The last time we considered inference rules

We will not consider the so called Replacement Rules that will also help us make deductions

However, as starting points we will specify some axioms

These axioms are not “self evident”

The axioms may seem strange but we will realize their motivation when we see them as the only missing pieces when trying to prove that only some “natural” lemmas

A **replacement rule**

A **replacement rule**

We will now define a replacement rule

A **replacement rule** is of the form $p \iff q$

and use this notation

A **replacement rule** is of the form $p \iff q$ and means that whenever $p$ is in a subexpression, it may be replaced with $q$

It allows us to replace anything on the left hand side of $\iff$ with whatever is on the right hand side

A **replacement rule** is of the form $p \iff q$ and means that whenever $p$ is in a subexpression, it may be replaced with $q$ and vice-versa.

but unlike inference rules, we can replace the right hand side with the left hand side

A **replacement rule** is of the form $p \iff q$ and means that whenever $p$ is in a subexpression, it may be replaced with $q$ and vice-versa.

Also note that unlike inference rules, replacememnt rules may be applied to sub-expressions

A **replacement rule** is of the form $p \iff q$ and means that whenever $p$ is in a subexpression, it may be replaced with $q$ and vice-versa.

Inference rules have to applied exactly as they are

A **replacement rule** is of the form $p \iff q$ and means that whenever $p$ is in a subexpression, it may be replaced with $q$ and vice-versa.

Try and find an example where performing an inference rule on a sub-expression,

A **replacement rule** is of the form $p \iff q$ and means that whenever $p$ is in a subexpression, it may be replaced with $q$ and vice-versa.

leads to deducing something that ought to be absurd

A **replacement rule** is of the form $p \iff q$ and means that whenever $p$ is in a subexpression, it may be replaced with $q$ and vice-versa.

Replacement rules are designed to give “syntactic meaning” to the symbols in logic.

A **replacement rule** is of the form $p \iff q$ and means that whenever $p$ is in a subexpression, it may be replaced with $q$ and vice-versa.

Let us consider the negation function and list out some of its properties.

A **replacement rule** is of the form $p \iff q$ and means that whenever $p$ is in a subexpression, it may be replaced with $q$ and vice-versa.

Until now it is merely a symbol in our language along with the other operators $\wedge$ etc.

A **replacement rule** is of the form $p \iff q$ and means that whenever $p$ is in a subexpression, it may be replaced with $q$ and vice-versa.

It will turn out that these properties are enough to "characterize" $\neg$, and other operators

A **replacement rule** is of the form $p \iff q$ and means that whenever $p$ is in a subexpression, it may be replaced with $q$ and vice-versa.

along with the FL axioms and Modus Ponens etc

A **replacement rule** is of the form $p \iff q$ and means that whenever $p$ is in a subexpression, it may be replaced with $q$ and vice-versa.

in the sense that any tautology will be provable from axioms + Modus Ponens + replacement rules.

A **replacement rule** is of the form $p \iff q$ and means that whenever $p$ is in a subexpression, it may be replaced with $q$ and vice-versa.

$p \iff \neg\neg p$

That a negation should negate a negation seems obvious and so we demand it

A **replacement rule** is of the form $p \iff q$ and means that whenever $p$ is in a subexpression, it may be replaced with $q$ and vice-versa.

$$p \iff \neg\neg p$$

$$\neg(p \vee q) \iff \neg p \wedge \neg q$$

We also demand the DeMorgan's Laws, which encodes how $\neg$ interacts with $\wedge$ and $\vee$

A **replacement rule** is of the form $p \iff q$ and means that whenever $p$ is in a subexpression, it may be replaced with $q$ and vice-versa.

$p \iff \neg\neg p$

$\neg(p \vee q) \iff \neg p \wedge \neg q$
$\neg(p \wedge q) \iff \neg p \vee \neg q$

In fact, this one can be used to define $\wedge$ in terms of $\vee$ and $\neg$

A **replacement rule** is of the form $p \iff q$ and means that whenever $p$ is in a subexpression, it may be replaced with $q$ and vice-versa.

$$p \iff \neg\neg p$$

$$\neg(p \vee q) \iff \neg p \wedge \neg q$$
$$\neg(p \wedge q) \iff \neg p \vee \neg q$$

$$p \wedge p \iff p$$

This encodes how $\wedge$ interacts with itself

A **replacement rule** is of the form $p \iff q$ and means that whenever $p$ is in a subexpression, it may be replaced with $q$ and vice-versa.

$$p \iff \neg\neg p$$

$$\neg(p \vee q) \iff \neg p \wedge \neg q$$
$$\neg(p \wedge q) \iff \neg p \vee \neg q$$

$$p \wedge p \iff p$$
$$p \vee p \iff p$$

This encodes how $\vee$ interacts with itself

A **replacement rule** is of the form $p \iff q$ and means that whenever $p$ is in a subexpression, it may be replaced with $q$ and vice-versa.

$$p \iff \neg\neg p$$

$$\neg(p \lor q) \iff \neg p \land \neg q$$
$$\neg(p \land q) \iff \neg p \lor \neg q$$

$$p \land p \iff p$$
$$p \lor p \iff p$$

$$p \land q \iff q \land p$$

And this tells us that order of $\land$ does not matter

A **replacement rule** is of the form $p \iff q$ and means that whenever $p$ is in a subexpression, it may be replaced with $q$ and vice-versa.

$$p \iff \neg\neg p$$

$$\neg(p \lor q) \iff \neg p \land \neg q$$
$$\neg(p \land q) \iff \neg p \lor \neg q$$

$$p \land p \iff p$$
$$p \lor p \iff p$$

$$p \land q \iff q \land p$$
$$p \lor q \iff q \lor p$$

and this does the same for $\lor$

A **replacement rule** is of the form $p \iff q$ and means that whenever $p$ is in a subexpression, it may be replaced with $q$ and vice-versa.

$$p \iff \neg\neg p$$

$$\neg(p \vee q) \iff \neg p \wedge \neg q$$
$$\neg(p \wedge q) \iff \neg p \vee \neg q$$

$$p \wedge p \iff p$$
$$p \vee p \iff p$$

$$p \wedge q \iff q \wedge p$$
$$p \vee q \iff q \vee p$$

$$(p \rightarrow q) \iff (\neg p \vee q)$$

Here is a rule that can taken to be a definition of $\rightarrow$

A **replacement rule** is of the form $p \iff q$ and means that whenever $p$ is in a subexpression, it may be replaced with $q$ and vice-versa.

$$p \iff \neg\neg p$$

$$\neg(p \lor q) \iff \neg p \land \neg q$$
$$\neg(p \land q) \iff \neg p \lor \neg q$$

$$p \land p \iff p$$
$$p \lor p \iff p$$

$$p \land q \iff q \land p$$
$$p \lor q \iff q \lor p$$

$$(p \to q) \iff (\neg p \lor q)$$

it is defined in terms of $\neg$ and $\lor$ just like $\land$

A **replacement rule** is of the form $p \iff q$ and means that whenever $p$ is in a subexpression, it may be replaced with $q$ and vice-versa.

$$p \iff \neg\neg p$$

$$\neg(p \vee q) \iff \neg p \wedge \neg q$$
$$\neg(p \wedge q) \iff \neg p \vee \neg q$$

$$p \wedge p \iff p$$
$$p \vee p \iff p$$

$$p \wedge q \iff q \wedge p$$
$$p \vee q \iff q \vee p$$

$$(p \to q) \iff (\neg p \vee q)$$
$$(p \to q) \iff (\neg q \to \neg p)$$

Here is a convenient rule that can actually be derived by the others

A **replacement rule** is of the form $p \iff q$ and means that whenever $p$ is in a subexpression, it may be replaced with $q$ and vice-versa.

$$p \iff \neg\neg p$$

$$\neg(p \vee q) \iff \neg p \wedge \neg q$$
$$\neg(p \wedge q) \iff \neg p \vee \neg q$$

$$p \wedge p \iff p$$
$$p \vee p \iff p$$

$$p \wedge q \iff q \wedge p$$
$$p \vee q \iff q \vee p$$

$$(p \rightarrow q) \iff (\neg p \vee q)$$
$$(p \rightarrow q) \iff (\neg q \rightarrow \neg p)$$

$$(p \leftrightarrow q) \iff ((p \rightarrow q) \wedge (q \leftarrow p))$$

This one introduces $\leftrightarrow$ in terms of earlier operators

A **replacement rule** is of the form $p \iff q$ and means that whenever $p$ is in a subexpression, it may be replaced with $q$ and vice-versa.

$$p \iff \neg\neg p$$

$$\neg(p \vee q) \iff \neg p \wedge \neg q$$
$$\neg(p \wedge q) \iff \neg p \vee \neg q$$

$$p \wedge p \iff p$$
$$p \vee p \iff p$$

$$p \wedge q \iff q \wedge p$$
$$p \vee q \iff q \vee p$$

$$(p \to q) \iff (\neg p \vee q)$$
$$(p \to q) \iff (\neg q \to \neg p)$$

$$(p \leftrightarrow q) \iff ((p \to q) \wedge (q \leftarrow p))$$

$$(p \wedge q \to r) \iff (p \to (q \to r))$$

This one is also a replacement rule in the sense that it can be used on subexpressions

A **replacement rule** is of the form $p \iff q$ and means that whenever $p$ is in a subexpression, it may be replaced with $q$ and vice-versa.

$$p \iff \neg\neg p$$

$$\neg(p \vee q) \iff \neg p \wedge \neg q$$
$$\neg(p \wedge q) \iff \neg p \vee \neg q$$

$$p \wedge p \iff p$$
$$p \vee p \iff p$$

$$p \wedge q \iff q \wedge p$$
$$p \vee q \iff q \vee p$$

$$(p \to q) \iff (\neg p \vee q)$$
$$(p \to q) \iff (\neg q \to \neg p)$$

$$(p \leftrightarrow q) \iff ((p \to q) \wedge (q \leftarrow p))$$

$$(p \wedge q \to r) \iff (p \to (q \to r))$$

it works both ways

A **replacement rule** is of the form $p \iff q$ and means that whenever $p$ is in a subexpression, it may be replaced with $q$ and vice-versa.

$$p \iff \neg\neg p$$

$$\neg(p \vee q) \iff \neg p \wedge \neg q$$
$$\neg(p \wedge q) \iff \neg p \vee \neg q$$

$$p \wedge p \iff p$$
$$p \vee p \iff p$$

$$p \wedge q \iff q \wedge p$$
$$p \vee q \iff q \vee p$$

$$(p \rightarrow q) \iff (\neg p \vee q)$$
$$(p \rightarrow q) \iff (\neg q \rightarrow \neg p)$$

$$(p \leftrightarrow q) \iff ((p \rightarrow q) \wedge (q \leftarrow p))$$

$$(p \wedge q \rightarrow r) \iff (p \rightarrow (q \rightarrow r))$$

and if we turned $\leftrightarrow$ into $\iff$, it would be a tautology

A **replacement rule** is of the form $p \iff q$ and means that whenever $p$ is in a subexpression, it may be replaced with $q$ and vice-versa.

$(p \wedge (q \wedge r)) \iff ((p \wedge q) \wedge r)$

$p \iff \neg\neg p$

$\neg(p \vee q) \iff \neg p \wedge \neg q$
$\neg(p \wedge q) \iff \neg p \vee \neg q$

$p \wedge p \iff p$
$p \vee p \iff p$

$p \wedge q \iff q \wedge p$
$p \vee q \iff q \vee p$

$(p \to q) \iff (\neg p \vee q)$
$(p \to q) \iff (\neg q \to \neg p)$

$(p \leftrightarrow q) \iff ((p \to q) \wedge (q \leftarrow p))$

$(p \wedge q \to r) \iff (p \to (q \to r))$

This rule is associativity. it allows us to apply $\wedge$ multiple times

A **replacement rule** is of the form $p \iff q$ and means that whenever $p$ is in a subexpression, it may be replaced with $q$ and vice-versa.

$$(p \wedge (q \wedge r)) \iff ((p \wedge q) \wedge r)$$
$$(p \vee (q \vee r)) \iff ((p \vee q) \vee r)$$

$$p \iff \neg\neg p$$

$$\neg(p \vee q) \iff \neg p \wedge \neg q$$
$$\neg(p \wedge q) \iff \neg p \vee \neg q$$

$$p \wedge p \iff p$$
$$p \vee p \iff p$$

$$p \wedge q \iff q \wedge p$$
$$p \vee q \iff q \vee p$$

$$(p \to q) \iff (\neg p \vee q)$$
$$(p \to q) \iff (\neg q \to \neg p)$$

$$(p \leftrightarrow q) \iff ((p \to q) \wedge (q \leftarrow p))$$

$$(p \wedge q \to r) \iff (p \to (q \to r))$$

and this allows us to apply $\vee$ multiple times

A **replacement rule** is of the form $p \iff q$ and means that whenever $p$ is in a subexpression, it may be replaced with $q$ and vice-versa.

$$p \iff \neg\neg p$$

$$\neg(p \lor q) \iff \neg p \land \neg q$$
$$\neg(p \land q) \iff \neg p \lor \neg q$$

$$p \land p \iff p$$
$$p \lor p \iff p$$

$$p \land q \iff q \land p$$
$$p \lor q \iff q \lor p$$

$$(p \to q) \iff (\neg p \lor q)$$
$$(p \to q) \iff (\neg q \to \neg p)$$

$$(p \leftrightarrow q) \iff ((p \to q) \land (q \leftarrow p))$$

$$(p \land q \to r) \iff (p \to (q \to r))$$

$$(p \land (q \land r)) \iff ((p \land q) \land r)$$
$$(p \lor (q \lor r)) \iff ((p \lor q) \lor r)$$

$$(p \land (q \lor r)) \iff ((p \land q) \lor (p \land r))$$

We also have rules like this that tell us how $\land$ and $\lor$ interact

A **replacement rule** is of the form $p \iff q$ and means that whenever $p$ is in a subexpression, it may be replaced with $q$ and vice-versa.

$$p \iff \neg\neg p$$

$$\neg(p \vee q) \iff \neg p \wedge \neg q$$
$$\neg(p \wedge q) \iff \neg p \vee \neg q$$

$$p \wedge p \iff p$$
$$p \vee p \iff p$$

$$p \wedge q \iff q \wedge p$$
$$p \vee q \iff q \vee p$$

$$(p \to q) \iff (\neg p \vee q)$$
$$(p \to q) \iff (\neg q \to \neg p)$$

$$(p \leftrightarrow q) \iff ((p \to q) \wedge (q \leftarrow p))$$

$$(p \wedge q \to r) \iff (p \to (q \to r))$$

$$(p \wedge (q \wedge r)) \iff ((p \wedge q) \wedge r)$$
$$(p \vee (q \vee r)) \iff ((p \vee q) \vee r)$$

$$(p \wedge (q \vee r)) \iff ((p \wedge q) \vee (p \wedge r))$$
$$(p \vee (q \wedge r)) \iff ((p \vee q) \wedge (p \vee r))$$

as does this one

A **replacement rule** is of the form $p \iff q$ and means that whenever $p$ is in a subexpression, it may be replaced with $q$ and vice-versa.

$$p \iff \neg\neg p$$

$$\neg(p \vee q) \iff \neg p \wedge \neg q$$
$$\neg(p \wedge q) \iff \neg p \vee \neg q$$

$$p \wedge p \iff p$$
$$p \vee p \iff p$$

$$p \wedge q \iff q \wedge p$$
$$p \vee q \iff q \vee p$$

$$(p \rightarrow q) \iff (\neg p \vee q)$$
$$(p \rightarrow q) \iff (\neg q \rightarrow \neg p)$$

$$(p \leftrightarrow q) \iff ((p \rightarrow q) \wedge (q \leftarrow p))$$

$$(p \wedge q \rightarrow r) \iff (p \rightarrow (q \rightarrow r))$$

$$(p \wedge (q \wedge r)) \iff ((p \wedge q) \wedge r)$$
$$(p \vee (q \vee r)) \iff ((p \vee q) \vee r)$$

$$(p \wedge (q \vee r)) \iff ((p \wedge q) \vee (p \wedge r))$$
$$(p \vee (q \wedge r)) \iff ((p \vee q) \wedge (p \vee r))$$

**Example** By double negation, from $\neg\neg p \rightarrow \neg\neg q$ we can infer $p \rightarrow q$

Note that we are applying it to the sub-expressions here.

A **replacement rule** is of the form $p \iff q$ and means that whenever $p$ is in a subexpression, it may be replaced with $q$ and vice-versa.

$p \iff \neg\neg p$

$\neg(p \vee q) \iff \neg p \wedge \neg q$
$\neg(p \wedge q) \iff \neg p \vee \neg q$

$p \wedge p \iff p$
$p \vee p \iff p$

$p \wedge q \iff q \wedge p$
$p \vee q \iff q \vee p$

$(p \rightarrow q) \iff (\neg p \vee q)$
$(p \rightarrow q) \iff (\neg q \rightarrow \neg p)$

$(p \leftrightarrow q) \iff ((p \rightarrow q) \wedge (q \leftarrow p))$

$(p \wedge q \rightarrow r) \iff (p \rightarrow (q \rightarrow r))$

$(p \wedge (q \wedge r)) \iff ((p \wedge q) \wedge r)$
$(p \vee (q \vee r)) \iff ((p \vee q) \vee r)$

$(p \wedge (q \vee r)) \iff ((p \wedge q) \vee (p \wedge r))$
$(p \vee (q \wedge r)) \iff ((p \vee q) \wedge (p \vee r))$

**Example** By double negation, from $\neg\neg p \rightarrow \neg\neg q$ we can infer $p \rightarrow q$

Also, note that here we are applying the rule backward because $\iff$ applies both ways

A **replacement rule** is of the form $p \iff q$ and means that whenever $p$ is in a subexpression, it may be replaced with $q$ and vice-versa.

$p \iff \neg\neg p$

$\neg(p \vee q) \iff \neg p \wedge \neg q$
$\neg(p \wedge q) \iff \neg p \vee \neg q$

$p \wedge p \iff p$
$p \vee p \iff p$

$p \wedge q \iff q \wedge p$
$p \vee q \iff q \vee p$

$(p \to q) \iff (\neg p \vee q)$
$(p \to q) \iff (\neg q \to \neg p)$

$(p \leftrightarrow q) \iff ((p \to q) \wedge (q \leftarrow p))$

$(p \wedge q \to r) \iff (p \to (q \to r))$

$(p \wedge (q \wedge r)) \iff ((p \wedge q) \wedge r)$
$(p \vee (q \vee r)) \iff ((p \vee q) \vee r)$

$(p \wedge (q \vee r)) \iff ((p \wedge q) \vee (p \wedge r))$
$(p \vee (q \wedge r)) \iff ((p \vee q) \wedge (p \vee r))$

**Example** By double negation, from $\neg\neg p \to \neg\neg q$ we can infer $p \to q$

Let us demonstrate the use of replacement rules in a proof

A **replacement rule** is of the form $p \iff q$ and means that whenever $p$ is in a subexpression, it may be replaced with $q$ and vice-versa.

$$p \iff \neg\neg p$$

$$\neg(p \lor q) \iff \neg p \land \neg q$$
$$\neg(p \land q) \iff \neg p \lor \neg q$$

$$p \land p \iff p$$
$$p \lor p \iff p$$

$$p \land q \iff q \land p$$
$$p \lor q \iff q \lor p$$

$$(p \to q) \iff (\neg p \lor q)$$
$$(p \to q) \iff (\neg q \to \neg p)$$

$$(p \leftrightarrow q) \iff ((p \to q) \land (q \leftarrow p))$$

$$(p \land q \to r) \iff (p \to (q \to r))$$

$$(p \land (q \land r)) \iff ((p \land q) \land r)$$
$$(p \lor (q \lor r)) \iff ((p \lor q) \lor r)$$

$$(p \land (q \lor r)) \iff ((p \land q) \lor (p \land r))$$
$$(p \lor (q \land r)) \iff ((p \lor q) \land (p \lor r))$$

**Example** By double negation, from $\neg\neg p \to \neg\neg q$ we can infer $p \to q$

Theorem 1 $p \vdash p \lor q$

Let us prove the addition rule

A **replacement rule** is of the form $p \iff q$ and means that whenever $p$ is in a subexpression, it may be replaced with $q$ and vice-versa.

$$p \iff \neg\neg p$$

$$\neg(p \vee q) \iff \neg p \wedge \neg q$$
$$\neg(p \wedge q) \iff \neg p \vee \neg q$$

$$p \wedge p \iff p$$
$$p \vee p \iff p$$

$$p \wedge q \iff q \wedge p$$
$$p \vee q \iff q \vee p$$

$$(p \to q) \iff (\neg p \vee q)$$
$$(p \to q) \iff (\neg q \to \neg p)$$

$$(p \leftrightarrow q) \iff ((p \to q) \wedge (q \leftarrow p))$$

$$(p \wedge q \to r) \iff (p \to (q \to r))$$

$$(p \wedge (q \wedge r)) \iff ((p \wedge q) \wedge r)$$
$$(p \vee (q \vee r)) \iff ((p \vee q) \vee r)$$

$$(p \wedge (q \vee r)) \iff ((p \wedge q) \vee (p \wedge r))$$
$$(p \vee (q \wedge r)) \iff ((p \vee q) \wedge (p \vee r))$$

**Example** By double negation, from $\neg\neg p \to \neg\neg q$ we can infer $p \to q$

Theorem 1 $p \vdash p \vee q$

The strategy of the proof will involve the following observations

A **replacement rule** is of the form $p \iff q$ and means that whenever $p$ is in a subexpression, it may be replaced with $q$ and vice-versa.

$$p \iff \neg\neg p$$

$$\neg(p \vee q) \iff \neg p \wedge \neg q$$
$$\neg(p \wedge q) \iff \neg p \vee \neg q$$

$$p \wedge p \iff p$$
$$p \vee p \iff p$$

$$p \wedge q \iff q \wedge p$$
$$p \vee q \iff q \vee p$$

$$(p \to q) \iff (\neg p \vee q)$$
$$(p \to q) \iff (\neg q \to \neg p)$$

$$(p \leftrightarrow q) \iff ((p \to q) \wedge (q \leftarrow p))$$

$$(p \wedge q \to r) \iff (p \to (q \to r))$$

$$(p \wedge (q \wedge r)) \iff ((p \wedge q) \wedge r)$$
$$(p \vee (q \vee r)) \iff ((p \vee q) \vee r)$$

$$(p \wedge (q \vee r)) \iff ((p \wedge q) \vee (p \wedge r))$$
$$(p \vee (q \wedge r)) \iff ((p \vee q) \wedge (p \vee r))$$

**Example** By double negation, from $\neg\neg p \to \neg\neg q$ we can infer $p \to q$

Theorem 1 $p \vdash p \vee q$

The first is that there is a (highlighted) replacement rule to convert $\to$ in terms of $\vee$

A **replacement rule** is of the form $p \iff q$ and means that whenever $p$ is in a subexpression, it may be replaced with $q$ and vice-versa.

$p \iff \neg\neg p$

$\neg(p \vee q) \iff \neg p \wedge \neg q$
$\neg(p \wedge q) \iff \neg p \vee \neg q$

$p \wedge p \iff p$
$p \vee p \iff p$

$p \wedge q \iff q \wedge p$
$p \vee q \iff q \vee p$

$(p \to q) \iff (\neg p \vee q)$
$(p \to q) \iff (\neg q \to \neg p)$

$(p \leftrightarrow q) \iff ((p \to q) \wedge (q \leftarrow p))$

$(p \wedge q \to r) \iff (p \to (q \to r))$

$(p \wedge (q \wedge r)) \iff ((p \wedge q) \wedge r)$
$(p \vee (q \vee r)) \iff ((p \vee q) \vee r)$

$(p \wedge (q \vee r)) \iff ((p \wedge q) \vee (p \wedge r))$
$(p \vee (q \wedge r)) \iff ((p \vee q) \wedge (p \vee r))$

**Example** By double negation, from $\neg\neg p \to \neg\neg q$ we can infer $p \to q$

Theorem 1 $p \vdash p \vee q$

But we need to apply it on $\neg p \to q$ so that the $\neg$ will be eliminated

A **replacement rule** is of the form $p \iff q$ and means that whenever $p$ is in a subexpression, it may be replaced with $q$ and vice-versa.

$p \iff \neg\neg p$

$\neg(p \vee q) \iff \neg p \wedge \neg q$
$\neg(p \wedge q) \iff \neg p \vee \neg q$

$p \wedge p \iff p$
$p \vee p \iff p$

$p \wedge q \iff q \wedge p$
$p \vee q \iff q \vee p$

$(p \to q) \iff (\neg p \vee q)$
$(p \to q) \iff (\neg q \to \neg p)$

$(p \leftrightarrow q) \iff ((p \to q) \wedge (q \leftarrow p))$

$(p \wedge q \to r) \iff (p \to (q \to r))$

$(p \wedge (q \wedge r)) \iff ((p \wedge q) \wedge r)$
$(p \vee (q \vee r)) \iff ((p \vee q) \vee r)$

$(p \wedge (q \vee r)) \iff ((p \wedge q) \vee (p \wedge r))$
$(p \vee (q \wedge r)) \iff ((p \vee q) \wedge (p \vee r))$

**Example** By double negation, from $\neg\neg p \to \neg\neg q$ we can infer $p \to q$

Theorem 1 $p \vdash p \vee q$

by double negation

A **replacement rule** is of the form $p \iff q$ and means that whenever $p$ is in a subexpression, it may be replaced with $q$ and vice-versa.

$p \iff \neg\neg p$

$\neg(p \vee q) \iff \neg p \wedge \neg q$
$\neg(p \wedge q) \iff \neg p \vee \neg q$

$p \wedge p \iff p$
$p \vee p \iff p$

$p \wedge q \iff q \wedge p$
$p \vee q \iff q \vee p$

$(p \rightarrow q) \iff (\neg p \vee q)$
$(p \rightarrow q) \iff (\neg q \rightarrow \neg p)$

$(p \leftrightarrow q) \iff ((p \rightarrow q) \wedge (q \leftarrow p))$

$(p \wedge q \rightarrow r) \iff (p \rightarrow (q \rightarrow r))$

$(p \wedge (q \wedge r)) \iff ((p \wedge q) \wedge r)$
$(p \vee (q \vee r)) \iff ((p \vee q) \vee r)$

$(p \wedge (q \vee r)) \iff ((p \wedge q) \vee (p \wedge r))$
$(p \vee (q \wedge r)) \iff ((p \vee q) \wedge (p \vee r))$

**Example** By double negation, from $\neg\neg p \rightarrow \neg\neg q$ we can infer $p \rightarrow q$

Theorem 1 $p \vdash p \vee q$

It will turn out that FL1 axiom will provide the needed $\neg p \rightarrow q$

A **replacement rule** is of the form $p \iff q$ and means that whenever $p$ is in a subexpression, it may be replaced with $q$ and vice-versa.

$p \iff \neg\neg p$

$\neg(p \vee q) \iff \neg p \wedge \neg q$
$\neg(p \wedge q) \iff \neg p \vee \neg q$

$p \wedge p \iff p$
$p \vee p \iff p$

$p \wedge q \iff q \wedge p$
$p \vee q \iff q \vee p$

$(p \to q) \iff (\neg p \vee q)$
$(p \to q) \iff (\neg q \to \neg p)$

$(p \leftrightarrow q) \iff ((p \to q) \wedge (q \leftarrow p))$

$(p \wedge q \to r) \iff (p \to (q \to r))$

$(p \wedge (q \wedge r)) \iff ((p \wedge q) \wedge r)$
$(p \vee (q \vee r)) \iff ((p \vee q) \vee r)$

$(p \wedge (q \vee r)) \iff ((p \wedge q) \vee (p \wedge r))$
$(p \vee (q \wedge r)) \iff ((p \vee q) \wedge (p \vee r))$

**Example** By double negation, from $\neg\neg p \to \neg\neg q$ we can infer $p \to q$

Theorem 1 $p \vdash p \vee q$

*Proof*

From now on I will shift to the following notation for proofs

A **replacement rule** is of the form $p \iff q$ and means that whenever $p$ is in a subexpression, it may be replaced with $q$ and vice-versa.

$p \iff \neg\neg p$

$\neg(p \lor q) \iff \neg p \land \neg q$
$\neg(p \land q) \iff \neg p \lor \neg q$

$p \land p \iff p$
$p \lor p \iff p$

$p \land q \iff q \land p$
$p \lor q \iff q \lor p$

$(p \to q) \iff (\neg p \lor q)$
$(p \to q) \iff (\neg q \to \neg p)$

$(p \leftrightarrow q) \iff ((p \to q) \land (q \leftarrow p))$

$(p \land q \to r) \iff (p \to (q \to r))$

$(p \land (q \land r)) \iff ((p \land q) \land r)$
$(p \lor (q \lor r)) \iff ((p \lor q) \lor r)$

$(p \land (q \lor r)) \iff ((p \land q) \lor (p \land r))$
$(p \lor (q \land r)) \iff ((p \lor q) \land (p \lor r))$

**Example** By double negation, from $\neg\neg p \to \neg\neg q$ we can infer $p \to q$

Theorem 1 $p \vdash p \lor q$

*Proof*

First we state the two hypotheses

A **replacement rule** is of the form $p \iff q$ and means that whenever $p$ is in a subexpression, it may be replaced with $q$ and vice-versa.

$p \iff \neg\neg p$

$\neg(p \vee q) \iff \neg p \wedge \neg q$
$\neg(p \wedge q) \iff \neg p \vee \neg q$

$p \wedge p \iff p$
$p \vee p \iff p$

$p \wedge q \iff q \wedge p$
$p \vee q \iff q \vee p$

$(p \to q) \iff (\neg p \vee q)$
$(p \to q) \iff (\neg q \to \neg p)$

$(p \leftrightarrow q) \iff ((p \to q) \wedge (q \leftarrow p))$

$(p \wedge q \to r) \iff (p \to (q \to r))$

$(p \wedge (q \wedge r)) \iff ((p \wedge q) \wedge r)$
$(p \vee (q \vee r)) \iff ((p \vee q) \vee r)$

$(p \wedge (q \vee r)) \iff ((p \wedge q) \vee (p \wedge r))$
$(p \vee (q \wedge r)) \iff ((p \vee q) \wedge (p \vee r))$

**Example** By double negation, from $\neg\neg p \to \neg\neg q$ we can infer $p \to q$

Theorem 1 $p \vdash p \vee q$

*Proof*

| | |
|---|---|
| 0 | $p$ (Given) |

which is this

A **replacement rule** is of the form $p \iff q$ and means that whenever $p$ is in a subexpression, it may be replaced with $q$ and vice-versa.

$$p \iff \neg\neg p$$

$$\neg(p \vee q) \iff \neg p \wedge \neg q$$
$$\neg(p \wedge q) \iff \neg p \vee \neg q$$

$$p \wedge p \iff p$$
$$p \vee p \iff p$$

$$p \wedge q \iff q \wedge p$$
$$p \vee q \iff q \vee p$$

$$(p \to q) \iff (\neg p \vee q)$$
$$(p \to q) \iff (\neg q \to \neg p)$$

$$(p \leftrightarrow q) \iff ((p \to q) \wedge (q \leftarrow p))$$

$$(p \wedge q \to r) \iff (p \to (q \to r))$$

$$(p \wedge (q \wedge r)) \iff ((p \wedge q) \wedge r)$$
$$(p \vee (q \vee r)) \iff ((p \vee q) \vee r)$$

$$(p \wedge (q \vee r)) \iff ((p \wedge q) \vee (p \wedge r))$$
$$(p \vee (q \wedge r)) \iff ((p \vee q) \wedge (p \vee r))$$

**Example** By double negation, from $\neg\neg p \to \neg\neg q$ we can infer $p \to q$

Theorem 1 $p \vdash p \vee q$

*Proof*

| | |
|---|---|
| 0 | $p$ (Given) |
| 1 | $p \to (\neg q \to p)$ (FL1) |

Notice that FL1 is precisely what is needed to use Modus Ponens

A **replacement rule** is of the form $p \iff q$ and means that whenever $p$ is in a subexpression, it may be replaced with $q$ and vice-versa.

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$\neg(p \vee q) \iff \neg p \wedge \neg q$
$\neg(p \wedge q) \iff \neg p \vee \neg q$

$p \wedge p \iff p$
$p \vee p \iff p$

$p \wedge q \iff q \wedge p$
$p \vee q \iff q \vee p$

$(p \to q) \iff (\neg p \vee q)$
$(p \to q) \iff (\neg q \to \neg p)$

$(p \leftrightarrow q) \iff ((p \to q) \wedge (q \leftarrow p))$

$(p \wedge q \to r) \iff (p \to (q \to r))$

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$(p \vee (q \wedge r)) \iff ((p \vee q) \wedge (p \vee r))$

**Example** By double negation, from $\neg\neg p \to \neg\neg q$ we can infer $p \to q$

Theorem 1 $p \vdash p \vee q$

*Proof*

| | |
|---|---|
| 0 | $p$ (Given) |
| 1 | $p \to (\neg q \to p)$ (FL1) |
| 3 | $\neg q \to p$ (by Modus Ponens) |

By Modus Ponens on it gives an expression

A **replacement rule** is of the form $p \iff q$ and means that whenever $p$ is in a subexpression, it may be replaced with $q$ and vice-versa.

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$\neg(p \wedge q) \iff \neg p \vee \neg q$

$p \wedge p \iff p$
$p \vee p \iff p$

$p \wedge q \iff q \wedge p$
$p \vee q \iff q \vee p$

$(p \to q) \iff (\neg p \vee q)$
$(p \to q) \iff (\neg q \to \neg p)$

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$(p \vee (q \vee r)) \iff ((p \vee q) \vee r)$

$(p \wedge (q \vee r)) \iff ((p \wedge q) \vee (p \wedge r))$
$(p \vee (q \wedge r)) \iff ((p \vee q) \wedge (p \vee r))$

**Example** By double negation, from $\neg\neg p \to \neg\neg q$ we can infer $p \to q$

Theorem 1 $p \vdash p \vee q$

*Proof*

0 | $p$ (Given)
1 | $p \to (\neg q \to p)$ (FL1)
3 | $\neg q \to p$ (by Modus Ponens)
5 | $\neg\neg q \vee p$

This is from the (highlighted) rule that defines $\to$ in terms of $\vee$ and $\neg$

A **replacement rule** is of the form $p \iff q$ and means that whenever $p$ is in a subexpression, it may be replaced with $q$ and vice-versa.

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$(p \vee (q \wedge r)) \iff ((p \vee q) \wedge (p \vee r))$

**Example** By double negation, from $\neg\neg p \to \neg\neg q$ we can infer $p \to q$

Theorem 1 $p \vdash p \vee q$

*Proof*

| | |
|---|---|
| 0 | $p$ (Given) |
| 1 | $p \to (\neg q \to p)$ (FL1) |
| 3 | $\neg q \to p$ (by Modus Ponens) |
| 5 | $\neg\neg q \vee p$ |
| 5 | $q \vee p$ (Double negation) |

This one again follows from double negation