Definition $p_0, p_1, \ldots, p_{n-1}$ and q are propositional forms

If we have a sequence of propositional forms p_i and a propositiona form q

Definition $p_0, p_1, \ldots, p_{n-1}$ and q are propositional forms

$p_0, p_1, \ldots, p_{n-1}$ **logically implies** q if

How do we say that the p_i's imply q?

Definition $p_0, p_1, \ldots, p_{n-1}$ and q are propositional forms

$p_0, p_1, \ldots, p_{n-1}$ **logically implies** q if

$$p_0 \wedge p_1 \wedge \ldots \wedge p_{n-1} \rightarrow q$$

It simply means that if each of the p_i's are true, then so is q

Definition $p_0, p_1, \ldots, p_{n-1}$ and q are propositional forms

$p_0, p_1, \ldots, p_{n-1}$ **logically implies** q if

$$p_0 \land p_1 \land \ldots \land p_{n-1} \to q$$

In other words, the above implication is always true

Definition $p_0, p_1, \ldots, p_{n-1}$ and q are propositional forms
$p_0, p_1, \ldots, p_{n-1}$ **logically implies** q if

$$\models p_0 \wedge p_1 \wedge \ldots \wedge p_{n-1} \to q$$

i.e. it is a tautology

Definition $p_0, p_1, \ldots, p_{n-1}$ and q are propositional forms

$p_0, p_1, \ldots, p_{n-1}$ **logically implies** q if

$$\models p_0 \wedge p_1 \wedge \ldots \wedge p_{n-1} \rightarrow q$$

Notation:

$$p_0, p_1, \ldots, p_{n-1} \models q$$

We use this notation

Definition $p_0, p_1, \ldots, p_{n-1}$ and q are propositional forms

$p_0, p_1, \ldots, p_{n-1}$ **logically implies** q if

$$\models p_0 \wedge p_1 \wedge \ldots \wedge p_{n-1} \to q$$

Notation:

$$p_0, p_1, \ldots, p_{n-1} \models q$$

Equivalently, if there is a valuation ν, so that $\nu(p_i) = T$,

In terms of valuations, it means that those valuations that assign true to p_i

Definition $p_0, p_1, \ldots, p_{n-1}$ and q are propositional forms

$p_0, p_1, \ldots, p_{n-1}$ **logically implies** q if

$$\models p_0 \wedge p_1 \wedge \ldots \wedge p_{n-1} \to q$$

Notation:

$$p_0, p_1, \ldots, p_{n-1} \models q$$

Equivalently, if there is a valuation ν, so that $\nu(p_i) = T$, then $\nu(q) = T$

must also assign true to q

Definition $p_0, p_1, \ldots, p_{n-1}$ and q are propositional forms

$p_0, p_1, \ldots, p_{n-1}$ **logically implies** q if

$$\models p_0 \wedge p_1 \wedge \ldots \wedge p_{n-1} \rightarrow q$$

Notation:

$$p_0, p_1, \ldots, p_{n-1} \models q$$

Equivalently, if there is a valuation ν, so that $\nu(p_i) = T$, then $\nu(q) = T$

Example $P \rightarrow Q, P \models Q$

For example, should not $P \rightarrow Q$, and P, logically imply Q if we have defined $\rightarrow$ correctly?

Definition $p_0, p_1, \ldots, p_{n-1}$ and q are propositional forms
$p_0, p_1, \ldots, p_{n-1}$ **logically implies** q if

$$\models p_0 \wedge p_1 \wedge \ldots \wedge p_{n-1} \rightarrow q$$

Notation:

$$p_0, p_1, \ldots, p_{n-1} \models q$$

Equivalently, if there is a valuation ν, so that $\nu(p_i) = T$, then $\nu(q) = T$

Example $P \rightarrow Q, P \models Q$
To show, $\models ((P \rightarrow Q) \wedge P) \rightarrow Q$

By the definition above, it means we need to check that this expression is a tuatology

Definition $p_0, p_1, \ldots, p_{n-1}$ and q are propositional forms

$p_0, p_1, \ldots, p_{n-1}$ **logically implies** q if

$$\models p_0 \wedge p_1 \wedge \ldots \wedge p_{n-1} \rightarrow q$$

Notation:

$$p_0, p_1, \ldots, p_{n-1} \models q$$

Equivalently, if there is a valuation ν, so that $\nu(p_i) = T$, then $\nu(q) = T$

Example $P \rightarrow Q, P \models Q$

To show, $\models ((P \rightarrow Q) \wedge P) \rightarrow Q$

Proof: Exercise using truth tables!

That should be a simple exercise

Definition $p_0, p_1, \ldots, p_{n-1}$ and q are propositional forms
$p_0, p_1, \ldots, p_{n-1}$ **logically implies** q if

$$\models p_0 \wedge p_1 \wedge \ldots \wedge p_{n-1} \rightarrow q$$

Notation:

$$p_0, p_1, \ldots, p_{n-1} \models q$$

Equivalently, if there is a valuation ν, so that $\nu(p_i) = T$, then $\nu(q) = T$

Example $P \rightarrow Q, P \models Q$
To show, $\models ((P \rightarrow Q) \wedge P) \rightarrow Q$
Proof: Exercise using truth tables!

Let us now turn to the syntactic side

Syntax

Definition $p_0, p_1, \ldots, p_{n-1}$ and q are propositional forms

$p_0, p_1, \ldots, p_{n-1}$ **logically implies** q if

$$\models p_0 \wedge p_1 \wedge \ldots \wedge p_{n-1} \rightarrow q$$

Notation:

$$p_0, p_1, \ldots, p_{n-1} \models q$$

Equivalently, if there is a valuation ν, so that $\nu(p_i) = T$, then $\nu(q) = T$

Example $P \rightarrow Q, P \models Q$
To show, $\models ((P \rightarrow Q) \wedge P) \rightarrow Q$
Proof: Exercise using truth tables!

In other words, we do not rely on the meaning any of the symbols

Definition $p_0, p_1, \ldots, p_{n-1}$ and q are propositional forms

$p_0, p_1, \ldots, p_{n-1}$ **logically implies** q if

$$\models p_0 \wedge p_1 \wedge \ldots \wedge p_{n-1} \rightarrow q$$

Notation:

$$p_0, p_1, \ldots, p_{n-1} \models q$$

Equivalently, if there is a valuation ν, so that $\nu(p_i) = T$, then $\nu(q) = T$

Example $P \rightarrow Q, P \models Q$
To show, $\models ((P \rightarrow Q) \wedge P) \rightarrow Q$
Proof: Exercise using truth tables!

We will not need to know that $\wedge$ means "and" or $\implies$ means "if...then"

Definition $p_0, p_1, \ldots, p_{n-1}$ and q are propositional forms
$p_0, p_1, \ldots, p_{n-1}$ **logically implies** q if

$$\models p_0 \wedge p_1 \wedge \ldots \wedge p_{n-1} \to q$$

Notation:

$$p_0, p_1, \ldots, p_{n-1} \models q$$

Equivalently, if there is a valuation ν, so that $\nu(p_i) = T$, then $\nu(q) = T$

Example $P \to Q, P \models Q$
To show, $\models ((P \to Q) \wedge P) \to Q$
Proof: Exercise using truth tables!

We will treat these as symbols and come up with a set of rules

Definition $p_0, p_1, \ldots, p_{n-1}$ and q are propositional forms

$p_0, p_1, \ldots, p_{n-1}$ **logically implies** q if

$$\models p_0 \wedge p_1 \wedge \ldots \wedge p_{n-1} \to q$$

Notation:

$$p_0, p_1, \ldots, p_{n-1} \models q$$

Equivalently, if there is a valuation ν, so that $\nu(p_i) = T$, then $\nu(q) = T$

Example $P \to Q, P \models Q$

To show, $\models ((P \to Q) \wedge P) \to Q$

Proof: Exercise using truth tables!

that will tell us when, given a propositional form

Definition $p_0, p_1, \ldots, p_{n-1}$ and q are propositional forms

$p_0, p_1, \ldots, p_{n-1}$ **logically implies** q if

$$\models p_0 \wedge p_1 \wedge \ldots \wedge p_{n-1} \to q$$

Notation:

$$p_0, p_1, \ldots, p_{n-1} \models q$$

Equivalently, if there is a valuation ν, so that $\nu(p_i) = T$, then $\nu(q) = T$

Example $P \to Q, P \models Q$

To show, $\models ((P \to Q) \wedge P) \to Q$

Proof: Exercise using truth tables!

i.e. strings in our language (with no meaning attached),

Definition $p_0, p_1, \ldots, p_{n-1}$ and q are propositional forms
$p_0, p_1, \ldots, p_{n-1}$ **logically implies** q if

$$\models p_0 \wedge p_1 \wedge \ldots \wedge p_{n-1} \to q$$

Notation:

$$p_0, p_1, \ldots, p_{n-1} \models q$$

Equivalently, if there is a valuation ν, so that $\nu(p_i) = T$, then $\nu(q) = T$

Example $P \to Q, P \models Q$
To show, $\models ((P \to Q) \wedge P) \to Q$
Proof: Exercise using truth tables!

Syntax

Axioms

We will have a starting point of assertions that we accept without proof

Definition $p_0, p_1, \ldots, p_{n-1}$ and q are propositional forms

$p_0, p_1, \ldots, p_{n-1}$ **logically implies** q if

$$\models p_0 \land p_1 \land \ldots \land p_{n-1} \to q$$

Notation:

$$p_0, p_1, \ldots, p_{n-1} \models q$$

Equivalently, if there is a valuation ν, so that $\nu(p_i) = T$, then $\nu(q) = T$

Example $P \to Q, P \models Q$
To show, $\models ((P \to Q) \land P) \to Q$
Proof: Exercise using truth tables!

Syntax

Axioms and inference rules

and syntactic rules to derive assertions from those axioms

Definition $p_0, p_1, \ldots, p_{n-1}$ and q are propositional forms

$p_0, p_1, \ldots, p_{n-1}$ **logically implies** q if

$$\models p_0 \wedge p_1 \wedge \ldots \wedge p_{n-1} \rightarrow q$$

Notation:

$$p_0, p_1, \ldots, p_{n-1} \models q$$

Equivalently, if there is a valuation ν, so that $\nu(p_i) = T$, then $\nu(q) = T$

Example $P \rightarrow Q, P \models Q$

To show, $\models ((P \rightarrow Q) \wedge P) \rightarrow Q$

Proof: Exercise using truth tables!

Syntax

Axioms and inference rules

These rules will merely involve manipulating the strings

Definition $p_0, p_1, \ldots, p_{n-1}$ and q are propositional forms
$p_0, p_1, \ldots, p_{n-1}$ **logically implies** q if

$$\models p_0 \wedge p_1 \wedge \ldots \wedge p_{n-1} \rightarrow q$$

Notation:

$$p_0, p_1, \ldots, p_{n-1} \models q$$

Equivalently, if there is a valuation ν, so that $\nu(p_i) = T$, then $\nu(q) = T$

Example $P \rightarrow Q, P \models Q$
To show, $\models ((P \rightarrow Q) \wedge P) \rightarrow Q$
Proof: Exercise using truth tables!

Syntax

Axioms and inference rules

The goal will be to come up with these string manipulation rules

Definition $p_0, p_1, \ldots, p_{n-1}$ and q are propositional forms

$p_0, p_1, \ldots, p_{n-1}$ **logically implies** q if

$$\models p_0 \wedge p_1 \wedge \ldots \wedge p_{n-1} \to q$$

Notation:

$$p_0, p_1, \ldots, p_{n-1} \models q$$

Equivalently, if there is a valuation ν, so that $\nu(p_i) = T$, then $\nu(q) = T$

Example $P \to Q, P \models Q$
To show, $\models ((P \to Q) \wedge P) \to Q$
Proof: Exercise using truth tables!

Syntax

Axioms and inference rules

And axioms so that by performing repeated string manipulations

Definition $p_0, p_1, \ldots, p_{n-1}$ and q are propositional forms
$p_0, p_1, \ldots, p_{n-1}$ **logically implies** q if

$$\vDash p_0 \wedge p_1 \wedge \ldots \wedge p_{n-1} \rightarrow q$$

Notation:

$$p_0, p_1, \ldots, p_{n-1} \vDash q$$

Equivalently, if there is a valuation ν, so that $\nu(p_i) = T$, then $\nu(q) = T$

Example $P \rightarrow Q, P \vDash Q$
To show, $\vDash ((P \rightarrow Q) \wedge P) \rightarrow Q$
Proof: Exercise using truth tables!

Syntax

Axioms and inference rules

allowed by the rules, we will be able to come up with statements that we judge to be true

Definition $p_0, p_1, \ldots, p_{n-1}$ and q are propositional forms
$p_0, p_1, \ldots, p_{n-1}$ **logically implies** q if

$$\models p_0 \wedge p_1 \wedge \ldots \wedge p_{n-1} \rightarrow q$$

Notation:

$$p_0, p_1, \ldots, p_{n-1} \models q$$

Equivalently, if there is a valuation ν, so that $\nu(p_i) = T$, then $\nu(q) = T$

Example $P \rightarrow Q, P \models Q$
To show, $\models ((P \rightarrow Q) \wedge P) \rightarrow Q$
Proof: Exercise using truth tables!

Syntax

Axioms and inference rules

Examples of inference rules:

Let us first look at an example of an inference rule

Definition $p_0, p_1, \ldots, p_{n-1}$ and q are propositional forms

$p_0, p_1, \ldots, p_{n-1}$ **logically implies** q if

$$\models p_0 \wedge p_1 \wedge \ldots \wedge p_{n-1} \rightarrow q$$

Notation:

$$p_0, p_1, \ldots, p_{n-1} \models q$$

Equivalently, if there is a valuation ν, so that $\nu(p_i) = T$, then $\nu(q) = T$

Example $P \rightarrow Q, P \models Q$

To show, $\models ((P \rightarrow Q) \wedge P) \rightarrow Q$

Proof: Exercise using truth tables!

Syntax

Axioms and inference rules

Examples of inference rules:

1. $p \rightarrow q, p \implies q$

Here p and q represent any propositional form

Definition $p_0, p_1, \ldots, p_{n-1}$ and q are propositional forms

$p_0, p_1, \ldots, p_{n-1}$ **logically implies** q if

$$\models p_0 \wedge p_1 \wedge \ldots \wedge p_{n-1} \rightarrow q$$

Notation:

$$p_0, p_1, \ldots, p_{n-1} \models q$$

Equivalently, if there is a valuation ν, so that $\nu(p_i) = T$, then $\nu(q) = T$

Example $P \rightarrow Q, P \models Q$

To show, $\models ((P \rightarrow Q) \wedge P) \rightarrow Q$

Proof: Exercise using truth tables!

Syntax

Axioms and inference rules

Examples of inference rules:

1. $p \rightarrow q, p \implies q$ (Modus Ponens)

It is called "Modus Ponens"

Definition $p_0, p_1, \ldots, p_{n-1}$ and q are propositional forms
$p_0, p_1, \ldots, p_{n-1}$ **logically implies** q if

$$\models p_0 \wedge p_1 \wedge \ldots \wedge p_{n-1} \to q$$

Notation:

$$p_0, p_1, \ldots, p_{n-1} \models q$$

Equivalently, if there is a valuation ν, so that $\nu(p_i) = T$, then $\nu(q) = T$

Example $P \to Q, P \models Q$
To show, $\models ((P \to Q) \wedge P) \to Q$
Proof: Exercise using truth tables!

Syntax

Axioms and inference rules

Examples of inference rules:

1. $p \to q, p \implies q$ (Modus Ponens)

Example application:
$(P \wedge Q \to R) \to (R \wedge S), (P \wedge Q \to R)$

Let us understand it with an application

Definition $p_0, p_1, \ldots, p_{n-1}$ and q are propositional forms

$p_0, p_1, \ldots, p_{n-1}$ **logically implies** q if

$$\models p_0 \wedge p_1 \wedge \ldots \wedge p_{n-1} \to q$$

Notation:

$$p_0, p_1, \ldots, p_{n-1} \models q$$

Equivalently, if there is a valuation ν, so that $\nu(p_i) = T$, then $\nu(q) = T$

Example $P \to Q, P \models Q$

To show, $\models ((P \to Q) \wedge P) \to Q$

Proof: Exercise using truth tables!

Syntax

Axioms and inference rules

Examples of inference rules:

1. $p \to q, p \implies q$ (Modus Ponens)

Example application:

$(P \wedge Q \to R) \to (R \wedge S), (P \wedge Q \to R)$

Note that that $P \wedge Q \to R$ matches exactly p in the rule

Definition $p_0, p_1, \ldots, p_{n-1}$ and q are propositional forms
$p_0, p_1, \ldots, p_{n-1}$ **logically implies** q if

$$\models p_0 \wedge p_1 \wedge \ldots \wedge p_{n-1} \rightarrow q$$

Notation:

$$p_0, p_1, \ldots, p_{n-1} \models q$$

Equivalently, if there is a valuation ν, so that $\nu(p_i) = T$, then $\nu(q) = T$

Example $P \rightarrow Q, P \models Q$
To show, $\models ((P \rightarrow Q) \wedge P) \rightarrow Q$
Proof: Exercise using truth tables!

Syntax

Axioms and inference rules

Examples of inference rules:

1. $p \rightarrow q, p \implies q$ (Modus Ponens)

Example application:
$(P \wedge Q \rightarrow R) \rightarrow (R \wedge S), (P \wedge Q \rightarrow R)$

Similarly, even $R \wedge S$ is q

Definition $p_0, p_1, \ldots, p_{n-1}$ and q are propositional forms
$p_0, p_1, \ldots, p_{n-1}$ **logically implies** q if

$$\models p_0 \wedge p_1 \wedge \ldots \wedge p_{n-1} \rightarrow q$$

Notation:

$$p_0, p_1, \ldots, p_{n-1} \models q$$

Equivalently, if there is a valuation ν, so that $\nu(p_i) = T$, then $\nu(q) = T$

Example $P \rightarrow Q, P \models Q$
To show, $\models ((P \rightarrow Q) \wedge P) \rightarrow Q$
Proof: Exercise using truth tables!

Syntax

Axioms and inference rules

Examples of inference rules:

1. $p \rightarrow q, p \implies q$ (Modus Ponens)

Example application:
$(P \wedge Q \rightarrow R) \rightarrow (R \wedge S), (P \wedge Q \rightarrow R) \implies (R \wedge S)$

The rule Modus Ponens tells us that we can infer the highlighted expression

Definition $p_0, p_1, \ldots, p_{n-1}$ and q are propositional forms

$p_0, p_1, \ldots, p_{n-1}$ **logically implies** q if

$$\models p_0 \wedge p_1 \wedge \ldots \wedge p_{n-1} \rightarrow q$$

Notation:

$$p_0, p_1, \ldots, p_{n-1} \models q$$

Equivalently, if there is a valuation ν, so that $\nu(p_i) = T$, then $\nu(q) = T$

Example $P \rightarrow Q, P \models Q$

To show, $\models ((P \rightarrow Q) \wedge P) \rightarrow Q$

Proof: Exercise using truth tables!

Syntax

Axioms and inference rules

Definition $p_0, p_1, \ldots, p_{n-1}$ and q are propositional forms

Examples of inference rules:

1. $p \rightarrow q, p \implies q$ (Modus Ponens)

Example application:

$(P \wedge Q \rightarrow R) \rightarrow (R \wedge S), (P \wedge Q \rightarrow R) \implies (R \wedge S)$

Rules of this kind are the building blocks of a proof

Definition $p_0, p_1, \ldots, p_{n-1}$ and q are propositional forms
$p_0, p_1, \ldots, p_{n-1}$ **logically implies** q if

$$\models p_0 \wedge p_1 \wedge \ldots \wedge p_{n-1} \rightarrow q$$

Notation:

$$p_0, p_1, \ldots, p_{n-1} \models q$$

Equivalently, if there is a valuation ν, so that $\nu(p_i) = T$, then $\nu(q) = T$

Example $P \rightarrow Q, P \models Q$
To show, $\models ((P \rightarrow Q) \wedge P) \rightarrow Q$
Proof: Exercise using truth tables!

Syntax

Axioms and inference rules

Definition $p_0, p_1, \ldots, p_{n-1}$ and q are propositional forms
$p_0, p_1, \ldots, p_{n-1}$ **infer** q if

Examples of inference rules:

1. $p \rightarrow q, p \implies q$ (Modus Ponens)

Example application:
$(P \wedge Q \rightarrow R) \rightarrow (R \wedge S), (P \wedge Q \rightarrow R) \implies (R \wedge S)$

Modus Ponens was an example of this.

Definition $p_0, p_1, \ldots, p_{n-1}$ and q are propositional forms
$p_0, p_1, \ldots, p_{n-1}$ **logically implies** q if

$$\vDash p_0 \wedge p_1 \wedge \ldots \wedge p_{n-1} \rightarrow q$$

Notation:

$$p_0, p_1, \ldots, p_{n-1} \vDash q$$

Equivalently, if there is a valuation ν, so that $\nu(p_i) = T$, then $\nu(q) = T$

Example $P \rightarrow Q, P \vDash Q$
To show, $\vDash ((P \rightarrow Q) \wedge P) \rightarrow Q$
Proof: Exercise using truth tables!

Syntax

Axioms and inference rules

Definition $p_0, p_1, \ldots, p_{n-1}$ and q are propositional forms
$p_0, p_1, \ldots, p_{n-1}$ **infer** q if q can be written whenever $p_0, p_1, \ldots, p_{n-1}$ can be written.

Examples of inference rules:

1. $p \rightarrow q, p \implies q$ (Modus Ponens)

Example application:
$(P \wedge Q \rightarrow R) \rightarrow (R \wedge S), (P \wedge Q \rightarrow R) \implies (R \wedge S)$

It declares that if propositions of patterns specified by the p_i exist, then you can infer q

Definition $p_0, p_1, \ldots, p_{n-1}$ and q are propositional forms

$p_0, p_1, \ldots, p_{n-1}$ **logically implies** q if

$$\models p_0 \wedge p_1 \wedge \ldots \wedge p_{n-1} \to q$$

Notation:

$$p_0, p_1, \ldots, p_{n-1} \models q$$

Equivalently, if there is a valuation ν, so that $\nu(p_i) = T$, then $\nu(q) = T$

Example $P \to Q, P \models Q$
To show, $\models ((P \to Q) \wedge P) \to Q$
Proof: Exercise using truth tables!

Syntax

Axioms and inference rules

Definition $p_0, p_1, \ldots, p_{n-1}$ and q are propositional forms

$p_0, p_1, \ldots, p_{n-1}$ **infer** q if q can be written whenever $p_0, p_1, \ldots, p_{n-1}$ can be written.

Notation:

$$p_0, p_1, \ldots, p_{n-1} \implies q$$

Examples of inference rules:

1. $p \to q, p \implies q$ (Modus Ponens)

Example application:
$(P \wedge Q \to R) \to (R \wedge S), (P \wedge Q \to R) \implies (R \wedge S)$

We denote it with $\implies$ as shown in the Modus Ponens example.

A **formal proof**

Syntax

Axioms and inference rules

Definition $p_0, p_1, \ldots, p_{n-1}$ and q are propositional forms
$p_0, p_1, \ldots, p_{n-1}$ **infer** q if q can be written whenever $p_0, p_1, \ldots, p_{n-1}$ can be written.

Notation:

$$p_0, p_1, \ldots, p_{n-1} \implies q$$

Examples of inference rules:

1. $p \to q, p \implies q$ (Modus Ponens)

Example application:
$(P \wedge Q \to R) \to (R \wedge S), (P \wedge Q \to R) \implies (R \wedge S)$

If this is not clear yet, it should be when we use this to define a formal proof

A **formal proof** of the propositional form q

Syntax

Axioms and inference rules

Definition $p_0, p_1, \ldots, p_{n-1}$ and q are propositional forms
$p_0, p_1, \ldots, p_{n-1}$ **infer** q if q can be written whenever $p_0, p_1, \ldots, p_{n-1}$ can be written.

Notation:

$$p_0, p_1, \ldots, p_{n-1} \implies q$$

Examples of inference rules:

1. $p \to q, p \implies q$ (Modus Ponens)

Example application:
$(P \wedge Q \to R) \to (R \wedge S), (P \wedge Q \to R) \implies (R \wedge S)$

We try to prove some propositional form q

A **formal proof** of the propositional form q from propostional forms $p_0, p_1, \ldots p_{n-1}$

Syntax

Axioms and inference rules

Definition $p_0, p_1, \ldots, p_{n-1}$ and q are propositional forms
$p_0, p_1, \ldots, p_{n-1}$ **infer** q if q can be written whenever $p_0, p_1, \ldots, p_{n-1}$ can be written.

Notation:

$$p_0, p_1, \ldots, p_{n-1} \implies q$$

Examples of inference rules:

1. $p \to q, p \implies q$ (Modus Ponens)

Example application:
$(P \wedge Q \to R) \to (R \wedge S), (P \wedge Q \to R) \implies (R \wedge S)$

assuming some propostions p_i as assumptions or hypotheses

A **formal proof** of the propositional form q from propostional forms $p_0, p_1, \ldots p_{n-1}$ is a sequence of propositional forms,

Syntax

Axioms and inference rules

Definition $p_0, p_1, \ldots, p_{n-1}$ and q are propositional forms
$p_0, p_1, \ldots, p_{n-1}$ **infer** q if q can be written whenever $p_0, p_1, \ldots, p_{n-1}$ can be written.

Notation:

$$p_0, p_1, \ldots, p_{n-1} \implies q$$

Examples of inference rules:

1. $p \to q, p \implies q$ (Modus Ponens)

Example application:
$(P \wedge Q \to R) \to (R \wedge S), (P \wedge Q \to R) \implies (R \wedge S)$

A proof involves some intermediate steps q_i

A **formal proof** of the propositional form q from propostional forms $p_0, p_1, \ldots p_{n-1}$ is a sequence of propositional forms,

$$p_0, p_1, \ldots p_{n-1}, q_0, q_1, \ldots, q_{m-1}$$

Syntax

Axioms and inference rules

Definition $p_0, p_1, \ldots, p_{n-1}$ and q are propositional forms
$p_0, p_1, \ldots, p_{n-1}$ **infer** q if q can be written whenever $p_0, p_1, \ldots, p_{n-1}$ can be written.

Notation:

$$p_0, p_1, \ldots, p_{n-1} \implies q$$

Examples of inference rules:

1. $p \to q, p \implies q$ (Modus Ponens)

Example application:
$(P \wedge Q \to R) \to (R \wedge S), (P \wedge Q \to R) \implies (R \wedge S)$

that connects the hypotheses to the conclusion

A **formal proof** of the propositional form q from propostional forms $p_0, p_1, \dots p_{n-1}$ is a sequence of propositional forms,

$$p_0, p_1, \dots p_{n-1}, q_0, q_1, \dots, q_{m-1}$$

such that $q_{m-1} = q$,

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the last statement of the proof must be the conclusion

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such that $q_{m-1} = q$, and each q_i is either an axiom

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some of these intermediate steps may quote axioms (agreed to be true without proof)

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Axioms and inference rules

Definition $p_0, p_1, \ldots, p_{n-1}$ and q are propositional forms
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or they must be an inference rule specified in the previous definition

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$$p_0, p_1, \ldots, p_{n-1} \vdash q$$

Syntax

Axioms and inference rules

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We use $\vdash$ to denote the existence of a proof connection the hypothesis to the conclusion

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Example $P \vee Q \to Q \wedge R, P \vdash$

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Here is an example of statement in logic which we will prove formally

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Examples of inference rules:

1. $p \to q, p \implies q$ (Modus Ponens)

Example application:

$(P \wedge Q \to R) \to (R \wedge S), (P \wedge Q \to R) \implies (R \wedge S)$

It is easy to check that this is a tautology but that is semantics

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Syntax

Axioms and inference rules

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$(P \wedge Q \to R) \to (R \wedge S), (P \wedge Q \to R) \implies (R \wedge S)$

We want to show this by using an proof system alone

A **formal proof** of the propositional form q from propostional forms $p_0, p_1, \ldots p_{n-1}$ is a sequence of propositional forms,

$$p_0, p_1, \ldots p_{n-1}, q_0, q_1, \ldots, q_{m-1}$$

such that $q_{m-1} = q$, and each q_i is either an axiom or $p_0, p_1, \ldots p_{n-1}, q_0, q_1, \ldots, q_{i-1} \implies q_i$

Notation:

$$p_0, p_1, \ldots, p_{n-1} \vdash q$$

Example $P \vee Q \to Q \wedge R, P \vdash Q$
$p_0 = P \vee Q \to Q \wedge R$

Syntax

Axioms and inference rules

Definition $p_0, p_1, \ldots, p_{n-1}$ and q are propositional forms
$p_0, p_1, \ldots, p_{n-1}$ **infer** q if q can be written whenever $p_0, p_1, \ldots, p_{n-1}$ can be written.

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Examples of inference rules:

1. $p \to q, p \implies q$ (Modus Ponens)

Example application:
$(P \wedge Q \to R) \to (R \wedge S), (P \wedge Q \to R) \implies (R \wedge S)$

We denote this by p because it is the given hypothesis

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Notation:

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Example $P \vee Q \to Q \wedge R, P \vdash Q$
$p_0 = P \vee Q \to Q \wedge R$
$p_1 = P$

Syntax

Axioms and inference rules

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$p_0, p_1, \ldots, p_{n-1}$ **infer** q if q can be written whenever $p_0, p_1, \ldots, p_{n-1}$ can be written.

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Example application:
$(P \wedge Q \to R) \to (R \wedge S), (P \wedge Q \to R) \implies (R \wedge S)$

along with this

A **formal proof** of the propositional form q from propostional forms $p_0, p_1, \ldots p_{n-1}$ is a sequence of propositional forms,

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Example $P \vee Q \to Q \wedge R, P \vdash Q$
$p_0 = P \vee Q \to Q \wedge R$
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Axioms and inference rules

Definition $p_0, p_1, \ldots, p_{n-1}$ and q are propositional forms
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$(P \wedge Q \to R) \to (R \wedge S), (P \wedge Q \to R) \implies (R \wedge S)$

We now need to come up with a sequence of steps leading to the conclusion

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Axioms and inference rules

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1. $p \to q, p \implies q$ (Modus Ponens)

Example application:
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We are not yet ready to use Modus Ponens

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Notation:

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Example $P \vee Q \to Q \wedge R, P \vdash Q$
$p_0 = P \vee Q \to Q \wedge R$
$p_1 = P$
$q_0 = P \vee Q$

Syntax

Axioms and inference rules

Definition $p_0, p_1, \dots, p_{n-1}$ and q are propositional forms
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Examples of inference rules:

1. $p \to q, p \implies q$ (Modus Ponens)

Example application:
$(P \wedge Q \to R) \to (R \wedge S), (P \wedge Q \to R) \implies (R \wedge S)$

But we intuitively know that $p \vee q$ must imply p

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Example $P \vee Q \to Q \wedge R, P \vdash Q$
$p_0 = P \vee Q \to Q \wedge R$
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Syntax

Axioms and inference rules

Definition $p_0, p_1, \ldots, p_{n-1}$ and q are propositional forms
$p_0, p_1, \ldots, p_{n-1}$ **infer** q if q can be written whenever $p_0, p_1, \ldots, p_{n-1}$ can be written.

Notation:

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Examples of inference rules:

1. $p \to q, p \implies q$ (Modus Ponens)

Example application:
$(P \wedge Q \to R) \to (R \wedge S), (P \wedge Q \to R) \implies (R \wedge S)$

But we cannot deduce it from anything else.

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Notation:

$$p_0, p_1, \dots, p_{n-1} \vdash q$$

Example $P \vee Q \to Q \wedge R, P \vdash Q$
$p_0 = P \vee Q \to Q \wedge R$
$p_1 = P$
$q_0 = P \vee Q$

Syntax

Axioms and inference rules

Definition $p_0, p_1, \dots, p_{n-1}$ and q are propositional forms
$p_0, p_1, \dots, p_{n-1}$ **infer** q if q can be written whenever $p_0, p_1, \dots, p_{n-1}$ can be written.

Notation:

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Examples of inference rules:

1. $p \to q, p \implies q$ (Modus Ponens)
2. $p \implies p \vee q$ (Addition)

Example application:
$(P \wedge Q \to R) \to (R \wedge S), (P \wedge Q \to R) \implies (R \wedge S)$

So we add this as a rule to the list of rules

A **formal proof** of the propositional form q from propostional forms $p_0, p_1, \ldots p_{n-1}$ is a sequence of propositional forms,

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Notation:

$$p_0, p_1, \ldots, p_{n-1} \vdash q$$

Example $P \vee Q \to Q \wedge R, P \vdash Q$
$p_0 = P \vee Q \to Q \wedge R$
$p_1 = P$
$q_0 = P \vee Q$ (by Addition rule)

Syntax

Axioms and inference rules

Definition $p_0, p_1, \ldots, p_{n-1}$ and q are propositional forms
$p_0, p_1, \ldots, p_{n-1}$ **infer** q if q can be written whenever $p_0, p_1, \ldots, p_{n-1}$ can be written.

Notation:

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Examples of inference rules:

1. $p \to q, p \implies q$ (Modus Ponens)
2. $p \implies p \vee q$ (Addition)

Example application:
$(P \wedge Q \to R) \to (R \wedge S), (P \wedge Q \to R) \implies (R \wedge S)$

and quote it here in the proof

A **formal proof** of the propositional form q from propostional forms $p_0, p_1, \ldots p_{n-1}$ is a sequence of propositional forms,

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Notation:

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Example $P \vee Q \to Q \wedge R, P \vdash Q$
$p_0 = P \vee Q \to Q \wedge R$
$p_1 = P$
$q_0 = P \vee Q$ (by Addition rule)
$q_1 = Q \wedge R$

Syntax

Axioms and inference rules

Definition $p_0, p_1, \ldots, p_{n-1}$ and q are propositional forms
$p_0, p_1, \ldots, p_{n-1}$ **infer** q if q can be written whenever $p_0, p_1, \ldots, p_{n-1}$ can be written.

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Examples of inference rules:

1. $p \to q, p \implies q$ (Modus Ponens)
2. $p \implies p \vee q$ (Addition)

Example application:
$(P \wedge Q \to R) \to (R \wedge S), (P \wedge Q \to R) \implies (R \wedge S)$

Now we are ready to use Modus Ponens on p_0 and q_0

A **formal proof** of the propositional form q from propostional forms $p_0, p_1, \dots p_{n-1}$ is a sequence of propositional forms,

$$p_0, p_1, \dots p_{n-1}, q_0, q_1, \dots, q_{m-1}$$

such that $q_{m-1} = q$, and each q_i is either an axiom or $p_0, p_1, \dots p_{n-1}, q_0, q_1, \dots, q_{i-1} \implies q_i$

Notation:

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Example $P \vee Q \to Q \wedge R, P \vdash Q$
$p_0 = P \vee Q \to Q \wedge R$
$p_1 = P$
$q_0 = P \vee Q$ (by Addition rule)
$q_1 = Q \wedge R$ (by Modus Ponens)

Syntax

Axioms and inference rules

Definition $p_0, p_1, \dots, p_{n-1}$ and q are propositional forms
$p_0, p_1, \dots, p_{n-1}$ **infer** q if q can be written whenever $p_0, p_1, \dots, p_{n-1}$ can be written.

Notation:

$$p_0, p_1, \dots, p_{n-1} \implies q$$

Examples of inference rules:

1. $p \to q, p \implies q$ (Modus Ponens)
2. $p \implies p \vee q$ (Addition)

Example application:
$(P \wedge Q \to R) \to (R \wedge S), (P \wedge Q \to R) \implies (R \wedge S)$

A **formal proof** of the propositional form q from propostional forms $p_0, p_1, \ldots p_{n-1}$ is a sequence of propositional forms,

$$p_0, p_1, \ldots p_{n-1}, q_0, q_1, \ldots, q_{m-1}$$

such that $q_{m-1} = q$, and each q_i is either an axiom or $p_0, p_1, \ldots p_{n-1}, q_0, q_1, \ldots, q_{i-1} \implies q_i$

Notation:

$$p_0, p_1, \ldots, p_{n-1} \vdash q$$

Example $P \vee Q \to Q \wedge R, P \vdash Q$
$p_0 = P \vee Q \to Q \wedge R$
$p_1 = P$
$q_0 = P \vee Q$ (by Addition rule)
$q_1 = Q \wedge R$ (by Modus Ponens)
$q_2 = R$

Syntax

Axioms and inference rules

Definition $p_0, p_1, \ldots, p_{n-1}$ and q are propositional forms
$p_0, p_1, \ldots, p_{n-1}$ **infer** q if q can be written whenever $p_0, p_1, \ldots, p_{n-1}$ can be written.

Notation:

$$p_0, p_1, \ldots, p_{n-1} \implies q$$

Examples of inference rules:

1. $p \to q, p \implies q$ (Modus Ponens)
2. $p \implies p \vee q$ (Addition)

Example application:
$(P \wedge Q \to R) \to (R \wedge S), (P \wedge Q \to R) \implies (R \wedge S)$

Now once again, $Q \wedge R$, i.e. Q *and* R should mean Q

A **formal proof** of the propositional form q from propostional forms $p_0, p_1, \dots p_{n-1}$ is a sequence of propositional forms,

$$p_0, p_1, \dots p_{n-1}, q_0, q_1, \dots, q_{m-1}$$

such that $q_{m-1} = q$, and each q_i is either an axiom or $p_0, p_1, \dots p_{n-1}, q_0, q_1, \dots, q_{i-1} \implies q_i$

Notation:

$$p_0, p_1, \dots, p_{n-1} \vdash q$$

Example $P \vee Q \to Q \wedge R, P \vdash Q$
$p_0 = P \vee Q \to Q \wedge R$
$p_1 = P$
$q_0 = P \vee Q$ (by Addition rule)
$q_1 = Q \wedge R$ (by Modus Ponens)
$q_2 = R$

Syntax

Axioms and inference rules

Definition $p_0, p_1, \dots, p_{n-1}$ and q are propositional forms
$p_0, p_1, \dots, p_{n-1}$ **infer** q if q can be written whenever $p_0, p_1, \dots, p_{n-1}$ can be written.

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Examples of inference rules:

1. $p \to q, p \implies q$ (Modus Ponens)
2. $p \implies p \vee q$ (Addition)

Example application:
$(P \wedge Q \to R) \to (R \wedge S), (P \wedge Q \to R) \implies (R \wedge S)$

Once again while we can check with the truth table, we want to separate the syntactic from the semantic

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So we add this as an inference rule

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Now we need to ensure that there finitely many such inference rules are enough