
$n + 1$	$q \rightarrow (p \rightarrow q)$ (FL1)
$n + 2$	$p \rightarrow q$ (MP on n & $n + 1$)

So, assuming a proof of q , we can extend it to a proof of $p \rightarrow q$ by FL1 and Modus Ponens \square

Right now, do not worry about the justification of induction

Theorem 3 (Direct proof) $p \vdash_* q$ *implies* $\vdash_* p \rightarrow q$

Proof Assume true for proofs of length $\leq k$

Lemma 4 $\vdash_* q$ *implies* $\vdash_* p \rightarrow q$

Proof.

$n - 1$	$\dots (\dots)$
n	$q (\dots)$
$n + 1$	$q \rightarrow (p \rightarrow q)$ (FL1)
$n + 2$	$p \rightarrow q$ (MP on n & $n + 1$)

So, assuming a proof of q , we can extend it to a proof of $p \rightarrow q$ by FL1 and Modus Ponens \square

and focus on how it can be used to prove what we want

Theorem 3 (Direct proof) $p \vdash_* q$ *implies* $\vdash_* p \rightarrow q$

Proof Assume true for proofs of length $\leq k$

Case 1: Concluded q using Modus Ponens

Lemma 4 $\vdash_* q$ *implies* $\vdash_* p \rightarrow q$

Proof.

$n - 1$	$\dots (\dots)$
n	$q (\dots)$
$n + 1$	$q \rightarrow (p \rightarrow q)$ (FL1)
$n + 2$	$p \rightarrow q$ (MP on n & $n + 1$)

So, assuming a proof of q , we can extend it to a proof of $p \rightarrow q$ by FL1 and Modus Ponens \square

Let us first consider the case where in the proof of $p \vdash q$,

Theorem 3 (Direct proof) $p \vdash_* q$ *implies* $\vdash_* p \rightarrow q$

Proof Assume true for proofs of length $\leq k$

Case 1: Concluded q using Modus Ponens

Lemma 4 $\vdash_* q$ *implies* $\vdash_* p \rightarrow q$

Proof.

$n - 1$	$\dots (\dots)$
n	$q (\dots)$
$n + 1$	$q \rightarrow (p \rightarrow q)$ (FL1)
$n + 2$	$p \rightarrow q$ (MP on n & $n + 1$)

So, assuming a proof of q , we can extend it to a proof of $p \rightarrow q$ by FL1 and Modus Ponens \square

the last statement, i.e. q , follows from Modus Ponens

Theorem 3 (Direct proof) $p \vdash_* q$ *implies* $\vdash_* p \rightarrow q$

Proof Assume true for proofs of length $\leq k$

Case 1: Concluded q using Modus Ponens

Lemma 4 $\vdash_* q$ *implies* $\vdash_* p \rightarrow q$

Proof.

$n - 1$	$\dots (\dots)$
n	$q (\dots)$
$n + 1$	$q \rightarrow (p \rightarrow q)$ (FL1)
$n + 2$	$p \rightarrow q$ (MP on n & $n + 1$)

So, assuming a proof of q , we can extend it to a proof of $p \rightarrow q$ by FL1 and Modus Ponens \square

(rather than a replacement rule)

Theorem 3 (Direct proof) $p \vdash_* q$ *implies* $\vdash_* p \rightarrow q$

Proof Assume true for proofs of length $\leq k$

Case 1: Concluded q using Modus Ponens

Lemma 4 $\vdash_* q$ *implies* $\vdash_* p \rightarrow q$

Proof.

$n - 1$	\vdots	(\dots)
n	q	(\dots)
$n + 1$	$q \rightarrow (p \rightarrow q)$	(FL1)
$n + 2$	$p \rightarrow q$	(MP on n & $n + 1$)

So, assuming a proof of q , we can extend it to a proof of $p \rightarrow q$ by FL1 and Modus Ponens \square

Theorem 3 (Direct proof) $p \vdash_* q$ *implies* $\vdash_* p \rightarrow q$

Proof Assume true for proofs of length $\leq k$

Case 1: Concluded q using Modus Ponens

$0 \mid p$ (Given)

Lemma 4 $\vdash_* q$ *implies* $\vdash_* p \rightarrow q$

Proof.

$n - 1$	$\mid \dots (\dots)$
n	$\mid q (\dots)$
$n + 1$	$\mid q \rightarrow (p \rightarrow q) \text{ (FL1)}$
$n + 2$	$\mid p \rightarrow q \text{ (MP on } n \text{ \& } n + 1)$

So, assuming a proof of q , we can extend it to a proof of $p \rightarrow q$ by FL1 and Modus Ponens \square

So we have some proof of q beginning with the given p

Theorem 3 (Direct proof) $p \vdash_* q$ *implies* $\vdash_* p \rightarrow q$

Proof Assume true for proofs of length $\leq k$

Case 1: Concluded q using Modus Ponens

0		p (Given)
...		... (...)
...		...
...		...

Lemma 4 $\vdash_* q$ *implies* $\vdash_* p \rightarrow q$

Proof.

$n - 1$... (...)
n		q (...)
$n + 1$		$q \rightarrow (p \rightarrow q)$ (FL1)
$n + 2$		$p \rightarrow q$ (MP on n & $n + 1$)

So, assuming a proof of q , we can extend it to a proof of $p \rightarrow q$ by FL1 and Modus Ponens □

Now we have some other intermediate steps

Theorem 3 (Direct proof) $p \vdash_* q$ *implies* $\vdash_* p \rightarrow q$

Proof Assume true for proofs of length $\leq k$

Case 1: Concluded q using Modus Ponens

0		p (Given)
...		... (...)
...		... (...)
...		...
k		q

Lemma 4 $\vdash_* q$ *implies* $\vdash_* p \rightarrow q$

Proof.

$n - 1$... (...)
n		q (...)
$n + 1$		$q \rightarrow (p \rightarrow q)$ (FL1)
$n + 2$		$p \rightarrow q$ (MP on n & $n + 1$)

So, assuming a proof of q , we can extend it to a proof of $p \rightarrow q$ by FL1 and Modus Ponens □

Eventually proving q

Theorem 3 (Direct proof) $p \vdash_* q$ *implies* $\vdash_* p \rightarrow q$

Proof Assume true for proofs of length $\leq k$

Case 1: Concluded q using Modus Ponens

0		p (Given)
...		... (...)
...		...
...		...
...		...
k		q

Lemma 4 $\vdash_* q$ *implies* $\vdash_* p \rightarrow q$

Proof.

$n - 1$... (...)
n		q (...)
$n + 1$		$q \rightarrow (p \rightarrow q)$ (FL1)
$n + 2$		$p \rightarrow q$ (MP on n & $n + 1$)

So, assuming a proof of q , we can extend it to a proof of $p \rightarrow q$ by FL1 and Modus Ponens □

Since we are assuming in this case that it follows from Modus Ponens

Theorem 3 (Direct proof) $p \vdash_* q$ *implies* $\vdash_* p \rightarrow q$

Proof Assume true for proofs of length $\leq k$

Case 1: Concluded q using Modus Ponens

0		p (Given)
...		... (...)
...		... (...)
...		...
k		q

Lemma 4 $\vdash_* q$ *implies* $\vdash_* p \rightarrow q$

Proof.

$n - 1$... (...)
n		q (...)
$n + 1$		$q \rightarrow (p \rightarrow q)$ (FL1)
$n + 2$		$p \rightarrow q$ (MP on n & $n + 1$)

So, assuming a proof of q , we can extend it to a proof of $p \rightarrow q$ by FL1 and Modus Ponens \square

there will be some intermediate steps that Modus Ponens is used on

Theorem 3 (Direct proof) $p \vdash_* q$ *implies* $\vdash_* p \rightarrow q$

Proof Assume true for proofs of length $\leq k$

Case 1: Concluded q using Modus Ponens

0		p (Given)
...		... (...)
...		... (...)
n		$r \rightarrow q$ (...)
...		... (...)
k		q

Lemma 4 $\vdash_* q$ *implies* $\vdash_* p \rightarrow q$

Proof.

$n - 1$... (...)
n		q (...)
$n + 1$		$q \rightarrow (p \rightarrow q)$ (FL1)
$n + 2$		$p \rightarrow q$ (MP on n & $n + 1$)

So, assuming a proof of q , we can extend it to a proof of $p \rightarrow q$ by FL1 and Modus Ponens □

One of those steps, say the n th, will be of the form $r \rightarrow q$ for some q

Theorem 3 (Direct proof) $p \vdash_* q$ *implies* $\vdash_* p \rightarrow q$

Proof Assume true for proofs of length $\leq k$

Case 1: Concluded q using Modus Ponens

0	p (Given)
...	... (...)
m	r (...)
...	... (...)
n	$r \rightarrow q$ (...)
...	... (...)
k	q

Lemma 4 $\vdash_* q$ *implies* $\vdash_* p \rightarrow q$

Proof.

$n - 1$... (...)
n	q (...)
$n + 1$	$q \rightarrow (p \rightarrow q)$ (FL1)
$n + 2$	$p \rightarrow q$ (MP on n & $n + 1$)

So, assuming a proof of q , we can extend it to a proof of $p \rightarrow q$ by FL1 and Modus Ponens □

while the m th statement would be r

Theorem 3 (Direct proof) $p \vdash_* q$ *implies* $\vdash_* p \rightarrow q$

Proof Assume true for proofs of length $\leq k$

Case 1: Concluded q using Modus Ponens

0	p (Given)
...	...
m	r (...)
...	...
n	$r \rightarrow q$ (...)
...	...
k	q (MP on m & n)

Lemma 4 $\vdash_* q$ *implies* $\vdash_* p \rightarrow q$

Proof.

$n - 1$...
n	q (...)
$n + 1$	$q \rightarrow (p \rightarrow q)$ (FL1)
$n + 2$	$p \rightarrow q$ (MP on n & $n + 1$)

So, assuming a proof of q , we can extend it to a proof of $p \rightarrow q$ by FL1 and Modus Ponens □

so that Modus Ponens can be applied to it.

Theorem 3 (Direct proof) $p \vdash_* q$ *implies* $\vdash_* p \rightarrow q$

Proof Assume true for proofs of length $\leq k$

Case 1: Concluded q using Modus Ponens

0	p (Given)
...	... (...)
m	r (...)
...	... (...)
n	$r \rightarrow q$ (...)
...	... (...)
k	q (MP on m & n)

Lemma 4 $\vdash_* q$ *implies* $\vdash_* p \rightarrow q$

Proof.

$n - 1$... (...)
n	q (...)
$n + 1$	$q \rightarrow (p \rightarrow q)$ (FL1)
$n + 2$	$p \rightarrow q$ (MP on n & $n + 1$)

So, assuming a proof of q , we can extend it to a proof of $p \rightarrow q$ by FL1 and Modus Ponens □

Notice that the steps from p to r is fewer than k steps

Theorem 3 (Direct proof) $p \vdash_* q$ *implies* $\vdash_* p \rightarrow q$

Proof Assume true for proofs of length $\leq k$

Case 1: Concluded q using Modus Ponens

0	p (Given)
...	... (...)
m	r (...)
...	... (...)
n	$r \rightarrow q$ (...)
...	... (...)
k	q (MP on m & n)
$k + 1$	$p \rightarrow r$ (Induction)

Lemma 4 $\vdash_* q$ *implies* $\vdash_* p \rightarrow q$

Proof.

$n - 1$... (...)
n	q (...)
$n + 1$	$q \rightarrow (p \rightarrow q)$ (FL1)
$n + 2$	$p \rightarrow q$ (MP on n & $n + 1$)

So, assuming a proof of q , we can extend it to a proof of $p \rightarrow q$ by FL1 and Modus Ponens \square

So we can deduce this by induction

Theorem 3 (Direct proof) $p \vdash_* q$ implies $\vdash_* p \rightarrow q$

Proof Assume true for proofs of length $\leq k$

Case 1: Concluded q using Modus Ponens

0	p (Given)
...	... (...)
m	r (...)
...	... (...)
n	$r \rightarrow q$ (...)
...	... (...)
k	q (MP on m & n)
$k + 1$	$p \rightarrow r$ (Induction)

Lemma 4 $\vdash_* q$ implies $\vdash_* p \rightarrow q$

Proof.

$n - 1$... (...)
n	q (...)
$n + 1$	$q \rightarrow (p \rightarrow q)$ (FL1)
$n + 2$	$p \rightarrow q$ (MP on n & $n + 1$)

So, assuming a proof of q , we can extend it to a proof of $p \rightarrow q$ by FL1 and Modus Ponens \square

Similarly notice that the steps from p to $r \rightarrow q$ is fewer than k steps

Theorem 3 (Direct proof) $p \vdash_* q$ *implies* $\vdash_* p \rightarrow q$

Proof Assume true for proofs of length $\leq k$

Case 1: Concluded q using Modus Ponens

0	p (Given)
...	... (...)
m	r (...)
...	... (...)
n	$r \rightarrow q$ (...)
...	... (...)
k	q (MP on m & n)
$k+1$	$p \rightarrow r$ (Induction)
$k+2$	$p \rightarrow (r \rightarrow q)$ (Induction)

Lemma 4 $\vdash_* q$ *implies* $\vdash_* p \rightarrow q$

Proof.

$n-1$... (...)
n	q (...)
$n+1$	$q \rightarrow (p \rightarrow q)$ (FL1)
$n+2$	$p \rightarrow q$ (MP on n & $n+1$)

So, assuming a proof of q , we can extend it to a proof of $p \rightarrow q$ by FL1 and Modus Ponens □

So we have can also infer this

Theorem 3 (Direct proof) $p \vdash_* q$ *implies* $\vdash_* p \rightarrow q$

Lemma 4 $\vdash_* q$ *implies* $\vdash_* p \rightarrow q$

Proof Assume true for proofs of length $\leq k$

Case 1: Concluded q using Modus Ponens

0	p (Given)
...	... (...)
m	r (...)
...	... (...)
n	$r \rightarrow q$ (...)
...	... (...)
k	q (MP on m & n)
$k+1$	$p \rightarrow r$ (Induction)
$k+2$	$p \rightarrow (r \rightarrow q)$ (Induction)
$k+3$	$p \rightarrow (r \rightarrow q) \rightarrow (p \rightarrow r \rightarrow (p \rightarrow q))$ (FL2)

Proof.

$n-1$... (...)
n	q (...)
$n+1$	$q \rightarrow (p \rightarrow q)$ (FL1)
$n+2$	$p \rightarrow q$ (MP on n & $n+1$)

So, assuming a proof of q , we can extend it to a proof of $p \rightarrow q$ by FL1 and Modus Ponens □

Now we are ready to use FL3

Theorem 3 (Direct proof) $p \vdash_* q$ *implies* $\vdash_* p \rightarrow q$

Lemma 4 $\vdash_* q$ *implies* $\vdash_* p \rightarrow q$

Proof Assume true for proofs of length $\leq k$

Case 1: Concluded q using Modus Ponens

0	p (Given)
...	... (...)
m	r (...)
...	... (...)
n	$r \rightarrow q$ (...)
...	... (...)
k	q (MP on m & n)
$k+1$	$p \rightarrow r$ (Induction)
$k+2$	$p \rightarrow (r \rightarrow q)$ (Induction)
$k+3$	$p \rightarrow (r \rightarrow q) \rightarrow (p \rightarrow r \rightarrow (p \rightarrow q))$ (FL2)
$k+4$	$p \rightarrow r \rightarrow (p \rightarrow q)$ (MP on $k+1, k+3$)

Proof.

$n-1$... (...)
n	q (...)
$n+1$	$q \rightarrow (p \rightarrow q)$ (FL1)
$n+2$	$p \rightarrow q$ (MP on n & $n+1$)

So, assuming a proof of q , we can extend it to a proof of $p \rightarrow q$ by FL1 and Modus Ponens □

Notice that FL3 is designed to allow us to use Modus Ponens

Theorem 3 (Direct proof) $p \vdash_* q$ *implies* $\vdash_* p \rightarrow q$

Lemma 4 $\vdash_* q$ *implies* $\vdash_* p \rightarrow q$

Proof Assume true for proofs of length $\leq k$

Case 1: Concluded q using Modus Ponens

0	p (Given)
...	... (...)
m	r (...)
...	... (...)
n	$r \rightarrow q$ (...)
...	... (...)
k	q (MP on m & n)
$k+1$	$p \rightarrow r$ (Induction)
$k+2$	$p \rightarrow (r \rightarrow q)$ (Induction)
$k+3$	$p \rightarrow (r \rightarrow q) \rightarrow (p \rightarrow r \rightarrow (p \rightarrow q))$ (FL2)
$k+4$	$p \rightarrow r \rightarrow (p \rightarrow q)$ (MP on $k+1, k+3$)

Proof.

$n-1$... (...)
n	q (...)
$n+1$	$q \rightarrow (p \rightarrow q)$ (FL1)
$n+2$	$p \rightarrow q$ (MP on n & $n+1$)

So, assuming a proof of q , we can extend it to a proof of $p \rightarrow q$ by FL1 and Modus Ponens □

to give us $p \rightarrow r \rightarrow (p \rightarrow q)$

Theorem 3 (Direct proof) $p \vdash_* q$ implies $\vdash_* p \rightarrow q$

Lemma 4 $\vdash_* q$ implies $\vdash_* p \rightarrow q$

Proof Assume true for proofs of length $\leq k$

Case 1: Concluded q using Modus Ponens

0	p (Given)
...	... (...)
m	r (...)
...	... (...)
n	$r \rightarrow q$ (...)
...	... (...)
k	q (MP on m & n)
$k+1$	$p \rightarrow r$ (Induction)
$k+2$	$p \rightarrow (r \rightarrow q)$ (Induction)
$k+3$	$p \rightarrow (r \rightarrow q) \rightarrow (p \rightarrow r \rightarrow (p \rightarrow q))$ (FL2)
$k+4$	$p \rightarrow r \rightarrow (p \rightarrow q)$ (MP on $k+1, k+3$)
$k+5$	$p \rightarrow q$ (MP on $k+1, k+4$)

Proof.

$n-1$... (...)
n	q (...)
$n+1$	$q \rightarrow (p \rightarrow q)$ (FL1)
$n+2$	$p \rightarrow q$ (MP on n & $n+1$)

So, assuming a proof of q , we can extend it to a proof of $p \rightarrow q$ by FL1 and Modus Ponens □

Which in turn is precisely what Modus Ponens needs to give us what we seek

Theorem 3 (Direct proof) $p \vdash_* q$ implies $\vdash_* p \rightarrow q$

Lemma 4 $\vdash_* q$ implies $\vdash_* p \rightarrow q$

Proof Assume true for proofs of length $\leq k$

Case 1: Concluded q using Modus Ponens

0	p (Given)
...	...
m	r (...)
...	...
n	$r \rightarrow q$ (...)
...	...
k	q (MP on m & n)
$k+1$	$p \rightarrow r$ (Induction)
$k+2$	$p \rightarrow (r \rightarrow q)$ (Induction)
$k+3$	$p \rightarrow (r \rightarrow q) \rightarrow (p \rightarrow r \rightarrow (p \rightarrow q))$ (FL2)
$k+4$	$p \rightarrow r \rightarrow (p \rightarrow q)$ (MP on $k+1, k+3$)
$k+5$	$p \rightarrow q$ (MP on $k+1, k+4$)

Proof.

$n-1$...
n	q (...)
$n+1$	$q \rightarrow (p \rightarrow q)$ (FL1)
$n+2$	$p \rightarrow q$ (MP on n & $n+1$)

So, assuming a proof of q , we can extend it to a proof of $p \rightarrow q$ by FL1 and Modus Ponens □

In the next lecture we will consider the case where the last step used a replacement rule

Theorem 3 (Direct proof) $p \vdash_* q$ implies $\vdash_* p \rightarrow q$

Lemma 4 $\vdash_* q$ implies $\vdash_* p \rightarrow q$

Proof Assume true for proofs of length $\leq k$

Case 1: Concluded q using Modus Ponens

0	p (Given)
...	... (...)
m	r (...)
...	... (...)
n	$r \rightarrow q$ (...)
...	... (...)
k	q (MP on m & n)
$k+1$	$p \rightarrow r$ (Induction)
$k+2$	$p \rightarrow (r \rightarrow q)$ (Induction)
$k+3$	$p \rightarrow (r \rightarrow q) \rightarrow (p \rightarrow r \rightarrow (p \rightarrow q))$ (FL2)
$k+4$	$p \rightarrow r \rightarrow (p \rightarrow q)$ (MP on $k+1, k+3$)
$k+5$	$p \rightarrow q$ (MP on $k+1, k+4$)

Proof.

$n-1$... (...)
n	q (...)
$n+1$	$q \rightarrow (p \rightarrow q)$ (FL1)
$n+2$	$p \rightarrow q$ (MP on n & $n+1$)

So, assuming a proof of q , we can extend it to a proof of $p \rightarrow q$ by FL1 and Modus Ponens □

We will also look at the starting cases of the induction hypotheses