

## Exercise sheet 1

1. Let  $f : X \rightarrow Y$  be a continuous map. Show that:
  - a) For any  $\sigma \in C_n(X)$ ,  $f \circ \sigma \in C_n(Y)$ . Therefore, one obtains a map  $f_{\#} : C_n(X) \rightarrow C_n(Y)$ .
  - b) Show that  $\partial'_n \circ f_{\#} = f_{\#} \circ \partial_n$ , where  $\partial'_n : C_n(Y) \rightarrow C_{n-1}(Y)$  is the boundary map on  $C_n(Y)$ .
  - c) Show that  $f_{\#}(Z_n(X)) \subset Z_n(Y)$
  - d) Show that  $f_{\#}(B_n(X)) \subset B_n(Y)$
  - e) The previous two parts allow us to define  $f_* : H_n(X) \rightarrow H_n(Y)$ . Prove that  $(f \circ g)_* = f_* \circ g_*$  and  $(id_X)_* = id_{H_n(X)}$
2. Prove that a set map  $f : X \rightarrow Y$  is injective if and only if it has a left inverse.
3. Prove that a set map  $f : X \rightarrow Y$  is surjective if and only if it has a right inverse.
4. Prove that if  $f : X \rightarrow Y$  is a homeomorphism, then  $f_*$  is an isomorphism.
5. Prove that if  $r : X \rightarrow A$  is a retract, then  $r_*$  is surjective.
6. Consider the subset  $A = S^1 \times x_0$  of  $X = S^1 \times S^1$ . Prove that  $A$  is a retract of  $X$ .
7. Show that for any  $p \in \mathbb{R}^n$ , there is a retract  $r : \mathbb{R}^n \rightarrow \{p\}$
8. Show that  $\mathbb{R}^n \setminus \{p\}$ , where  $p$  is the origin, retracts onto  $S^{n-1}$ .
9. Prove that  $\partial_n \circ \partial_{n+1} = 0$ .
10. For any topological space,  $X$ , define the map  $\epsilon : C_0(X) \rightarrow \mathbb{Z}$  by  $\epsilon(\sum_i n_i \sigma_i) = \sum_i n_i$ . Prove that:
  - a)  $\epsilon$  is a homomorphism.
  - b)  $\epsilon \circ \partial_1 = 0$ . We can, therefore, define  $\epsilon_* : H_0(X) \rightarrow \mathbb{Z}$
  - c) If  $X$  is path-connected,  $\epsilon_*$  is an isomorphism.