

Exercise sheet 4

1. The Jordan curve theorem says that if $f : S^1 \rightarrow S^2$ is a continuous map which is a homomorphism onto its image, then $S^2 \setminus f(S^1)$ has two components (equivalently, $\tilde{H}_0(S^2 \setminus f(S^1)) = \mathbb{Z}$).
- a) Prove that the Jordan curve theorem follows from the following, if $f : [0, 1] \rightarrow S^2$ is a continuous map that is homeomorphic onto its image, then $\tilde{H}_0(S^2 \setminus f([0, 1])) = 0$
1. For pairs (X_α, x_α) where, for each α , X_α is a topological space and $x_\alpha \in X_\alpha$ is a chosen point, the wedge sum $\vee_\alpha X_\alpha := \sqcup_\alpha X_\alpha / \{x_\alpha\}$ is constructed by taking the disjoint union of the X_α and identifying all the base points x_α . If (X_α, x_α) are good pairs, and prove the following:
 - a) The inclusion, $i_\alpha : X_\alpha \rightarrow \vee_\alpha X_\alpha$ (defined by composing the inclusion to the disjoint union with the quotient map) induces $i_{\alpha*} : \tilde{H}_n(X_\alpha) \rightarrow \tilde{H}_n(\vee_\alpha X_\alpha)$.
 - b) Prove that $H_n(\vee_\alpha X_\alpha) = \oplus_\alpha i_{\alpha*}(H_n(X_\alpha))$
1. Let $X_0 \subset X_1 \subset \dots \subset X_n = X$ be a nested finite sequence of topological spaces. Assume that (X_{i+1}, X_i) form a good pair for all i and that the reduced homologies, $\tilde{H}_k(X_i/X_{i-1})$ are non-trivial only when $k = i$ (*such a situation occurs if X_k denotes the k -skeleton of a simplicial complex, or, as we shall see later in this course, of a CW-complex*).
 - a) Prove that a k -dimensional homology class in $H_k(X)$ has a representative in $H_k(X_k)$ (i.e. $H_k(X_k) \xrightarrow{i_*} H_k(X)$ is surjective).
 - b) Prove that $H_k(X) \cong H_k(X_{k+1})$ (i.e. the k th homology of X depends only on the homology of X_{k+1}).
1. For a finite subset, $A := \{x_1, x_2, \dots, x_n\} \subset \mathbb{R}^{N+1}$
 - a) Prove that $H_N(\mathbb{R}^{N+1} \setminus A) \cong \oplus_{i=1}^n \mathbb{Z}$
 - b) Find maps $f_i : S^N \rightarrow \mathbb{R}^{N+1} \setminus A$ so that f_{i*} is injective and maps a generator of $H_N(S^N) \cong \mathbb{Z}$ to a generator of the i th component of $H_N(\mathbb{R}^{N+1} \setminus A) \cong \oplus_{i=1}^n \mathbb{Z}$
 - c) Use the f_i defined above to prove that the inclusion, $\mathbb{R}^{N+1} \setminus A \hookrightarrow \mathbb{R}^{N+1} \setminus x_j$ induces a map from $\oplus_{i=1}^n \mathbb{Z} \rightarrow \mathbb{Z}$ that is a projection onto the i th copy of \mathbb{Z} .
1. Prove the following properties of the degree of a map $f : S^n \rightarrow S^n$:

- a) $\deg Id = 1$
 - b) $\deg f = 0$ if f is not-surjective
 - c) Homotopic maps have the same degrees
 - d) Compute the degree of the antipodal map
1. Realize S^1 as the subspace $\{z \mid |z| = 1\} \subset \mathbb{C}$ and prove that the map $\theta : S^1 \rightarrow S^1$ defined as $\theta(z) = z^k$ has degree k .
 2. Let A denote the complement of k disjoint open subsets of S^n that are each homeomorphic to open discs of dimension n .
 - a) Show that S^n/A is homeomorphic to a wedge sum of k n -spheres.
 - b) Prove that it is possible to glue the homeomorphisms from each sphere in the wedge to S^n to define a surjection, $p : S^n/A \rightarrow S^n$.
 - c) Define the map $f : S^n \rightarrow S^n$ as the composition, $S^n \xrightarrow{q} S^n/A \xrightarrow{p} S^n$. What is its degree (*Hint: use the local degree formulation*)?