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unique upto chain homotopy

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$$
\begin{tikzcd}
\cdots \arrow[r] & C_2 \arrow[r] \arrow[d, "\alpha_2{,}\beta_2"] & C_1 \arrow[r] \arrow[d, "\alpha_1{,}\beta_1"] & C_0 \arrow[d, "\alpha_0{,}\beta_0"] \\
\cdots \arrow[r] & C_2' \arrow[r] & C_1' \arrow[r] & C_0'
\end{tikzcd}
$$

$$
\begin{array}{ccccccc}
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\cdots & \longrightarrow & C'_2 & \longrightarrow & C'_1 & \longrightarrow & C'_0
\end{array}
$$

$$\alpha_n - \beta_n = \partial'_{n+1} \circ D_n + D_{n-1} \circ \partial_n$$

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Can we define D_n so that,

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Assume, D_k defined for $k < n$

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & C_2 & \longrightarrow & C_1 & \longrightarrow & C_0 \\
 & & \downarrow{\scriptstyle \alpha_2,\beta_2} & & \downarrow{\scriptstyle \alpha_1,\beta_1} & & \downarrow{\scriptstyle \alpha_0,\beta_0} \\
\cdots & \longrightarrow & C_2' & \longrightarrow & C_1' & \longrightarrow & C_0'
\end{array}$$

$\alpha_n - \beta_n = \partial'_{n+1} \circ D_n + D_{n-1} \circ \partial_n$

Can we define D_n so that,
$\partial'_{n+1}(D_n(x)) = \alpha_n(x) - \beta_n(x) - D_{n-1}(\partial_n(x))$?

Assume, D_k defined for $k < n$. Observe,

$$\partial'_n(\alpha_n(x) - \beta_n(x) - D_{n-1}(\partial_n(x)))$$

$$
\begin{tikzcd}
\cdots \arrow[r] & C_2 \arrow[r] \arrow[d, "\alpha_2{,}\beta_2"] & C_1 \arrow[r] \arrow[d, "\alpha_1{,}\beta_1"] & C_0 \arrow[d, "\alpha_0{,}\beta_0"] \\
\cdots \arrow[r] & C_2' \arrow[r] & C_1' \arrow[r] & C_0'
\end{tikzcd}
$$

$\alpha_n - \beta_n = \partial'_{n+1} \circ D_n + D_{n-1} \circ \partial_n$

Can we define D_n so that,
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Assume, D_k defined for $k < n$. Observe,

$$
\begin{aligned}
& \partial'_n(\alpha_n(x) - \beta_n(x) - D_{n-1}(\partial_n(x))) \\
= \; & \partial'_n\alpha_n(x) - \partial'_n\beta_n(x) - \partial'_n D_{n-1}(\partial_n(x))
\end{aligned}
$$

$$
\begin{tikzcd}
\cdots \arrow[r] & C_2 \arrow[r] \arrow[d, "\alpha_2{,}\beta_2"] & C_1 \arrow[r] \arrow[d, "\alpha_1{,}\beta_1"] & C_0 \arrow[d, "\alpha_0{,}\beta_0"] \\
\cdots \arrow[r] & C_2' \arrow[r] & C_1' \arrow[r] & C_0'
\end{tikzcd}
$$

$\alpha_n - \beta_n = \partial'_{n+1} \circ D_n + D_{n-1} \circ \partial_n$

Can we define D_n so that,
$\partial'_{n+1}(D_n(x)) = \alpha_n(x) - \beta_n(x) - D_{n-1}(\partial_n(x))$?

Assume, D_k defined for $k < n$. Observe,

$$
\begin{aligned}
& \partial'_n(\alpha_n(x) - \beta_n(x) - D_{n-1}(\partial_n(x))) \\
= \ & \partial'_n\alpha_n(x) - \partial'_n\beta_n(x) - \partial'_n D_{n-1}(\partial_n(x)) \\
= \ & \alpha_{n-1}(\partial_n x) - \beta_{n-1}(\partial_n x) - \partial'_n D_{n-1}(\partial_n(x))
\end{aligned}
$$

$$\begin{tikzcd}
\cdots \arrow[r] & C_2 \arrow[r] \arrow[d, "\alpha_2{,}\beta_2"] & C_1 \arrow[r] \arrow[d, "\alpha_1{,}\beta_1"] & C_0 \arrow[d, "\alpha_0{,}\beta_0"] \\
\cdots \arrow[r] & C_2' \arrow[r] & C_1' \arrow[r] & C_0'
\end{tikzcd}$$

$\alpha_n - \beta_n = \partial'_{n+1} \circ D_n + D_{n-1} \circ \partial_n$

Can we define D_n so that,
$\partial'_{n+1}(D_n(x)) = \alpha_n(x) - \beta_n(x) - D_{n-1}(\partial_n(x))$?

Assume, D_k defined for $k < n$. Observe,

$$\begin{aligned}
& \partial'_n(\alpha_n(x) - \beta_n(x) - D_{n-1}(\partial_n(x))) \\
= \ & \partial'_n\alpha_n(x) - \partial'_n\beta_n(x) - \partial'_n D_{n-1}(\partial_n(x)) \\
= \ & \alpha_{n-1}(\partial_n x) - \beta_{n-1}(\partial_n x) - \partial'_n D_{n-1}(\partial_n(x)) \\
= \ & \alpha_{n-1}(\partial_n x) - \beta_{n-1}(\partial_n x) - (\alpha_{n-1}(\partial_n x) - \beta_{n-1}(\partial_n x) - D_{n-2}(\partial_{n-1}\partial_n(x)))
\end{aligned}$$

$$
\begin{array}{ccccccc}
\cdots & \longrightarrow & C_2 & \longrightarrow & C_1 & \longrightarrow & C_0 \\
 & & \Big\downarrow{\scriptstyle \alpha_2,\beta_2} & & \Big\downarrow{\scriptstyle \alpha_1,\beta_1} & & \Big\downarrow{\scriptstyle \alpha_0,\beta_0} \\
\cdots & \longrightarrow & C_2' & \longrightarrow & C_1' & \longrightarrow & C_0'
\end{array}
$$

$\alpha_n - \beta_n = \partial'_{n+1} \circ D_n + D_{n-1} \circ \partial_n$

Can we define D_n so that,
$\partial'_{n+1}(D_n(x)) = \alpha_n(x) - \beta_n(x) - D_{n-1}(\partial_n(x))$?

Assume, D_k defined for $k < n$. Observe,

$$
\begin{aligned}
& \partial'_n(\alpha_n(x) - \beta_n(x) - D_{n-1}(\partial_n(x))) \\
= \ & \partial'_n\alpha_n(x) - \partial'_n\beta_n(x) - \partial'_n D_{n-1}(\partial_n(x)) \\
= \ & \alpha_{n-1}(\partial_n x) - \beta_{n-1}(\partial_n x) - \partial'_n D_{n-1}(\partial_n(x)) \\
= \ & \alpha_{n-1}(\partial_n x) - \beta_{n-1}(\partial_n x) - (\alpha_{n-1}(\partial_n x) - \beta_{n-1}(\partial_n x) - D_{n-2}(\partial_{n-1}\partial_n(x))) \\
= \ & 0
\end{aligned}
$$

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & C_2 & \longrightarrow & C_1 & \longrightarrow & C_0 \\
 & & \Big\downarrow{\scriptstyle \alpha_2,\beta_2} & & \Big\downarrow{\scriptstyle \alpha_1,\beta_1} & & \Big\downarrow{\scriptstyle \alpha_0,\beta_0} \\
\cdots & \longrightarrow & C'_2 & \longrightarrow & C'_1 & \longrightarrow & C'_0
\end{array}$$

$\alpha_n - \beta_n = \partial'_{n+1} \circ D_n + D_{n-1} \circ \partial_n$

Can we define D_n so that,
$\partial'_{n+1}(D_n(x)) = \alpha_n(x) - \beta_n(x) - D_{n-1}(\partial_n(x))$?

Assume, D_k defined for $k < n$. Observe,

$$\begin{aligned}
& \partial'_n(\alpha_n(x) - \beta_n(x) - D_{n-1}(\partial_n(x))) \\
= \ & \partial'_n\alpha_n(x) - \partial'_n\beta_n(x) - \partial'_n D_{n-1}(\partial_n(x)) \\
= \ & \alpha_{n-1}(\partial_n x) - \beta_{n-1}(\partial_n x) - \partial'_n D_{n-1}(\partial_n(x)) \\
= \ & \alpha_{n-1}(\partial_n x) - \beta_{n-1}(\partial_n x) - (\alpha_{n-1}(\partial_n x) - \beta_{n-1}(\partial_n x) - D_{n-2}(\partial_{n-1}\partial_n(x))) \\
= \ & 0
\end{aligned}$$

Therefore, if $H_n(C'_*) = 0$, then $\partial'_{n+1}(x') = \alpha_n(x) - \beta_n(x) - D_{n-1}(\partial_n(x))$ for some x'

$$\begin{tikzcd}
\cdots \arrow[r] & C_2 \arrow[r] \arrow[d, "\alpha_2{,}\beta_2"] & C_1 \arrow[r] \arrow[d, "\alpha_1{,}\beta_1"] & C_0 \arrow[d, "\alpha_0{,}\beta_0"] \\
\cdots \arrow[r] & C'_2 \arrow[r] & C'_1 \arrow[r] & C'_0
\end{tikzcd}$$

$\alpha_n - \beta_n = \partial'_{n+1} \circ D_n + D_{n-1} \circ \partial_n$

Can we define D_n so that,
$\partial'_{n+1}(D_n(x)) = \alpha_n(x) - \beta_n(x) - D_{n-1}(\partial_n(x))$?

Assume, D_k defined for $k < n$. Observe,

$$\begin{aligned}
& \partial'_n(\alpha_n(x) - \beta_n(x) - D_{n-1}(\partial_n(x))) \\
= \ & \partial'_n\alpha_n(x) - \partial'_n\beta_n(x) - \partial'_n D_{n-1}(\partial_n(x)) \\
= \ & \alpha_{n-1}(\partial_n x) - \beta_{n-1}(\partial_n x) - \partial'_n D_{n-1}(\partial_n(x)) \\
= \ & \alpha_{n-1}(\partial_n x) - \beta_{n-1}(\partial_n x) - (\alpha_{n-1}(\partial_n x) - \beta_{n-1}(\partial_n x) - D_{n-2}(\partial_{n-1}\partial_n(x))) \\
= \ & 0
\end{aligned}$$

Therefore, if $H_n(C'_*) = 0$, then $\partial'_{n+1}(x') = \alpha_n(x) - \beta_n(x) - D_{n-1}(\partial_n(x))$ for some x'. Define, $D_n(x) = x'$

$$\begin{tikzcd}
\cdots \arrow[r] & C_2 \arrow[r] \arrow[d, "\alpha_2{,}\beta_2"] & C_1 \arrow[r] \arrow[d, "\alpha_1{,}\beta_1"] & C_0 \arrow[d, "\alpha_0{,}\beta_0"] \\
\cdots \arrow[r] & C_2' \arrow[r] & C_1' \arrow[r] & C_0'
\end{tikzcd}$$

Lemma. *If C_* and C_*' are chain complexes and $H_k(C_*') = 0$ for all k, then any chain maps $\alpha, \beta : C_* \to C_*'$ are chain homotopic.*

$\alpha_n - \beta_n = \partial_{n+1}' \circ D_n + D_{n-1} \circ \partial_n$

Can we define D_n so that,
$\partial_{n+1}'(D_n(x)) = \alpha_n(x) - \beta_n(x) - D_{n-1}(\partial_n(x))$?

Assume, D_k defined for $k < n$. Observe,

$$\begin{aligned}
& \partial_n'(\alpha_n(x) - \beta_n(x) - D_{n-1}(\partial_n(x))) \\
= \; & \partial_n'\alpha_n(x) - \partial_n'\beta_n(x) - \partial_n' D_{n-1}(\partial_n(x)) \\
= \; & \alpha_{n-1}(\partial_n x) - \beta_{n-1}(\partial_n x) - \partial_n' D_{n-1}(\partial_n(x)) \\
= \; & \alpha_{n-1}(\partial_n x) - \beta_{n-1}(\partial_n x) - (\alpha_{n-1}(\partial_n x) - \beta_{n-1}(\partial_n x) - D_{n-2}(\partial_{n-1}\partial_n(x))) \\
= \; & 0
\end{aligned}$$

Therefore, if $H_n(C_*') = 0$, then $\partial_{n+1}'(x') = \alpha_n(x) - \beta_n(x) - D_{n-1}(\partial_n(x))$ for some x'. Define, $D_n(x) = x'$

$$\cdots \to C_n(X) \xrightarrow{\partial_n} C_{n-1}(X) \to \cdots \to C_1(X) \xrightarrow{\partial_1} C_0(X)$$

$$\cdots \to C_n(X) \otimes G \xrightarrow{\partial_n \otimes Id} C_{n-1}(X) \otimes G \to \cdots \to C_1(X) \otimes G \xrightarrow{\partial_1 \otimes Id} C_0(X) \otimes G$$

G abelian

$$\cdots \to C_n(X) \otimes G \xrightarrow{\partial_n \otimes Id} C_{n-1}(X) \otimes G \to \cdots \to C_1(X) \otimes G \xrightarrow{\partial_1 \otimes Id} C_0(X) \otimes G$$

G abelian

$$(\partial_n \otimes Id) \circ (\partial_{n-1} \otimes Id) = 0$$

$$\cdots \to C_n(X) \otimes G \xrightarrow{\partial_n \otimes Id} C_{n-1}(X) \otimes G \to \cdots \to C_1(X) \otimes G \xrightarrow{\partial_1 \otimes Id} C_0(X) \otimes G$$

G abelian

$$(\partial_n \otimes Id) \circ (\partial_{n-1} \otimes Id) = 0$$

$$Z_n(X;G) := ker\ (\partial_n \otimes Id)$$

$$\cdots \to C_n(X) \otimes G \xrightarrow{\partial_n \otimes Id} C_{n-1}(X) \otimes G \to \cdots \to C_1(X) \otimes G \xrightarrow{\partial_1 \otimes Id} C_0(X) \otimes G$$

G abelian

$$(\partial_n \otimes Id) \circ (\partial_{n-1} \otimes Id) = 0$$

$$Z_n(X;G) := ker\ (\partial_n \otimes Id)$$

$$B_n(X;G) := Im\ (\partial_{n+1} \otimes Id)$$

$$\cdots \to C_n(X) \otimes G \xrightarrow{\partial_n \otimes Id} C_{n-1}(X) \otimes G \to \cdots \to C_1(X) \otimes G \xrightarrow{\partial_1 \otimes Id} C_0(X) \otimes G$$

G abelian

$$(\partial_n \otimes Id) \circ (\partial_{n-1} \otimes Id) = 0$$

$$Z_n(X;G) := ker\ (\partial_n \otimes Id)$$

$$B_n(X;G) := Im\ (\partial_{n+1} \otimes Id)$$

$$H_n(X;G) := Z_n(X;G)/B_n(X;G)$$

$$\cdots \to C_n(X) \otimes G \xrightarrow{\partial_n \otimes Id} C_{n-1}(X) \otimes G \to \cdots \to C_1(X) \otimes G \xrightarrow{\partial_1 \otimes Id} C_0(X) \otimes G$$

G abelian

$$(\partial_n \otimes Id) \circ (\partial_{n-1} \otimes Id) = 0$$

$$Z_n(X;G) := ker\ (\partial_n \otimes Id)$$

$$B_n(X;G) := Im\ (\partial_{n+1} \otimes Id)$$

$$H_n(X;G) := Z_n(X;G)/B_n(X;G)$$

$$H_n(X) \otimes G \to H_n(X;G)$$

$$\cdots \to C_n(X) \otimes G \xrightarrow{\partial_n \otimes Id} C_{n-1}(X) \otimes G \to \cdots \to C_1(X) \otimes G \xrightarrow{\partial_1 \otimes Id} C_0(X) \otimes G$$

G abelian

$$(\partial_n \otimes Id) \circ (\partial_{n-1} \otimes Id) = 0$$

$$Z_n(X;G) := ker\ (\partial_n \otimes Id)$$

$$B_n(X;G) := Im\ (\partial_{n+1} \otimes Id)$$

$$H_n(X;G) := Z_n(X;G)/B_n(X;G)$$

$$H_n(X) \otimes G \to H_n(X;G)\ [c] \otimes g \to [c \otimes g]$$

$$0 \longrightarrow Z_{n+1} \longrightarrow C_{n+1} \longrightarrow B_n \longrightarrow 0$$

$$0 \longrightarrow Z_{n+1} \longrightarrow C_{n+1} \longrightarrow B_n \longrightarrow 0$$

$$0 \longrightarrow Z_n \longrightarrow C_n \longrightarrow B_{n-1} \longrightarrow 0$$

$$\begin{tikzcd}
0 \arrow[r] & Z_{n+1} \arrow[r] & C_{n+1} \arrow[r] \arrow[d, "\partial"] & B_n \arrow[r] & 0 \\
0 \arrow[r] & Z_n \arrow[r] & C_n \arrow[r] & B_{n-1} \arrow[r] & 0
\end{tikzcd}$$

$$
\begin{tikzcd}
0 \arrow[r] & Z_{n+1} \arrow[r] \arrow[d, "\partial"] & C_{n+1} \arrow[r] \arrow[d, "\partial"] & B_n \arrow[r] & 0 \\
0 \arrow[r] & Z_n \arrow[r] & C_n \arrow[r] & B_{n-1} \arrow[r] & 0
\end{tikzcd}
$$

$$
\begin{tikzcd}
0 \arrow[r] & Z_{n+1} \arrow[r] \arrow[d, "\partial=0"] & C_{n+1} \arrow[r] \arrow[d, "\partial"] & B_n \arrow[r] & 0 \\
0 \arrow[r] & Z_n \arrow[r] & C_n \arrow[r] & B_{n-1} \arrow[r] & 0
\end{tikzcd}
$$

$$\begin{tikzcd}
0 \arrow[r] & Z_{n+1} \arrow[r] \arrow[d, "\partial=0"] & C_{n+1} \arrow[r] \arrow[d, "\partial"] & B_n \arrow[r] \arrow[d, "\partial"] & 0 \\
0 \arrow[r] & Z_n \arrow[r] & C_n \arrow[r] & B_{n-1} \arrow[r] & 0
\end{tikzcd}$$

$$
\begin{tikzcd}
0 \arrow[r] & Z_{n+1} \arrow[r] \arrow[d, "\partial=0"] & C_{n+1} \arrow[r] \arrow[d, "\partial"] & B_n \arrow[r] \arrow[d, "\partial=0"] & 0 \\
0 \arrow[r] & Z_n \arrow[r] & C_n \arrow[r] & B_{n-1} \arrow[r] & 0
\end{tikzcd}
$$

$$
\begin{tikzcd}
0 \arrow[r] & Z_{n+1} \arrow[r] \arrow[d, "\partial=0"] & C_{n+1} \arrow[r] \arrow[d, "\partial"] & B_n \arrow[r] \arrow[d, "\partial=0"] & 0 \\
0 \arrow[r] & Z_n \arrow[r] & C_n \arrow[r] & B_{n-1} \arrow[r] & 0
\end{tikzcd}
$$

$$\cdots \to B_n \to Z_n \xrightarrow{j} H_n(C_*) \to B_{n-1} \to \cdots$$

$$b_n = \partial c_{n+1}$$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Z_{n+1} & \longrightarrow & C_{n+1} & \longrightarrow & B_n & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle \partial=0} & & \Big\downarrow{\scriptstyle \partial} & & \Big\downarrow{\scriptstyle \partial=0} & & \\ 0 & \longrightarrow & Z_n & \longrightarrow & C_n & \longrightarrow & B_{n-1} & \longrightarrow & 0 \end{array}$$

$$\cdots \to B_n \to Z_n \xrightarrow{j} H_n(C_*) \to B_{n-1} \to \cdots$$

$b_n = \partial c_{n+1}$

$b_n \in B_n \to \partial c_{n+1} = b_n \in Z_n$

$$
\begin{tikzcd}
0 \arrow[r] & Z_{n+1} \arrow[r] \arrow[d, "\partial=0"] & C_{n+1} \arrow[r] \arrow[d, "\partial"] & B_n \arrow[r] \arrow[d, "\partial=0"] & 0 \\
0 \arrow[r] & Z_n \arrow[r] & C_n \arrow[r] & B_{n-1} \arrow[r] & 0
\end{tikzcd}
$$

$$\cdots \to B_n \to Z_n \xrightarrow{j} H_n(C_*) \to B_{n-1} \to \cdots$$

$b_n = \partial c_{n+1}$

$b_n \in B_n \to \partial c_{n+1} = b_n \in Z_n$

Therefore, $b_n \in B_n \to b_n \in Z_n$

$$\begin{tikzcd}
0 \arrow[r] & Z_{n+1} \arrow[r] \arrow[d, "\partial=0"] & C_{n+1} \arrow[r] \arrow[d, "\partial"] & B_n \arrow[r] \arrow[d, "\partial=0"] & 0 \\
0 \arrow[r] & Z_n \arrow[r] & C_n \arrow[r] & B_{n-1} \arrow[r] & 0
\end{tikzcd}$$

$$\cdots \to B_n \xrightarrow{i} Z_n \xrightarrow{j} H_n(C_*) \to B_{n-1} \to \cdots$$

$b_n = \partial c_{n+1}$
$b_n \in B_n \to \partial c_{n+1} = b_n \in Z_n$

Therefore, $b_n \in B_n \to b_n \in Z_n$

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Z_{n+1} & \longrightarrow & C_{n+1} & \longrightarrow & B_n & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle \partial=0} & & \big\downarrow{\scriptstyle \partial} & & \big\downarrow{\scriptstyle \partial=0} & & \\
0 & \longrightarrow & Z_n & \longrightarrow & C_n & \longrightarrow & B_{n-1} & \longrightarrow & 0
\end{array}$$

$$\cdots \to B_n \xrightarrow{i} Z_n \xrightarrow{j} H_n(C_*) \to B_{n-1} \to \cdots$$

$b_n = \partial c_{n+1}$

$b_n \in B_n \to \partial c_{n+1} = b_n \in Z_n$

Therefore, $b_n \in B_n \to b_n \in Z_n$

$$Z_n \xrightarrow{j_*} H_n(C_*) \to ker\ i \to 0$$

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & Z_{n+1} & \longrightarrow & C_{n+1} & \longrightarrow & B_n & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \partial=0} & & \Big\downarrow{\scriptstyle \partial} & & \Big\downarrow{\scriptstyle \partial=0} & & \\
0 & \longrightarrow & Z_n & \longrightarrow & C_n & \longrightarrow & B_{n-1} & \longrightarrow & 0
\end{array}
$$

$$\cdots \to B_n \xrightarrow{i} Z_n \xrightarrow{j} H_n(C_*) \to B_{n-1} \to \cdots$$

$b_n = \partial c_{n+1}$

$b_n \in B_n \to \partial c_{n+1} = b_n \in Z_n$

Therefore, $b_n \in B_n \to b_n \in Z_n$

$$Z_n \xrightarrow{j_*} H_n(C_*) \to 0$$

$$\begin{tikzcd}
0 \arrow[r] & Z_{n+1} \arrow[r] \arrow[d, "\partial=0"] & C_{n+1} \arrow[r] \arrow[d, "\partial"] & B_n \arrow[r] \arrow[d, "\partial=0"] & 0 \\
0 \arrow[r] & Z_n \arrow[r] & C_n \arrow[r] & B_{n-1} \arrow[r] & 0
\end{tikzcd}$$

$$\cdots \to B_n \xrightarrow{i} Z_n \xrightarrow{j} H_n(C_*) \to B_{n-1} \to \cdots$$

$b_n = \partial c_{n+1}$
$b_n \in B_n \to \partial c_{n+1} = b_n \in Z_n$

Therefore, $b_n \in B_n \to b_n \in Z_n$

$$0 \to Z_n/ker\ j \xrightarrow{j_*} H_n(C_*) \to 0$$

$$\begin{tikzcd}
0 \arrow[r] & Z_{n+1} \arrow[r] \arrow[d, "\partial=0"] & C_{n+1} \arrow[r] \arrow[d, "\partial"] & B_n \arrow[r] \arrow[d, "\partial=0"] & 0 \\
0 \arrow[r] & Z_n \arrow[r] & C_n \arrow[r] & B_{n-1} \arrow[r] & 0
\end{tikzcd}$$

$$\cdots \to B_n \xrightarrow{i} Z_n \xrightarrow{j} H_n(C_*) \to B_{n-1} \to \cdots$$

$b_n = \partial c_{n+1}$
$b_n \in B_n \to \partial c_{n+1} = b_n \in Z_n$

Therefore, $b_n \in B_n \to b_n \in Z_n$

$$0 \to Z_n/i(B_n) \xrightarrow{j_*} H_n(C_*) \to 0$$

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & Z_{n+1} & \longrightarrow & C_{n+1} & \longrightarrow & B_n & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \partial=0} & & \Big\downarrow{\scriptstyle \partial} & & \Big\downarrow{\scriptstyle \partial=0} & & \\
0 & \longrightarrow & Z_n & \longrightarrow & C_n & \longrightarrow & B_{n-1} & \longrightarrow & 0
\end{array}
$$

$$\cdots \to B_n \xrightarrow{i} Z_n \xrightarrow{j} H_n(C_*) \to B_{n-1} \to \cdots$$

$b_n = \partial c_{n+1}$

$b_n \in B_n \to \partial c_{n+1} = b_n \in Z_n$

Therefore, $b_n \in B_n \to b_n \in Z_n$

$0 \to H_n(C_*) \xrightarrow{j_*} H_n(C_*) \to 0$

$$0 \longrightarrow Z_{n+1} \xrightarrow{j} C_{n+1} \xrightarrow{\partial} B_n \longrightarrow 0$$

$$0 \longrightarrow Z_{n+1} \xrightarrow{j} C_{n+1} \xrightarrow{\partial} B_n \longrightarrow 0$$

$$0 \longrightarrow Z_n \xrightarrow{j} C_n \xrightarrow{\partial} B_{n-1} \longrightarrow 0$$

$$
\begin{tikzcd}
0 \arrow[r] & Z_{n+1} \arrow[r, "j"] & C_{n+1} \arrow[r, "\partial"] \arrow[d, "\partial"] & B_n \arrow[r] & 0 \\
0 \arrow[r] & Z_n \arrow[r, "j"] & C_n \arrow[r, "\partial"] & B_{n-1} \arrow[r] & 0
\end{tikzcd}
$$

$$\begin{tikzcd}
0 \arrow[r] & Z_{n+1} \arrow[r, "j"] \arrow[d, "\partial"] & C_{n+1} \arrow[r, "\partial"] \arrow[d, "\partial"] & B_n \arrow[r] & 0 \\
0 \arrow[r] & Z_n \arrow[r, "j"] & C_n \arrow[r, "\partial"] & B_{n-1} \arrow[r] & 0
\end{tikzcd}$$

$$
\begin{tikzcd}
0 \arrow[r] & Z_{n+1} \arrow[r, "j"] \arrow[d, "\partial=0"] & C_{n+1} \arrow[r, "\partial"] \arrow[d, "\partial"] & B_n \arrow[r] & 0 \\
0 \arrow[r] & Z_n \arrow[r, "j"] & C_n \arrow[r, "\partial"] & B_{n-1} \arrow[r] & 0
\end{tikzcd}
$$

$$\begin{tikzcd}
0 \arrow[r] & Z_{n+1} \arrow[r, "j"] \arrow[d, "\partial=0"] & C_{n+1} \arrow[r, "\partial"] \arrow[d, "\partial"] & B_n \arrow[r] \arrow[d, "\partial"] & 0 \\
0 \arrow[r] & Z_n \arrow[r, "j"] & C_n \arrow[r, "\partial"] & B_{n-1} \arrow[r] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[r] & Z_{n+1} \arrow[r, "j"] \arrow[d, "\partial=0"] & C_{n+1} \arrow[r, "\partial"] \arrow[d, "\partial"] & B_n \arrow[r] \arrow[d, "\partial=0"] & 0 \\
0 \arrow[r] & Z_n \arrow[r, "j"] & C_n \arrow[r, "\partial"] & B_{n-1} \arrow[r] & 0
\end{tikzcd}$$

$$
\begin{tikzcd}
0 \arrow[r] & Z_{n+1} \otimes G \arrow[r, "j\otimes Id"] \arrow[d, "\partial\otimes Id=0"] & C_{n+1} \otimes G \arrow[r, "\partial\otimes Id"] \arrow[d, "\partial\otimes Id"] & B_n \otimes G \arrow[r] \arrow[d, "\partial\otimes Id=0"] & 0 \\
0 \arrow[r] & Z_n \otimes G \arrow[r, "j\otimes Id"] & C_n \otimes G \arrow[r, "\partial\otimes Id"] & B_{n-1} \otimes G \arrow[r] & 0
\end{tikzcd}
$$

$$\begin{tikzcd}
0 \arrow[r] & Z_{n+1} \otimes G \arrow[r, "j\otimes Id"] \arrow[d, "\partial\otimes Id=0"] & C_{n+1} \otimes G \arrow[r, "\partial\otimes Id"] \arrow[d, "\partial\otimes Id"] & B_n \otimes G \arrow[r] \arrow[d, "\partial\otimes Id=0"] & 0 \\
0 \arrow[r] & Z_n \otimes G \arrow[r, "j\otimes Id"] & C_n \otimes G \arrow[r, "\partial\otimes Id"] & B_{n-1} \otimes G \arrow[r] & 0
\end{tikzcd}$$

$$\cdots H_n(Z_* \otimes G)$$

$$\begin{tikzcd}
0 \arrow[r] & Z_{n+1} \otimes G \arrow[r, "j\otimes Id"] \arrow[d, "\partial\otimes Id=0"] & C_{n+1} \otimes G \arrow[r, "\partial\otimes Id"] \arrow[d, "\partial\otimes Id"] & B_n \otimes G \arrow[r] \arrow[d, "\partial\otimes Id=0"] & 0 \\
0 \arrow[r] & Z_n \otimes G \arrow[r, "j\otimes Id"] & C_n \otimes G \arrow[r, "\partial\otimes Id"] & B_{n-1} \otimes G \arrow[r] & 0
\end{tikzcd}$$

$$\cdots H_n(Z_* \otimes G) \to H_n(C_* \otimes G)$$

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Z_{n+1} \otimes G & \xrightarrow{j \otimes Id} & C_{n+1} \otimes G & \xrightarrow{\partial \otimes Id} & B_n \otimes G & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \partial \otimes Id = 0} & & \Big\downarrow{\scriptstyle \partial \otimes Id} & & \Big\downarrow{\scriptstyle \partial \otimes Id = 0} & & \\
0 & \longrightarrow & Z_n \otimes G & \xrightarrow{j \otimes Id} & C_n \otimes G & \xrightarrow{\partial \otimes Id} & B_{n-1} \otimes G & \longrightarrow & 0
\end{array}$$

$$\cdots H_n(Z_* \otimes G) \to H_n(C_* \otimes G) \to H_n(B_{*-1} \otimes G)$$

$$
\begin{tikzcd}
0 \arrow[r] & Z_{n+1} \otimes G \arrow[r, "j\otimes Id"] \arrow[d, "\partial\otimes Id=0"] & C_{n+1} \otimes G \arrow[r, "\partial\otimes Id"] \arrow[d, "\partial\otimes Id"] & B_n \otimes G \arrow[r] \arrow[d, "\partial\otimes Id=0"] & 0 \\
0 \arrow[r] & Z_n \otimes G \arrow[r, "j\otimes Id"] & C_n \otimes G \arrow[r, "\partial\otimes Id"] & B_{n-1} \otimes G \arrow[r] & 0
\end{tikzcd}
$$

$$\cdots H_n(Z_* \otimes G) \to H_n(C_* \otimes G) \to H_n(B_{*-1} \otimes G) \to H_{n-1}(Z_* \otimes G) \to \cdots$$

$$
\begin{tikzcd}
0 \arrow[r] & Z_{n+1} \otimes G \arrow[r, "j\otimes Id"] \arrow[d, "\partial\otimes Id=0"] & C_{n+1} \otimes G \arrow[r, "\partial\otimes Id"] \arrow[d, "\partial\otimes Id"] & B_n \otimes G \arrow[r] \arrow[d, "\partial\otimes Id=0"] & 0 \\
0 \arrow[r] & Z_n \otimes G \arrow[r, "j\otimes Id"] & C_n \otimes G \arrow[r, "\partial\otimes Id"] & B_{n-1} \otimes G \arrow[r] & 0
\end{tikzcd}
$$

$$\cdots Z_n \otimes G \to H_n(C_* \otimes G) \to H_n(B_{*-1} \otimes G) \to H_{n-1}(Z_* \otimes G) \to \cdots$$

$$\begin{tikzcd}
0 \arrow[r] & Z_{n+1} \otimes G \arrow[r, "j\otimes Id"] \arrow[d, "\partial\otimes Id=0"] & C_{n+1} \otimes G \arrow[r, "\partial\otimes Id"] \arrow[d, "\partial\otimes Id"] & B_n \otimes G \arrow[r] \arrow[d, "\partial\otimes Id=0"] & 0 \\
0 \arrow[r] & Z_n \otimes G \arrow[r, "j\otimes Id"] & C_n \otimes G \arrow[r, "\partial\otimes Id"] & B_{n-1} \otimes G \arrow[r] & 0
\end{tikzcd}$$

$$\cdots Z_n \otimes G \to H_n(C;G) \to H_n(B_{*-1} \otimes G) \to H_{n-1}(Z_* \otimes G) \to \cdots$$

$$\begin{tikzcd}
0 \arrow[r] & Z_{n+1} \otimes G \arrow[r, "j\otimes Id"] \arrow[d, "\partial\otimes Id=0"] & C_{n+1} \otimes G \arrow[r, "\partial\otimes Id"] \arrow[d, "\partial\otimes Id"] & B_n \otimes G \arrow[r] \arrow[d, "\partial\otimes Id=0"] & 0 \\
0 \arrow[r] & Z_n \otimes G \arrow[r, "j\otimes Id"] & C_n \otimes G \arrow[r, "\partial\otimes Id"] & B_{n-1} \otimes G \arrow[r] & 0
\end{tikzcd}$$

$$\cdots Z_n \otimes G \to H_n(C;G) \to B_{n-1} \otimes G \to H_{n-1}(Z_* \otimes G) \to \cdots$$

$$\begin{tikzcd}
0 \arrow[r] & Z_{n+1} \otimes G \arrow[r, "j\otimes Id"] \arrow[d, "\partial\otimes Id=0"] & C_{n+1} \otimes G \arrow[r, "\partial\otimes Id"] \arrow[d, "\partial\otimes Id"] & B_n \otimes G \arrow[r] \arrow[d, "\partial\otimes Id=0"] & 0 \\
0 \arrow[r] & Z_n \otimes G \arrow[r, "j\otimes Id"] & C_n \otimes G \arrow[r, "\partial\otimes Id"] & B_{n-1} \otimes G \arrow[r] & 0
\end{tikzcd}$$

$$\cdots Z_n \otimes G \to H_n(C;G) \to B_{n-1} \otimes G \to Z_{n-1} \otimes G \to \cdots$$

$$
\begin{tikzcd}
0 \arrow[r] & Z_{n+1} \otimes G \arrow[r, "j\otimes Id"] \arrow[d, "\partial\otimes Id=0"] & C_{n+1} \otimes G \arrow[r, "\partial\otimes Id"] \arrow[d, "\partial\otimes Id"] & B_n \otimes G \arrow[r] \arrow[d, "\partial\otimes Id=0"] & 0 \\
0 \arrow[r] & Z_n \otimes G \arrow[r, "j\otimes Id"] & C_n \otimes G \arrow[r, "\partial\otimes Id"] & B_{n-1} \otimes G \arrow[r] & 0
\end{tikzcd}
$$

$$
\cdots Z_n \otimes G \to H_n(C;G) \to B_{n-1} \otimes G \xrightarrow{??} Z_{n-1} \otimes G \to \cdots
$$

$$
\begin{tikzcd}
0 \arrow[r] & Z_{n+1} \otimes G \arrow[r, "j\otimes Id"] \arrow[d, "\partial\otimes Id=0"] & C_{n+1} \otimes G \arrow[r, "\partial\otimes Id"] \arrow[d, "\partial\otimes Id"] & \overset{b\otimes g}{B_n \otimes G} \arrow[r] \arrow[d, "\partial\otimes Id=0"] & 0 \\
0 \arrow[r] & Z_n \otimes G \arrow[r, "j\otimes Id"] & C_n \otimes G \arrow[r, "\partial\otimes Id"] & B_{n-1} \otimes G \arrow[r] & 0
\end{tikzcd}
$$

$$
\cdots Z_n \otimes G \to H_n(C;G) \to B_{n-1} \otimes G \xrightarrow{??} Z_{n-1} \otimes G \to \cdots
$$

$$
\begin{tikzcd}
0 \arrow[r] & Z_{n+1}\otimes G \arrow[r, "j\otimes Id"] \arrow[d, "\partial\otimes Id=0"] & \overset{c\otimes g}{C_{n+1}\otimes G} \arrow[r, "\partial\otimes Id"] \arrow[d, "\partial\otimes Id"] & \overset{b\otimes g=\partial c\otimes g}{B_n\otimes G} \arrow[r] \arrow[d, "\partial\otimes Id=0"] & 0 \\
0 \arrow[r] & Z_n\otimes G \arrow[r, "j\otimes Id"] & C_n\otimes G \arrow[r, "\partial\otimes Id"] & B_{n-1}\otimes G \arrow[r] & 0
\end{tikzcd}
$$

$$\cdots Z_n\otimes G \to H_n(C;G) \to B_{n-1}\otimes G \xrightarrow{??} Z_{n-1}\otimes G \to \cdots$$

$$
\begin{tikzcd}
0 \arrow[r] & Z_{n+1}\otimes G \arrow[r, "j\otimes Id"] \arrow[d, "\partial\otimes Id=0"] & \overset{c\otimes g}{C_{n+1}\otimes G} \arrow[r, "\partial\otimes Id"] \arrow[d, "\partial\otimes Id"] & \overset{b\otimes g=\partial c\otimes g}{B_n\otimes G} \arrow[r] \arrow[d, "\partial\otimes Id=0"] & 0 \\
0 \arrow[r] & Z_n\otimes G \arrow[r, "j\otimes Id"] & \overset{\partial c\otimes g}{C_n\otimes G} \arrow[r, "\partial\otimes Id"] & B_{n-1}\otimes G \arrow[r] & 0
\end{tikzcd}
$$

$$
\cdots Z_n\otimes G \to H_n(C;G) \to B_{n-1}\otimes G \xrightarrow{??} Z_{n-1}\otimes G \to \cdots
$$

$$
\begin{tikzcd}
0 \arrow[r] & Z_{n+1}\otimes G \arrow[r, "j\otimes Id"] \arrow[d, "\partial\otimes Id=0"] & \overset{c\otimes g}{C_{n+1}\otimes G} \arrow[r, "\partial\otimes Id"] \arrow[d, "\partial\otimes Id"] & \overset{b\otimes g=\partial c\otimes g}{B_n\otimes G} \arrow[r] \arrow[d, "\partial\otimes Id=0"] & 0 \\
0 \arrow[r] & Z_n\otimes G \arrow[r, "j\otimes Id"] & \overset{\partial c\otimes g=b\otimes g}{C_n\otimes G} \arrow[r, "\partial\otimes Id"] & B_{n-1}\otimes G \arrow[r] & 0
\end{tikzcd}
$$

$$
\cdots Z_n\otimes G \to H_n(C;G) \to B_{n-1}\otimes G \xrightarrow{??} Z_{n-1}\otimes G \to \cdots
$$

$$
\begin{tikzcd}
0 \arrow[r] & Z_{n+1} \otimes G \arrow[r, "j\otimes Id"] \arrow[d, "\partial\otimes Id=0"] & \overset{c\otimes g}{C_{n+1} \otimes G} \arrow[r, "\partial\otimes Id"] \arrow[d, "\partial\otimes Id"] & \overset{b\otimes g=\partial c\otimes g}{B_n \otimes G} \arrow[r] \arrow[d, "\partial\otimes Id=0"] & 0 \\
0 \arrow[r] & \overset{b\otimes g}{Z_n \otimes G} \arrow[r, "j\otimes Id"] & \overset{\partial c\otimes g=b\otimes g}{C_n \otimes G} \arrow[r, "\partial\otimes Id"] & B_{n-1} \otimes G \arrow[r] & 0
\end{tikzcd}
$$

$$
\cdots Z_n \otimes G \to H_n(C;G) \to B_{n-1} \otimes G \xrightarrow{??} Z_{n-1} \otimes G \to \cdots
$$

$$
\begin{tikzcd}
0 \arrow[r] & Z_{n+1}\otimes G \arrow[r, "j\otimes Id"] \arrow[d, "\partial\otimes Id=0"] & \overset{\color{blue}c\otimes g}{C_{n+1}\otimes G} \arrow[r, "\partial\otimes Id"] \arrow[d, "\partial\otimes Id"] & \overset{\color{blue}b\otimes g=\partial c\otimes g}{B_n\otimes G} \arrow[r] \arrow[d, "\partial\otimes Id=0"] & 0 \\
0 \arrow[r] & \overset{\color{blue}b\otimes g}{Z_n\otimes G} \arrow[r, "j\otimes Id"] & \overset{\color{blue}\partial c\otimes g=b\otimes g}{C_n\otimes G} \arrow[r, "\partial\otimes Id"] & B_{n-1}\otimes G \arrow[r] & 0
\end{tikzcd}
$$

$$
\cdots Z_n\otimes G \to H_n(C;G) \to B_{n-1}\otimes G \xrightarrow{i\otimes Id} Z_{n-1}\otimes G \to \cdots
$$

$$
\begin{tikzcd}
0 \arrow[r] & Z_{n+1}\otimes G \arrow[r, "j\otimes Id"] \arrow[d, "\partial\otimes Id=0"] & \overset{c\otimes g}{C_{n+1}\otimes G} \arrow[r, "\partial\otimes Id"] \arrow[d, "\partial\otimes Id"] & \overset{b\otimes g=\partial c\otimes g}{B_n\otimes G} \arrow[r] \arrow[d, "\partial\otimes Id=0"] & 0 \\
0 \arrow[r] & \overset{b\otimes g}{Z_n\otimes G} \arrow[r, "j\otimes Id"] & \overset{\partial c\otimes g=b\otimes g}{C_n\otimes G} \arrow[r, "\partial\otimes Id"] & B_{n-1}\otimes G \arrow[r] & 0
\end{tikzcd}
$$

$$
\cdots B_n\otimes G \xrightarrow{i\otimes Id} Z_n\otimes G \to H_n(C;G) \to B_{n-1}\otimes G \xrightarrow{i\otimes Id} Z_{n-1}\otimes G \to \cdots
$$

$$
\begin{tikzcd}
0 \arrow[r] & Z_{n+1} \otimes G \arrow[r, "j\otimes Id"] \arrow[d, "\partial\otimes Id=0"] & \overset{c\otimes g}{C_{n+1} \otimes G} \arrow[r, "\partial\otimes Id"] \arrow[d, "\partial\otimes Id"] & \overset{b\otimes g=\partial c\otimes g}{B_n \otimes G} \arrow[r] \arrow[d, "\partial\otimes Id=0"] & 0 \\
0 \arrow[r] & \overset{b\otimes g}{Z_n \otimes G} \arrow[r, "j\otimes Id"] & \overset{\partial c\otimes g=b\otimes g}{C_n \otimes G} \arrow[r, "\partial\otimes Id"] & B_{n-1} \otimes G \arrow[r] & 0
\end{tikzcd}
$$

$$
\cdots B_n \otimes G \xrightarrow{i\otimes Id} Z_n \otimes G \to H_n(C;G) \to B_{n-1} \otimes G \xrightarrow{i\otimes Id} Z_{n-1} \otimes G \to \cdots
$$

$$
\begin{tikzcd}
 & & c\otimes g & b\otimes g=\partial c\otimes g & \\
0 \arrow[r] & Z_{n+1}\otimes G \arrow[r, "j\otimes Id"] \arrow[dd, "\partial\otimes Id=0"] & C_{n+1}\otimes G \arrow[r, "\partial\otimes Id"] \arrow[dd, "\partial\otimes Id"] & B_n\otimes G \arrow[r] \arrow[dd, "\partial\otimes Id=0"] & 0 \\
 & b\otimes g & \partial c\otimes g=b\otimes g & & \\
0 \arrow[r] & Z_n\otimes G \arrow[r, "j\otimes Id"] & C_n\otimes G \arrow[r, "\partial\otimes Id"] & B_{n-1}\otimes G \arrow[r] & 0
\end{tikzcd}
$$

$$
\cdots B_n\otimes G \xrightarrow{i\otimes Id} Z_n\otimes G \to H_n(C;G) \to B_{n-1}\otimes G \xrightarrow{i\otimes Id} Z_{n-1}\otimes G \to \cdots
$$

$$
\begin{tikzcd}
0 \arrow[r] & Z_{n+1} \otimes G \arrow[r, "j\otimes Id"] \arrow[d, "\partial\otimes Id=0"] & \overset{c\otimes g}{C_{n+1} \otimes G} \arrow[r, "\partial\otimes Id"] \arrow[d, "\partial\otimes Id"] & \overset{b\otimes g=\partial c\otimes g}{B_n \otimes G} \arrow[r] \arrow[d, "\partial\otimes Id=0"] & 0 \\
0 \arrow[r] & \overset{b\otimes g}{Z_n \otimes G} \arrow[r, "j\otimes Id"] & \overset{\partial c\otimes g=b\otimes g}{C_n \otimes G} \arrow[r, "\partial\otimes Id"] & B_{n-1} \otimes G \arrow[r] & 0
\end{tikzcd}
$$

$$
\cdots B_n \otimes G \xrightarrow{i\otimes Id} Z_n \otimes G \to H_n(C;G) \to B_{n-1} \otimes G \xrightarrow{i\otimes Id} Z_{n-1} \otimes G \to \cdots
$$

$$\begin{tikzcd}
0 \arrow[r] & Z_{n+1}\otimes G \arrow[r, "j\otimes Id"] \arrow[d, "\partial\otimes Id=0"] & \overset{c\otimes g}{C_{n+1}\otimes G} \arrow[r, "\partial\otimes Id"] \arrow[d, "\partial\otimes Id"] & \overset{b\otimes g=\partial c\otimes g}{B_n\otimes G} \arrow[r] \arrow[d, "\partial\otimes Id=0"] & 0 \\
0 \arrow[r] & \overset{b\otimes g}{Z_n\otimes G} \arrow[r, "j\otimes Id"] & \overset{\partial c\otimes g=b\otimes g}{C_n\otimes G} \arrow[r, "\partial\otimes Id"] & B_{n-1}\otimes G \arrow[r] & 0
\end{tikzcd}$$

$$\cdots B_n\otimes G \xrightarrow{i\otimes Id} Z_n\otimes G \to H_n(C;G) \to B_{n-1}\otimes G \xrightarrow{i\otimes Id} Z_{n-1}\otimes G \to \cdots$$

$$
\begin{tikzcd}
0 \arrow[r] & Z_{n+1}\otimes G \arrow[r, "j\otimes Id"] \arrow[d, "\partial\otimes Id=0"] & \overset{c\otimes g}{C_{n+1}\otimes G} \arrow[r, "\partial\otimes Id"] \arrow[d, "\partial\otimes Id"] & \overset{b\otimes g=\partial c\otimes g}{B_n\otimes G} \arrow[r] \arrow[d, "\partial\otimes Id=0"] & 0 \\
0 \arrow[r] & \overset{b\otimes g}{Z_n\otimes G} \arrow[r, "j\otimes Id"] & \overset{\partial c\otimes g=b\otimes g}{C_n\otimes G} \arrow[r, "\partial\otimes Id"] & B_{n-1}\otimes G \arrow[r] & 0
\end{tikzcd}
$$

$$\cdots B_n\otimes G \xrightarrow{i\otimes Id} Z_n\otimes G \to H_n(C;G) \to B_{n-1}\otimes G \xrightarrow{i\otimes Id} Z_{n-1}\otimes G \to \cdots$$

$$0 \to Ker\ (\partial\otimes Id)_* \to H_n(C;G) \to Im\ (\partial\otimes Id)_* \to 0$$

$$
\begin{tikzcd}
0 \arrow[r] & Z_{n+1}\otimes G \arrow[r, "j\otimes Id"] \arrow[d, "\partial\otimes Id=0"] & \overset{c\otimes g}{C_{n+1}\otimes G} \arrow[r, "\partial\otimes Id"] \arrow[d, "\partial\otimes Id"] & \overset{b\otimes g=\partial c\otimes g}{B_n\otimes G} \arrow[r] \arrow[d, "\partial\otimes Id=0"] & 0 \\
0 \arrow[r] & \overset{b\otimes g}{Z_n\otimes G} \arrow[r, "j\otimes Id"] & \overset{\partial c\otimes g=b\otimes g}{C_n\otimes G} \arrow[r, "\partial\otimes Id"] & B_{n-1}\otimes G \arrow[r] & 0
\end{tikzcd}
$$

$$\cdots B_n\otimes G \xrightarrow{i\otimes Id} Z_n\otimes G \to H_n(C;G) \to B_{n-1}\otimes G \xrightarrow{i\otimes Id} Z_{n-1}\otimes G \to \cdots$$

$$0 \to Ker\ (\partial\otimes Id)_* \to H_n(C;G) \to Ker\ (i\otimes Id) \to 0$$

$$
\begin{tikzcd}
0 \arrow[r] & Z_{n+1}\otimes G \arrow[r, "j\otimes Id"] \arrow[d, "\partial\otimes Id=0"] & \overset{\color{blue}{c\otimes g}}{C_{n+1}\otimes G} \arrow[r, "\partial\otimes Id"] \arrow[d, "\partial\otimes Id"] & \overset{\color{blue}{b\otimes g=\partial c\otimes g}}{B_n\otimes G} \arrow[r] \arrow[d, "\partial\otimes Id=0"] & 0 \\
0 \arrow[r] & \overset{\color{blue}{b\otimes g}}{Z_n\otimes G} \arrow[r, "j\otimes Id"] & \overset{\color{blue}{\partial c\otimes g=b\otimes g}}{C_n\otimes G} \arrow[r, "\partial\otimes Id"] & B_{n-1}\otimes G \arrow[r] & 0
\end{tikzcd}
$$

$$\cdots B_n \otimes G \xrightarrow{i\otimes Id} Z_n \otimes G \to H_n(C;G) \to B_{n-1}\otimes G \xrightarrow{i\otimes Id} Z_{n-1}\otimes G \to \cdots$$

$$0 \to Im\ (j\otimes Id)_* \to H_n(C;G) \to Ker\ (i\otimes Id) \to 0$$

$$
\begin{tikzcd}
0 \arrow[r] & Z_{n+1}\otimes G \arrow[r, "j\otimes Id"] \arrow[d, "\partial\otimes Id=0"] & \overset{c\otimes g}{C_{n+1}\otimes G} \arrow[r, "\partial\otimes Id"] \arrow[d, "\partial\otimes Id"] & \overset{b\otimes g=\partial c\otimes g}{B_n\otimes G} \arrow[r] \arrow[d, "\partial\otimes Id=0"] & 0 \\
0 \arrow[r] & \overset{b\otimes g}{Z_n\otimes G} \arrow[r, "j\otimes Id"] & \overset{\partial c\otimes g=b\otimes g}{C_n\otimes G} \arrow[r, "\partial\otimes Id"] & B_{n-1}\otimes G \arrow[r] & 0
\end{tikzcd}
$$

$$\cdots B_n\otimes G \xrightarrow{i\otimes Id} Z_n\otimes G \to H_n(C;G) \to B_{n-1}\otimes G \xrightarrow{i\otimes Id} Z_{n-1}\otimes G \to \cdots$$

$$0 \to Z_n\otimes G/Im\ (i\otimes Id) \to H_n(C;G) \to Ker\ (i\otimes Id) \to 0$$

$$
\begin{tikzcd}
0 \arrow[r] & Z_{n+1}\otimes G \arrow[r, "j\otimes Id"] \arrow[d, "\partial\otimes Id=0"] & \overset{c\otimes g}{C_{n+1}\otimes G} \arrow[r, "\partial\otimes Id"] \arrow[d, "\partial\otimes Id"] & \overset{b\otimes g=\partial c\otimes g}{B_n\otimes G} \arrow[r] \arrow[d, "\partial\otimes Id=0"] & 0 \\
0 \arrow[r] & \overset{b\otimes g}{Z_n\otimes G} \arrow[r, "j\otimes Id"] & \overset{\partial c\otimes g=b\otimes g}{C_n\otimes G} \arrow[r, "\partial\otimes Id"] & B_{n-1}\otimes G \arrow[r] & 0
\end{tikzcd}
$$

$$\cdots B_n\otimes G \xrightarrow{i\otimes Id} Z_n\otimes G \to H_n(C;G) \to B_{n-1}\otimes G \xrightarrow{i\otimes Id} Z_{n-1}\otimes G \to \cdots$$

$$0 \to (Z_n/i(B_n))\otimes G \to H_n(C;G) \to Ker\ (i\otimes Id) \to 0$$

because,

$$\begin{tikzcd}
0 \arrow[r] & Z_{n+1}\otimes G \arrow[r, "j\otimes Id"] \arrow[d, "\partial\otimes Id=0"] & \overset{c\otimes g}{C_{n+1}\otimes G} \arrow[r, "\partial\otimes Id"] \arrow[d, "\partial\otimes Id"] & \overset{b\otimes g=\partial c\otimes g}{B_n\otimes G} \arrow[r] \arrow[d, "\partial\otimes Id=0"] & 0 \\
0 \arrow[r] & \overset{b\otimes g}{Z_n\otimes G} \arrow[r, "j\otimes Id"] & \overset{\partial c\otimes g=b\otimes g}{C_n\otimes G} \arrow[r, "\partial\otimes Id"] & B_{n-1}\otimes G \arrow[r] & 0
\end{tikzcd}$$

$$\cdots B_n\otimes G \xrightarrow{i\otimes Id} Z_n\otimes G \to H_n(C;G) \to B_{n-1}\otimes G \xrightarrow{i\otimes Id} Z_{n-1}\otimes G \to \cdots$$

$$0 \to (Z_n/i(B_n))\otimes G \to H_n(C;G) \to Ker\ (i\otimes Id) \to 0$$

because,

$$0 \to B_n \xrightarrow{i} Z_n \to Z_n/B_n \to 0$$

$$\begin{tikzcd}
0 \arrow[r] & Z_{n+1} \otimes G \arrow[r, "j\otimes Id"] \arrow[d, "\partial\otimes Id=0"] & \overset{c\otimes g}{C_{n+1} \otimes G} \arrow[r, "\partial\otimes Id"] \arrow[d, "\partial\otimes Id"] & \overset{b\otimes g=\partial c\otimes g}{B_n \otimes G} \arrow[r] \arrow[d, "\partial\otimes Id=0"] & 0 \\
0 \arrow[r] & \overset{b\otimes g}{Z_n \otimes G} \arrow[r, "j\otimes Id"] & \overset{\partial c\otimes g=b\otimes g}{C_n \otimes G} \arrow[r, "\partial\otimes Id"] & B_{n-1} \otimes G \arrow[r] & 0
\end{tikzcd}$$

$$\cdots B_n \otimes G \xrightarrow{i\otimes Id} Z_n \otimes G \to H_n(C;G) \to B_{n-1} \otimes G \xrightarrow{i\otimes Id} Z_{n-1} \otimes G \to \cdots$$

$$0 \to (Z_n/i(B_n)) \otimes G \to H_n(C;G) \to Ker\ (i \otimes Id) \to 0$$

because,

$$0 \to B_n \xrightarrow{i} Z_n \to Z_n/B_n \to 0$$

$$\implies B_n \otimes G \xrightarrow{i\otimes Id} Z_n \otimes G \to (Z_n/B_n) \otimes G \to 0$$

$$
\begin{tikzcd}
0 \arrow[r] & Z_{n+1}\otimes G \arrow[r, "j\otimes Id"] \arrow[d, "\partial\otimes Id=0"] & \overset{c\otimes g}{C_{n+1}\otimes G} \arrow[r, "\partial\otimes Id"] \arrow[d, "\partial\otimes Id"] & \overset{b\otimes g=\partial c\otimes g}{B_n\otimes G} \arrow[r] \arrow[d, "\partial\otimes Id=0"] & 0 \\
0 \arrow[r] & \overset{b\otimes g}{Z_n\otimes G} \arrow[r, "j\otimes Id"] & \overset{\partial c\otimes g=b\otimes g}{C_n\otimes G} \arrow[r, "\partial\otimes Id"] & B_{n-1}\otimes G \arrow[r] & 0
\end{tikzcd}
$$

$$\cdots B_n\otimes G \xrightarrow{i\otimes Id} Z_n\otimes G \to H_n(C;G) \to B_{n-1}\otimes G \xrightarrow{i\otimes Id} Z_{n-1}\otimes G \to \cdots$$

$$0 \to (Z_n/i(B_n))\otimes G \to H_n(C;G) \to Ker\ (i\otimes Id) \to 0$$

because,

$$0 \to B_n \xrightarrow{i} Z_n \to Z_n/B_n \to 0$$

$$\implies B_n\otimes G \xrightarrow{i\otimes Id} Z_n\otimes G \to (Z_n/B_n)\otimes G \to 0$$

$$\implies Z_n\otimes G/Im\ (i\otimes Id) \cong (Z_n/B_n)\otimes G$$

$$
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0 \arrow[r] & Z_{n+1} \otimes G \arrow[r, "j\otimes Id"] \arrow[d, "\partial\otimes Id=0"] & \overset{c\otimes g}{C_{n+1} \otimes G} \arrow[r, "\partial\otimes Id"] \arrow[d, "\partial\otimes Id"] & \overset{b\otimes g=\partial c\otimes g}{B_n \otimes G} \arrow[r] \arrow[d, "\partial\otimes Id=0"] & 0 \\
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$$
\begin{tikzcd}
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$$0 \to H_n(C) \otimes G \to H_n(C;G) \to Ker\ (i \otimes Id) \to 0$$

Note, $0 \to B_{n-1} \to Z_{n-1} \to H_{n-1}(C) \to 0$ (exact and a free resolution of $H_{n-1}(C)$)

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Note, $0 \to \underbrace{B_{n-1}}_{F_1} \to \underbrace{Z_{n-1}}_{F_0} \to H_{n-1}(C) \to 0$ (exact and a free resolution of $H_{n-1}(C)$)

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$0 \to B_{n-1} \otimes G \xrightarrow{i_{n-1}\otimes Id} Z_{n-1} \otimes G \to H_{n-1}(C) \otimes G \to 0$ (only a complex. **not** exact)

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$ker\ (i_{n-1}\otimes Id) = Tor_1(H_{n-1}(C), G)$

$$\begin{tikzcd}
0 \arrow[r] & Z_{n+1}\otimes G \arrow[r, "j\otimes Id"] \arrow[d, "\partial\otimes Id=0"] & \overset{\color{blue}{c\otimes g}}{C_{n+1}\otimes G} \arrow[r, "\partial\otimes Id"] \arrow[d, "\partial\otimes Id"] & \overset{\color{blue}{b\otimes g=\partial c\otimes g}}{B_n\otimes G} \arrow[r] \arrow[d, "\partial\otimes Id=0"] & 0 \\
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Theorem. *$f : C \to C'$ a chain map, then,*

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Proof. $0 \longrightarrow Z_n \longrightarrow C_n \longrightarrow B_{n-1} \longrightarrow 0$

□

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commutes

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Naturality of long exact sequence... $\square$

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Proof.

$$\begin{array}{ccccccccc}
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 & & \downarrow & & \downarrow & & \downarrow & & \\
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0 & \longrightarrow & Z_n \otimes G & \longrightarrow & C_n \otimes G & \longrightarrow & B_{n-1} \otimes G & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
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\end{array}$$

Naturality of long exact sequence...(exercise!) $\square$

Theorem. *The short exact sequence,*
$0 \to H_n(C_*) \otimes G \to H_n(C_*; G) \to Tor_1(H_{n-1}(C), G) \to 0$ *splits.*

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Proof. $0 \to Z_n \to C_n \to B_{n-1} \to 0$ splits. $\square$

Theorem. *The short exact sequence,*
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Proof. $0 \to Z_n \to C_n \to B_{n-1} \to 0$ splits.
There exists, $a : C_n \to Z_n$, such that $a \circ i = Id$

$\square$

Theorem. *The short exact sequence,*
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Proof. $0 \to Z_n \to C_n \to B_{n-1} \to 0$ splits.
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To show f_* is an isomorphism:

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H_2(S^2;\mathbb{Z}/2) \arrow[r] & H_2(S^2, \mathbb{RP}^2 \setminus p;\mathbb{Z}/2) \arrow[r, "excision"] & H_2(U, U \setminus p)
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Example.

$$H_k(\mathbb{RP}^2) = \begin{cases} \mathbb{Z} & k = 0 \\ \mathbb{Z}/2 & k = 1 \\ 0 & otherwise \end{cases}$$

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$$\begin{tikzcd} C_k(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\tau"] \arrow[d, "f'_\#"] & C_k(S^n;\mathbb{Z}/2) \arrow[r, "p_\#"] \arrow[d, "f_\#"] & C_k(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[d, "f'_\#"] \\ C_k(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\tau"] & C_k(S^n;\mathbb{Z}/2) \arrow[r, "p_\#"] & C_k(\mathbb{RP}^n;\mathbb{Z}/2) \end{tikzcd}$$

because $\sigma : \Delta_k \to X$ lifts to $\tilde{\sigma} : \Delta_k \to \tilde{X}$, $p \circ \tilde{\sigma} = \sigma$, i.e. $p_\#(\tilde{\sigma}) = \sigma$

$$p_\#(\tilde{\sigma}_1 + \tilde{\sigma}_2 + \cdots + \tilde{\sigma}_m) = 0$$
$$\implies p_\#(\tilde{\sigma}_1) + p_\#(\tilde{\sigma}_2) + \cdots + p_\#(\tilde{\sigma}_m) = 0$$

$p_\#(\tilde{\sigma}_1) = p_\#(\tilde{\sigma}_2)$ etc., i.e. $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ are lifts of the same simplex

Implies exactness of,
$< \quad injt \quad > \quad C_k(X;\mathbb{Z}/2) \quad \xrightarrow{\tau} \quad C_k(\tilde{X};\mathbb{Z}/2) \quad \xrightarrow{p_\#}$
$C_k(X;\mathbb{Z}/2) \to 0$
where, $\tau(\sigma) = \tilde{\sigma_1} + \tilde{\sigma_2}$, $\tilde{\sigma_1}$, where $\tilde{\sigma_2}$ are the lifts of σ

$\tilde{X} \xrightarrow{p} X$ two sheeted covering

$f : S^n \to S^n,\ f(-x) = -f(x)$

$C_k(\tilde{X}; \mathbb{Z}/2) \xrightarrow{p_\#} C_k(X; \mathbb{Z}/2) \to 0$

because $\sigma : \Delta_k \to X$ lifts to $\tilde{\sigma} : \Delta_k \to \tilde{X}$, $p \circ \tilde{\sigma} = \sigma$, i.e. $p_\#(\tilde{\sigma}) = \sigma$

$$\begin{tikzcd}
0 \arrow[r] & C_k(\mathbb{RP}^n; \mathbb{Z}/2) \arrow[r, "\tau"] \arrow[d, "f'_\#"] & C_k(S^n; \mathbb{Z}/2) \arrow[r, "p_\#"] \arrow[d, "f_\#"] & C_k(\mathbb{RP}^n; \mathbb{Z}/2) \arrow[r] \arrow[d, "f'_\#"] & 0 \\
0 \arrow[r] & C_k(\mathbb{RP}^n; \mathbb{Z}/2) \arrow[r, "\tau"] & C_k(S^n; \mathbb{Z}/2) \arrow[r, "p_\#"] & C_k(\mathbb{RP}^n; \mathbb{Z}/2) \arrow[r] & 0
\end{tikzcd}$$

$$p_\#(\tilde{\sigma}_1 + \tilde{\sigma}_2 + \cdots + \tilde{\sigma}_m) = 0$$
$$\implies p_\#(\tilde{\sigma}_1) + p_\#(\tilde{\sigma}_2) + \cdots + p_\#(\tilde{\sigma}_m) = 0$$

$p_\#(\tilde{\sigma}_1) = p_\#(\tilde{\sigma}_2)$ etc., i.e. $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ are lifts of the same simplex

Implies exactness of,

$< \quad injt \quad > \quad C_k(X; \mathbb{Z}/2) \quad \xrightarrow{\tau} \quad C_k(\tilde{X}; \mathbb{Z}/2) \quad \xrightarrow{p_\#} C_k(X; \mathbb{Z}/2) \to 0$

where, $\tau(\sigma) = \tilde{\sigma_1} + \tilde{\sigma_2}$, $\tilde{\sigma_1}$, where $\tilde{\sigma_2}$ are the lifts of σ

$$
\begin{tikzcd}
0 \arrow[r] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\tau_*"] \arrow[d, "f'_*"] & H_n(S^n;\mathbb{Z}/2) \arrow[r, "p_*"] \arrow[d, "f_*"] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "<iso>"] \arrow[d, "f'_*"] & H_{n-1}(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\tau_*"] \arrow[d, "f'_*"] & H_{n-1}(S^n;\mathbb{Z}/2) \arrow[d, "f_*"] \\
0 \arrow[r] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\tau_*"] & H_n(S^n;\mathbb{Z}/2) \arrow[r, "p_*"] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "<iso>"] & H_{n-1}(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\tau_*"] & H_{n-1}(S^n;\mathbb{Z}/2)
\end{tikzcd}
$$

$$
\begin{tikzcd}
0 \arrow[r] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\tau_*"] \arrow[d, "f'_*"] & H_n(S^n;\mathbb{Z}/2) \arrow[r, "p_*"] \arrow[d, "f_*"] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "<iso>"] \arrow[d, "f'_*"] & H_{n-1}(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\tau_*"] \arrow[d, "f'_*"] & 0 \\
0 \arrow[r] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\tau_*"] & H_n(S^n;\mathbb{Z}/2) \arrow[r, "p_*"] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "<iso>"] & H_{n-1}(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\tau_*"] & 0
\end{tikzcd}
$$

$$
\begin{tikzcd}
0 \arrow[r] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\tau_*"] \arrow[d, "f'_*"] & H_n(S^n;\mathbb{Z}/2) \arrow[r, "p_*"] \arrow[d, "f_*"] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "<iso>"] \arrow[d, "f'_*"] & H_{n-1}(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\tau_*"] \arrow[d, "f'_*"] & 0 \\
0 \arrow[r] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\tau_*"] & H_n(S^n;\mathbb{Z}/2) \arrow[r, "p_*"] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "<iso>"] & H_{n-1}(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\tau_*"] & 0
\end{tikzcd}
$$

θ is the (only) non-trivial deck transformation

$$\begin{tikzcd}
0 \arrow[r] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\tau_*"] \arrow[d, "f'_*"] & H_n(S^n;\mathbb{Z}/2) \arrow[r, "p_*"] \arrow[d, "f_*"] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "<iso>"] \arrow[d, "f'_*"] & H_{n-1}(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\tau_*"] \arrow[d, "f'_*"] & 0 \\
0 \arrow[r] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\tau_*"] & H_n(S^n;\mathbb{Z}/2) \arrow[r, "p_*"] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "<iso>"] & H_{n-1}(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\tau_*"] & 0
\end{tikzcd}$$

θ is the (only) non-trivial deck transformation

$p_*([z]) = [p_\#(z)]$??

$$
\begin{tikzcd}
0 \arrow[r] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\tau_*"] \arrow[d, "f'_*"] & H_n(S^n;\mathbb{Z}/2) \arrow[r, "p_*"] \arrow[d, "f_*"] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "<iso>"] \arrow[d, "f'_*"] & H_{n-1}(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\tau_*"] \arrow[d, "f'_*"] & 0 \\
0 \arrow[r] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\tau_*"] & H_n(S^n;\mathbb{Z}/2) \arrow[r, "p_*"] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "<iso>"] & H_{n-1}(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\tau_*"] & 0
\end{tikzcd}
$$

θ is the (only) non-trivial deck transformation

$p_*([z]) = [p_\#(z)]??$

$\tau_*(p_*([z]))$

$$
\begin{tikzcd}
0 \arrow[r] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\tau_*"] \arrow[d, "f'_*"] & H_n(S^n;\mathbb{Z}/2) \arrow[r, "p_*"] \arrow[d, "f_*"] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "<iso>"] \arrow[d, "f'_*"] & H_{n-1}(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\tau_*"] \arrow[d, "f'_*"] & 0 \\
0 \arrow[r] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\tau_*"] & H_n(S^n;\mathbb{Z}/2) \arrow[r, "p_*"] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "<iso>"] & H_{n-1}(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\tau_*"] & 0
\end{tikzcd}
$$

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$p_*([z]) = [p_\#(z)]??$

$\tau_*(p_*([z])) = \tau_*([p_\#(z)])$

$$\begin{tikzcd}
0 \arrow[r] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\tau_*"] \arrow[d, "f'_*"] & H_n(S^n;\mathbb{Z}/2) \arrow[r, "p_*"] \arrow[d, "f_*"] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "<iso>"] \arrow[d, "f'_*"] & H_{n-1}(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\tau_*"] \arrow[d, "f'_*"] & 0 \\
0 \arrow[r] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\tau_*"] & H_n(S^n;\mathbb{Z}/2) \arrow[r, "p_*"] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "<iso>"] & H_{n-1}(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\tau_*"] & 0
\end{tikzcd}$$

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$p_*([z]) = [p_\#(z)]??$

$\tau_*(p_*([z])) = \tau_*([p_\#(z)]) = [z + \theta_\#(z)]$

$$
\begin{tikzcd}
0 \arrow[r] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\tau_*"] \arrow[d, "f'_*"] & H_n(S^n;\mathbb{Z}/2) \arrow[r, "p_*"] \arrow[d, "f_*"] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "<iso>"] \arrow[d, "f'_*"] & H_{n-1}(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\tau_*"] \arrow[d, "f'_*"] & 0 \\
0 \arrow[r] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\tau_*"] & H_n(S^n;\mathbb{Z}/2) \arrow[r, "p_*"] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "<iso>"] & H_{n-1}(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\tau_*"] & 0
\end{tikzcd}
$$

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$p_*([z]) = [p_\#(z)]??$

$\tau_*(p_*([z])) = \tau_*([p_\#(z)]) = [z + \theta_\#(z)] = [z] + [\theta_\#(z)]$

$$
\begin{tikzcd}
0 \arrow[r] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\tau_*"] \arrow[d, "f'_*"] & H_n(S^n;\mathbb{Z}/2) \arrow[r, "p_*"] \arrow[d, "f_*"] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "<iso>"] \arrow[d, "f'_*"] & H_{n-1}(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\tau_*"] \arrow[d, "f'_*"] & 0 \\
0 \arrow[r] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\tau_*"] & H_n(S^n;\mathbb{Z}/2) \arrow[r, "p_*"] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "<iso>"] & H_{n-1}(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\tau_*"] & 0
\end{tikzcd}
$$

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$$\begin{tikzcd}
0 \arrow[r] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\tau_*"] \arrow[d, "f'_*"] & H_n(S^n;\mathbb{Z}/2) \arrow[r, "p_*"] \arrow[d, "f_*"] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "<iso>"] \arrow[d, "f'_*"] & H_{n-1}(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\tau_*"] \arrow[d, "f'_*"] & 0 \\
0 \arrow[r] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\tau_*"] & H_n(S^n;\mathbb{Z}/2) \arrow[r, "p_*"] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "<iso>"] & H_{n-1}(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\tau_*"] & 0
\end{tikzcd}$$

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$\tau_*(p_*([z])) = \tau_*([p_\#(z)]) = [z + \theta_\#(z)] = [z] + [\theta_\#(z)] = [z] + \theta_*[(z)] = 2[z]$

$$
\begin{tikzcd}
0 \arrow[r] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\tau_*"] \arrow[d, "f'_*"] & H_n(S^n;\mathbb{Z}/2) \arrow[r, "p_*"] \arrow[d, "f_*"] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "<iso>"] \arrow[d, "f'_*"] & H_{n-1}(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\tau_*"] \arrow[d, "f'_*"] & 0 \\
0 \arrow[r] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\tau_*"] & H_n(S^n;\mathbb{Z}/2) \arrow[r, "p_*"] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "<iso>"] & H_{n-1}(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\tau_*"] & 0
\end{tikzcd}
$$

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$\tau_*(p_*([z])) = \tau_*([p_\#(z)]) = [z + \theta_\#(z)] = [z] + [\theta_\#(z)] = [z] + \theta_*[(z)] = 2[z] = 0$

But τ_* is injective.

$$\begin{tikzcd}
0 \arrow[r] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\tau_*"] \arrow[d, "f'_*"] & H_n(S^n;\mathbb{Z}/2) \arrow[r, "p_*"] \arrow[d, "f_*"] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "<iso>"] \arrow[d, "f'_*"] & H_{n-1}(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\tau_*"] \arrow[d, "f'_*"] & 0 \\
0 \arrow[r] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\tau_*"] & H_n(S^n;\mathbb{Z}/2) \arrow[r, "p_*"] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "<iso>"] & H_{n-1}(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\tau_*"] & 0
\end{tikzcd}$$

θ is the (only) non-trivial deck transformation

$p_*([z]) = [p_\#(z)]??$

$\tau_*(p_*([z])) = \tau_*([p_\#(z)]) = [z + \theta_\#(z)] = [z] + [\theta_\#(z)] = [z] + \theta_*[(z)] = 2[z] = 0$

But τ_* is injective. Therefore, $p_* = 0$

$$\begin{tikzcd}
0 \arrow[r] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\tau_*"] \arrow[d, "f'_*"] & H_n(S^n;\mathbb{Z}/2) \arrow[r, "0"] \arrow[d, "f_*"] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "<iso>"] \arrow[d, "f'_*"] & H_{n-1}(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\tau_*"] \arrow[d, "f'_*"] & 0 \\
0 \arrow[r] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\tau_*"] & H_n(S^n;\mathbb{Z}/2) \arrow[r, "0"] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "<iso>"] & H_{n-1}(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\tau_*"] & 0
\end{tikzcd}$$

θ is the (only) non-trivial deck transformation

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But τ_* is injective. Therefore, $p_* = 0$

$$
\begin{tikzcd}
0 \arrow[r] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\sim"] \arrow[d, "f'_*"] & H_n(S^n;\mathbb{Z}/2) \arrow[r, "0"] \arrow[d, "f_*"] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "<iso>"] \arrow[d, "f'_*"] & H_{n-1}(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\tau_*"] \arrow[d, "f'_*"] & 0 \\
0 \arrow[r] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\sim"] & H_n(S^n;\mathbb{Z}/2) \arrow[r, "0"] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "<iso>"] & H_{n-1}(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\tau_*"] & 0
\end{tikzcd}
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$$\begin{array}{ccccccccccc}
0 & \longrightarrow & H_n(\mathbb{RP}^n;\mathbb{Z}/2) & \xrightarrow{\sim} & H_n(S^n;\mathbb{Z}/2) & \xrightarrow{0} & H_n(\mathbb{RP}^n;\mathbb{Z}/2) & \xrightarrow{<iso>} & H_{n-1}(\mathbb{RP}^n;\mathbb{Z}/2) & \xrightarrow{\tau_*} & 0 \\
 & & \downarrow f'_* & & \downarrow f_* & & \downarrow f'_* & & \downarrow f'_* & & \\
0 & \longrightarrow & H_n(\mathbb{RP}^n;\mathbb{Z}/2) & \xrightarrow{\sim} & H_n(S^n;\mathbb{Z}/2) & \xrightarrow{0} & H_n(\mathbb{RP}^n;\mathbb{Z}/2) & \xrightarrow{<iso>} & H_{n-1}(\mathbb{RP}^n;\mathbb{Z}/2) & \xrightarrow{\tau_*} & 0
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f has odd degree

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & H_n(\mathbb{RP}^n;\mathbb{Z}/2) & \xrightarrow{\sim} & H_n(S^n;\mathbb{Z}/2) & \xrightarrow{0} & H_n(\mathbb{RP}^n;\mathbb{Z}/2) & \xrightarrow{<iso>} & H_{n-1}(\mathbb{RP}^n;\mathbb{Z}/2) & \xrightarrow{\tau_*} & 0 \\
 & & \downarrow f'_* & & \downarrow f_* & & \downarrow f'_* & & \downarrow f'_* & & \\
0 & \longrightarrow & H_n(\mathbb{RP}^n;\mathbb{Z}/2) & \xrightarrow{\sim} & H_n(S^n;\mathbb{Z}/2) & \xrightarrow{0} & H_n(\mathbb{RP}^n;\mathbb{Z}/2) & \xrightarrow{<iso>} & H_{n-1}(\mathbb{RP}^n;\mathbb{Z}/2) & \xrightarrow{\tau_*} & 0
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f has odd degree

Theorem (Borsuk-Ulam)**.** *$f : S^n \to \mathbb{R}^n$, then $f(-x) = f(x)$ for some $x \in S^n$*

$$\begin{tikzcd}
0 \arrow[r] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\sim"] \arrow[d, "f'_*"] & H_n(S^n;\mathbb{Z}/2) \arrow[r, "0"] \arrow[d, "f_*"] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "<iso>"] \arrow[d, "f'_*"] & H_{n-1}(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\tau_*"] \arrow[d, "f'_*"] & 0 \\
0 \arrow[r] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\sim"] & H_n(S^n;\mathbb{Z}/2) \arrow[r, "0"] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "<iso>"] & H_{n-1}(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\tau_*"] & 0
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θ is the (only) non-trivial deck transformation

$p_*([z]) = [p_\#(z)]??$

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f has odd degree

Theorem (Borsuk-Ulam)**.** *$f : S^n \to \mathbb{R}^n$, then $f(-x) = f(x)$ for some $x \in S^n$*

Proof. Assume $f(x) - f(-x) \neq 0$ for any x

$\square$

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & H_n(\mathbb{RP}^n;\mathbb{Z}/2) & \xrightarrow{\sim} & H_n(S^n;\mathbb{Z}/2) & \xrightarrow{0} & H_n(\mathbb{RP}^n;\mathbb{Z}/2) & \xrightarrow{<iso>} & H_{n-1}(\mathbb{RP}^n;\mathbb{Z}/2) & \xrightarrow{\tau_*} & 0 \\
 & & \downarrow f'_* & & \downarrow f_* & & \downarrow f'_* & & \downarrow f'_* & & \\
0 & \longrightarrow & H_n(\mathbb{RP}^n;\mathbb{Z}/2) & \xrightarrow{\sim} & H_n(S^n;\mathbb{Z}/2) & \xrightarrow{0} & H_n(\mathbb{RP}^n;\mathbb{Z}/2) & \xrightarrow{<iso>} & H_{n-1}(\mathbb{RP}^n;\mathbb{Z}/2) & \xrightarrow{\tau_*} & 0
\end{array}$$

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f has odd degree

Theorem (Borsuk-Ulam)**.** *$f : S^n \to \mathbb{R}^n$, then $f(-x) = f(x)$ for some $x \in S^n$*

Proof. Assume $f(x) - f(-x) \neq 0$ for any x

$h(x) = \frac{f(x)-f(-x)}{|f(x)-f(-x)|}$ is well defined and $h(-x) = -h(x)$. $\square$

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & H_n(\mathbb{RP}^n;\mathbb{Z}/2) & \xrightarrow{\sim} & H_n(S^n;\mathbb{Z}/2) & \xrightarrow{0} & H_n(\mathbb{RP}^n;\mathbb{Z}/2) & \xrightarrow{<iso>} & H_{n-1}(\mathbb{RP}^n;\mathbb{Z}/2) & \xrightarrow{\tau_*} & 0 \\ & & \downarrow f'_* & & \downarrow f_* & & \downarrow f'_* & & \downarrow f'_* & & \\ 0 & \longrightarrow & H_n(\mathbb{RP}^n;\mathbb{Z}/2) & \xrightarrow{\sim} & H_n(S^n;\mathbb{Z}/2) & \xrightarrow{0} & H_n(\mathbb{RP}^n;\mathbb{Z}/2) & \xrightarrow{<iso>} & H_{n-1}(\mathbb{RP}^n;\mathbb{Z}/2) & \xrightarrow{\tau_*} & 0 \end{array}$$

θ is the (only) non-trivial deck transformation

$p_*([z]) = [p_\#(z)]??$

$\tau_*(p_*([z])) = \tau_*([p_\#(z)]) = [z + \theta_\#(z)] = [z] + [\theta_\#(z)] = [z] + \theta_*[(z)] = 2[z] = 0$

But τ_* is injective. Therefore, $p_* = 0$

f has odd degree

Theorem (Borsuk-Ulam)**.** *$f : S^n \to \mathbb{R}^n$, then $f(-x) = f(x)$ for some $x \in S^n$*

Proof. Assume $f(x) - f(-x) \neq 0$ for any x

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$S^n \xrightarrow{h} S^{n-1}$ $\square$

$$\begin{tikzcd}
0 \arrow[r] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\sim"] \arrow[d, "f'_*"] & H_n(S^n;\mathbb{Z}/2) \arrow[r, "0"] \arrow[d, "f_*"] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "<iso>"] \arrow[d, "f'_*"] & H_{n-1}(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\tau_*"] \arrow[d, "f'_*"] & 0 \\
0 \arrow[r] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\sim"] & H_n(S^n;\mathbb{Z}/2) \arrow[r, "0"] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "<iso>"] & H_{n-1}(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\tau_*"] & 0
\end{tikzcd}$$

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