

## Exercise sheet 6

1. Use the universal coefficient theorem to compute the cohomologies of  $S^n$ , the torus, and  $\mathbb{RP}^2$ , each with coefficients  $\mathbb{Z}$ ,  $\mathbb{Z}/2$ , and  $\mathbb{Q}$ . Then redo them using the Mayer-Vietoris and, for  $S^n$ , using the long exact sequence of pairs.

2. Prove that the short exact sequence

$$0 \rightarrow \text{Ext}(H_{n-1}(X), G) \rightarrow H^n(X; G) \rightarrow \text{Hom}(H_n(X), G) \rightarrow 0$$

is natural and splits.

3. Prove that a degree  $d$  map  $f : S^n \rightarrow S^n$  induces a map  $f^* : H^n(S^n) \rightarrow H^n(S^n)$  which is multiplication by  $d$ .
4. Prove that following:

$$\delta(\phi \smile \tau) = \delta\phi \smile \tau + (-1)^m \phi \smile \delta\tau$$

5. Show that if  $f : X \rightarrow Y$  is a continuous map then  $f^*(x \smile y) = f^*(x) \smile f^*(y)$ .
6. Show that we can define a relative version of the cup product:

$$\smile : H^m(X, A; R) \times H^n(X, B; R) \rightarrow H^{m+n}(X, A \cup B; R)$$

7. Compute the cohomology ring of

- (a) The projective plane.
- (b) The Klein bottle