

Definition p_0, p_1, \dots, p_{n-1} and q are propositional forms

If we have a sequence of propositional forms p_i and a propositional form q

Definition p_0, p_1, \dots, p_{n-1} and q are propositional forms

p_0, p_1, \dots, p_{n-1} **logically implies** q if

How do we say that the p_i 's imply q ?

Definition p_0, p_1, \dots, p_{n-1} and q are propositional forms

p_0, p_1, \dots, p_{n-1} **logically implies** q if

$$p_0 \wedge p_1 \wedge \dots \wedge p_{n-1} \rightarrow q$$

It simply means that if each of the p_i 's are true, then so is q

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In other words, the above implication is always true

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i.e. it is a tautology

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Notation:

$$p_0, p_1, \dots, p_{n-1} \models q$$

We use this notation

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Equivalently, if there is a valuation ν , so that $\nu(p_i) = T$,

In terms of valuations, it means that those valuations that assign true to p_i

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must also assign true to q

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Example $P \rightarrow Q, P \models Q$

For example, should not $P \rightarrow Q$, and P , logically imply Q if we have defined \rightarrow correctly?

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Example $P \rightarrow Q, P \models Q$

To show, $\models ((P \rightarrow Q) \wedge P) \rightarrow Q$

By the definition above, it means we need to check that this expression is a tautology

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To show, $\models ((P \rightarrow Q) \wedge P) \rightarrow Q$

Proof: Exercise using truth tables!

That should be a simple exercise

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that will tell us when, given a propositional form

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To show, $\models ((P \rightarrow Q) \wedge P) \rightarrow Q$

Proof: Exercise using truth tables!

Axioms

We will have a starting point of assertions that we accept without proof

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Example $P \rightarrow Q, P \models Q$

To show, $\models ((P \rightarrow Q) \wedge P) \rightarrow Q$

Proof: Exercise using truth tables!

Axioms and inference rules

and syntactic rules to derive assertions from those axioms

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Proof: Exercise using truth tables!

Axioms and inference rules

These rules will merely involve manipulating the strings

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Axioms and inference rules

The goal will be to come up with these string manipulation rules

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To show, $\models ((P \rightarrow Q) \wedge P) \rightarrow Q$

Proof: Exercise using truth tables!

Axioms and inference rules

And axioms so that by performing repeated string manipulations

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To show, $\models ((P \rightarrow Q) \wedge P) \rightarrow Q$

Proof: Exercise using truth tables!

Axioms and inference rules

allowed by the rules, we will be able to come up with statements that we judge to be true

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To show, $\models ((P \rightarrow Q) \wedge P) \rightarrow Q$

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Axioms and inference rules

Examples of inference rules:

Let us first look at an example of an inference rule

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To show, $\models ((P \rightarrow Q) \wedge P) \rightarrow Q$

Proof: Exercise using truth tables!

Axioms and inference rules

Examples of inference rules:

1. $p \rightarrow q, p \implies q$

Here p and q represent any propositional form

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Example $P \rightarrow Q, P \models Q$

To show, $\models ((P \rightarrow Q) \wedge P) \rightarrow Q$

Proof: Exercise using truth tables!

Axioms and inference rules

Examples of inference rules:

1. $p \rightarrow q, p \implies q$ (Modus Ponens)

It is called “Modus Ponens”

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Example $P \rightarrow Q, P \models Q$

To show, $\models ((P \rightarrow Q) \wedge P) \rightarrow Q$

Proof: Exercise using truth tables!

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Axioms and inference rules

Examples of inference rules:

1. $p \rightarrow q, p \implies q$ (Modus Ponens)

Example application:

$$(P \wedge Q \rightarrow R) \rightarrow (R \wedge S), (P \wedge Q \rightarrow R)$$

Let us understand it with an application

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To show, $\models ((P \rightarrow Q) \wedge P) \rightarrow Q$

Proof: Exercise using truth tables!

Axioms and inference rules

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Note that that $P \wedge Q \rightarrow R$ matches exactly p in the rule

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Similarly, even $R \wedge S$ is q

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Examples of inference rules:

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The rule Modus Ponens tells us that we can infer the highlighted expression

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Axioms and inference rules

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Rules of this kind are the building blocks of a proof

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Axioms and inference rules

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p_0, p_1, \dots, p_{n-1} **infer** q if

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Example application:

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Modus Ponens was an example of this.

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Proof: Exercise using truth tables!

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Axioms and inference rules

Definition p_0, p_1, \dots, p_{n-1} and q are propositional forms

p_0, p_1, \dots, p_{n-1} **infer** q if q can be written whenever p_0, p_1, \dots, p_{n-1} can be written.

Examples of inference rules:

1. $p \rightarrow q, p \implies q$ (Modus Ponens)

Example application:

$$(P \wedge Q \rightarrow R) \rightarrow (R \wedge S), (P \wedge Q \rightarrow R) \implies (R \wedge S)$$

It declares that if propositions of patterns specified by the p_i exist, then you can infer q

Definition p_0, p_1, \dots, p_{n-1} and q are propositional forms

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Notation:

$$p_0, p_1, \dots, p_{n-1} \Longrightarrow q$$

Examples of inference rules:

1. $p \rightarrow q, p \Longrightarrow q$ (Modus Ponens)

Example application:

$$(P \wedge Q \rightarrow R) \rightarrow (R \wedge S), (P \wedge Q \rightarrow R) \Longrightarrow (R \wedge S)$$

We denote it with \Longrightarrow as shown in the Modus Ponens example.

Axioms and inference rules

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If this is not clear yet, it should be when we use this to define a formal proof

A **formal proof** of the propositional form q

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Axioms and inference rules

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Example application:

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We try to prove some propositional form q

A **formal proof** of the propositional form q from propositional forms p_0, p_1, \dots, p_{n-1}

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Axioms and inference rules

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Example application:

$$(P \wedge Q \rightarrow R) \rightarrow (R \wedge S), (P \wedge Q \rightarrow R) \implies (R \wedge S)$$

assuming some propositions p_i as assumptions or hypotheses

A **formal proof** of the propositional form q from propositional forms p_0, p_1, \dots, p_{n-1} is a sequence of propositional forms,

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Axioms and inference rules

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Example application:

$$(P \wedge Q \rightarrow R) \rightarrow (R \wedge S), (P \wedge Q \rightarrow R) \implies (R \wedge S)$$

A proof involves some intermediate steps q_i

A **formal proof** of the propositional form q from propositional forms p_0, p_1, \dots, p_{n-1} is a sequence of propositional forms,

$$p_0, p_1, \dots, p_{n-1}, q_0, q_1, \dots, q_{m-1}$$

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Axioms and inference rules

Definition p_0, p_1, \dots, p_{n-1} and q are propositional forms

p_0, p_1, \dots, p_{n-1} **infer** q if q can be written whenever p_0, p_1, \dots, p_{n-1} can be written.

Notation:

$$p_0, p_1, \dots, p_{n-1} \implies q$$

Examples of inference rules:

$$1. p \rightarrow q, p \implies q \text{ (Modus Ponens)}$$

Example application:

$$(P \wedge Q \rightarrow R) \rightarrow (R \wedge S), (P \wedge Q \rightarrow R) \implies (R \wedge S)$$

that connects the hypotheses to the conclusion

A **formal proof** of the propositional form q from propositional forms p_0, p_1, \dots, p_{n-1} is a sequence of propositional forms,

$$p_0, p_1, \dots, p_{n-1}, q_0, q_1, \dots, q_{m-1}$$

such that $q_{m-1} = q$,

Syntax

Axioms and inference rules

Definition p_0, p_1, \dots, p_{n-1} and q are propositional forms

p_0, p_1, \dots, p_{n-1} **infer** q if q can be written whenever p_0, p_1, \dots, p_{n-1} can be written.

Notation:

$$p_0, p_1, \dots, p_{n-1} \implies q$$

Examples of inference rules:

1. $p \rightarrow q, p \implies q$ (Modus Ponens)

Example application:

$$(P \wedge Q \rightarrow R) \rightarrow (R \wedge S), (P \wedge Q \rightarrow R) \implies (R \wedge S)$$

the last statement of the proof must be the conclusion

A **formal proof** of the propositional form q from propositional forms p_0, p_1, \dots, p_{n-1} is a sequence of propositional forms,

$$p_0, p_1, \dots, p_{n-1}, q_0, q_1, \dots, q_{m-1}$$

such that $q_{m-1} = q$, and each q_i is either an axiom

Syntax

Axioms and inference rules

Definition p_0, p_1, \dots, p_{n-1} and q are propositional forms

p_0, p_1, \dots, p_{n-1} **infer** q if q can be written whenever p_0, p_1, \dots, p_{n-1} can be written.

Notation:

$$p_0, p_1, \dots, p_{n-1} \implies q$$

Examples of inference rules:

1. $p \rightarrow q, p \implies q$ (Modus Ponens)

Example application:

$$(P \wedge Q \rightarrow R) \rightarrow (R \wedge S), (P \wedge Q \rightarrow R) \implies (R \wedge S)$$

some of these intermediate steps may quote axioms (agreed to be true without proof)

A **formal proof** of the propositional form q from propositional forms p_0, p_1, \dots, p_{n-1} is a sequence of propositional forms,

$$p_0, p_1, \dots, p_{n-1}, q_0, q_1, \dots, q_{m-1}$$

such that $q_{m-1} = q$, and each q_i is either an axiom or $p_0, p_1, \dots, p_{n-1}, q_0, q_1, \dots, q_{i-1} \implies q_i$

Syntax

Axioms and inference rules

Definition p_0, p_1, \dots, p_{n-1} and q are propositional forms

p_0, p_1, \dots, p_{n-1} **infer** q if q can be written whenever p_0, p_1, \dots, p_{n-1} can be written.

Notation:

$$p_0, p_1, \dots, p_{n-1} \implies q$$

Examples of inference rules:

1. $p \rightarrow q, p \implies q$ (Modus Ponens)

Example application:

$$(P \wedge Q \rightarrow R) \rightarrow (R \wedge S), (P \wedge Q \rightarrow R) \implies (R \wedge S)$$

or they must be an inference rule specified in the previous definition

A **formal proof** of the propositional form q from propositional forms p_0, p_1, \dots, p_{n-1} is a sequence of propositional forms,

$$p_0, p_1, \dots, p_{n-1}, q_0, q_1, \dots, q_{m-1}$$

such that $q_{m-1} = q$, and each q_i is either an axiom or $p_0, p_1, \dots, p_{n-1}, q_0, q_1, \dots, q_{i-1} \implies q_i$

Notation:

$$p_0, p_1, \dots, p_{n-1} \vdash q$$

Syntax

Axioms and inference rules

Definition p_0, p_1, \dots, p_{n-1} and q are propositional forms

p_0, p_1, \dots, p_{n-1} **infer** q if q can be written whenever p_0, p_1, \dots, p_{n-1} can be written.

Notation:

$$p_0, p_1, \dots, p_{n-1} \implies q$$

Examples of inference rules:

1. $p \rightarrow q, p \implies q$ (Modus Ponens)

Example application:

$$(P \wedge Q \rightarrow R) \rightarrow (R \wedge S), (P \wedge Q \rightarrow R) \implies (R \wedge S)$$

We use \vdash to denote the existence of a proof connection the hypothesis to the conclusion

A **formal proof** of the propositional form q from propositional forms p_0, p_1, \dots, p_{n-1} is a sequence of propositional forms,

$$p_0, p_1, \dots, p_{n-1}, q_0, q_1, \dots, q_{m-1}$$

such that $q_{m-1} = q$, and each q_i is either an axiom or $p_0, p_1, \dots, p_{n-1}, q_0, q_1, \dots, q_{i-1} \implies q_i$

Notation:

$$p_0, p_1, \dots, p_{n-1} \vdash q$$

Example $P \vee Q \rightarrow Q \wedge R, P \vdash$

Syntax

Axioms and inference rules

Definition p_0, p_1, \dots, p_{n-1} and q are propositional forms

p_0, p_1, \dots, p_{n-1} **infer** q if q can be written whenever p_0, p_1, \dots, p_{n-1} can be written.

Notation:

$$p_0, p_1, \dots, p_{n-1} \implies q$$

Examples of inference rules:

1. $p \rightarrow q, p \implies q$ (Modus Ponens)

Example application:

$$(P \wedge Q \rightarrow R) \rightarrow (R \wedge S), (P \wedge Q \rightarrow R) \implies (R \wedge S)$$

Here is an example of statement in logic which we will prove formally

A **formal proof** of the propositional form q from propositional forms p_0, p_1, \dots, p_{n-1} is a sequence of propositional forms,

$$p_0, p_1, \dots, p_{n-1}, q_0, q_1, \dots, q_{m-1}$$

such that $q_{m-1} = q$, and each q_i is either an axiom or $p_0, p_1, \dots, p_{n-1}, q_0, q_1, \dots, q_{i-1} \implies q_i$

Notation:

$$p_0, p_1, \dots, p_{n-1} \vdash q$$

Example $P \vee Q \rightarrow Q \wedge R, P \vdash Q$

Syntax

Axioms and inference rules

Definition p_0, p_1, \dots, p_{n-1} and q are propositional forms

p_0, p_1, \dots, p_{n-1} **infer** q if q can be written whenever p_0, p_1, \dots, p_{n-1} can be written.

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Examples of inference rules:

1. $p \rightarrow q, p \implies q$ (Modus Ponens)

Example application:

$$(P \wedge Q \rightarrow R) \rightarrow (R \wedge S), (P \wedge Q \rightarrow R) \implies (R \wedge S)$$

A **formal proof** of the propositional form q from propositional forms p_0, p_1, \dots, p_{n-1} is a sequence of propositional forms,

$$p_0, p_1, \dots, p_{n-1}, q_0, q_1, \dots, q_{m-1}$$

such that $q_{m-1} = q$, and each q_i is either an axiom or $p_0, p_1, \dots, p_{n-1}, q_0, q_1, \dots, q_{i-1} \implies q_i$

Notation:

$$p_0, p_1, \dots, p_{n-1} \vdash q$$

Example $P \vee Q \rightarrow Q \wedge R, P \vdash Q$

Syntax

Axioms and inference rules

Definition p_0, p_1, \dots, p_{n-1} and q are propositional forms

p_0, p_1, \dots, p_{n-1} **infer** q if q can be written whenever p_0, p_1, \dots, p_{n-1} can be written.

Notation:

$$p_0, p_1, \dots, p_{n-1} \implies q$$

Examples of inference rules:

1. $p \rightarrow q, p \implies q$ (Modus Ponens)

Example application:

$$(P \wedge Q \rightarrow R) \rightarrow (R \wedge S), (P \wedge Q \rightarrow R) \implies (R \wedge S)$$

It is easy to check that this is a tautology but that is semantics

A **formal proof** of the propositional form q from propositional forms p_0, p_1, \dots, p_{n-1} is a sequence of propositional forms,

$$p_0, p_1, \dots, p_{n-1}, q_0, q_1, \dots, q_{m-1}$$

such that $q_{m-1} = q$, and each q_i is either an axiom or $p_0, p_1, \dots, p_{n-1}, q_0, q_1, \dots, q_{i-1} \implies q_i$

Notation:

$$p_0, p_1, \dots, p_{n-1} \vdash q$$

Example $P \vee Q \rightarrow Q \wedge R, P \vdash Q$

Syntax

Axioms and inference rules

Definition p_0, p_1, \dots, p_{n-1} and q are propositional forms

p_0, p_1, \dots, p_{n-1} **infer** q if q can be written whenever p_0, p_1, \dots, p_{n-1} can be written.

Notation:

$$p_0, p_1, \dots, p_{n-1} \implies q$$

Examples of inference rules:

1. $p \rightarrow q, p \implies q$ (Modus Ponens)

Example application:

$$(P \wedge Q \rightarrow R) \rightarrow (R \wedge S), (P \wedge Q \rightarrow R) \implies (R \wedge S)$$

We want to show this by using an proof system alone

A **formal proof** of the propositional form q from propositional forms p_0, p_1, \dots, p_{n-1} is a sequence of propositional forms,

$$p_0, p_1, \dots, p_{n-1}, q_0, q_1, \dots, q_{m-1}$$

such that $q_{m-1} = q$, and each q_i is either an axiom or $p_0, p_1, \dots, p_{n-1}, q_0, q_1, \dots, q_{i-1} \implies q_i$

Notation:

$$p_0, p_1, \dots, p_{n-1} \vdash q$$

Example $P \vee Q \rightarrow Q \wedge R, P \vdash Q$

$$p_0 = P \vee Q \rightarrow Q \wedge R$$

Syntax

Axioms and inference rules

Definition p_0, p_1, \dots, p_{n-1} and q are propositional forms

p_0, p_1, \dots, p_{n-1} **infer** q if q can be written whenever p_0, p_1, \dots, p_{n-1} can be written.

Notation:

$$p_0, p_1, \dots, p_{n-1} \implies q$$

Examples of inference rules:

1. $p \rightarrow q, p \implies q$ (Modus Ponens)

Example application:

$$(P \wedge Q \rightarrow R) \rightarrow (R \wedge S), (P \wedge Q \rightarrow R) \implies (R \wedge S)$$

We denote this by p because it is the given hypothesis

A **formal proof** of the propositional form q from propositional forms p_0, p_1, \dots, p_{n-1} is a sequence of propositional forms,

$$p_0, p_1, \dots, p_{n-1}, q_0, q_1, \dots, q_{m-1}$$

such that $q_{m-1} = q$, and each q_i is either an axiom or $p_0, p_1, \dots, p_{n-1}, q_0, q_1, \dots, q_{i-1} \implies q_i$

Notation:

$$p_0, p_1, \dots, p_{n-1} \vdash q$$

Example $P \vee Q \rightarrow Q \wedge R, P \vdash Q$

$$p_0 = P \vee Q \rightarrow Q \wedge R$$

$$p_1 = P$$

Syntax

Axioms and inference rules

Definition p_0, p_1, \dots, p_{n-1} and q are propositional forms

p_0, p_1, \dots, p_{n-1} **infer** q if q can be written whenever p_0, p_1, \dots, p_{n-1} can be written.

Notation:

$$p_0, p_1, \dots, p_{n-1} \implies q$$

Examples of inference rules:

1. $p \rightarrow q, p \implies q$ (Modus Ponens)

Example application:

$$(P \wedge Q \rightarrow R) \rightarrow (R \wedge S), (P \wedge Q \rightarrow R) \implies (R \wedge S)$$

along with this

A **formal proof** of the propositional form q from propositional forms p_0, p_1, \dots, p_{n-1} is a sequence of propositional forms,

$$p_0, p_1, \dots, p_{n-1}, q_0, q_1, \dots, q_{m-1}$$

such that $q_{m-1} = q$, and each q_i is either an axiom or $p_0, p_1, \dots, p_{n-1}, q_0, q_1, \dots, q_{i-1} \implies q_i$

Notation:

$$p_0, p_1, \dots, p_{n-1} \vdash q$$

Example $P \vee Q \rightarrow Q \wedge R, P \vdash Q$

$$p_0 = P \vee Q \rightarrow Q \wedge R$$

$$p_1 = P$$

Syntax

Axioms and inference rules

Definition p_0, p_1, \dots, p_{n-1} and q are propositional forms

p_0, p_1, \dots, p_{n-1} **infer** q if q can be written whenever p_0, p_1, \dots, p_{n-1} can be written.

Notation:

$$p_0, p_1, \dots, p_{n-1} \implies q$$

Examples of inference rules:

1. $p \rightarrow q, p \implies q$ (Modus Ponens)

Example application:

$$(P \wedge Q \rightarrow R) \rightarrow (R \wedge S), (P \wedge Q \rightarrow R) \implies (R \wedge S)$$

We now need to come up with a sequence of steps leading to the conclusion

A **formal proof** of the propositional form q from propositional forms p_0, p_1, \dots, p_{n-1} is a sequence of propositional forms,

$$p_0, p_1, \dots, p_{n-1}, q_0, q_1, \dots, q_{m-1}$$

such that $q_{m-1} = q$, and each q_i is either an axiom or $p_0, p_1, \dots, p_{n-1}, q_0, q_1, \dots, q_{i-1} \implies q_i$

Notation:

$$p_0, p_1, \dots, p_{n-1} \vdash q$$

Example $P \vee Q \rightarrow Q \wedge R, P \vdash Q$

$$p_0 = P \vee Q \rightarrow Q \wedge R$$

$$p_1 = P$$

Syntax

Axioms and inference rules

Definition p_0, p_1, \dots, p_{n-1} and q are propositional forms

p_0, p_1, \dots, p_{n-1} **infer** q if q can be written whenever p_0, p_1, \dots, p_{n-1} can be written.

Notation:

$$p_0, p_1, \dots, p_{n-1} \implies q$$

Examples of inference rules:

1. $p \rightarrow q, p \implies q$ (Modus Ponens)

Example application:

$$(P \wedge Q \rightarrow R) \rightarrow (R \wedge S), (P \wedge Q \rightarrow R) \implies (R \wedge S)$$

We are not yet ready to use Modus Ponens

A **formal proof** of the propositional form q from propositional forms p_0, p_1, \dots, p_{n-1} is a sequence of propositional forms,

$$p_0, p_1, \dots, p_{n-1}, q_0, q_1, \dots, q_{m-1}$$

such that $q_{m-1} = q$, and each q_i is either an axiom or $p_0, p_1, \dots, p_{n-1}, q_0, q_1, \dots, q_{i-1} \implies q_i$

Notation:

$$p_0, p_1, \dots, p_{n-1} \vdash q$$

Example $P \vee Q \rightarrow Q \wedge R, P \vdash Q$

$$p_0 = P \vee Q \rightarrow Q \wedge R$$

$$p_1 = P$$

$$q_0 = P \vee Q$$

Syntax

Axioms and inference rules

Definition p_0, p_1, \dots, p_{n-1} and q are propositional forms

p_0, p_1, \dots, p_{n-1} **infer** q if q can be written whenever p_0, p_1, \dots, p_{n-1} can be written.

Notation:

$$p_0, p_1, \dots, p_{n-1} \implies q$$

Examples of inference rules:

1. $p \rightarrow q, p \implies q$ (Modus Ponens)

Example application:

$$(P \wedge Q \rightarrow R) \rightarrow (R \wedge S), (P \wedge Q \rightarrow R) \implies (R \wedge S)$$

But we intuitively know that $p \vee q$ must imply p

A **formal proof** of the propositional form q from propositional forms p_0, p_1, \dots, p_{n-1} is a sequence of propositional forms,

$$p_0, p_1, \dots, p_{n-1}, q_0, q_1, \dots, q_{m-1}$$

such that $q_{m-1} = q$, and each q_i is either an axiom or $p_0, p_1, \dots, p_{n-1}, q_0, q_1, \dots, q_{i-1} \implies q_i$

Notation:

$$p_0, p_1, \dots, p_{n-1} \vdash q$$

Example $P \vee Q \rightarrow Q \wedge R, P \vdash Q$

$$p_0 = P \vee Q \rightarrow Q \wedge R$$

$$p_1 = P$$

$$q_0 = P \vee Q$$

Syntax

Axioms and inference rules

Definition p_0, p_1, \dots, p_{n-1} and q are propositional forms

p_0, p_1, \dots, p_{n-1} **infer** q if q can be written whenever p_0, p_1, \dots, p_{n-1} can be written.

Notation:

$$p_0, p_1, \dots, p_{n-1} \implies q$$

Examples of inference rules:

1. $p \rightarrow q, p \implies q$ (Modus Ponens)

Example application:

$$(P \wedge Q \rightarrow R) \rightarrow (R \wedge S), (P \wedge Q \rightarrow R) \implies (R \wedge S)$$

But we cannot deduce it from anything else.

A **formal proof** of the propositional form q from propositional forms p_0, p_1, \dots, p_{n-1} is a sequence of propositional forms,

$$p_0, p_1, \dots, p_{n-1}, q_0, q_1, \dots, q_{m-1}$$

such that $q_{m-1} = q$, and each q_i is either an axiom or $p_0, p_1, \dots, p_{n-1}, q_0, q_1, \dots, q_{i-1} \implies q_i$

Notation:

$$p_0, p_1, \dots, p_{n-1} \vdash q$$

Example $P \vee Q \rightarrow Q \wedge R, P \vdash Q$

$$p_0 = P \vee Q \rightarrow Q \wedge R$$

$$p_1 = P$$

$$q_0 = P \vee Q$$

Syntax

Axioms and inference rules

Definition p_0, p_1, \dots, p_{n-1} and q are propositional forms

p_0, p_1, \dots, p_{n-1} **infer** q if q can be written whenever p_0, p_1, \dots, p_{n-1} can be written.

Notation:

$$p_0, p_1, \dots, p_{n-1} \implies q$$

Examples of inference rules:

1. $p \rightarrow q, p \implies q$ (Modus Ponens)
2. $p \implies p \vee q$ (Addition)

Example application:

$$(P \wedge Q \rightarrow R) \rightarrow (R \wedge S), (P \wedge Q \rightarrow R) \implies (R \wedge S)$$

So we add this as a rule to the list of rules

A **formal proof** of the propositional form q from propositional forms p_0, p_1, \dots, p_{n-1} is a sequence of propositional forms,

$$p_0, p_1, \dots, p_{n-1}, q_0, q_1, \dots, q_{m-1}$$

such that $q_{m-1} = q$, and each q_i is either an axiom or $p_0, p_1, \dots, p_{n-1}, q_0, q_1, \dots, q_{i-1} \implies q_i$

Notation:

$$p_0, p_1, \dots, p_{n-1} \vdash q$$

Example $P \vee Q \rightarrow Q \wedge R, P \vdash Q$

$$p_0 = P \vee Q \rightarrow Q \wedge R$$

$$p_1 = P$$

$$q_0 = P \vee Q \text{ (by Addition rule)}$$

Syntax

Axioms and inference rules

Definition p_0, p_1, \dots, p_{n-1} and q are propositional forms

p_0, p_1, \dots, p_{n-1} **infer** q if q can be written whenever p_0, p_1, \dots, p_{n-1} can be written.

Notation:

$$p_0, p_1, \dots, p_{n-1} \implies q$$

Examples of inference rules:

$$1. p \rightarrow q, p \implies q \text{ (Modus Ponens)}$$

$$2. p \implies p \vee q \text{ (Addition)}$$

Example application:

$$(P \wedge Q \rightarrow R) \rightarrow (R \wedge S), (P \wedge Q \rightarrow R) \implies (R \wedge S)$$

and quote it here in the proof

A **formal proof** of the propositional form q from propositional forms p_0, p_1, \dots, p_{n-1} is a sequence of propositional forms,

$$p_0, p_1, \dots, p_{n-1}, q_0, q_1, \dots, q_{m-1}$$

such that $q_{m-1} = q$, and each q_i is either an axiom or $p_0, p_1, \dots, p_{n-1}, q_0, q_1, \dots, q_{i-1} \implies q_i$

Notation:

$$p_0, p_1, \dots, p_{n-1} \vdash q$$

Example $P \vee Q \rightarrow Q \wedge R, P \vdash Q$

$$p_0 = P \vee Q \rightarrow Q \wedge R$$

$$p_1 = P$$

$$q_0 = P \vee Q \text{ (by Addition rule)}$$

$$q_1 = Q \wedge R$$

Syntax

Axioms and inference rules

Definition p_0, p_1, \dots, p_{n-1} and q are propositional forms

p_0, p_1, \dots, p_{n-1} **infer** q if q can be written whenever p_0, p_1, \dots, p_{n-1} can be written.

Notation:

$$p_0, p_1, \dots, p_{n-1} \implies q$$

Examples of inference rules:

$$1. p \rightarrow q, p \implies q \text{ (Modus Ponens)}$$

$$2. p \implies p \vee q \text{ (Addition)}$$

Example application:

$$(P \wedge Q \rightarrow R) \rightarrow (R \wedge S), (P \wedge Q \rightarrow R) \implies (R \wedge S)$$

Now we are ready to use Modus Ponens on p_0 and q_0

A **formal proof** of the propositional form q from propositional forms p_0, p_1, \dots, p_{n-1} is a sequence of propositional forms,

$$p_0, p_1, \dots, p_{n-1}, q_0, q_1, \dots, q_{m-1}$$

such that $q_{m-1} = q$, and each q_i is either an axiom or $p_0, p_1, \dots, p_{n-1}, q_0, q_1, \dots, q_{i-1} \implies q_i$

Notation:

$$p_0, p_1, \dots, p_{n-1} \vdash q$$

Example $P \vee Q \rightarrow Q \wedge R, P \vdash Q$

$$p_0 = P \vee Q \rightarrow Q \wedge R$$

$$p_1 = P$$

$$q_0 = P \vee Q \text{ (by Addition rule)}$$

$$q_1 = Q \wedge R \text{ (by Modus Ponens)}$$

Syntax

Axioms and inference rules

Definition p_0, p_1, \dots, p_{n-1} and q are propositional forms

p_0, p_1, \dots, p_{n-1} **infer** q if q can be written whenever p_0, p_1, \dots, p_{n-1} can be written.

Notation:

$$p_0, p_1, \dots, p_{n-1} \implies q$$

Examples of inference rules:

$$1. \rightarrow p \rightarrow q, p \implies q \text{ (Modus Ponens)}$$

$$2. \rightarrow p \implies p \vee q \text{ (Addition)}$$

Example application:

$$(P \wedge Q \rightarrow R) \rightarrow (R \wedge S), (P \wedge Q \rightarrow R) \implies (R \wedge S)$$

A **formal proof** of the propositional form q from propositional forms p_0, p_1, \dots, p_{n-1} is a sequence of propositional forms,

$$p_0, p_1, \dots, p_{n-1}, q_0, q_1, \dots, q_{m-1}$$

such that $q_{m-1} = q$, and each q_i is either an axiom or $p_0, p_1, \dots, p_{n-1}, q_0, q_1, \dots, q_{i-1} \implies q_i$

Notation:

$$p_0, p_1, \dots, p_{n-1} \vdash q$$

Example $P \vee Q \rightarrow Q \wedge R, P \vdash Q$

$$p_0 = P \vee Q \rightarrow Q \wedge R$$

$$p_1 = P$$

$$q_0 = P \vee Q \text{ (by Addition rule)}$$

$$q_1 = Q \wedge R \text{ (by Modus Ponens)}$$

$$q_2 = R$$

Syntax

Axioms and inference rules

Definition p_0, p_1, \dots, p_{n-1} and q are propositional forms

p_0, p_1, \dots, p_{n-1} **infer** q if q can be written whenever p_0, p_1, \dots, p_{n-1} can be written.

Notation:

$$p_0, p_1, \dots, p_{n-1} \implies q$$

Examples of inference rules:

$$1. \rightarrow p \rightarrow q, p \implies q \text{ (Modus Ponens)}$$

$$2. \rightarrow p \implies p \vee q \text{ (Addition)}$$

Example application:

$$(P \wedge Q \rightarrow R) \rightarrow (R \wedge S), (P \wedge Q \rightarrow R) \implies (R \wedge S)$$

Now once again, $Q \wedge R$, i.e. Q and R should mean Q

A **formal proof** of the propositional form q from propositional forms p_0, p_1, \dots, p_{n-1} is a sequence of propositional forms,

$$p_0, p_1, \dots, p_{n-1}, q_0, q_1, \dots, q_{m-1}$$

such that $q_{m-1} = q$, and each q_i is either an axiom or $p_0, p_1, \dots, p_{n-1}, q_0, q_1, \dots, q_{i-1} \implies q_i$

Notation:

$$p_0, p_1, \dots, p_{n-1} \vdash q$$

Example $P \vee Q \rightarrow Q \wedge R, P \vdash Q$

$$p_0 = P \vee Q \rightarrow Q \wedge R$$

$$p_1 = P$$

$$q_0 = P \vee Q \text{ (by Addition rule)}$$

$$q_1 = Q \wedge R \text{ (by Modus Ponens)}$$

$$q_2 = R$$

Syntax

Axioms and inference rules

Definition p_0, p_1, \dots, p_{n-1} and q are propositional forms

p_0, p_1, \dots, p_{n-1} **infer** q if q can be written whenever p_0, p_1, \dots, p_{n-1} can be written.

Notation:

$$p_0, p_1, \dots, p_{n-1} \implies q$$

Examples of inference rules:

$$1. \rightarrow p \rightarrow q, p \implies q \text{ (Modus Ponens)}$$

$$2. \rightarrow p \implies p \vee q \text{ (Addition)}$$

Example application:

$$(P \wedge Q \rightarrow R) \rightarrow (R \wedge S), (P \wedge Q \rightarrow R) \implies (R \wedge S)$$

Once again while we can check with the truth table, we want to separate the syntactic from the semantic

A **formal proof** of the propositional form q from propositional forms p_0, p_1, \dots, p_{n-1} is a sequence of propositional forms,

$$p_0, p_1, \dots, p_{n-1}, q_0, q_1, \dots, q_{m-1}$$

such that $q_{m-1} = q$, and each q_i is either an axiom or $p_0, p_1, \dots, p_{n-1}, q_0, q_1, \dots, q_{i-1} \implies q_i$

Notation:

$$p_0, p_1, \dots, p_{n-1} \vdash q$$

Example $P \vee Q \rightarrow Q \wedge R, P \vdash Q$

$$p_0 = P \vee Q \rightarrow Q \wedge R$$

$$p_1 = P$$

$$q_0 = P \vee Q \text{ (by Addition rule)}$$

$$q_1 = Q \wedge R \text{ (by Modus Ponens)}$$

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Example application:

$$(P \wedge Q \rightarrow R) \rightarrow (R \wedge S), (P \wedge Q \rightarrow R) \implies (R \wedge S)$$

So we add this as an inference rule

A **formal proof** of the propositional form q from propositional forms p_0, p_1, \dots, p_{n-1} is a sequence of propositional forms,

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and quote it here

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Example application:

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Now we need to ensure that there finitely many such inference rules are enough