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$a_n = A'\frac{1}{\alpha^n} + B'\frac{1}{\beta^n}$
α, β are roots of $1 - x - x^2$
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$\alpha' := 1/\alpha, \beta' := 1/\beta$ are roots of $x^2 - x - 1$

$\alpha' = \frac{1+\sqrt{5}}{2}$
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In general:

$\alpha' = \frac{1+\sqrt{5}}{2}$
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In general:

Theorem.

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In general:

Theorem. *Let* $(a_0, a_1, a_2, a_3, \ldots)$ *be a sequence,*

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In general:

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Theorem. *Let* $(a_0, a_1, a_2, a_3, \ldots)$ *be a sequence, satisfying a recurrence relation of the form*

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Define the associated characteristic polynomial,

$$c_0 x^r + c_1 x^{r-1} + \cdots + c_r = 0$$

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Let α_i *be its roots.*

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Define the associated characteristic polynomial,

$$c_0 x^r + c_1 x^{r-1} + \cdots + c_r = 0$$

Let α_i be its roots. If they are distinct,

$\alpha' = \frac{1+\sqrt{5}}{2}$
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Theorem. *Let* $(a_0, a_1, a_2, a_3, \ldots)$ *be a sequence, satisfying a recurrence relation of the form*

$$c_0 a_n + c_1 a_{n-1} + \cdots + c_r a_{n-r} = 0$$

Define the associated characteristic polynomial,

$$c_0 x^r + c_1 x^{r-1} + \cdots + c_r = 0$$

Let α_i *be its roots. If they are distinct,*

$$a_n = A_1 \alpha_1^n + \cdots + A_r \alpha_r^n$$

for some fixed $A_1, \ldots, A_r$

Example.

In general:

Theorem. *Let* $(a_0, a_1, a_2, a_3, \ldots)$ *be a sequence, satisfying a recurrence relation of the form*

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Example (Returning to Fibonacci).

In general:

Theorem. *Let* $(a_0, a_1, a_2, a_3, \ldots)$ *be a sequence, satisfying a recurrence relation of the form*

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Example (Returning to Fibonacci).

$$a_n = a_{n-1} + a_{n-2}$$

In general:

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for some fixed $A_1, \ldots, A_r$

Example (Returning to Fibonacci)**.**

$$a_n = a_{n-1} + a_{n-2}$$

$$a_n - a_{n-1} - a_{n-2} = 0$$

In general:

Theorem. *Let* $(a_0, a_1, a_2, a_3, \ldots)$ *be a sequence, satisfying a recurrence relation of the form*

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Example (Returning to Fibonacci).

$$a_n = a_{n-1} + a_{n-2}$$

$$a_n - a_{n-1} - a_{n-2} = 0$$

Characteristic polynomial:

In general:

Theorem. *Let $(a_0, a_1, a_2, a_3, \ldots)$ be a sequence, satisfying a recurrence relation of the form*

$$c_0 a_n + c_1 a_{n-1} + \cdots + c_r a_{n-r} = 0$$

Define the associated characteristic polynomial,

$$c_0 x^r + c_1 x^{r-1} + \cdots + c_r = 0$$

Let α_i be its roots. If they are distinct,

$$a_n = A_1 \alpha_1^n + \cdots + A_r \alpha_r^n$$

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Example (Returning to Fibonacci)**.**

$$a_n = a_{n-1} + a_{n-2}$$

$$a_n - a_{n-1} - a_{n-2} = 0$$

Characteristic polynomial:

$$x^2 - x - 1 = 0$$

In general:

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Example (Returning to Fibonacci)**.**

$$a_n = a_{n-1} + a_{n-2}$$

$$a_n - a_{n-1} - a_{n-2} = 0$$

Characteristic polynomial:

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Its roots,

In general:

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$$a_n = a_{n-1} + a_{n-2}$$

$$a_n - a_{n-1} - a_{n-2} = 0$$

Characteristic polynomial:

$$x^2 - x - 1 = 0$$

Its roots,
$\alpha_1 = \frac{1+\sqrt{5}}{2}$

In general:

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$$x^2 - x - 1 = 0$$

Its roots,
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Characteristic polynomial:

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Its roots,
$\alpha_1 = \frac{1+\sqrt{5}}{2}$
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Theorem. *Let $(a_0, a_1, a_2, a_3, \ldots)$ be a sequence, satisfying a recurrence relation of the form*

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