$$C_n(X) \xrightarrow{j} C_n(X)/C_n(A)$$

$$\boxed{C_n(X) \xrightarrow{j} C_n(X)/C_n(A)}$$

$$H_n(X) \xrightarrow{j_*} H_n(X, A)$$

$$C_n(A) \xrightarrow{i_\#} C_n(X) \xrightarrow{j} C_n(X)/C_n(A)$$

$$H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A)$$

$$0 \to C_n(A) \xrightarrow{i_\#} C_n(X) \xrightarrow{j} C_n(X)/C_n(A)$$

$$H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A)$$

$$0 \to C_n(A) \xrightarrow{i_\#} C_n(X) \xrightarrow{j} C_n(X)/C_n(A)$$

$$H_{n+1}(X, A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A)$$

$$0 \to C_n(A) \xrightarrow{i_\#} C_n(X) \xrightarrow{j} C_n(X)/C_n(A) \to 0$$

$$H_{n+1}(X, A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A)$$

$$0 \to C_n(A) \xrightarrow{i_\#} C_n(X) \xrightarrow{j} C_n(X)/C_n(A) \to 0$$

$$H_{n+1}(X, A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A)$$

$$0 \to C_n(A) \xrightarrow{i_\#} C_n(X) \xrightarrow{j} C_n(X)/C_n(A) \to 0$$

$$\cdots \to H_{n+1}(X, A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

$$0 \to C_n(A) \xrightarrow{i_\#} C_n(X) \xrightarrow{j} C_n(X)/C_n(A) \to 0$$

$$\cdots \to H_{n+1}(X, A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Exact sequence

$$\cdots \to A_{i+i} \xrightarrow{f_{i+1}} A_i \xrightarrow{f_i} A_{i-1} \to \cdots$$

$$0 \to C_n(A) \xrightarrow{i_\#} C_n(X) \xrightarrow{j} C_n(X)/C_n(A) \to 0$$

$$\cdots \to H_{n+1}(X, A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Exact sequence

$$\cdots \to A_{i+i} \xrightarrow{f_{i+1}} A_i \xrightarrow{f_i} A_{i-1} \to \cdots$$

$Ker\ f_i = Im\ f_{i+1}$

$$0 \to C_n(A) \xrightarrow{i_\#} C_n(X) \xrightarrow{j} C_n(X)/C_n(A) \to 0$$

$$\cdots \to H_{n+1}(X, A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Exact sequence

$$\cdots \to A_{i+i} \xrightarrow{f_{i+1}} \color{red}{A_i} \xrightarrow{f_i} A_{i-1} \to \cdots$$

$$Ker\ f_i = Im\ f_{i+1}$$

Short exact sequence

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

$$0 \to C_n(A) \xrightarrow{i_\#} C_n(X) \xrightarrow{j} C_n(X)/C_n(A) \to 0$$

$$\cdots \to H_{n+1}(X, A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Exact sequence

$$\cdots \to A_{i+i} \xrightarrow{f_{i+1}} {\color{red}A_i} \xrightarrow{f_i} A_{i-1} \to \cdots$$

$$Ker\ f_i = Im\ f_{i+1}$$

Short exact sequence

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

$$Ker\ g = Im\ f$$

$$0 \to C_n(A) \xrightarrow{i_\#} C_n(X) \xrightarrow{j} C_n(X)/C_n(A) \to 0$$

$$\cdots \to H_{n+1}(X, A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Exact sequence

$$\cdots \to A_{i+i} \xrightarrow{f_{i+1}} {\color{red}A_i} \xrightarrow{f_i} A_{i-1} \to \cdots$$

$Ker\ f_i = Im\ f_{i+1}$

Short exact sequence

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

$Ker\ g = Im\ f$

f injective

$$0 \to C_n(A) \xrightarrow{i_\#} C_n(X) \xrightarrow{j} C_n(X)/C_n(A) \to 0$$

$$\cdots \to H_{n+1}(X, A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Exact sequence

$$\cdots \to A_{i+i} \xrightarrow{f_{i+1}} A_i \xrightarrow{f_i} A_{i-1} \to \cdots$$

$Ker\ f_i = Im\ f_{i+1}$

Short exact sequence

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

$Ker\ g = Im\ f$

f injective

g surjective

Chain complexes

Chain complexes

$$\cdots \xrightarrow{\partial_{n+2}} A_{n+1} \xrightarrow{\partial_{n+1}} A_n \xrightarrow{\partial_n} A_{n-1} \xrightarrow{\partial_{n-1}} \cdots$$

Chain complexes

$$\cdots \xrightarrow{\partial_{n+2}} C_{n+1}(X) \xrightarrow{\partial_{n+1}} C_n(X) \xrightarrow{\partial_n} C_{n-1}(X) \xrightarrow{\partial_{n-1}} \cdots$$

Chain complexes

$$\cdots \xrightarrow{\partial_{n+2}} C_{n+1}(X,A) \xrightarrow{\partial_{n+1}} C_n(X,A) \xrightarrow{\partial_n} C_{n-1}(X,A) \xrightarrow{\partial_{n-1}} \cdots$$

Chain complexes

$$\cdots \xrightarrow{\partial_{n+2}} A_{n+1} \xrightarrow{\partial_{n+1}} A_n \xrightarrow{\partial_n} A_{n-1} \xrightarrow{\partial_{n-1}} \cdots$$

Chain complexes

$$\cdots \xrightarrow{\partial_{n+2}} A_{n+1} \xrightarrow{\partial_{n+1}} A_n \xrightarrow{\partial_n} A_{n-1} \xrightarrow{\partial_{n-1}} \cdots$$

$$\partial_n \circ \partial_{n+1} = 0$$

Chain complexes

$$A_* := \cdots \xrightarrow{\partial_{n+2}} A_{n+1} \xrightarrow{\partial_{n+1}} A_n \xrightarrow{\partial_n} A_{n-1} \xrightarrow{\partial_{n-1}} \cdots$$

$$\partial_n \circ \partial_{n+1} = 0$$

Chain complexes

$$A_* := \cdots \xrightarrow{\partial_{n+2}} A_{n+1} \xrightarrow{\partial_{n+1}} A_n \xrightarrow{\partial_n} A_{n-1} \xrightarrow{\partial_{n-1}} \cdots$$

$$\partial_n \circ \partial_{n+1} = 0$$

$$H_n(A) := \frac{ker\ \partial_n}{Im\ \partial_{n+1}}$$

Chain map

Chain complexes

$$A_* := \cdots \xrightarrow{\partial_{n+2}} A_{n+1} \xrightarrow{\partial_{n+1}} A_n \xrightarrow{\partial_n} A_{n-1} \xrightarrow{\partial_{n-1}} \cdots$$

$$\partial_n \circ \partial_{n+1} = 0$$

$$H_n(A) := \frac{ker\ \partial_n}{Im\ \partial_{n+1}}$$

Chain map

$$f : A_* \to B_*$$

Chain complexes

$A_* := \cdots \xrightarrow{\partial_{n+2}} A_{n+1} \xrightarrow{\partial_{n+1}} A_n \xrightarrow{\partial_n} A_{n-1} \xrightarrow{\partial_{n-1}} \cdots$

$\partial_n \circ \partial_{n+1} = 0$

$H_n(A) := \frac{ker\ \partial_n}{Im\ \partial_{n+1}}$

Chain map

$f : A_* \to B_*$

$$\cdots \xrightarrow{\partial_{n+2}} A_{n+1} \xrightarrow{\partial_{n+1}} A_n \xrightarrow{\quad\partial_n\quad} A_{n-1} \to \cdots$$

Chain complexes

$A_* := \cdots \xrightarrow{\partial_{n+2}} A_{n+1} \xrightarrow{\partial_{n+1}} A_n \xrightarrow{\partial_n} A_{n-1} \xrightarrow{\partial_{n-1}} \cdots$

$\partial_n \circ \partial_{n+1} = 0$

$H_n(A) := \frac{ker\ \partial_n}{Im\ \partial_{n+1}}$

Chain map

$f : A_* \to B_*$

$$\cdots \xrightarrow{\partial_{n+2}} A_{n+1} \xrightarrow{\partial_{n+1}} A_n \xrightarrow{\partial_n} A_{n-1} \to \cdots$$

$$\cdots \xrightarrow{\partial'_{n+2}} B_{n+1} \xrightarrow{\partial'_{n+1}} B_n \xrightarrow{\partial'_n} B_{n-1} \to \cdots$$

Chain complexes

$$A_* := \cdots \xrightarrow{\partial_{n+2}} A_{n+1} \xrightarrow{\partial_{n+1}} A_n \xrightarrow{\partial_n} A_{n-1} \xrightarrow{\partial_{n-1}} \cdots$$

$$\partial_n \circ \partial_{n+1} = 0$$

$$H_n(A) := \frac{ker\ \partial_n}{Im\ \partial_{n+1}}$$

Chain map

$$f : A_* \to B_*$$

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{\partial_{n+2}} & A_{n+1} & \xrightarrow{\partial_{n+1}} & A_n & \xrightarrow{\partial_n} & A_{n-1} & \to & \cdots \\
 & & \downarrow{\scriptstyle f_{n+1}} & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} & & \\
\cdots & \xrightarrow{\partial'_{n+2}} & B_{n+1} & \xrightarrow{\partial'_{n+1}} & B_n & \xrightarrow{\partial'_n} & B_{n-1} & \to & \cdots
\end{array}$$

Chain complexes

$$0 \to A_* \to B_* \to C_* \to 0$$

$$A_* := \cdots \xrightarrow{\partial_{n+2}} A_{n+1} \xrightarrow{\partial_{n+1}} A_n \xrightarrow{\partial_n} A_{n-1} \xrightarrow{\partial_{n-1}} \cdots$$

$$\partial_n \circ \partial_{n+1} = 0$$

$$H_n(A) := \frac{ker\ \partial_n}{Im\ \partial_{n+1}}$$

Chain map

$$f : A_* \to B_*$$

$$\begin{tikzcd}
\cdots \arrow[r, "\partial_{n+2}"] & A_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "f_{n+1}"] & A_n \arrow[r, "\partial_n"] \arrow[d, "f_n"] & A_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \\
\cdots \arrow[r, "\partial'_{n+2}"] & B_{n+1} \arrow[r, "\partial'_{n+1}"] & B_n \arrow[r, "\partial'_n"] & B_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

$$\partial'_n \circ f_n = f_{n-1} \circ \partial_n$$

Chain complexes

$$A_* := \cdots \xrightarrow{\partial_{n+2}} A_{n+1} \xrightarrow{\partial_{n+1}} A_n \xrightarrow{\partial_n} A_{n-1} \xrightarrow{\partial_{n-1}} \cdots$$

$$\partial_n \circ \partial_{n+1} = 0$$

$$H_n(A) := \frac{ker\ \partial_n}{Im\ \partial_{n+1}}$$

Chain map

$$f : A_* \to B_*$$

$$\begin{tikzcd}
\cdots \arrow[r, "\partial_{n+2}"] & A_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "f_{n+1}"] & A_n \arrow[r, "\partial_n"] \arrow[d, "f_n"] & A_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \\
\cdots \arrow[r, "\partial'_{n+2}"] & B_{n+1} \arrow[r, "\partial'_{n+1}"] & B_n \arrow[r, "\partial'_n"] & B_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

$$\partial'_n \circ f_n = f_{n-1} \circ \partial_n$$

$0 \to A_* \to B_* \to C_* \to 0$ implies

$H_n(A) \to H_n(B) \to H_n(C)$

Chain complexes

$$A_* := \cdots \xrightarrow{\partial_{n+2}} A_{n+1} \xrightarrow{\partial_{n+1}} A_n \xrightarrow{\partial_n} A_{n-1} \xrightarrow{\partial_{n-1}} \cdots$$

$$\partial_n \circ \partial_{n+1} = 0$$

$$H_n(A) := \frac{ker\ \partial_n}{Im\ \partial_{n+1}}$$

Chain map

$$f : A_* \to B_*$$

$$\begin{tikzcd}
\cdots \arrow[r, "\partial_{n+2}"] & A_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "f_{n+1}"] & A_n \arrow[r, "\partial_n"] \arrow[d, "f_n"] & A_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \\
\cdots \arrow[r, "\partial'_{n+2}"] & B_{n+1} \arrow[r, "\partial'_{n+1}"] & B_n \arrow[r, "\partial'_n"] & B_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

$$\partial'_n \circ f_n = f_{n-1} \circ \partial_n$$

$0 \to A_* \to B_* \to C_* \to 0$ implies

$$\to H_{n+1}(A) \to H_n(A) \to H_n(B) \to H_n(C)$$

Chain complexes

$$A_* := \cdots \xrightarrow{\partial_{n+2}} A_{n+1} \xrightarrow{\partial_{n+1}} A_n \xrightarrow{\partial_n} A_{n-1} \xrightarrow{\partial_{n-1}} \cdots$$

$$\partial_n \circ \partial_{n+1} = 0$$

$$H_n(A) := \frac{ker\ \partial_n}{Im\ \partial_{n+1}}$$

Chain map

$f : A_* \to B_*$

$$\begin{tikzcd}
\cdots \arrow[r, "\partial_{n+2}"] & A_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "f_{n+1}"] & A_n \arrow[r, "\partial_n"] \arrow[d, "f_n"] & A_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \\
\cdots \arrow[r, "\partial'_{n+2}"] & B_{n+1} \arrow[r, "\partial'_{n+1}"] & B_n \arrow[r, "\partial'_n"] & B_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

$$\partial'_n \circ f_n = f_{n-1} \circ \partial_n$$

$0 \to A_* \to B_* \to C_* \to 0$ implies
$\to H_{n+1}(A) \to H_n(A) \to H_n(B) \to H_n(C) \to H_{n-1}(A) \to$

Chain complexes

$$A_* := \cdots \xrightarrow{\partial_{n+2}} A_{n+1} \xrightarrow{\partial_{n+1}} A_n \xrightarrow{\partial_n} A_{n-1} \xrightarrow{\partial_{n-1}} \cdots$$

$$\partial_n \circ \partial_{n+1} = 0$$

$$H_n(A) := \frac{ker\ \partial_n}{Im\ \partial_{n+1}}$$

Chain map

$$f : A_* \to B_*$$

$$\begin{tikzcd}
\cdots \arrow[r, "\partial_{n+2}"] & A_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "f_{n+1}"] & A_n \arrow[r, "\partial_n"] \arrow[d, "f_n"] & A_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \\
\cdots \arrow[r, "\partial'_{n+2}"] & B_{n+1} \arrow[r, "\partial'_{n+1}"] & B_n \arrow[r, "\partial'_n"] & B_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

$$\partial'_n \circ f_n = f_{n-1} \circ \partial_n$$

$0 \to A_* \to B_* \to C_* \to 0$ implies
$\to H_{n+1}(A) \to H_n(A) \to H_n(B) \to H_n(C) \to H_{n-1}(A) \to H_{n-1}(B) \to H_{n-1}(C) \to \cdots$

Defining ∂

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C_{n+1}(A) & \longrightarrow & C_{n+1}(X) & \longrightarrow & C_{n+1}(X,A) & \longrightarrow & 0 \\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\
0 & \longrightarrow & C_{n}(A) & \longrightarrow & C_{n}(X) & \longrightarrow & C_{n}(X,A) & \longrightarrow & 0 \\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\
0 & \longrightarrow & C_{n-1}(A) & \longrightarrow & C_{n-1}(X) & \longrightarrow & C_{n-1}(X,A) & \longrightarrow & 0
\end{array}$$

$$\cdots \to H_{n+1}(X,A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Defining ∂

$$
\begin{tikzcd}
0 \arrow[r] & C_{n+1}(A) \arrow[r] \arrow[d] & C_{n+1}(X) \arrow[r] \arrow[d] & C_{n+1}(X, A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_n(A) \arrow[r] \arrow[d] & C_n(X) \arrow[r] \arrow[d] & \overset{c_n}{C_n(X, A)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_{n-1}(A) \arrow[r] & C_{n-1}(X) \arrow[r] & C_{n-1}(X, A) \arrow[r] & 0
\end{tikzcd}
$$

$$\cdots \to H_{n+1}(X, A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Defining ∂

$$\begin{tikzcd}
0 \arrow[r] & C_{n+1}(A) \arrow[r] \arrow[d] & C_{n+1}(X) \arrow[r] \arrow[d] & C_{n+1}(X,A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_n(A) \arrow[r] \arrow[d] & C_n(X) \arrow[r] \arrow[d] & \overset{c_n}{C_n(X,A)} \arrow[r] \arrow[d, red] & 0 \\
0 \arrow[r] & C_{n-1}(A) \arrow[r] & C_{n-1}(X) \arrow[r] & \overset{\partial c_n}{C_{n-1}(X,A)} \arrow[r] & 0
\end{tikzcd}$$

$$\cdots \to H_{n+1}(X,A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Defining ∂

$$
\begin{tikzcd}
0 \arrow[r] & C_{n+1}(A) \arrow[r] \arrow[d] & C_{n+1}(X) \arrow[r] \arrow[d] & C_{n+1}(X,A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_n(A) \arrow[r] \arrow[d] & C_n(X) \arrow[r] \arrow[d] & \overset{c_n}{C_n(X,A)} \arrow[r] \arrow[d, red] & 0 \\
0 \arrow[r] & C_{n-1}(A) \arrow[r] & C_{n-1}(X) \arrow[r] & \overset{\partial c_n = 0}{C_{n-1}(X,A)} \arrow[r] & 0
\end{tikzcd}
$$

$$\cdots \to H_{n+1}(X,A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Defining ∂

$$
\begin{tikzcd}
0 \arrow[r] & C_{n+1}(A) \arrow[r] \arrow[d] & C_{n+1}(X) \arrow[r] \arrow[d] & C_{n+1}(X, A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_n(A) \arrow[r] \arrow[d] & \overset{b_n}{C_n(X)} \arrow[r] \arrow[d] & \overset{j(b_n)}{C_n(X, A)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_{n-1}(A) \arrow[r] & C_{n-1}(X) \arrow[r] & \overset{\partial j(b_n)=0}{C_{n-1}(X, A)} \arrow[r] & 0
\end{tikzcd}
$$

$$\cdots \to H_{n+1}(X, A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Defining ∂

$$
\begin{tikzcd}
0 \arrow[r] & C_{n+1}(A) \arrow[r] \arrow[d] & C_{n+1}(X) \arrow[r] \arrow[d] & C_{n+1}(X,A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_n(A) \arrow[r] \arrow[d] & \overset{b_n}{C_n(X)} \arrow[r, red] \arrow[d, red] & \overset{j(b_n)}{C_n(X,A)} \arrow[r] \arrow[d, red] & 0 \\
0 \arrow[r] & C_{n-1}(A) \arrow[r] & \overset{\partial b_n}{C_{n-1}(X)} \arrow[r] & \overset{\partial j(b_n)=0}{C_{n-1}(X,A)} \arrow[r] & 0
\end{tikzcd}
$$

$$\cdots \to H_{n+1}(X,A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Defining ∂

$$
\begin{tikzcd}
0 \arrow[r] & C_{n+1}(A) \arrow[r] \arrow[d] & C_{n+1}(X) \arrow[r] \arrow[d] & C_{n+1}(X,A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_n(A) \arrow[r] \arrow[d] & \overset{b_n}{C_n(X)} \arrow[r] \arrow[d] & \overset{j(b_n)}{C_n(X,A)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_{n-1}(A) \arrow[r] & \overset{\partial b_n}{C_{n-1}(X)} \arrow[r] & \overset{\partial j(b_n)=0=j(\partial b_n)}{C_{n-1}(X,A)} \arrow[r] & 0
\end{tikzcd}
$$

$$\cdots \to H_{n+1}(X,A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Defining ∂

$$
\begin{tikzcd}
0 \arrow[r] & C_{n+1}(A) \arrow[r] \arrow[d] & C_{n+1}(X) \arrow[r] \arrow[d] & C_{n+1}(X,A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_n(A) \arrow[r] \arrow[d] & \overset{b_n}{C_n(X)} \arrow[r, red] \arrow[d, red] & \overset{j(b_n)}{C_n(X,A)} \arrow[r] \arrow[d, red] & 0 \\
0 \arrow[r] & \overset{a_{n-1}}{C_{n-1}(A)} \arrow[r, red] & \overset{\partial b_n = j(a_n)}{C_{n-1}(X)} \arrow[r, red] & \overset{\partial j(b_n) = 0 = j(\partial b_n)}{C_{n-1}(X,A)} \arrow[r] & 0
\end{tikzcd}
$$

$$\cdots \to H_{n+1}(X,A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Checking well definedness

Independence of preimage of j

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C_{n+1}(A) & \longrightarrow & C_{n+1}(X) & \longrightarrow & C_{n+1}(X,A) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C_{n}(A) & \longrightarrow & C_{n}(X) & \longrightarrow & C_{n}(X,A) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C_{n-1}(A) & \longrightarrow & C_{n-1}(X) & \longrightarrow & C_{n-1}(X,A) & \longrightarrow & 0
\end{array}$$

$$\cdots \to H_{n+1}(X,A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Checking well definedness

Independence of preimage of j

$$\begin{tikzcd}
0 \arrow[r] & C_{n+1}(A) \arrow[r] \arrow[d] & C_{n+1}(X) \arrow[r] \arrow[d] & C_{n+1}(X, A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_n(A) \arrow[r] \arrow[d] & C_n(X) \arrow[r] \arrow[d] & \overset{0}{C_n(X, A)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_{n-1}(A) \arrow[r] & C_{n-1}(X) \arrow[r] & C_{n-1}(X, A) \arrow[r] & 0
\end{tikzcd}$$

$$\cdots \to H_{n+1}(X, A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Checking well definedness

Independence of preimage of j

$$
\begin{tikzcd}
0 \arrow[r] & C_{n+1}(A) \arrow[r] \arrow[d] & C_{n+1}(X) \arrow[r] \arrow[d] & C_{n+1}(X, A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_n(A) \arrow[r] \arrow[d] & \overset{b_n}{C_n(X)} \arrow[r, red] \arrow[d] & \overset{0}{C_n(X, A)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_{n-1}(A) \arrow[r] & C_{n-1}(X) \arrow[r] & C_{n-1}(X, A) \arrow[r] & 0
\end{tikzcd}
$$

$$\cdots \to H_{n+1}(X, A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Checking well definedness

Independence of preimage of j

$$\begin{tikzcd}
0 \arrow[r] & C_{n+1}(A) \arrow[r] \arrow[d] & C_{n+1}(X) \arrow[r] \arrow[d] & C_{n+1}(X,A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \overset{a_n}{C_n(A)} \arrow[r, red] \arrow[d] & \overset{i(a_n)}{C_n(X)} \arrow[r, red] \arrow[d] & \overset{0}{C_n(X,A)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_{n-1}(A) \arrow[r] & C_{n-1}(X) \arrow[r] & C_{n-1}(X,A) \arrow[r] & 0
\end{tikzcd}$$

$$\cdots \to H_{n+1}(X,A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Checking well definedness

Independence of preimage of j

$$
\begin{tikzcd}
0 \arrow[r] & C_{n+1}(A) \arrow[r] \arrow[d] & C_{n+1}(X) \arrow[r] \arrow[d] & C_{n+1}(X, A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \overset{a_n}{C_n(A)} \arrow[r, red] \arrow[d] & \overset{i(a_n)}{C_n(X)} \arrow[r, red] \arrow[d, red] & \overset{0}{C_n(X, A)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_{n-1}(A) \arrow[r] & \overset{\partial i(a_n)}{C_{n-1}(X)} \arrow[r] & C_{n-1}(X, A) \arrow[r] & 0
\end{tikzcd}
$$

$$\cdots \to H_{n+1}(X, A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Checking well definedness

Independence of preimage of j

$$\begin{tikzcd}
0 \arrow[r] & C_{n+1}(A) \arrow[r] \arrow[d] & C_{n+1}(X) \arrow[r] \arrow[d] & C_{n+1}(X,A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \overset{a_n}{C_n(A)} \arrow[r, red] \arrow[d] & \overset{i(a_n)}{C_n(X)} \arrow[r, red] \arrow[d, red] & \overset{0}{C_n(X,A)} \arrow[r] \arrow[d, red] & 0 \\
0 \arrow[r] & C_{n-1}(A) \arrow[r] & \overset{\partial i(a_n)}{C_{n-1}(X)} \arrow[r] & \overset{0}{C_{n-1}(X,A)} \arrow[r] & 0
\end{tikzcd}$$

$$\cdots \to H_{n+1}(X,A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Checking well definedness

Independence of preimage of j

$$\begin{tikzcd}
0 \arrow[r] & C_{n+1}(A) \arrow[r] \arrow[d] & C_{n+1}(X) \arrow[r] \arrow[d] & C_{n+1}(X,A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \overset{a_n}{C_n(A)} \arrow[r] \arrow[d] & \overset{i(a_n)}{C_n(X)} \arrow[r] \arrow[d] & \overset{0}{C_n(X,A)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \overset{a_{n-1}}{C_{n-1}(A)} \arrow[r] & \overset{\partial i(a_n)}{C_{n-1}(X)} \arrow[r] & \overset{0}{C_{n-1}(X,A)} \arrow[r] & 0
\end{tikzcd}$$

$$\cdots \to H_{n+1}(X,A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Checking well definedness

Independence of preimage of j

$$\begin{tikzcd}
0 \arrow[r] & C_{n+1}(A) \arrow[r] \arrow[d] & C_{n+1}(X) \arrow[r] \arrow[d] & C_{n+1}(X,A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \overset{a_n}{C_n(A)} \arrow[r, red] \arrow[d, red] & \overset{i(a_n)}{C_n(X)} \arrow[r, red] \arrow[d, red] & \overset{0}{C_n(X,A)} \arrow[r] \arrow[d, red] & 0 \\
0 \arrow[r] & \overset{a_{n-1}}{C_{n-1}(A)} \arrow[r, red] & \overset{\partial i(a_n)}{C_{n-1}(X)} \arrow[r] & \overset{0}{C_{n-1}(X,A)} \arrow[r] & 0
\end{tikzcd}$$

$$\cdots \to H_{n+1}(X,A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Checking well definedness

Independence of homology class

$$\begin{tikzcd}
0 \arrow[r] & C_{n+1}(A) \arrow[r] \arrow[d] & C_{n+1}(X) \arrow[r] \arrow[d] & C_{n+1}(X,A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_n(A) \arrow[r] \arrow[d] & C_n(X) \arrow[r] \arrow[d] & C_n(X,A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_{n-1}(A) \arrow[r] & C_{n-1}(X) \arrow[r] & C_{n-1}(X,A) \arrow[r] & 0
\end{tikzcd}$$

$$\cdots \to H_{n+1}(X,A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Checking well definedness

Independence of homology class

$$
\begin{tikzcd}
0 \arrow[r] & C_{n+1}(A) \arrow[r] \arrow[d] & C_{n+1}(X) \arrow[r] \arrow[d] & \overset{c}{C_{n+1}(X,A)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_n(A) \arrow[r] \arrow[d] & C_n(X) \arrow[r] \arrow[d] & \overset{\partial c}{C_n(X,A)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_{n-1}(A) \arrow[r] & C_{n-1}(X) \arrow[r] & C_{n-1}(X,A) \arrow[r] & 0
\end{tikzcd}
$$

$$\cdots \to H_{n+1}(X,A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Checking well definedness

Independence of homology class

$$
\begin{tikzcd}
0 \arrow[r] & C_{n+1}(A) \arrow[r] \arrow[d] & \overset{b}{C_{n+1}(X)} \arrow[r] \arrow[d] & \overset{j(b)}{C_{n+1}(X,A)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_n(A) \arrow[r] \arrow[d] & C_n(X) \arrow[r] \arrow[d] & \overset{\partial(j(b))}{C_n(X,A)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_{n-1}(A) \arrow[r] & C_{n-1}(X) \arrow[r] & C_{n-1}(X,A) \arrow[r] & 0
\end{tikzcd}
$$

$$\cdots \to H_{n+1}(X,A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Checking well definedness

Independence of homology class

$$
\begin{tikzcd}
0 \arrow[r] & C_{n+1}(A) \arrow[r] \arrow[d] & \overset{b}{C_{n+1}(X)} \arrow[r, red] \arrow[d, red] & \overset{j(b)}{C_{n+1}(X,A)} \arrow[r] \arrow[d, red] & 0 \\
0 \arrow[r] & C_n(A) \arrow[r] \arrow[d] & \overset{\partial b}{C_n(X)} \arrow[r] \arrow[d] & \overset{\partial(j(b))}{C_n(X,A)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_{n-1}(A) \arrow[r] & C_{n-1}(X) \arrow[r] & C_{n-1}(X,A) \arrow[r] & 0
\end{tikzcd}
$$

$$\cdots \to H_{n+1}(X,A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Checking well definedness

Independence of homology class

$$
\begin{tikzcd}
0 \arrow[r] & C_{n+1}(A) \arrow[r] \arrow[d] & \overset{b}{C_{n+1}(X)} \arrow[r, red] \arrow[d, red] & \overset{j(b)}{C_{n+1}(X,A)} \arrow[r] \arrow[d, red] & 0 \\
0 \arrow[r] & C_n(A) \arrow[r] \arrow[d] & \overset{\partial b}{C_n(X)} \arrow[r, red] \arrow[d] & \overset{\partial(j(b))}{C_n(X,A)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_{n-1}(A) \arrow[r] & C_{n-1}(X) \arrow[r] & C_{n-1}(X,A) \arrow[r] & 0
\end{tikzcd}
$$

$$\cdots \to H_{n+1}(X,A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Checking well definedness

Independence of homology class

$$
\begin{tikzcd}
0 \arrow[r] & C_{n+1}(A) \arrow[r] \arrow[d] & \overset{b}{C_{n+1}(X)} \arrow[r] \arrow[d] & \overset{j(b)}{C_{n+1}(X, A)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_n(A) \arrow[r] \arrow[d] & \overset{\partial b}{C_n(X)} \arrow[r] \arrow[d] & \overset{\partial(j(b))}{C_n(X, A)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_{n-1}(A) \arrow[r] & \overset{0}{C_{n-1}(X)} \arrow[r] & C_{n-1}(X, A) \arrow[r] & 0
\end{tikzcd}
$$

$$\cdots \to H_{n+1}(X, A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Checking well definedness

Independence of homology class

$$
\begin{tikzcd}
0 \arrow[r] & C_{n+1}(A) \arrow[r] \arrow[d] & \overset{b}{C_{n+1}(X)} \arrow[r, red] \arrow[d, red] & \overset{j(b)}{C_{n+1}(X,A)} \arrow[r] \arrow[d, red] & 0 \\
0 \arrow[r] & C_n(A) \arrow[r] \arrow[d] & \overset{\partial b}{C_n(X)} \arrow[r, red] \arrow[d, red] & \overset{\partial(j(b))}{C_n(X,A)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \overset{0}{C_{n-1}(A)} \arrow[r, red] & \overset{0}{C_{n-1}(X)} \arrow[r] & C_{n-1}(X,A) \arrow[r] & 0
\end{tikzcd}
$$

$$\cdots \to H_{n+1}(X,A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Checking exactness at $H_n(X, A)$

$Ker\ \partial \subset Im\ j_*$

$$
\begin{array}{ccccccccc}
 & & & & b & & j(b) & & \\
0 & \longrightarrow & C_{n+1}(A) & \longrightarrow & C_{n+1}(X) & \longrightarrow & C_{n+1}(X, A) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C_n(A) & \longrightarrow & C_n(X) & \longrightarrow & C_n(X, A) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C_{n-1}(A) & \longrightarrow & C_{n-1}(X) & \longrightarrow & C_{n-1}(X, A) & \longrightarrow & 0
\end{array}
$$

$$\cdots \to H_{n+1}(X, A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Checking exactness at $H_n(X, A)$

$Ker\ \partial \subset Im\ j_*$

$$
\begin{array}{ccccccccc}
 & & & & \overset{b}{} & & \overset{j(b)}{} & & \\
0 & \longrightarrow & C_{n+1}(A) & \longrightarrow & C_{n+1}(X) & \longrightarrow & C_{n+1}(X, A) & \longrightarrow & 0 \\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\
 & & & & & & \overset{j(b_n)}{} & & \\
0 & \longrightarrow & C_n(A) & \longrightarrow & C_n(X) & \longrightarrow & C_n(X, A) & \longrightarrow & 0 \\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\
0 & \longrightarrow & C_{n-1}(A) & \longrightarrow & C_{n-1}(X) & \longrightarrow & C_{n-1}(X, A) & \longrightarrow & 0
\end{array}
$$

$$\cdots \to H_{n+1}(X, A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Checking exactness at $H_n(X, A)$

$Ker\ \partial \subset Im\ j_*$

$$
\begin{tikzcd}
0 \arrow[r] & C_{n+1}(A) \arrow[r] \arrow[d] & \overset{b}{C_{n+1}(X)} \arrow[r] \arrow[d] & \overset{j(b)}{C_{n+1}(X, A)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_n(A) \arrow[r] \arrow[d] & \overset{b_n}{C_n(X)} \arrow[r, red] \arrow[d] & \overset{j(b_n)}{C_n(X, A)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_{n-1}(A) \arrow[r] & C_{n-1}(X) \arrow[r] & C_{n-1}(X, A) \arrow[r] & 0
\end{tikzcd}
$$

$$\cdots \to H_{n+1}(X, A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Checking exactness at $H_n(X, A)$

$Ker\ \partial \subset Im\ j_*$

$$
\begin{tikzcd}
 & & b & j(b) & \\
0 \arrow[r] & C_{n+1}(A) \arrow[r] \arrow[dd] & C_{n+1}(X) \arrow[r] \arrow[dd] & C_{n+1}(X, A) \arrow[r] \arrow[dd] & 0 \\
 & & b_n & j(b_n) & \\
0 \arrow[r] & C_n(A) \arrow[r] \arrow[dd] & C_n(X) \arrow[r, red] \arrow[dd] & C_n(X, A) \arrow[r] \arrow[d, red] & 0 \\
 & & & 0 & \\
0 \arrow[r] & C_{n-1}(A) \arrow[r] & C_{n-1}(X) \arrow[r] & C_{n-1}(X, A) \arrow[r] & 0
\end{tikzcd}
$$

$$\cdots \to H_{n+1}(X, A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Checking exactness at $H_n(X, A)$

$Ker\ \partial \subset Im\ j_*$

$$
\begin{tikzcd}
0 \arrow[r] & C_{n+1}(A) \arrow[r] \arrow[d] & \overset{b}{C_{n+1}(X)} \arrow[r] \arrow[d] & \overset{j(b)}{C_{n+1}(X, A)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_n(A) \arrow[r] \arrow[d] & \overset{b_n}{C_n(X)} \arrow[r] \arrow[d] & \overset{j(b_n)}{C_n(X, A)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_{n-1}(A) \arrow[r] & \overset{\partial b_n}{C_{n-1}(X)} \arrow[r] & \overset{0}{C_{n-1}(X, A)} \arrow[r] & 0
\end{tikzcd}
$$

$$\cdots \to H_{n+1}(X, A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Checking exactness at $H_n(X, A)$

$Ker\ \partial \subset Im\ j_*$

$$
\begin{tikzcd}
0 \arrow[r] & C_{n+1}(A) \arrow[r] \arrow[d] & \overset{b}{C_{n+1}(X)} \arrow[r] \arrow[d] & \overset{j(b)}{C_{n+1}(X, A)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_n(A) \arrow[r] \arrow[d] & \overset{b_n}{C_n(X)} \arrow[r, red] \arrow[d, red] & \overset{j(b_n)}{C_n(X, A)} \arrow[r] \arrow[d, red] & 0 \\
0 \arrow[r] & C_{n-1}(A) \arrow[r] & \overset{\partial b_n}{C_{n-1}(X)} \arrow[r, red] & \overset{0}{C_{n-1}(X, A)} \arrow[r] & 0
\end{tikzcd}
$$

$$\cdots \to H_{n+1}(X, A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Checking exactness at $H_n(X, A)$

$Ker\ \partial \subset Im\ j_*$

$$
\begin{tikzcd}
0 \arrow[r] & C_{n+1}(A) \arrow[r] \arrow[d] & \overset{b}{C_{n+1}(X)} \arrow[r] \arrow[d] & \overset{j(b)}{C_{n+1}(X, A)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_n(A) \arrow[r] \arrow[d] & \overset{b_n}{C_n(X)} \arrow[r] \arrow[d] & \overset{j(b_n)}{C_n(X, A)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \overset{a_{n-1}}{C_{n-1}(A)} \arrow[r] & \overset{\partial b_n = i(a_{n-1})}{C_{n-1}(X)} \arrow[r] & \overset{0}{C_{n-1}(X, A)} \arrow[r] & 0
\end{tikzcd}
$$

$$\cdots \to H_{n+1}(X, A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Checking exactness at $H_n(X, A)$

$Ker\ \partial \subset Im\ j_*$

$$\begin{tikzcd}
0 \arrow[r] & C_{n+1}(A) \arrow[r] \arrow[d] & \overset{b}{C_{n+1}(X)} \arrow[r] \arrow[d] & \overset{j(b)}{C_{n+1}(X, A)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \overset{a_n}{C_n(A)} \arrow[r] \arrow[d] & \overset{b_n}{C_n(X)} \arrow[r] \arrow[d] & \overset{j(b_n)}{C_n(X, A)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \overset{a_{n-1}=\partial a_n}{C_{n-1}(A)} \arrow[r] & \overset{\partial b_n=i(a_{n-1})}{C_{n-1}(X)} \arrow[r] & \overset{0}{C_{n-1}(X, A)} \arrow[r] & 0
\end{tikzcd}$$

$$\cdots \to H_{n+1}(X, A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Checking exactness at $H_n(X, A)$

$Ker\ \partial \subset Im\ j_*$

$$\begin{tikzcd}
0 \arrow[r] & C_{n+1}(A) \arrow[r] \arrow[d] & \overset{b}{C_{n+1}(X)} \arrow[r] \arrow[d] & \overset{j(b)}{C_{n+1}(X, A)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \overset{a_n}{C_n(A)} \arrow[r] \arrow[d] & \overset{b_n, b_n - i(a_n)}{C_n(X)} \arrow[r] \arrow[d] & \overset{j(b_n)}{C_n(X, A)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \overset{a_{n-1} = \partial a_n}{C_{n-1}(A)} \arrow[r] & \overset{\partial b_n = i(a_{n-1})}{C_{n-1}(X)} \arrow[r] & \overset{0}{C_{n-1}(X, A)} \arrow[r] & 0
\end{tikzcd}$$

$$\cdots \to H_{n+1}(X, A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Checking exactness at $H_n(X, A)$

$Im\ j_* \subset Ker\ \partial$

$$\begin{tikzcd}
& & b & j(b) & \\
0 \arrow[r] & C_{n+1}(A) \arrow[r] \arrow[d] & C_{n+1}(X) \arrow[r] \arrow[d] & C_{n+1}(X, A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_n(A) \arrow[r] \arrow[d] & C_n(X) \arrow[r] \arrow[d] & C_n(X, A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_{n-1}(A) \arrow[r] & C_{n-1}(X) \arrow[r] & C_{n-1}(X, A) \arrow[r] & 0
\end{tikzcd}$$

$$\cdots \to H_{n+1}(X, A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Checking exactness at $H_n(X, A)$

$Im\ j_* \subset Ker\ \partial$

$$
\begin{array}{ccccccccc}
 & & & & b & & j(b) & & \\
0 & \longrightarrow & C_{n+1}(A) & \longrightarrow & C_{n+1}(X) & \longrightarrow & C_{n+1}(X, A) & \longrightarrow & 0 \\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\
 & & & & b_n & & & & \\
0 & \longrightarrow & C_n(A) & \longrightarrow & C_n(X) & \longrightarrow & C_n(X, A) & \longrightarrow & 0 \\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\
0 & \longrightarrow & C_{n-1}(A) & \longrightarrow & C_{n-1}(X) & \longrightarrow & C_{n-1}(X, A) & \longrightarrow & 0
\end{array}
$$

$$\cdots \to H_{n+1}(X, A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Checking exactness at $H_n(X, A)$

$Im\ j_* \subset Ker\ \partial$

$$\begin{tikzcd}
0 \arrow[r] & C_{n+1}(A) \arrow[r] \arrow[d] & \overset{b}{C_{n+1}(X)} \arrow[r] \arrow[d] & \overset{j(b)}{C_{n+1}(X, A)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_n(A) \arrow[r] \arrow[d] & \overset{b_n}{C_n(X)} \arrow[r] \arrow[d, red] & C_n(X, A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_{n-1}(A) \arrow[r] & \overset{\partial b_n = 0}{C_{n-1}(X)} \arrow[r] & C_{n-1}(X, A) \arrow[r] & 0
\end{tikzcd}$$

$$\cdots \to H_{n+1}(X, A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Checking exactness at $H_n(X, A)$

$Im\ j_* \subset Ker\ \partial$

$$\begin{tikzcd}
0 \arrow[r] & C_{n+1}(A) \arrow[r] \arrow[d] & \overset{b}{C_{n+1}(X)} \arrow[r] \arrow[d] & \overset{j(b)}{C_{n+1}(X, A)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_n(A) \arrow[r] \arrow[d] & \overset{b_n}{C_n(X)} \arrow[r] \arrow[d] & \overset{j(b_n)}{C_n(X, A)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_{n-1}(A) \arrow[r] & \overset{\partial b_n = 0}{C_{n-1}(X)} \arrow[r] & C_{n-1}(X, A) \arrow[r] & 0
\end{tikzcd}$$

$$\cdots \to H_{n+1}(X, A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Checking exactness at $H_n(X, A)$

$Im\ j_* \subset Ker\ \partial$

$$
\begin{tikzcd}
0 \arrow[r] & C_{n+1}(A) \arrow[r] \arrow[d] & \overset{b}{C_{n+1}(X)} \arrow[r] \arrow[d] & \overset{j(b)}{C_{n+1}(X, A)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_n(A) \arrow[r] \arrow[d] & \overset{b_n}{C_n(X)} \arrow[r] \arrow[d] & \overset{j(b_n)}{C_n(X, A)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \overset{0}{C_{n-1}(A)} \arrow[r] & \overset{\partial b_n = 0}{C_{n-1}(X)} \arrow[r] & C_{n-1}(X, A) \arrow[r] & 0
\end{tikzcd}
$$

$$\cdots \to H_{n+1}(X, A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Checking exactness at $H_n(A)$

$Ker\ i \subset Im\ \partial$

$$\begin{tikzcd}
 & & b & j(b) & \\
0 \arrow[r] & C_{n+1}(A) \arrow[r] \arrow[d] & C_{n+1}(X) \arrow[r] \arrow[d] & C_{n+1}(X,A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_n(A) \arrow[r] \arrow[d] & C_n(X) \arrow[r] \arrow[d] & C_n(X,A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_{n-1}(A) \arrow[r] & C_{n-1}(X) \arrow[r] & C_{n-1}(X,A) \arrow[r] & 0
\end{tikzcd}$$

$$\cdots \to H_{n+1}(X,A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Checking exactness at $H_n(A)$

$Ker\ i \subset Im\ \partial$

$$
\begin{tikzcd}
0 \arrow[r] & C_{n+1}(A) \arrow[r] \arrow[d] & \overset{b}{C_{n+1}(X)} \arrow[r] \arrow[d] & \overset{j(b)}{C_{n+1}(X, A)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \overset{a_n}{C_n(A)} \arrow[r] \arrow[d] & C_n(X) \arrow[r] \arrow[d] & C_n(X, A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_{n-1}(A) \arrow[r] & C_{n-1}(X) \arrow[r] & C_{n-1}(X, A) \arrow[r] & 0
\end{tikzcd}
$$

$$\cdots \to H_{n+1}(X, A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Checking exactness at $H_n(A)$

$Ker\ i \subset Im\ \partial$

$$
\begin{tikzcd}
0 \arrow[r] & C_{n+1}(A) \arrow[r] \arrow[d] & \overset{b}{C_{n+1}(X)} \arrow[r] \arrow[d] & \overset{j(b)}{C_{n+1}(X,A)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \overset{a_n}{C_n(A)} \arrow[r] \arrow[d] & C_n(X) \arrow[r] \arrow[d] & C_n(X,A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \overset{\partial a_n = 0}{C_{n-1}(A)} \arrow[r] & C_{n-1}(X) \arrow[r] & C_{n-1}(X,A) \arrow[r] & 0
\end{tikzcd}
$$

$$\cdots \to H_{n+1}(X,A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Checking exactness at $H_n(A)$

$Ker\ i \subset Im\ \partial$

$$\begin{tikzcd}
0 \arrow[r] & C_{n+1}(A) \arrow[r] \arrow[d] & \overset{b}{C_{n+1}(X)} \arrow[r] \arrow[d] & \overset{j(b)}{C_{n+1}(X, A)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \overset{a_n}{C_n(A)} \arrow[r] \arrow[d] & \overset{i(a_n)}{C_n(X)} \arrow[r] \arrow[d] & C_n(X, A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \overset{\partial a_n = 0}{C_{n-1}(A)} \arrow[r] & C_{n-1}(X) \arrow[r] & C_{n-1}(X, A) \arrow[r] & 0
\end{tikzcd}$$

$$\cdots \to H_{n+1}(X, A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Checking exactness at $H_n(A)$

$Ker\ i \subset Im\ \partial$

$$
\begin{tikzcd}
0 \arrow[r] & C_{n+1}(A) \arrow[r] \arrow[d] & \overset{bb_{n+1}}{C_{n+1}(X)} \arrow[r] \arrow[d] & \overset{j(b)}{C_{n+1}(X, A)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \overset{a_n}{C_n(A)} \arrow[r] \arrow[d] & \overset{i(a_n)=\partial(b_{n+1})}{C_n(X)} \arrow[r] \arrow[d] & C_n(X, A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \overset{\partial a_n=0}{C_{n-1}(A)} \arrow[r] & C_{n-1}(X) \arrow[r] & C_{n-1}(X, A) \arrow[r] & 0
\end{tikzcd}
$$

$$\cdots \to H_{n+1}(X, A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Checking exactness at $H_n(A)$

$Ker\ i \subset Im\ \partial$

$$
\begin{tikzcd}
0 \arrow[r] & C_{n+1}(A) \arrow[r] \arrow[d] & \overset{bb_{n+1}}{C_{n+1}(X)} \arrow[r, red] \arrow[d, red] & \overset{j(b)j(b_{n+1})}{C_{n+1}(X,A)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \overset{a_n}{C_n(A)} \arrow[r, red] \arrow[d, red] & \overset{i(a_n)=\partial(b_{n+1})}{C_n(X)} \arrow[r] \arrow[d] & C_n(X,A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \overset{\partial a_n=0}{C_{n-1}(A)} \arrow[r] & C_{n-1}(X) \arrow[r] & C_{n-1}(X,A) \arrow[r] & 0
\end{tikzcd}
$$

$$\cdots \to H_{n+1}(X,A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Checking exactness at $H_n(A)$

$Im\ \partial \subset Ker\ i$

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C_{n+1}(A) & \longrightarrow & C_{n+1}(X) & \longrightarrow & C_{n+1}(X,A) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C_n(A) & \longrightarrow & C_n(X) & \longrightarrow & C_n(X,A) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C_{n-1}(A) & \longrightarrow & C_{n-1}(X) & \longrightarrow & C_{n-1}(X,A) & \longrightarrow & 0
\end{array}$$

$$\cdots \to H_{n+1}(X,A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Checking exactness at $H_n(A)$

$Im\ \partial \subset Ker\ i$

$$\begin{tikzcd}
0 \arrow[r] & C_{n+1}(A) \arrow[r] \arrow[d] & C_{n+1}(X) \arrow[r] \arrow[d] & \overset{j(b_{n+1})}{C_{n+1}(X,A)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_n(A) \arrow[r] \arrow[d] & C_n(X) \arrow[r] \arrow[d] & C_n(X,A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_{n-1}(A) \arrow[r] & C_{n-1}(X) \arrow[r] & C_{n-1}(X,A) \arrow[r] & 0
\end{tikzcd}$$

$$\cdots \to H_{n+1}(X,A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Checking exactness at $H_n(A)$

$Im\ \partial \subset Ker\ i$

$$
\begin{tikzcd}
0 \arrow[r] & C_{n+1}(A) \arrow[r] \arrow[d] & C_{n+1}(X) \arrow[r] \arrow[d] & \overset{j(b_{n+1})}{C_{n+1}(X,A)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_n(A) \arrow[r] \arrow[d] & C_n(X) \arrow[r] \arrow[d] & \overset{0}{C_n(X,A)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_{n-1}(A) \arrow[r] & C_{n-1}(X) \arrow[r] & C_{n-1}(X,A) \arrow[r] & 0
\end{tikzcd}
$$

$$\cdots \to H_{n+1}(X,A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Checking exactness at $H_n(A)$

$Im\ \partial \subset Ker\ i$

$$
\begin{tikzcd}
 & & b_{n+1} & j(b_{n+1}) & \\
0 \arrow[r] & C_{n+1}(A) \arrow[r] \arrow[d] & C_{n+1}(X) \arrow[r, red] \arrow[d] & C_{n+1}(X, A) \arrow[r] \arrow[d, red] & 0 \\
 & & & 0 & \\
0 \arrow[r] & C_n(A) \arrow[r] \arrow[d] & C_n(X) \arrow[r] \arrow[d] & C_n(X, A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_{n-1}(A) \arrow[r] & C_{n-1}(X) \arrow[r] & C_{n-1}(X, A) \arrow[r] & 0
\end{tikzcd}
$$

$$\cdots \to H_{n+1}(X, A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Checking exactness at $H_n(A)$

$Im\ \partial \subset Ker\ i$

$$
\begin{tikzcd}
0 \arrow[r] & C_{n+1}(A) \arrow[r] \arrow[d] & \overset{b_{n+1}}{C_{n+1}(X)} \arrow[r, red] \arrow[d, red] & \overset{j(b_{n+1})}{C_{n+1}(X,A)} \arrow[r] \arrow[d, red] & 0 \\
0 \arrow[r] & C_n(A) \arrow[r] \arrow[d] & \overset{\partial b_{n+1}}{C_n(X)} \arrow[r] \arrow[d] & \overset{0}{C_n(X,A)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_{n-1}(A) \arrow[r] & C_{n-1}(X) \arrow[r] & C_{n-1}(X,A) \arrow[r] & 0
\end{tikzcd}
$$

$$\cdots \to H_{n+1}(X,A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Checking exactness at $H_n(A)$

$Im\ \partial \subset Ker\ i$

$$\begin{tikzcd}
0 \arrow[r] & C_{n+1}(A) \arrow[r] \arrow[d] & \overset{b_{n+1}}{C_{n+1}(X)} \arrow[r] \arrow[d] & \overset{j(b_{n+1})}{C_{n+1}(X,A)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \overset{a_n}{C_n(A)} \arrow[r] \arrow[d] & \overset{\partial b_{n+1}}{C_n(X)} \arrow[r] \arrow[d] & \overset{0}{C_n(X,A)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_{n-1}(A) \arrow[r] & C_{n-1}(X) \arrow[r] & C_{n-1}(X,A) \arrow[r] & 0
\end{tikzcd}$$

$$\cdots \to H_{n+1}(X,A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Checking exactness at $H_n(X)$

$Ker\ j \subset Im\ i$

$$\begin{tikzcd}
0 \arrow[r] & C_{n+1}(A) \arrow[r] \arrow[d] & C_{n+1}(X) \arrow[r] \arrow[d] & C_{n+1}(X, A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_n(A) \arrow[r] \arrow[d] & C_n(X) \arrow[r] \arrow[d] & C_n(X, A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_{n-1}(A) \arrow[r] & C_{n-1}(X) \arrow[r] & C_{n-1}(X, A) \arrow[r] & 0
\end{tikzcd}$$

$$\cdots \to H_{n+1}(X, A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Checking exactness at $H_n(X)$

$Ker\ j \subset Im\ i$

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & C_{n+1}(A) & \longrightarrow & C_{n+1}(X) & \longrightarrow & C_{n+1}(X,A) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & & & b_n & & & & \\
0 & \longrightarrow & C_n(A) & \longrightarrow & C_n(X) & \longrightarrow & C_n(X,A) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C_{n-1}(A) & \longrightarrow & C_{n-1}(X) & \longrightarrow & C_{n-1}(X,A) & \longrightarrow & 0
\end{array}
$$

$$\cdots \to H_{n+1}(X,A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Checking exactness at $H_n(X)$

$Ker\ j \subset Im\ i$

$$
\begin{tikzcd}
0 \arrow[r] & C_{n+1}(A) \arrow[r] \arrow[d] & C_{n+1}(X) \arrow[r] \arrow[d] & C_{n+1}(X,A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_n(A) \arrow[r] \arrow[d] & \overset{b_n}{C_n(X)} \arrow[r] \arrow[d, red] & C_n(X,A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_{n-1}(A) \arrow[r] & \overset{0}{C_{n-1}(X)} \arrow[r] & C_{n-1}(X,A) \arrow[r] & 0
\end{tikzcd}
$$

$$\cdots \to H_{n+1}(X,A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Checking exactness at $H_n(X)$

$Ker\ j \subset Im\ i$

$$
\begin{tikzcd}
0 \arrow[r] & C_{n+1}(A) \arrow[r] \arrow[d] & C_{n+1}(X) \arrow[r] \arrow[d] & C_{n+1}(X,A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_n(A) \arrow[r] \arrow[d] & \overset{b_n}{C_n(X)} \arrow[r, red] \arrow[d, red] & \overset{j(b_n)}{C_n(X,A)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_{n-1}(A) \arrow[r] & \overset{0}{C_{n-1}(X)} \arrow[r] & C_{n-1}(X,A) \arrow[r] & 0
\end{tikzcd}
$$

$$\cdots \to H_{n+1}(X,A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Checking exactness at $H_n(X)$

$Ker\ j \subset Im\ i$

$$
\begin{tikzcd}
0 \arrow[r] & C_{n+1}(A) \arrow[r] \arrow[d] & C_{n+1}(X) \arrow[r] \arrow[d] & \overset{c_{n+1}}{C_{n+1}(X,A)} \arrow[r] \arrow[d, red] & 0 \\
0 \arrow[r] & C_n(A) \arrow[r] \arrow[d] & \overset{b_n}{C_n(X)} \arrow[r, red] \arrow[d, red] & \overset{j(b_n)=\partial c_{n+1}}{C_n(X,A)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_{n-1}(A) \arrow[r] & \overset{0}{C_{n-1}(X)} \arrow[r] & C_{n-1}(X,A) \arrow[r] & 0
\end{tikzcd}
$$

$$\cdots \to H_{n+1}(X,A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Checking exactness at $H_n(X)$

$Ker\ j \subset Im\ i$

$$\begin{tikzcd}
0 \arrow[r] & C_{n+1}(A) \arrow[r] \arrow[d] & \overset{b_{n+1}}{C_{n+1}(X)} \arrow[r, red] \arrow[d] & \overset{j(b_{n+1})}{C_{n+1}(X,A)} \arrow[r] \arrow[d, red] & 0 \\
0 \arrow[r] & C_n(A) \arrow[r] \arrow[d] & \overset{b_n}{C_n(X)} \arrow[r, red] \arrow[d, red] & \overset{j(b_n)=\partial j(b_{n+1})}{C_n(X,A)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_{n-1}(A) \arrow[r] & \overset{0}{C_{n-1}(X)} \arrow[r] & C_{n-1}(X,A) \arrow[r] & 0
\end{tikzcd}$$

$$\cdots \to H_{n+1}(X,A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Checking exactness at $H_n(X)$

$Ker\ j \subset Im\ i$

$$
\begin{tikzcd}
0 \arrow[r] & C_{n+1}(A) \arrow[r] \arrow[d] & \overset{b_{n+1}}{C_{n+1}(X)} \arrow[r] \arrow[d] & \overset{j(b_{n+1})}{C_{n+1}(X,A)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_n(A) \arrow[r] \arrow[d] & \overset{b_n \text{ homologous } b_n - \partial b_{n+1}}{C_n(X)} \arrow[r] \arrow[d] & \overset{j(b_n) = \partial j(b_{n+1})}{C_n(X,A)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_{n-1}(A) \arrow[r] & \overset{0}{C_{n-1}(X)} \arrow[r] & C_{n-1}(X,A) \arrow[r] & 0
\end{tikzcd}
$$

$$\cdots \to H_{n+1}(X,A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Checking exactness at $H_n(X)$

$Ker\ j \subset Im\ i$

$$\begin{tikzcd}
0 \arrow[r] & C_{n+1}(A) \arrow[r] \arrow[d] & \overset{b_{n+1}}{C_{n+1}(X)} \arrow[r] \arrow[d] & \overset{j(b_{n+1})}{C_{n+1}(X,A)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \overset{a_n}{C_n(A)} \arrow[r] \arrow[d] & \overset{b_n \text{ homologous } b_n - \partial b_{n+1} = i(a_n)}{C_n(X)} \arrow[r] \arrow[d] & \overset{j(b_n) = \partial j(b_{n+1})}{C_n(X,A)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_{n-1}(A) \arrow[r] & \overset{0}{C_{n-1}(X)} \arrow[r] & C_{n-1}(X,A) \arrow[r] & 0
\end{tikzcd}$$

$$\cdots \to H_{n+1}(X,A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Checking exactness at $H_n(X)$

$Im\ i \subset Ker\ j$

$$\begin{tikzcd}
0 \arrow[r] & C_{n+1}(A) \arrow[r] \arrow[d] & C_{n+1}(X) \arrow[r] \arrow[d] & C_{n+1}(X, A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_n(A) \arrow[r] \arrow[d] & C_n(X) \arrow[r] \arrow[d] & C_n(X, A) \arrow[r] \arrow[d] & 0 \\
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Checking exactness at $H_n(X)$

$Im\ i \subset Ker\ j$

$$
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$$

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$$C_n(A) \oplus C_n(B) \xrightarrow[i_A + i_B]{j} C_n(X)$$

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Reduced homology

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Exercise. *Prove that $\tilde{H}_n(A) \xrightarrow{i_*} \tilde{H}_n(X)$ is an isomorphism for all n if and only if $H_n(X, A) = 0$ for all n*

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H_n(X \setminus A, V \setminus A) \arrow[r, "i_* \text{ excision}"] \arrow[d, "q_*"] & H_n(X, V) \\
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\downarrow q_* & & \\
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Exercise. *If $A \subset V \subset X$ and V deformation retracts onto A, then $j_* : H_n(X, A) \to H_n(X, V)$ is an isomorphism.*

$$\begin{tikzcd}
H_n(X, A) \arrow[r, "j_*"] \arrow[d, "q''_*"] & H_n(X, V) \arrow[d, "q'_*"] \\
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\end{tikzcd}$$

$j' \circ q'' = q' \circ j$
$(j'(q''(x)) = j'(\{x\}) = \{x\} = q'(x) = q'(j(x))$

$j'_* \circ q''_* = q'_* \circ j_* \implies q''_* = {j'}_*^{-1} \circ q_* \circ j_*$

$q : X \setminus A \to X/A \setminus A/A$ is an isomorphism

If there exists $V \subset X$, such that $A \subset V$ and V deformation retracts onto A, then

$$\begin{tikzcd} H_n(X \setminus A, V \setminus A) \arrow[r, "i_* \text{ excision}"] \arrow[d, "q_*"] & H_n(X, V) \arrow[d, "q'_*"] \\ H_n(X/A \setminus A/A, V/A \setminus A/A) \arrow[r, "i'_*(excision)"] & H_n(X/A, V/A) \end{tikzcd}$$

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Cone over A, CA

Cone over A, $CA := A \times [0,1]/(A \times \{0\})$

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$\boxed{H_n(X, A) \text{ is isomorphic to } \tilde{H}_n(X \cup CA)}$

$f : X \to Y$

Not in the syllabus

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$$X \overset{i}{\hookrightarrow} M_f \xrightarrow{r} Y$$

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Not in the syllabus

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Assume X is also a CW complex and $\pi_1(X) = 0$, then $H_n(X) \xrightarrow{f_*} H_n(Y)$ isomorphism for all n $\implies H_n(M_f, X) = 0$ for *all* n

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Assume X is also a CW complex and $\pi_1(X) = 0$, then $H_n(X) \xrightarrow{f_*} H_n(Y)$ isomorphism for all $n \implies H_n(M_f, X) = 0$ for *all* $n \implies \pi_n(M_f, X) = 0$ for *all* $n \implies \pi_n(X) = \pi_n(M_f) = \pi_n(Y)$

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Theorem. *$\tilde{H}_i(S^n)$ is $\mathbb{Z}$ if $i = n$, otherwise, trivial*

Proof. $\tilde{H}_i(D^n) = 0$ for all i. ∎

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Proof. $H_k(U, U \setminus \{x\}) \xrightarrow{i_*} H_k(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\}) \simeq H_{k-1}(\mathbb{R}^n \setminus \{x\}) \simeq H_{k-1}(S^{n-1})$

$f : U \to V$ homeomorphism induces,

$\square$

Theorem. *$\tilde{H}_i(S^n)$ is $\mathbb{Z}$ if $i = n$, otherwise, trivial*

Proof. $\tilde{H}_i(D^n) = 0$ for all i.

$\tilde{H}_i(D^k) = 0$ for all i.

By the long exact sequence,
$\tilde{H}_i(S^n) = \tilde{H}_i(D^n/S^{n-1}) = H_i(D^n, S^{n-1}) = \tilde{H}_{i-1}(S^{n-1})$

$\tilde{H}_i(S^0) = \mathbb{Z}$ if $i = 0$, otherwise, trivial

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$f : U \to V$ homeomorphism induces,
$f_* : H_n(U, U \setminus \{x\}) \to H_n(U, U \setminus \{f(x)\})$ □

Jordan curve theorem

Theorem. *$f : S^1 \to S^2$ homeomorphism, then $S^2 \setminus f(S^1)$ has two components*

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$\tilde{H}_0(S^2 \setminus f(S^1))$

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Claim: $H_i(S^2 \setminus f(I)) = 0$, $\square$

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$\tilde{H}_1(U \cup V) = \tilde{H}_0(U \cap V)$

□

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Proof of claim: $I = I_1^{1/2} \cup I_2^{1/2}$ $\square$

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Proof of claim: $I = I_1^{1/2} \cup I_2^{1/2}$,
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$U = S^2 \setminus f(I_1^{1/2})$

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$U \cup V = S^2 \setminus f(p)$ □

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$U \cup V = S^2 \setminus f(p) = \mathbb{R}^2$, $\square$

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$V = S^2 \setminus f(I_2^{1/2})$
$U \cup V = S^2 \setminus f(p) = \mathbb{R}^2$, so $(\tilde{H}_i(U \cup V) = 0)$

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$V = S^2 \setminus f(I_2^{1/2})$
$U \cup V = S^2 \setminus f(p) = \mathbb{R}^2$, so $(\tilde{H}_i(U \cup V) = 0)$
$U \cap V = S^2 \setminus f(I)$

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Proof of claim: $I = I_1^{1/2} \cup I_2^{1/2}$,
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Jordan curve theorem

Theorem. *$f : S^1 \to S^2$ homeomorphism, then $S^2 \setminus f(S^1)$ has two components, i.e. $H_0(S^2 \setminus f(S^1)) - \mathbb{Z}$.*

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Repeatedly zooming in and subdividing, $x \in C_i(S^2 \setminus f(q))$
(Exercise!)

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Therefore, $x = \partial y$.

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Therefore, $[x] = [0] \in H_n(S^2 \setminus I_k^{1/2^m})$, contradicting $[x] \neq [0]$ $\square$