Exercise sheet 3

Set theory and Logic, MTH303

1. Show that $\{\forall x(\phi \to \psi), \forall x \phi\} \vDash \forall x \psi$
2. Prove that $\phi \nvDash \forall x \phi$ if x is free in ϕ but $\phi \vDash \forall x \phi$ if x is not free in ϕ
3. Show that $\alpha \vDash \forall x \alpha$ as long as x is *not* free in α.
4. Define a new quantifier $\exists!$, so that $\exists! \alpha$ which is satisfied by s in a structure $\mathcal{U}$ iff there exists exactly one $k \in \mathcal{U}$ such tht $\vDash_{\mathcal{U}} \alpha[s(x|k)]$. Show that there $\exists! x \alpha$ is always logically equivalent to a formula that involves the letters provided by a usual first order alphabet.
5. Consider a first order language with just a one binary predicate symbol and no function symbol. Up to isomorphism, how many structures does it have with just two elements?
6. Let P denote a predicate symbol. Use the axioms to show that $\vdash Px \to \exists Py$. What is this theorem telling you?
7. Show that if $\vdash \alpha \to \beta$ then $\vdash \forall x \alpha \to \forall x \beta$.