$$0 \to A \to 0 \implies A \cong 0$$

$$0 \to A \to B \to 0 \implies A \cong B$$

$$0 \to A \to B \to ?? \to 0 \implies ?? \cong B/i(A)$$

$$0 \to ?? \to B \xrightarrow{f} C \to 0 \implies ?? \cong Ker\ f$$

$$0 \to A \to ?? \to C \to 0$$

$0 \to A \to ?? \to C \to 0$

Example. $0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \to \mathbb{Z}/2 \to 0$

$0 \to A \to ?? \to C \to 0$

Example. $0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \to \mathbb{Z}/2 \to 0$
$\mathbb{Z} \ncong \mathbb{Z} \oplus \mathbb{Z}/2$

$0 \to A \to ?? \to C \to 0$

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Lemma. $0 \to A \xrightarrow{i} B \xrightarrow{q} C \to 0$
If there exists, $s : C \to B$, such that
$q \circ s = Id$,
Then, $B \cong i(A) \oplus s(C)$

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Proof. Consider $b \in B$ $\square$

$0 \to A \to ?? \to C \to 0$

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Proof. Consider $b \in B$
b

$\square$

$0 \to A \to ?? \to C \to 0$

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Lemma. $0 \to A \xrightarrow{i} B \xrightarrow{q} C \to 0$
If there exists, $s : C \to B$*, such that*
$q \circ s = Id$*,*
Then, $B \cong i(A) \oplus s(C) \cong A \oplus C$

Proof. Consider $b \in B$
$q(b)$

$\square$

$0 \to A \to ?? \to C \to 0$

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Proof. Consider $b \in B$
$s(q(b))$

$\square$

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Proof. Consider $b \in B$
$b - s(q(b))$

$\square$

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Proof. Consider $b \in B$
$q(b - s(q(b)))$

$\square$

$0 \to A \to ?? \to C \to 0$

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If there exists, $s : C \to B$*, such that*
$q \circ s = Id$*,*
Then, $B \cong i(A) \oplus s(C) \cong A \oplus C$

Proof. Consider $b \in B$
$q(b - s(q(b))) = 0$

$\square$

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Proof. Consider $b \in B$
$q(b - s(q(b))) = 0$

Therefore,
$b - s(q(b)) = i(a)$ for some $a \in A$

$\square$

$0 \to A \to ?? \to C \to 0$

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Proof. Consider $b \in B$
$q(b - s(q(b))) = 0$

Therefore,
$b - s(q(b)) = i(a)$ for some $a \in A$
$b = i(a) + s(q(b))$ $\square$

$0 \to A \to ?? \to C \to 0$

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Proof. Consider $b \in B$
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Therefore,
$b - s(q(b)) = i(a)$ for some $a \in A$
$b = i(a) + s(q(b)) = i(a) + s(c)$, where $c := i(b)$

$\square$

$0 \to A \to ?? \to C \to 0$

$B = i(A) + s(B)$ $\square$

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$$B = i(A) + s(B)$$

$$i(a) = s(c)$$

$\square$

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$B = i(A) + s(B)$

$$\begin{aligned} & i(a) = s(c) \\ \implies & q(i(a)) = q(s(c)) \end{aligned}$$

$\square$

$0 \to A \to ?? \to C \to 0$

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$B = i(A) + s(B)$

$$\begin{aligned} i(a) &= s(c) \\ \implies \quad q(i(a)) &= q(s(c)) \\ \implies \quad 0 &= c \end{aligned}$$

$\square$

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$\square$

$0 \to A \to ?? \to C \to 0$

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$B = i(A) + s(B)$

$$\begin{aligned} i(a) &= s(c) \\ \implies q(i(a)) &= q(s(c)) \\ \implies 0 &= c \\ \implies 0 &= s(c) = i(a) \end{aligned}$$

$\square$

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$B = i(A) + s(B)$

$$\begin{aligned} i(a) &= s(c) \\ \implies \quad q(i(a)) &= q(s(c)) \\ \implies \quad 0 &= c \\ \implies \quad 0 &= s(c) = i(a) \end{aligned}$$

Therefore,
$i(A) \cap s(C) = 0$

$\square$

$0 \to A \to ?? \to C \to 0$

Example. $0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \to \mathbb{Z}/2 \to 0$
$\mathbb{Z} \ncong \mathbb{Z} \oplus \mathbb{Z}/2$

Lemma. $0 \to A \xrightarrow{i} B \xrightarrow{q} C \to 0$
If there exists, $s : C \to B$, such that
$q \circ s = Id$ (the exact sequence splits),
Then, $B \cong i(A) \oplus s(C) \cong A \oplus C$

Proof. Consider $b \in B$
$q(b - s(q(b))) = 0$

Therefore,
$b - s(q(b)) = i(a)$ for some $a \in A$
$b = i(a) + s(q(b)) = i(a) + s(c)$, where $c := i(b)$

$B = i(A) + s(B)$

$$\begin{aligned} i(a) &= s(c) \\ \implies q(i(a)) &= q(s(c)) \\ \implies 0 &= c \\ \implies 0 &= s(c) = i(a) \end{aligned}$$

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$\square$

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$$\begin{aligned} & i(a) = s(c) \\ \implies & q(i(a)) = q(s(c)) \\ \implies & 0 = c \\ \implies & 0 = s(c) = i(a) \end{aligned}$$

Therefore,
$i(A) \cap s(C) = 0$

$\square$

Exercise. *If $r : X \to A$ is a retract,*
$H_i(X) \cong H_i(A) \oplus H_i(X, A)$

$X = S^1 \times S^1$

$X = S^1 \times S^1$
$A = S^1 \times I_1$

$X = S^1 \times S^1$
$A = S^1 \times I_1$
$B = S^1 \times I_2$

$X = S^1 \times S^1$
$A = S^1 \times I_1$
$B = S^1 \times I_2$
$A \cup B = X$

$$X = S^1 \times S^1$$
$$A = S^1 \times I_1$$
$$B = S^1 \times I_2$$
$$A \cup B = X$$
$$A \cap B = S^1 \times p \sqcup S^1 \times q$$

$$X = S^1 \times S^1$$
$$A = S^1 \times I_1$$
$$B = S^1 \times I_2$$
$$A \cup B = X$$
$$A \cap B = S^1 \times p \sqcup S^1 \times q$$

$$U := S^1 \times (S^1 \setminus x_0)$$

$X = S^1 \times S^1$
$A = S^1 \times I_1$
$B = S^1 \times I_2$
$A \cup B = X$
$A \cap B = S^1 \times p \sqcup S^1 \times q$

$U := S^1 \times (S^1 \setminus x_0)$ deformation retracts onto A

$X = S^1 \times S^1$
$A = S^1 \times I_1$
$B = S^1 \times I_2$
$A \cup B = X$
$A \cap B = S^1 \times p \sqcup S^1 \times q$

$U := S^1 \times (S^1 \setminus x_0)$ deformation retracts onto A
$V := S^1 \times (S^1 \setminus x_1)$

$X = S^1 \times S^1$
$A = S^1 \times I_1$
$B = S^1 \times I_2$
$A \cup B = X$
$A \cap B = S^1 \times p \sqcup S^1 \times q$

$U := S^1 \times (S^1 \setminus x_0)$ deformation retracts onto A
$V := S^1 \times (S^1 \setminus x_1)$ deformation retracts onto B

$X = S^1 \times S^1$
$A = S^1 \times I_1$
$B = S^1 \times I_2$
$A \cup B = X$
$A \cap B = S^1 \times p \sqcup S^1 \times q$

$U := S^1 \times (S^1 \setminus x_0)$ deformation retracts onto A
$V := S^1 \times (S^1 \setminus x_1)$ deformation retracts onto B
$V \cap V = S^1 \times (S^1 \setminus \{x_0, x_1\})$

$X = S^1 \times S^1$
$A = S^1 \times I_1$
$B = S^1 \times I_2$
$A \cup B = X$
$A \cap B = S^1 \times p \sqcup S^1 \times q$

$U := S^1 \times (S^1 \setminus x_0)$ deformation retracts onto A
$V := S^1 \times (S^1 \setminus x_1)$ deformation retracts onto B
$V \cap V = S^1 \times (S^1 \setminus \{x_0, x_1\})$ deformation retracts onto
$A \cap B$

$X = S^1 \times S^1$
$A = S^1 \times I_1$
$B = S^1 \times I_2$
$A \cup B = X$
$A \cap B = S^1 \times p \sqcup S^1 \times q$

$U := S^1 \times (S^1 \setminus x_0)$ deformation retracts onto A
$V := S^1 \times (S^1 \setminus x_1)$ deformation retracts onto B
$V \cap V = S^1 \times (S^1 \setminus \{x_0, x_1\})$ deformation retracts onto $A \cap B$

$i_p : S^1 \hookrightarrow S^1 \times I_1$, where $i_p(s) = (s, p)$

$X = S^1 \times S^1$
$A = S^1 \times I_1$
$B = S^1 \times I_2$
$A \cup B = X$
$A \cap B = S^1 \times p \sqcup S^1 \times q$

$U := S^1 \times (S^1 \setminus x_0)$ deformation retracts onto A
$V := S^1 \times (S^1 \setminus x_1)$ deformation retracts onto B
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$A \cap B$

$i_p : S^1 \hookrightarrow S^1 \times I_1$, where $i_p(s) = (s, p)$
$i_q : S^1 \hookrightarrow S^1 \times I_1$, where $i_q(s) = (s, q)$

$X = S^1 \times S^1$
$A = S^1 \times I_1$
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$A \cup B = X$
$A \cap B = S^1 \times p \sqcup S^1 \times q$

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$V \cap V = S^1 \times (S^1 \setminus \{x_0, x_1\})$ deformation retracts onto $A \cap B$

$i_p : S^1 \hookrightarrow S^1 \times I_1$, where $i_p(s) = (s, p)$
$i_q : S^1 \hookrightarrow S^1 \times I_1$, where $i_q(s) = (s, q)$
$i_{p_*}(1) = i_{q_*}(1)$

$X = S^1 \times S^1$
$A = S^1 \times I_1$
$B = S^1 \times I_2$
$A \cup B = X$
$A \cap B = S^1 \times p \sqcup S^1 \times q$

$U := S^1 \times (S^1 \setminus x_0)$ deformation retracts onto A
$V := S^1 \times (S^1 \setminus x_1)$ deformation retracts onto B
$V \cap V = S^1 \times (S^1 \setminus \{x_0, x_1\})$ deformation retracts onto $A \cap B$

$i_p : S^1 \hookrightarrow S^1 \times I_1$, where $i_p(s) = (s, p)$
$i_q : S^1 \hookrightarrow S^1 \times I_1$, where $i_q(s) = (s, q)$
$i_{p_*}(1) = i_{q_*}(1)$ because i_p is homotopic to i_q

$X = S^1 \times S^1$
$A = S^1 \times I_1$
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$A \cap B = S^1 \times p \sqcup S^1 \times q$

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$i_q : S^1 \hookrightarrow S^1 \times I_1$, where $i_q(s) = (s, q)$
$i_{p_*}(1) = i_{q_*}(1)$ because i_p is homotopic to i_q
$y_A := i_{p_*}(1)$

$X = S^1 \times S^1$
$A = S^1 \times I_1$
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$A \cup B = X$
$A \cap B = S^1 \times p \sqcup S^1 \times q$

$U := S^1 \times (S^1 \setminus x_0)$ deformation retracts onto A
$V := S^1 \times (S^1 \setminus x_1)$ deformation retracts onto B
$V \cap V = S^1 \times (S^1 \setminus \{x_0, x_1\})$ deformation retracts onto
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$i_p : S^1 \hookrightarrow S^1 \times I_1$, where $i_p(s) = (s, p)$
$i_q : S^1 \hookrightarrow S^1 \times I_1$, where $i_q(s) = (s, q)$
$i_{p_*}(1) = i_{q_*}(1)$ because i_p is homotopic to i_q
$y_A := i_{p_*}(1)(= i_{q_*}(1))$ generates $H_1(A) = \mathbb{Z}$

$X = S^1 \times S^1$
$A = S^1 \times I_1$
$B = S^1 \times I_2$
$A \cup B = X$
$A \cap B = S^1 \times p \sqcup S^1 \times q$

$U := S^1 \times (S^1 \setminus x_0)$ deformation retracts onto A
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$i_p : S^1 \hookrightarrow S^1 \times I_1$, where $i_p(s) = (s, p)$
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$i_{p_*}(1) = i_{q_*}(1)$ because i_p is homotopic to i_q
$y_A := i_{p_*}(1)(= i_{q_*}(1))$ generates $H_1(A) = \mathbb{Z}$

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$x_p = j_{p_*}(1)$ and $x_q = j_{q_*}(1)$ generate $H_1(A \cap B) = \mathbb{Z} \oplus \mathbb{Z}$

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Similarly,

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$k_{A*}(x_p) = y_A = k_{A*}(x_q)$
Similarly,
$k_{B*}(x_p) = y_B = k_{B*}(x_q)$, where y_B generates $H_1(B) = \mathbb{Z}$

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$x_p = j_{p_*}(1)$ and $x_q = j_{q_*}(1)$ generate $H_1(A \cap B) = \mathbb{Z} \oplus \mathbb{Z}$

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$H_1(A \cap B) \underset{i_*}{\longrightarrow} H_1(A) \oplus H_1(B)$

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$x_p = j_{p_*}(1)$ and $x_q = j_{q_*}(1)$ generate $H_1(A \cap B) = \mathbb{Z} \oplus \mathbb{Z}$

$A \cap B = S^1 \times p \sqcup S^1 \times q \xhookrightarrow{k_A} S^1 \times I_p = A$
$k_A \circ j_p = i_p$ and $k_A \circ j_q = i_q$
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Similarly,
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$$\underbrace{H_1(A \cap B)}_{\mathbb{Z}\oplus\mathbb{Z} \cong <x_p, x_q>} \underset{i_*}{\longrightarrow} H_1(A) \oplus H_1(B)$$

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$$\underbrace{H_1(A \cap B)}_{\mathbb{Z}\oplus\mathbb{Z}\cong<x_p,x_q>} \underset{i_*}{\longrightarrow} \underbrace{H_1(A) \oplus H_1(B)}_{\mathbb{Z}\oplus\mathbb{Z}\cong<y_A,y_B>}$$

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$$\underbrace{H_1(A \cap B)}_{\mathbb{Z}\oplus\mathbb{Z} \cong <x_p, x_q>} \xrightarrow[i_*]{x_p \to (y_A, y_B),} \underbrace{H_1(A) \oplus H_1(B)}_{\mathbb{Z}\oplus\mathbb{Z} \cong <y_A, y_B>}$$

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$A \cup B = X$
$A \cap B = S^1 \times p \sqcup S^1 \times q$

$U := S^1 \times (S^1 \setminus x_0)$ deformation retracts onto A
$V := S^1 \times (S^1 \setminus x_1)$ deformation retracts onto B
$V \cap V = S^1 \times (S^1 \setminus \{x_0, x_1\})$ deformation retracts onto $A \cap B$

$i_p : S^1 \hookrightarrow S^1 \times I_1$, where $i_p(s) = (s, p)$
$i_q : S^1 \hookrightarrow S^1 \times I_1$, where $i_q(s) = (s, q)$
$i_{p_*}(1) = i_{q_*}(1)$ because i_p is homotopic to i_q
$y_A := i_{p_*}(1)(= i_{q_*}(1))$ generates $H_1(A) = \mathbb{Z}$

$j_p : S^1 \hookrightarrow S^1 \times p \sqcup S^1 \times q$, where $j_p(s) = (s, p)$
$j_q : S^1 \hookrightarrow S^1 \times p \sqcup S^1 \times q$, where $j_q(s) = (s, q)$
$x_p = j_{p_*}(1)$ and $x_q = j_{q_*}(1)$ generate $H_1(A \cap B) = \mathbb{Z} \oplus \mathbb{Z}$

$A \cap B = S^1 \times p \sqcup S^1 \times q \overset{k_A}{\hookrightarrow} S^1 \times I_p = A$
$k_A \circ j_p = i_p$ and $k_A \circ j_q = i_q$
$k_{A*} \circ j_{p_*} = i_{p_*}$ and $k_{A*} \circ j_{q_*} = i_{q_*}$
$k_{A*}(j_{p_*}(1)) = i_{p_*}(1) = i_{q_*}(1) = k_{A*}(j_{q_*}(1))$

$k_{A*}(x_p) = y_A = k_{A*}(x_q)$
Similarly,
$k_{B*}(x_p) = y_B = k_{B*}(x_q)$, where y_B generates $H_1(B) = \mathbb{Z}$

$$\underbrace{H_1(A \cap B)}_{\mathbb{Z}\oplus\mathbb{Z} \cong <x_p, x_q>} \xrightarrow[i_*]{x_p \to (y_A, y_B), x_q \to (y_A, y_B)} \underbrace{H_1(A) \oplus H_1(B)}_{\mathbb{Z}\oplus\mathbb{Z} \cong <y_A, y_B>}$$

$X = S^1 \times S^1$
$A = S^1 \times I_1$
$B = S^1 \times I_2$
$A \cup B = X$
$A \cap B = S^1 \times p \sqcup S^1 \times q$

$U := S^1 \times (S^1 \setminus x_0)$ deformation retracts onto A
$V := S^1 \times (S^1 \setminus x_1)$ deformation retracts onto B
$V \cap V = S^1 \times (S^1 \setminus \{x_0, x_1\})$ deformation retracts onto $A \cap B$

$i_p : S^1 \hookrightarrow S^1 \times I_1$, where $i_p(s) = (s, p)$
$i_q : S^1 \hookrightarrow S^1 \times I_1$, where $i_q(s) = (s, q)$
$i_{p*}(1) = i_{q*}(1)$ because i_p is homotopic to i_q
$y_A := i_{p*}(1)(= i_{q*}(1))$ generates $H_1(A) = \mathbb{Z}$

$j_p : S^1 \hookrightarrow S^1 \times p \sqcup S^1 \times q$, where $j_p(s) = (s, p)$
$j_q : S^1 \hookrightarrow S^1 \times p \sqcup S^1 \times q$, where $j_q(s) = (s, q)$
$x_p = j_{p*}(1)$ and $x_q = j_{q*}(1)$ generate $H_1(A \cap B) = \mathbb{Z} \oplus \mathbb{Z}$

$A \cap B = S^1 \times p \sqcup S^1 \times q \xhookrightarrow{k_A} S^1 \times I_p = A$
$k_A \circ j_p = i_p$ and $k_A \circ j_q = i_q$
$k_{A*} \circ j_{p*} = i_{p*}$ and $k_{A*} \circ j_{q*} = i_{q*}$
$k_{A*}(j_{p*}(1)) = i_{p*}(1) = i_{q*}(1) = k_{A*}(j_{q*}(1))$

$k_{A*}(x_p) = y_A = k_{A*}(x_q)$
Similarly,
$k_{B*}(x_p) = y_B = k_{B*}(x_q)$, where y_B generates $H_1(B) = \mathbb{Z}$

$$\underbrace{H_1(A \cap B)}_{\mathbb{Z}\oplus\mathbb{Z}\cong<x_p,x_q>} \xrightarrow[i_*]{x_p \to (y_A, y_B),\, x_q \to (y_A, y_B)} \underbrace{H_1(A) \oplus H_1(B)}_{\mathbb{Z}\oplus\mathbb{Z}\cong<y_A,y_B>}$$

$Im\ i_* = \mathbb{Z}$

$X = S^1 \times S^1$
$A = S^1 \times I_1$
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$A \cap B = S^1 \times p \sqcup S^1 \times q$

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$V \cap V = S^1 \times (S^1 \setminus \{x_0, x_1\})$ deformation retracts onto $A \cap B$

$i_p : S^1 \hookrightarrow S^1 \times I_1$, where $i_p(s) = (s, p)$
$i_q : S^1 \hookrightarrow S^1 \times I_1$, where $i_q(s) = (s, q)$
$i_{p_*}(1) = i_{q_*}(1)$ because i_p is homotopic to i_q
$y_A := i_{p_*}(1)(= i_{q_*}(1))$ generates $H_1(A) = \mathbb{Z}$

$j_p : S^1 \hookrightarrow S^1 \times p \sqcup S^1 \times q$, where $j_p(s) = (s, p)$
$j_q : S^1 \hookrightarrow S^1 \times p \sqcup S^1 \times q$, where $j_q(s) = (s, q)$
$x_p = j_{p_*}(1)$ and $x_q = j_{q_*}(1)$ generate $H_1(A \cap B) = \mathbb{Z} \oplus \mathbb{Z}$

$A \cap B = S^1 \times p \sqcup S^1 \times q \xhookrightarrow{k_A} S^1 \times I_p = A$
$k_A \circ j_p = i_p$ and $k_A \circ j_q = i_q$
$k_{A*} \circ j_{p_*} = i_{p_*}$ and $k_{A*} \circ j_{q_*} = i_{q_*}$
$k_{A*}(j_{p_*}(1)) = i_{p_*}(1) = i_{q_*}(1) = k_{A*}(j_{q_*}(1))$

$k_{A*}(x_p) = y_A = k_{A*}(x_q)$
Similarly,
$k_{B*}(x_p) = y_B = k_{B*}(x_q)$, where y_B generates $H_1(B) = \mathbb{Z}$

$$\underbrace{H_1(A \cap B)}_{\mathbb{Z} \oplus \mathbb{Z} \cong <x_p, x_q>} \xrightarrow[i_*]{x_p \to (y_A, y_B),\, x_q \to (y_A, y_B)} \underbrace{H_1(A) \oplus H_1(B)}_{\mathbb{Z} \oplus \mathbb{Z} \cong <y_A, y_B>}$$

$Im\ i_* = \mathbb{Z}$
$Ker\ i_* = \mathbb{Z}$

$X = S^1 \times S^1$
$A = S^1 \times I_1$
$B = S^1 \times I_2$
$A \cup B = X$
$A \cap B = S^1 \times p \sqcup S^1 \times q$

$$\widetilde{H}_1(A \cap B) \longrightarrow \widetilde{H}_1(A) \oplus \widetilde{H}_1(B) \longrightarrow \widetilde{H}_1(X) \longrightarrow \widetilde{H}_0(A \cap B) \longrightarrow \widetilde{H}_0(A) \oplus \widetilde{H}_0(B) \qquad \cdots$$

$X = S^1 \times S^1$
$A = S^1 \times I_1$
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$$\widetilde{H}_1(A \cap B) \longrightarrow \widetilde{H}_1(A) \oplus \widetilde{H}_1(B) \longrightarrow \widetilde{H}_1(X) \longrightarrow \widetilde{H}_0(A \cap B) \longrightarrow \widetilde{H}_0(A) \oplus \widetilde{H}_0(B) \qquad \cdots$$

$$0$$

$X = S^1 \times S^1$
$A = S^1 \times I_1$
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$$\mathbb{Z} \longrightarrow 0$$

$X = S^1 \times S^1$
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$$\widetilde{H}_1(A \cap B) \longrightarrow \widetilde{H}_1(A) \oplus \widetilde{H}_1(B) \longrightarrow \widetilde{H}_1(X) \longrightarrow \widetilde{H}_0(A \cap B) \longrightarrow \widetilde{H}_0(A) \oplus \widetilde{H}_0(B) \qquad \cdots$$

$$?? \longrightarrow \mathbb{Z} \longrightarrow 0$$

$X = S^1 \times S^1$
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$$\mathbb{Z} \oplus \mathbb{Z} \xrightarrow[j_*]{} ?? \longrightarrow \mathbb{Z} \longrightarrow 0$$

$X = S^1 \times S^1$
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$$\widetilde{H}_1(A \cap B) \longrightarrow \widetilde{H}_1(A) \oplus \widetilde{H}_1(B) \longrightarrow \widetilde{H}_1(X) \longrightarrow \widetilde{H}_0(A \cap B) \longrightarrow \widetilde{H}_0(A) \oplus \widetilde{H}_0(B) \qquad \cdots$$

$$\mathbb{Z} \oplus \mathbb{Z} \xrightarrow[i_*]{} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow[j_*]{} ?? \longrightarrow \mathbb{Z} \longrightarrow 0$$

$X = S^1 \times S^1$
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$A \cup B = X$
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$$\widetilde{H}_1(A \cap B) \longrightarrow \widetilde{H}_1(A) \oplus \widetilde{H}_1(B) \longrightarrow \widetilde{H}_1(X) \longrightarrow \widetilde{H}_0(A \cap B) \longrightarrow \widetilde{H}_0(A) \oplus \widetilde{H}_0(B) \qquad \cdots$$

$$\mathbb{Z} \oplus \mathbb{Z} \xrightarrow[i_*]{x_p, x_q \to (y_A, y_B)} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow[j_*]{} ?? \longrightarrow \mathbb{Z} \longrightarrow 0$$

$X = S^1 \times S^1$
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$Im\ i_* = \mathbb{Z}$

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$$\mathbb{Z} \oplus \mathbb{Z} \xrightarrow[i_*]{x_p, x_q \to (y_A, y_B)} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow[j_*]{\qquad} ?? \longrightarrow \mathbb{Z} \longrightarrow 0$$

$Im\ i_* = \mathbb{Z} = ker\ j_*$

$X = S^1 \times S^1$
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$$\widetilde{H}_1(A \cap B) \longrightarrow \widetilde{H}_1(A) \oplus \widetilde{H}_1(B) \longrightarrow \widetilde{H}_1(X) \longrightarrow \widetilde{H}_0(A \cap B) \longrightarrow \widetilde{H}_0(A) \oplus \widetilde{H}_0(B) \qquad \cdots$$

$$\mathbb{Z} \oplus \mathbb{Z} \xrightarrow[i_*]{x_p, x_q \to (y_A, y_B)} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow[j_*]{\qquad\qquad} ?? \xrightarrow{\qquad\qquad} \mathbb{Z} \xrightarrow{\qquad\qquad} 0$$

$Im\ i_* = \mathbb{Z} = ker\ j_*$
$0 \to \mathbb{Z} \to H_1(X) \to \mathbb{Z} \to 0$

$X = S^1 \times S^1$
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$$\widetilde{H}_1(A \cap B) \longrightarrow \widetilde{H}_1(A) \oplus \widetilde{H}_1(B) \longrightarrow \widetilde{H}_1(X) \longrightarrow \widetilde{H}_0(A \cap B) \longrightarrow \widetilde{H}_0(A) \oplus \widetilde{H}_0(B) \qquad \cdots$$

$$\mathbb{Z} \oplus \mathbb{Z} \xrightarrow[i_*]{x_p, x_q \to (y_A, y_B)} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow[j_*]{\qquad} ?? \xrightarrow{\qquad} \mathbb{Z} \xrightarrow{\qquad} 0$$

$Im\ i_* = \mathbb{Z} = ker\ j_*$
$0 \to \mathbb{Z} \to H_1(X) \to \mathbb{Z} \to 0$ splits.

$X = S^1 \times S^1$
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$$\widetilde{H}_1(A \cap B) \longrightarrow \widetilde{H}_1(A) \oplus \widetilde{H}_1(B) \longrightarrow \widetilde{H}_1(X) \longrightarrow \widetilde{H}_0(A \cap B) \longrightarrow \widetilde{H}_0(A) \oplus \widetilde{H}_0(B) \qquad \cdots$$

$$\mathbb{Z} \oplus \mathbb{Z} \xrightarrow[i_*]{x_p, x_q \to (y_A, y_B)} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow[j_*]{\qquad\qquad} ?? \xrightarrow{\qquad\qquad} \mathbb{Z} \xrightarrow{\qquad\qquad} 0$$

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Therefore,
$H_1(X) = \mathbb{Z} \oplus \mathbb{Z}$

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$$\mathbb{Z} \oplus \mathbb{Z} \xrightarrow[i_*]{x_p, x_q \to (y_A, y_B)} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow[j_*]{\qquad} ?? \longrightarrow \mathbb{Z} \longrightarrow 0$$

$Im\ i_* = \mathbb{Z} = ker\ j_*$
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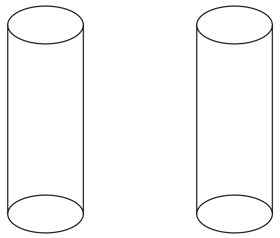

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$$\widetilde{H}_1(A \cap B) \longrightarrow \widetilde{H}_1(A) \oplus \widetilde{H}_1(B) \longrightarrow \widetilde{H}_1(X) \longrightarrow \widetilde{H}_0(A \cap B) \longrightarrow \widetilde{H}_0(A) \oplus \widetilde{H}_0(B) \qquad \cdots$$

$$\mathbb{Z} \oplus \mathbb{Z} \xrightarrow[i_*]{x_p, x_q \to (y_A, y_B)} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow[j_*]{\qquad\qquad} ?? \xrightarrow{\qquad\qquad} \mathbb{Z} \xrightarrow{\qquad\qquad} 0$$

$Im\ i_* = \mathbb{Z} = ker\ j_*$
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Therefore,
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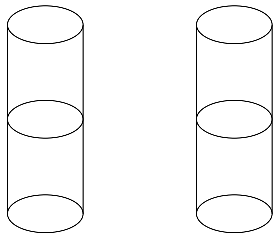

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$$\widetilde{H}_1(A\cap B) \longrightarrow \widetilde{H}_1(A) \oplus \widetilde{H}_1(B) \longrightarrow \widetilde{H}_1(X) \longrightarrow \widetilde{H}_0(A\cap B) \longrightarrow \widetilde{H}_0(A) \oplus \widetilde{H}_0(B) \qquad \cdots$$

$$\mathbb{Z}\oplus\mathbb{Z} \xrightarrow[i_*]{x_p,x_q \to (y_A,y_B)} \mathbb{Z}\oplus\mathbb{Z} \xrightarrow[j_*]{} ?? \longrightarrow \mathbb{Z} \longrightarrow 0$$

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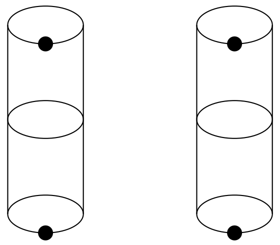

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$$\widetilde{H}_1(A \cap B) \longrightarrow \widetilde{H}_1(A) \oplus \widetilde{H}_1(B) \longrightarrow \widetilde{H}_1(X) \longrightarrow \widetilde{H}_0(A \cap B) \longrightarrow \widetilde{H}_0(A) \oplus \widetilde{H}_0(B) \qquad \cdots$$

$$\mathbb{Z} \oplus \mathbb{Z} \xrightarrow[i_*]{x_p, x_q \to (y_A, y_B)} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow[j_*]{} ?? \longrightarrow \mathbb{Z} \longrightarrow 0$$

$Im\ i_* = \mathbb{Z} = ker\ j_*$
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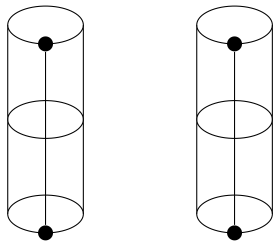

$$\widetilde{H}_2(A \cap B) \longrightarrow \widetilde{H}_2(A) \oplus \widetilde{H}_2(B) \longrightarrow \widetilde{H}_2(X) \longrightarrow \widetilde{H}_1(A \cap B) \longrightarrow \widetilde{H}_1(A) \oplus \widetilde{H}_1(B) \qquad \cdots$$

$$\widetilde{H}_2(A\cap B) \longrightarrow \widetilde{H}_2(A)\oplus\widetilde{H}_2(B) \longrightarrow \widetilde{H}_2(X) \longrightarrow \widetilde{H}_1(A\cap B) \longrightarrow \widetilde{H}_1(A)\oplus\widetilde{H}_1(B) \qquad \cdots$$

$$\mathbb{Z}\oplus\mathbb{Z}$$

$$
\begin{array}{ccccccccc}
\widetilde{H}_2(A\cap B) & \longrightarrow & \widetilde{H}_2(A)\oplus\widetilde{H}_2(B) & \longrightarrow & \widetilde{H}_2(X) & \longrightarrow & \widetilde{H}_1(A\cap B) & \longrightarrow & \widetilde{H}_1(A)\oplus\widetilde{H}_1(B) \qquad \cdots \\
 & & & & & & & & \\
 & & & & & & \mathbb{Z}\oplus\mathbb{Z} & & \mathbb{Z}\oplus\mathbb{Z}
\end{array}
$$

$$\widetilde{H}_2(A\cap B) \longrightarrow \widetilde{H}_2(A)\oplus\widetilde{H}_2(B) \longrightarrow \widetilde{H}_2(X) \longrightarrow \widetilde{H}_1(A\cap B) \longrightarrow \widetilde{H}_1(A)\oplus\widetilde{H}_1(B) \qquad \cdots$$

$$?? \qquad \mathbb{Z}\oplus\mathbb{Z} \qquad \mathbb{Z}\oplus\mathbb{Z}$$

$$\widetilde{H}_2(A\cap B) \longrightarrow \widetilde{H}_2(A)\oplus\widetilde{H}_2(B) \longrightarrow \widetilde{H}_2(X) \longrightarrow \widetilde{H}_1(A\cap B) \longrightarrow \widetilde{H}_1(A)\oplus\widetilde{H}_1(B) \qquad \cdots$$

$$0 \qquad\qquad ?? \qquad\qquad \mathbb{Z}\oplus\mathbb{Z} \qquad\qquad \mathbb{Z}\oplus\mathbb{Z}$$

$$\widetilde{H}_2(A \cap B) \longrightarrow \widetilde{H}_2(A) \oplus \widetilde{H}_2(B) \longrightarrow \widetilde{H}_2(X) \longrightarrow \widetilde{H}_1(A \cap B) \longrightarrow \widetilde{H}_1(A) \oplus \widetilde{H}_1(B) \qquad \cdots$$

$$0 \qquad\qquad 0 \qquad\qquad ?? \qquad\qquad \mathbb{Z} \oplus \mathbb{Z} \qquad\qquad \mathbb{Z} \oplus \mathbb{Z}$$

$$\widetilde{H}_2(A\cap B) \longrightarrow \widetilde{H}_2(A)\oplus\widetilde{H}_2(B) \longrightarrow \widetilde{H}_2(X) \longrightarrow \widetilde{H}_1(A\cap B) \longrightarrow \widetilde{H}_1(A)\oplus\widetilde{H}_1(B) \qquad \cdots$$

$$0 \qquad\qquad 0 \longrightarrow \,??\, \xrightarrow[\partial]{} \mathbb{Z}\oplus\mathbb{Z} \xrightarrow[i_*]{} \mathbb{Z}\oplus\mathbb{Z}$$

$$\widetilde{H}_2(A\cap B) \longrightarrow \widetilde{H}_2(A)\oplus\widetilde{H}_2(B) \longrightarrow \widetilde{H}_2(X) \longrightarrow \widetilde{H}_1(A\cap B) \longrightarrow \widetilde{H}_1(A)\oplus\widetilde{H}_1(B) \qquad \cdots$$

$$0 \qquad\qquad 0 \longrightarrow\ ?? \xrightarrow[\partial]{} \mathbb{Z}\oplus\mathbb{Z} \xrightarrow[i_*]{x_p,x_q\to(y_A,y_B)} \mathbb{Z}\oplus\mathbb{Z}$$

$$\widetilde{H}_2(A\cap B) \longrightarrow \widetilde{H}_2(A)\oplus\widetilde{H}_2(B) \longrightarrow \widetilde{H}_2(X) \longrightarrow \widetilde{H}_1(A\cap B) \longrightarrow \widetilde{H}_1(A)\oplus\widetilde{H}_1(B) \qquad \cdots$$

$$0 \qquad\qquad 0 \longrightarrow ?? \xrightarrow[\partial]{} \mathbb{Z}\oplus\mathbb{Z} \xrightarrow[i_*]{x_p, x_q \to (y_A, y_B)} \mathbb{Z}\oplus\mathbb{Z}$$

$Im\ \partial$

$$\begin{array}{ccccccccc}
\widetilde{H}_2(A\cap B) & \longrightarrow & \widetilde{H}_2(A)\oplus\widetilde{H}_2(B) & \longrightarrow & \widetilde{H}_2(X) & \longrightarrow & \widetilde{H}_1(A\cap B) & \longrightarrow & \widetilde{H}_1(A)\oplus\widetilde{H}_1(B) \quad \cdots \\
0 & & 0 & \xrightarrow{\hspace{3em}} & ?? & \xrightarrow[\partial]{} & \mathbb{Z}\oplus\mathbb{Z} & \xrightarrow[i_*]{x_p,x_q\to(y_A,y_B)} & \mathbb{Z}\oplus\mathbb{Z}
\end{array}$$

$$Im\ \partial = Ker\ i_*$$

$$\widetilde{H}_2(A\cap B) \longrightarrow \widetilde{H}_2(A)\oplus\widetilde{H}_2(B) \longrightarrow \widetilde{H}_2(X) \longrightarrow \widetilde{H}_1(A\cap B) \longrightarrow \widetilde{H}_1(A)\oplus\widetilde{H}_1(B) \qquad \cdots$$

$$0 \qquad\qquad 0 \longrightarrow ?? \xrightarrow[\partial]{} \mathbb{Z}\oplus\mathbb{Z} \xrightarrow[i_*]{x_p,x_q\to(y_A,y_B)} \mathbb{Z}\oplus\mathbb{Z}$$

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But $[c_1 - c_0] \neq 0 \in \widetilde{H}_0(\{0,1\})$

$0 \to \underbrace{H_1([0,1], \{0,1\})}_{<Id_{[0,1]}>} \xrightarrow{\partial} \underbrace{\widetilde{H}_0(\{0,1\}}_{<[c_1-c_0]=\partial Id_{[0,1]}>}) \to 0$

$[Id_{\Delta_2}] \neq 0 \in H_2(\Delta_2, \partial\Delta_2)$?
$H_2(\Delta_2, \partial\Delta_2) \xrightarrow{\partial} H_1(\partial\Delta_2, \wedge) \xrightarrow{\sim} \widetilde{H}(\partial\Delta_1/\partial\Delta_1) \xleftarrow{\sim} H_1(\Delta_1, \partial\Delta_1)$

Representative of a homology class

$\sigma : \Delta_1 \to S^1$
$\sigma(s) = (\cos s, \sin s)$ is a cycle
$[\sigma] = [0] \in H_1(S^1)$?

$q : ([0,1], \{0,1\}) \to (S^1, p)$
$\sigma = q_\#(Id_{[0,1]})$
$q_* : H_1([0,1], \{0,1\}) \xrightarrow{\sim} H_1(S^1, p) \cong H_1(S^1)$

$Id_{[0,1]} = 0$?
$Id_{[0,1]} = \partial \underbrace{x}_{\in[0,1]} + \underbrace{a}_{\in\{0,1\}}$
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$\wedge \subset \partial\Delta_2 \subset \Delta_2$

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$f : S^n \to S^n$

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Lemma. $H_n(S^n, S^n \setminus \{x_1, x_2, \ldots, x_k\}) \xrightarrow{p_*} H_n(S^n, S^n \setminus x_i)$ *is a projection onto the ith factor.*

Proof. The composition:
$H_n(U_i, U_i \setminus x_i)$ $\square$

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Is equal to:

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Is equal to:

$$H_n(U_i, U_i \setminus x_i) \to \underbrace{H_n(U_i, U_i \setminus x_i)}_{0} \to H_n(S^n, S^n \setminus x_j)$$

$$\begin{tikzcd} H_n(S^n, S^n \setminus \{x_1, \ldots, x_k\}) \arrow[r, "i_*"] & H_n(S^n, S^n \setminus x_i) \\ H_n(U_i, U_i \setminus x_i) \arrow[u, "\alpha_*"] \arrow[ur, "\beta_*"'] & \end{tikzcd}$$

$\square$

Theorem. $f : S^n \to S^n$
$f^{-1}(y) = \{x_1, x_2, \ldots, x_n\}$ for some y
$\deg\ f = \Sigma_i \deg\ f|_{x_i}$

$$\begin{tikzcd}
H_n(S^n, S^n \setminus f^{-1}(y)) \arrow[r, "f'_*"] & H_n(S^n, S^n \setminus y) \\
H_n(S^n) \arrow[u, "j'_*"] \arrow[r, "f_*"] & H_n(S^n) \arrow[u, "j_*"]
\end{tikzcd}$$

$$H_n(S^n, S^n \setminus f^{-1}(y)) \overset{\sim}{\leftarrow} H_n(\sqcup U_i, \sqcup(U_i \setminus x_i)) \cong \oplus\mathbb{Z}$$

$$H_n(S^n, S^n \setminus y) \overset{\sim}{\leftarrow} H_n(V, V \setminus y) \cong \mathbb{Z}$$

Lemma. $\deg\ f = f'_*(j'_*(1))$

Lemma. *$H_n(S^n, S^n \setminus \{x_1, x_2, \ldots, x_k\}) \xrightarrow{p_*} H_n(S^n, S^n \setminus x_i)$ is a projection onto the ith factor.*

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$$H_n(S^n, S^n \setminus f^{-1}(y)) \xleftarrow{\sim} H_n(\sqcup U_i, \sqcup(U_i \setminus x_i)) \cong \oplus\mathbb{Z}$$

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Lemma. $H_n(S^n) \xrightarrow{j'_*} H_n(S^n, S^n \setminus \{x_1, x_2, \ldots, x_k\})$ *maps the generator to the sum of the generators.*

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Proof.

$$\begin{tikzcd} H_n(S^n) \arrow[r, "j'_*"] \arrow[d] & H_n(S^n, S^n \setminus \{x_1, x_2, \dots, \\ H_n(S^n, S^n \setminus x_i) & \end{tikzcd}$$

□

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∎

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\end{tikzcd}$$