

MTH439 - Homological methods in Algebraic
Topology
Final examination

18th of April, 2021

Duration: 3 hours

1. Suppose that A is a closed subspace of **hausdorff** space X and there exists a finite set of points, B , so that $X \setminus B$ deformation retracts onto A . Prove that $H_1(X/A) = 0$ if and only if the following two conditions hold: *(5 points)*
 - i Every homology class of $H_1(X)$ can be represented by a cycle coming from A .
 - ii If two points in A can be joined by a path in X , then they can be joined by a path in A .
2. Consider an n -dimensional Δ -complex. Suppose we delete exactly one point from the **interior** of each n -simplex to obtain a subspace A , then prove that $H_{n-1}(A)$ has no torsion elements. *(5 points)*
3. (a) Find a 2-dimensional CW complex which has the following homology *(5 points)*

$$H_i(X) = \begin{cases} \mathbb{Z} & i = 0 \\ \mathbb{Z}/2 \oplus \mathbb{Z}/2 & i = 1 \\ \mathbb{Z} \oplus \mathbb{Z} & i = 2 \\ 0 & i > 2 \end{cases}$$

(Remember to give complete justifications for the degree computations).

- (b) Is it possible to find a 2-dimensional complex that is also a **connected and compact** manifold (**without boundary**) and with those homologies? *(5 points)*
 - (c) Compute all the cohomologies with $\mathbb{Z}/2$ coefficients. *(5 points)*
4. Consider the covering map $M_{\mathbb{Z}} \rightarrow M$ defined during the lecture and assume that M is connected and compact **manifold without boundary**.
 - (a) What are the possible cardinalities of its set of continuous sections? Under what circumstance does each possibility occur (i.e. can you find a criterion in terms of M)? *(5 points)*
 - (b) Prove that each continuous section is completely determined by its value at one point. *(5 points)*