MTH349 - Homological methods in Algebraic Topology

Quizzes / assignments / presentations	20%
Mid-sem	40%
Final	40%

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3. *Algebraic Topology: An introduction* by William S. Massey

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4. *An introduction to Algebraic Topology* by Joseph J. Rotman

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Why homology?

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$\pi_1(X)$ ~~abelian~~ group

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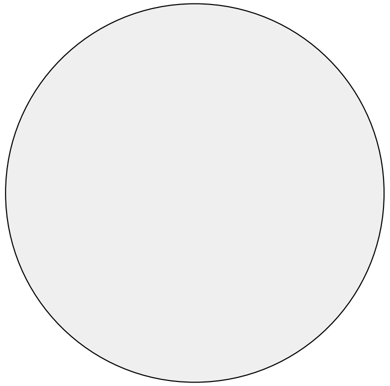

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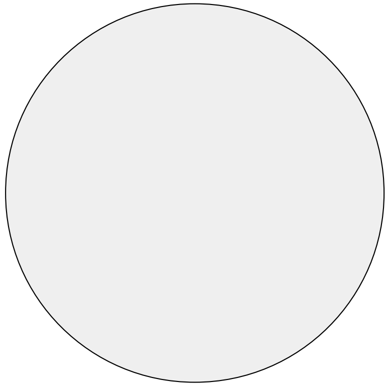

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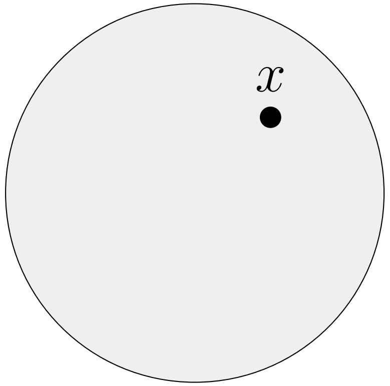

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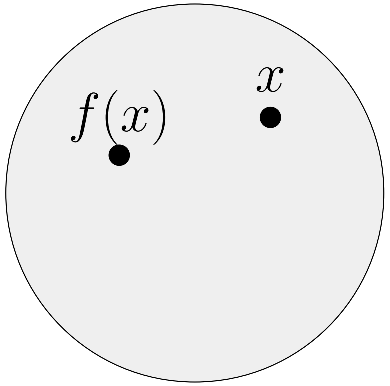

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2. $(\mathrm{id}_X)_* = \mathrm{id}_{\mathrm{H}_n(X)}$

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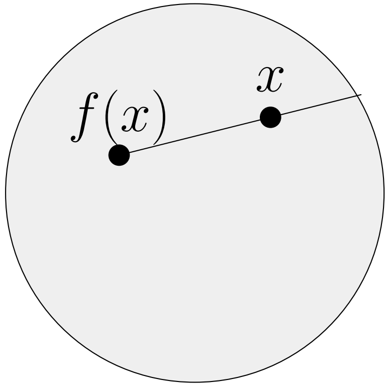
if $r(a) = a$ when $a \in A$.

Lemma. *$r : X \to A$ retract $\implies$ i_* is injective*

Fixed points

Given $f : B^2 \to B^2$, If f has no fixed point



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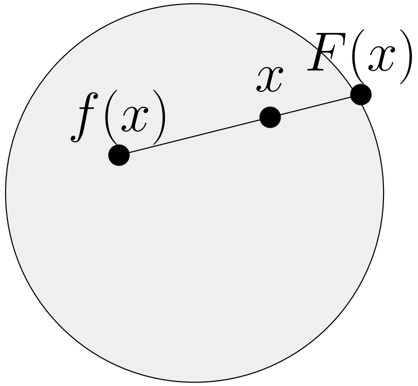

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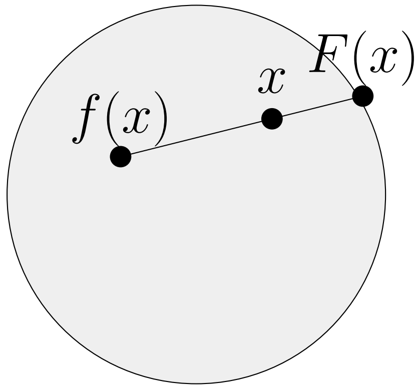

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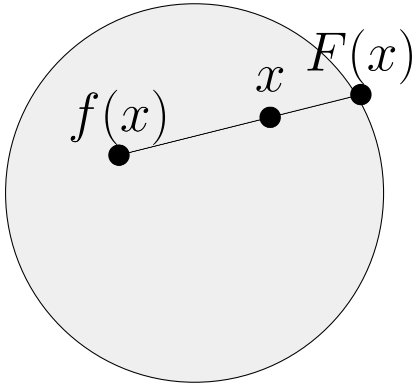

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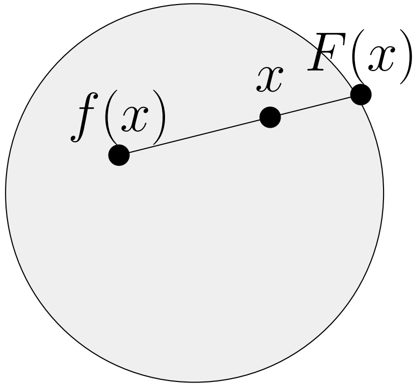

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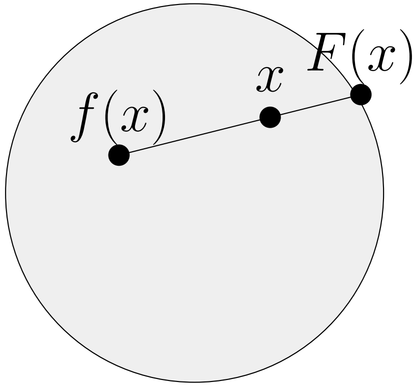

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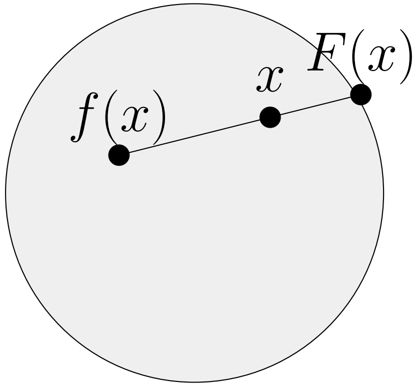

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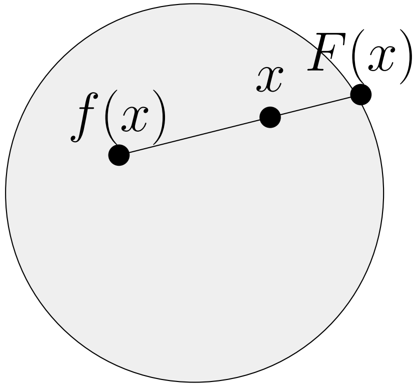

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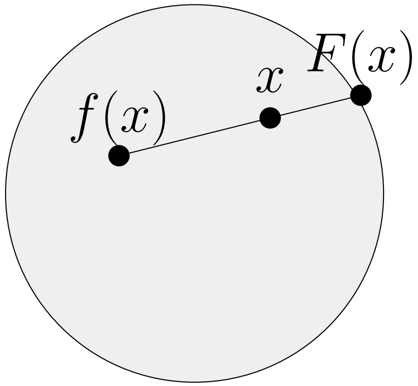
Fixed points

Theorem. *Any continuous $f : B^2 \to B^2$ has a fixed point.*



Proof: $F : B^2 \to \partial B^2$ is a retract
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Informal sketch of simplicial homology

Euler characteristic:

Informal sketch of simplicial homology

Euler characteristic: $V - E + F$

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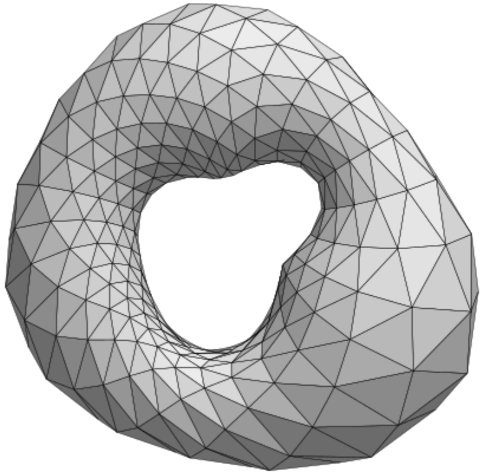

Simplex

Informal sketch of simplicial homology

Euler characteristic: $V - E + F$

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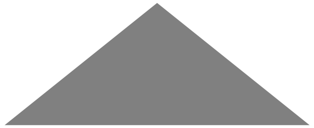

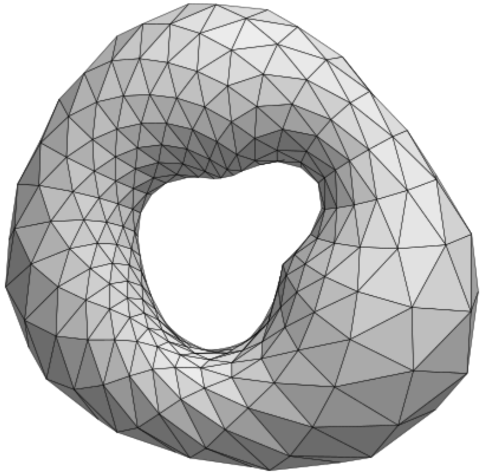

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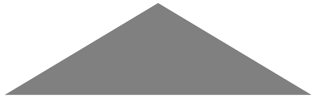

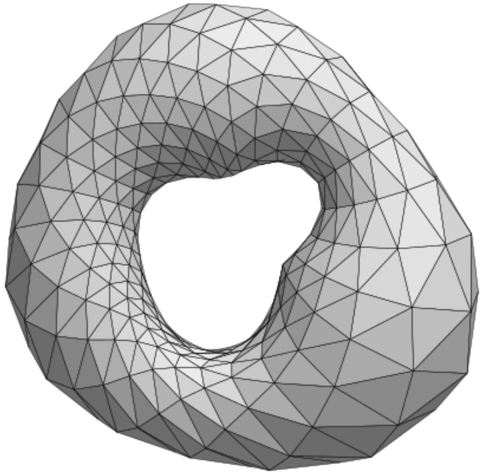

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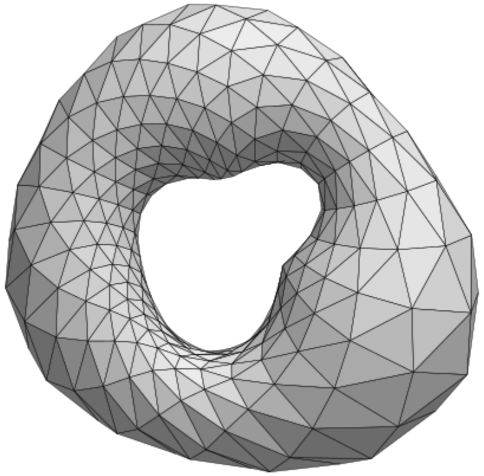

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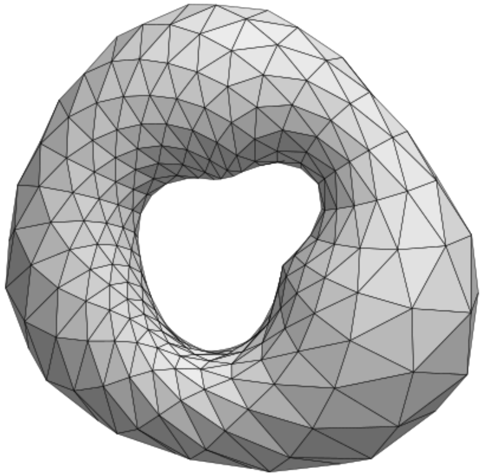

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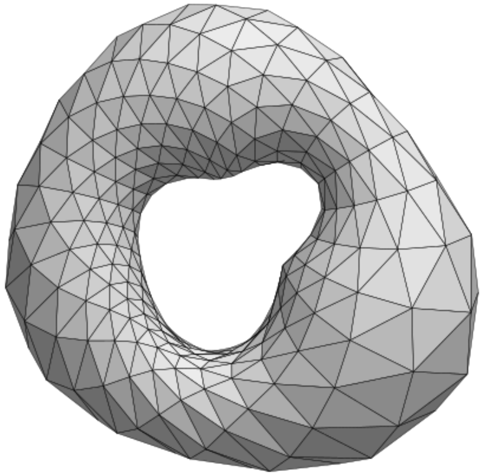

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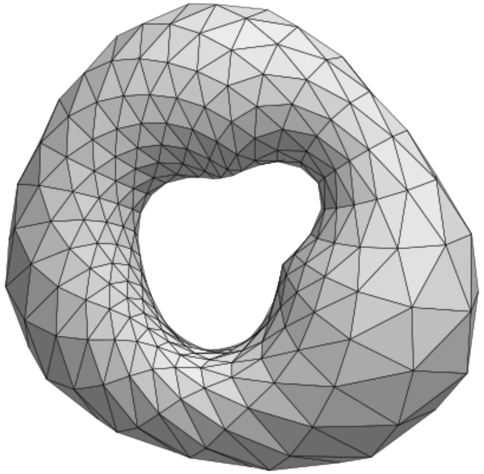

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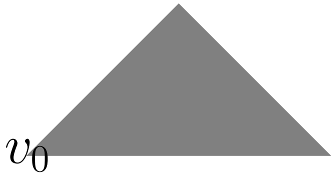

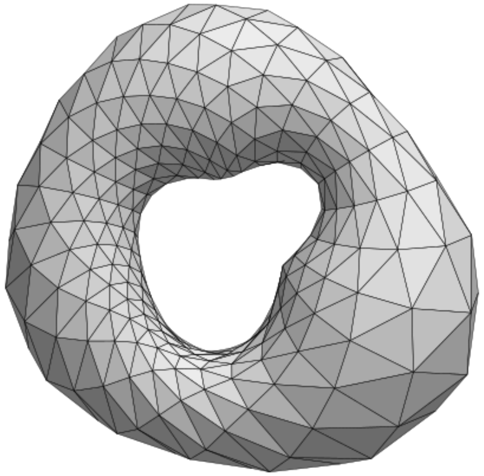

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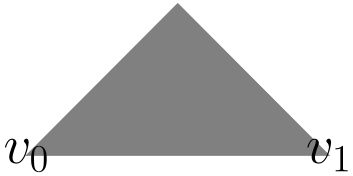

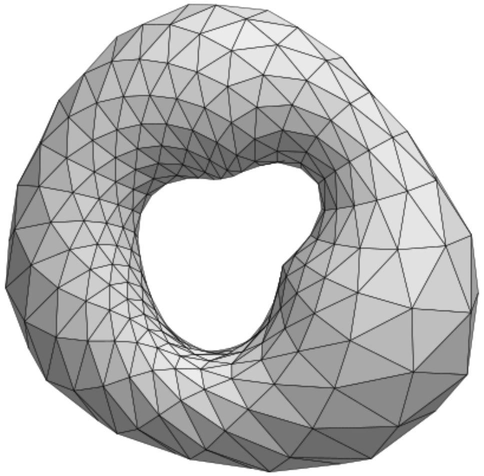

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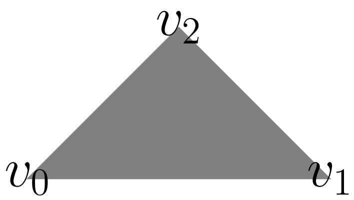

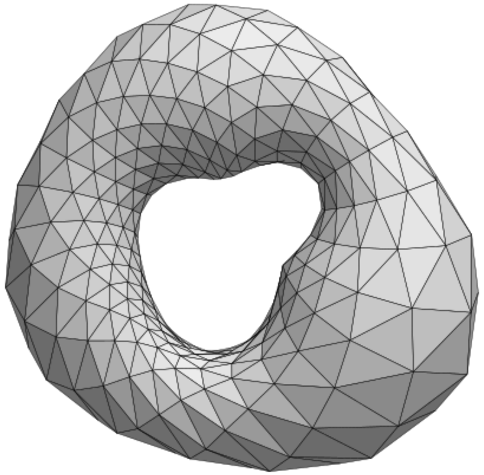

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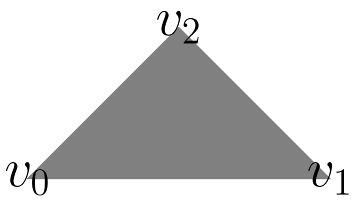
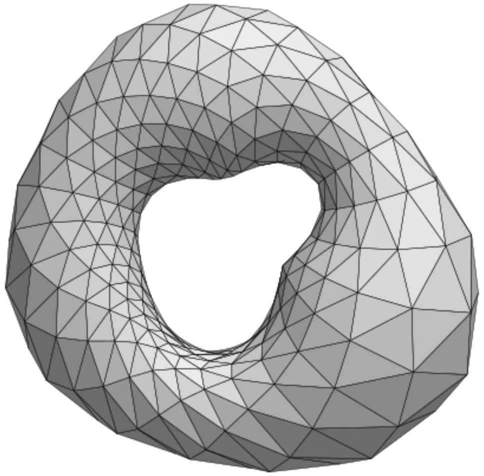


Definition. *Given "geometrically independent" points* $v_0,\, v_1,\, \ldots,\, v_n \in \mathbb{R}^N$,



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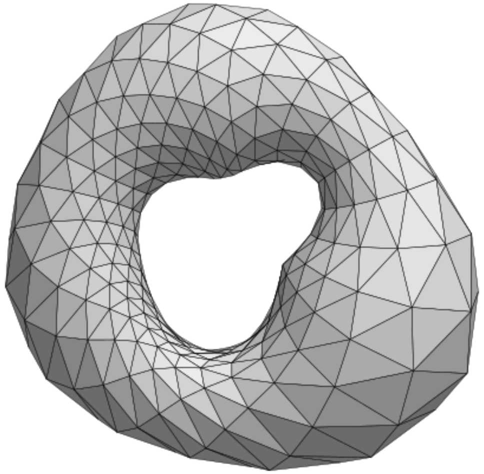

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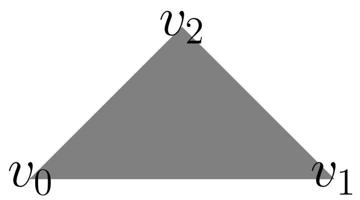

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Informal sketch of simplicial homology

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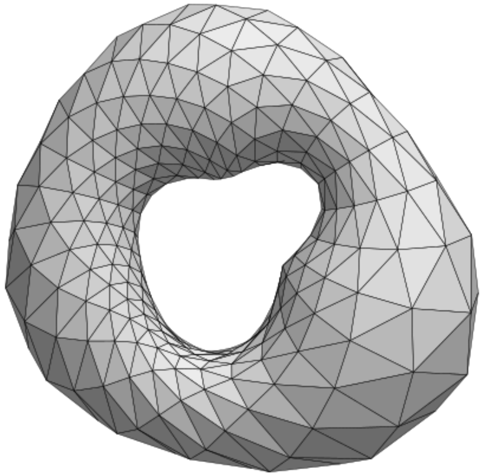

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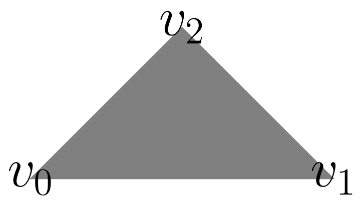

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Informal sketch of simplicial homology

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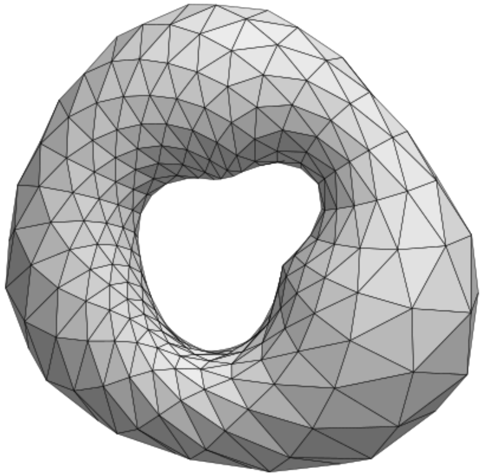

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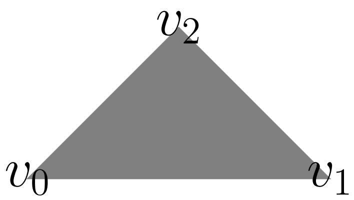

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Image credit:

Informal sketch of simplicial homology

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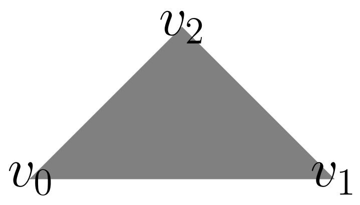

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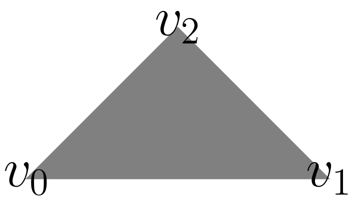
Simplicial complex

Simplicial complex K: collection of simplices such that:

Informal sketch of simplicial homology

Euler characteristic: $V - E + F$

Simplex



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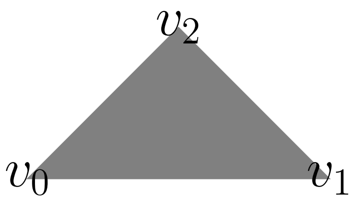

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Simplicial complex

Simplicial complex K: collection of simplices such that:

1. All faces of a simplex is in the collection
2. Intersection of two simplices is empty or a face

$C_n(K)$: formal integer linear combination of n-simplices of K

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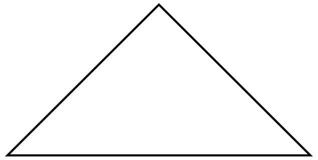
Example. *Circle:* 

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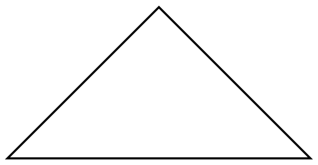

n_1

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$\partial_n[v_0, v_1, \ldots, v_n] = \Sigma_{i=0}^{n}(-1)^i[v_0, v_1, \ldots, \hat{v}_i, \ldots, v_n]$

Extend linearly, $\partial_n : C_n(X) \to C_{n-1}(X)$

$Z_n(X) :=$ ker ∂_n
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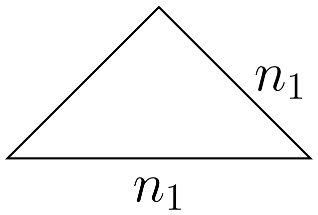

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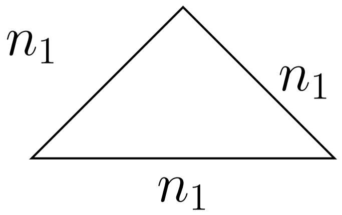

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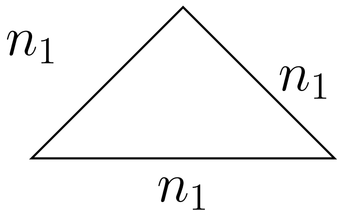

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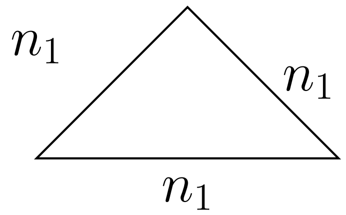

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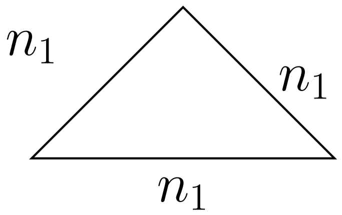

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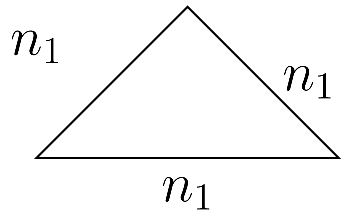

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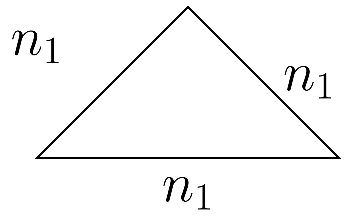

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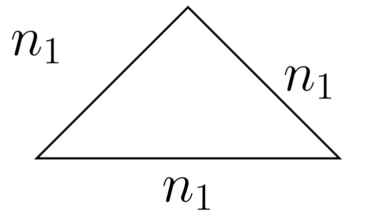

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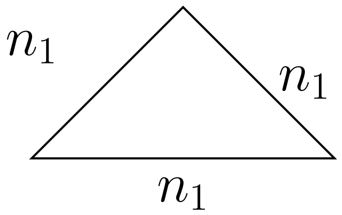

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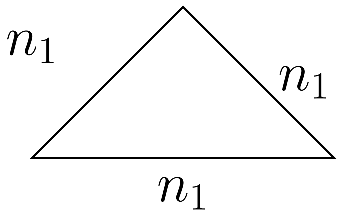

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(i.e. $\{\Sigma_i \lambda_i e_i \mid \Sigma_i \lambda = 1, 0 \leq \lambda_i \leq 1\} \subset \mathbb{R}^{n+1}$)

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span$\{e_0, \ldots, \hat{e}_i, \ldots, e^n\} \xrightarrow{\sigma} X$

Singular homology
Δ_n: standard n-simplex
(i.e. $\{\Sigma_i \lambda_i e_i \mid \Sigma_i \lambda = 1, 0 \leq \lambda_i \leq 1\} \subset \mathbb{R}^{n+1}$)
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(Set of 0-simplices of $X \leftrightarrow X$)

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$\sigma : [0,1] \simeq$ convex span$\{e_0, e_1\} \to X$
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Example 1: $H_n(pt)$

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Example 1: $H_n(pt)$

$C_n(pt) = \mathbb{Z}$

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$C_n(pt) = \mathbb{Z}$ (generated by constant functions)

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$C_n(pt) = \mathbb{Z}$ (generated by constant functions)

$\partial_n : C_n(pt) \to C_{n-1}(pt)$

is constant 0 if n is even, otherwise id.

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$\boxed{\partial_n \circ \partial_{n+1} = 0}$

$Z_n(X) := \ker \partial_n$

$B_n(X) := \mathrm{Im} \partial_{n+1}$

$B_n(X) \subset Z_n(X)$

$H_n(X) := \frac{Z_n(X)}{B_n(X)}$

Example 2: $H_0(X)$

Any path $\gamma : [0,1] \to X$ is a (singular) 1-simplex.

$\partial\gamma = \gamma(1) - \gamma(0)$

Therefore, $[\Sigma_i n_i \sigma_i] = [(\Sigma_i n_i)\sigma_0]$

$[n\sigma_0] = [0]$??, i.e. $n\sigma_0 = \partial c$??

$\epsilon : C_0 \to \mathbb{Z}$

$\epsilon(\Sigma_i n_i \sigma_i) = \Sigma_i n_i$

ϵ is a homomorphism

$\epsilon \circ \partial = 0$

$n\sigma_0 = \partial c \implies \epsilon(n\sigma_0) = \epsilon(\partial c) \implies n = 0$

$H_0(X) = \mathbb{Z}$ if X is path-connected

Singular homology

Δ_n: standard n-simplex

$\sigma : \Delta_n \to X$: singular simplex in X

$C_n(X) := \{\Sigma_i n_i \sigma_i \mid \sigma_i : \delta_n \to X, n_i \in \mathbb{Z}\}$

$\sigma^{(i)}$: ith face

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$H_0(X) = \mathbb{Z}$ if X is path-connected

$H_0(X) = \mathbb{Z}^{n+1}$ if X has n connected components

Singular homology

Δ_n: standard n-simplex

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$H_0(X) = \mathbb{Z}^{n+1}$ if X has n connected components (Exercise)

Maps

Maps

Chain level

$f : X \to Y$

Maps

$\boxed{X \to Y}$

Chain level

$f : X \to Y$

Maps

$$\Delta_n \to X \to Y$$

Chain level

$f : X \to Y$

Maps

$$\boxed{\Delta_n \to Y}$$

Chain level

$f : X \to Y$
$\sigma : \Delta_n \to X$

Maps

$\Delta_n \to Y$

Chain level

$f : X \to Y$
$\sigma : \Delta_n \to X$
$f \circ \sigma : \Delta_n \to Y$

Maps

$\Delta_n \to Y$

Chain level

$f : X \to Y$
$\sigma : \Delta_n \to X$
$f \circ \sigma : \Delta_n \to Y$

$f_{\#} : C_n(X) \to C_n(Y)$

Maps

$\Delta_n \to Y$

Chain level

$f : X \to Y$
$\sigma : \Delta_n \to X$
$f \circ \sigma : \Delta_n \to Y$

$f_\# : C_n(X) \to C_n(Y)$ extended linearly

Maps

Chain level

$f : X \to Y$
$\sigma : \Delta_n \to X$
$f \circ \sigma : \Delta_n \to Y$

$f_\# : C_n(X) \to C_n(Y)$ extended linearly

Exercise. *Prove:*

1. $(Id_X)_\# = Id_{C_n(X)}$

Maps

Chain level

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Exercise. *Prove:*

1. $(Id_X)_\# = Id_{C_n(X)}$
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Maps

Homology level

$\partial \circ f_\# = f_\# \circ \partial$

Chain level

$f : X \to Y$
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Homology level

$\partial \circ f_\# = f_\# \circ \partial$ (Exercise!)
$f_\#(Z_n(X)) \subset Z_n(Y)$

Maps

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Homology level

$\partial \circ f_\# = f_\# \circ \partial$ (Exercise!)
$f_\#(Z_n(X)) \subset Z_n(Y)$ (Exercise!!)

Maps

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$f_\#(Z_n(X)) \subset Z_n(Y)$ (Exercise!!)
$f_\#(B_n(X)) \subset Z_n(Y)$

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Chain level

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$f_\# : C_n(X) \to C_n(Y)$ extended linearly

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$f_\#(B_n(X)) \subset Z_n(Y)$ (Exercise!!!)

$f_* : H_n(X) \to H_n(Y)$

Maps

Chain level

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$\sigma : \Delta_n \to X$
$f \circ \sigma : \Delta_n \to Y$

$f_\# : C_n(X) \to C_n(Y)$ extended linearly

Exercise. *Prove:*

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$\partial \circ f_\# = f_\# \circ \partial$ (Exercise!)
$f_\#(Z_n(X)) \subset Z_n(Y)$ (Exercise!!)
$f_\#(B_n(X)) \subset Z_n(Y)$ (Exercise!!!)

$f_* : H_n(X) \to H_n(Y)$
$f_*([\sigma]) =$

Maps

Chain level

$f : X \to Y$
$\sigma : \Delta_n \to X$
$f \circ \sigma : \Delta_n \to Y$

$f_\# : C_n(X) \to C_n(Y)$ extended linearly

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Homology level

$\partial \circ f_\# = f_\# \circ \partial$ (Exercise!)
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$f_*([\sigma]) = [f \circ \sigma]$

Maps

Chain level

$f : X \to Y$
$\sigma : \Delta_n \to X$
$f \circ \sigma : \Delta_n \to Y$

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Homology level

$\partial \circ f_\# = f_\# \circ \partial$ (Exercise!)
$f_\#(Z_n(X)) \subset Z_n(Y)$ (Exercise!!)
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$f_*([\sigma]) = [f \circ \sigma]$

$$\cdots \xrightarrow{\partial_{n+2}} C_{n+1}(X) \xrightarrow{\partial_{n+1}} C_n(X) \xrightarrow{\partial_n} C_{n-1}(X) \to$$

Maps

Chain level

$f : X \to Y$
$\sigma : \Delta_n \to X$
$f \circ \sigma : \Delta_n \to Y$

$f_\# : C_n(X) \to C_n(Y)$ extended linearly

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Homology level

$\partial \circ f_\# = f_\# \circ \partial$ (Exercise!)
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Maps

Chain level

$f : X \to Y$
$\sigma : \Delta_n \to X$
$f \circ \sigma : \Delta_n \to Y$

$f_\# : C_n(X) \to C_n(Y)$ extended linearly

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Homology level

$\partial \circ f_\# = f_\# \circ \partial$ (Exercise!)
$f_\#(Z_n(X)) \subset Z_n(Y)$ (Exercise!!)
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$f_*([\sigma]) = [f \circ \sigma]$

$$\begin{tikzcd}
\cdots \arrow[r, "\partial_{n+2}"] & C_{n+1}(X) \arrow[r, "\partial_{n+1}"] \arrow[d, "f_\#"] & C_n(X) \arrow[r, "\partial_n"] \arrow[d, "f_\#"] & C_{n-1}(X) \arrow[r] \arrow[d, "f_\#"] & {} \\
\cdots \arrow[r, "\partial_{n+2}"] & C_{n+1}(Y) \arrow[r, "\partial_{n+1}"] & C_n(Y) \arrow[r, "\partial_n"] & C_{n-1}(Y) \arrow[r] & {}
\end{tikzcd}$$

Maps

Chain level

$f : X \to Y$
$\sigma : \Delta_n \to X$
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Exercise. *Prove:*

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Homology level

$\partial \circ f_\# = f_\# \circ \partial$ (Exercise!)
$f_\#(Z_n(X)) \subset Z_n(Y)$ (Exercise!!)
$f_\#(B_n(X)) \subset Z_n(Y)$ (Exercise!!!)

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$f_*([\sigma]) = [f \circ \sigma]$

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{\partial_{n+2}} & C_{n+1}(X) & \xrightarrow{\partial_{n+1}} & C_n(X) & \xrightarrow{\partial_n} & C_{n-1}(X) & \to \\
 & & \downarrow f_\# & & \downarrow f_\# & & \downarrow f_\# & \\
\cdots & \xrightarrow{\partial_{n+2}} & C_{n+1}(Y) & \xrightarrow{\partial_{n+1}} & C_n(Y) & \xrightarrow{\partial_n} & C_{n-1}(Y) & \to
\end{array}$$

Exercise. *Prove:*

1. $(Id_X)_* = Id_{H_n(X)}$

Maps

Chain level

$f : X \to Y$
$\sigma : \Delta_n \to X$
$f \circ \sigma : \Delta_n \to Y$

$f_\# : C_n(X) \to C_n(Y)$ extended linearly

Exercise. *Prove:*

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Homology level

$\partial \circ f_\# = f_\# \circ \partial$ (Exercise!)
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\cdots \arrow[r, "\partial_{n+2}"] & C_{n+1}(Y) \arrow[r, "\partial_{n+1}"] & C_n(Y) \arrow[r, "\partial_n"] & C_{n-1}(Y) \arrow[r] & {}
\end{tikzcd}$$

Exercise. *Prove:*

1. $(Id_X)_* = Id_{H_n(X)}$
2. $(f \circ g)_* = f_* \circ g_*$

Digression: homotopy

$f_0, f_1 : X \to Y$

Digression: homotopy

$f_0, f_1 : X \to Y$
$F : X \times [0,1] \to Y,$

Digression: homotopy

$f_0, f_1 : X \to Y$
$F : X \times [0,1] \to Y,$
$f_0(x) = F(x, 0)$

Digression: homotopy

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$f_0(x) = F(x,0)$
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Digression: homotopy

$f_0, f_1 : X \to Y$
$F : X \times [0,1] \to Y$, “homotopy”
$f_0(x) = F(x, 0)$
$f_1(x) = F(x, 1)$
“f_0 and f_1 are homotopic”

Digression: homotopy

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Lemma. $Id_{B^n} \simeq const$

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Lemma. $Id_{B^n} \simeq const$

Proof. $F(x,t) = tx \dots$ $\square$

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Back to homology

Digression: homotopy

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Back to homology

Theorem. $f_0 \simeq f_1$

Digression: homotopy

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Back to homology

Theorem. $f_0 \simeq f_1 \implies f_* = g_*$

Corollary. *If,*
$f : X \to Y$

Digression: homotopy

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Theorem. $f_0 \simeq f_1 \implies f_* = g_*$

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$f : X \to Y$, $g : Y \to X$,
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Digression: homotopy

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Proof. $F(x,t) = tx \ldots$ □

Back to homology

Theorem. $f_0 \simeq f_1 \implies f_* = g_*$

Corollary. *If,*
$f : X \to Y$, $g : Y \to X$,
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then, $H_n(X) \cong H_n(Y)$

Digression: homotopy

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Lemma. *$Id_{B^n} \simeq const$*

Proof. $F(x, t) = tx \ldots$ $\square$

Definition. *If, $f : X \to Y$*

Back to homology

Theorem. $f_0 \simeq f_1 \implies f_* = g_*$

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Definition. *If, $f : X \to Y$, $g : Y \to X$,*
$f \circ g \simeq id_Y$ and $g \circ f \simeq id_X$

Back to homology

Theorem. $f_0 \simeq f_1 \implies f_* = g_*$

Corollary. *If,*
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Definition. *If,* $f : X \to Y$, $g : Y \to X$,
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then, X *is homotopically equivalent to* Y

Back to homology

Theorem. $f_0 \simeq f_1 \implies f_* = g_*$

Corollary. *If,*
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Digression: homotopy

$f_0, f_1 : X \to Y$
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Lemma. *$Id_{B^n} \simeq const$*

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Back to homology

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Corollary. *If* $X \simeq Y$ *then,* $H_n(X) \cong H_n(Y)$

Corollary. $H_k(B^n) = 0$ *if* $n > 0$

Homotopy invariance

$F : X \times [0,1] \to Y$

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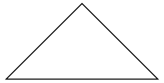
Homotopy invariance



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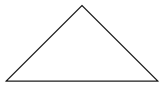

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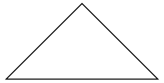

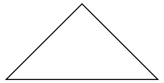

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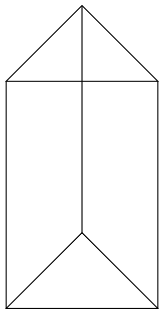

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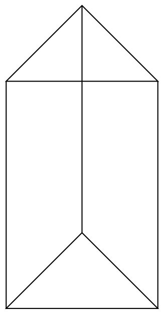

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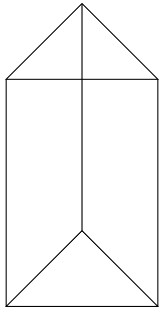

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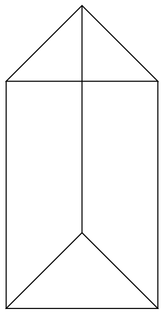

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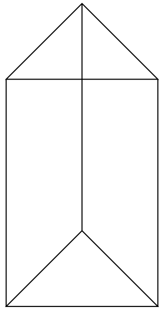
Homotopy invariance



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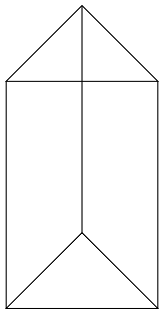

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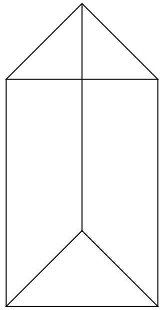
Homotopy invariance



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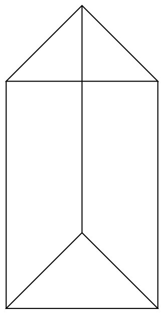
Homotopy invariance



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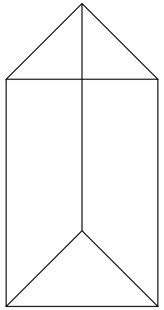

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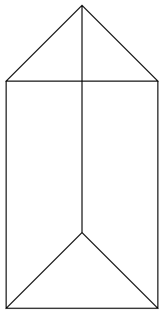
Homotopy invariance

If $c \in Z_n(X)$,



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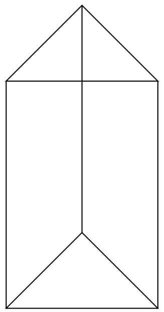

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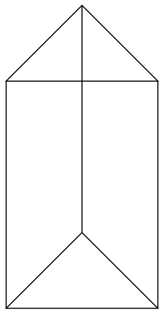
Homotopy invariance



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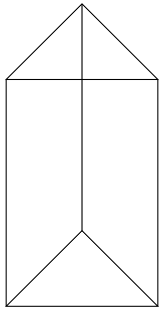

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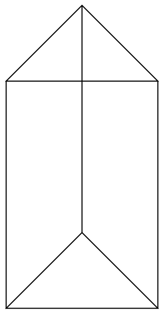
Homotopy invariance



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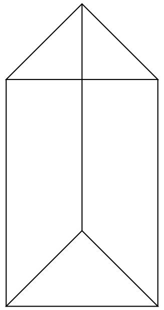

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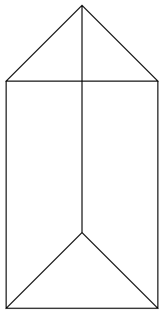

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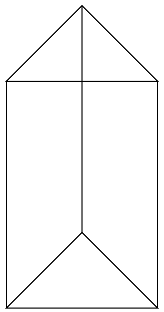

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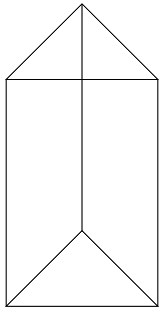

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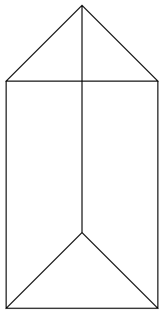

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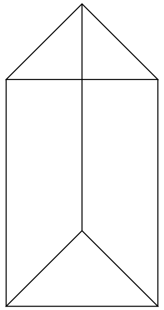

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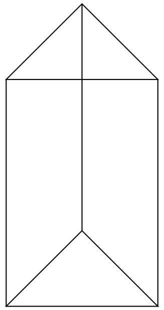

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f_0 homotopic f_1

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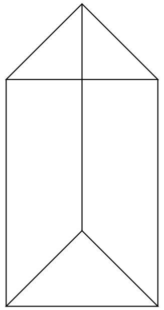

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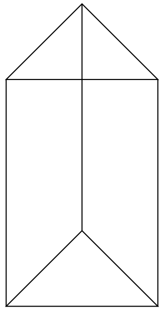

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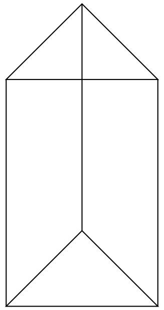
Homotopy invariance



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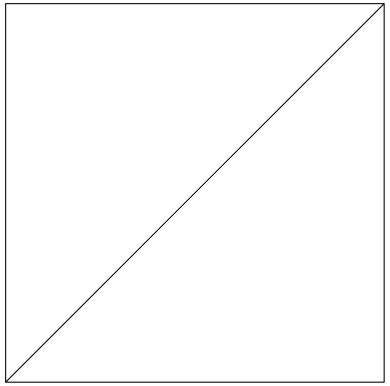

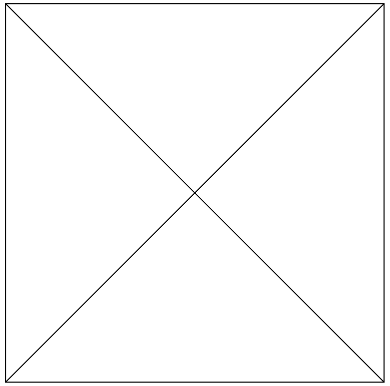

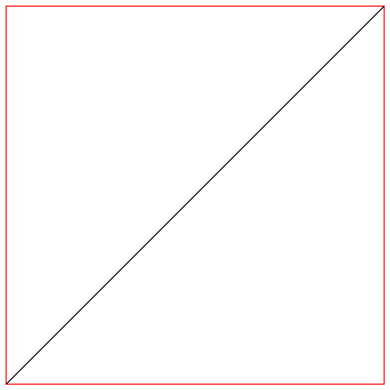

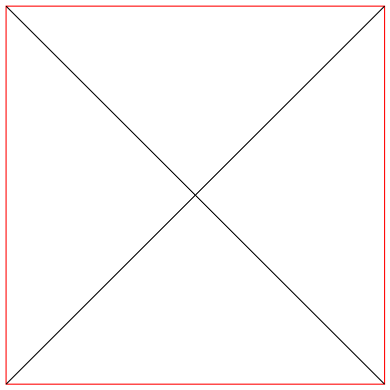

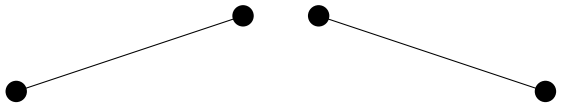

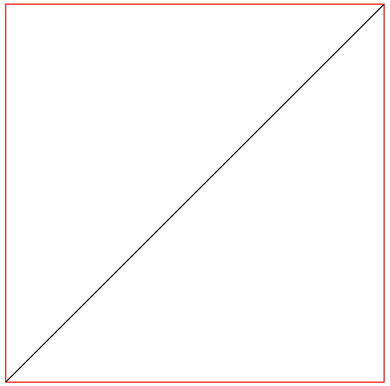

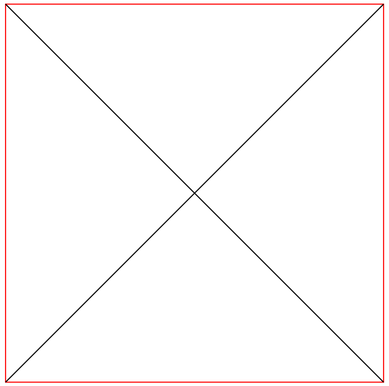

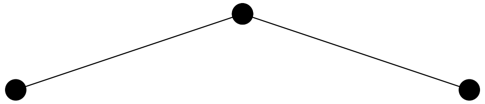

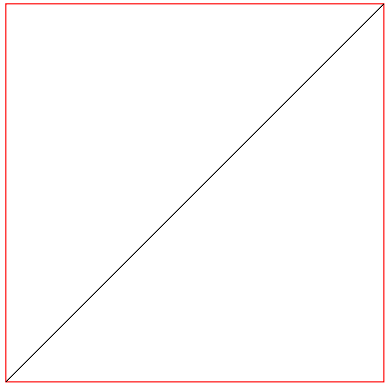

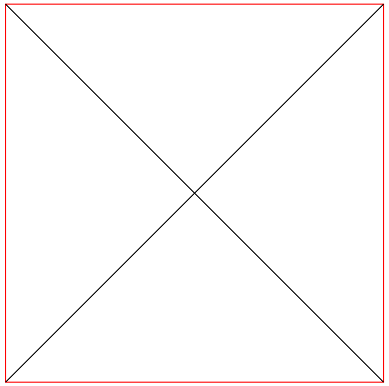

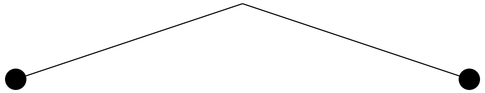

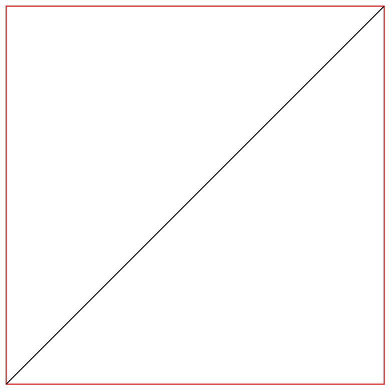

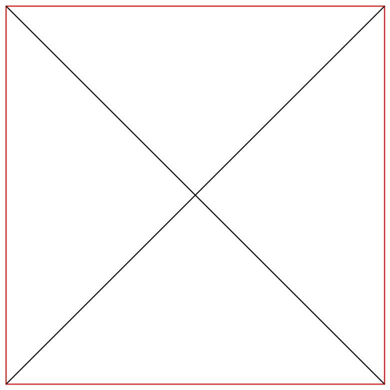

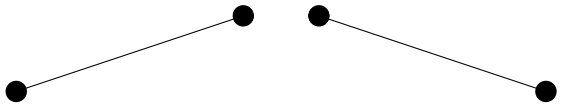

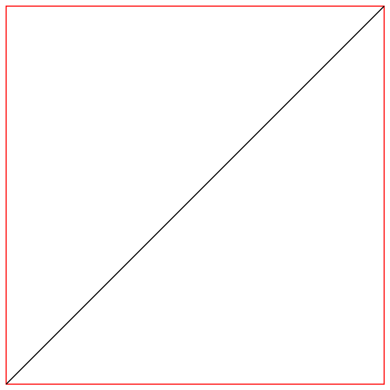

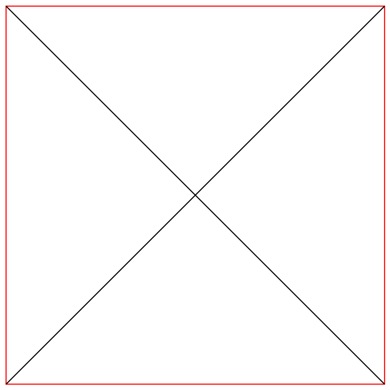

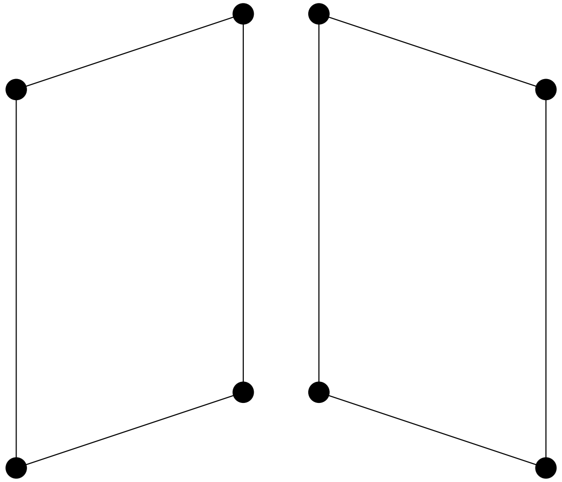

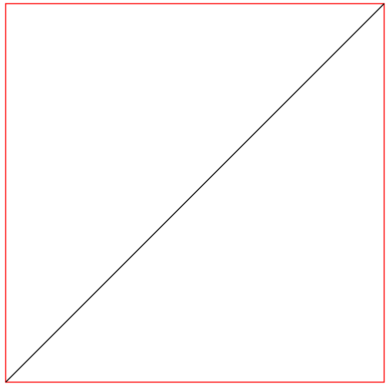

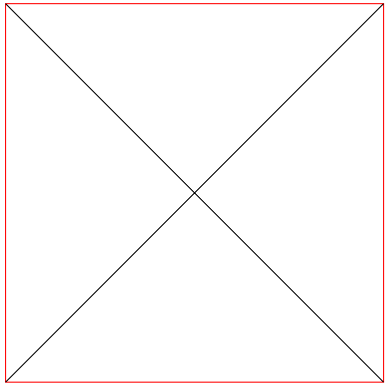

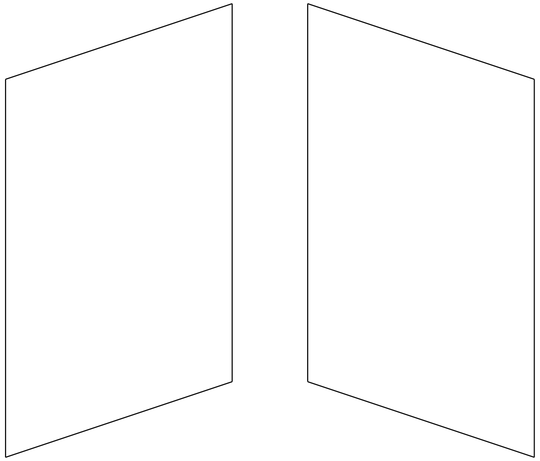

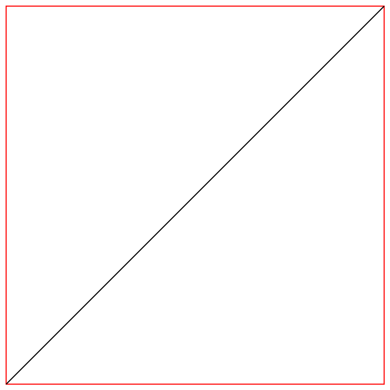

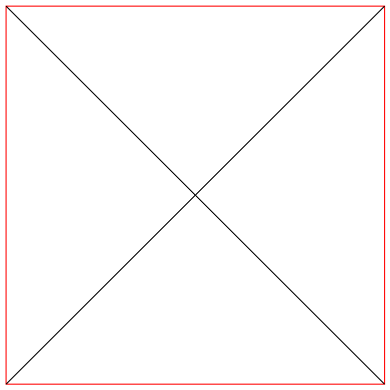

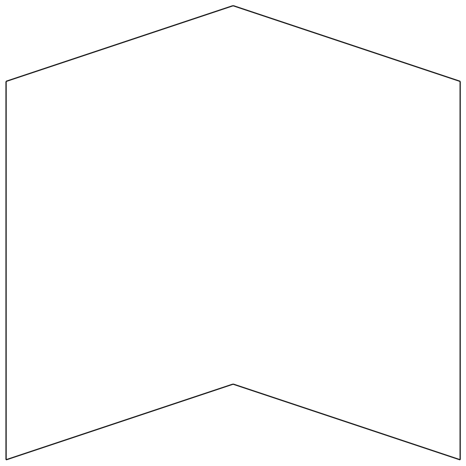

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$v_0, v_1, \ldots, v_n \in \mathbb{R}^k$

$\Delta_n \xrightarrow{\theta}$ convex span$\{v_0, v_1, \ldots, v_n\} \xrightarrow{h} X$
where $\theta : [e_0, e_1, \ldots, e_n] \to [v_0, v_1, \ldots, v_n]$

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v_0 ——— v_1

Notation:

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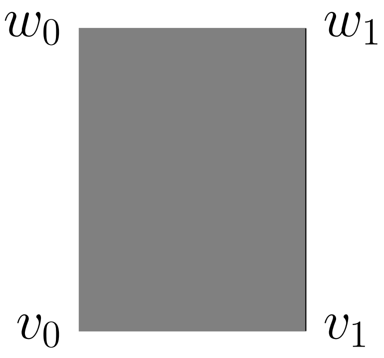

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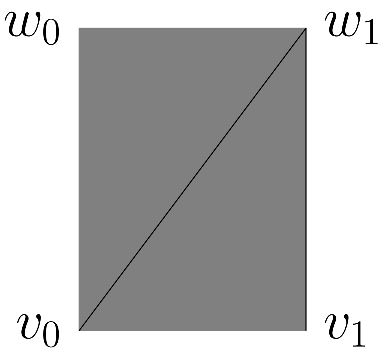

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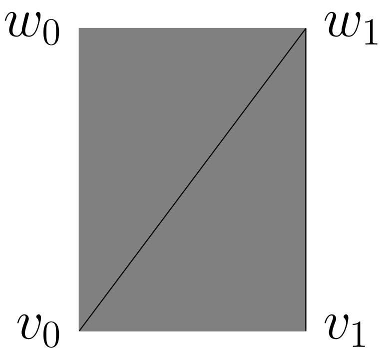

$$P(\sigma) := F \circ (\sigma \times Id) \restriction_{[v_0, v_1, w_1]} + F \circ (\sigma \times Id) \restriction_{[v_0, w_1, w_0]}$$

Notation:

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$\Delta_n \xrightarrow{\theta} \text{convex span}\{v_0, v_1, \ldots, v_n\} \xrightarrow{h} X$
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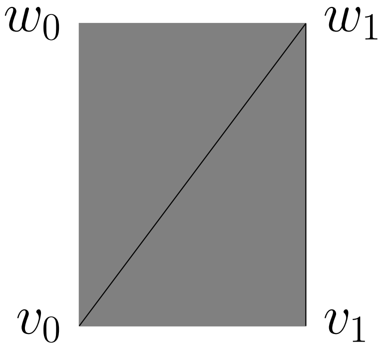

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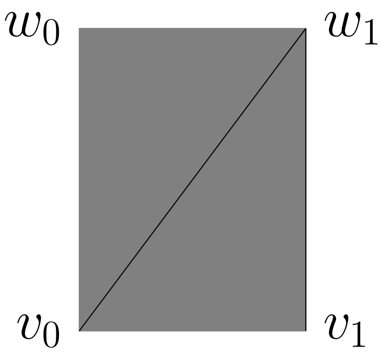


$$P(\sigma) := F \circ (\sigma \times Id) \restriction_{[v_0,v_1,w_1]} - F \circ (\sigma \times Id) \restriction_{[v_0,w_0,w_1]}$$

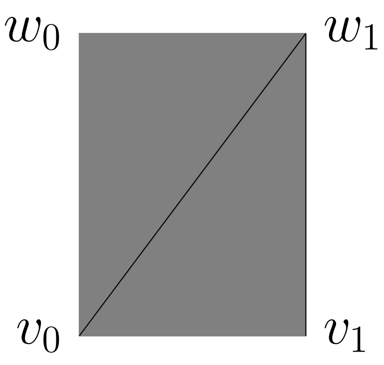
$$\partial(P(\sigma)) = \partial F \circ (\sigma \times Id) \restriction_{[v_0,v_1,w_1]} - \partial F \circ (\sigma \times Id) \restriction_{[v_0,w_0,w_1]}$$

Notation:

$v_0, v_1, \dots, v_n \in \mathbb{R}^k$

$\Delta_n \xrightarrow{\theta} \text{convex span}\{v_0, v_1, \dots, v_n\} \xrightarrow{h} X$
where $\theta : [e_0, e_1, \dots, e_n] \to [v_0, v_1, \dots, v_n]$

$\boxed{h \restriction_{[v_0,v_1,\dots,v_n]} := h \circ \theta : \Delta_n \to X}$



$P(\sigma) := F \circ (\sigma \times Id) \restriction_{[v_0,v_1,w_1]} - F \circ (\sigma \times Id) \restriction_{[v_0,w_0,w_1]}$

$\partial(P(\sigma)) = F \circ (\sigma \times Id) \restriction_{[v_1,w_1]} - F \circ (\sigma \times Id) \restriction_{[v_0,w_1]} + F \circ (\sigma \times Id) \restriction_{[v_0,v_1]} -$
$\partial F \circ (\sigma \times Id) \restriction_{[v_0,w_0,w_1]}$

Notation:

$v_0, v_1, \ldots, v_n \in \mathbb{R}^k$

$\Delta_n \xrightarrow{\theta} \text{convex span}\{v_0, v_1, \ldots, v_n\} \xrightarrow{h} X$
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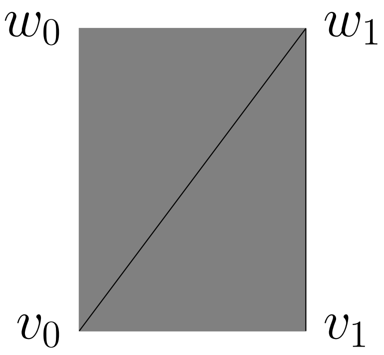

$$P(\sigma) := F \circ (\sigma \times Id) \restriction_{[v_0,v_1,w_1]} - F \circ (\sigma \times Id) \restriction_{[v_0,w_0,w_1]}$$

$$\partial(P(\sigma)) = F \circ (\sigma \times Id) \restriction_{[v_1,w_1]} - F \circ (\sigma \times Id) \restriction_{[v_0,w_1]} + F \circ (\sigma \times Id) \restriction_{[v_0,v_1]} - F \circ (\sigma \times Id) \restriction_{[w_0,w_1]} + F \circ (\sigma \times Id) \restriction_{[v_0,w_1]} - F \circ (\sigma \times Id) \restriction_{[v_0,w_0]}$$

Notation:

$v_0, v_1, \ldots, v_n \in \mathbb{R}^k$

$\Delta_n \xrightarrow{\theta} \text{convex span}\{v_0, v_1, \ldots, v_n\} \xrightarrow{h} X$
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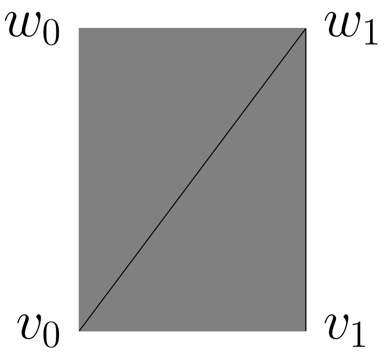
$$\boxed{h \restriction_{[v_0,v_1,\ldots,v_n]} := h \circ \theta : \Delta_n \to X}$$



$$P(\sigma) := F \circ (\sigma \times Id) \restriction_{[v_0,v_1,w_1]} - F \circ (\sigma \times Id) \restriction_{[v_0,w_0,w_1]}$$

$$\partial(P(\sigma)) = F \circ (\sigma \times Id) \restriction_{[v_1,w_1]} \cancel{- F \circ (\sigma \times Id) \restriction_{[v_0,w_1]}} + F \circ (\sigma \times Id) \restriction_{[v_0,v_1]} - F \circ (\sigma \times Id) \restriction_{[w_0,w_1]} \cancel{+ F \circ (\sigma \times Id) \restriction_{[v_0,w_1]}} - F \circ (\sigma \times Id) \restriction_{[v_0,w_0]}$$

Notation:

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$\Delta_n \xrightarrow{\theta} \text{convex span}\{v_0, v_1, \ldots, v_n\} \xrightarrow{h} X$
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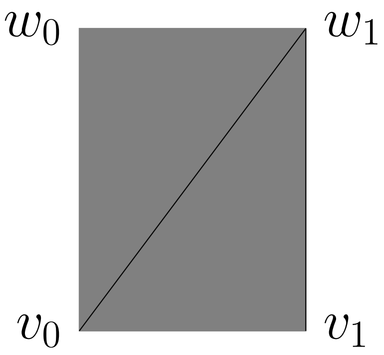

$$P(\sigma) := F \circ (\sigma \times Id) \restriction_{[v_0, v_1, w_1]} - F \circ (\sigma \times Id) \restriction_{[v_0, w_0, w_1]}$$

$$\partial(P(\sigma)) = F \circ (\sigma \times Id) \restriction_{[v_1, w_1]} + F \circ (\sigma \times Id) \restriction_{[v_0, v_1]}$$
$$- F \circ (\sigma \times Id) \restriction_{[w_0, w_1]} - F \circ (\sigma \times Id) \restriction_{[v_0, w_0]}$$

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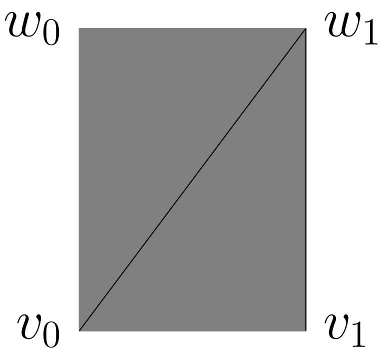

$$P(\sigma) := F \circ (\sigma \times Id) \restriction_{[v_0,v_1,w_1]} - F \circ (\sigma \times Id) \restriction_{[v_0,w_0,w_1]}$$

$$\partial(P(\sigma)) = F \circ (\sigma \times Id) \restriction_{[v_1,w_1]} + \underline{F \circ (\sigma \times Id) \restriction_{[v_0,v_1]}} - F \circ (\sigma \times Id) \restriction_{[w_0,w_1]} - F \circ (\sigma \times Id) \restriction_{[v_0,w_0]}$$

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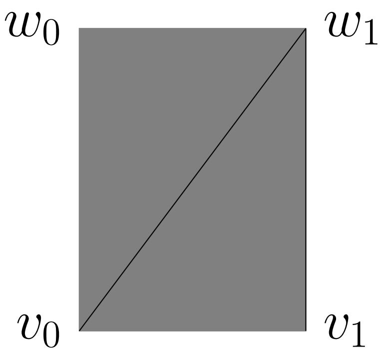

$$P(\sigma) := F \circ (\sigma \times Id) \upharpoonleft_{[v_0, v_1, w_1]} - F \circ (\sigma \times Id) \upharpoonleft_{[v_0, w_0, w_1]}$$

$$\partial(P(\sigma)) = F \circ (\sigma \times Id) \upharpoonleft_{[v_1, w_1]} + (f_0)_{\#}(\sigma) - F \circ (\sigma \times Id) \upharpoonleft_{[w_0, w_1]} - F \circ (\sigma \times Id) \upharpoonleft_{[v_0, w_0]}$$

Notation:

$v_0, v_1, \ldots, v_n \in \mathbb{R}^k$

$\Delta_n \xrightarrow{\theta} \text{convex span}\{v_0, v_1, \ldots, v_n\} \xrightarrow{h} X$
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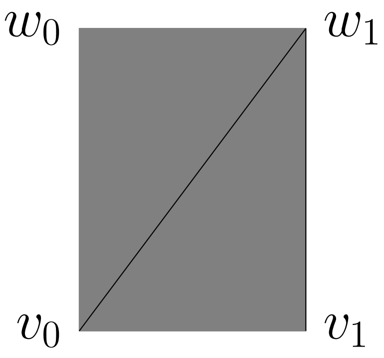

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$$\partial(P(\sigma)) = F \circ (\sigma \times Id) \restriction_{[v_1,w_1]} + (f_0)_{\#}(\sigma) - \underline{F \circ (\sigma \times Id) \restriction_{[w_0,w_1]}} - F \circ (\sigma \times Id) \restriction_{[v_0,w_0]}$$

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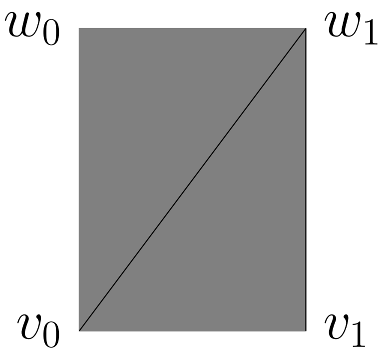

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$\partial(P(\sigma)) = F \circ (\sigma \times Id) \restriction_{[v_1,w_1]} + (f_0)_\#(\sigma) - (f_1)_\#(\sigma) -$
$F \circ (\sigma \times Id) \restriction_{[v_0,w_0]}$

Notation:

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$\Delta_n \xrightarrow{\theta} \text{convex span}\{v_0, v_1, \ldots, v_n\} \xrightarrow{h} X$
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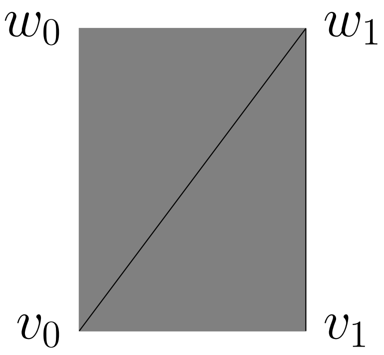

$$P(\sigma) := F \circ (\sigma \times Id) \restriction_{[v_0, v_1, w_1]} - F \circ (\sigma \times Id) \restriction_{[v_0, w_0, w_1]}$$

$$\partial(P(\sigma)) = P(\partial\sigma) + (f_0)_\#(\sigma) - (f_1)_\#(\sigma)$$

Notation:

$v_0, v_1, \ldots, v_n \in \mathbb{R}^k$

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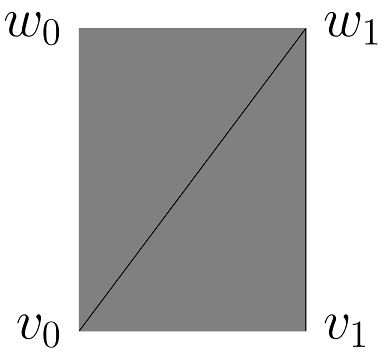

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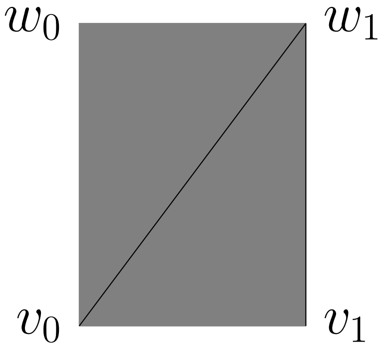

$$P(\sigma) := \Sigma_i (-1)^i F \circ (\sigma \times Id) \restriction_{[v_0,v_1,\ldots,v_i,w_i,\ldots,w_n]}$$

Notation:

$v_0, v_1, \ldots, v_n \in \mathbb{R}^k$

$\Delta_n \xrightarrow{\theta} \text{convex span}\{v_0, v_1, \ldots, v_n\} \xrightarrow{h} X$
where $\theta : [e_0, e_1, \ldots, e_n] \to [v_0, v_1, \ldots, v_n]$

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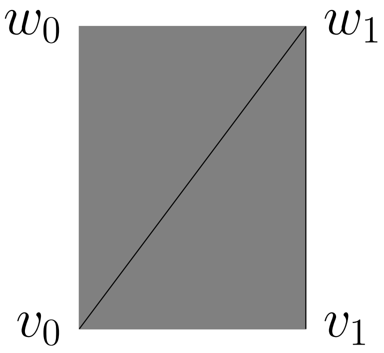

$$\cdots \xrightarrow{\partial_{n+2}} C_{n+1}(X) \xrightarrow{\partial_{n+1}} C_n(X) \xrightarrow{\partial_n} C_{n-1}(X) \to$$

$P(\sigma) := \Sigma_i (-1)^i F \circ (\sigma \times Id) \restriction_{[v_0, v_1, \ldots, v_i, w_i, \ldots, w_n]}$

Exercise. $\partial(P(\sigma)) = (f_1)_\#(\sigma) - (f_0)_\#(\sigma) - P(\partial(\sigma))$

Notation:

$v_0, v_1, \ldots, v_n \in \mathbb{R}^k$

$\Delta_n \xrightarrow{\theta} \text{convex span}\{v_0, v_1, \ldots, v_n\} \xrightarrow{h} X$
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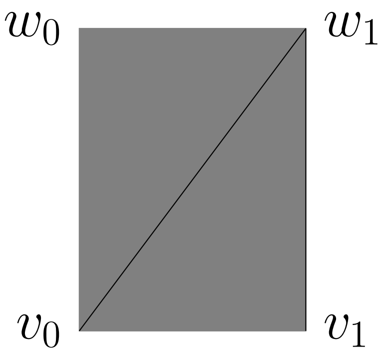
Exercise. $\partial(P(\sigma)) = (f_1)_\#(\sigma) - (f_0)_\#(\sigma) - P(\partial(\sigma))$



$$\cdots \xrightarrow{\partial_{n+2}} C_{n+1}(X) \xrightarrow{\partial_{n+1}} C_n(X) \xrightarrow{\partial_n} C_{n-1}(X) \to$$

$$\cdots \xrightarrow{\partial_{n+2}} C_{n+1}(Y) \xrightarrow{\partial_{n+1}} C_n(Y) \xrightarrow{\partial_n} C_{n-1}(Y) \to$$

Notation:

$v_0, v_1, \ldots, v_n \in \mathbb{R}^k$

$\Delta_n \xrightarrow{\theta} \text{convex span}\{v_0, v_1, \ldots, v_n\} \xrightarrow{h} X$
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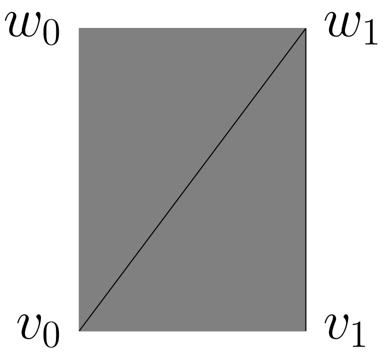
$$P(\sigma) := \Sigma_i (-1)^i F \circ (\sigma \times Id) \restriction_{[v_0,v_1,\ldots,v_i,w_i,\ldots,w_n]}$$

Exercise. $\partial(P(\sigma)) = (f_1)_\#(\sigma) - (f_0)_\#(\sigma) - P(\partial(\sigma))$



$$\begin{tikzcd}
\cdots \arrow[r, "\partial_{n+2}"] & C_{n+1}(X) \arrow[r, "\partial_{n+1}"] \arrow[d, "f_\#, g_\#"] & C_n(X) \arrow[r, "\partial_n"] \arrow[d, "f_\#, g_\#"] & C_{n-1}(X) \arrow[r] \arrow[d, "f_\#, g_\#"] & {} \\
\cdots \arrow[r, "\partial_{n+2}"] & C_{n+1}(Y) \arrow[r, "\partial_{n+1}"] & C_n(Y) \arrow[r, "\partial_n"] & C_{n-1}(Y) \arrow[r] & {}
\end{tikzcd}$$

Notation:

$v_0, v_1, \ldots, v_n \in \mathbb{R}^k$

$\Delta_n \xrightarrow{\theta} \text{convex span}\{v_0, v_1, \ldots, v_n\} \xrightarrow{h} X$
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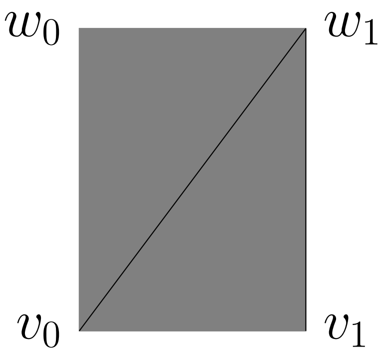
$\boxed{h \restriction_{[v_0,v_1,\ldots,v_n]} := h \circ \theta : \Delta_n \to X}$



$$\begin{tikzcd}
\cdots \arrow[r, "\partial_{n+2}"] & C_{n+1}(X) \arrow[r, "\partial_{n+1}"] \arrow[d, "f_\#, g_\#"] \arrow[dl, "P"] & C_n(X) \arrow[r, "\partial_n"] \arrow[d, "f_\#, g_\#"] \arrow[dl, "P"] & C_{n-1}(X) \arrow[r] \arrow[d, "f_\#, g_\#"] \arrow[dl, "P"] & {} \\
\cdots \arrow[r, "\partial_{n+2}"] & C_{n+1}(Y) \arrow[r, "\partial_{n+1}"] & C_n(Y) \arrow[r, "\partial_n"] & C_{n-1}(Y) \arrow[r] & {}
\end{tikzcd}$$

$$P(\sigma) := \Sigma_i (-1)^i F \circ (\sigma \times Id) \restriction_{[v_0,v_1,\ldots,v_i,w_i,\ldots,w_n]}$$

Exercise. $\partial(P(\sigma)) = (f_1)_\#(\sigma) - (f_0)_\#(\sigma) - P(\partial(\sigma))$