For the next case, we will need the following lemma.

which proves that if a statement is true, it remains true even if you add some hypothesis

In terms of semantics, if q is true, then $p \rightarrow q$ is also true because if the right hand side of $\rightarrow$ is true, then the implication is true

Lemma 1 $\vdash_* q$

If we have a proof of q

Lemma 1 $\vdash_* q$ *implies* $\vdash_* p \to q$

we can modify it to a proof of $p \to q$

Lemma 1 $\vdash_* q$ *implies* $\vdash_* p \to q$

Proof Assuming a proof of q,

Lemma 1 $\vdash_* q$ *implies* $\vdash_* p \to q$

Proof Assuming a proof of q, extend to a proof of $p \to q$

Lemma 1 $\vdash_* q$ *implies* $\vdash_* p \to q$

Proof Assuming a proof of q, extend to a proof of $p \to q$

Lemma 1 $\vdash_* q$ *implies* $\vdash_* p \to q$

Proof Assuming a proof of q, extend to a proof of $p \to q$

$n-1 \mid \ \ldots\ (\ldots)$

A proof of q would have many steps

Lemma 1 $\vdash_* q$ *implies* $\vdash_* p \to q$

Proof Assuming a proof of q, extend to a proof of $p \to q$

$$\begin{array}{r|l} n-1 & \ldots\ (\ldots) \\ n & q\ (\ldots) \end{array}$$

of which the last line would be q

Lemma 1 $\vdash_* q$ *implies* $\vdash_* p \to q$

Proof Assuming a proof of q, extend to a proof of $p \to q$

$$\begin{array}{r|l} n-1 & \ldots\ (\ldots) \\ n & q\ (\ldots) \\ n+1 & q \to (p \to q)\ \text{(FL1)} \end{array}$$

Now we simply use FL1

Lemma 1 $\vdash_* q$ *implies* $\vdash_* p \to q$

Proof Assuming a proof of q, extend to a proof of $p \to q$

$$\begin{array}{r|l} n-1 & \ldots\ (\ldots) \\ n & q\ (\ldots) \\ n+1 & q \to (p \to q)\ \text{(FL1)} \\ n+2 & p \to q\ (\text{MP on } n\ \&\ n+1) \end{array}$$

□

Continuing with proof of $p \vdash q \iff \vdash p \to q$

to give what we need using Modus Ponens

Lemma 1 $\vdash_* q$ *implies* $\vdash_* p \to q$

Proof Assuming a proof of q, extend to a proof of $p \to q$

$$\begin{array}{r|l} n-1 & \ldots\ (\ldots) \\ n & q\ (\ldots) \\ n+1 & q \to (p \to q)\ \text{(FL1)} \\ n+2 & p \to q\ (\text{MP on } n \ \& \ n+1) \end{array}$$

$\square$

Continuing with proof of $p \vdash q \iff \vdash p \to q$

We now return to the proof of case 2

Lemma 1 $\vdash_* q$ *implies* $\vdash_* p \to q$

Proof Assuming a proof of q, extend to a proof of $p \to q$

$$\begin{array}{r|l} n-1 & \dots\ (\dots) \\ n & q\ (\dots) \\ n+1 & q \to (p \to q)\ \text{(FL1)} \\ n+2 & p \to q\ (\text{MP on } n \ \&\ n+1) \end{array}$$

$\square$

Continuing with proof of $p \vdash q \iff \vdash p \to q$
Case 2: Concluded q using replacement rule,

We consider the case where q was deduced using a replacement rule instead of Modus Ponens

Lemma 1 $\vdash_* q$ *implies* $\vdash_* p \to q$

Proof Assuming a proof of q, extend to a proof of $p \to q$

$$\begin{array}{r|l} n-1 & \ldots\ (\ldots) \\ n & q\ (\ldots) \\ n+1 & q \to (p \to q)\ \text{(FL1)} \\ n+2 & p \to q\ (\text{MP on } n\ \&\ n+1) \end{array}$$

$\square$

Continuing with proof of $p \vdash q \iff \vdash p \to q$
Case 2: Concluded q using replacement rule, i.e. proof looks like

So a proof would look like this

Lemma 1 $\vdash_* q$ *implies* $\vdash_* p \to q$

Proof Assuming a proof of q, extend to a proof of $p \to q$

$$
\begin{array}{r|l}
n-1 & \ldots\ (\ldots) \\
n & q\ (\ldots) \\
n+1 & q \to (p \to q)\ \text{(FL1)} \\
n+2 & p \to q\ \text{(MP on } n \text{ \& } n+1\text{)}
\end{array}
$$

□

Continuing with proof of $p \vdash q \iff \vdash p \to q$

Case 2: Concluded q using replacement rule, i.e. proof looks like

Lemma 1 $\vdash_* q$ *implies* $\vdash_* p \to q$

Proof Assuming a proof of q, extend to a proof of $p \to q$

$$\begin{array}{r|l} n-1 & \dots\ (\dots) \\ n & q\ (\dots) \\ n+1 & q \to (p \to q)\ \text{(FL1)} \\ n+2 & p \to q\ (\text{MP on } n\ \&\ n+1) \end{array}$$

□

Continuing with proof of $p \vdash q \iff \vdash p \to q$

Case 2: Concluded q using replacement rule, i.e. proof looks like

$$\begin{array}{r|l} 0 & p\ \text{(Given)} \end{array}$$

It would start with the hypothesis

Lemma 1 $\vdash_* q$ *implies* $\vdash_* p \to q$

Proof Assuming a proof of q, extend to a proof of $p \to q$

$$\begin{array}{r|l} n-1 & \ldots\ (\ldots) \\ n & q\ (\ldots) \\ n+1 & q \to (p \to q)\ \text{(FL1)} \\ n+2 & p \to q\ (\text{MP on } n\ \&\ n+1) \end{array}$$

$\square$

Continuing with proof of $p \vdash q \iff \vdash p \to q$

Case 2: Concluded q using replacement rule, i.e. proof looks like

$$\begin{array}{r|l} 0 & p\ \text{(Given)} \\ \ldots & \ldots\ (\ldots) \\ m & r\ (\ldots) \end{array}$$

and arrive at a proof of some statement r

Lemma 1 $\vdash_* q$ *implies* $\vdash_* p \to q$

Proof Assuming a proof of q, extend to a proof of $p \to q$

$$\begin{array}{r|l} n-1 & \ldots\ (\ldots) \\ n & q\ (\ldots) \\ n+1 & q \to (p \to q)\ \text{(FL1)} \\ n+2 & p \to q\ (\text{MP on } n\ \&\ n+1) \end{array}$$

□

Continuing with proof of $p \vdash q \iff \vdash p \to q$

Case 2: Concluded q using replacement rule, i.e. proof looks like

$$\begin{array}{r|l} 0 & p\ \text{(Given)} \\ \ldots & \ldots\ (\ldots) \\ m & r\ (\ldots) \\ \ldots & \ldots\ (\ldots) \\ \ldots & \ldots\ (\ldots) \\ k & q\ (\text{Replacement rule on} m) \end{array}$$

which would have been used to prove q by some replacement rule

Lemma 1 $\vdash_* q$ *implies* $\vdash_* p \to q$

Proof Assuming a proof of q, extend to a proof of $p \to q$

$$\begin{array}{r|l} n-1 & \dots\ (\dots) \\ n & q\ (\dots) \\ n+1 & q \to (p \to q)\ \text{(FL1)} \\ n+2 & p \to q\ (\text{MP on } n \ \&\ n+1) \end{array}$$

$\square$

Continuing with proof of $p \vdash q \iff \vdash p \to q$

Case 2: Concluded q using replacement rule, i.e. proof looks like

$$\begin{array}{r|l} 0 & p\ \text{(Given)} \\ \dots & \dots\ (\dots) \\ m & r\ (\dots) \\ \dots & \dots\ (\dots) \\ \dots & \dots\ (\dots) \\ k & q\ (\text{Replacement rule on} m) \end{array}$$

We will use this proof and induction to generate a proof for $\vdash p \to q$

Lemma 1 $\vdash_* q$ *implies* $\vdash_* p \to q$

Proof Assuming a proof of q, extend to a proof of $p \to q$

$$\begin{array}{r|l} n-1 & \ldots\ (\ldots) \\ n & q\ (\ldots) \\ n+1 & q \to (p \to q)\ \text{(FL1)} \\ n+2 & p \to q\ \text{(MP on } n\ \&\ n+1) \end{array}$$

$\square$

Continuing with proof of $p \vdash q \iff \vdash p \to q$

Case 2: Concluded q using replacement rule, i.e. proof looks like

$$\begin{array}{r|l} 0 & p\ \text{(Given)} \\ \ldots & \ldots\ (\ldots) \\ m & r\ (\ldots) \\ \ldots & \ldots\ (\ldots) \\ \ldots & \ldots\ (\ldots) \\ k & q\ \text{(Replacement rule on} m) \end{array}$$

By induction hypothesis: $\vdash_* p \to r$

Notice that r follows from p

Lemma 1 $\vdash_* q$ *implies* $\vdash_* p \to q$

Proof Assuming a proof of q, extend to a proof of $p \to q$

$n-1$	$\ldots$ $(\ldots)$
n	q $(\ldots)$
$n+1$	$q \to (p \to q)$ (FL1)
$n+2$	$p \to q$ (MP on n & $n+1$)

$\square$

Continuing with proof of $p \vdash q \iff \vdash p \to q$

Case 2: Concluded q using replacement rule, i.e. proof looks like

0	p (Given)
$\ldots$	$\ldots$ $(\ldots)$
m	r $(\ldots)$
$\ldots$	$\ldots$ $(\ldots)$
$\ldots$	$\ldots$ $(\ldots)$
k	q (Replacement rule on$m$)

By induction hypothesis: $\vdash_* p \to r$& $\vdash_* r \to q$

and q follows from r

Lemma 1 $\vdash_* q$ *implies* $\vdash_* p \to q$

Proof Assuming a proof of q, extend to a proof of $p \to q$

$$\begin{array}{r|l}
n-1 & \ldots\ (\ldots) \\
n & q\ (\ldots) \\
n+1 & q \to (p \to q)\ \text{(FL1)} \\
n+2 & p \to q\ (\text{MP on } n \ \&\ n+1)
\end{array}$$

□

Continuing with proof of $p \vdash q \iff \vdash p \to q$

Case 2: Concluded q using replacement rule, i.e. proof looks like

$$\begin{array}{r|l}
0 & p\ \text{(Given)} \\
\ldots & \ldots\ (\ldots) \\
m & r\ (\ldots) \\
\ldots & \ldots\ (\ldots) \\
\ldots & \ldots\ (\ldots) \\
k & q\ (\text{Replacement rule on} m)
\end{array}$$

By induction hypothesis: $\vdash_* p \to r \&\ \vdash_* r \to q$

$$\begin{array}{r|l}
\ldots & \ldots \\
m & p \to r
\end{array}$$

because the highlighted subproof of $p \vdash r$ is smaller than k by Induction we know that there is a proof of $p \to r$

Lemma 1 $\vdash_* q$ *implies* $\vdash_* p \to q$

Proof Assuming a proof of q, extend to a proof of $p \to q$

$$\begin{array}{r|l} n-1 & \ldots\ (\ldots) \\ n & q\ (\ldots) \\ n+1 & q \to (p \to q)\ \text{(FL1)} \\ n+2 & p \to q\ (\text{MP on } n \ \&\ n+1) \end{array}$$

$\square$

Continuing with proof of $p \vdash q \iff \vdash p \to q$

Case 2: Concluded q using replacement rule, i.e. proof looks like

$$\begin{array}{r|l} 0 & p\ \text{(Given)} \\ \ldots & \ldots\ (\ldots) \\ m & r\ (\ldots) \\ \ldots & \ldots\ (\ldots) \\ \ldots & \ldots\ (\ldots) \\ k & q\ (\text{Replacement rule on} m) \end{array}$$

By induction hypothesis: $\vdash_* p \to r \&\ \vdash_* r \to q$

$$\begin{array}{r|l} \ldots & \ldots \\ m & p \to r \end{array}$$

$$\begin{array}{r|l} \ldots & \ldots \\ n & r \to q\ \text{(Induction)} \end{array}$$

Similarly, we know by induction that there is a proof of $r \to q$

Lemma 1 $\vdash_* q$ *implies* $\vdash_* p \to q$

Proof Assuming a proof of q, extend to a proof of $p \to q$

$$\begin{array}{r|l} n-1 & \ldots\ (\ldots) \\ n & q\ (\ldots) \\ n+1 & q \to (p \to q)\ \text{(FL1)} \\ n+2 & p \to q\ (\text{MP on } n \ \&\ n+1) \end{array}$$

□

Continuing with proof of $p \vdash q \iff \vdash p \to q$

Case 2: Concluded q using replacement rule, i.e. proof looks like

$$\begin{array}{r|l} 0 & p\ \text{(Given)} \\ \ldots & \ldots\ (\ldots) \\ m & r\ (\ldots) \\ \ldots & \ldots\ (\ldots) \\ \ldots & \ldots\ (\ldots) \\ k & q\ (\text{Replacement rule on} m) \end{array}$$

By induction hypothesis: $\vdash_* p \to r \&\ \vdash_* r \to q$

$$\begin{array}{r|l} \ldots & \ldots \\ m & p \to r \end{array}$$

$$\begin{array}{r|l} \ldots & \ldots \\ n & r \to q\ \text{(Induction)} \end{array}$$

Again, the highlighted part shows $r \vdash q$ so by induction we have some proof of $r \to q$

Lemma 1 $\vdash_* q$ *implies* $\vdash_* p \to q$

Proof Assuming a proof of q, extend to a proof of $p \to q$

$n-1$	$\ldots$ $(\ldots)$
n	q $(\ldots)$
$n+1$	$q \to (p \to q)$ (FL1)
$n+2$	$p \to q$ (MP on n & $n+1$)

$\square$

Continuing with proof of $p \vdash q \iff \vdash p \to q$

Case 2: Concluded q using replacement rule, i.e. proof looks like

0	p (Given)
$\ldots$	$\ldots$ $(\ldots)$
m	r $(\ldots)$
$\ldots$	$\ldots$ $(\ldots)$
$\ldots$	$\ldots$ $(\ldots)$
k	q (Replacement rule on$m$)

By induction hypothesis: $\vdash_* p \to r$& $\vdash_* r \to q$

$\ldots$	$\ldots$
m	$p \to r$
$\ldots$	$\ldots$
n	$r \to q$ (Induction)
$n+1$	$p \to (r \to q)$ (Lemma)

This time, to apply FL2, we will need to use the lemma to claim that if $r \to q$ is true then $p \to (r \to q)$ is true

Lemma 1 $\vdash_* q$ *implies* $\vdash_* p \to q$

Proof Assuming a proof of q, extend to a proof of $p \to q$

$$\begin{array}{r|l} n-1 & \dots\ (\dots) \\ n & q\ (\dots) \\ n+1 & q \to (p \to q)\ \text{(FL1)} \\ n+2 & p \to q\ (\text{MP on } n\ \&\ n+1) \end{array}$$

□

Continuing with proof of $p \vdash q \iff \vdash p \to q$

Case 2: Concluded q using replacement rule, i.e. proof looks like

$$\begin{array}{r|l} 0 & p\ \text{(Given)} \\ \dots & \dots\ (\dots) \\ m & r\ (\dots) \\ \dots & \dots\ (\dots) \\ \dots & \dots\ (\dots) \\ k & q\ (\text{Replacement rule on} m) \end{array}$$

By induction hypothesis: $\vdash_* p \to r \&\ \vdash_* r \to q$

$$\begin{array}{r|l} \dots & \dots \\ m & p \to r \\ \dots & \dots \\ n & r \to q\ \text{(Induction)} \\ n+1 & p \to (r \to q)\ \text{(Lemma)} \\ k+2 & p \to (r \to q) \to (p \to r \to (p \to q))\ \text{(FL2)} \end{array}$$

We quote the version of FL2 we will need

Lemma 1 $\vdash_* q$ *implies* $\vdash_* p \to q$

Proof Assuming a proof of q, extend to a proof of $p \to q$

$n-1$	$\ldots$ $(\ldots)$
n	q $(\ldots)$
$n+1$	$q \to (p \to q)$ (FL1)
$n+2$	$p \to q$ (MP on n & $n+1$)

□

Continuing with proof of $p \vdash q \iff \vdash p \to q$

Case 2: Concluded q using replacement rule, i.e. proof looks like

0	p (Given)
$\ldots$	$\ldots$ $(\ldots)$
m	r $(\ldots)$
$\ldots$	$\ldots$ $(\ldots)$
$\ldots$	$\ldots$ $(\ldots)$
k	q (Replacement rule on$m$)

By induction hypothesis: $\vdash_* p \to r$& $\vdash_* r \to q$

$\ldots$	$\ldots$
m	$p \to r$
$\ldots$	$\ldots$
n	$r \to q$ (Induction)
$n+1$	$p \to (r \to q)$ (Lemma)
$k+2$	$p \to (r \to q) \to (p \to r \to (p \to q))$ (FL2)
$n+3$	$p \to r \to (p \to q)$ (MP on $k+1, k+3$)

Then use Modus Ponens to get something that allows us to use the highlighted part

Lemma 1 $\vdash_* q$ *implies* $\vdash_* p \to q$

Proof Assuming a proof of q, extend to a proof of $p \to q$

$$\begin{array}{r|l} n-1 & \ldots\ (\ldots) \\ n & q\ (\ldots) \\ n+1 & q \to (p \to q)\ \text{(FL1)} \\ n+2 & p \to q\ (\text{MP on } n \ \&\ n+1) \end{array}$$

□

Continuing with proof of $p \vdash q \iff \vdash p \to q$

Case 2: Concluded q using replacement rule, i.e. proof looks like

$$\begin{array}{r|l} 0 & p\ \text{(Given)} \\ \ldots & \ldots\ (\ldots) \\ m & r\ (\ldots) \\ \ldots & \ldots\ (\ldots) \\ \ldots & \ldots\ (\ldots) \\ k & q\ (\text{Replacement rule on} m) \end{array}$$

By induction hypothesis: $\vdash_* p \to r \& \vdash_* r \to q$

$$\begin{array}{r|l} \ldots & \ldots \\ m & p \to r \end{array}$$

$$\begin{array}{r|l} \ldots & \ldots \\ n & r \to q\ \text{(Induction)} \\ n+1 & p \to (r \to q)\ \text{(Lemma)} \\ k+2 & p \to (r \to q) \to (p \to r \to (p \to q))\ \text{(FL2)} \\ n+3 & p \to r \to (p \to q)\ (\text{MP on } k+1, k+3) \\ n+4 & p \to q\ (\text{MP on } k+1, k+4) \end{array}$$

by applying Modus Ponens again to the last statement

Lemma 1 $\vdash_* q$ *implies* $\vdash_* p \to q$

Proof Assuming a proof of q, extend to a proof of $p \to q$

$n-1$	$\ldots$ $(\ldots)$
n	q $(\ldots)$
$n+1$	$q \to (p \to q)$ (FL1)
$n+2$	$p \to q$ (MP on n & $n+1$)

$\square$

Continuing with proof of $p \vdash q \iff \vdash p \to q$

Case 2: Concluded q using replacement rule, i.e. proof looks like

0	p (Given)
$\ldots$	$\ldots$ $(\ldots)$
m	r $(\ldots)$
$\ldots$	$\ldots$ $(\ldots)$
$\ldots$	$\ldots$ $(\ldots)$
k	q (Replacement rule on$m$)

By induction hypothesis: $\vdash_* p \to r$& $\vdash_* r \to q$

$\ldots$	$\ldots$
m	$p \to r$
$\ldots$	$\ldots$
n	$r \to q$ (Induction)
$n+1$	$p \to (r \to q)$ (Lemma)
$k+2$	$p \to (r \to q) \to (p \to r \to (p \to q))$ (FL2)
$n+3$	$p \to r \to (p \to q)$ (MP on $k+1, k+3$)
$n+4$	$p \to q$ (MP on $k+1, k+4$)

If you check carefully, you will find that we have implicitly assumed that the proofs have at least 3 statements

Lemma 1 $\vdash_* q$ *implies* $\vdash_* p \to q$

Proof Assuming a proof of q, extend to a proof of $p \to q$

$$\begin{array}{r|l} n-1 & \ldots\ (\ldots) \\ n & q\ (\ldots) \\ n+1 & q \to (p \to q)\ \text{(FL1)} \\ n+2 & p \to q\ (\text{MP on } n\ \&\ n+1) \end{array}$$

□

Continuing with proof of $p \vdash q \iff \vdash p \to q$

Case 2: Concluded q using replacement rule, i.e. proof looks like

$$\begin{array}{r|l} 0 & p\ \text{(Given)} \\ \ldots & \ldots\ (\ldots) \\ m & r\ (\ldots) \\ \ldots & \ldots\ (\ldots) \\ \ldots & \ldots\ (\ldots) \\ k & q\ (\text{Replacement rule on} m) \end{array}$$

By induction hypothesis: $\vdash_* p \to r \&\ \vdash_* r \to q$

$$\begin{array}{r|l} \ldots & \ldots \\ m & p \to r \\ \ldots & \ldots \\ n & r \to q\ \text{(Induction)} \\ n+1 & p \to (r \to q)\ \text{(Lemma)} \\ k+2 & p \to (r \to q) \to (p \to r \to (p \to q))\ \text{(FL2)} \\ n+3 & p \to r \to (p \to q)\ (\text{MP on } k+1, k+3) \\ n+4 & p \to q\ (\text{MP on } k+1, k+4) \end{array}$$

Case 3: Concluded q with 2 steps

Let us consider the case where the proof has 2 steps

Lemma 1 $\vdash_* q$ *implies* $\vdash_* p \to q$

Proof Assuming a proof of q, extend to a proof of $p \to q$

$n-1$	$\ldots$ $(\ldots)$
n	q $(\ldots)$
$n+1$	$q \to (p \to q)$ (FL1)
$n+2$	$p \to q$ (MP on n & $n+1$)

□

Continuing with proof of $p \vdash q \iff \vdash p \to q$

Case 2: Concluded q using replacement rule, i.e. proof looks like

0	p (Given)
$\ldots$	$\ldots$ $(\ldots)$
m	r $(\ldots)$
$\ldots$	$\ldots$ $(\ldots)$
$\ldots$	$\ldots$ $(\ldots)$
k	q (Replacement rule on$m$)

By induction hypothesis: $\vdash_* p \to r$& $\vdash_* r \to q$

$\ldots$	$\ldots$
m	$p \to r$
$\ldots$	$\ldots$
n	$r \to q$ (Induction)
$n+1$	$p \to (r \to q)$ (Lemma)
$k+2$	$p \to (r \to q) \to (p \to r \to (p \to q))$ (FL2)
$n+3$	$p \to r \to (p \to q)$ (MP on $k+1, k+3$)
$n+4$	$p \to q$ (MP on $k+1, k+4$)

Case 3: Concluded q with 2 steps

0	p (Given)
1	q (Replacement rule on 0)

so the proof will look like this

Lemma 1 $\vdash_* q$ *implies* $\vdash_* p \to q$

Proof Assuming a proof of q, extend to a proof of $p \to q$

$n-1$	$\dots$ $(\dots)$
n	q $(\dots)$
$n+1$	$q \to (p \to q)$ (FL1)
$n+2$	$p \to q$ (MP on n & $n+1$)

□

Continuing with proof of $p \vdash q \iff \vdash p \to q$

Case 2: Concluded q using replacement rule, i.e. proof looks like

0	p (Given)
$\dots$	$\dots$ $(\dots)$
m	r $(\dots)$
$\dots$	$\dots$ $(\dots)$
$\dots$	$\dots$ $(\dots)$
k	q (Replacement rule on$m$)

By induction hypothesis: $\vdash_* p \to r$& $\vdash_* r \to q$

$\dots$	$\dots$
m	$p \to r$
$\dots$	$\dots$
n	$r \to q$ (Induction)
$n+1$	$p \to (r \to q)$ (Lemma)
$k+2$	$p \to (r \to q) \to (p \to r \to (p \to q))$ (FL2)
$n+3$	$p \to r \to (p \to q)$ (MP on $k+1, k+3$)
$n+4$	$p \to q$ (MP on $k+1, k+4$)

Case 3: Concluded q with 2 steps

0	p (Given)
1	q (Replacement rule on 0)

Note that there is no chance of Modus Ponens with only 2 steps

Lemma 1 $\vdash_* q$ *implies* $\vdash_* p \to q$

Proof Assuming a proof of q, extend to a proof of $p \to q$

$$
\begin{array}{r|l}
n-1 & \dots\ (\dots) \\
n & q\ (\dots) \\
n+1 & q \to (p \to q)\ \text{(FL1)} \\
n+2 & p \to q\ (\text{MP on } n\ \&\ n+1)
\end{array}
$$

□

Continuing with proof of $p \vdash q \iff \vdash p \to q$

Case 2: Concluded q using replacement rule, i.e. proof looks like

$$
\begin{array}{r|l}
0 & p\ \text{(Given)} \\
\dots & \dots\ (\dots) \\
m & r\ (\dots) \\
\dots & \dots\ (\dots) \\
\dots & \dots\ (\dots) \\
k & q\ (\text{Replacement rule on} m)
\end{array}
$$

By induction hypothesis: $\vdash_* p \to r \& \ \vdash_* r \to q$

$$
\begin{array}{r|l}
\dots & \dots \\
m & p \to r \\
\dots & \dots \\
n & r \to q\ \text{(Induction)} \\
n+1 & p \to (r \to q)\ \text{(Lemma)} \\
k+2 & p \to (r \to q) \to (p \to r \to (p \to q))\ \text{(FL2)} \\
n+3 & p \to r \to (p \to q)\ (\text{MP on } k+1, k+3) \\
n+4 & p \to q\ (\text{MP on } k+1, k+4)
\end{array}
$$

Case 3: Concluded q with 2 steps

$$
\begin{array}{r|l}
0 & p\ \text{(Given)} \\
1 & q\ (\text{Replacement rule on } 0)
\end{array}
$$

Note that this is a very special case.

Lemma 1 $\vdash_* q$ *implies* $\vdash_* p \to q$

Proof Assuming a proof of q, extend to a proof of $p \to q$

$n-1$	$\ldots$ $(\ldots)$
n	q $(\ldots)$
$n+1$	$q \to (p \to q)$ (FL1)
$n+2$	$p \to q$ (MP on n & $n+1$)

□

Continuing with proof of $p \vdash q \iff \vdash p \to q$

Case 2: Concluded q using replacement rule, i.e. proof looks like

0	p (Given)
$\ldots$	$\ldots$ $(\ldots)$
m	r $(\ldots)$
$\ldots$	$\ldots$ $(\ldots)$
$\ldots$	$\ldots$ $(\ldots)$
k	q (Replacement rule on$m$)

By induction hypothesis: $\vdash_* p \to r$& $\vdash_* r \to q$

$\ldots$	$\ldots$
m	$p \to r$
$\ldots$	$\ldots$
n	$r \to q$ (Induction)
$n+1$	$p \to (r \to q)$ (Lemma)
$k+2$	$p \to (r \to q) \to (p \to r \to (p \to q))$ (FL2)
$n+3$	$p \to r \to (p \to q)$ (MP on $k+1, k+3$)
$n+4$	$p \to q$ (MP on $k+1, k+4$)

Case 3: Concluded q with 2 steps

0	p (Given)
1	q (Replacement rule on 0)

q is in some senses equivalent to p

Lemma 1 $\vdash_* q$ *implies* $\vdash_* p \to q$

Proof Assuming a proof of q, extend to a proof of $p \to q$

$$\begin{array}{r|l} n-1 & \ldots\ (\ldots) \\ n & q\ (\ldots) \\ n+1 & q \to (p \to q)\ \text{(FL1)} \\ n+2 & p \to q\ (\text{MP on } n\ \&\ n+1) \end{array}$$

□

Continuing with proof of $p \vdash q \iff \vdash p \to q$

Case 2: Concluded q using replacement rule, i.e. proof looks like

$$\begin{array}{r|l} 0 & p\ \text{(Given)} \\ \ldots & \ldots\ (\ldots) \\ m & r\ (\ldots) \\ \ldots & \ldots\ (\ldots) \\ \ldots & \ldots\ (\ldots) \\ k & q\ (\text{Replacement rule on} m) \end{array}$$

By induction hypothesis: $\vdash_* p \to r \&\ \vdash_* r \to q$

$$\begin{array}{r|l} \ldots & \ldots \\ m & p \to r \\ \ldots & \ldots \\ n & r \to q\ \text{(Induction)} \\ n+1 & p \to (r \to q)\ \text{(Lemma)} \\ k+2 & p \to (r \to q) \to (p \to r \to (p \to q))\ \text{(FL2)} \\ n+3 & p \to r \to (p \to q)\ (\text{MP on } k+1, k+3) \\ n+4 & p \to q\ (\text{MP on } k+1, k+4) \end{array}$$

Case 3: Concluded q with 2 steps

$$\begin{array}{r|l} 0 & p\ \text{(Given)} \\ 1 & q\ (\text{Replacement rule on } 0) \end{array}$$

because it is obtained by replacing a sub-expression of p with something allowed by the replacement rule

Lemma 1 $\vdash_* q$ *implies* $\vdash_* p \to q$

Proof Assuming a proof of q, extend to a proof of $p \to q$

$n-1$	$\ldots$ $(\ldots)$
n	q $(\ldots)$
$n+1$	$q \to (p \to q)$ (FL1)
$n+2$	$p \to q$ (MP on n & $n+1$)

$\square$

Continuing with proof of $p \vdash q \iff \vdash p \to q$

Case 2: Concluded q using replacement rule, i.e. proof looks like

0	p (Given)
$\ldots$	$\ldots$ $(\ldots)$
m	r $(\ldots)$
$\ldots$	$\ldots$ $(\ldots)$
$\ldots$	$\ldots$ $(\ldots)$
k	q (Replacement rule on$m$)

By induction hypothesis: $\vdash_* p \to r$& $\vdash_* r \to q$

$\ldots$	$\ldots$
m	$p \to r$
$\ldots$	$\ldots$
n	$r \to q$ (Induction)
$n+1$	$p \to (r \to q)$ (Lemma)
$k+2$	$p \to (r \to q) \to (p \to r \to (p \to q))$ (FL2)
$n+3$	$p \to r \to (p \to q)$ (MP on $k+1, k+3$)
$n+4$	$p \to q$ (MP on $k+1, k+4$)

Case 3: Concluded q with 2 steps

0	p (Given)
1	q (Replacement rule on 0)

So note that we will prove this from the axioms alone

Lemma 1 $\vdash_* q$ *implies* $\vdash_* p \to q$

Proof Assuming a proof of q, extend to a proof of $p \to q$

$$\begin{array}{r|l} n-1 & \ldots\ (\ldots) \\ n & q\ (\ldots) \\ n+1 & q \to (p \to q)\ \text{(FL1)} \\ n+2 & p \to q\ \text{(MP on } n \text{ \& } n+1\text{)} \end{array}$$

□

Continuing with proof of $p \vdash q \iff \vdash p \to q$

Case 2: Concluded q using replacement rule, i.e. proof looks like

$$\begin{array}{r|l} 0 & p\ \text{(Given)} \\ \ldots & \ldots\ (\ldots) \\ m & r\ (\ldots) \\ \ldots & \ldots\ (\ldots) \\ \ldots & \ldots\ (\ldots) \\ k & q\ \text{(Replacement rule on} m\text{)} \end{array}$$

By induction hypothesis: $\vdash_* p \to r \&\ \vdash_* r \to q$

$$\begin{array}{r|l} \ldots & \ldots \\ m & p \to r \\ \ldots & \ldots \\ n & r \to q\ \text{(Induction)} \\ n+1 & p \to (r \to q)\ \text{(Lemma)} \\ k+2 & p \to (r \to q) \to (p \to r \to (p \to q))\ \text{(FL2)} \\ n+3 & p \to r \to (p \to q)\ \text{(MP on } k+1, k+3\text{)} \\ n+4 & p \to q\ \text{(MP on } k+1, k+4\text{)} \end{array}$$

Case 3: Concluded q with 2 steps

$$\begin{array}{r|l} 0 & p\ \text{(Given)} \\ 1 & q\ \text{(Replacement rule on } 0\text{)} \end{array}$$

$$\begin{array}{r|l} 0 & p \to (\neg q \to p)\ \text{(FL1)} \end{array}$$

Particularly the FL1 axiom, but with $\neg q$ so that double negation will get rid of it

Lemma 1 $\vdash_* q$ *implies* $\vdash_* p \to q$

Proof Assuming a proof of q, extend to a proof of $p \to q$

$$\begin{array}{r|l} n-1 & \dots\ (\dots) \\ n & q\ (\dots) \\ n+1 & q \to (p \to q)\ \text{(FL1)} \\ n+2 & p \to q\ (\text{MP on } n\ \&\ n+1) \end{array}$$

□

Continuing with proof of $p \vdash q \iff \vdash p \to q$

Case 2: Concluded q using replacement rule, i.e. proof looks like

$$\begin{array}{r|l} 0 & p\ \text{(Given)} \\ \dots & \dots\ (\dots) \\ m & r\ (\dots) \\ \dots & \dots\ (\dots) \\ \dots & \dots\ (\dots) \\ k & q\ (\text{Replacement rule on} m) \end{array}$$

By induction hypothesis: $\vdash_* p \to r \&\ \vdash_* r \to q$

$$\begin{array}{r|l} \dots & \dots \\ m & p \to r \\ \dots & \dots \\ n & r \to q\ \text{(Induction)} \\ n+1 & p \to (r \to q)\ \text{(Lemma)} \\ k+2 & p \to (r \to q) \to (p \to r \to (p \to q))\ \text{(FL2)} \\ n+3 & p \to r \to (p \to q)\ (\text{MP on } k+1, k+3) \\ n+4 & p \to q\ (\text{MP on } k+1, k+4) \end{array}$$

Case 3: Concluded q with 2 steps

$$\begin{array}{r|l} 0 & p\ \text{(Given)} \\ 1 & q\ (\text{Replacement rule on } 0) \end{array}$$

$$\begin{array}{r|l} 0 & p \to (\neg q \to p)\ \text{(FL1)} \\ 1 & p \to (\neg q \to q)\ \text{(Same replacement rule as in 1)} \end{array}$$

Now the same replacement rule that turned p into q

Lemma 1 $\vdash_* q$ *implies* $\vdash_* p \to q$

Proof Assuming a proof of q, extend to a proof of $p \to q$

$n-1$	$\ldots\ (\ldots)$
n	$q\ (\ldots)$
$n+1$	$q \to (p \to q)$ (FL1)
$n+2$	$p \to q$ (MP on n & $n+1$)

□

Continuing with proof of $p \vdash q \iff \vdash p \to q$

Case 2: Concluded q using replacement rule, i.e. proof looks like

0	p (Given)
$\ldots$	$\ldots\ (\ldots)$
m	$r\ (\ldots)$
$\ldots$	$\ldots\ (\ldots)$
$\ldots$	$\ldots\ (\ldots)$
k	q (Replacement rule on$m$)

By induction hypothesis: $\vdash_* p \to r$& $\vdash_* r \to q$

$\ldots$	$\ldots$
m	$p \to r$
$\ldots$	$\ldots$
n	$r \to q$ (Induction)
$n+1$	$p \to (r \to q)$ (Lemma)
$k+2$	$p \to (r \to q) \to (p \to r \to (p \to q))$ (FL2)
$n+3$	$p \to r \to (p \to q)$ (MP on $k+1, k+3$)
$n+4$	$p \to q$ (MP on $k+1, k+4$)

Case 3: Concluded q with 2 steps

0	p (Given)
1	q (Replacement rule on 0)

0	$p \to (\neg q \to p)$ (FL1)
1	$p \to (\neg q \to q)$ (Same replacement rule as in 1)

can change one of the p's to q but we leave the other one intact

Lemma 1 $\vdash_* q$ *implies* $\vdash_* p \to q$

Proof Assuming a proof of q, extend to a proof of $p \to q$

$n-1$	$\ldots\ (\ldots)$
n	$q\ (\ldots)$
$n+1$	$q \to (p \to q)$ (FL1)
$n+2$	$p \to q$ (MP on n & $n+1$)

$\square$

Continuing with proof of $p \vdash q \iff \vdash p \to q$

Case 2: Concluded q using replacement rule, i.e. proof looks like

0	p (Given)
$\ldots$	$\ldots\ (\ldots)$
m	$r\ (\ldots)$
$\ldots$	$\ldots\ (\ldots)$
$\ldots$	$\ldots\ (\ldots)$
k	q (Replacement rule on$m$)

By induction hypothesis: $\vdash_* p \to r$& $\vdash_* r \to q$

$\ldots$	$\ldots$
m	$p \to r$
$\ldots$	$\ldots$
n	$r \to q$ (Induction)
$n+1$	$p \to (r \to q)$ (Lemma)
$k+2$	$p \to (r \to q) \to (p \to r \to (p \to q))$ (FL2)
$n+3$	$p \to r \to (p \to q)$ (MP on $k+1, k+3$)
$n+4$	$p \to q$ (MP on $k+1, k+4$)

Case 3: Concluded q with 2 steps

0	p (Given)
1	q (Replacement rule on 0)

0	$p \to (\neg q \to p)$ (FL1)
1	$p \to (\neg q \to q)$ (Same replacement rule as in 1)
2	$p \to (\neg\neg q \vee q)$ (Material implication)

Then we use the rule that convert $\to$ in terms of $\vee$

Lemma 1 $\vdash_* q$ *implies* $\vdash_* p \to q$

Proof Assuming a proof of q, extend to a proof of $p \to q$

$n-1$	$\ldots\ (\ldots)$
n	$q\ (\ldots)$
$n+1$	$q \to (p \to q)$ (FL1)
$n+2$	$p \to q$ (MP on n & $n+1$)

□

Continuing with proof of $p \vdash q \iff \vdash p \to q$

Case 2: Concluded q using replacement rule, i.e. proof looks like

0	p (Given)
$\ldots$	$\ldots\ (\ldots)$
m	$r\ (\ldots)$
$\ldots$	$\ldots\ (\ldots)$
$\ldots$	$\ldots\ (\ldots)$
k	q (Replacement rule on$m$)

By induction hypothesis: $\vdash_* p \to r$& $\vdash_* r \to q$

$\ldots$	$\ldots$
m	$p \to r$
$\ldots$	$\ldots$
n	$r \to q$ (Induction)
$n+1$	$p \to (r \to q)$ (Lemma)
$k+2$	$p \to (r \to q) \to (p \to r \to (p \to q))$ (FL2)
$n+3$	$p \to r \to (p \to q)$ (MP on $k+1, k+3$)
$n+4$	$p \to q$ (MP on $k+1, k+4$)

Case 3: Concluded q with 2 steps

0	p (Given)
1	q (Replacement rule on 0)

0	$p \to (\neg q \to p)$ (FL1)
1	$p \to (\neg q \to q)$ (Same replacement rule as in 1)
2	$p \to (\neg\neg q \vee q)$ (Material implication)
3	$p \to (q \vee q)$ (Double negation)

and finally with double negation

Lemma 1 $\vdash_* q$ *implies* $\vdash_* p \to q$

Proof Assuming a proof of q, extend to a proof of $p \to q$

$$\begin{array}{r|l} n-1 & \ldots\ (\ldots) \\ n & q\ (\ldots) \\ n+1 & q \to (p \to q)\ \text{(FL1)} \\ n+2 & p \to q\ \text{(MP on } n \ \& \ n+1\text{)} \end{array}$$

□

Continuing with proof of $p \vdash q \iff \vdash p \to q$

Case 2: Concluded q using replacement rule, i.e. proof looks like

$$\begin{array}{r|l} 0 & p\ \text{(Given)} \\ \ldots & \ldots\ (\ldots) \\ m & r\ (\ldots) \\ \ldots & \ldots\ (\ldots) \\ \ldots & \ldots\ (\ldots) \\ k & q\ \text{(Replacement rule on} m\text{)} \end{array}$$

By induction hypothesis: $\vdash_* p \to r \& \ \vdash_* r \to q$

$$\begin{array}{r|l} \ldots & \ldots \\ m & p \to r \\ \ldots & \ldots \\ n & r \to q\ \text{(Induction)} \\ n+1 & p \to (r \to q)\ \text{(Lemma)} \\ k+2 & p \to (r \to q) \to (p \to r \to (p \to q))\ \text{(FL2)} \\ n+3 & p \to r \to (p \to q)\ \text{(MP on } k+1, k+3\text{)} \\ n+4 & p \to q\ \text{(MP on } k+1, k+4\text{)} \end{array}$$

Case 3: Concluded q with 2 steps

$$\begin{array}{r|l} 0 & p\ \text{(Given)} \\ 1 & q\ \text{(Replacement rule on } 0\text{)} \end{array}$$

$$\begin{array}{r|l} 0 & p \to (\neg q \to p)\ \text{(FL1)} \\ 1 & p \to (\neg q \to q)\ \text{(Same replacement rule as in 1)} \\ 2 & p \to (\neg\neg q \vee q)\ \text{(Material implication)} \\ 3 & p \to (q \vee q)\ \text{(Double negation)} \\ 4 & p \to q \end{array}$$

and simplification will yeild this

Lemma 1 $\vdash_* q$ *implies* $\vdash_* p \to q$

Proof Assuming a proof of q, extend to a proof of $p \to q$

$$\begin{array}{r|l} n-1 & \dots\ (\dots) \\ n & q\ (\dots) \\ n+1 & q \to (p \to q)\ \text{(FL1)} \\ n+2 & p \to q\ \text{(MP on } n \text{ \& } n+1) \end{array}$$

□

Continuing with proof of $p \vdash q \iff \vdash p \to q$

Case 2: Concluded q using replacement rule, i.e. proof looks like

$$\begin{array}{r|l} 0 & p\ \text{(Given)} \\ \dots & \dots\ (\dots) \\ m & r\ (\dots) \\ \dots & \dots\ (\dots) \\ \dots & \dots\ (\dots) \\ k & q\ \text{(Replacement rule on} m) \end{array}$$

By induction hypothesis: $\vdash_* p \to r \&\ \vdash_* r \to q$

$$\begin{array}{r|l} \dots & \dots \\ m & p \to r \\ \dots & \dots \\ n & r \to q\ \text{(Induction)} \\ n+1 & p \to (r \to q)\ \text{(Lemma)} \\ k+2 & p \to (r \to q) \to (p \to r \to (p \to q))\ \text{(FL2)} \\ n+3 & p \to r \to (p \to q)\ \text{(MP on } k+1, k+3) \\ n+4 & p \to q\ \text{(MP on } k+1, k+4) \end{array}$$

Case 3: Concluded q with 2 steps

$$\begin{array}{r|l} 0 & p\ \text{(Given)} \\ 1 & q\ \text{(Replacement rule on } 0) \end{array}$$

$$\begin{array}{r|l} 0 & p \to (\neg q \to p)\ \text{(FL1)} \\ 1 & p \to (\neg q \to q)\ \text{(Same replacement rule as in 1)} \\ 2 & p \to (\neg\neg q \vee q)\ \text{(Material implication)} \\ 3 & p \to (q \vee q)\ \text{(Double negation)} \\ 4 & p \to q \end{array}$$

Case 4: Concluded q with 1 step,

Now we have only one case left, where there is just one step

Lemma 1 $\vdash_* q$ *implies* $\vdash_* p \to q$

Proof Assuming a proof of q, extend to a proof of $p \to q$

$n-1$	$\dots\ (\dots)$
n	$q\ (\dots)$
$n+1$	$q \to (p \to q)$ (FL1)
$n+2$	$p \to q$ (MP on n & $n+1$)

□

Continuing with proof of $p \vdash q \iff \vdash p \to q$

Case 2: Concluded q using replacement rule, i.e. proof looks like

0	p (Given)
$\dots$	$\dots\ (\dots)$
m	$r\ (\dots)$
$\dots$	$\dots\ (\dots)$
$\dots$	$\dots\ (\dots)$
k	q (Replacement rule on$m$)

By induction hypothesis: $\vdash_* p \to r$& $\vdash_* r \to q$

$\dots$	$\dots$
m	$p \to r$
$\dots$	$\dots$
n	$r \to q$ (Induction)
$n+1$	$p \to (r \to q)$ (Lemma)
$k+2$	$p \to (r \to q) \to (p \to r \to (p \to q))$ (FL2)
$n+3$	$p \to r \to (p \to q)$ (MP on $k+1, k+3$)
$n+4$	$p \to q$ (MP on $k+1, k+4$)

Case 3: Concluded q with 2 steps

0	p (Given)
1	q (Replacement rule on 0)

0	$p \to (\neg q \to p)$ (FL1)
1	$p \to (\neg q \to q)$ (Same replacement rule as in 1)
2	$p \to (\neg\neg q \vee q)$ (Material implication)
3	$p \to (q \vee q)$ (Double negation)
4	$p \to q$

Case 4: Concluded q with 1 step,i.e. $p = q$

in that case q has to be p

Lemma 1 $\vdash_* q$ *implies* $\vdash_* p \to q$

Proof Assuming a proof of q, extend to a proof of $p \to q$

$$\begin{array}{r|l} n-1 & \dots\ (\dots) \\ n & q\ (\dots) \\ n+1 & q \to (p \to q)\ \text{(FL1)} \\ n+2 & p \to q\ (\text{MP on } n \ \&\ n+1) \end{array}$$

□

Continuing with proof of $p \vdash q \iff \vdash p \to q$

Case 2: Concluded q using replacement rule, i.e. proof looks like

$$\begin{array}{r|l} 0 & p\ \text{(Given)} \\ \dots & \dots\ (\dots) \\ m & r\ (\dots) \\ \dots & \dots\ (\dots) \\ \dots & \dots\ (\dots) \\ k & q\ (\text{Replacement rule on} m) \end{array}$$

By induction hypothesis: $\vdash_* p \to r \& \ \vdash_* r \to q$

$$\begin{array}{r|l} \dots & \dots \\ m & p \to r \\ \dots & \dots \\ n & r \to q\ \text{(Induction)} \\ n+1 & p \to (r \to q)\ \text{(Lemma)} \\ k+2 & p \to (r \to q) \to (p \to r \to (p \to q))\ \text{(FL2)} \\ n+3 & p \to r \to (p \to q)\ (\text{MP on } k+1, k+3) \\ n+4 & p \to q\ (\text{MP on } k+1, k+4) \end{array}$$

Case 3: Concluded q with 2 steps

$$\begin{array}{r|l} 0 & p\ \text{(Given)} \\ 1 & q\ (\text{Replacement rule on } 0) \end{array}$$

$$\begin{array}{r|l} 0 & p \to (\neg q \to p)\ \text{(FL1)} \\ 1 & p \to (\neg q \to q)\ \text{(Same replacement rule as in 1)} \\ 2 & p \to (\neg\neg q \vee q)\ \text{(Material implication)} \\ 3 & p \to (q \vee q)\ \text{(Double negation)} \\ 4 & p \to q \end{array}$$

Case 4: Concluded q with 1 step,i.e. $p = q$

$$\begin{array}{r|l} 0 & p \to (\neg p \to p)\ \text{(FL1)} \end{array}$$

so plugging it in FL1 yields

Lemma 1 $\vdash_* q$ *implies* $\vdash_* p \to q$

Proof Assuming a proof of q, extend to a proof of $p \to q$

$n-1$	$\dots$ $(\dots)$
n	q $(\dots)$
$n+1$	$q \to (p \to q)$ (FL1)
$n+2$	$p \to q$ (MP on n & $n+1$)

□

Continuing with proof of $p \vdash q \iff \vdash p \to q$

Case 2: Concluded q using replacement rule, i.e. proof looks like

0	p (Given)
$\dots$	$\dots$ $(\dots)$
m	r $(\dots)$
$\dots$	$\dots$ $(\dots)$
$\dots$	$\dots$ $(\dots)$
k	q (Replacement rule on$m$)

By induction hypothesis: $\vdash_* p \to r$& $\vdash_* r \to q$

$\dots$	$\dots$
m	$p \to r$
$\dots$	$\dots$
n	$r \to q$ (Induction)
$n+1$	$p \to (r \to q)$ (Lemma)
$k+2$	$p \to (r \to q) \to (p \to r \to (p \to q))$ (FL2)
$n+3$	$p \to r \to (p \to q)$ (MP on $k+1, k+3$)
$n+4$	$p \to q$ (MP on $k+1, k+4$)

Case 3: Concluded q with 2 steps

0	p (Given)
1	q (Replacement rule on 0)

0	$p \to (\neg q \to p)$ (FL1)
1	$p \to (\neg q \to q)$ (Same replacement rule as in 1)
2	$p \to (\neg\neg q \vee q)$ (Material implication)
3	$p \to (q \vee q)$ (Double negation)
4	$p \to q$

Case 4: Concluded q with 1 step,i.e. $p = q$

0	$p \to (\neg p \to p)$ (FL1)
1	$p \to (\neg\neg p \vee p)$ (Material implication)

and then using the usual material implication

Lemma 1 $\vdash_* q$ *implies* $\vdash_* p \to q$

Proof Assuming a proof of q, extend to a proof of $p \to q$

$$\begin{array}{r|l} n-1 & \ldots\ (\ldots) \\ n & q\ (\ldots) \\ n+1 & q \to (p \to q)\ \text{(FL1)} \\ n+2 & p \to q\ (\text{MP on } n\ \&\ n+1) \end{array}$$

□

Continuing with proof of $p \vdash q \iff \vdash p \to q$

Case 2: Concluded q using replacement rule, i.e. proof looks like

$$\begin{array}{r|l} 0 & p\ \text{(Given)} \\ \ldots & \ldots\ (\ldots) \\ m & r\ (\ldots) \\ \ldots & \ldots\ (\ldots) \\ \ldots & \ldots\ (\ldots) \\ k & q\ (\text{Replacement rule on} m) \end{array}$$

By induction hypothesis: $\vdash_* p \to r \&\ \vdash_* r \to q$

$$\begin{array}{r|l} \ldots & \ldots \\ m & p \to r \\ \ldots & \ldots \\ n & r \to q\ \text{(Induction)} \\ n+1 & p \to (r \to q)\ \text{(Lemma)} \\ k+2 & p \to (r \to q) \to (p \to r \to (p \to q))\ \text{(FL2)} \\ n+3 & p \to r \to (p \to q)\ (\text{MP on } k+1, k+3) \\ n+4 & p \to q\ (\text{MP on } k+1, k+4) \end{array}$$

Case 3: Concluded q with 2 steps

$$\begin{array}{r|l} 0 & p\ \text{(Given)} \\ 1 & q\ (\text{Replacement rule on } 0) \end{array}$$

$$\begin{array}{r|l} 0 & p \to (\neg q \to p)\ \text{(FL1)} \\ 1 & p \to (\neg q \to q)\ \text{(Same replacement rule as in 1)} \\ 2 & p \to (\neg\neg q \vee q)\ \text{(Material implication)} \\ 3 & p \to (q \vee q)\ \text{(Double negation)} \\ 4 & p \to q \end{array}$$

Case 4: Concluded q with 1 step,i.e. $p = q$

$$\begin{array}{r|l} 0 & p \to (\neg p \to p)\ \text{(FL1)} \\ 1 & p \to (\neg\neg p \vee p)\ \text{(Material implication)} \\ 2 & p \to (p \vee p)\ \text{(Double negation)} \end{array}$$

and double negation

Lemma 1 $\vdash_* q$ *implies* $\vdash_* p \to q$

Proof Assuming a proof of q, extend to a proof of $p \to q$

$$\begin{array}{r|l}
n-1 & \dots\ (\dots) \\
n & q\ (\dots) \\
n+1 & q \to (p \to q)\ \text{(FL1)} \\
n+2 & p \to q\ (\text{MP on } n\ \&\ n+1)
\end{array}$$

□

Continuing with proof of $p \vdash q \iff \vdash p \to q$

Case 2: Concluded q using replacement rule, i.e. proof looks like

$$\begin{array}{r|l}
0 & p\ \text{(Given)} \\
\dots & \dots\ (\dots) \\
m & r\ (\dots) \\
\dots & \dots\ (\dots) \\
\dots & \dots\ (\dots) \\
k & q\ (\text{Replacement rule on} m)
\end{array}$$

By induction hypothesis: $\vdash_* p \to r \& \ \vdash_* r \to q$

$$\begin{array}{r|l}
\dots & \dots \\
m & p \to r \\
\dots & \dots \\
n & r \to q\ \text{(Induction)} \\
n+1 & p \to (r \to q)\ \text{(Lemma)} \\
k+2 & p \to (r \to q) \to (p \to r \to (p \to q))\ \text{(FL2)} \\
n+3 & p \to r \to (p \to q)\ (\text{MP on } k+1, k+3) \\
n+4 & p \to q\ (\text{MP on } k+1, k+4)
\end{array}$$

Case 3: Concluded q with 2 steps

$$\begin{array}{r|l}
0 & p\ \text{(Given)} \\
1 & q\ (\text{Replacement rule on } 0)
\end{array}$$

$$\begin{array}{r|l}
0 & p \to (\neg q \to p)\ \text{(FL1)} \\
1 & p \to (\neg q \to q)\ \text{(Same replacement rule as in 1)} \\
2 & p \to (\neg\neg q \vee q)\ \text{(Material implication)} \\
3 & p \to (q \vee q)\ \text{(Double negation)} \\
4 & p \to q
\end{array}$$

Case 4: Concluded q with 1 step,i.e. $p = q$

$$\begin{array}{r|l}
0 & p \to (\neg p \to p)\ \text{(FL1)} \\
1 & p \to (\neg\neg p \vee p)\ \text{(Material implication)} \\
2 & p \to (p \vee p)\ \text{(Double negation)} \\
3 & p \to p\ \text{(Simplification)}
\end{array}$$

and simplifcaiton yields the final answer