Exercise sheet 4

Knots and Braids, MTH436

1. Prove that two vector fields X and Y on a smooth manifold M are equal if and only if $X(f) = Y(f)$ for every continuous function $f : M \to \mathbb{R}$

2. Prove that given a smooth vector field X on a smooth manifold M so that $X_p \neq 0$ for some point $p \in M$, there exists a chart $\phi : U \to \mathbb{R}^n$ for some open neighbourhood U of p so that $X|_U = \frac{\partial}{\partial x_1}$. Here x_1 denotes the first coordinate function of the chart.

3. Consider a Lie Group, i.e. a smooth manifold G which is also a group and the multiplication and inverse is smooth, i.e. the map $\mu : G \times G \to G$ defined by $\mu(x, y) = xy$ and the map $\iota : G \to G$ defined by $\iota(x) = x^{-1}$ are smooth. Let $l_g : G \to G$ denote the (smooth) map $l_g(x) = gx$. Define a vector field X on G to be left invariant if ${l_g}_*(X) = X$.

   (a) Prove that there is a natural bijective correspondence between the tangent space at the identity element, $T_e(G)$, and left invariant vector fields on G.

   (b) Prove that a vector field is left invariant if and only if it is l_g-related to itself.

   (c) Prove that if X and Y are left invariant vector fields, then so is $[X, Y]$.