Exercise sheet 5

1. Define $\mathbb{RP}^n = (\mathbb{R}^{n+1} \setminus 0)/\sim$ where $(x_0, x_1, \ldots, x_n) \sim (y_0, y_1, \ldots, y_n)$ if and only if there is an $\alpha \in \mathbb{R}$ such that $(x_0, x_1, \ldots, x_n) = \alpha(y_0, y_1, \ldots, y_n)$. Show that it is a CW-complex for each n and compute its homology.

2. Define $\mathbb{CP}^n = (\mathbb{C}^{n+1} \setminus 0)/\sim$ where $(x_0, x_1, \ldots, x_n) \sim (y_0, y_1, \ldots, y_n)$ if and only if there is an $\alpha \in \mathbb{C}$ such that $(x_0, x_1, \ldots, x_n) = \alpha(y_0, y_1, \ldots, y_n)$. Show that it is a CW-complex for each n and compute its homology.

3. Prove that $H_n(X_n)$ is always free, where X_n denotes the n skeleton of a CW-complex X.

4. Compute $H_k(S^n; G)$ for any abelian group, G.

5. Prove that if $f : S^n \to S^n$ has degree d, then the induced map $f : H_k(S^n; G) \to H_k(S^n; G)$ is multiplication by n.

6. If X is a topological space such that $H_k(X)$ is always finitely generated, then prove that if F is a field, then the $\chi(X) = \Sigma_i(-1)^i \text{dim } H_n(X; F)$.

7. Prove the following properties of Tor_1:

   (a) $Tor_1(A, B) = Tor_1(B, A)$

   (b) $Tor(A, G) = 0$ if G is free

   (c) If $0 \to A \to B \to C \to 0$ is a short exact sequence then, the following sequence is exact: $0 \to Tor_1(A, G) \to Tor_1(B, G) \to Tor_1(C, G) \to A \otimes G \to B \otimes G \to C \otimes G \to 0$