Exercise sheet 4

1. The Jordan curve theorem says that if $f : S^1 \to S^2$ is a continuous map which is a homemorphism onto its image, then $S^2 \setminus f(S^1)$ has two components (equivalently, $\tilde{H}_0(S^2 \setminus f(S^1)) = \mathbb{Z}$).

   a) Prove that the Jordan curve theorem follows from the following, if $f : [0,1] \to S^2$ is a continuous map that is homeomorphic onto its image, then $\tilde{H}_0(S^2 \setminus f([0,1])) = 0$)
   b) Prove that $\tilde{H}_k(S^2 \setminus f([0,1])) = 0$

2. For pairs (X_α, x_α) where, for each α, X_α is a topological space and $x_\alpha \in X_\alpha$ is a chosen point, the wedge sum $\vee_\alpha X_\alpha := \sqcup_\alpha X_\alpha / \{x_\alpha\}$ is constructed by taking the disjoint union of the X_α and identifying all the base points x_α. If (X_α, x_α) are good pairs, and prove the following:

   a) The inclusion, $i_\alpha : X_\alpha \to \vee_\alpha X_\alpha$ (defined by composing the inclusion to the disjoint union with the quotient map) induces an injection $i_{\alpha *} : \widetilde{H}_n(X_\alpha) \to \widetilde{H}_n(\vee_\alpha X_\alpha)$.

   b. Prove that $H_n(\vee_\alpha X_\alpha) = \oplus_\alpha i_{\alpha *}(H_n(X_\alpha))$

3. Let $X_0 \subset X_1 \subset \dots X_n = X$ be a nested finite sequence of topological spaces. Assume that (X_{i+1}, X_i) form a good pair for all i and that the the reduced homologies, $\widetilde{H}_k(X_i/X_{i-1})$ are non-trivial only when $k = i$ *(such a situation occurs if X_k denotes the k-skeleton of a simplicial complex, or, as we shall see later in this course, of a CW-complex).*

   a) Prove that a k-dimensional homology class in $H_k(X)$ has a representative in $H_k(X_k)$ (i.e. $H_k(X_k) \xrightarrow{i_*} H_k(X)$ is surjective).
   b) Prove that $H_k(X) \cong H_k(X_{k+1})$ (i.e. the kth homology of X depends only on the homology of X_{k+1}).

4. For a finite subset, $A := \{x_1, x_2, \dots, x_n\} \subset \mathbb{R}^{N+1}$

   a) Prove that $H_N(\mathbb{R}^{N+1} \setminus A) \cong \oplus_{i=1}^n \mathbb{Z}$
   b) Find maps $f_i : S^N \to \mathbb{R}^{N+1} \setminus A$ so that f_{i*} is injective and maps a generator of $H_N(S^N) \cong \mathbb{Z}$ to a generator of the ith component of $H_N(\mathbb{R}^{N+1} \setminus A) \cong \oplus_{i=1}^n \mathbb{Z}$
   c) Use the f_i defined above to prove that the inclusion, $\mathbb{R}^{N+1} \setminus A \hookrightarrow \mathbb{R}^{N+1} \setminus x_j$ induces a map from $\oplus_{i=1}^n \mathbb{Z} \to \mathbb{Z}$ that is a projection onto

the ith copy of $\mathbb{Z}$.

5. Prove the following properties of the degree of a map $f : S^n \to S^n$:

   a) $\deg Id = 1$
   b) $\deg f = 0$ if f is not-surjective
   c) Homotopic maps have the same degrees
   d) Compute the degree of the antipodal map

6. Realize S^1 as the subspace $\{z \mid |z| = 1\} \subset \mathbb{C}$ and prove that the map $\theta : S^1 \to S^1$ defined as $\theta(z) = z^k$ has degree k.

7. Let A denote the complement of k disjoint open subsets of S^n that are each homeomorphic to open dics of dimension n.

   a) Show that S^n/A is homeomorphic to a wedge sum of k n-spheres.
   b) Prove that it is possible to glue the homeorphisms from each sphere in the wedge to S^n to define a surjection, $p : S^n/A \to S^n$.
   c) Define the map $f : S^n \to S^n$ as the composition, $S^n \xrightarrow{q} S^n/A \xrightarrow{p} S^n$. What is its degree *(Hint: use the local degree formulation)*?