Observation on local consistency :

$$\begin{tikzcd}
 & H_n(M, M \setminus B) \arrow[dl] \arrow[dr] & \\
H_n(M, M \setminus p) & & H_n(M, M \setminus q))
\end{tikzcd}$$

Observation on local consistency $(B' \subset B)$:

$$\begin{tikzcd}[column sep=small]
& H_n(M, M \setminus B) \arrow[ddl] \arrow[ddr] & \\
& H_n(M, M \setminus B') \arrow[dl] \arrow[dr] & \\
H_n(M, M \setminus p) & & H_n(M, M \setminus q))
\end{tikzcd}$$

Observation on local consistency $(B' \subset B)$:

$$\begin{tikzcd}
& H_n(M, M \setminus B) \arrow[d, "i_{B'}"] \arrow[ddl] \arrow[ddr] & \\
& H_n(M, M \setminus B') \arrow[dl] \arrow[dr] & \\
H_n(M, M \setminus p) & & H_n(M, M \setminus q))
\end{tikzcd}$$

Observation on local consistency $(B' \subset B)$:

$$\begin{tikzcd}
 & H_n(M, M \setminus B) \arrow[dl] \arrow[d, "i_{B'}"] \arrow[dr] & \\
 & H_n(M, M \setminus B') \arrow[dl] \arrow[dr] & \\
H_n(M, M \setminus p) & & H_n(M, M \setminus q))
\end{tikzcd}$$

An orientation on M needs the assignment $p \to \mu_p$ to satisfy compatibily criterion.

$$\begin{tikzcd}
 & \mu_B \arrow[dl] \arrow[dr] & \\
\mu_p & & \mu_q
\end{tikzcd}$$

Observation on local consistency $(B' \subset B)$:

$$
\begin{tikzcd}
 & H_n(M, M \setminus B) \arrow[d, "i_{B'}"] \arrow[ddl] \arrow[ddr] & \\
 & H_n(M, M \setminus B') \arrow[dl] \arrow[dr] & \\
H_n(M, M \setminus p) & & H_n(M, M \setminus q))
\end{tikzcd}
$$

An orientation on M needs the assignment $p \to \mu_p$ to satisfy compatibily criterion.

$$
\begin{tikzcd}
 & \mu_B \arrow[ddl] \arrow[ddr] & \\
 & \mu_{B'} \arrow[dl] \arrow[dr] & \\
\mu_p & & \mu_q
\end{tikzcd}
$$

Observation on local consistency $(B' \subset B)$:

$$
\begin{tikzcd}
& H_n(M, M \setminus B) \arrow[d, "i_{B'}"] \arrow[ddl] \arrow[ddr] & \\
& H_n(M, M \setminus B') \arrow[dl] \arrow[dr] & \\
H_n(M, M \setminus p) & & H_n(M, M \setminus q))
\end{tikzcd}
$$

An orientation on M needs the assignment $p \to \mu_p$ to satisfy compatibily criterion.

$$
\begin{tikzcd}
& \mu_B \arrow[d] \arrow[ddl] \arrow[ddr] & \\
& \mu_{B'} \arrow[dl] \arrow[dr] & \\
\mu_p & & \mu_q
\end{tikzcd}
$$

if $\mu_p = i_{p_*}(\mu_B)$ and $\mu_q = i_{q_*}(\mu_B)$,

Observation on local consistency $(B' \subset B)$:

$$\begin{tikzcd}
& H_n(M, M \setminus B) \arrow[d, "i_{B'}"] \arrow[ddl] \arrow[ddr] & \\
& H_n(M, M \setminus B') \arrow[dl] \arrow[dr] & \\
H_n(M, M \setminus p) & & H_n(M, M \setminus q))
\end{tikzcd}$$

An orientation on M needs the assignment $p \to \mu_p$ to satisfy compatibily criterion.

$$\begin{tikzcd}
& \mu_B \arrow[d] \arrow[ddl] \arrow[ddr] & \\
& \mu_{B'} \arrow[dl] \arrow[dr] & \\
\mu_p & & \mu_q
\end{tikzcd}$$

if $\mu_p = i_{p_*}(\mu_B)$ and $\mu_q = i_{q_*}(\mu_B)$, then for any ball $B' \subset B$,

Observation on local consistency $(B' \subset B)$:

$$\begin{tikzcd}
 & H_n(M, M \setminus B) \arrow[d, "i_{B'}"] \arrow[ddl] \arrow[ddr] & \\
 & H_n(M, M \setminus B') \arrow[dl] \arrow[dr] & \\
H_n(M, M \setminus p) & & H_n(M, M \setminus q))
\end{tikzcd}$$

An orientation on M needs the assignment $p \to \mu_p$ to satisfy compatibily criterion.

$$\begin{tikzcd}
 & \mu_B \arrow[d] \arrow[ddl] \arrow[ddr] & \\
 & \mu_{B'} \arrow[dl] \arrow[dr] & \\
\mu_p & & \mu_q
\end{tikzcd}$$

if $\mu_p = i_{p_*}(\mu_B)$ and $\mu_q = i_{q_*}(\mu_B)$, then for any ball $B' \subset B$, $\mu_p = i_{p_*}(\mu_{B'})$ and $\mu_q = i_{q_*}(\mu_{B'})$ and $\mu_{B'} = i_{B'*}(\mu_B)$.

Informally, if a ball is responsible for compatible local orientations at two points,

Observation on local consistency $(B' \subset B)$:

$$\begin{tikzcd}
 & H_n(M, M \setminus B) \arrow[d, "i_{B'}"] \arrow[ddl] \arrow[ddr] & \\
 & H_n(M, M \setminus B') \arrow[dl] \arrow[dr] & \\
H_n(M, M \setminus p) & & H_n(M, M \setminus q))
\end{tikzcd}$$

An orientation on M needs the assignment $p \to \mu_p$ to satisfy compatibily criterion.

$$\begin{tikzcd}
 & \mu_B \arrow[d] \arrow[ddl] \arrow[ddr] & \\
 & \mu_{B'} \arrow[dl] \arrow[dr] & \\
\mu_p & & \mu_q
\end{tikzcd}$$

if $\mu_p = i_{p_*}(\mu_B)$ and $\mu_q = i_{q_*}(\mu_B)$, then for any ball $B' \subset B$, $\mu_p = i_{p_*}(\mu_{B'})$ and $\mu_q = i_{q_*}(\mu_{B'})$ and $\mu_{B'} = i_{B'*}(\mu_B)$.

Informally, if a ball is responsible for compatible local orientations at two points, then any smaller ball that contains those two points

Observation on local consistency $(B' \subset B)$:

$$
\begin{tikzcd}
 & H_n(M, M \setminus B) \arrow[d, "i_{B'}"] \arrow[ddl] \arrow[ddr] & \\
 & H_n(M, M \setminus B') \arrow[dl] \arrow[dr] & \\
H_n(M, M \setminus p) & & H_n(M, M \setminus q))
\end{tikzcd}
$$

Informally, if a ball is responsible for compatible local orientations at two points, then any smaller ball that contains those two points may also be used to ensure compatibility of the orientations at these points.

An orientation on M needs the assignment $p \to \mu_p$ to satisfy compatibily criterion.

$$
\begin{tikzcd}
 & \mu_B \arrow[d] \arrow[ddl] \arrow[ddr] & \\
 & \mu_{B'} \arrow[dl] \arrow[dr] & \\
\mu_p & & \mu_q
\end{tikzcd}
$$

if $\mu_p = i_{p_*}(\mu_B)$ and $\mu_q = i_{q_*}(\mu_B)$, then for any ball $B' \subset B$, $\mu_p = i_{p_*}(\mu_{B'})$ and $\mu_q = i_{q_*}(\mu_{B'})$ and $\mu_{B'} = i_{B'*}(\mu_B)$.

Theorem. *There is a covering* $\pi : M_{\mathbb{Z}} \to M$

Theorem. *There is a covering* $\pi : M_{\mathbb{Z}} \to M$ *such that*

Theorem. *There is a covering $\pi : M_{\mathbb{Z}} \to M$ such that (Here, $n :=$ dim M)*

1. $H_n(M, M \setminus p)$ *is the fibre at* p.

Theorem. *There is a covering $\pi : M_{\mathbb{Z}} \to M$ such that (Here, $n :=$ dim M)*

1. $H_n(M, M \setminus p) \cong \mathbb{Z}$ *is the fibre at p.*

Theorem. *There is a covering* $\pi : M_{\mathbb{Z}} \to M$ *such that (Here,* $n :=$ *dim* M*)*

1. $H_n(M, M \setminus p) \cong \mathbb{Z}$ *is the fibre at* p.

2. $M_{\mathbb{Z}}$ *is a disjoint union of a copy of* M *and copies of* $\widetilde{M}$

Theorem. *There is a covering $\pi : M_{\mathbb{Z}} \to M$ such that (Here, $n :=$ dim M)*

1. *$H_n(M, M \setminus p) \cong \mathbb{Z}$ is the fibre at p.*

2. *$M_{\mathbb{Z}}$ is a disjoint union of a copy of M and copies of $\widetilde{M}$*

3. *Any continuous section $s : M \to M_{\mathbb{Z}}$ is an integer multiple of a section $s_1 : M \to \widetilde{M}$ defining an orientation.*

Theorem. *There is a covering* $\pi : M_{\mathbb{Z}} \to M$ *such that (Here,* $n :=$ *dim* M*)*

1. $H_n(M, M \setminus p) \cong \mathbb{Z}$ *is the fibre at* p*.*
2. $M_{\mathbb{Z}}$ *is a disjoint union of a copy of* M *and copies of* $\widetilde{M}$
3. *Any continuous section* $s : M \to M_{\mathbb{Z}}$ *is an integer multiple of a section* $s_1 : M \to \widetilde{M}$ *defining an orientation.*

Sketch of the proof. $M_{\mathbb{Z}} := \{\alpha_p \mid \alpha_p \in H_n(M, M \setminus p)\}$

□

Theorem. *There is a covering $\pi : M_{\mathbb{Z}} \to M$ such that (Here, $n :=$ dim M)*

1. *$H_n(M, M \setminus p) \cong \mathbb{Z}$ is the fibre at p.*
2. *$M_{\mathbb{Z}}$ is a disjoint union of a copy of M and copies of $\widetilde{M}$*
3. *Any continuous section $s : M \to M_{\mathbb{Z}}$ is an integer multiple of a section $s_1 : M \to \widetilde{M}$ defining an orientation.*

Sketch of the proof. $M_{\mathbb{Z}} := \{\alpha_p \mid \alpha_p \in H_n(M, M \setminus p)\}$ $\pi : M_{\mathbb{Z}} \to M$, defined by $\pi(\alpha_p) := p$. $\square$

Theorem. *There is a covering $\pi : M_{\mathbb{Z}} \to M$ such that (Here, $n :=$ dim M)*

1. *$H_n(M, M \setminus p) \cong \mathbb{Z}$ is the fibre at p.*
2. *$M_{\mathbb{Z}}$ is a disjoint union of a copy of M and copies of $\widetilde{M}$*
3. *Any continuous section $s : M \to M_{\mathbb{Z}}$ is an integer multiple of a section $s_1 : M \to \widetilde{M}$ defining an orientation.*

Sketch of the proof. $M_{\mathbb{Z}} := \{\alpha_p \mid \alpha_p \in H_n(M, M \setminus p)\}$
$\pi : M_{\mathbb{Z}} \to M$, defined by $\pi(\alpha_p) := p$.

Given $B \subset U \subset M$, where U is homeomorphic to $\mathbb{R}^n$ and B homeomorphic to an open ball in $\mathbb{R}^n$,
and $\alpha_B \in H_n(M, M \setminus B)$, define
$U_{\alpha_B} := \{i_{p_*}(\alpha_B) \mid p \in B\}$ $\square$

Theorem. *There is a covering* $\pi : M_{\mathbb{Z}} \to M$ *such that (Here,* $n :=$ *dim* M*)*

1. $H_n(M, M \setminus p) \cong \mathbb{Z}$ *is the fibre at* p*.*
2. $M_{\mathbb{Z}}$ *is a disjoint union of a copy of* M *and copies of* $\widetilde{M}$
3. *Any continuous section* $s : M \to M_{\mathbb{Z}}$ *is an integer multiple of a section* $s_1 : M \to \widetilde{M}$ *defining an orientation.*

Sketch of the proof. $M_{\mathbb{Z}} := \{\alpha_p \mid \alpha_p \in H_n(M, M \setminus p)\}$
$\pi : M_{\mathbb{Z}} \to M$, defined by $\pi(\alpha_p) := p$.

Given $B \subset U \subset M$, where U is homeomorphic to $\mathbb{R}^n$ and B homeomorphic to an open ball in $\mathbb{R}^n$,
and $\alpha_B \in H_n(M, M \setminus B)$, define
$U_{\alpha_B} := \{i_{p_*}(\alpha_B) \mid p \in B\}$

The rest of the proof that it is a covering is an exercise, and follows as for the oriented covering.

$\square$

Theorem. *There is a covering $\pi : M_{\mathbb{Z}} \to M$ such that (Here, $n := \dim M$)*

1. *$H_n(M, M \setminus p) \cong \mathbb{Z}$ is the fibre at p.*
2. *$M_{\mathbb{Z}}$ is a disjoint union of a copy of M and copies of $\widetilde{M}$*
3. *Any continuous section $s : M \to M_{\mathbb{Z}}$ is an integer multiple of a section $s_1 : M \to \widetilde{M}$ defining an orientation.*

Sketch of the proof. $M_{\mathbb{Z}} := \{\alpha_p \mid \alpha_p \in H_n(M, M \setminus p)\}$
$\pi : M_{\mathbb{Z}} \to M$, defined by $\pi(\alpha_p) := p$.

Given $B \subset U \subset M$, where U is homeomorphic to $\mathbb{R}^n$
and B homeomorphic to an open ball in $\mathbb{R}^n$,
and $\alpha_B \in H_n(M, M \setminus B)$, define
$U_{\alpha_B} := \{i_{p_*}(\alpha_B) \mid p \in B\}$

The rest of the proof that it is a covering is an exercise, and follows as for the oriented covering.

For the other parts, observe that $M_{\mathbb{Z}} = \sqcup M_k$, $k > 0$, where $\square$

Theorem. *There is a covering $\pi : M_{\mathbb{Z}} \to M$ such that (Here, $n := \dim M$)*

1. *$H_n(M, M \setminus p) \cong \mathbb{Z}$ is the fibre at p.*
2. *$M_{\mathbb{Z}}$ is a disjoint union of a copy of M and copies of $\widetilde{M}$*
3. *Any continuous section $s : M \to M_{\mathbb{Z}}$ is an integer multiple of a section $s_1 : M \to \widetilde{M}$ defining an orientation.*

Sketch of the proof. $M_{\mathbb{Z}} := \{\alpha_p \mid \alpha_p \in H_n(M, M \setminus p)\}$
$\pi : M_{\mathbb{Z}} \to M$, defined by $\pi(\alpha_p) := p$.

Given $B \subset U \subset M$, where U is homeomorphic to $\mathbb{R}^n$ and B homeomorphic to an open ball in $\mathbb{R}^n$,
and $\alpha_B \in H_n(M, M \setminus B)$, define
$U_{\alpha_B} := \{i_{p_*}(\alpha_B) \mid p \in B\}$

The rest of the proof that it is a covering is an exercise, and follows as for the oriented covering.

For the other parts, observe that $M_{\mathbb{Z}} = \sqcup M_k$, $k > 0$, where

$$M_k := \{k\mu_p \mid \mu_p \in H_n(M, M \setminus p) \text{ is a generator}\}.$$

$\square$

Theorem. *There is a covering* $\pi : M_{\mathbb{Z}} \to M$ *such that (Here,* $n := \dim M$*)*

1. $H_n(M, M \setminus p) \cong \mathbb{Z}$ *is the fibre at* p.
2. $M_{\mathbb{Z}}$ *is a disjoint union of a copy of* M *and copies of* $\widetilde{M}$
3. *Any continuous section* $s : M \to M_{\mathbb{Z}}$ *is an integer multiple of a section* $s_1 : M \to \widetilde{M}$ *defining an orientation.*

Sketch of the proof. $M_{\mathbb{Z}} := \{\alpha_p \mid \alpha_p \in H_n(M, M \setminus p)\}$
$\pi : M_{\mathbb{Z}} \to M$, defined by $\pi(\alpha_p) := p$.

Given $B \subset U \subset M$, where U is homeomorphic to $\mathbb{R}^n$ and B homeomorphic to an open ball in $\mathbb{R}^n$,
and $\alpha_B \in H_n(M, M \setminus B)$, define
$U_{\alpha_B} := \{i_{p_*}(\alpha_B) \mid p \in B\}$

The rest of the proof that it is a covering is an exercise, and follows as for the oriented covering.

For the other parts, observe that $M_{\mathbb{Z}} = \sqcup M_k$, $k > 0$, where

$$\begin{aligned} M_k &:= \{k\mu_p \mid \mu_p \in H_n(M, M \setminus p) \text{ is a generator}\}. \\ &:= \{k\mu_p \mid \mu_p \in \widetilde{M}\}. \end{aligned}$$

and π restricts to coverings on each of them. $\square$

Theorem. *There is a covering $\pi : M_{\mathbb{Z}} \to M$ such that (Here, $n := \dim M$)*

1. *$H_n(M, M \setminus p) \cong \mathbb{Z}$ is the fibre at p.*
2. *$M_{\mathbb{Z}}$ is a disjoint union of a copy of M and copies of $\widetilde{M}$*
3. *Any continuous section $s : M \to M_{\mathbb{Z}}$ is an integer multiple of a section $s_1 : M \to \widetilde{M}$ defining an orientation.*

Sketch of the proof. $M_{\mathbb{Z}} := \{\alpha_p \mid \alpha_p \in H_n(M, M \setminus p)\}$
$\pi : M_{\mathbb{Z}} \to M$, defined by $\pi(\alpha_p) := p$.

Given $B \subset U \subset M$, where U is homeomorphic to $\mathbb{R}^n$ and B homeomorphic to an open ball in $\mathbb{R}^n$,
and $\alpha_B \in H_n(M, M \setminus B)$, define
$U_{\alpha_B} := \{i_{p_*}(\alpha_B) \mid p \in B\}$

The rest of the proof that it is a covering is an exercise, and follows as for the oriented covering.

For the other parts, observe that $M_{\mathbb{Z}} = \sqcup M_k$, $k > 0$, where

$$\begin{aligned} M_k &:= \{k\mu_p \mid \mu_p \in H_n(M, M \setminus p) \text{ is a generator}\}. \\ &:= \{k\mu_p \mid \mu_p \in \widetilde{M}\}. \end{aligned}$$

and π restricts to coverings on each of them.

Note, M_0 is a 1 sheeted cover so homeormorphic to M. □

Theorem. *There is a covering $\pi : M_{\mathbb{Z}} \to M$ such that (Here, $n := \dim M$)*

1. *$H_n(M, M \setminus p) \cong \mathbb{Z}$ is the fibre at p.*
2. *$M_{\mathbb{Z}}$ is a disjoint union of a copy of M and copies of $\widetilde{M}$*
3. *Any continuous section $s : M \to M_{\mathbb{Z}}$ is an integer multiple of a section $s_1 : M \to \widetilde{M}$ defining an orientation.*

Sketch of the proof. $M_{\mathbb{Z}} := \{\alpha_p \mid \alpha_p \in H_n(M, M \setminus p)\}$
$\pi : M_{\mathbb{Z}} \to M$, defined by $\pi(\alpha_p) := p$.

Given $B \subset U \subset M$, where U is homeomorphic to $\mathbb{R}^n$ and B homeomorphic to an open ball in $\mathbb{R}^n$,
and $\alpha_B \in H_n(M, M \setminus B)$, define
$U_{\alpha_B} := \{i_{p_*}(\alpha_B) \mid p \in B\}$

The rest of the proof that it is a covering is an exercise, and follows as for the oriented covering.

For the other parts, observe that $M_{\mathbb{Z}} = \sqcup M_k$, $k > 0$, where

$$\begin{aligned} M_k &:= \{k\mu_p \mid \mu_p \in H_n(M, M \setminus p) \text{ is a generator}\}. \\ &:= \{k\mu_p \mid \mu_p \in \widetilde{M}\}. \end{aligned}$$

and π restricts to coverings on each of them.

Note, M_0 is a 1 sheeted cover so homeormorphic to M.

If M is connnected, $\square$

Theorem. *There is a covering* $\pi : M_{\mathbb{Z}} \to M$ *such that (Here,* $n := \dim M$*)*

1. $H_n(M, M \setminus p) \cong \mathbb{Z}$ *is the fibre at* p.
2. $M_{\mathbb{Z}}$ *is a disjoint union of a copy of* M *and copies of* $\widetilde{M}$
3. *Any continuous section* $s : M \to M_{\mathbb{Z}}$ *is an integer multiple of a section* $s_1 : M \to \widetilde{M}$ *defining an orientation.*

Sketch of the proof. $M_{\mathbb{Z}} := \{\alpha_p \mid \alpha_p \in H_n(M, M \setminus p)\}$
$\pi : M_{\mathbb{Z}} \to M$, defined by $\pi(\alpha_p) := p$.

Given $B \subset U \subset M$, where U is homeomorphic to $\mathbb{R}^n$ and B homeomorphic to an open ball in $\mathbb{R}^n$,
and $\alpha_B \in H_n(M, M \setminus B)$, define
$U_{\alpha_B} := \{i_{p_*}(\alpha_B) \mid p \in B\}$

The rest of the proof that it is a covering is an exercise, and follows as for the oriented covering.

For the other parts, observe that $M_{\mathbb{Z}} = \sqcup M_k$, $k > 0$, where

$$\begin{aligned} M_k &:= \{k\mu_p \mid \mu_p \in H_n(M, M \setminus p) \text{ is a generator}\}. \\ &:= \{k\mu_p \mid \mu_p \in \widetilde{M}\}. \end{aligned}$$

and π restricts to coverings on each of them.

Note, M_0 is a 1 sheeted cover so homeormorphic to M.

If M is connnected, the image of section has to lie in M_n for some n. $\square$

Theorem. *There is a covering $\pi : M_{\mathbb{Z}} \to M$ such that (Here, $n := \dim M$)*

1. *$H_n(M, M \setminus p) \cong \mathbb{Z}$ is the fibre at p.*
2. *$M_{\mathbb{Z}}$ is a disjoint union of a copy of M and copies of $\widetilde{M}$*
3. *Any continuous section $s : M \to M_{\mathbb{Z}}$ is an integer multiple of a section $s_1 : M \to \widetilde{M}$ defining an orientation.*

Sketch of the proof. $M_{\mathbb{Z}} := \{\alpha_p \mid \alpha_p \in H_n(M, M \setminus p)\}$
$\pi : M_{\mathbb{Z}} \to M$, defined by $\pi(\alpha_p) := p$.

Given $B \subset U \subset M$, where U is homeomorphic to $\mathbb{R}^n$ and B homeomorphic to an open ball in $\mathbb{R}^n$,
and $\alpha_B \in H_n(M, M \setminus B)$, define
$U_{\alpha_B} := \{i_{p_*}(\alpha_B) \mid p \in B\}$

The rest of the proof that it is a covering is an exercise, and follows as for the oriented covering.

For the other parts, observe that $M_{\mathbb{Z}} = \sqcup M_k$, $k > 0$, where

$$\begin{aligned} M_k &:= \{k\mu_p \mid \mu_p \in H_n(M, M \setminus p) \text{ is a generator}\}. \\ &:= \{k\mu_p \mid \mu_p \in \widetilde{M}\}. \end{aligned}$$

and π restricts to coverings on each of them.

Note, M_0 is a 1 sheeted cover so homeormorphic to M.

If M is connnected, the image of section has to lie in M_n for some n. $\square$

$\Gamma_{\mathbb{Z}}(M) := \{s : M \to M_{\mathbb{Z}} \mid \pi \circ s = Id, \ s \text{ continuous}\}$

Theorem. *There is a covering $\pi : M_{\mathbb{Z}} \to M$ such that (Here, $n := \dim M$)*

1. *$H_n(M, M \setminus p) \cong \mathbb{Z}$ is the fibre at p.*
2. *$M_{\mathbb{Z}}$ is a disjoint union of a copy of M and copies of $\widetilde{M}$*
3. *Any continuous section $s : M \to M_{\mathbb{Z}}$ is an integer multiple of a section $s_1 : M \to \widetilde{M}$ defining an orientation.*

Sketch of the proof. $M_{\mathbb{Z}} := \{\alpha_p \mid \alpha_p \in H_n(M, M \setminus p)\}$
$\pi : M_{\mathbb{Z}} \to M$, defined by $\pi(\alpha_p) := p$.

Given $B \subset U \subset M$, where U is homeomorphic to $\mathbb{R}^n$ and B homeomorphic to an open ball in $\mathbb{R}^n$,
and $\alpha_B \in H_n(M, M \setminus B)$, define
$U_{\alpha_B} := \{i_{p_*}(\alpha_B) \mid p \in B\}$

The rest of the proof that it is a covering is an exercise, and follows as for the oriented covering.

For the other parts, observe that $M_{\mathbb{Z}} = \sqcup M_k$, $k > 0$, where

$$\begin{aligned} M_k &:= \{k\mu_p \mid \mu_p \in H_n(M, M \setminus p) \text{ is a generator}\}. \\ &:= \{k\mu_p \mid \mu_p \in \widetilde{M}\}. \end{aligned}$$

and π restricts to coverings on each of them.

Note, M_0 is a 1 sheeted cover so homeormorphic to M.

If M is connnected, the image of section has to lie in M_n for some n. $\square$

$\Gamma_{\mathbb{Z}}(M) := \{s : M \to M_{\mathbb{Z}} \mid \pi \circ s = Id,\ s \text{ continuous}\}$
(called, space of sections)

Theorem. *There is a covering $\pi : M_{\mathbb{Z}} \to M$ such that (Here, $n := \dim M$)*

1. *$H_n(M, M \setminus p) \cong \mathbb{Z}$ is the fibre at p.*
2. *$M_{\mathbb{Z}}$ is a disjoint union of a copy of M and copies of $\widetilde{M}$*
3. *Any continuous section $s : M \to M_{\mathbb{Z}}$ is an integer multiple of a section $s_1 : M \to \widetilde{M}$ defining an orientation.*

Sketch of the proof. $M_{\mathbb{Z}} := \{\alpha_p \mid \alpha_p \in H_n(M, M \setminus p)\}$
$\pi : M_{\mathbb{Z}} \to M$, defined by $\pi(\alpha_p) := p$.

Given $B \subset U \subset M$, where U is homeomorphic to $\mathbb{R}^n$ and B homeomorphic to an open ball in $\mathbb{R}^n$,
and $\alpha_B \in H_n(M, M \setminus B)$, define
$U_{\alpha_B} := \{i_{p_*}(\alpha_B) \mid p \in B\}$

The rest of the proof that it is a covering is an exercise, and follows as for the oriented covering.

For the other parts, observe that $M_{\mathbb{Z}} = \sqcup M_k$, $k > 0$, where

$$\begin{aligned} M_k &:= \{k\mu_p \mid \mu_p \in H_n(M, M \setminus p) \text{ is a generator}\}. \\ &:= \{k\mu_p \mid \mu_p \in \widetilde{M}\}. \end{aligned}$$

and π restricts to coverings on each of them.

Note, M_0 is a 1 sheeted cover so homeormorphic to M.

If M is connnected, the image of section has to lie in M_n for some n. □

$\Gamma_{\mathbb{Z}}(M) := \{s : M \to M_{\mathbb{Z}} \mid \pi \circ s = Id,\ s \text{ continuous}\}$ (called, space of sections)

$\Gamma_{\mathbb{Z}}(M)$ is an abelian group (0 section assigns the 0 homology class to each point)

Example. $\alpha \to \alpha_x$

Example. $\underbrace{\alpha}_{\in H_n(M)} \to \alpha_x$

Example. $\underbrace{\alpha}_{\in H_n(M)} \to \underbrace{\alpha_x}_{\in H_n(M, M\setminus x)}$

Example. $\underbrace{\alpha}_{\in H_n(M)} \to \underbrace{\alpha_x}_{\in H_n(M, M \setminus x)}$ *section defined by a homology class of M*

Example. $\underbrace{\alpha}_{\in H_n(M)} \to \underbrace{\alpha_x}_{\in H_n(M, M\setminus x)}$ *section defined by a homology class of* M

Example. $s \in \Gamma_{\mathbb{Z}}(\mathbb{R}^n)$

Example. $\underbrace{\alpha}_{\in H_n(M)} \to \underbrace{\alpha_x}_{\in H_n(M,M\backslash x)}$ *section defined by a homology class of M*

Example. $s \in \Gamma_{\mathbb{Z}}(\mathbb{R}^n)$
$s = ms_0$, *where* s_0 *is the orientation section*

Example. $\underbrace{\alpha}_{\in H_n(M)} \rightarrow \underbrace{\alpha_x}_{\in H_n(M,M\backslash x)}$ *section defined by a homology class of M*

Example. $s \in \Gamma_{\mathbb{Z}}(\mathbb{R}^n)$
$s = ms_0$, *where* s_0 *is the orientation section and* $m \in \mathbb{Z}$. *For any ball B*

Question:
A compact.

Question:
A compact. How is $\alpha \in H_n(M, M \setminus A)$ characterized by $\alpha_x \in H_n(M, M \setminus x)$ for $x \in A$?

Already seen: $\alpha_x = 0$ for all $x \in A$ is equivalent to $\alpha = 0$

Question:
A compact. How is $\alpha \in H_n(M, M \setminus A)$ characterized by $\alpha_x \in H_n(M, M \setminus x)$ for $x \in A$?

Already seen: $\alpha_x = 0$ for all $x \in A$ is equivalent to $\alpha = 0$

Given any $s \in \Gamma_{\mathbb{Z}}(M)$,

Question:
A compact. How is $\alpha \in H_n(M, M \setminus A)$ characterized by $\alpha_x \in H_n(M, M \setminus x)$ for $x \in A$?

Already seen: $\alpha_x = 0$ for all $x \in A$ is equivalent to $\alpha = 0$

Given any $s \in \Gamma_{\mathbb{Z}}(M)$, is there an $\alpha \in H_n(M, M \setminus A)$ such that $\alpha_x = s(x)$ for all $x \in A$?

Question:
A compact. How is $\alpha \in H_n(M, M \setminus A)$ characterized by $\alpha_x \in H_n(M, M \setminus x)$ for $x \in A$?

Already seen: $\alpha_x = 0$ for all $x \in A$ is equivalent to $\alpha = 0$

Theorem. *Given any* $s \in \Gamma_{\mathbb{Z}}(M)$,

Question:
A compact. How is $\alpha \in H_n(M, M \setminus A)$ characterized by $\alpha_x \in H_n(M, M \setminus x)$ for $x \in A$?

Already seen: $\alpha_x = 0$ for all $x \in A$ is equivalent to $\alpha = 0$

Theorem. *Given any $s \in \Gamma_{\mathbb{Z}}(M)$, there is a $\alpha \in H_n(M, M \setminus A)$ such that $\alpha_x = s(x)$ for all $x \in A$.*

Question:
A compact. How is $\alpha \in H_n(M, M \setminus A)$ characterized by $\alpha_x \in H_n(M, M \setminus x)$ for $x \in A$?

Already seen: $\alpha_x = 0$ for all $x \in A$ is equivalent to $\alpha = 0$

Theorem. *Given any $s \in \Gamma_{\mathbb{Z}}(M)$, there is a **unique** $\alpha \in H_n(M, M \setminus A)$ such that $\alpha_x = s(x)$ for all $x \in A$.*

Uniqueness follows from the previous statement.

Question:
A compact. How is $\alpha \in H_n(M, M \setminus A)$ characterized by $\alpha_x \in H_n(M, M \setminus x)$ for $x \in A$?

Already seen: $\alpha_x = 0$ for all $x \in A$ is equivalent to $\alpha = 0$

Theorem. *Given any $s \in \Gamma_{\mathbb{Z}}(M)$, there is a* ***unique*** *$\alpha \in H_n(M, M \setminus A)$ such that $\alpha_x = s(x)$ for all $x \in A$.*

Uniqueness follows from the previous statement.

Sketch of proof. Can we patch it up from simpler pieces? $\square$

Question:
A compact. How is $\alpha \in H_n(M, M \setminus A)$ characterized by $\alpha_x \in H_n(M, M \setminus x)$ for $x \in A$?

Already seen: $\alpha_x = 0$ for all $x \in A$ is equivalent to $\alpha = 0$

Theorem. *Given any $s \in \Gamma_{\mathbb{Z}}(M)$, there is a* ***unique*** *$\alpha \in H_n(M, M \setminus A)$ such that $\alpha_x = s(x)$ for all $x \in A$.*

Uniqueness follows from the previous statement.

Sketch of proof. Can we patch it up from simpler pieces?i.e.

Claim: Assume true for A, $\square$

Question:
A compact. How is $\alpha \in H_n(M, M \setminus A)$ characterized by $\alpha_x \in H_n(M, M \setminus x)$ for $x \in A$?

Already seen: $\alpha_x = 0$ for all $x \in A$ is equivalent to $\alpha = 0$

Theorem. *Given any $s \in \Gamma_{\mathbb{Z}}(M)$, there is a* ***unique*** *$\alpha \in H_n(M, M \setminus A)$ such that $\alpha_x = s(x)$ for all $x \in A$.*

Uniqueness follows from the previous statement.

Sketch of proof. Can we patch it up from simpler pieces?i.e.

Claim: Assume true for A, B, $\square$

Question:
A compact. How is $\alpha \in H_n(M, M \setminus A)$ characterized by $\alpha_x \in H_n(M, M \setminus x)$ for $x \in A$?

Already seen: $\alpha_x = 0$ for all $x \in A$ is equivalent to $\alpha = 0$

Theorem. *Given any $s \in \Gamma_{\mathbb{Z}}(M)$, there is a* ***unique*** *$\alpha \in H_n(M, M \setminus A)$ such that $\alpha_x = s(x)$ for all $x \in A$.*

Uniqueness follows from the previous statement.

Sketch of proof. Can we patch it up from simpler pieces?i.e.

Claim: Assume true for A, B, and $A \cap B$. $\square$

Question:
A compact. How is $\alpha \in H_n(M, M \setminus A)$ characterized by $\alpha_x \in H_n(M, M \setminus x)$ for $x \in A$?

Already seen: $\alpha_x = 0$ for all $x \in A$ is equivalent to $\alpha = 0$

Theorem. *Given any $s \in \Gamma_{\mathbb{Z}}(M)$, there is a* ***unique*** *$\alpha \in H_n(M, M \setminus A)$ such that $\alpha_x = s(x)$ for all $x \in A$.*

Uniqueness follows from the previous statement.

Sketch of proof. Can we patch it up from simpler pieces?i.e.

Claim: Assume true for A, B, and $A \cap B$. To show that it is true for $A \cup B$. $\square$

Question:
A compact. How is $\alpha \in H_n(M, M \setminus A)$ characterized by $\alpha_x \in H_n(M, M \setminus x)$ for $x \in A$?

Already seen: $\alpha_x = 0$ for all $x \in A$ is equivalent to $\alpha = 0$

Theorem. *Given any $s \in \Gamma_{\mathbb{Z}}(M)$, there is a* ***unique*** *$\alpha \in H_n(M, M \setminus A)$ such that $\alpha_x = s(x)$ for all $x \in A$.*

Uniqueness follows from the previous statement.

Sketch of proof. Can we patch it up from simpler pieces?i.e.

Claim: Assume true for A, B, and $A \cap B$. To show that it is true for $A \cup B$.

For $s \in \Gamma_{\mathbb{Z}}(M)$, $\square$

Question:
A compact. How is $\alpha \in H_n(M, M \setminus A)$ characterized by $\alpha_x \in H_n(M, M \setminus x)$ for $x \in A$?

Already seen: $\alpha_x = 0$ for all $x \in A$ is equivalent to $\alpha = 0$

Theorem. *Given any $s \in \Gamma_{\mathbb{Z}}(M)$, there is a **unique** $\alpha \in H_n(M, M \setminus A)$ such that $\alpha_x = s(x)$ for all $x \in A$.*

Uniqueness follows from the previous statement.

Sketch of proof. Can we patch it up from simpler pieces?i.e.

Claim: Assume true for A, B, and $A \cap B$. To show that it is true for $A \cup B$.

For $s \in \Gamma_{\mathbb{Z}}(M)$, there exists unique $\alpha_A \in H_n(M, M \setminus A)$, $\alpha_B \in H_n(M, M \setminus B)$, and $\alpha_{A\cap B} \in H_n(M, M \setminus (A \cap B))$ so that $\square$

Question:
A compact. How is $\alpha \in H_n(M, M \setminus A)$ characterized by $\alpha_x \in H_n(M, M \setminus x)$ for $x \in A$?

Already seen: $\alpha_x = 0$ for all $x \in A$ is equivalent to $\alpha = 0$

Theorem. *Given any $s \in \Gamma_{\mathbb{Z}}(M)$, there is a **unique** $\alpha \in H_n(M, M \setminus A)$ such that $\alpha_x = s(x)$ for all $x \in A$.*

Uniqueness follows from the previous statement.

Sketch of proof. Can we patch it up from simpler pieces?i.e.

Claim: Assume true for A, B, and $A \cap B$. To show that it is true for $A \cup B$.

For $s \in \Gamma_{\mathbb{Z}}(M)$, there exists unique $\alpha_A \in H_n(M, M \setminus A)$, $\alpha_B \in H_n(M, M \setminus B)$, and $\alpha_{A\cap B} \in H_n(M, M \setminus (A \cap B))$ so that $\alpha_{A_x} = \alpha_{B_x} = \alpha_{A\cap B_x} = s(x)$ $\square$

Question:
A compact. How is $\alpha \in H_n(M, M \setminus A)$ characterized by $\alpha_x \in H_n(M, M \setminus x)$ for $x \in A$?

Already seen: $\alpha_x = 0$ for all $x \in A$ is equivalent to $\alpha = 0$

Theorem. *Given any $s \in \Gamma_{\mathbb{Z}}(M)$, there is a* ***unique*** *$\alpha \in H_n(M, M \setminus A)$ such that $\alpha_x = s(x)$ for all $x \in A$.*

Uniqueness follows from the previous statement.

Sketch of proof. Can we patch it up from simpler pieces?i.e.

Claim: Assume true for A, B, and $A \cap B$. To show that it is true for $A \cup B$.

For $s \in \Gamma_{\mathbb{Z}}(M)$, there exists unique $\alpha_A \in H_n(M, M \setminus A)$, $\alpha_B \in H_n(M, M \setminus B)$, and $\alpha_{A\cap B} \in H_n(M, M \setminus (A \cap B))$ so that $\alpha_{Ax} = \alpha_{Bx} = \alpha_{A\cap Bx} = s(x)$

By commutativity of the appropriate diagram, $\square$

Question:
A compact. How is $\alpha \in H_n(M, M \setminus A)$ characterized by $\alpha_x \in H_n(M, M \setminus x)$ for $x \in A$?

Already seen: $\alpha_x = 0$ for all $x \in A$ is equivalent to $\alpha = 0$

Theorem. *Given any $s \in \Gamma_{\mathbb{Z}}(M)$, there is a* ***unique*** *$\alpha \in H_n(M, M \setminus A)$ such that $\alpha_x = s(x)$ for all $x \in A$.*

Uniqueness follows from the previous statement.

Sketch of proof. Can we patch it up from simpler pieces?i.e.

Claim: Assume true for A, B, and $A \cap B$. To show that it is true for $A \cup B$.

For $s \in \Gamma_{\mathbb{Z}}(M)$, there exists unique $\alpha_A \in H_n(M, M \setminus A)$, $\alpha_B \in H_n(M, M \setminus B)$, and $\alpha_{A\cap B} \in H_n(M, M \setminus (A \cap B))$ so that $\alpha_{Ax} = \alpha_{Bx} = \alpha_{A\cap Bx} = s(x)$

By commutativity of the appropriate diagram, $(j_{A*}(\alpha_A))_x = \alpha_{A\cap Bx} \forall x \in A$, $\square$

Question:
A compact. How is $\alpha \in H_n(M, M \setminus A)$ characterized by $\alpha_x \in H_n(M, M \setminus x)$ for $x \in A$?

Already seen: $\alpha_x = 0$ for all $x \in A$ is equivalent to $\alpha = 0$

Theorem. *Given any $s \in \Gamma_{\mathbb{Z}}(M)$, there is a* ***unique*** *$\alpha \in H_n(M, M \setminus A)$ such that $\alpha_x = s(x)$ for all $x \in A$.*

Uniqueness follows from the previous statement.

Sketch of proof. Can we patch it up from simpler pieces?i.e.

Claim: Assume true for A, B, and $A \cap B$. To show that it is true for $A \cup B$.

For $s \in \Gamma_{\mathbb{Z}}(M)$, there exists unique $\alpha_A \in H_n(M, M \setminus A)$, $\alpha_B \in H_n(M, M \setminus B)$, and $\alpha_{A\cap B} \in H_n(M, M \setminus (A \cap B))$ so that $\alpha_{Ax} = \alpha_{Bx} = \alpha_{A\cap Bx} = s(x)$

By commutativity of the appropriate diagram, $(j_{A*}(\alpha_A))_x = \alpha_{A\cap Bx} \forall x \in A$, so by the uniqueness assumption $j_{A*}(\alpha_A) = \alpha_{A\cap B}$. $\square$

Question:
A compact. How is $\alpha \in H_n(M, M \setminus A)$ characterized by $\alpha_x \in H_n(M, M \setminus x)$ for $x \in A$?

Already seen: $\alpha_x = 0$ for all $x \in A$ is equivalent to $\alpha = 0$

Theorem. *Given any $s \in \Gamma_{\mathbb{Z}}(M)$, there is a* ***unique*** *$\alpha \in H_n(M, M \setminus A)$ such that $\alpha_x = s(x)$ for all $x \in A$.*

Uniqueness follows from the previous statement.

Sketch of proof. Can we patch it up from simpler pieces?i.e.

Claim: Assume true for A, B, and $A \cap B$. To show that it is true for $A \cup B$.

For $s \in \Gamma_{\mathbb{Z}}(M)$, there exists unique $\alpha_A \in H_n(M, M \setminus A)$, $\alpha_B \in H_n(M, M \setminus B)$, and $\alpha_{A\cap B} \in H_n(M, M \setminus (A \cap B))$ so that $\alpha_{Ax} = \alpha_{Bx} = \alpha_{A\cap Bx} = s(x)$

By commutativity of the appropriate diagram, $(j_{A*}(\alpha_A))_x = \alpha_{A\cap Bx} \forall x \in A$, so by the uniqueness assumption $j_{A*}(\alpha_A) = \alpha_{A\cap B}$. Similarly, $j_{B*}(\alpha_B) = \alpha_{A\cap B}$. □

Question:
A compact. How is $\alpha \in H_n(M, M \setminus A)$ characterized by $\alpha_x \in H_n(M, M \setminus x)$ for $x \in A$?

Already seen: $\alpha_x = 0$ for all $x \in A$ is equivalent to $\alpha = 0$

Theorem. *Given any $s \in \Gamma_{\mathbb{Z}}(M)$, there is a* ***unique*** *$\alpha \in H_n(M, M \setminus A)$ such that $\alpha_x = s(x)$ for all $x \in A$.*

Uniqueness follows from the previous statement.

Sketch of proof. Can we patch it up from simpler pieces?i.e.

Claim: Assume true for A, B, and $A \cap B$. To show that it is true for $A \cup B$.

For $s \in \Gamma_{\mathbb{Z}}(M)$, there exists unique $\alpha_A \in H_n(M, M \setminus A)$, $\alpha_B \in H_n(M, M \setminus B)$, and $\alpha_{A\cap B} \in H_n(M, M \setminus (A \cap B))$ so that $\alpha_{A_x} = \alpha_{B_x} = \alpha_{A\cap B_x} = s(x)$

By commutativity of the appropriate diagram, $(j_{A*}(\alpha_A))_x = \alpha_{A\cap B_x} \forall x \in A$, so by the uniqueness assumption $j_{A*}(\alpha_A) = \alpha_{A\cap B}$. Similarly, $j_{B*}(\alpha_B) = \alpha_{A\cap B}$. Therefore, the map,

$$H_n(M, M\setminus A) \oplus H_n(M, M\setminus B) \xrightarrow{\Psi} H_n(M, M\setminus(A\cap B))$$

, $\Psi(\alpha, \beta) = j_{A*}(\alpha) - j_{B*}(\beta)$ sends (α_A, α_B) to 0. $\square$

Question:
A compact. How is $\alpha \in H_n(M, M \setminus A)$ characterized by $\alpha_x \in H_n(M, M \setminus x)$ for $x \in A$?

Already seen: $\alpha_x = 0$ for all $x \in A$ is equivalent to $\alpha = 0$

Theorem. *Given any $s \in \Gamma_{\mathbb{Z}}(M)$, there is a* ***unique*** *$\alpha \in H_n(M, M \setminus A)$ such that $\alpha_x = s(x)$ for all $x \in A$.*

Uniqueness follows from the previous statement.

Sketch of proof. Can we patch it up from simpler pieces?i.e.

Claim: Assume true for A, B, and $A \cap B$. To show that it is true for $A \cup B$.

For $s \in \Gamma_{\mathbb{Z}}(M)$, there exists unique $\alpha_A \in H_n(M, M \setminus A)$, $\alpha_B \in H_n(M, M \setminus B)$, and $\alpha_{A \cap B} \in H_n(M, M \setminus (A \cap B))$ so that $\alpha_{Ax} = \alpha_{Bx} = \alpha_{A \cap Bx} = s(x)$

By commutativity of the appropriate diagram, $(j_{A*}(\alpha_A))_x = \alpha_{A \cap Bx} \forall x \in A$, so by the uniqueness assumption $j_{A*}(\alpha_A) = \alpha_{A \cap B}$. Similarly, $j_{B*}(\alpha_B) = \alpha_{A \cap B}$. Therefore, the map,

$$H_n(M, M \setminus A) \oplus H_n(M, M \setminus B) \xrightarrow{\Psi} H_n(M, M \setminus (A \cap B))$$

, $\Psi(\alpha, \beta) = j_{A*}(\alpha) - j_{B*}(\beta)$ sends (α_A, α_B) to 0. By Mayer Vietoris, there is $\alpha_{A \cup B}$ such that $(i_{A*}(\alpha_{A \cup B}), i_{A*}(\alpha_{A \cup B})) = (\alpha_A, \alpha_B)$ $\square$

Question:
A compact. How is $\alpha \in H_n(M, M \setminus A)$ characterized by $\alpha_x \in H_n(M, M \setminus x)$ for $x \in A$?

Already seen: $\alpha_x = 0$ for all $x \in A$ is equivalent to $\alpha = 0$

Theorem. *Given any $s \in \Gamma_{\mathbb{Z}}(M)$, there is a* ***unique*** *$\alpha \in H_n(M, M \setminus A)$ such that $\alpha_x = s(x)$ for all $x \in A$.*

Uniqueness follows from the previous statement.

Sketch of proof. Can we patch it up from simpler pieces?i.e.

Claim: Assume true for A, B, and $A \cap B$. To show that it is true for $A \cup B$.

For $s \in \Gamma_{\mathbb{Z}}(M)$, there exists unique $\alpha_A \in H_n(M, M \setminus A)$, $\alpha_B \in H_n(M, M \setminus B)$, and $\alpha_{A\cap B} \in H_n(M, M \setminus (A \cap B))$ so that $\alpha_{Ax} = \alpha_{Bx} = \alpha_{A\cap Bx} = s(x)$

By commutativity of the appropriate diagram, $(j_{A*}(\alpha_A))_x = \alpha_{A\cap Bx} \forall x \in A$, so by the uniqueness assumption $j_{A*}(\alpha_A) = \alpha_{A\cap B}$. Similarly, $j_{B*}(\alpha_B) = \alpha_{A\cap B}$. Therefore, the map,

$$H_n(M, M\setminus A) \oplus H_n(M, M \setminus B) \xrightarrow{\Psi} H_n(M, M \setminus (A \cap B))$$

, $\Psi(\alpha, \beta) = j_{A*}(\alpha) - j_{B*}(\beta)$ sends (α_A, α_B) to 0. By Mayer Vietoris, there is $\alpha_{A\cup B}$ such that $(i_{A*}(\alpha_{A\cup B}), i_{A*}(\alpha_{A\cup B})) = (\alpha_A, \alpha_B)$

By commutativity of the appropriate diagram, $\square$

Question:
A compact. How is $\alpha \in H_n(M, M \setminus A)$ characterized by $\alpha_x \in H_n(M, M \setminus x)$ for $x \in A$?

Already seen: $\alpha_x = 0$ for all $x \in A$ is equivalent to $\alpha = 0$

Theorem. *Given any $s \in \Gamma_{\mathbb{Z}}(M)$, there is a* ***unique*** *$\alpha \in H_n(M, M \setminus A)$ such that $\alpha_x = s(x)$ for all $x \in A$.*

Uniqueness follows from the previous statement.

Sketch of proof. Can we patch it up from simpler pieces?i.e.

Claim: Assume true for A, B, and $A \cap B$. To show that it is true for $A \cup B$.

For $s \in \Gamma_{\mathbb{Z}}(M)$, there exists unique $\alpha_A \in H_n(M, M \setminus A)$, $\alpha_B \in H_n(M, M \setminus B)$, and $\alpha_{A\cap B} \in H_n(M, M \setminus (A \cap B))$ so that $\alpha_{Ax} = \alpha_{Bx} = \alpha_{A\cap Bx} = s(x)$

By commutativity of the appropriate diagram, $(j_{A*}(\alpha_A))_x = \alpha_{A\cap Bx} \forall x \in A$, so by the uniqueness assumption $j_{A*}(\alpha_A) = \alpha_{A\cap B}$. Similarly, $j_{B*}(\alpha_B) = \alpha_{A\cap B}$. Therefore, the map,

$$H_n(M, M \setminus A) \oplus H_n(M, M \setminus B) \xrightarrow{\Psi} H_n(M, M \setminus (A \cap B))$$

, $\Psi(\alpha, \beta) = j_{A*}(\alpha) - j_{B*}(\beta)$ sends (α_A, α_B) to 0. By Mayer Vietoris, there is $\alpha_{A\cup B}$ such that $(i_{A*}(\alpha_{A\cup B}), i_{A*}(\alpha_{A\cup B})) = (\alpha_A, \alpha_B)$

By commutativity of the appropriate diagram, if $x \in A$, $\square$

Question:
A compact. How is $\alpha \in H_n(M, M \setminus A)$ characterized by $\alpha_x \in H_n(M, M \setminus x)$ for $x \in A$?

Already seen: $\alpha_x = 0$ for all $x \in A$ is equivalent to $\alpha = 0$

Theorem. *Given any $s \in \Gamma_{\mathbb{Z}}(M)$, there is a* ***unique*** *$\alpha \in H_n(M, M \setminus A)$ such that $\alpha_x = s(x)$ for all $x \in A$.*

Uniqueness follows from the previous statement.

Sketch of proof. Can we patch it up from simpler pieces?i.e.

Claim: Assume true for A, B, and $A \cap B$. To show that it is true for $A \cup B$.

For $s \in \Gamma_{\mathbb{Z}}(M)$, there exists unique $\alpha_A \in H_n(M, M \setminus A)$, $\alpha_B \in H_n(M, M \setminus B)$, and $\alpha_{A\cap B} \in H_n(M, M \setminus (A \cap B))$ so that $\alpha_{A_x} = \alpha_{B_x} = \alpha_{A\cap B_x} = s(x)$

By commutativity of the appropriate diagram, $(j_{A*}(\alpha_A))_x = \alpha_{A\cap B_x} \forall x \in A$, so by the uniqueness assumption $j_{A*}(\alpha_A) = \alpha_{A\cap B}$. Similarly, $j_{B*}(\alpha_B) = \alpha_{A\cap B}$. Therefore, the map,

$$H_n(M, M \setminus A) \oplus H_n(M, M \setminus B) \xrightarrow{\Psi} H_n(M, M \setminus (A \cap B))$$

, $\Psi(\alpha, \beta) = j_{A*}(\alpha) - j_{B*}(\beta)$ sends (α_A, α_B) to 0. By Mayer Vietoris, there is $\alpha_{A\cup B}$ such that $(i_{A*}(\alpha_{A\cup B}), i_{A*}(\alpha_{A\cup B})) = (\alpha_A, \alpha_B)$

By commutativity of the appropriate diagram, if $x \in A$, then $(\alpha_{A\cup B})_x = (i_{A*}(\alpha_{A\cup B}))_x$ □

Question:
A compact. How is $\alpha \in H_n(M, M \setminus A)$ characterized by $\alpha_x \in H_n(M, M \setminus x)$ for $x \in A$?

Already seen: $\alpha_x = 0$ for all $x \in A$ is equivalent to $\alpha = 0$

Theorem. *Given any $s \in \Gamma_{\mathbb{Z}}(M)$, there is a* ***unique*** *$\alpha \in H_n(M, M \setminus A)$ such that $\alpha_x = s(x)$ for all $x \in A$.*

Uniqueness follows from the previous statement.

Sketch of proof. Can we patch it up from simpler pieces?i.e.

Claim: Assume true for A, B, and $A \cap B$. To show that it is true for $A \cup B$.

For $s \in \Gamma_{\mathbb{Z}}(M)$, there exists unique $\alpha_A \in H_n(M, M \setminus A)$, $\alpha_B \in H_n(M, M \setminus B)$, and $\alpha_{A\cap B} \in H_n(M, M \setminus (A \cap B))$ so that $\alpha_{Ax} = \alpha_{Bx} = \alpha_{A\cap Bx} = s(x)$

By commutativity of the appropriate diagram, $(j_{A*}(\alpha_A))_x = \alpha_{A\cap Bx} \forall x \in A$, so by the uniqueness assumption $j_{A*}(\alpha_A) = \alpha_{A\cap B}$. Similarly, $j_{B*}(\alpha_B) = \alpha_{A\cap B}$. Therefore, the map,

$$H_n(M, M\setminus A) \oplus H_n(M, M \setminus B) \xrightarrow{\Psi} H_n(M, M \setminus (A \cap B))$$

, $\Psi(\alpha, \beta) = j_{A*}(\alpha) - j_{B*}(\beta)$ sends (α_A, α_B) to 0. By Mayer Vietoris, there is $\alpha_{A\cup B}$ such that $(i_{A*}(\alpha_{A\cup B}), i_{A*}(\alpha_{A\cup B})) = (\alpha_A, \alpha_B)$

By commutativity of the appropriate diagram, if $x \in A$, then $(\alpha_{A\cup B})_x = (i_{A*}(\alpha_{A\cup B}))_x = (\alpha_A)_x$ □

Question:
A compact. How is $\alpha \in H_n(M, M \setminus A)$ characterized by $\alpha_x \in H_n(M, M \setminus x)$ for $x \in A$?

Already seen: $\alpha_x = 0$ for all $x \in A$ is equivalent to $\alpha = 0$

Theorem. *Given any $s \in \Gamma_{\mathbb{Z}}(M)$, there is a* ***unique*** *$\alpha \in H_n(M, M \setminus A)$ such that $\alpha_x = s(x)$ for all $x \in A$.*

Uniqueness follows from the previous statement.

Sketch of proof. Can we patch it up from simpler pieces?i.e.

Claim: Assume true for A, B, and $A \cap B$. To show that it is true for $A \cup B$.

For $s \in \Gamma_{\mathbb{Z}}(M)$, there exists unique $\alpha_A \in H_n(M, M \setminus A)$, $\alpha_B \in H_n(M, M \setminus B)$, and $\alpha_{A\cap B} \in H_n(M, M \setminus (A \cap B))$ so that $\alpha_{Ax} = \alpha_{Bx} = \alpha_{A\cap Bx} = s(x)$

By commutativity of the appropriate diagram, $(j_{A*}(\alpha_A))_x = \alpha_{A\cap Bx} \forall x \in A$, so by the uniqueness assumption $j_{A*}(\alpha_A) = \alpha_{A\cap B}$. Similarly, $j_{B*}(\alpha_B) = \alpha_{A\cap B}$. Therefore, the map,

$$H_n(M, M\setminus A) \oplus H_n(M, M\setminus B) \xrightarrow{\Psi} H_n(M, M\setminus (A\cap B))$$

, $\Psi(\alpha, \beta) = j_{A*}(\alpha) - j_{B*}(\beta)$ sends (α_A, α_B) to 0. By Mayer Vietoris, there is $\alpha_{A\cup B}$ such that $(i_{A*}(\alpha_{A\cup B}), i_{A*}(\alpha_{A\cup B})) = (\alpha_A, \alpha_B)$

By commutativity of the appropriate diagram, if $x \in A$, then $(\alpha_{A\cup B})_x = (i_{A*}(\alpha_{A\cup B}))_x = (\alpha_A)_x = s(x)$. $\square$

Question:
A compact. How is $\alpha \in H_n(M, M \setminus A)$ characterized by $\alpha_x \in H_n(M, M \setminus x)$ for $x \in A$?

Already seen: $\alpha_x = 0$ for all $x \in A$ is equivalent to $\alpha = 0$

Theorem. *Given any $s \in \Gamma_{\mathbb{Z}}(M)$, there is a* ***unique*** *$\alpha \in H_n(M, M \setminus A)$ such that $\alpha_x = s(x)$ for all $x \in A$.*

Uniqueness follows from the previous statement.

Sketch of proof. Can we patch it up from simpler pieces?i.e.

Claim: Assume true for A, B, and $A \cap B$. To show that it is true for $A \cup B$.

For $s \in \Gamma_{\mathbb{Z}}(M)$, there exists unique $\alpha_A \in H_n(M, M \setminus A)$, $\alpha_B \in H_n(M, M \setminus B)$, and $\alpha_{A\cap B} \in H_n(M, M \setminus (A \cap B))$ so that $\alpha_{A_x} = \alpha_{B_x} = \alpha_{A\cap B_x} = s(x)$

By commutativity of the appropriate diagram, $(j_{A*}(\alpha_A))_x = \alpha_{A\cap B_x} \forall x \in A$, so by the uniqueness assumption $j_{A*}(\alpha_A) = \alpha_{A\cap B}$. Similarly, $j_{B*}(\alpha_B) = \alpha_{A\cap B}$. Therefore, the map,

$$H_n(M, M \setminus A) \oplus H_n(M, M \setminus B) \xrightarrow{\Psi} H_n(M, M \setminus (A \cap B))$$

, $\Psi(\alpha, \beta) = j_{A*}(\alpha) - j_{B*}(\beta)$ sends (α_A, α_B) to 0. By Mayer Vietoris, there is $\alpha_{A\cup B}$ such that $(i_{A*}(\alpha_{A\cup B}), i_{A*}(\alpha_{A\cup B})) = (\alpha_A, \alpha_B)$

By commutativity of the appropriate diagram, if $x \in A$, then $(\alpha_{A\cup B})_x = (i_{A*}(\alpha_{A\cup B}))_x = (\alpha_A)_x = s(x)$. Similarly for B. $\square$

Question:
A compact. How is $\alpha \in H_n(M, M \setminus A)$ characterized by $\alpha_x \in H_n(M, M \setminus x)$ for $x \in A$?

Already seen: $\alpha_x = 0$ for all $x \in A$ is equivalent to $\alpha = 0$

Theorem. *Given any $s \in \Gamma_{\mathbb{Z}}(M)$, there is a **unique** $\alpha \in H_n(M, M \setminus A)$ such that $\alpha_x = s(x)$ for all $x \in A$.*

Uniqueness follows from the previous statement.

Sketch of proof. Can we patch it up from simpler pieces?i.e.

Claim: Assume true for A, B, and $A \cap B$. To show that it is true for $A \cup B$.

For $s \in \Gamma_{\mathbb{Z}}(M)$, there exists unique $\alpha_A \in H_n(M, M \setminus A)$, $\alpha_B \in H_n(M, M \setminus B)$, and $\alpha_{A\cap B} \in H_n(M, M \setminus (A \cap B))$ so that $\alpha_{A_x} = \alpha_{B_x} = \alpha_{A\cap B_x} = s(x)$

By commutativity of the appropriate diagram, $(j_{A*}(\alpha_A))_x = \alpha_{A\cap B_x} \forall x \in A$, so by the uniqueness assumption $j_{A*}(\alpha_A) = \alpha_{A\cap B}$. Similarly, $j_{B*}(\alpha_B) = \alpha_{A\cap B}$. Therefore, the map,

$$H_n(M, M\setminus A) \oplus H_n(M, M\setminus B) \xrightarrow{\Psi} H_n(M, M\setminus(A\cap B))$$

, $\Psi(\alpha, \beta) = j_{A*}(\alpha) - j_{B*}(\beta)$ sends (α_A, α_B) to 0. By Mayer Vietoris, there is $\alpha_{A\cup B}$ such that $(i_{A*}(\alpha_{A\cup B}), i_{A*}(\alpha_{A\cup B})) = (\alpha_A, \alpha_B)$

By commutativity of the appropriate diagram, if $x \in A$, then $(\alpha_{A\cup B})_x = (i_{A*}(\alpha_{A\cup B}))_x = (\alpha_A)_x = s(x)$. Similarly for B. This proves the claim. $\square$

Question:
A compact. How is $\alpha \in H_n(M, M \setminus A)$ characterized by $\alpha_x \in H_n(M, M \setminus x)$ for $x \in A$?

Already seen: $\alpha_x = 0$ for all $x \in A$ is equivalent to $\alpha = 0$

Theorem. *Given any $s \in \Gamma_{\mathbb{Z}}(M)$, there is a* ***unique*** *$\alpha \in H_n(M, M \setminus A)$ such that $\alpha_x = s(x)$ for all $x \in A$.*

Uniqueness follows from the previous statement.

Sketch of proof. Can we patch it up from simpler pieces?i.e.

Claim: Assume true for A, B, and $A \cap B$. To show that it is true for $A \cup B$.

For $s \in \Gamma_{\mathbb{Z}}(M)$, there exists unique $\alpha_A \in H_n(M, M \setminus A)$, $\alpha_B \in H_n(M, M \setminus B)$, and $\alpha_{A\cap B} \in H_n(M, M \setminus (A \cap B))$ so that $\alpha_{A x} = \alpha_{B x} = \alpha_{A \cap B x} = s(x)$

By commutativity of the appropriate diagram, $(j_{A*}(\alpha_A))_x = \alpha_{A\cap B x} \forall x \in A$, so by the uniqueness assumption $j_{A*}(\alpha_A) = \alpha_{A \cap B}$. Similarly, $j_{B*}(\alpha_B) = \alpha_{A\cap B}$. Therefore, the map,

$$H_n(M, M \setminus A) \oplus H_n(M, M \setminus B) \xrightarrow{\Psi} H_n(M, M \setminus (A \cap B))$$

, $\Psi(\alpha, \beta) = j_{A*}(\alpha) - j_{B*}(\beta)$ sends (α_A, α_B) to 0. By Mayer Vietoris, there is $\alpha_{A\cup B}$ such that $(i_{A*}(\alpha_{A\cup B}), i_{A*}(\alpha_{A\cup B})) = (\alpha_A, \alpha_B)$

By commutativity of the appropriate diagram, if $x \in A$, then $(\alpha_{A\cup B})_x = (i_{A*}(\alpha_{A\cup B}))_x = (\alpha_A)_x = s(x)$. Similarly for B. This proves the claim.

$\square$

Rest of the steps as before:

Rest of the steps as before:

1. Realize a compact set as union of compact sets in coordinate neighbourhoods

Rest of the steps as before:

1. Realize a compact set as union of compact sets in coordinate neighbourhoods and use the above to reduce to the case where A is in a coordinate neighbourhood.

Rest of the steps as before:

1. Realize a compact set as union of compact sets in coordinate neighbourhoods and use the above to reduce to the case where A is in a coordinate neighbourhood.

2. Use excision to reduce to $H_n(\mathbb{R}^n, \mathbb{R}^n \setminus A)$.

Rest of the steps as before:

1. Realize a compact set as union of compact sets in coordinate neighbourhoods and use the above to reduce to the case where A is in a coordinate neighbourhood.

2. Use excision to reduce to $H_n(\mathbb{R}^n, \mathbb{R}^n \setminus A)$.

3. Can replace A by *any* “sane” bigger compact set

Rest of the steps as before:

1. Realize a compact set as union of compact sets in coordinate neighbourhoods and use the above to reduce to the case where A is in a coordinate neighbourhood.

2. Use excision to reduce to $H_n(\mathbb{R}^n, \mathbb{R}^n \setminus A)$.

3. Can replace A by *any* "sane" bigger compact set , eg. closed ball B, such that $A \subset B$ (exercise!).

Rest of the steps as before:

1. Realize a compact set as union of compact sets in coordinate neighbourhoods and use the above to reduce to the case where A is in a coordinate neighbourhood.

2. Use excision to reduce to $H_n(\mathbb{R}^n, \mathbb{R}^n \setminus A)$.

3. Can replace A by *any* "sane" bigger compact set , eg. closed ball B, such that $A \subset B$ (exercise!). The theorem is true if A is a closed ball (exercise!

Rest of the steps as before:

1. Realize a compact set as union of compact sets in coordinate neighbourhoods and use the above to reduce to the case where A is in a coordinate neighbourhood.

2. Use excision to reduce to $H_n(\mathbb{R}^n, \mathbb{R}^n \setminus A)$.

3. Can replace A by *any* "sane" bigger compact set , eg. closed ball B, such that $A \subset B$ (exercise!). The theorem is true if A is a closed ball (exercise! Use the observation on local consistency described at the beginning)

We have proved:

Theorem. *There is a covering* $\pi : M_{\mathbb{Z}} \to M$ *such that (Here,* $n :=$ *dim* M*)*

1. $H_n(M, M \setminus p) \cong \mathbb{Z}$ *is the fibre at* p.
2. $M_{\mathbb{Z}}$ *is a disjoint union of a copy of* M *and copies of* $\widetilde{M}$
3. *Any continuous section* $s : M \to M_{\mathbb{Z}}$ *is an integer multiple of a section* $s_1 : M \to \widetilde{M}$ *defining an orientation.*

Theorem. *Given any* $s \in \Gamma_{\mathbb{Z}}(M)$, *there is a* ***unique*** $\alpha \in H_n(M, M \setminus A)$ *such that* $\alpha_x = s(x)$ *for all* $x \in A$.

We have proved:

Theorem. *There is a covering* $\pi : M_{\mathbb{Z}} \to M$ *such that (Here,* $n :=$ *dim* M*)*

1. $H_n(M, M \setminus p) \cong \mathbb{Z}$ *is the fibre at* p.
2. $M_{\mathbb{Z}}$ *is a disjoint union of a copy of* M *and copies of* $\widetilde{M}$
3. *Any continuous section* $s : M \to M_{\mathbb{Z}}$ *is an integer multiple of a section* $s_1 : M \to \widetilde{M}$ *defining an orientation.*

Theorem. *Given any* $s \in \Gamma_{\mathbb{Z}}(M)$, *there is a* ***unique*** $\alpha \in H_n(M, M \setminus A)$ *such that* $\alpha_x = s(x)$ *for all* $x \in A$.

Corollary. *M compact,*

We have proved:

Theorem. *There is a covering $\pi : M_{\mathbb{Z}} \to M$ such that (Here, $n :=$ dim M)*

1. $H_n(M, M \setminus p) \cong \mathbb{Z}$ *is the fibre at* p.
2. $M_{\mathbb{Z}}$ *is a disjoint union of a copy of* M *and copies of* $\widetilde{M}$
3. *Any continuous section* $s : M \to M_{\mathbb{Z}}$ *is an integer multiple of a section* $s_1 : M \to \widetilde{M}$ *defining an orientation.*

Theorem. *Given any $s \in \Gamma_{\mathbb{Z}}(M)$, there is a **unique** $\alpha \in H_n(M, M \setminus A)$ such that $\alpha_x = s(x)$ for all $x \in A$.*

Corollary. *M compact, connected manifold,*

We have proved:

Theorem. *There is a covering $\pi : M_{\mathbb{Z}} \to M$ such that (Here, $n :=$ dim M)*

1. *$H_n(M, M \setminus p) \cong \mathbb{Z}$ is the fibre at p.*
2. *$M_{\mathbb{Z}}$ is a disjoint union of a copy of M and copies of $\widetilde{M}$*
3. *Any continuous section $s : M \to M_{\mathbb{Z}}$ is an integer multiple of a section $s_1 : M \to \widetilde{M}$ defining an orientation.*

Theorem. *Given any $s \in \Gamma_{\mathbb{Z}}(M)$, there is a* ***unique*** *$\alpha \in H_n(M, M \setminus A)$ such that $\alpha_x = s(x)$ for all $x \in A$.*

Corollary. *M compact, connected manifold, without boundary,*

We have proved:

Theorem. *There is a covering* $\pi : M_{\mathbb{Z}} \to M$ *such that (Here,* $n :=$ *dim* M*)*

1. $H_n(M, M \setminus p) \cong \mathbb{Z}$ *is the fibre at* p.
2. $M_{\mathbb{Z}}$ *is a disjoint union of a copy of* M *and copies of* $\widetilde{M}$
3. *Any continuous section* $s : M \to M_{\mathbb{Z}}$ *is an integer multiple of a section* $s_1 : M \to \widetilde{M}$ *defining an orientation.*

Theorem. *Given any* $s \in \Gamma_{\mathbb{Z}}(M)$*, there is a* ***unique*** $\alpha \in H_n(M, M \setminus A)$ *such that* $\alpha_x = s(x)$ *for all* $x \in A$.

Corollary. M *compact, connected manifold, without boundary, then*

We have proved:

Theorem. *There is a covering $\pi : M_{\mathbb{Z}} \to M$ such that (Here, $n :=$ dim M)*

1. $H_n(M, M \setminus p) \cong \mathbb{Z}$ *is the fibre at* p.
2. $M_{\mathbb{Z}}$ *is a disjoint union of a copy of* M *and copies of* $\widetilde{M}$
3. *Any continuous section* $s : M \to M_{\mathbb{Z}}$ *is an integer multiple of a section* $s_1 : M \to \widetilde{M}$ *defining an orientation.*

Theorem. *Given any* $s \in \Gamma_{\mathbb{Z}}(M)$, *there is a* ***unique*** $\alpha \in H_n(M, M \setminus A)$ *such that* $\alpha_x = s(x)$ *for all* $x \in A$.

Corollary. *M compact, connected manifold, without boundary, then*

$$H_n(M) = \begin{cases} \mathbb{Z} & M \text{ is orientable} \end{cases}$$

We have proved:

Theorem. *There is a covering $\pi : M_{\mathbb{Z}} \to M$ such that (Here, $n :=$ dim M)*

1. *$H_n(M, M \setminus p) \cong \mathbb{Z}$ is the fibre at p.*
2. *$M_{\mathbb{Z}}$ is a disjoint union of a copy of M and copies of $\widetilde{M}$*
3. *Any continuous section $s : M \to M_{\mathbb{Z}}$ is an integer multiple of a section $s_1 : M \to \widetilde{M}$ defining an orientation.*

Theorem. *Given any $s \in \Gamma_{\mathbb{Z}}(M)$, there is a* ***unique*** *$\alpha \in H_n(M, M \setminus A)$ such that $\alpha_x = s(x)$ for all $x \in A$.*

Corollary. *M compact, connected manifold, without boundary, then*

$$H_n(M) = \begin{cases} \mathbb{Z} & M \text{ is orientable} \\ 0 & M \text{ is not orientable} \end{cases}$$

We have proved:

Theorem. *There is a covering $\pi : M_{\mathbb{Z}} \to M$ such that (Here, $n :=$ dim M)*

1. *$H_n(M, M \setminus p) \cong \mathbb{Z}$ is the fibre at p.*

2. *$M_{\mathbb{Z}}$ is a disjoint union of a copy of M and copies of $\widetilde{M}$*

3. *Any continuous section $s : M \to M_{\mathbb{Z}}$ is an integer multiple of a section $s_1 : M \to \widetilde{M}$ defining an orientation.*

Theorem. *Given any $s \in \Gamma_{\mathbb{Z}}(M)$, there is a **unique** $\alpha \in H_n(M, M \setminus A)$ such that $\alpha_x = s(x)$ for all $x \in A$.*

Corollary. *M compact, connected manifold, without boundary, then*

$$H_n(M) = \begin{cases} \mathbb{Z} & M \text{ is orientable} \\ 0 & M \text{ is not orientable} \end{cases}$$

Proof. Taking $A = M$ above, $\square$

We have proved:

Theorem. *There is a covering $\pi : M_{\mathbb{Z}} \to M$ such that (Here, $n := \dim M$)*

1. $H_n(M, M \setminus p) \cong \mathbb{Z}$ *is the fibre at* p.
2. $M_{\mathbb{Z}}$ *is a disjoint union of a copy of* M *and copies of* $\widetilde{M}$
3. *Any continuous section* $s : M \to M_{\mathbb{Z}}$ *is an integer multiple of a section* $s_1 : M \to \widetilde{M}$ *defining an orientation.*

Theorem. *Given any* $s \in \Gamma_{\mathbb{Z}}(M)$, *there is a* ***unique*** $\alpha \in H_n(M, M \setminus A)$ *such that* $\alpha_x = s(x)$ *for all* $x \in A$.

Corollary. *M compact, connected manifold, without boundary, then*

$$H_n(M) = \begin{cases} \mathbb{Z} & M \text{ is orientable} \\ 0 & M \text{ is not orientable} \end{cases}$$

Proof. Taking $A = M$ above, the map $H_n(M) \to \Gamma_{\mathbb{Z}}(M)$ defined by $\alpha \to \alpha_x$ is an isomorphism. $\square$

We have proved:

Theorem. *There is a covering $\pi : M_{\mathbb{Z}} \to M$ such that (Here, $n := \dim M$)*

1. *$H_n(M, M \setminus p) \cong \mathbb{Z}$ is the fibre at p.*
2. *$M_{\mathbb{Z}}$ is a disjoint union of a copy of M and copies of $\widetilde{M}$*
3. *Any continuous section $s : M \to M_{\mathbb{Z}}$ is an integer multiple of a section $s_1 : M \to \widetilde{M}$ defining an orientation.*

Theorem. *Given any $s \in \Gamma_{\mathbb{Z}}(M)$, there is a **unique** $\alpha \in H_n(M, M \setminus A)$ such that $\alpha_x = s(x)$ for all $x \in A$.*

Corollary. *M compact, connected manifold, without boundary, then*

$$H_n(M) = \begin{cases} \mathbb{Z} & M \text{ is orientable} \\ 0 & M \text{ is not orientable} \end{cases}$$

Proof. Taking $A = M$ above, the map $H_n(M) \to \Gamma_{\mathbb{Z}}(M)$ defined by $\alpha \to \alpha_x$ is an isomorphism. Corollary follows from,

$$\Gamma_{\mathbb{Z}}(M) = \begin{cases} \mathbb{Z} & M \text{ is orientable} \end{cases}$$

□

We have proved:

Theorem. *There is a covering $\pi : M_{\mathbb{Z}} \to M$ such that (Here, $n := \dim M$)*

1. $H_n(M, M \setminus p) \cong \mathbb{Z}$ *is the fibre at* p.
2. $M_{\mathbb{Z}}$ *is a disjoint union of a copy of* M *and copies of* $\widetilde{M}$
3. *Any continuous section* $s : M \to M_{\mathbb{Z}}$ *is an integer multiple of a section* $s_1 : M \to \widetilde{M}$ *defining an orientation.*

Theorem. *Given any* $s \in \Gamma_{\mathbb{Z}}(M)$, *there is a* ***unique*** $\alpha \in H_n(M, M \setminus A)$ *such that* $\alpha_x = s(x)$ *for all* $x \in A$.

Corollary. *M compact, connected manifold, without boundary, then*

$$H_n(M) = \begin{cases} \mathbb{Z} & M \textit{ is orientable} \\ 0 & M \textit{ is not orientable} \end{cases}$$

Proof. Taking $A = M$ above, the map $H_n(M) \to \Gamma_{\mathbb{Z}}(M)$ defined by $\alpha \to \alpha_x$ is an isomorphism. Corollary follows from,

$$\Gamma_{\mathbb{Z}}(M) = \begin{cases} \mathbb{Z} & M \text{ is orientable} \\ 0 & M \text{ is not orientable} \end{cases}$$

□

We have proved:

Theorem. *There is a covering $\pi : M_{\mathbb{Z}} \to M$ such that (Here, $n := \dim M$)*

1. $H_n(M, M \setminus p) \cong \mathbb{Z}$ *is the fibre at p.*
2. $M_{\mathbb{Z}}$ *is a disjoint union of a copy of M and copies of $\widetilde{M}$*
3. *Any continuous section $s : M \to M_{\mathbb{Z}}$ is an integer multiple of a section $s_1 : M \to \widetilde{M}$ defining an orientation.*

Theorem. *Given any $s \in \Gamma_{\mathbb{Z}}(M)$, there is a* ***unique*** *$\alpha \in H_n(M, M \setminus A)$ such that $\alpha_x = s(x)$ for all $x \in A$.*

Corollary. *M compact, connected manifold, without boundary, then*

$$H_n(M) = \begin{cases} \mathbb{Z} & M \textit{ is orientable} \\ 0 & M \textit{ is not orientable} \end{cases}$$

Proof. Taking $A = M$ above, the map $H_n(M) \to \Gamma_{\mathbb{Z}}(M)$ defined by $\alpha \to \alpha_x$ is an isomorphism. Corollary follows from,

$$\Gamma_{\mathbb{Z}}(M) = \begin{cases} \mathbb{Z} & M \text{ is orientable} \\ 0 & M \text{ is not orientable} \end{cases}$$

□