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Eg.
$A_1 \cap A_2 \cap A_3^c \cap A_4$
$A_1^c \cap A_2 \cap A_3^c \cap A_4$

$S(i_1, i_2, \ldots, i_k) \cap S(j_1, j_2, \ldots, j_k) = \emptyset$
$S(k) := \cup S(i_1, i_2, \ldots, i_k)$ set of elements with *exactly* k belongs to exactly k sets, but does not belong to the others .
$E(k) := |S(k)| = \Sigma |S(i_1, i_2, \ldots, i_k)|$

$\Omega(i_1, i_2, \ldots, i_k) = \cap A_{i_l}$
$\omega(i_1, i_2, \ldots, i_k) = |\Omega(i_1, i_2, \ldots, i_k)|$

$\omega(k) := \Sigma |\omega(i_1, i_2, \ldots, i_k)| = \Sigma | \cap A_{i_l}|$

$x \in S(k)$, $n > k$
$x \in S(j_1, j_2, \ldots, j_k)$ for some $j_1, j_2, \ldots, j_k$
$x \in \cap A_{i_l}$ for any $i_1, i_2, \ldots, i_l$ chosen out of $j_1, j_2, \ldots, j_k$.
x is counted $\binom{l}{k}$ times in $\omega(k)$.

Inclusion exclusions principle, equivalent to
$\omega(1) - \omega(2) + \omega(3) + \ldots + (-1)^{l+1}\omega(l) = 1$
Every element is counted once!

In general,
$E(k) = \omega(k) - \binom{k+1}{k}\omega(k+1) + \binom{k+2}{k}\omega(k+1) - \ldots + (-1)^{n-k}\binom{n}{k}\omega(n)$

Theorem. *Consider a set S such that $|S| = N$.*
Let $A_1, A_2, \ldots, A_k$ be subsets of S.
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$$E(n) = \omega(n) - \binom{n+1}{n}\omega(n+1) + \cdots + (-1)^{k-n}\binom{k}{n}\omega(k)$$

Proof. For any $x \in S$,
$x \in S(t)$ for some t.
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Case 1: $t < n$.
x is in *exactly* t intersections
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$$\binom{n+r}{n}\binom{t}{n+r} = \frac{(n+r)!}{n!r!}\frac{t!}{(n+r)!(t-n-r)!}$$

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So, RHS is

$$0 = \binom{t}{n} - \binom{t}{n}\binom{t-n}{1}$$
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Example. Find the number of non-negative integer solutions of

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0 = &\binom{t}{n} - \binom{t}{n}\binom{t-n}{1} \\
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Example. Find the number of non-negative integer solutions of

$$x + y + z = 20$$

$$
\begin{aligned}
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Example. Find the number of non-negative integer solutions of

$$x + y + z = 20$$

So that $x \leq 5$,

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Example. Find the number of non-negative integer solutions of

$$x + y + z = 20$$

So that $x \leq 5$, $y \leq 7$,

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Example. Find the number of non-negative integer solutions of

$$x + y + z = 20$$

So that $x \leq 5$, $y \leq 7$, and $z \leq 9$

$$
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Solution.