CW-complex

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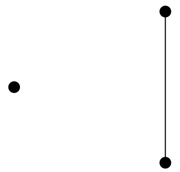
CW-complex



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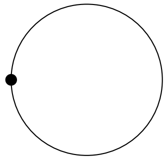

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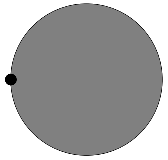
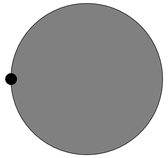


CW-complex

- 
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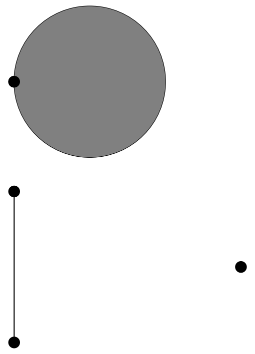
CW-complex



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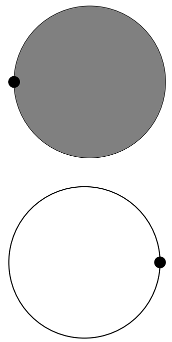

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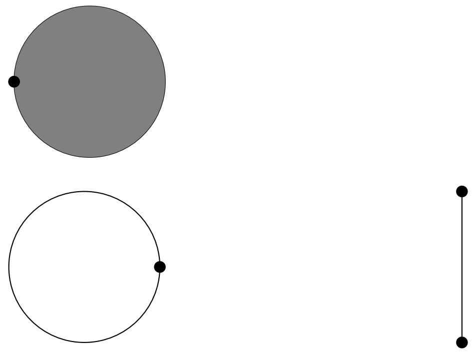

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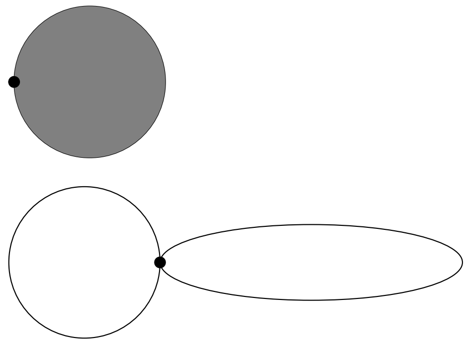

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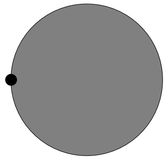

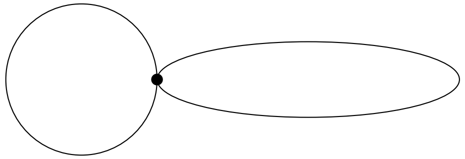

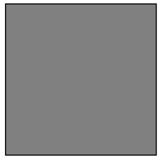

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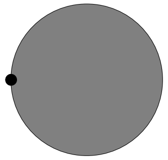

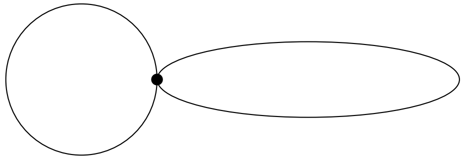

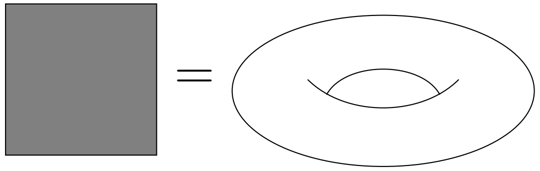

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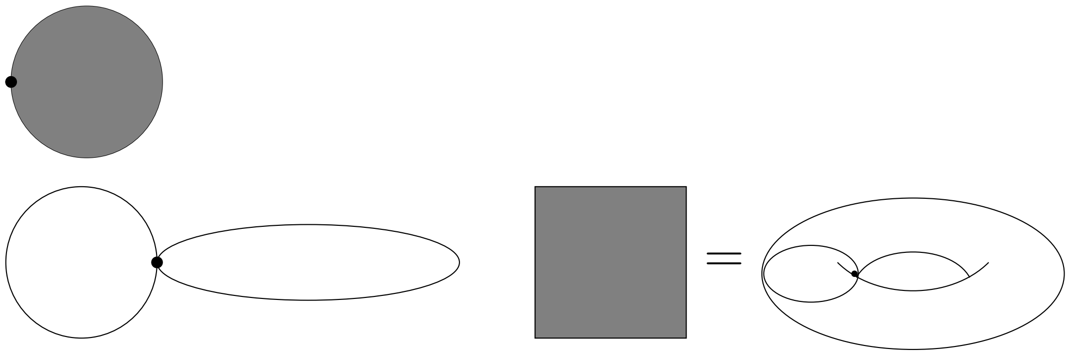
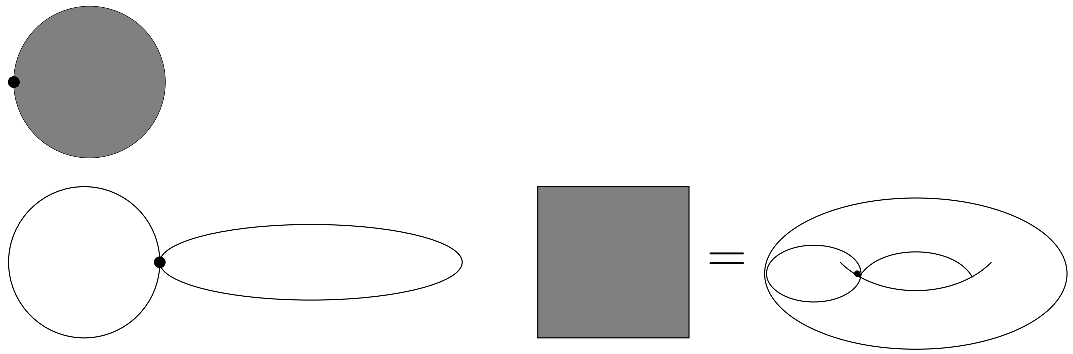
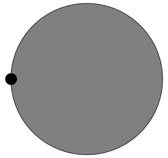


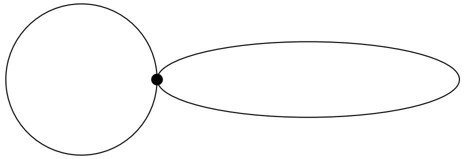
CW-complex

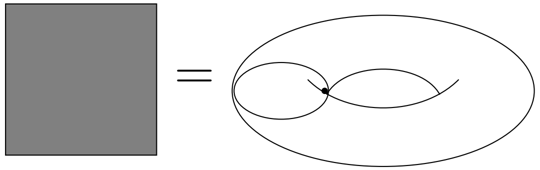


X_0: set of points

CW-complex



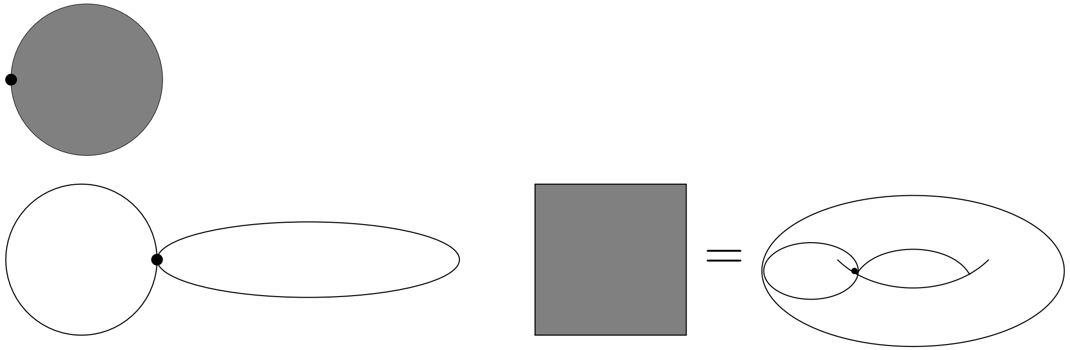




X_0: set of points

$$\phi_\alpha : S^{n-1} = \partial D^n_\alpha \to X_{n-1}$$

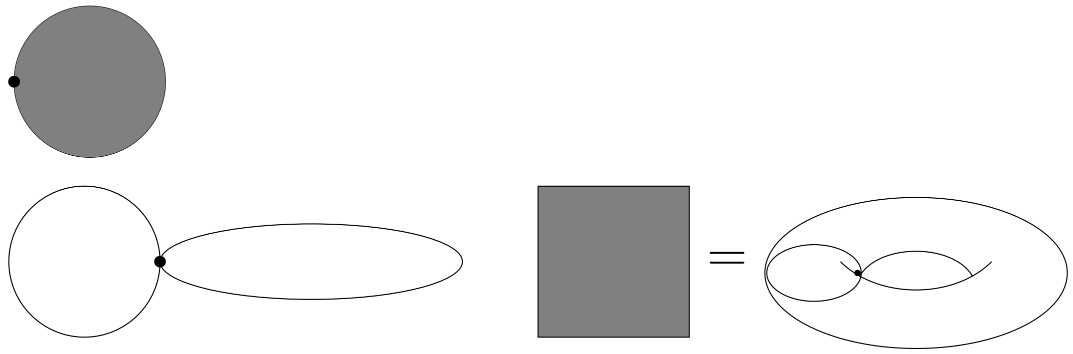
CW-complex



X_0: set of points

$$\phi_\alpha : S^{n-1} = \partial D^n_\alpha \to X_{n-1}$$
$$X_n = X_{n-1} \sqcup D^n_\alpha / \sim, \text{ where } y \sim \phi_\alpha(x)$$

CW-complex



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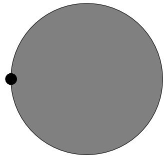
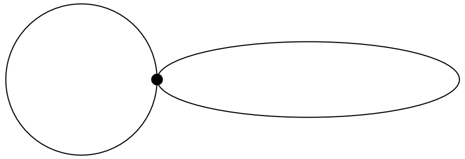
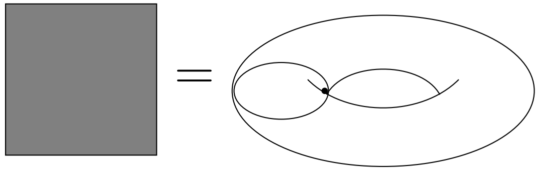
$X = \cup X_i$

CW-complex







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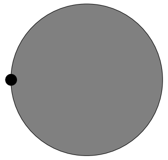
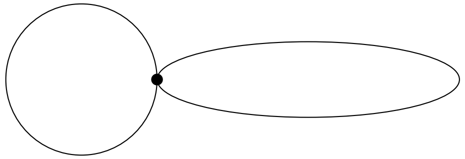
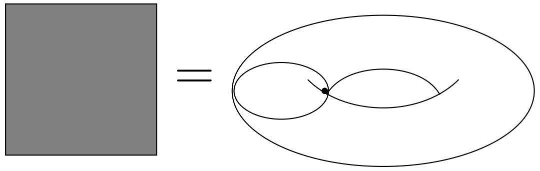
$X = \cup X_i$, and U open in X if and only if $U \cap X_i$ open in each X_i.

CW-complex







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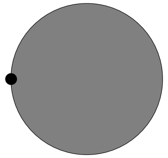
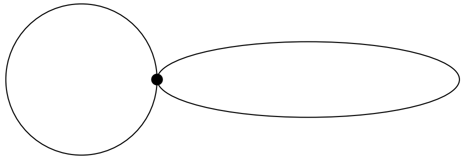
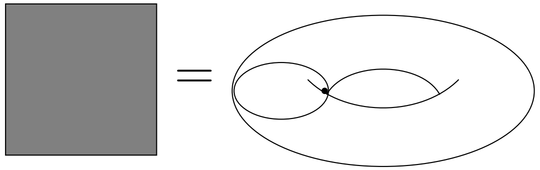
Exercise. *Prove that the spheres (any dimension), torus, projective plane, Klein bottle, and any compact surface are all CW-complexes*

CW-complex $e^n_\alpha = D^n_\alpha \setminus \partial D^n_\alpha$







X_0: set of points

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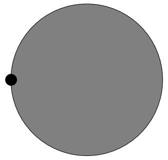
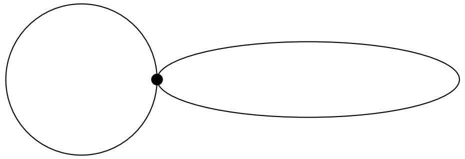
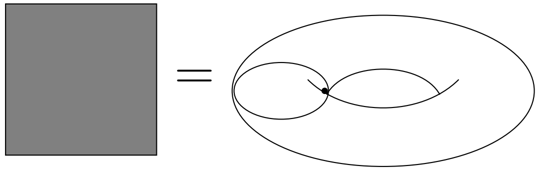
Exercise. *Prove that the spheres (any dimension), torus, projective plane, Klein bottle, and any compact surface are all CW-complexes*

CW-complex

$e_\alpha^n = D_\alpha^n \setminus \partial D_\alpha^n$
$X_n \setminus X_{n-1} = \sqcup e_\alpha^n$







X_0: set of points

$\phi_\alpha : S^{n-1} = \partial D_\alpha^n \to X_{n-1}$
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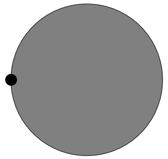
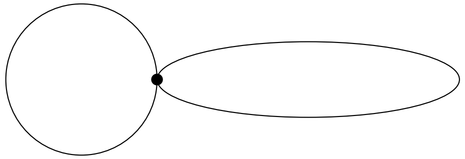
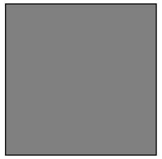
CW-complex



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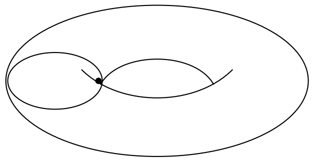

X_0: set of points

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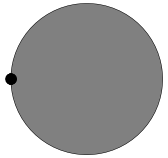
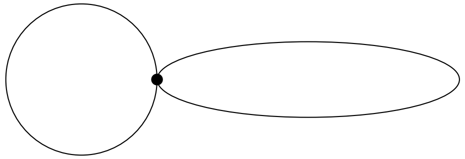
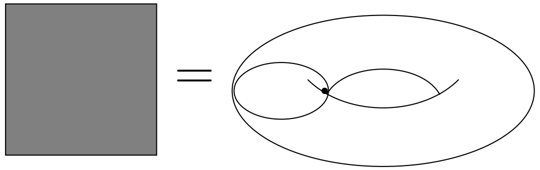
Exercise. *Prove that the spheres (any dimension), torus, projective plane, Klein bottle, and any compact surface are all CW-complexes*

$e^n_\alpha = D^n_\alpha \setminus \partial D^n_\alpha$
$X_n \setminus X_{n-1} = \sqcup e^n_\alpha$
$\Phi_\alpha : D^n_\alpha \to X$

CW-complex







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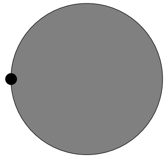
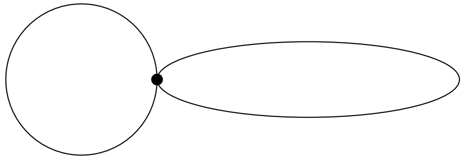
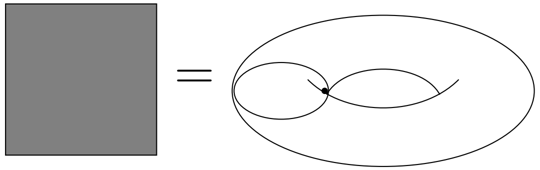
$e^n_\alpha = D^n_\alpha \setminus \partial D^n_\alpha$
$X_n \setminus X_{n-1} = \sqcup e^n_\alpha$
$\Phi_\alpha : D^n_\alpha \to X$ (characteristic map)

CW-complex







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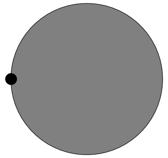
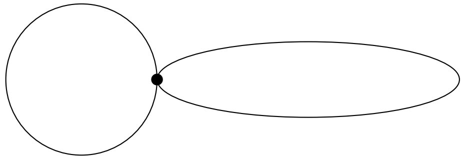
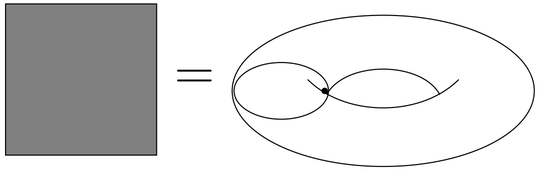
$\Phi_\alpha|_{\partial D^n_\alpha} = \phi$

CW-complex







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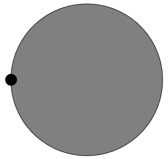
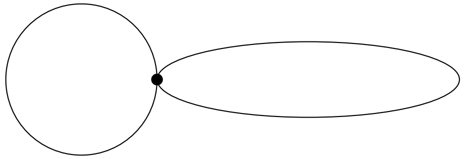
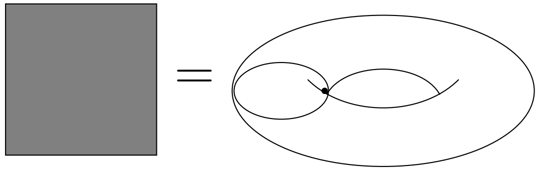
$X_n \setminus X_{n-1} = \sqcup e^n_\alpha$
$\Phi_\alpha : D^n_\alpha \to X$ (characteristic map)
$\Phi_\alpha|_{\partial D^n_\alpha} = \phi$
$\Phi_\alpha|_{Int\ D^n_\alpha}$ is a homeomorphism onto e^N_α

CW-complex







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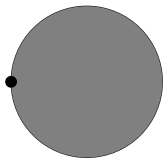
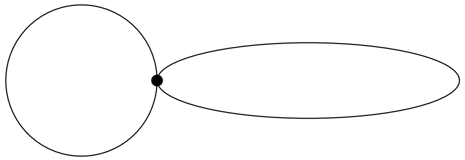
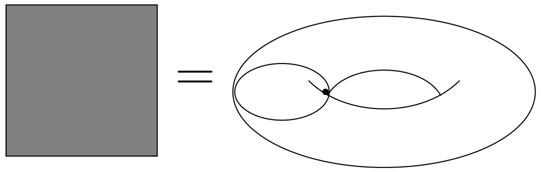
$\Phi_\alpha|_{\partial D^n_\alpha} = \phi$
$\Phi_\alpha|_{Int\ D^n_\alpha}$ is a homeomorphism onto e^N_α

(X_n, X_{n-1}) is a good pair?

CW-complex







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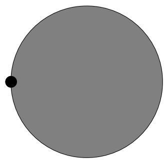
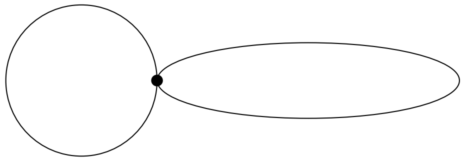
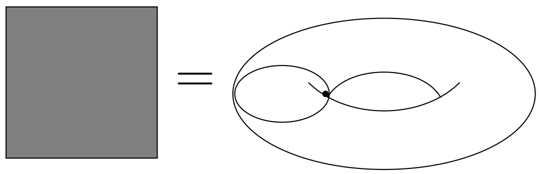
$\Phi_\alpha|_{\partial D^n_\alpha} = \phi$
$\Phi_\alpha|_{Int\ D^n_\alpha}$ is a homeomorphism onto e^N_α

(X_n, X_{n-1}) is a good pair.

CW-complex







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Exercise. *Prove that the spheres (any dimension), torus, projective plane, Klein bottle, and any compact surface are all CW-complexes*

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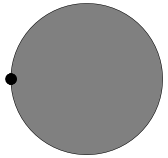
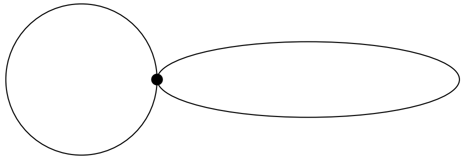
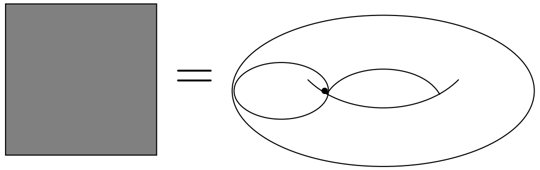
(X_n, X_{n-1}) is a good pair.
$H_k(X_n, X_{n-1})$

CW-complex







X_0: set of points

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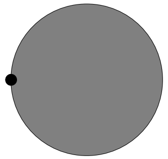

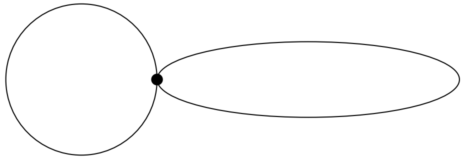

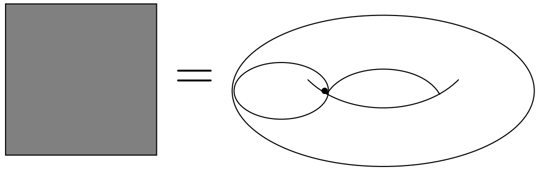

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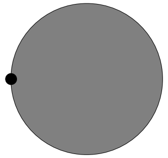
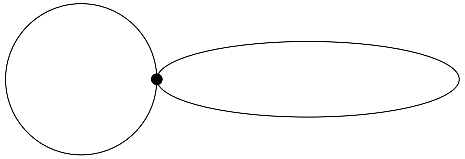
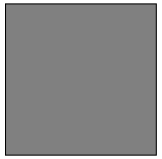
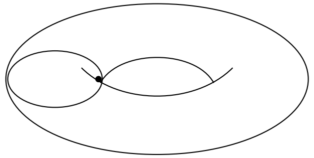
CW-complex





 = 

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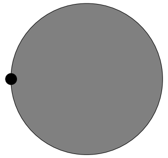
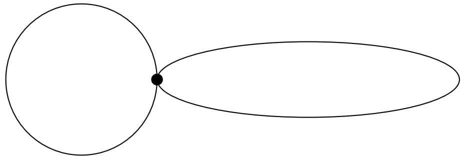
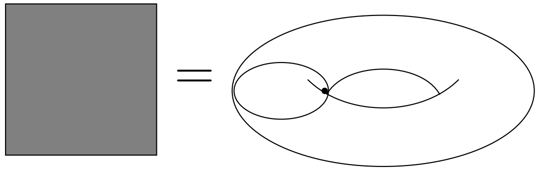
CW-complex







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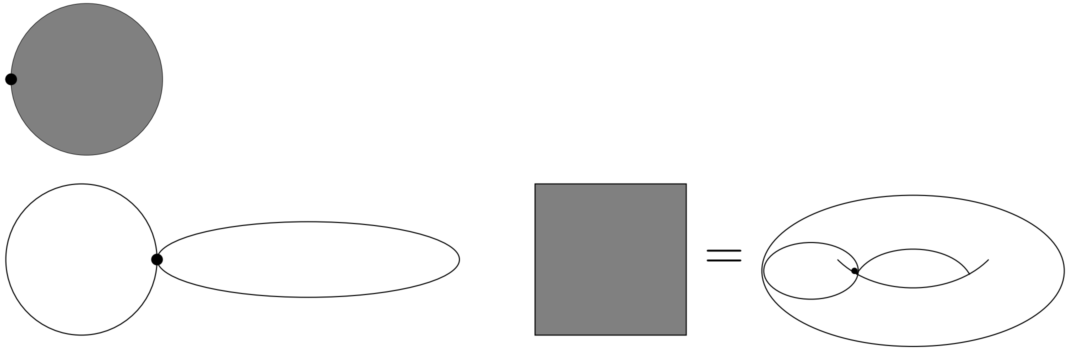

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$$H_{n+1}(X_k, X_{k-1}) \to H_n(X_{k-1}) \to H_n(X_k) \to H_n(X_k, X_{k-1})$$

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(if $n > 0$)

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$$\begin{tikzcd}
H_{n+1}(X_{n+1}, X_n) \arrow[r, "\partial_{n+1}"] & H_n(X_n) \arrow[r] \arrow[d, "j_n"] & H_n(X_{n+1}) \cong H_n(X) \to 0 \\
& H_n(X_n, X_{n-1}) &
\end{tikzcd}$$

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H_{n-1}(X_{n-2}) \cong 0 \arrow[r] & H_{n-1}(X_{n-1}) \arrow[r, "j_{n-1}"] & H_{n-1}(X_{n-1}, X_{n-2})
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$$\begin{tikzcd}
 & H_n(X_{n-1}) = 0 \arrow[d] & \\
H_{n+1}(X_{n+1}, X_n) \arrow[r, "\partial_{n+1}"] \arrow[dr, "d_{n+1}"] & H_n(X_n) \arrow[r] \arrow[d, "j_n"] & H_n(X_{n+1}) \cong H_n(X) \to 0 \\
 & H_n(X_n, X_{n-1}) \arrow[d, "\partial_n"] \arrow[dr, "d_n"] & \\
H_{n-1}(X_{n-2}) \cong 0 \arrow[r] & H_{n-1}(X_{n-1}) \arrow[r, "j_{n-1}"] & H_{n-1}(X_{n-1}, X_{n-2})
\end{tikzcd}$$

$H_n(X) \cong H_n(X_n)/Im\ \partial_{n+1} \cong j_n(H_n(X_n))/Im\ (j_n \circ \partial_{n+1}) \cong ker\ \partial_n/Im\ (j_n \circ \partial_{n+1}) \cong ker\ (j_{n-1} \circ \partial_n)/Im\ (j_n \circ \partial_{n+1})$

$$\boxed{H_n(X_{n+1}) \cong H_n(X_{n+2}) \cong H_n(X_{n+3}) \cong \ldots \cong H_n(X) \text{ exercise!}}$$

$$\boxed{H_n(X_{n-1}) \cong H_n(X_{n-2}) \cong H_n(X_{n-3}) \cong \ldots \cong H_n(X_0) \cong 0 \text{ (if } n > 0)}$$

$$\begin{tikzcd}
 & H_n(X_{n-1}) = 0 \arrow[d] & \\
H_{n+1}(X_{n+1}, X_n) \arrow[r, "\partial_{n+1}"] \arrow[dr, "d_{n+1}"] & H_n(X_n) \arrow[r] \arrow[d, "j_n"] & H_n(X_{n+1}) \cong H_n(X) \to 0 \\
 & H_n(X_n, X_{n-1}) \arrow[d, "\partial_n"] \arrow[dr, "d_n"] & \\
H_{n-1}(X_{n-2}) \cong 0 \arrow[r] & H_{n-1}(X_{n-1}) \arrow[r, "j_{n-1}"] & H_{n-1}(X_{n-1}, X_{n-2})
\end{tikzcd}$$

$H_n(X) \cong H_n(X_n)/Im\ \partial_{n+1} \cong j_n(H_n(X_n))/Im\ (j_n \circ \partial_{n+1}) \cong ker\ \partial_n/Im\ (j_n \circ \partial_{n+1}) \cong ker\ (j_{n-1} \circ \partial_n)/Im\ (j_n \circ \partial_{n+1}) \cong ker\ d_n/Im\ d_{n+1}$

$$\begin{tikzcd}
H_n(X_n, X_{n-1}) \arrow[r, "\partial_n"] \arrow[dr, "d_n"] & H_{n-1}(X_{n-1}) \arrow[d, "j"] \\
& H_{n-1}(X_{n-1}, X_{n-2})
\end{tikzcd}$$

$$\begin{tikzcd}
H_n(D^n, \partial D^n) \arrow[d, "\Phi_*"] & & \\
H_n(X_n, X_{n-1}) \arrow[r, "\partial_n"] \arrow[dr, "d_n"'] & H_{n-1}(X_{n-1}) \arrow[d, "j"] \\
& H_{n-1}(X_{n-1}, X_{n-2})
\end{tikzcd}$$

$d_n(\Phi_*(1))$

$$
\begin{tikzcd}
H_n(D^n, \partial D^n) \arrow[r, "\partial'_n"] \arrow[d, "\Phi_*"] & H_{n-1}(\partial D^n) \\
H_n(X_n, X_{n-1}) \arrow[r, "\partial_n"] \arrow[dr, "d_n"] & H_{n-1}(X_{n-1}) \arrow[d, "j"] \\
& H_{n-1}(X_{n-1}, X_{n-2})
\end{tikzcd}
$$

$$d_n(\Phi_*(\partial_n'^{\,-1}(1)))$$

$$
\begin{tikzcd}
H_n(D^n, \partial D^n) \arrow[r, "\partial'_n"] \arrow[d, "\Phi_*"] & H_{n-1}(\partial D^n) \\
H_n(X_n, X_{n-1}) \arrow[r, "\partial_n"] \arrow[dr, "d_n"] & H_{n-1}(X_{n-1}) \arrow[d, "j"] \\
& H_{n-1}(X_{n-1}, X_{n-2})
\end{tikzcd}
$$

$$d_n(\Phi_*({\partial'_n}^{-1}(1))) = j(\partial_n(\Phi_*({\partial'_n}^{-1}(1))))$$

$$\begin{tikzcd}
H_n(D^n, \partial D^n) \arrow[r, "\partial_n'"] \arrow[d, "\Phi_*"] & H_{n-1}(\partial D^n) \arrow[d, "\phi_*"] \\
H_n(X_n, X_{n-1}) \arrow[r, "\partial_n"] \arrow[dr, "d_n"] & H_{n-1}(X_{n-1}) \arrow[d, "j"] \\
& H_{n-1}(X_{n-1}, X_{n-2})
\end{tikzcd}$$

$$d_n(\Phi_*({\partial_n'}^{-1}(1))) = j(\partial_n(\Phi_*({\partial_n'}^{-1}(1)))) = j(\phi_*(1))$$

$$\begin{tikzcd}
H_n(D^n, \partial D^n) \arrow[r, "\partial'_n"] \arrow[d, "\Phi_*"] & H_{n-1}(\partial D^n) \arrow[d, "\phi_*"] \\
H_n(X_n, X_{n-1}) \arrow[r, "\partial_n"] \arrow[dr, "d_n"] & H_{n-1}(X_{n-1}) \arrow[d, "j"] \\
& H_{n-1}(X_{n-1}, X_{n-2})
\end{tikzcd}$$

$$d_n(\Phi_*(\partial_n'^{-1}(1))) = j(\partial_n(\Phi_*(\partial_n'^{-1}(1)))) = j(\phi_*(1))$$

$$j(\phi_*(1)) = \Sigma_i n_i g_i$$

$$\begin{tikzcd}
H_n(D^n, \partial D^n) \arrow[r, "\partial'_n"] \arrow[d, "\Phi_*"] & H_{n-1}(\partial D^n) \arrow[d, "\phi_*"] & \\
H_n(X_n, X_{n-1}) \arrow[r, "\partial_n"] \arrow[dr, "d_n"] & H_{n-1}(X_{n-1}) \arrow[d, "j"] & \\
& H_{n-1}(X_{n-1}, X_{n-2}) \arrow[r, "q_*"] & H_{n-1}(X_{n-1}/X_{n-2}, X_{n-2}/X_{n-2})
\end{tikzcd}$$

$$d_n(\Phi_*({\partial'_n}^{-1}(1))) = j(\partial_n(\Phi_*({\partial'_n}^{-1}(1)))) = j(\phi_*(1))$$

$$j(\phi_*(1)) = \Sigma_i n_i g_i$$
$$q_*(j(\phi_*(1))) = \Sigma_i n_i q_*(g_i)$$

$$
\begin{tikzcd}
H_n(D^n, \partial D^n) \arrow[r, "\partial_n'"] \arrow[d, "\Phi_*"] & H_{n-1}(\partial D^n) \arrow[d, "\phi_*"] & \\
H_n(X_n, X_{n-1}) \arrow[r, "\partial_n"] \arrow[dr, "d_n"] & H_{n-1}(X_{n-1}) \arrow[d, "j"] & H_{n-1}(X_{n-1}/X_{n-2}) \arrow[d, "j'"] \\
& H_{n-1}(X_{n-1}, X_{n-2}) \arrow[r, "q_*"] & H_{n-1}(X_{n-1}/X_{n-2}, X_{n-2}/X_{n-2})
\end{tikzcd}
$$

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$$
\begin{tikzcd}
H_n(D^n, \partial D^n) \arrow[r, "\partial'_n"] \arrow[d, "\Phi_*"] & H_{n-1}(\partial D^n) \arrow[d, "\phi_*"] & \\
H_n(X_n, X_{n-1}) \arrow[r, "\partial_n"] \arrow[dr, "d_n"] & H_{n-1}(X_{n-1}) \arrow[r, "q'_*"] \arrow[d, "j"] & H_{n-1}(X_{n-1}/X_{n-2}) \arrow[d, "j'"] \\
& H_{n-1}(X_{n-1}, X_{n-2}) \arrow[r, "q_*"] & H_{n-1}(X_{n-1}/X_{n-2}, X_{n-2}/X_{n-2})
\end{tikzcd}
$$

$$d_n(\Phi_*({\partial'_n}^{-1}(1))) = j(\partial_n(\Phi_*({\partial'_n}^{-1}(1)))) = j(\phi_*(1))$$

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$$
\begin{tikzcd}
H_n(D^n, \partial D^n) \arrow[r, "\partial'_n"] \arrow[d, "\Phi_*"] & H_{n-1}(\partial D^n) \arrow[d, "\phi_*"] & \\
H_n(X_n, X_{n-1}) \arrow[r, "\partial_n"] \arrow[dr, "d_n"] & H_{n-1}(X_{n-1}) \arrow[r, "q'_*"] \arrow[d, "j"] & H_{n-1}(X_{n-1}/X_{n-2}) \arrow[d, "j'"] \\
& H_{n-1}(X_{n-1}, X_{n-2}) \arrow[r, "q_*"] & H_{n-1}(X_{n-1}/X_{n-2}, X_{n-2}/X_{n-2})
\end{tikzcd}
$$

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$$q'_*(\phi_*(1)) = \Sigma_i n_i {j'_*}^{-1}(q_*(g_i))$$

Lemma. $H_{n-1}(X_{n-1}/X_{n-2}) \cong \vee S_\alpha^{n-1}$

$$\begin{tikzcd}
H_n(D^n, \partial D^n) \arrow[r, "\partial'_n"] \arrow[d, "\Phi_*"] & H_{n-1}(\partial D^n) \arrow[d, "\phi_*"] & \\
H_n(X_n, X_{n-1}) \arrow[r, "\partial_n"] \arrow[dr, "d_n"] & H_{n-1}(X_{n-1}) \arrow[r, "q'_*"] \arrow[d, "j"] & H_{n-1}(X_{n-1}/X_{n-2}) \arrow[d, "j'"] \\
& H_{n-1}(X_{n-1}, X_{n-2}) \arrow[r, "q_*"] & H_{n-1}(X_{n-1}/X_{n-2}, X_{n-2}/X_{n-2})
\end{tikzcd}$$

$$d_n(\Phi_*({\partial'_n}^{-1}(1))) = j(\partial_n(\Phi_*({\partial'_n}^{-1}(1)))) = j(\phi_*(1))$$

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$${j'_*}^{-1}(q_*(j(\phi_*(1)))) = \Sigma_i n_i {j'_*}^{-1}(q_*(g_i))$$
$$q'_*(\phi_*(1)) = \Sigma_i n_i {j'_*}^{-1}(q_*(g_i))$$

Lemma. $H_{n-1}(X_{n-1}/X_{n-2}) \cong \vee S_\alpha^{n-1} \xrightarrow{q''} S_\beta^{n-1}$

$$\begin{tikzcd}
H_n(D^n, \partial D^n) \arrow[r, "\partial'_n"] \arrow[d, "\Phi_*"] & H_{n-1}(\partial D^n) \arrow[d, "\phi_*"] & \\
H_n(X_n, X_{n-1}) \arrow[r, "\partial_n"] \arrow[dr, "d_n"] & H_{n-1}(X_{n-1}) \arrow[r, "q'_*"] \arrow[d, "j"] & H_{n-1}(X_{n-1}/X_{n-2}) \arrow[d, "j'"] \\
& H_{n-1}(X_{n-1}, X_{n-2}) \arrow[r, "q_*"] & H_{n-1}(X_{n-1}/X_{n-2}, X_{n-2}/X_{n-2})
\end{tikzcd}$$

$$d_n(\Phi_*({\partial'_n}^{-1}(1))) = j(\partial_n(\Phi_*({\partial'_n}^{-1}(1)))) = j(\phi_*(1))$$

$$j(\phi_*(1)) = \Sigma_i n_i g_i$$
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Lemma. $H_{n-1}(X_{n-1}/X_{n-2}) \cong \vee S^{n-1}_\alpha \xrightarrow{q''} S^{n-1}_\beta$ *induces* $q''_* : \oplus_\alpha \mathbb{Z} \to \mathbb{Z}$,

$$\begin{tikzcd}
H_n(D^n, \partial D^n) \arrow[r, "\partial_n'"] \arrow[d, "\Phi_*"] & H_{n-1}(\partial D^n) \arrow[d, "\phi_*"] & & \\
H_n(X_n, X_{n-1}) \arrow[r, "\partial_n"] \arrow[dr, "d_n"] & H_{n-1}(X_{n-1}) \arrow[rr, "q_*'"] \arrow[d, "j"] & & H_{n-1}(X_{n-1}/X_{n-2}) \arrow[d, "j'"] \\
& H_{n-1}(X_{n-1}, X_{n-2}) \arrow[rr, "q_*"] & & H_{n-1}(X_{n-1}/X_{n-2}, X_{n-2}/X_{n-2})
\end{tikzcd}$$

$$d_n(\Phi_*({\partial_n'}^{-1}(1))) = j(\partial_n(\Phi_*({\partial_n'}^{-1}(1)))) = j(\phi_*(1))$$

$$j(\phi_*(1)) = \Sigma_i n_i g_i$$
$$q_*(j(\phi_*(1))) = \Sigma_i n_i q_*(g_i)$$
$${j_*'}^{-1}(q_*(j(\phi_*(1)))) = \Sigma_i n_i {j_*'}^{-1}(q_*(g_i))$$
$$q_*'(\phi_*(1)) = \Sigma_i n_i {j_*'}^{-1}(q_*(g_i))$$

Lemma. *$H_{n-1}(X_{n-1}/X_{n-2}) \cong \vee S_\alpha^{n-1} \xrightarrow{q''} S_\beta^{n-1}$ induces $q_*'' : \oplus_\alpha \mathbb{Z} \to \mathbb{Z}$, which is a projection onto the βth term.*

$$
\begin{tikzcd}
H_n(D^n, \partial D^n) \arrow[r, "\partial'_n"] \arrow[d, "\Phi_*"] & H_{n-1}(\partial D^n) \arrow[d, "\phi_*"] & S_\beta^{n-1} \\
H_n(X_n, X_{n-1}) \arrow[r, "\partial_n"] \arrow[dr, "d_n"] & H_{n-1}(X_{n-1}) \arrow[r, "q'_*"] \arrow[d, "j"] & H_{n-1}(X_{n-1}/X_{n-2}) \arrow[u, "q''"'] \arrow[d, "j'"] \\
& H_{n-1}(X_{n-1}, X_{n-2}) \arrow[r, "q_*"] & H_{n-1}(X_{n-1}/X_{n-2}, X_{n-2}/X_{n-2})
\end{tikzcd}
$$

$$d_n(\Phi_*({\partial'_n}^{-1}(1))) = j(\partial_n(\Phi_*({\partial'_n}^{-1}(1)))) = j(\phi_*(1))$$

$$j(\phi_*(1)) = \Sigma_i n_i g_i$$
$$q_*(j(\phi_*(1))) = \Sigma_i n_i q_*(g_i)$$
$${j'_*}^{-1}(q_*(j(\phi_*(1)))) = \Sigma_i n_i {j'_*}^{-1}(q_*(g_i))$$
$$q'_*(\phi_*(1)) = \Sigma_i n_i {j'_*}^{-1}(q_*(g_i))$$

Lemma. $H_{n-1}(X_{n-1}/X_{n-2}) \cong \vee S_\alpha^{n-1} \xrightarrow{q''} S_\beta^{n-1}$ *induces* $q''_* : \oplus_\alpha \mathbb{Z} \to \mathbb{Z}$*, which is a projection onto the* β*th term.*

$$
\begin{tikzcd}
H_n(D^n, \partial D^n) \arrow[r, "\partial_n'"] \arrow[d, "\Phi_*"] & H_{n-1}(\partial D^n) \arrow[r] \arrow[d, "\phi_*"] & S_\beta^{n-1} \\
H_n(X_n, X_{n-1}) \arrow[r, "\partial_n"] \arrow[dr, "d_n"] & H_{n-1}(X_{n-1}) \arrow[r, "q_*'"] \arrow[d, "j"] & H_{n-1}(X_{n-1}/X_{n-2}) \arrow[u, "q''"] \arrow[d, "j'"] \\
& H_{n-1}(X_{n-1}, X_{n-2}) \arrow[r, "q_*"] & H_{n-1}(X_{n-1}/X_{n-2}, X_{n-2}/X_{n-2})
\end{tikzcd}
$$

$$d_n(\Phi_*({\partial_n'}^{-1}(1))) = j(\partial_n(\Phi_*({\partial_n'}^{-1}(1)))) = j(\phi_*(1))$$

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Lemma. *$H_{n-1}(X_{n-1}/X_{n-2}) \cong \vee S_\alpha^{n-1} \xrightarrow{q''} S_\beta^{n-1}$ induces $q_*'' : \oplus_\alpha \mathbb{Z} \to \mathbb{Z}$, which is a projection onto the βth term.*

$$
\begin{tikzcd}
H_n(D^n, \partial D^n) \arrow[r, "\partial_n'"] \arrow[d, "\Phi_*"] & H_{n-1}(\partial D^n) \arrow[r] \arrow[d, "\phi_*"] & S_\beta^{n-1} \\
H_n(X_n, X_{n-1}) \arrow[r, "\partial_n"] \arrow[dr, "d_n"] & H_{n-1}(X_{n-1}) \arrow[r, "q_*'"] \arrow[d, "j"] & H_{n-1}(X_{n-1}/X_{n-2}) \arrow[u, "q''"] \arrow[d, "j'"] \\
& H_{n-1}(X_{n-1}, X_{n-2}) \arrow[r, "q_*"] & H_{n-1}(X_{n-1}/X_{n-2}, X_{n-2}/X_{n-2})
\end{tikzcd}
$$

$$
\begin{tikzcd}
\partial D^n \arrow[d, "\phi"] & S_\beta^{n-1} \\
X_{n-1} \arrow[r, "q'"] & X_{n-1}/X_{n-2} \arrow[u, "q''"]
\end{tikzcd}
$$

Summary of cellular homology

$H_n(X_n, X_{n-1}) = \oplus_\alpha \mathbb{Z}$

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Genarators g_α^n correspond to n-cells e_α

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$H_n(X_n, X_{n-1}) = \oplus_\alpha \mathbb{Z}$

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$$\to H_{n+1}(X_{n+1}, X_n) \xrightarrow{d_{n+1}} H_n(X_n, X_{n-1}) \xrightarrow{d_n} H_n(X_n, X_{n-1}) \to$$

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$d_n \circ d_{n+1} = 0$

$H_n(X) = ker\ d_n / Im\ d_{n+1}$

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$d_n(g^n_\alpha) = \Sigma n_{\alpha,\beta} g^{n-1}_\beta$

Summary of cellular homology

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Genarators g_α^n correspond to n-cells e_α

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$d_n \circ d_{n+1} = 0$

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$d_n(g_\alpha^n) = \Sigma n_{\alpha,\beta} g_\beta^{n-1}$

To find $n_{\alpha,\beta}$,

Summary of cellular homology

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To find $n_{\alpha,\beta}$,

Let $\phi_\alpha : \partial D_\alpha^n \to X_{n-1}$ be the gluing map.

Summary of cellular homology

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Compose with $q : X_{n-1} \to X_{n-1}/(X_{n-1} \setminus e_\beta) \cong S^{n-1}$,

Summary of cellular homology

$H_n(X_n, X_{n-1}) = \oplus_\alpha \mathbb{Z}$

Genarators g_α^n correspond to n-cells e_α

$\rightarrow H_{n+1}(X_{n+1}, X_n) \xrightarrow{d_{n+1}} H_n(X_n, X_{n-1}) \xrightarrow{d_n} H_n(X_n, X_{n-1}) \rightarrow$

$d_n \circ d_{n+1} = 0$

$H_n(X) = ker\ d_n / Im\ d_{n+1}$

$d_n(g_\alpha^n) = \Sigma n_{\alpha,\beta} g_\beta^{n-1}$

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Compose with $q : X_{n-1} \rightarrow X_{n-1}/(X_{n-1} \setminus e_\beta) \cong S^{n-1}$,

$q \circ \phi_\alpha : S^{n-1} \cong \partial D^{n-1} \rightarrow X_{n-1}/(X_{n-1} \setminus e_\beta) \cong S^{n-1}$

Summary of cellular homology

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Compose with $q : X_{n-1} \to X_{n-1}/(X_{n-1} \setminus e_\beta) \cong S^{n-1}$,

$q \circ \phi_\alpha : S^{n-1} \cong \partial D^{n-1} \to X_{n-1}/(X_{n-1} \setminus e_\beta) \cong S^{n-1}$

$n_{\alpha,\beta} = \text{degree } (q \circ \phi_\alpha)$

Recall:

$\phi : S^n \to S^n$

Recall:

$\phi : S^n \to S^n$

For $y \in S^n$, $\phi^{-1}(y) = \{x_1, \ldots, x_k\}$

Recall:

$\phi : S^n \to S^n$

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degree $\phi = \Sigma_i$ degree $\phi|_{x_i}$

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$$\begin{tikzcd}
 & H_n(S^n, S^n \setminus x_i) & H_n(U_i, U_i \setminus x_i) \arrow[l, "\sim"'] \\
H_n(S^n) \arrow[ur] \arrow[dr] & & \\
 & H_n(S^n, S^n \setminus x_j) & H_n(U_j, U_j \setminus x_j) \arrow[l, "\sim"']
\end{tikzcd}$$

Recall: $\theta : S^n \to S^n$,

$\phi : S^n \to S^n$

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H_n(S^n) \arrow[r] & H_n(S^n, S^n \setminus x_j) & H_n(U_j, U_j \setminus x_j) \arrow[l, "\sim"']
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H_n(S^n) \arrow[r] & H_n(S^n, S^n \setminus x_j) & H_n(U_j, U_j \setminus x_j) \arrow[l, "\sim"']
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Solution:

$I := [-1/2, 1/2]$

$$\widetilde{H}_1(I, \partial I) \xrightarrow{i_*} \widetilde{H}_1(V, V \setminus 0)$$

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$$\tau'_*(g) = ng$$

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$\tau' : (I, \partial I) \to (I, \partial I)$

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$\tau' : \widetilde{H}_1(I, \partial I) \to \widetilde{H}_1(I, \partial I)$

$\tau'_* =?$

$$\widetilde{H}_1(I, \partial I) \xrightarrow{\partial} \widetilde{H}_0(\partial I)$$

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$\tau' : (I, \partial I) \to (I, \partial I)$

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$\tau'_* = -Id$

$V := (-1,1) \subset \mathbb{R}$

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$\tau' : (I, \partial I) \to (I, \partial I)$

$\tau'(t) = -t$

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$\tau'_* = -Id$

Therefore, $\tau_* = -Id$

Example: S^n

Example: S^n

0 cells 1

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0 cells $1\ (H_0(X_0) = \mathbb{Z})$

Example: S^n

0 cells 1 $(H_0(X_0) = \mathbb{Z})$

n cells 1

Example: S^n

0 cells $1 \; (H_0(X_0) = \mathbb{Z})$

n cells $1 \; (H_n(X_n, X_{n-1}) = \mathbb{Z})$

Example: S^n

0 cells 1 $(H_0(X_0) = \mathbb{Z})$

n cells 1 $(H_n(X_n, X_{n-1}) = \mathbb{Z})$

$$H_n(X_{n+1}, X_n) \longrightarrow H_n(X_n, X_{n-1}) \longrightarrow H_n(X_{n-1}, X_{n-2}) \longrightarrow \dots \longrightarrow H_0(X_0)$$

Example: S^n

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$$0 \longrightarrow \mathbb{Z} \longrightarrow 0 \longrightarrow \ldots \longrightarrow \mathbb{Z}$$

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n cells $1 \; (H_n(X_n, X_{n-1}) = \mathbb{Z})$

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$$0 \longrightarrow \mathbb{Z} \longrightarrow 0 \longrightarrow \dots \longrightarrow \mathbb{Z}$$

$$H_k(S^n) = \begin{cases} \mathbb{Z} & k = 0, n \end{cases}$$

Example: S^n

0 cells 1 $(H_0(X_0) = \mathbb{Z})$

n cells 1 $(H_n(X_n, X_{n-1}) = \mathbb{Z})$

$$H_n(X_{n+1}, X_n) \longrightarrow H_n(X_n, X_{n-1}) \longrightarrow H_n(X_{n-1}, X_{n-2}) \longrightarrow \dots \longrightarrow H_0(X_0)$$

$$0 \longrightarrow \mathbb{Z} \longrightarrow 0 \longrightarrow \dots \longrightarrow \mathbb{Z}$$

$$H_k(S^n) = \begin{cases} \mathbb{Z} & k = 0, n \\ 0 & \text{otherwise} \end{cases}$$

Example: $\mathbb{RP}^2$

Example: $\mathbb{RP}^2$

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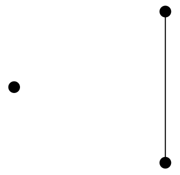
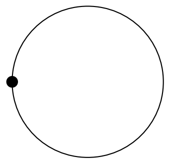
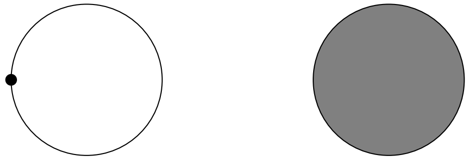
Example: $\mathbb{RP}^2$



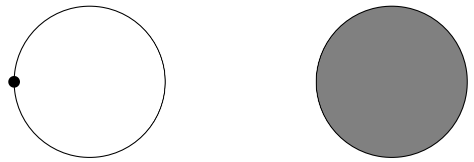
Example: $\mathbb{RP}^2$



Example: $\mathbb{RP}^2$

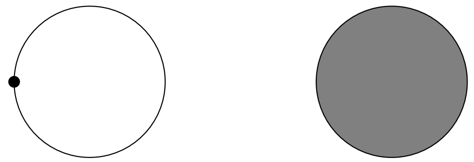


Example: $\mathbb{RP}^2$



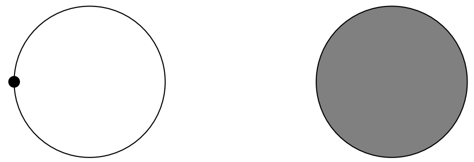
0 cells 1

Example: $\mathbb{RP}^2$



0 cells 1 $(H_0(X_0) = \mathbb{Z})$

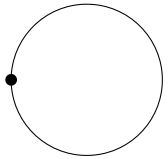
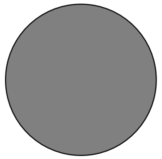
Example: $\mathbb{RP}^2$



0 cells $1\ (H_0(X_0) = \mathbb{Z})$

1 cells 1

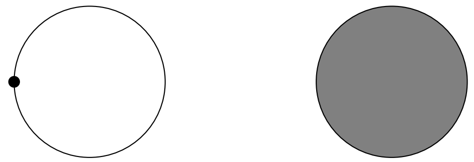
Example: $\mathbb{RP}^2$

0 cells $1 \; (H_0(X_0) = \mathbb{Z})$

1 cells $1 \; (H_1(X_1, X_0) = \mathbb{Z})$

Example: $\mathbb{RP}^2$

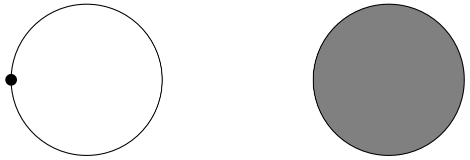


0 cells $1\ (H_0(X_0) = \mathbb{Z})$

1 cells $1\ (H_1(X_1, X_0) = \mathbb{Z})$

2 cells 1

Example: $\mathbb{RP}^2$



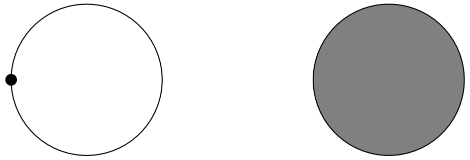
0 cells 1 $(H_0(X_0) = \mathbb{Z})$

1 cells 1 $(H_1(X_1, X_0) = \mathbb{Z})$

2 cells 1 $(H_2(X_2, X_1) = \mathbb{Z})$

Example: $\mathbb{RP}^2$

$$H_2(X_2, X_1) \longrightarrow H_1(X_1, X_0) \longrightarrow H_0(X_0)$$



0 cells $1\ (H_0(X_0) = \mathbb{Z})$

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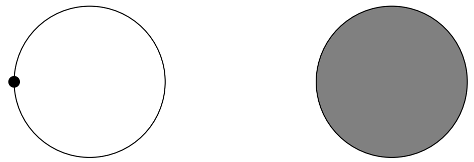
2 cells $1\ (H_2(X_2, X_1) = \mathbb{Z})$

Example: $\mathbb{RP}^2$



$$H_2(X_2, X_1) \longrightarrow H_1(X_1, X_0) \longrightarrow H_0(X_0)$$

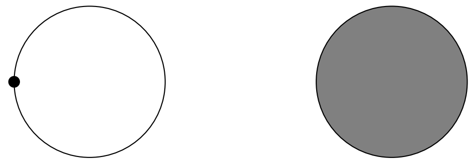
$$\mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}$$

0 cells $1\ (H_0(X_0) = \mathbb{Z})$

1 cells $1\ (H_1(X_1, X_0) = \mathbb{Z})$

2 cells $1\ (H_2(X_2, X_1) = \mathbb{Z})$

Example: $\mathbb{RP}^2$



$$H_2(X_2, X_1) \longrightarrow H_1(X_1, X_0) \longrightarrow H_0(X_0)$$

$$\mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}$$

0 cells $1\ (H_0(X_0) = \mathbb{Z})$

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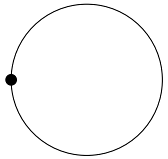
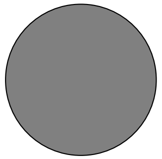
2 cells $1\ (H_2(X_2, X_1) = \mathbb{Z})$

Example: $\mathbb{RP}^2$





$$H_2(X_2, X_1) \longrightarrow H_1(X_1, X_0) \longrightarrow H_0(X_0)$$

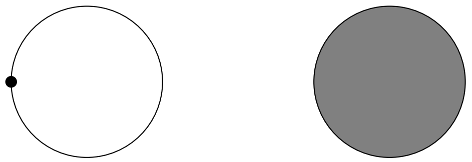
$$\mathbb{Z} \longrightarrow \mathbb{Z} \xrightarrow{0} \mathbb{Z}$$

0 cells $1\ (H_0(X_0) = \mathbb{Z})$

1 cells $1\ (H_1(X_1, X_0) = \mathbb{Z})$

2 cells $1\ (H_2(X_2, X_1) = \mathbb{Z})$

Example: $\mathbb{RP}^2$



$$H_2(X_2, X_1) \longrightarrow H_1(X_1, X_0) \longrightarrow H_0(X_0)$$

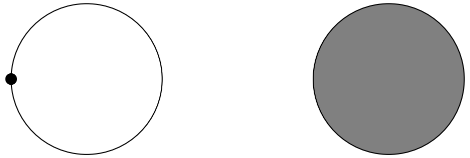
$$\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{0} \mathbb{Z}$$

0 cells 1 $(H_0(X_0) = \mathbb{Z})$

1 cells 1 $(H_1(X_1, X_0) = \mathbb{Z})$

2 cells 1 $(H_2(X_2, X_1) = \mathbb{Z})$

Example: $\mathbb{RP}^2$



0 cells 1 $(H_0(X_0) = \mathbb{Z})$

1 cells 1 $(H_1(X_1, X_0) = \mathbb{Z})$

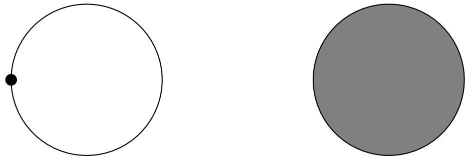
2 cells 1 $(H_2(X_2, X_1) = \mathbb{Z})$

$$H_2(X_2, X_1) \longrightarrow H_1(X_1, X_0) \longrightarrow H_0(X_0)$$

$$\mathbb{Z} \xrightarrow{\quad \times 2 \quad} \mathbb{Z} \xrightarrow{\quad 0 \quad} \mathbb{Z}$$

$$H_k(\mathbb{RP}^2) = \begin{cases} \mathbb{Z} & k = 0 \end{cases}$$

Example: $\mathbb{RP}^2$



0 cells $1 \ (H_0(X_0) = \mathbb{Z})$

1 cells $1 \ (H_1(X_1, X_0) = \mathbb{Z})$

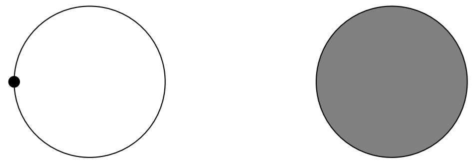
2 cells $1 \ (H_2(X_2, X_1) = \mathbb{Z})$

$$H_2(X_2, X_1) \longrightarrow H_1(X_1, X_0) \longrightarrow H_0(X_0)$$

$$\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{0} \mathbb{Z}$$

$$H_k(\mathbb{RP}^2) = \begin{cases} \mathbb{Z} & k = 0 \\ \mathbb{Z}/2 & k = 1 \end{cases}$$

Example: $\mathbb{RP}^2$



0 cells 1 $(H_0(X_0) = \mathbb{Z})$

1 cells 1 $(H_1(X_1, X_0) = \mathbb{Z})$

2 cells 1 $(H_2(X_2, X_1) = \mathbb{Z})$

$$H_2(X_2, X_1) \longrightarrow H_1(X_1, X_0) \longrightarrow H_0(X_0)$$

$$\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{0} \mathbb{Z}$$

$$H_k(\mathbb{RP}^2) = \begin{cases} \mathbb{Z} & k = 0 \\ \mathbb{Z}/2 & k = 1 \\ 0 & \text{otherwise} \end{cases}$$

Example: $S^1 \times S^1$

Example: $S^1 \times S^1$

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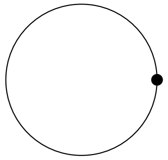
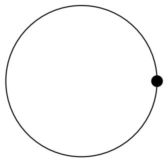
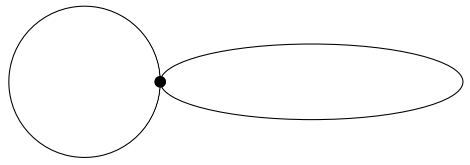


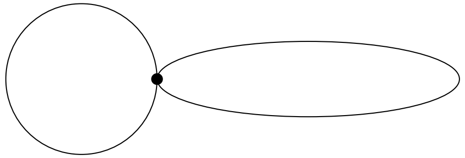
Example: $S^1 \times S^1$

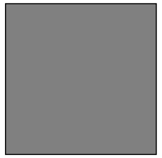


Example: $S^1 \times S^1$

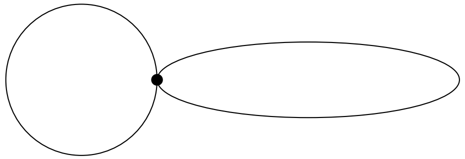


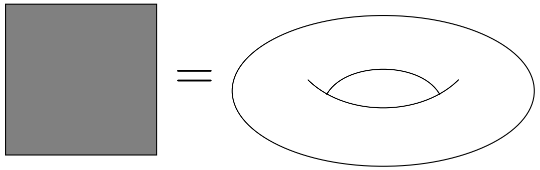
Example: $S^1 \times S^1$



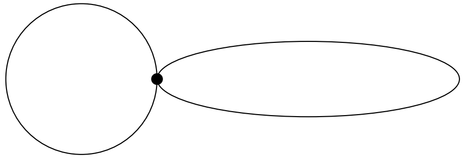


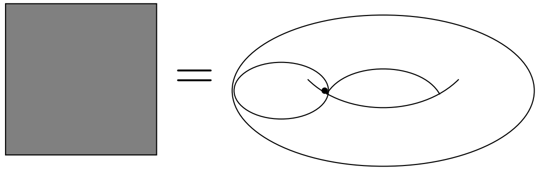
Example: $S^1 \times S^1$



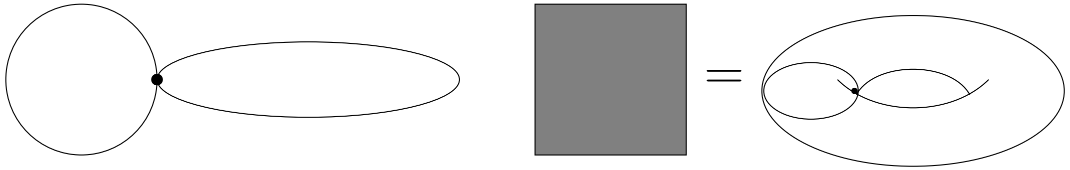


Example: $S^1 \times S^1$



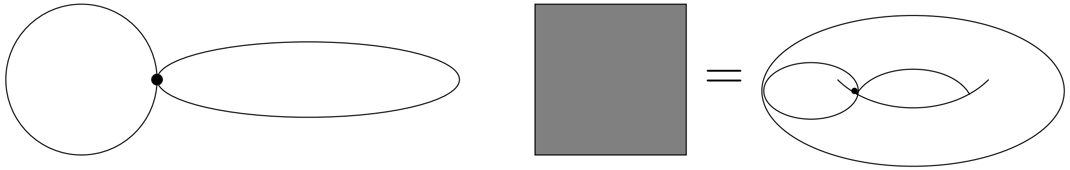


Example: $S^1 \times S^1$



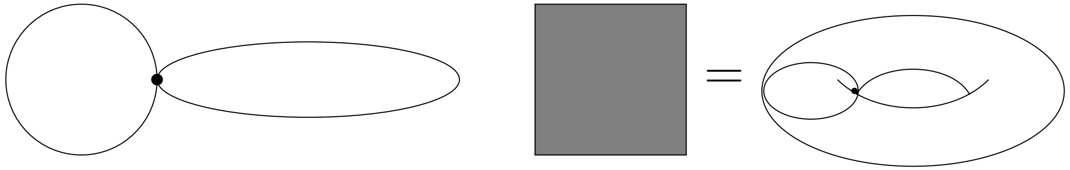
0 cells 1

Example: $S^1 \times S^1$



0 cells 1 $(H_0(X_0) = \mathbb{Z})$

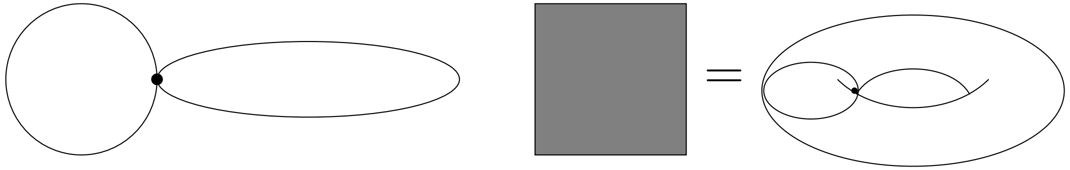
Example: $S^1 \times S^1$



0 cells 1 $(H_0(X_0) = \mathbb{Z})$

1 cells 2

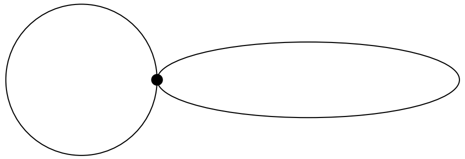
Example: $S^1 \times S^1$

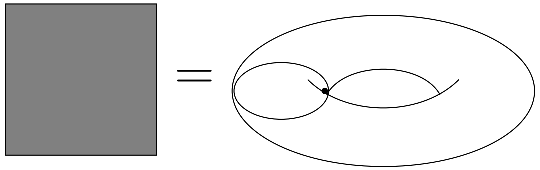


0 cells $1\ (H_0(X_0) = \mathbb{Z})$

1 cells $2\ (H_1(X_1, X_0) = \mathbb{Z} \oplus \mathbb{Z})$

Example: $S^1 \times S^1$



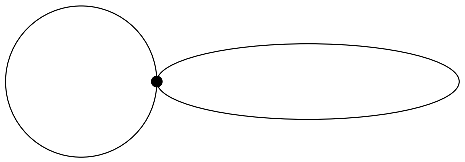


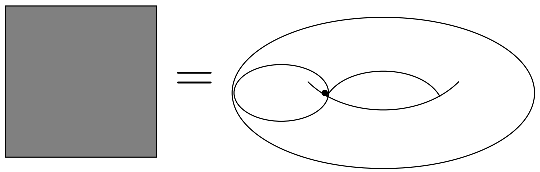
0 cells $1 \ (H_0(X_0) = \mathbb{Z})$

1 cells $2 \ (H_1(X_1, X_0) = \mathbb{Z} \oplus \mathbb{Z})$

2 cells 1

Example: $S^1 \times S^1$





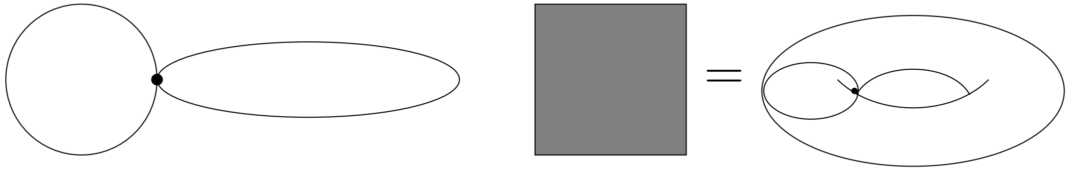
0 cells $1\ (H_0(X_0) = \mathbb{Z})$

1 cells $2\ (H_1(X_1, X_0) = \mathbb{Z} \oplus \mathbb{Z})$

2 cells $1\ (H_2(X_2, X_1) = \mathbb{Z})$

Example: $S^1 \times S^1$

$$H_2(X_2, X_1) \longrightarrow H_1(X_1, X_0) \longrightarrow H_0(X_0)$$



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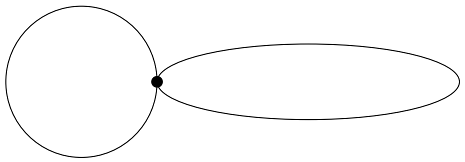
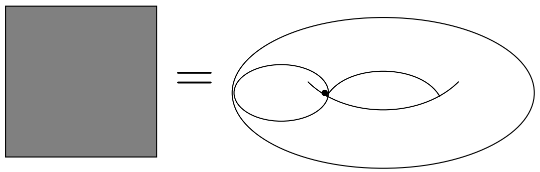
Example: $S^1 \times S^1$





0 cells 1 $(H_0(X_0) = \mathbb{Z})$

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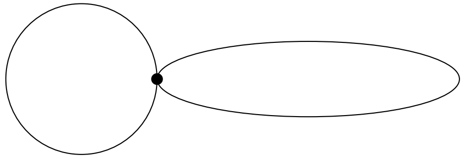
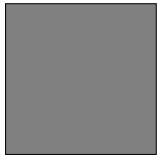
2 cells 1 $(H_2(X_2, X_1) = \mathbb{Z})$

$$H_2(X_2, X_1) \longrightarrow H_1(X_1, X_0) \longrightarrow H_0(X_0)$$

$$\mathbb{Z} \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \longrightarrow \mathbb{Z}$$

Example: $S^1 \times S^1$





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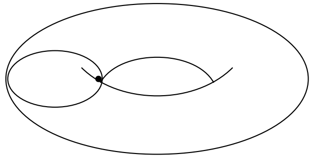


0 cells 1 $(H_0(X_0) = \mathbb{Z})$

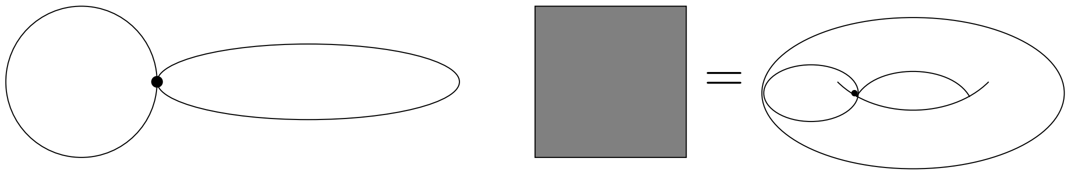
1 cells 2 $(H_1(X_1, X_0) = \mathbb{Z} \oplus \mathbb{Z})$

2 cells 1 $(H_2(X_2, X_1) = \mathbb{Z})$

$$H_2(X_2, X_1) \longrightarrow H_1(X_1, X_0) \longrightarrow H_0(X_0)$$

$$\mathbb{Z} \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \longrightarrow \mathbb{Z}$$

Example: $S^1 \times S^1$



0 cells 1 $(H_0(X_0) = \mathbb{Z})$

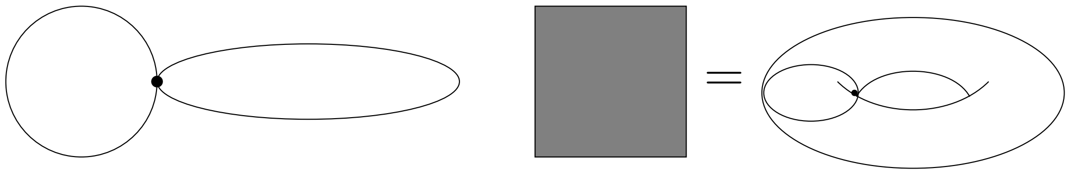
1 cells 2 $(H_1(X_1, X_0) = \mathbb{Z} \oplus \mathbb{Z})$

2 cells 1 $(H_2(X_2, X_1) = \mathbb{Z})$

$$H_2(X_2, X_1) \longrightarrow H_1(X_1, X_0) \longrightarrow H_0(X_0)$$

$$\mathbb{Z} \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{0} \mathbb{Z}$$

Example: $S^1 \times S^1$



0 cells $1 \; (H_0(X_0) = \mathbb{Z})$

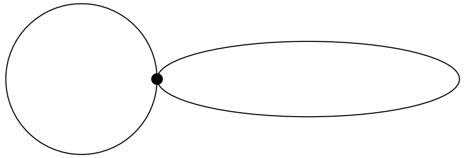
1 cells $2 \; (H_1(X_1, X_0) = \mathbb{Z} \oplus \mathbb{Z})$

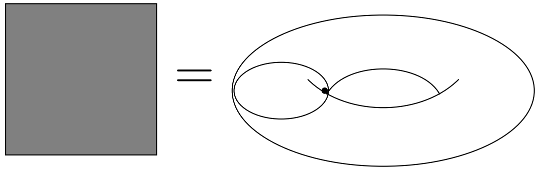
2 cells $1 \; (H_2(X_2, X_1) = \mathbb{Z})$

$$H_2(X_2, X_1) \longrightarrow H_1(X_1, X_0) \longrightarrow H_0(X_0)$$

$$\mathbb{Z} \xrightarrow{\quad 0 \quad} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\quad 0 \quad} \mathbb{Z}$$

Example: $S^1 \times S^1$





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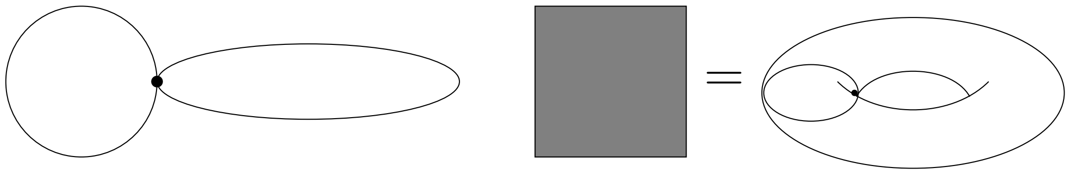
2 cells $1\ (H_2(X_2, X_1) = \mathbb{Z})$

$$H_2(X_2, X_1) \longrightarrow H_1(X_1, X_0) \longrightarrow H_0(X_0)$$

$$\mathbb{Z} \xrightarrow{\quad 0 \quad} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\quad 0 \quad} \mathbb{Z}$$

$$H_k(S^1 \times S^1) = \begin{cases} \mathbb{Z} & k = 0 \end{cases}$$

Example: $S^1 \times S^1$



0 cells 1 $(H_0(X_0) = \mathbb{Z})$

1 cells 2 $(H_1(X_1, X_0) = \mathbb{Z} \oplus \mathbb{Z})$

2 cells 1 $(H_2(X_2, X_1) = \mathbb{Z})$

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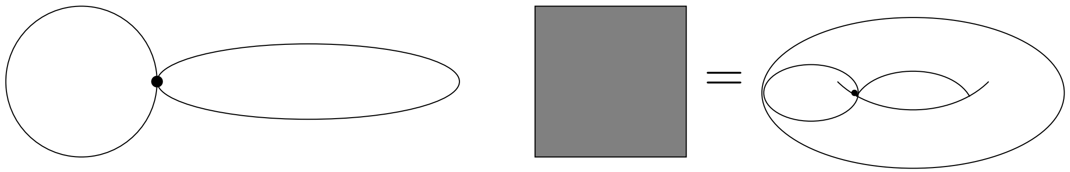
$$\mathbb{Z} \xrightarrow{\quad 0 \quad} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\quad 0 \quad} \mathbb{Z}$$

$$H_k(S^1 \times S^1) = \begin{cases} \mathbb{Z} & k = 0 \\ \mathbb{Z} \oplus \mathbb{Z} & k = 1 \end{cases}$$

Example: $S^1 \times S^1$



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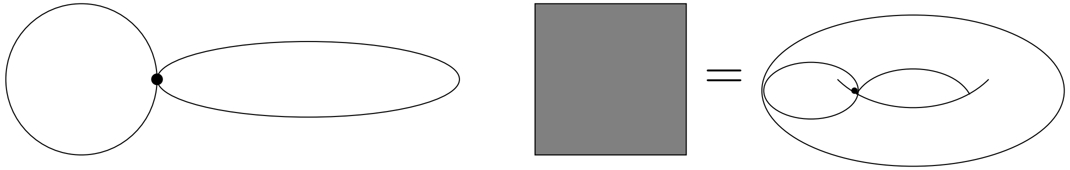

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Example: Klein bottle

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-

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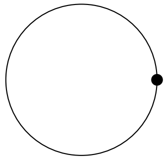

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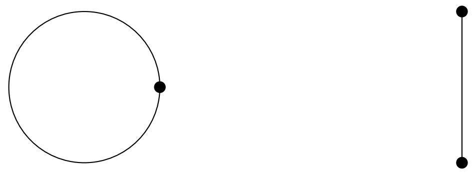

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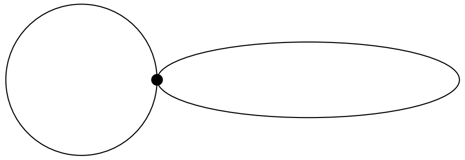

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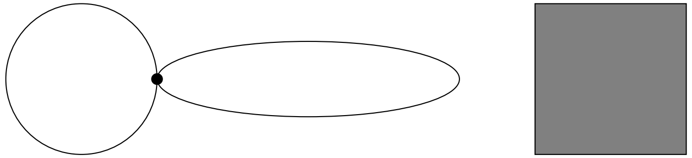

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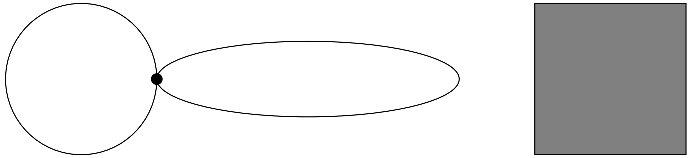
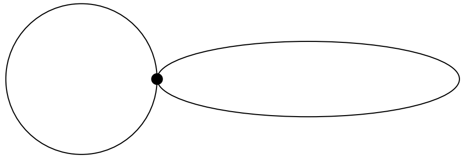
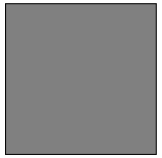


0 cells 1

Example: Klein bottle





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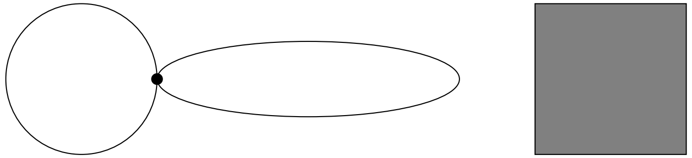

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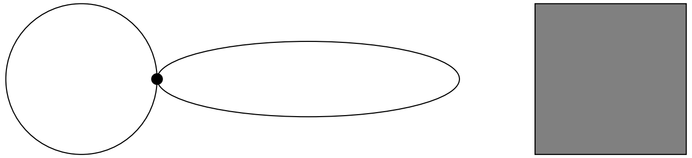
1 cells 2

Example: Klein bottle



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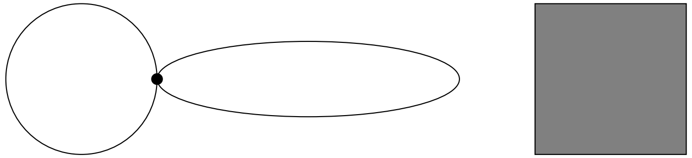

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2 cells 1

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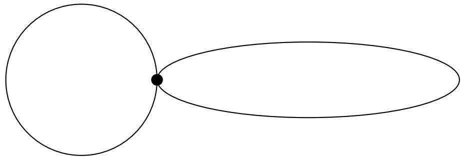

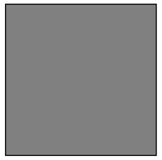

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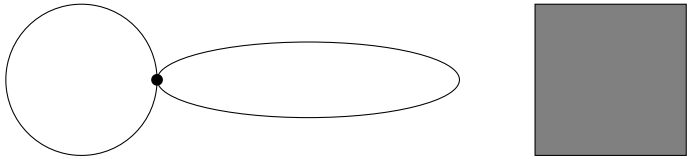

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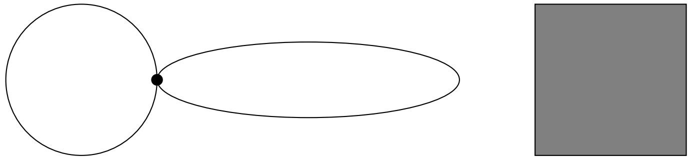

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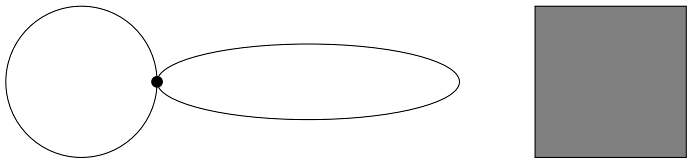

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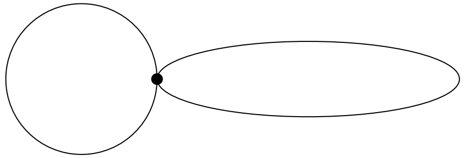

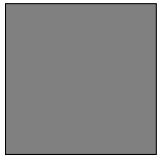

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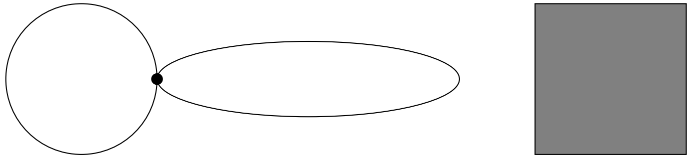

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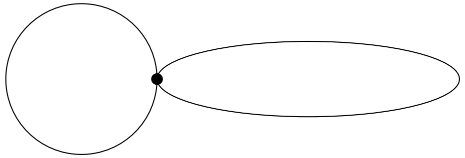

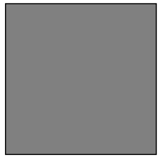

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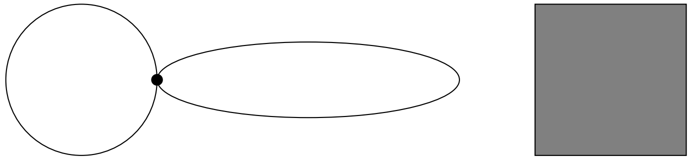

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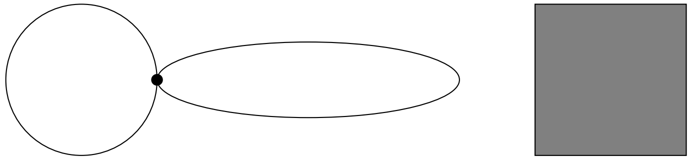

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then, rank B = rank A + rank C

Proof. $0 \to \oplus_{i=1}^{m} \mathbb{Z} \oplus T \xrightarrow{i} \oplus_{i=1}^{n} \mathbb{Z} \oplus T' \to F \oplus T'' \to 0$ $\square$

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Proof. $0 \to \oplus_{i=1}^{m} \mathbb{Z} \oplus T \xrightarrow{i} \oplus_{i=1}^{n} \mathbb{Z} \oplus T' \to F \oplus T'' \to 0$

Injectivity of $i \implies$ free maps to free, each generator maps to a different copy of $\mathbb{Z}$.

Those copies will become torsion in $B/i(A)$ $\square$

$C_n := H_n(X_n, X_{n-1})$ (number of cells of dimension n)

rank $H_n(X) =$ rank $ker\ d_n -$ rank $Im\ d_{n+1}$
rank $ker\ d_n =$ rank $C_n -$ rank $Im\ d_n$
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$\ldots$

$\Sigma(-1)^n$rank $H_n(X) = \Sigma(-1)^n$rank C_n

G abelian, $G \cong \oplus_{i=1}^{k} \mathbb{Z} \oplus$ torsion part
rank $G = k$

Theorem. *A, B, C, finitely generated, abelian*
$0 \to A \to B \to C \to 0$ *short exact*
then, rank $B =$ *rank* $A +$ *rank* C

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$\chi(X) = \Sigma(-1)^n$rank $H_n(X) = \Sigma(-1)^n$rank C_n
(Euler characteristic)

Prove that the quotient map $q : S^1 \times S^1 \to (S^1 \times S^1)/(x_0 \times S^1)$, where $x_0 \in S^1$, induces on isomorphism $q_* : H_2(S^1 \times S^1) \to H_2((S^1 \times S^1)/(x_0 \times S^1))$.

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i_* injective

q_* is an isomorphism

Let p_1, p_2, and p_3 denote 3 distinct points chosen on S^2.

1. Compute $H_i(S^2 \setminus \{p_1, p_2, p_3\})$ for all $i \geq 0$.

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   Solution: $\widetilde{H}_i(S^2) \xrightarrow{\sim} \widetilde{H}_i(S^2/\{p_1, p_2, p_3\})$, if $i > 1$

   $0 \to \widetilde{H}_1(S^2) \to H_1(S^2, \{p_1, p_2, p_3\}) \to \mathbb{Z} \oplus \mathbb{Z} \to 0$

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Let $r : S^1 \times S^1 \to S^1$ denote the map $r(s,t) = s$. Prove that $r_* : H_1(S^1 \times S^1) \cong \mathbb{Z} \oplus \mathbb{Z} \to H_1(S^1) \cong \mathbb{Z}$ is the projection onto the first factor.