Exercise sheet 6

1. Use the universal coefficient theorem to compute the cohomologies of S^n, the torus, and $\mathbb{RP}^2$, each with coeffients $\mathbb{Z}$, $\mathbb{Z}/2$, and $\mathbb{Q}$. Then redo them using the Mayer-Vietoris and, for S^n, using the long exact sequence of pairs.

2. Prove that the short exact sequence

$$0 \to Ext(H_{n-1}(X), G) \to H^n(X; G) \to Hom(H_n(X), G) \to 0$$

   is natural and splits.

3. Prove that a degree d map $f : S^n \to S^n$ induces a map $f^* : H^n(S^n) \to H^n(S^n)$ which is multiplication by d.

4. Prove that following:

$$\delta(\phi \smile \tau) = \delta\phi \smile \tau + (-1)^m \phi \smile \delta\tau$$

5. Show that if $f : X \to Y$ is a continuous map then $f^*(x \smile y) = f^*(x) \smile f^*(y)$.

6. Show that we can define a relative version of the cup product:

$$\smile : H^m(X, A; R) \times H^n(X, B; R) \to H^{m+n}(X, A \cup B; R)$$

7. Compute the cohomology ring of
   (a) The projective plane.
   (b) The Klein bottle