$$C_n(X) \xrightarrow{j} C_n(X)/C_n(A)$$

$$C_n(X) \xrightarrow{j} C_n(X)/C_n(A)$$

$$H_n(X) \xrightarrow{j_*} H_n(X, A)$$

$$\boxed{C_n(A) \xrightarrow{i_\#} C_n(X) \xrightarrow{j} C_n(X)/C_n(A)}$$

$$H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A)$$

$$0 \to C_n(A) \xrightarrow{i_\#} C_n(X) \xrightarrow{j} C_n(X)/C_n(A)$$

$$H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A)$$

$$0 \to C_n(A) \xrightarrow{i_\#} C_n(X) \xrightarrow{j} C_n(X)/C_n(A)$$

$$H_{n+1}(X, A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A)$$

$$0 \to C_n(A) \xrightarrow{i_\#} C_n(X) \xrightarrow{j} C_n(X)/C_n(A) \to 0$$

$$H_{n+1}(X, A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A)$$

$$0 \to C_n(A) \xrightarrow{i_\#} C_n(X) \xrightarrow{j} C_n(X)/C_n(A) \to 0$$

$$H_{n+1}(X, A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A)$$

$$0 \to C_n(A) \xrightarrow{i_\#} C_n(X) \xrightarrow{j} C_n(X)/C_n(A) \to 0$$

$$\cdots \to H_{n+1}(X,A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

$$0 \to C_n(A) \xrightarrow{i_\#} C_n(X) \xrightarrow{j} C_n(X)/C_n(A) \to 0$$

$$\cdots \to H_{n+1}(X,A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Exact sequence

$$\cdots \to A_{i+i} \xrightarrow{f_{i+1}} A_i \xrightarrow{f_i} A_{i-1} \to \cdots$$

$$0 \to C_n(A) \xrightarrow{i_\#} C_n(X) \xrightarrow{j} C_n(X)/C_n(A) \to 0$$

$$\cdots \to H_{n+1}(X,A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Exact sequence

$$\cdots \to A_{i+i} \xrightarrow{f_{i+1}} A_i \xrightarrow{f_i} A_{i-1} \to \cdots$$

$$Ker\ f_i = Im\ f_{i+1}$$

$$0 \to C_n(A) \xrightarrow{i_\#} C_n(X) \xrightarrow{j} C_n(X)/C_n(A) \to 0$$

$$\cdots \to H_{n+1}(X, A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Exact sequence

$$\cdots \to A_{i+i} \xrightarrow{f_{i+1}} {\color{red}A_i} \xrightarrow{f_i} A_{i-1} \to \cdots$$

$$Ker\ f_i = Im\ f_{i+1}$$

Short exact sequence

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

$$0 \to C_n(A) \xrightarrow{i_\#} C_n(X) \xrightarrow{j} C_n(X)/C_n(A) \to 0$$

$$\cdots \to H_{n+1}(X, A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Exact sequence

$$\cdots \to A_{i+i} \xrightarrow{f_{i+1}} {\color{red}A_i} \xrightarrow{f_i} A_{i-1} \to \cdots$$

$$Ker\ f_i = Im\ f_{i+1}$$

Short exact sequence

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

$$Ker\ g = Im\ f$$

$$0 \to C_n(A) \xrightarrow{i_\#} C_n(X) \xrightarrow{j} C_n(X)/C_n(A) \to 0$$

$$\cdots \to H_{n+1}(X, A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Exact sequence

$$\cdots \to A_{i+i} \xrightarrow{f_{i+1}} {\color{red}A_i} \xrightarrow{f_i} A_{i-1} \to \cdots$$

$Ker\ f_i = Im\ f_{i+1}$

Short exact sequence

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

$Ker\ g = Im\ f$

f injective

$$0 \to C_n(A) \xrightarrow{i_\#} C_n(X) \xrightarrow{j} C_n(X)/C_n(A) \to 0$$

$$\cdots \to H_{n+1}(X, A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Exact sequence

$$\cdots \to A_{i+i} \xrightarrow{f_{i+1}} {\color{red}A_i} \xrightarrow{f_i} A_{i-1} \to \cdots$$

$Ker\ f_i = Im\ f_{i+1}$

Short exact sequence

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

$Ker\ g = Im\ f$

f injective

g surjective

Chain complexes

Chain complexes

$$\cdots \xrightarrow{\partial_{n+2}} A_{n+1} \xrightarrow{\partial_{n+1}} A_n \xrightarrow{\partial_n} A_{n-1} \xrightarrow{\partial_{n-1}} \cdots$$

Chain complexes

$$\cdots \xrightarrow{\partial_{n+2}} C_{n+1}(X) \xrightarrow{\partial_{n+1}} C_n(X) \xrightarrow{\partial_n} C_{n-1}(X) \xrightarrow{\partial_{n-1}} \cdots$$

Chain complexes

$$\cdots \xrightarrow{\partial_{n+2}} C_{n+1}(X,A) \xrightarrow{\partial_{n+1}} C_n(X,A) \xrightarrow{\partial_n} C_{n-1}(X,A) \xrightarrow{\partial_{n-1}} \cdots$$

Chain complexes

$$\cdots \xrightarrow{\partial_{n+2}} A_{n+1} \xrightarrow{\partial_{n+1}} A_n \xrightarrow{\partial_n} A_{n-1} \xrightarrow{\partial_{n-1}} \cdots$$

Chain complexes

$$\cdots \xrightarrow{\partial_{n+2}} A_{n+1} \xrightarrow{\partial_{n+1}} A_n \xrightarrow{\partial_n} A_{n-1} \xrightarrow{\partial_{n-1}} \cdots$$

$$\partial_n \circ \partial_{n+1} = 0$$

Chain complexes

$$A_* := \cdots \xrightarrow{\partial_{n+2}} A_{n+1} \xrightarrow{\partial_{n+1}} A_n \xrightarrow{\partial_n} A_{n-1} \xrightarrow{\partial_{n-1}} \cdots$$

$$\partial_n \circ \partial_{n+1} = 0$$

Chain complexes

$$A_* := \cdots \xrightarrow{\partial_{n+2}} A_{n+1} \xrightarrow{\partial_{n+1}} A_n \xrightarrow{\partial_n} A_{n-1} \xrightarrow{\partial_{n-1}} \cdots$$

$$\partial_n \circ \partial_{n+1} = 0$$

$$H_n(A) := \frac{ker\ \partial_n}{Im\ \partial_{n+1}}$$

Chain map

Chain complexes

$$A_* := \cdots \xrightarrow{\partial_{n+2}} A_{n+1} \xrightarrow{\partial_{n+1}} A_n \xrightarrow{\partial_n} A_{n-1} \xrightarrow{\partial_{n-1}} \cdots$$

$$\partial_n \circ \partial_{n+1} = 0$$

$$H_n(A) := \frac{ker\ \partial_n}{Im\ \partial_{n+1}}$$

Chain map

$$f : A_* \to B_*$$

Chain complexes

$$A_* := \cdots \xrightarrow{\partial_{n+2}} A_{n+1} \xrightarrow{\partial_{n+1}} A_n \xrightarrow{\partial_n} A_{n-1} \xrightarrow{\partial_{n-1}} \cdots$$

$$\partial_n \circ \partial_{n+1} = 0$$

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Chain map

$$f : A_* \to B_*$$

$$\cdots \xrightarrow{\partial_{n+2}} A_{n+1} \xrightarrow{\partial_{n+1}} A_n \xrightarrow{\partial_n} A_{n-1} \to \cdots$$

Chain complexes

$$A_* := \cdots \xrightarrow{\partial_{n+2}} A_{n+1} \xrightarrow{\partial_{n+1}} A_n \xrightarrow{\partial_n} A_{n-1} \xrightarrow{\partial_{n-1}} \cdots$$

$$\partial_n \circ \partial_{n+1} = 0$$

$$H_n(A) := \frac{ker\ \partial_n}{Im\ \partial_{n+1}}$$

Chain map

$$f : A_* \to B_*$$

$$\cdots \xrightarrow{\partial_{n+2}} A_{n+1} \xrightarrow{\partial_{n+1}} A_n \xrightarrow{\partial_n} A_{n-1} \to \cdots$$

$$\cdots \xrightarrow{\partial'_{n+2}} B_{n+1} \xrightarrow{\partial'_{n+1}} B_n \xrightarrow{\partial'_n} B_{n-1} \to \cdots$$

Chain complexes

$$A_* := \cdots \xrightarrow{\partial_{n+2}} A_{n+1} \xrightarrow{\partial_{n+1}} A_n \xrightarrow{\partial_n} A_{n-1} \xrightarrow{\partial_{n-1}} \cdots$$

$$\partial_n \circ \partial_{n+1} = 0$$

$$H_n(A) := \frac{ker\ \partial_n}{Im\ \partial_{n+1}}$$

Chain map

$$f : A_* \to B_*$$

$$\begin{tikzcd}
\cdots \arrow[r, "\partial_{n+2}"] & A_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "f_{n+1}"] & A_n \arrow[r, "\partial_n"] \arrow[d, "f_n"] & A_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \\
\cdots \arrow[r, "\partial'_{n+2}"] & B_{n+1} \arrow[r, "\partial'_{n+1}"] & B_n \arrow[r, "\partial'_n"] & B_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

Chain complexes

$$0 \to A_* \to B_* \to C_* \to 0$$

$$A_* := \cdots \xrightarrow{\partial_{n+2}} A_{n+1} \xrightarrow{\partial_{n+1}} A_n \xrightarrow{\partial_n} A_{n-1} \xrightarrow{\partial_{n-1}} \cdots$$

$$\partial_n \circ \partial_{n+1} = 0$$

$$H_n(A) := \frac{ker\ \partial_n}{Im\ \partial_{n+1}}$$

Chain map

$$f : A_* \to B_*$$

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{\partial_{n+2}} & A_{n+1} & \xrightarrow{\partial_{n+1}} & A_n & \xrightarrow{\partial_n} & A_{n-1} & \to & \cdots \\
 & & \downarrow f_{n+1} & & \downarrow f_n & & \downarrow f_{n-1} & & \\
\cdots & \xrightarrow{\partial'_{n+2}} & B_{n+1} & \xrightarrow{\partial'_{n+1}} & B_n & \xrightarrow{\partial'_n} & B_{n-1} & \to & \cdots
\end{array}$$

$$\partial'_n \circ f_n = f_{n-1} \circ \partial_n$$

Chain complexes

$$A_* := \cdots \xrightarrow{\partial_{n+2}} A_{n+1} \xrightarrow{\partial_{n+1}} A_n \xrightarrow{\partial_n} A_{n-1} \xrightarrow{\partial_{n-1}} \cdots$$

$$\partial_n \circ \partial_{n+1} = 0$$

$$H_n(A) := \frac{ker\ \partial_n}{Im\ \partial_{n+1}}$$

Chain map

$$f : A_* \to B_*$$

$$\begin{tikzcd}
\cdots \arrow[r, "\partial_{n+2}"] & A_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "f_{n+1}"] & A_n \arrow[r, "\partial_n"] \arrow[d, "f_n"] & A_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \\
\cdots \arrow[r, "\partial'_{n+2}"] & B_{n+1} \arrow[r, "\partial'_{n+1}"] & B_n \arrow[r, "\partial'_n"] & B_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

$$\partial'_n \circ f_n = f_{n-1} \circ \partial_n$$

$0 \to A_* \to B_* \to C_* \to 0$ implies

$H_n(A) \to H_n(B) \to H_n(C)$

Chain complexes

$$A_* := \cdots \xrightarrow{\partial_{n+2}} A_{n+1} \xrightarrow{\partial_{n+1}} A_n \xrightarrow{\partial_n} A_{n-1} \xrightarrow{\partial_{n-1}} \cdots$$

$$\partial_n \circ \partial_{n+1} = 0$$

$$H_n(A) := \frac{ker\ \partial_n}{Im\ \partial_{n+1}}$$

Chain map

$$f : A_* \to B_*$$

$$\begin{tikzcd}
\cdots \arrow[r, "\partial_{n+2}"] & A_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "f_{n+1}"] & A_n \arrow[r, "\partial_n"] \arrow[d, "f_n"] & A_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \\
\cdots \arrow[r, "\partial'_{n+2}"] & B_{n+1} \arrow[r, "\partial'_{n+1}"] & B_n \arrow[r, "\partial'_n"] & B_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

$$\partial'_n \circ f_n = f_{n-1} \circ \partial_n$$

$0 \to A_* \to B_* \to C_* \to 0$ implies

$\to H_{n+1}(A) \to H_n(A) \to H_n(B) \to H_n(C)$

Chain complexes

$$A_* := \cdots \xrightarrow{\partial_{n+2}} A_{n+1} \xrightarrow{\partial_{n+1}} A_n \xrightarrow{\partial_n} A_{n-1} \xrightarrow{\partial_{n-1}} \cdots$$

$$\partial_n \circ \partial_{n+1} = 0$$

$$H_n(A) := \frac{ker\ \partial_n}{Im\ \partial_{n+1}}$$

Chain map

$$f : A_* \to B_*$$

$$\begin{array}{ccccccccc} \cdots & \xrightarrow{\partial_{n+2}} & A_{n+1} & \xrightarrow{\partial_{n+1}} & A_n & \xrightarrow{\partial_n} & A_{n-1} & \to & \cdots \\ & & \downarrow f_{n+1} & & \downarrow f_n & & \downarrow f_{n-1} & & \\ \cdots & \xrightarrow{\partial'_{n+2}} & B_{n+1} & \xrightarrow{\partial'_{n+1}} & B_n & \xrightarrow{\partial'_n} & B_{n-1} & \to & \cdots \end{array}$$

$$\partial'_n \circ f_n = f_{n-1} \circ \partial_n$$

$0 \to A_* \to B_* \to C_* \to 0$ implies

$$\to H_{n+1}(A) \to H_n(A) \to H_n(B) \to H_n(C) \to H_{n-1}(A) \to$$

Chain complexes

$$A_* := \cdots \xrightarrow{\partial_{n+2}} A_{n+1} \xrightarrow{\partial_{n+1}} A_n \xrightarrow{\partial_n} A_{n-1} \xrightarrow{\partial_{n-1}} \cdots$$

$$\partial_n \circ \partial_{n+1} = 0$$

$$H_n(A) := \frac{ker\ \partial_n}{Im\ \partial_{n+1}}$$

Chain map

$$f : A_* \to B_*$$

$$\begin{tikzcd}
\cdots \arrow[r, "\partial_{n+2}"] & A_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "f_{n+1}"] & A_n \arrow[r, "\partial_n"] \arrow[d, "f_n"] & A_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \\
\cdots \arrow[r, "\partial'_{n+2}"] & B_{n+1} \arrow[r, "\partial'_{n+1}"] & B_n \arrow[r, "\partial'_n"] & B_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

$$\partial'_n \circ f_n = f_{n-1} \circ \partial_n$$

$0 \to A_* \to B_* \to C_* \to 0$ implies
$\to H_{n+1}(A) \to H_n(A) \to H_n(B) \to H_n(C) \to H_{n-1}(A) \to H_{n-1}(B) \to H_{n-1}(C) \to \cdots$

Defining ∂

$$
\begin{tikzcd}
0 \arrow[r] & C_{n+1}(A) \arrow[r] \arrow[d] & C_{n+1}(X) \arrow[r] \arrow[d] & C_{n+1}(X,A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_n(A) \arrow[r] \arrow[d] & C_n(X) \arrow[r] \arrow[d] & C_n(X,A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_{n-1}(A) \arrow[r] & C_{n-1}(X) \arrow[r] & C_{n-1}(X,A) \arrow[r] & 0
\end{tikzcd}
$$

$$\cdots \to H_{n+1}(X,A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Defining ∂

$$
\begin{tikzcd}
0 \arrow[r] & C_{n+1}(A) \arrow[r] \arrow[d] & C_{n+1}(X) \arrow[r] \arrow[d] & C_{n+1}(X, A) \arrow[r] \arrow[d, "c_n"] & 0 \\
0 \arrow[r] & C_n(A) \arrow[r] \arrow[d] & C_n(X) \arrow[r] \arrow[d] & C_n(X, A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_{n-1}(A) \arrow[r] & C_{n-1}(X) \arrow[r] & C_{n-1}(X, A) \arrow[r] & 0
\end{tikzcd}
$$

$$\cdots \to H_{n+1}(X, A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Defining ∂

$$
\begin{tikzcd}
0 \arrow[r] & C_{n+1}(A) \arrow[r] \arrow[d] & C_{n+1}(X) \arrow[r] \arrow[d] & C_{n+1}(X,A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_n(A) \arrow[r] \arrow[d] & C_n(X) \arrow[r] \arrow[d] & \overset{c_n}{C_n(X,A)} \arrow[r] \arrow[d, red] & 0 \\
0 \arrow[r] & C_{n-1}(A) \arrow[r] & C_{n-1}(X) \arrow[r] & \overset{\partial c_n}{C_{n-1}(X,A)} \arrow[r] & 0
\end{tikzcd}
$$

$$\cdots \to H_{n+1}(X,A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Defining ∂

$$
\begin{tikzcd}
0 \arrow[r] & C_{n+1}(A) \arrow[r] \arrow[d] & C_{n+1}(X) \arrow[r] \arrow[d] & C_{n+1}(X,A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_n(A) \arrow[r] \arrow[d] & C_n(X) \arrow[r] \arrow[d] & \overset{c_n}{C_n(X,A)} \arrow[r] \arrow[d, red] & 0 \\
0 \arrow[r] & C_{n-1}(A) \arrow[r] & C_{n-1}(X) \arrow[r] & \overset{\partial c_n = 0}{C_{n-1}(X,A)} \arrow[r] & 0
\end{tikzcd}
$$

$$\cdots \to H_{n+1}(X,A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Defining ∂

$$\begin{tikzcd}
0 \arrow[r] & C_{n+1}(A) \arrow[r] \arrow[d] & C_{n+1}(X) \arrow[r] \arrow[d] & C_{n+1}(X, A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_n(A) \arrow[r] \arrow[d] & \overset{b_n}{C_n(X)} \arrow[r] \arrow[d] & \overset{j(b_n)}{C_n(X, A)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_{n-1}(A) \arrow[r] & C_{n-1}(X) \arrow[r] & \overset{\partial j(b_n)=0}{C_{n-1}(X, A)} \arrow[r] & 0
\end{tikzcd}$$

$$\cdots \to H_{n+1}(X, A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Defining ∂

$$
\begin{tikzcd}
0 \arrow[r] & C_{n+1}(A) \arrow[r] \arrow[d] & C_{n+1}(X) \arrow[r] \arrow[d] & C_{n+1}(X,A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_n(A) \arrow[r] \arrow[d] & \overset{b_n}{C_n(X)} \arrow[r, red] \arrow[d, red] & \overset{j(b_n)}{C_n(X,A)} \arrow[r] \arrow[d, red] & 0 \\
0 \arrow[r] & C_{n-1}(A) \arrow[r] & \overset{\partial b_n}{C_{n-1}(X)} \arrow[r] & \overset{\partial j(b_n)=0}{C_{n-1}(X,A)} \arrow[r] & 0
\end{tikzcd}
$$

$$\cdots \to H_{n+1}(X,A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Defining ∂

$$
\begin{tikzcd}
0 \arrow[r] & C_{n+1}(A) \arrow[r] \arrow[d] & C_{n+1}(X) \arrow[r] \arrow[d] & C_{n+1}(X,A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_n(A) \arrow[r] \arrow[d] & \overset{b_n}{C_n(X)} \arrow[r] \arrow[d] & \overset{j(b_n)}{C_n(X,A)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_{n-1}(A) \arrow[r] & \overset{\partial b_n}{C_{n-1}(X)} \arrow[r] & \overset{\partial j(b_n)=0=j(\partial b_n)}{C_{n-1}(X,A)} \arrow[r] & 0
\end{tikzcd}
$$

$$\cdots \to H_{n+1}(X,A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Defining ∂

$$
\begin{tikzcd}
0 \arrow[r] & C_{n+1}(A) \arrow[r] \arrow[d] & C_{n+1}(X) \arrow[r] \arrow[d] & C_{n+1}(X, A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_n(A) \arrow[r] \arrow[d] & \overset{b_n}{C_n(X)} \arrow[r] \arrow[d] & \overset{j(b_n)}{C_n(X, A)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \overset{a_{n-1}}{C_{n-1}(A)} \arrow[r] & \overset{\partial b_n = j(a_n)}{C_{n-1}(X)} \arrow[r] & \overset{\partial j(b_n) = 0 = j(\partial b_n)}{C_{n-1}(X, A)} \arrow[r] & 0
\end{tikzcd}
$$

$$\cdots \to H_{n+1}(X, A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Checking well definedness

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C_{n+1}(A) & \longrightarrow & C_{n+1}(X) & \longrightarrow & C_{n+1}(X,A) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C_{n}(A) & \longrightarrow & C_{n}(X) & \longrightarrow & C_{n}(X,A) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C_{n-1}(A) & \longrightarrow & C_{n-1}(X) & \longrightarrow & C_{n-1}(X,A) & \longrightarrow & 0
\end{array}$$

$$\cdots \to H_{n+1}(X,A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Checking well definedness

$$
\begin{tikzcd}
0 \arrow[r] & C_{n+1}(A) \arrow[r] \arrow[d] & C_{n+1}(X) \arrow[r] \arrow[d] & C_{n+1}(X,A) \arrow[r] \arrow[d, "0"'] & 0 \\
0 \arrow[r] & C_n(A) \arrow[r] \arrow[d] & C_n(X) \arrow[r] \arrow[d] & C_n(X,A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_{n-1}(A) \arrow[r] & C_{n-1}(X) \arrow[r] & C_{n-1}(X,A) \arrow[r] & 0
\end{tikzcd}
$$

$$\cdots \to H_{n+1}(X,A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Checking well definedness

$$
\begin{tikzcd}
0 \arrow[r] & C_{n+1}(A) \arrow[r] \arrow[d] & C_{n+1}(X) \arrow[r] \arrow[d] & C_{n+1}(X,A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_n(A) \arrow[r] \arrow[d] & \overset{b_n}{C_n(X)} \arrow[r, red] \arrow[d] & \overset{0}{C_n(X,A)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_{n-1}(A) \arrow[r] & C_{n-1}(X) \arrow[r] & C_{n-1}(X,A) \arrow[r] & 0
\end{tikzcd}
$$

$$\cdots \to H_{n+1}(X,A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Checking well definedness

$$\begin{tikzcd}
0 \arrow[r] & C_{n+1}(A) \arrow[r] \arrow[d] & C_{n+1}(X) \arrow[r] \arrow[d] & C_{n+1}(X,A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \overset{a_n}{C_n(A)} \arrow[r, red] \arrow[d] & \overset{i(a_n)}{C_n(X)} \arrow[r, red] \arrow[d] & \overset{0}{C_n(X,A)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_{n-1}(A) \arrow[r] & C_{n-1}(X) \arrow[r] & C_{n-1}(X,A) \arrow[r] & 0
\end{tikzcd}$$

$$\cdots \to H_{n+1}(X,A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Checking well definedness

$$
\begin{tikzcd}
0 \arrow[r] & C_{n+1}(A) \arrow[r] \arrow[d] & C_{n+1}(X) \arrow[r] \arrow[d] & C_{n+1}(X, A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \overset{a_n}{C_n(A)} \arrow[r] \arrow[d] & \overset{i(a_n)}{C_n(X)} \arrow[r] \arrow[d] & \overset{0}{C_n(X, A)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_{n-1}(A) \arrow[r] & \overset{b_{n-1}}{C_{n-1}(X)} \arrow[r] & C_{n-1}(X, A) \arrow[r] & 0
\end{tikzcd}
$$

$$\cdots \to H_{n+1}(X, A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Checking well definedness

$$
\begin{tikzcd}
0 \arrow[r] & C_{n+1}(A) \arrow[r] \arrow[d] & C_{n+1}(X) \arrow[r] \arrow[d] & C_{n+1}(X, A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \overset{a_n}{C_n(A)} \arrow[r] \arrow[d] & \overset{i(a_n)}{C_n(X)} \arrow[r] \arrow[d] & \overset{0}{C_n(X, A)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_{n-1}(A) \arrow[r] & \overset{b_{n-1}}{C_{n-1}(X)} \arrow[r] & \overset{0}{C_{n-1}(X, A)} \arrow[r] & 0
\end{tikzcd}
$$

$$\cdots \to H_{n+1}(X, A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Checking well definedness

$$\begin{tikzcd}
0 \arrow[r] & C_{n+1}(A) \arrow[r] \arrow[d] & C_{n+1}(X) \arrow[r] \arrow[d] & C_{n+1}(X, A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \overset{a_n}{C_n(A)} \arrow[r] \arrow[d] & \overset{i(a_n)}{C_n(X)} \arrow[r] \arrow[d] & \overset{0}{C_n(X, A)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \overset{a_{n-1}}{C_{n-1}(A)} \arrow[r] & \overset{b_{n-1}}{C_{n-1}(X)} \arrow[r] & \overset{0}{C_{n-1}(X, A)} \arrow[r] & 0
\end{tikzcd}$$

$$\cdots \to H_{n+1}(X, A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Checking exactness at $H_n(X, A)$

$Ker\ \partial \subset Im\ j_*$

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & C_{n+1}(A) & \longrightarrow & C_{n+1}(X) & \longrightarrow & C_{n+1}(X, A) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C_n(A) & \longrightarrow & C_n(X) & \longrightarrow & C_n(X, A) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C_{n-1}(A) & \longrightarrow & C_{n-1}(X) & \longrightarrow & C_{n-1}(X, A) & \longrightarrow & 0
\end{array}
$$

$$\cdots \to H_{n+1}(X, A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Checking exactness at $H_n(X, A)$

$Ker\ \partial \subset Im\ j_*$

$$
\begin{tikzcd}
0 \arrow[r] & C_{n+1}(A) \arrow[r] \arrow[d] & C_{n+1}(X) \arrow[r] \arrow[d] & C_{n+1}(X, A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_n(A) \arrow[r] \arrow[d] & C_n(X) \arrow[r] \arrow[d] & \overset{j(b_n)}{C_n(X, A)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_{n-1}(A) \arrow[r] & C_{n-1}(X) \arrow[r] & C_{n-1}(X, A) \arrow[r] & 0
\end{tikzcd}
$$

$$\cdots \to H_{n+1}(X, A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Checking exactness at $H_n(X, A)$

$Ker\ \partial \subset Im\ j_*$

$$\begin{tikzcd}
0 \arrow[r] & C_{n+1}(A) \arrow[r] \arrow[d] & C_{n+1}(X) \arrow[r] \arrow[d] & C_{n+1}(X,A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_n(A) \arrow[r] \arrow[d] & \overset{b_n}{C_n(X)} \arrow[r, red] \arrow[d] & \overset{j(b_n)}{C_n(X,A)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_{n-1}(A) \arrow[r] & C_{n-1}(X) \arrow[r] & C_{n-1}(X,A) \arrow[r] & 0
\end{tikzcd}$$

$$\cdots \to H_{n+1}(X, A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Checking exactness at $H_n(X, A)$

$Ker\ \partial \subset Im\ j_*$

$$
\begin{tikzcd}
0 \arrow[r] & C_{n+1}(A) \arrow[r] \arrow[d] & C_{n+1}(X) \arrow[r] \arrow[d] & C_{n+1}(X, A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_n(A) \arrow[r] \arrow[d] & \overset{b_n}{C_n(X)} \arrow[r] \arrow[d] & \overset{j(b_n)}{C_n(X, A)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_{n-1}(A) \arrow[r] & C_{n-1}(X) \arrow[r] & \overset{0}{C_{n-1}(X, A)} \arrow[r] & 0
\end{tikzcd}
$$

$$\cdots \to H_{n+1}(X, A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Checking exactness at $H_n(X, A)$

$Ker\ \partial \subset Im\ j_*$

$$\begin{tikzcd}
0 \arrow[r] & C_{n+1}(A) \arrow[r] \arrow[d] & C_{n+1}(X) \arrow[r] \arrow[d] & C_{n+1}(X, A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_n(A) \arrow[r] \arrow[d] & \overset{b_n}{C_n(X)} \arrow[r] \arrow[d] & \overset{j(b_n)}{C_n(X, A)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_{n-1}(A) \arrow[r] & \overset{\partial b_n}{C_{n-1}(X)} \arrow[r] & \overset{0}{C_{n-1}(X, A)} \arrow[r] & 0
\end{tikzcd}$$

$$\cdots \to H_{n+1}(X, A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Checking exactness at $H_n(X, A)$

$Ker\ \partial \subset Im\ j_*$

$$\begin{tikzcd}
0 \arrow[r] & C_{n+1}(A) \arrow[r] \arrow[d] & C_{n+1}(X) \arrow[r] \arrow[d] & C_{n+1}(X, A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_n(A) \arrow[r] \arrow[d] & \overset{b_n}{C_n(X)} \arrow[r, red] \arrow[d, red] & \overset{j(b_n)}{C_n(X, A)} \arrow[r] \arrow[d, red] & 0 \\
0 \arrow[r] & C_{n-1}(A) \arrow[r] & \overset{\partial b_n}{C_{n-1}(X)} \arrow[r, red] & \overset{0}{C_{n-1}(X, A)} \arrow[r] & 0
\end{tikzcd}$$

$$\cdots \to H_{n+1}(X, A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Checking exactness at $H_n(X, A)$

$Ker\ \partial \subset Im\ j_*$

$$
\begin{tikzcd}
0 \arrow[r] & C_{n+1}(A) \arrow[r] \arrow[d] & C_{n+1}(X) \arrow[r] \arrow[d] & C_{n+1}(X, A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_n(A) \arrow[r] \arrow[d] & \overset{b_n}{C_n(X)} \arrow[r] \arrow[d] & \overset{j(b_n)}{C_n(X, A)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \overset{a_{n-1}}{C_{n-1}(A)} \arrow[r] & \overset{\partial b_n = i(a_{n-1})}{C_{n-1}(X)} \arrow[r] & \overset{0}{C_{n-1}(X, A)} \arrow[r] & 0
\end{tikzcd}
$$

$$\cdots \to H_{n+1}(X, A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Checking exactness at $H_n(X, A)$

$Ker\ \partial \subset Im\ j_*$

$$
\begin{tikzcd}
0 \arrow[r] & C_{n+1}(A) \arrow[r] \arrow[d] & C_{n+1}(X) \arrow[r] \arrow[d] & C_{n+1}(X, A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \overset{a_n}{C_n(A)} \arrow[r] \arrow[d] & \overset{b_n}{C_n(X)} \arrow[r] \arrow[d] & \overset{j(b_n)}{C_n(X, A)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \overset{a_{n-1}=\partial a_n}{C_{n-1}(A)} \arrow[r] & \overset{\partial b_n = i(a_{n-1})}{C_{n-1}(X)} \arrow[r] & \overset{0}{C_{n-1}(X, A)} \arrow[r] & 0
\end{tikzcd}
$$

$$\cdots \to H_{n+1}(X, A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Checking exactness at $H_n(X, A)$

$Ker\ \partial \subset Im\ j_*$

$$
\begin{tikzcd}
0 \arrow[r] & C_{n+1}(A) \arrow[r] \arrow[d] & C_{n+1}(X) \arrow[r] \arrow[d] & C_{n+1}(X, A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \overset{a_n}{C_n(A)} \arrow[r] \arrow[d] & \overset{b_n, b_n - i(a_n)}{C_n(X)} \arrow[r] \arrow[d] & \overset{j(b_n)}{C_n(X, A)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \overset{a_{n-1} = \partial a_n}{C_{n-1}(A)} \arrow[r] & \overset{\partial b_n = i(a_{n-1})}{C_{n-1}(X)} \arrow[r] & \overset{0}{C_{n-1}(X, A)} \arrow[r] & 0
\end{tikzcd}
$$

$$\cdots \to H_{n+1}(X, A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Checking exactness at $H_n(X,A)$

$Im\ j_* \subset Ker\ \partial$

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C_{n+1}(A) & \longrightarrow & C_{n+1}(X) & \longrightarrow & C_{n+1}(X,A) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C_n(A) & \longrightarrow & C_n(X) & \longrightarrow & C_n(X,A) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C_{n-1}(A) & \longrightarrow & C_{n-1}(X) & \longrightarrow & C_{n-1}(X,A) & \longrightarrow & 0
\end{array}$$

$$\cdots \to H_{n+1}(X,A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Checking exactness at $H_n(X, A)$

$Im\ j_* \subset Ker\ \partial$

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C_{n+1}(A) & \longrightarrow & C_{n+1}(X) & \longrightarrow & C_{n+1}(X,A) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & & & b_n & & & & \\
0 & \longrightarrow & C_n(A) & \longrightarrow & C_n(X) & \longrightarrow & C_n(X,A) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C_{n-1}(A) & \longrightarrow & C_{n-1}(X) & \longrightarrow & C_{n-1}(X,A) & \longrightarrow & 0
\end{array}$$

$$\cdots \to H_{n+1}(X, A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Checking exactness at $H_n(X, A)$

$Im\ j_* \subset Ker\ \partial$

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C_{n+1}(A) & \longrightarrow & C_{n+1}(X) & \longrightarrow & C_{n+1}(X, A) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & & & b_n & & & & \\
0 & \longrightarrow & C_n(A) & \longrightarrow & C_n(X) & \longrightarrow & C_n(X, A) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & & & \partial b_n = 0 & & & & \\
0 & \longrightarrow & C_{n-1}(A) & \longrightarrow & C_{n-1}(X) & \longrightarrow & C_{n-1}(X, A) & \longrightarrow & 0
\end{array}$$

$$\cdots \to H_{n+1}(X, A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Checking exactness at $H_n(X, A)$

$Im\ j_* \subset Ker\ \partial$

$$\begin{tikzcd}
0 \arrow[r] & C_{n+1}(A) \arrow[r] \arrow[d] & C_{n+1}(X) \arrow[r] \arrow[d] & C_{n+1}(X, A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_n(A) \arrow[r] \arrow[d] & \overset{b_n}{C_n(X)} \arrow[r] \arrow[d] & \overset{j(b_n)}{C_n(X, A)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_{n-1}(A) \arrow[r] & \overset{\partial b_n = 0}{C_{n-1}(X)} \arrow[r] & C_{n-1}(X, A) \arrow[r] & 0
\end{tikzcd}$$

$$\cdots \to H_{n+1}(X, A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Checking exactness at $H_n(X, A)$

$Im\ j_* \subset Ker\ \partial$

$$\begin{tikzcd}
0 \arrow[r] & C_{n+1}(A) \arrow[r] \arrow[d] & C_{n+1}(X) \arrow[r] \arrow[d] & C_{n+1}(X, A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_n(A) \arrow[r] \arrow[d] & \overset{b_n}{C_n(X)} \arrow[r, red] \arrow[d, red] & \overset{j(b_n)}{C_n(X, A)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \overset{0}{C_{n-1}(A)} \arrow[r, red] & \overset{\partial b_n = 0}{C_{n-1}(X)} \arrow[r] & C_{n-1}(X, A) \arrow[r] & 0
\end{tikzcd}$$

$$\cdots \to H_{n+1}(X, A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Checking exactness at $H_n(A)$

$Ker\ i \subset Im\ \partial$

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C_{n+1}(A) & \longrightarrow & C_{n+1}(X) & \longrightarrow & C_{n+1}(X, A) & \longrightarrow & 0 \\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\
0 & \longrightarrow & C_n(A) & \longrightarrow & C_n(X) & \longrightarrow & C_n(X, A) & \longrightarrow & 0 \\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\
0 & \longrightarrow & C_{n-1}(A) & \longrightarrow & C_{n-1}(X) & \longrightarrow & C_{n-1}(X, A) & \longrightarrow & 0
\end{array}$$

$$\cdots \to H_{n+1}(X, A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Checking exactness at $H_n(A)$

$Ker\ i \subset Im\ \partial$

$$\begin{tikzcd}
0 \arrow[r] & C_{n+1}(A) \arrow[r] \arrow[d] & C_{n+1}(X) \arrow[r] \arrow[d] & C_{n+1}(X,A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_n(A) \arrow[r] \arrow[d] & C_n(X) \arrow[r] \arrow[d] & C_n(X,A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_{n-1}(A) \arrow[r] & C_{n-1}(X) \arrow[r] & C_{n-1}(X,A) \arrow[r] & 0
\end{tikzcd}$$

a_n

$$\cdots \to H_{n+1}(X,A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Checking exactness at $H_n(A)$

$Ker\ i \subset Im\ \partial$

$$\begin{tikzcd}
0 \arrow[r] & C_{n+1}(A) \arrow[r] \arrow[d] & C_{n+1}(X) \arrow[r] \arrow[d] & C_{n+1}(X,A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \overset{a_n}{C_n(A)} \arrow[r] \arrow[d, red] & C_n(X) \arrow[r] \arrow[d] & C_n(X,A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \overset{\partial a_n=0}{C_{n-1}(A)} \arrow[r] & C_{n-1}(X) \arrow[r] & C_{n-1}(X,A) \arrow[r] & 0
\end{tikzcd}$$

$$\cdots \to H_{n+1}(X,A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Checking exactness at $H_n(A)$

$Ker\ i \subset Im\ \partial$

$$
\begin{tikzcd}
0 \arrow[r] & C_{n+1}(A) \arrow[r] \arrow[d] & C_{n+1}(X) \arrow[r] \arrow[d] & C_{n+1}(X,A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \overset{a_n}{C_n(A)} \arrow[r] \arrow[d] & \overset{i(a_n)}{C_n(X)} \arrow[r] \arrow[d] & C_n(X,A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \overset{\partial a_n = 0}{C_{n-1}(A)} \arrow[r] & C_{n-1}(X) \arrow[r] & C_{n-1}(X,A) \arrow[r] & 0
\end{tikzcd}
$$

$$\cdots \to H_{n+1}(X,A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Checking exactness at $H_n(A)$

$Ker\ i \subset Im\ \partial$

$$\begin{tikzcd}
0 \arrow[r] & C_{n+1}(A) \arrow[r] \arrow[d] & \overset{b_{n+1}}{C_{n+1}(X)} \arrow[r] \arrow[d] & C_{n+1}(X,A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \overset{a_n}{C_n(A)} \arrow[r] \arrow[d] & \overset{i(a_n)=\partial(b_{n+1})}{C_n(X)} \arrow[r] \arrow[d] & C_n(X,A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \overset{\partial a_n=0}{C_{n-1}(A)} \arrow[r] & C_{n-1}(X) \arrow[r] & C_{n-1}(X,A) \arrow[r] & 0
\end{tikzcd}$$

$$\cdots \to H_{n+1}(X,A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Checking exactness at $H_n(A)$

$Ker\ i \subset Im\ \partial$

$$\begin{tikzcd}
0 \arrow[r] & C_{n+1}(A) \arrow[r] \arrow[d] & \overset{b_{n+1}}{C_{n+1}(X)} \arrow[r, red] \arrow[d, red] & \overset{j(b_{n+1})}{C_{n+1}(X,A)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \overset{a_n}{C_n(A)} \arrow[r, red] \arrow[d, red] & \overset{i(a_n)=\partial(b_{n+1})}{C_n(X)} \arrow[r] \arrow[d] & C_n(X,A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \overset{\partial a_n=0}{C_{n-1}(A)} \arrow[r] & C_{n-1}(X) \arrow[r] & C_{n-1}(X,A) \arrow[r] & 0
\end{tikzcd}$$

$$\cdots \to H_{n+1}(X,A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Checking exactness at $H_n(A)$

$Im\ \partial \subset Ker\ i$

$$\begin{tikzcd}
0 \arrow[r] & C_{n+1}(A) \arrow[r] \arrow[d] & C_{n+1}(X) \arrow[r] \arrow[d] & C_{n+1}(X,A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_n(A) \arrow[r] \arrow[d] & C_n(X) \arrow[r] \arrow[d] & C_n(X,A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_{n-1}(A) \arrow[r] & C_{n-1}(X) \arrow[r] & C_{n-1}(X,A) \arrow[r] & 0
\end{tikzcd}$$

$$\cdots \to H_{n+1}(X,A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Checking exactness at $H_n(A)$

$Im\ \partial \subset Ker\ i$

$$\begin{tikzcd}
 & & & j(b_{n+1}) & \\
0 \arrow[r] & C_{n+1}(A) \arrow[r] \arrow[d] & C_{n+1}(X) \arrow[r] \arrow[d] & C_{n+1}(X,A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_n(A) \arrow[r] \arrow[d] & C_n(X) \arrow[r] \arrow[d] & C_n(X,A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_{n-1}(A) \arrow[r] & C_{n-1}(X) \arrow[r] & C_{n-1}(X,A) \arrow[r] & 0
\end{tikzcd}$$

$$\cdots \to H_{n+1}(X,A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Checking exactness at $H_n(A)$

$Im\ \partial \subset Ker\ i$

$$\begin{tikzcd}
0 \arrow[r] & C_{n+1}(A) \arrow[r] \arrow[d] & C_{n+1}(X) \arrow[r] \arrow[d] & \overset{j(b_{n+1})}{C_{n+1}(X,A)} \arrow[r] \arrow[d, red] & 0 \\
0 \arrow[r] & C_n(A) \arrow[r] \arrow[d] & C_n(X) \arrow[r] \arrow[d] & \overset{0}{C_n(X,A)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_{n-1}(A) \arrow[r] & C_{n-1}(X) \arrow[r] & C_{n-1}(X,A) \arrow[r] & 0
\end{tikzcd}$$

$$\cdots \to H_{n+1}(X,A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Checking exactness at $H_n(A)$

$Im\ \partial \subset Ker\ i$

$$
\begin{tikzcd}
0 \arrow[r] & C_{n+1}(A) \arrow[r] \arrow[d] & \overset{b_{n+1}}{C_{n+1}(X)} \arrow[r, red] \arrow[d] & \overset{j(b_{n+1})}{C_{n+1}(X,A)} \arrow[r] \arrow[d, red] & 0 \\
0 \arrow[r] & C_n(A) \arrow[r] \arrow[d] & C_n(X) \arrow[r] \arrow[d] & \overset{0}{C_n(X,A)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_{n-1}(A) \arrow[r] & C_{n-1}(X) \arrow[r] & C_{n-1}(X,A) \arrow[r] & 0
\end{tikzcd}
$$

$$\cdots \to H_{n+1}(X,A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Checking exactness at $H_n(A)$

$Im\ \partial \subset Ker\ i$

$$\begin{tikzcd}
0 \arrow[r] & C_{n+1}(A) \arrow[r] \arrow[d] & \overset{b_{n+1}}{C_{n+1}(X)} \arrow[r] \arrow[d] & \overset{j(b_{n+1})}{C_{n+1}(X,A)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_n(A) \arrow[r] \arrow[d] & \overset{\partial b_{n+1}}{C_n(X)} \arrow[r] \arrow[d] & \overset{0}{C_n(X,A)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_{n-1}(A) \arrow[r] & C_{n-1}(X) \arrow[r] & C_{n-1}(X,A) \arrow[r] & 0
\end{tikzcd}$$

$$\cdots \to H_{n+1}(X,A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Checking exactness at $H_n(A)$

$Im\ \partial \subset Ker\ i$

$$
\begin{tikzcd}
 & & b_{n+1} & j(b_{n+1}) & \\
0 \arrow[r] & C_{n+1}(A) \arrow[r] \arrow[d] & C_{n+1}(X) \arrow[r] \arrow[d] & C_{n+1}(X,A) \arrow[r] \arrow[d] & 0 \\
 & a_n & \partial b_{n+1} & 0 & \\
0 \arrow[r] & C_n(A) \arrow[r] \arrow[d] & C_n(X) \arrow[r] \arrow[d] & C_n(X,A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_{n-1}(A) \arrow[r] & C_{n-1}(X) \arrow[r] & C_{n-1}(X,A) \arrow[r] & 0
\end{tikzcd}
$$

$$\cdots \to H_{n+1}(X,A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Checking exactness at $H_n(X)$

$Ker\ j \subset Im\ i$

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C_{n+1}(A) & \longrightarrow & C_{n+1}(X) & \longrightarrow & C_{n+1}(X,A) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C_n(A) & \longrightarrow & C_n(X) & \longrightarrow & C_n(X,A) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C_{n-1}(A) & \longrightarrow & C_{n-1}(X) & \longrightarrow & C_{n-1}(X,A) & \longrightarrow & 0
\end{array}$$

$$\cdots \to H_{n+1}(X,A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Checking exactness at $H_n(X)$

$Ker\ j \subset Im\ i$

$$\begin{tikzcd}
0 \arrow[r] & C_{n+1}(A) \arrow[r] \arrow[d] & C_{n+1}(X) \arrow[r] \arrow[d, "b_n"] & C_{n+1}(X,A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_n(A) \arrow[r] \arrow[d] & C_n(X) \arrow[r] \arrow[d] & C_n(X,A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_{n-1}(A) \arrow[r] & C_{n-1}(X) \arrow[r] & C_{n-1}(X,A) \arrow[r] & 0
\end{tikzcd}$$

$$\cdots \to H_{n+1}(X,A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Checking exactness at $H_n(X)$

$Ker\ j \subset Im\ i$

$$\begin{tikzcd}
0 \arrow[r] & C_{n+1}(A) \arrow[r] \arrow[d] & C_{n+1}(X) \arrow[r] \arrow[d] & C_{n+1}(X,A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_n(A) \arrow[r] \arrow[d] & \overset{b_n}{C_n(X)} \arrow[r] \arrow[d, red] & C_n(X,A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_{n-1}(A) \arrow[r] & \overset{0}{C_{n-1}(X)} \arrow[r] & C_{n-1}(X,A) \arrow[r] & 0
\end{tikzcd}$$

$$\cdots \to H_{n+1}(X,A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Checking exactness at $H_n(X)$

$Ker\ j \subset Im\ i$

$$
\begin{tikzcd}
0 \arrow[r] & C_{n+1}(A) \arrow[r] \arrow[d] & C_{n+1}(X) \arrow[r] \arrow[d] & C_{n+1}(X, A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_n(A) \arrow[r] \arrow[d] & \overset{b_n}{C_n(X)} \arrow[r, red] \arrow[d, red] & \overset{j(b_n)}{C_n(X, A)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_{n-1}(A) \arrow[r] & \overset{0}{C_{n-1}(X)} \arrow[r] & C_{n-1}(X, A) \arrow[r] & 0
\end{tikzcd}
$$

$$\cdots \to H_{n+1}(X, A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Checking exactness at $H_n(X)$

$Ker\ j \subset Im\ i$

$$\begin{tikzcd}
0 \arrow[r] & C_{n+1}(A) \arrow[r] \arrow[d] & C_{n+1}(X) \arrow[r] \arrow[d] & \overset{c_{n+1}}{C_{n+1}(X,A)} \arrow[r] \arrow[d, red] & 0 \\
0 \arrow[r] & C_n(A) \arrow[r] \arrow[d] & \overset{b_n}{C_n(X)} \arrow[r, red] \arrow[d, red] & \overset{j(b_n)=\partial c_{n+1}}{C_n(X,A)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_{n-1}(A) \arrow[r] & \overset{0}{C_{n-1}(X)} \arrow[r] & C_{n-1}(X,A) \arrow[r] & 0
\end{tikzcd}$$

$$\cdots \to H_{n+1}(X,A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Checking exactness at $H_n(X)$

$Ker\ j \subset Im\ i$

$$
\begin{tikzcd}
0 \arrow[r] & C_{n+1}(A) \arrow[r] \arrow[d] & \overset{b_{n+1}}{C_{n+1}(X)} \arrow[r] \arrow[d] & \overset{j(b_{n+1})}{C_{n+1}(X,A)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_n(A) \arrow[r] \arrow[d] & \overset{b_n}{C_n(X)} \arrow[r] \arrow[d] & \overset{j(b_n)=\partial j(b_{n+1})}{C_n(X,A)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_{n-1}(A) \arrow[r] & \overset{0}{C_{n-1}(X)} \arrow[r] & C_{n-1}(X,A) \arrow[r] & 0
\end{tikzcd}
$$

$$\cdots \to H_{n+1}(X,A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Checking exactness at $H_n(X)$

$Ker\ j \subset Im\ i$

$$
\begin{tikzcd}
0 \arrow[r] & C_{n+1}(A) \arrow[r] \arrow[d] & \overset{b_{n+1}}{C_{n+1}(X)} \arrow[r, red] \arrow[d, red] & \overset{j(b_{n+1})}{C_{n+1}(X,A)} \arrow[r] \arrow[d, red] & 0 \\
0 \arrow[r] & C_n(A) \arrow[r] \arrow[d] & \overset{b_n \text{ homologous } b_n - \partial b_{n+1}}{C_n(X)} \arrow[r, red] \arrow[d, red] & \overset{j(b_n)=\partial j(b_{n+1})}{C_n(X,A)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_{n-1}(A) \arrow[r] & \overset{0}{C_{n-1}(X)} \arrow[r] & C_{n-1}(X,A) \arrow[r] & 0
\end{tikzcd}
$$

$$\cdots \to H_{n+1}(X,A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Checking exactness at $H_n(X)$

$Ker\ j \subset Im\ i$

$$
\begin{tikzcd}
0 \arrow[r] & C_{n+1}(A) \arrow[r] \arrow[d] & \overset{b_{n+1}}{C_{n+1}(X)} \arrow[r] \arrow[d] & \overset{j(b_{n+1})}{C_{n+1}(X,A)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \overset{a_n}{C_n(A)} \arrow[r] \arrow[d] & \overset{b_n \text{ homologous } b_n - \partial b_{n+1} = i(a_n)}{C_n(X)} \arrow[r] \arrow[d] & \overset{j(b_n) = \partial j(b_{n+1})}{C_n(X,A)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_{n-1}(A) \arrow[r] & \overset{0}{C_{n-1}(X)} \arrow[r] & C_{n-1}(X,A) \arrow[r] & 0
\end{tikzcd}
$$

$$\cdots \to H_{n+1}(X,A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Checking exactness at $H_n(X)$

$Im\ i \subset Ker\ j$

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & C_{n+1}(A) & \longrightarrow & C_{n+1}(X) & \longrightarrow & C_{n+1}(X,A) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C_{n}(A) & \longrightarrow & C_{n}(X) & \longrightarrow & C_{n}(X,A) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C_{n-1}(A) & \longrightarrow & C_{n-1}(X) & \longrightarrow & C_{n-1}(X,A) & \longrightarrow & 0
\end{array}
$$

$$\cdots \to H_{n+1}(X,A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Checking exactness at $H_n(X)$

$Im\ i \subset Ker\ j$

$$\begin{tikzcd}
0 \arrow[r] & C_{n+1}(A) \arrow[r] \arrow[d] & C_{n+1}(X) \arrow[r] \arrow[d] & C_{n+1}(X,A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \overset{a_n}{C_n(A)} \arrow[r] \arrow[d] & C_n(X) \arrow[r] \arrow[d] & C_n(X,A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_{n-1}(A) \arrow[r] & C_{n-1}(X) \arrow[r] & C_{n-1}(X,A) \arrow[r] & 0
\end{tikzcd}$$

$$\cdots \to H_{n+1}(X,A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Checking exactness at $H_n(X)$

$Im\ i \subset Ker\ j$

$$\begin{tikzcd}
0 \arrow[r] & C_{n+1}(A) \arrow[r] \arrow[d] & C_{n+1}(X) \arrow[r] \arrow[d] & C_{n+1}(X,A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \overset{a_n}{C_n(A)} \arrow[r] \arrow[d] & \overset{i(a_n)}{C_n(X)} \arrow[r] \arrow[d] & C_n(X,A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_{n-1}(A) \arrow[r] & C_{n-1}(X) \arrow[r] & C_{n-1}(X,A) \arrow[r] & 0
\end{tikzcd}$$

$$\cdots \to H_{n+1}(X,A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Checking exactness at $H_n(X)$

$Im\ i \subset Ker\ j$

$$\begin{tikzcd}
0 \arrow[r] & C_{n+1}(A) \arrow[r] \arrow[d] & C_{n+1}(X) \arrow[r] \arrow[d] & C_{n+1}(X, A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \overset{a_n}{C_n(A)} \arrow[r] \arrow[d] & \overset{i(a_n)}{C_n(X)} \arrow[r] \arrow[d] & \overset{j(i(a_n))}{C_n(X, A)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_{n-1}(A) \arrow[r] & C_{n-1}(X) \arrow[r] & C_{n-1}(X, A) \arrow[r] & 0
\end{tikzcd}$$

$$\cdots \to H_{n+1}(X, A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Checking exactness at $H_n(X)$

$Im\ i \subset Ker\ j$

$$
\begin{tikzcd}
0 \arrow[r] & C_{n+1}(A) \arrow[r] \arrow[d] & C_{n+1}(X) \arrow[r] \arrow[d] & C_{n+1}(X,A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \overset{a_n}{C_n(A)} \arrow[r, red] \arrow[d] & \overset{i(a_n)}{C_n(X)} \arrow[r, red] \arrow[d] & \overset{j(i(a_n))=0}{C_n(X,A)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_{n-1}(A) \arrow[r] & C_{n-1}(X) \arrow[r] & C_{n-1}(X,A) \arrow[r] & 0
\end{tikzcd}
$$

$$\cdots \to H_{n+1}(X,A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

$B \subset A \subset X$

$B \subset A \subset X$

$0 \to C_n(A)/C_n(B) \xhookrightarrow{i} C_n(X)/C_n(B)$

$B \subset A \subset X$

$0 \to C_n(A)/C_n(B) \xhookrightarrow{i} C_n(X)/C_n(B) \xrightarrow{j} C_n(X)/C_n(A)$

$B \subset A \subset X$

$0 \to C_n(A)/C_n(B) \xhookrightarrow{i} C_n(X)/C_n(B) \xrightarrow{j} C_n(X)/C_n(A) \to 0$

$B \subset A \subset X$

$0 \to C_n(A, B) \xhookrightarrow{i} C_n(X)/C_n(B) \xrightarrow{j} C_n(X)/C_n(A) \to 0$

$$B \subset A \subset X$$

$$0 \to C_n(A, B) \xhookrightarrow{i} C_n(X, B) \xrightarrow{j} C_n(X)/C_n(A) \to 0$$

$B \subset A \subset X$

$0 \to C_n(A, B) \xhookrightarrow{i} C_n(X, B) \xrightarrow{j} C_n(X, A) \to 0$

$B \subset A \subset X$

$0 \to C_n(A, B) \xhookrightarrow{i} C_n(X, B) \xrightarrow{j} C_n(X, A) \to 0$

$H_n(A, B) \xhookrightarrow{i_*} H_n(X, B) \xrightarrow{j_*} H_n(X, A)$

$B \subset A \subset X$

$0 \to C_n(A,B) \xhookrightarrow{i} C_n(X,B) \xrightarrow{j} C_n(X,A) \to 0$

$H_n(A,B) \xhookrightarrow{i_*} H_n(X,B) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A,B) \to \cdots$

$B \subset A \subset X$

$0 \to C_n(A, B) \xhookrightarrow{i} C_n(X, B) \xrightarrow{j} C_n(X, A) \to 0$

$H_n(A, B) \xhookrightarrow{i_*} H_n(X, B) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A, B) \to \cdots$

$B \subset A \subset X$

$$0 \to C_n(A,B) \xhookrightarrow{i} C_n(X,B) \xrightarrow{j} C_n(X,A) \to 0$$

$$\cdots \to H_{n+1}(X,A) \xrightarrow{\partial} H_n(A,B) \xhookrightarrow{i_*} H_n(X,B) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A,B) \to \cdots$$

Long exact sequence of triple

$B \subset A \subset X$

$0 \to C_n(A,B) \xhookrightarrow{i} C_n(X,B) \xrightarrow{j} C_n(X,A) \to 0$

$\cdots \to H_{n+1}(X,A) \xrightarrow{\partial} H_n(A,B) \xhookrightarrow{i_*} H_n(X,B) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A,B) \to \cdots$

Long exact sequence of triple

$B \subset A \subset X$

$0 \to C_n(A,B) \xhookrightarrow{i} C_n(X,B) \xrightarrow{j} C_n(X,A) \to 0$

$$\cdots \to H_{n+1}(X,A) \xrightarrow{\partial} H_n(A,B) \xhookrightarrow{i_*} H_n(X,B) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A,B) \to \cdots$$

$$C_n(A) \oplus C_n(B) \xrightarrow[i_A + i_B]{j} C_n(X)$$

Long exact sequence of triple

$B \subset A \subset X$

$0 \to C_n(A,B) \overset{i}{\hookrightarrow} C_n(X,B) \overset{j}{\to} C_n(X,A) \to 0$

$$\cdots \to H_{n+1}(X,A) \overset{\partial}{\to} H_n(A,B) \overset{i_*}{\hookrightarrow} H_n(X,B) \overset{j_*}{\to} H_n(X,A) \overset{\partial}{\to} H_{n-1}(A,B) \to \cdots$$

$$C_n(A) \oplus C_n(B) \xrightarrow[i_A+i_B]{j} C_n^{A,B}(X)$$

Long exact sequence of triple

$B \subset A \subset X$

$0 \to C_n(A, B) \xhookrightarrow{i} C_n(X, B) \xrightarrow{j} C_n(X, A) \to 0$

$$\cdots \to H_{n+1}(X, A) \xrightarrow{\partial} H_n(A, B) \xhookrightarrow{i_*} H_n(X, B) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A, B) \to \cdots$$

$$C_n(A) \oplus C_n(B) \xrightarrow[i_A + i_B]{j} C_n^{A,B}(X) \to 0$$

Long exact sequence of triple

$B \subset A \subset X$

$0 \to C_n(A, B) \xhookrightarrow{i} C_n(X, B) \xrightarrow{j} C_n(X, A) \to 0$

$$\cdots \to H_{n+1}(X, A) \xrightarrow{\partial} H_n(A, B) \xhookrightarrow{i_*} H_n(X, B) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A, B) \to \cdots$$

$$0 \to C_n(A \cap B) \xrightarrow[c \to (i(c), -i(c))]{i} C_n(A) \oplus C_n(B) \xrightarrow[i_A + i_B]{j} C_n^{A,B}(X) \to 0$$

Long exact sequence of triple

$B \subset A \subset X$

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Cone over A, CA

Cone over A, $CA := A \times [0,1]/(A \times \{0\})$

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$\boxed{H_n(X, A) \text{ is isomorphic to } \tilde{H}_n(X \cup CA)}$

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Not in the syllabus

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$$X \xhookrightarrow{i} M_f \xrightarrow{r} Y$$

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$\cdots \to \pi_n(A) \to \pi_n(X) \to \pi_n(X, A) \to \pi_{n-1}(A) \to \cdots$

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If $\pi_1(A) = 0$, then first non-zero $H_n(X, A)$ is the first non-zero $\pi_n(X, A)$ and they are isomorphic.

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If $\pi_1(A) = 0$, then first non-zero $H_n(X, A)$ is the first non-zero $\pi_n(X, A)$ and they are isomorphic.

Fact 3:
$f : X \to Y$, X and Y ”CW complexes”, and $f_* : \pi_n(X) \to \pi(Y)$ isomorphisms, then X is homotopically equivalent to Y.

Not in the syllabus

$f : X \to Y$

Assume X is also a CW complex

$X \xhookrightarrow{i} M_f \xrightarrow{r} Y$
$f = r \circ i$
r_* an isomorphism (r deformation retract)

$H_n(X) \xrightarrow{f_*} H_n(Y)$ isomorphism $\implies$ $H_n(X) \xrightarrow{i_*} H_n(M_f)$ isomorphism $\implies$ $H_n(M_f, X) = 0$

Fact 1:
$\cdots \to \pi_n(A) \to \pi_n(X) \to \pi_n(X, A) \to \pi_{n-1}(A) \to \cdots$

Fact 2:
If $\pi_1(A) = 0$, then first non-zero $H_n(X, A)$ is the first non-zero $\pi_n(X, A)$ and they are isomorphic.

Fact 3:
$f : X \to Y$, X and Y "CW complexes", and $f_* : \pi_n(X) \to \pi(Y)$ isomorphisms, then X is homotopically equivalent to Y.

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Theorem. *$\tilde{H}_i(S^n)$ is $\mathbb{Z}$ if $i = n$, otherwise, trivial*

Proof. $\tilde{H}_i(D^n) = 0$ for all i. $\square$

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