MTH439 - Homological methods in Algebraic Topology

Final examination

18th of April, 2021

Duration: 3 hours

1. Suppose that A is a closed subspace of hausdorff space X and there exists a finite set of points, B, so that $X \setminus B$ deformation retracts onto A. Prove that $H_1(X/A) = 0$ if and only if the following two conditions hold: *(5 points)*

   i Every homology class of $H_1(X)$ can be represented by a cycle coming from A.

   ii If two points in A can be joined by a path in X, then they can be joined by a path in A.

2. Consider an n-dimensional Δ-complex. Suppose we delete exactly one point from the interior of each n-simplex to obtain a subspace A, then prove that $H_{n-1}(A)$ has no torsion elements. *(5 points)*

3. (a) Find a 2-dimensional CW complex which has the following homology *(5 points)*

$$H_i(X) = \begin{cases} \mathbb{Z} & i = 0 \\ \mathbb{Z}/2 \oplus \mathbb{Z}/2 & i = 1 \\ \mathbb{Z} \oplus \mathbb{Z} & i = 2 \\ 0 & i > 2 \end{cases}$$

   (Remember to give complete justifications for the degree computations).

   (b) Is it possible to find a 2-dimensional complex that is also a **connected and compact** manifold and with those homologies? *(5 points)*

   (c) Compute all the cohomologies with $\mathbb{Z}/2$ coefficients. *(5 points)*

4. Consider the covering map $M_{\mathbb{Z}} \to M$ defined during the lecture and assume that M is connected and compact.

   (a) What are the possible cardinalities of its set of continuous sections? Under what circumstance does each possibility occur (i.e. can you find a criterion in terms of M)? *(5 points)*

   (b) Prove that each continuous section is completely determined by its value at one point. *(5 points)*