Exercise sheet 1

1. Prove that $\partial_n \circ \partial_{n+1} = 0$.
2. For any topological space, X, define the map $\epsilon : C_0(X) \to \mathbb{Z}$ by $\epsilon(\Sigma_i n_i \sigma_i) = \Sigma_i n_i$. Prove that:
   a) ϵ is a homomorphism.
   b) $\epsilon \circ \partial_1 = 0$. We can, therefore, define $\epsilon_* : H_0(X) \to \mathbb{Z}$
   c) If X is path-connected, ϵ_* is an isomorphism.
3. Let $f : X \to Y$ be a continuous map. Show that:
   a) For any $\sigma \in C_n(X)$, $f \circ \sigma \in C_n(Y)$. Therefore, one obtains a map $f_\# : C_n(X) \to C_n(Y)$.
   b) Show that $\partial'_n \circ f_\# = f_\# \circ \partial_n$, where $\partial'_n : C_n(Y) \to C_{n-1}(Y)$ is the boundary map on $C_n(Y)$.
   c) Show that $f_\#(Z_n(X)) \subset Z_n(Y)$
   d) Show that $f_\#(B_n(X)) \subset B_n(Y)$
   e) The previous two parts allow us to define $f_* : H_n(X) \to H_n(Y)$. Prove that $(f \circ g)_* = f_* \circ g_*$ and $(id_X)* = id_{H_n(X)}$
4. Prove that a set map $f : X \to Y$ is injective if and only if it has a left inverse.
5. Prove that a set map $f : X \to Y$ is surjective if and only if it has a right inverse.
6. Prove that if $f : X \to Y$ is a homeomorphism, then f_* is an isomorphism.
7. Prove that if $r : X \to A$ is a retract, then r_* is surjective.
8. Consider the subset $A = S^1 \times x_0$ of $X = S^1 \times S^1$. Prove that A is a retract of X.
9. Show that for any $p \in \mathbb{R}^n$, there is a retract $r : \mathbb{R}^n \to \{p\}$
10. Show that $\mathbb{R}^n \setminus \{p\}$, where p is the origin, retracts onto S^{n-1}.