Exercise sheet 1

Manifolds, MTH406

1. Prove that any open subset of a smooth manifold is smooth.
2. Prove that the product of smooth manifolds is smooth.
3. Prove that any chart $\phi : U \to \mathbb{R}^n$ on a smooth manifold is a smooth map.
4. Prove that if $F : M \to N$ and $G : N \to P$ are smooth maps between manifolds, then the composition $G \circ F$ is also smooth.
5. Prove that if X_p is a derivation at $p \in M$, then $X_p(c) = 0$ for any constant function c.
6. Given a point p on a smooth manifold M, let F_p denote the ideal of germs that vanish at p. Prove that the dual of the vector space F_p/F_p^2 is isomorphic to $T_p(M)$.
7. Given a smooth manifold M and a curve $\gamma : (-\epsilon, \epsilon) \to M$ passing through a point $p \in M$ (i.e. $\gamma(0) = p$), let $[\gamma]_p$ denote the equivalence class of paths under the following equivalence relation. $\gamma_1 \sim \gamma_2$ if and only if $\frac{\mathrm{d}}{dt}(\phi \circ \gamma_1)|_{t=0} = \frac{\mathrm{d}}{dt}(\phi \circ \gamma_2)|_{t=0}$.
   (a) Prove that the equivalence relation is independent of the chart ϕ.
   (b) Given an equivalence class of paths, $[\gamma]_p$, we can define a map $\nu_{[\gamma]_p} : C_p^\infty(M) \to \mathbb{R}$, $\nu_{[\gamma]_p}(f) = \frac{\mathrm{d}}{dt} f(\gamma(t))$. Prove that $\nu_{[\gamma]_p}$ is a derivation. Prove that the map $[\gamma]_p \to \nu_{[\gamma]_p}$ is a bijection.