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any $x, y \in V(G) = V(T)$ is connected by a path

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any $x, y \in V(G) = V(T)$ is connected by a path
in T, and therefore in G.

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If G is connected, $|E(G)| \geq |V(G)| - 1$ edges.

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Then $|E(G')| = |V(G)| - 1$ so G' is a tree.
But $V(G') = V(G)$, i.e. G' spans G

□

Assume that if $|E(G)| = |V(G)| + k$

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then G has a spanning tree. If $|E(G)| = |V(G)| + k + 1$,
then it is not a tree.

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Theorem. *A graph G, has at least $|V(G)| - |E(G)|$ components.*

Main observation: adding an edge may decrease the number of components but by at most 1.

Proof. If $|E(G)| = 0$ then every vertex is a component.
Assume the result holds if $|E(G)| = k$
If $|E(G)| = k + 1$, delete an edge to get G'.
$V(G') = V(G)$ and $|E(G')| = |E(G)| - 1 = k$
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G, has at least $|V(G')| - |E(G')| - 1$ components. □

Definition. A cut vertex $v \in V(G)$ splits G into more components

Theorem. *$v \in V(G)$ is a cut vertex if and only if there exist $x, y \in V(G)$, such that every path joining x and y contains v.*

Assume that if $|E(G)| = |V(G)| + k$
for some $k \geq 0$
then G has a spanning tree. If $|E(G)| = |V(G)| + k + 1$,
then it is not a tree. i.e. it has a cycle
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Theorem. *If $|V(G)| \geq 2$ then G has at least two vertices which are not cut vertices.*