Exercise sheet 3

1. **(for submission)** If $D_n : C_n(X) \to C_{n+1}$ are a family of homomorphisms, and $\rho := Id - D_{n-1} \circ \partial_n - \partial_{n+1} \circ D_n$, then prove that ρ is a chain map.
2. **(for submission)** Recall that for v_i in some convex set K, $[v_0, \dots, v_n]$ denotes the linear simplex $\lambda(\Sigma_i \alpha_i e_i) = \Sigma_i \alpha_i v_i$, and $\hat{b}([v_0, \dots, v_n]) := [b, v_0, \dots, v_n]$, where b is $\lambda(\frac{e_0 + \dots + e_n}{n+1})$. Prove that if a family of maps. is defined inductively as, $S_n(\lambda) := \hat{b}(S_{n-1}(\partial \lambda))$, where $S_0 = Id$, then $\lambda_\#(S_n(Id_{\Delta_n})) = S_n(\lambda)$.
3. Let $f_0, f_i : [0,1] \to X$ be two loops, i.e. continuous maps such that $f_i(0) = f_i(1)$. Therefore, both maps can be realized as singular 1-simplices in X, that also happen to be cycles (because $f_i(0) = f_i(1)$). Let $H : [0,1] \times [0,1] \to X$ define a homotopy between f_0 and f_1 (i.e. $H(s,0) = f_0(s)$ and $H(s,1) = f_1(s)$) such that $H(0,t)$ and $H(1,t)$ are constant, i.e. the end points of the loop are fixed throughout the homotopy. This is called a path homotopy between the two loops. Prove that $[f_0] = [f_1] \in H_1(X)$, i.e. both loops represent the same homology class.
4. Prove the equivalence of these two versions of the excision theorem:

a) If X is a topological space and $A, B \subset X$ such that $X = Int\ A \cup Int\ B$, then $k : (B, A \cap B) \hookrightarrow (X, A)$ induces an isomorphism $k_* : H_n(B, A \cap B) \to H_n(X, A)$

b) If X is a topological space, $A \subset X$, and $Z \subset A$ such that $\bar{Z} \subset Int\ A$ $i : (X \setminus Z, A \setminus Z) \hookrightarrow (X, A)$ induces, $i_* : H_n(X \setminus Z, A \setminus Z) \hookrightarrow H_n(X, A)$ is an isomorphism