Exercise sheet 3

Manifolds, MTH406

1. Consider a smooth frame $s_1, s_2, \ldots, s_k$ of a smooth vector bundle $\pi : E \to B$ over an open set $U \subset B$. Prove that any other section $s(p) = \lambda_1(p)s_1(p) + \cdots + \lambda_k(p)s_k(p)$ over U is a smooth vector field if and only if $\lambda_i : U \to \mathbb{R}$ are smooth functions.

2. Consider a (not-necessarily smooth) section $s : M \to T(M)$ such that $\pi \circ s = Id$, where $T(M)$ denotes the tangent bundle and $\pi : T(M) \to M$ the natural projection. Prove that the following are equivalent (and, therefore,they are all equivalent definitions of a smooth vector field):

   (a) $s : M \to T(M)$ is a smooth map (considering the manifold structure on $T(M)$).

   (b) Given a chart $\phi : U \to \mathbb{R}^n$, let x_i be the coordinate functions, i.e. $\phi(p) = (x_1(p), x_2(p), \ldots, x_n(p))$. Consider the smooth (local) frame on U given by $\frac{\partial}{\partial y_i}$. Let $\lambda_i : U \to \mathbb{R}$ define smooth functions so that $s(p) = \Sigma_i \lambda_i(p) \frac{\partial}{\partial y_i}$, then λ_i are smooth.

   (c) Since $s(p)$ is a tangent vector, it is a derivation. So given a smooth function $f : M \to \mathbb{R}$, $s(p)(f) \in \mathbb{R}$. Therefore, we get a map $p \to s(p)(f)$ which is smooth for every (global!) smooth function $f : M \to \mathbb{R}$ (Be careful, we are only considering global functions f, not local ones).

3. Prove that (global) vector fields on a smooth manifold form a vector space.

4. Given a vector field $p \to X_p$ where X_p may be regarded as a derivation on the germs of smooth functions at p, let $X(f) := X_p(f)$. Prove that X is as a linear map $X : C^\infty(M) \to C^\infty(M)$ which satisfies Leibnitz's rule, i.e. $X(fg) = fX(g) + gX(f)$. Conversely, any linear map which satisfies Leibnitz rule is equal to $X(f)$, for some vector field $p \to X_p$.

5. Given two vector fields X and Y, define the *Lie Bracket* of X and Y (denoted $[X, Y]$) as $[X, Y](f) := X(Y(f)) - Y(X(f))$.

   (a) Prove that $[X, Y]$ is vector field.

   (b) Prove that the Lie Bracket satisfies these properties

      i. $[X, Y+Z] = [X, Y]+[X, Z]$ and $[X, \lambda Y] = \lambda[X, Y]$ for any $\lambda \in \mathbb{R}$.

      ii. $[X, Y] = -[Y, X]$

      iii. $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]]$. This is called the *Jacobi identity*.

(c) Given a chart $\phi : U \to \mathbb{R}^n$, with its coordinate functions $x_i(p)$ defined so that $\phi(p) = (x_1(p), x_2(p), \ldots, x_n(p)$, prove that $[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}] = 0$

(d) Derive a local expression for the Lie Bracket. In other words, given a chart $\phi : U \to \mathbb{R}^n$, with its coordinate functions $x_i(p)$ defined so that $\phi(p) = (x_1(p), x_2(p), \ldots, x_n(p)$, express the vector fields X and Y in terms of the coordinate frame on U, i.e. $X_p = \Sigma_i a_i(p) \frac{\partial}{\partial x_i}$ and $Y_p = \Sigma_i b_i(p) \frac{\partial}{\partial x_i}$, then write an expression for $[X, Y]$ in terms of a_i, b_i, and $\frac{\partial}{\partial x_i}$