

MTH439 - Homological methods in Algebraic  
Topology  
Final examination

18th of April, 2021

*Duration:* 3 hours

1. Suppose that  $A$  is a closed subspace of  $X$  and there exists a finite set of points,  $B$ , so that  $X \setminus B$  deformation retracts onto  $A$ . Prove that  $H_1(X/A) = 0$  if and only if the following two conditions hold: *(5 points)*
  - i Every homology class of  $H_1(X)$  can be represented by a cycle coming from  $A$ .
  - ii If two points in  $A$  can be joined by a path in  $X$ , then they can be joined by a path in  $A$ .
2. Consider an  $n$ -dimensional  $\Delta$ -complex. Suppose we delete exactly one point from each  $n$ -simplex to obtain a subspace  $A$ , then prove that  $H_{n-1}(A)$  has no torsion elements. *(5 points)*
3. (a) Find a 2-dimensional CW complex which has the following homology *(5 points)*

$$H_i(X) = \begin{cases} \mathbb{Z} & i = 0 \\ \mathbb{Z}/2 \oplus \mathbb{Z}/2 & i = 1 \\ \mathbb{Z} \oplus \mathbb{Z} & i = 2 \\ 0 & i > 2 \end{cases}$$

- (Remember to give complete justifications for the degree computations).
- (b) Is it possible to find a 2-dimensional complex that is also a **connected** manifold and with those homologies? *(5 points)*
  - (c) Compute all the cohomologies with  $\mathbb{Z}/2$  coefficients. *(5 points)*
4. Consider the covering map  $M_{\mathbb{Z}} \rightarrow M$  defined during the lecture and assume that  $M$  is connected and compact. *(5 points)*
    - (a) What are the possible cardinalities of its set of continuous sections? Under what circumstance does each possibility occur (i.e. can you find a criterion in terms of  $M$ )? *(5 points)*
    - (b) Prove that each continuous section is completely determined by its value at one point. *(5 points)*