R ring,

R ring, $\phi \in C^m(X;R)$

R ring, $\phi \in C^m(X;R)$ $\tau \in C^n(X;R)$

R ring, $\phi \in C^m(X; R)$ $\tau \in C^n(X; R)$

Define, $\smile: C^m(X;R) \times C^n(X;R) \to C^{m+n}(X;R)$

$$R \text{ ring},$$

 $\phi \in C^m(X;R)$
 $\tau \in C^n(X;R)$

Define,
$$\smile: C^m(X; R) \times C^n(X; R) \to C^{m+n}(X; R)$$

 $(\phi \smile \tau)(\sigma) = \phi(\sigma|_{[e_0, \dots, e_m]}) \tau(\sigma|_{[e_m, \dots, e_n]})$

R ring, $\phi \in C^m(X; R)$ $\tau \in C^n(X; R)$

Define,
$$\smile: C^m(X; R) \times C^n(X; R) \to C^{m+n}(X; R)$$

 $(\phi \smile \tau)(\sigma) = \phi(\sigma|_{[e_0, \dots, e_m]}) \tau(\sigma|_{[e_m, \dots, e_n]})$

$$\delta(\phi \smile \tau) = \delta\phi \smile \tau + (-1)^m \phi \smile \delta\tau$$

$$R \text{ ring},$$

 $\phi \in C^m(X;R)$
 $\tau \in C^n(X;R)$

Define,
$$\smile: C^m(X; R) \times C^n(X; R) \to C^{m+n}(X; R)$$

 $(\phi \smile \tau)(\sigma) = \phi(\sigma|_{[e_0, \dots, e_m]}) \tau(\sigma|_{[e_m, \dots, e_n]})$

$$\delta(\phi\smile\tau)=\delta\phi\smile\tau+(-1)^m\phi\smile\delta\tau$$

$$\delta \phi = 0, \, \delta \tau = 0 \text{ implies } \delta(\phi \smile \tau) = 0$$

R ring, $\phi \in C^m(X;R)$ $\tau \in C^n(X;R)$

Define,
$$\smile: C^m(X; R) \times C^n(X; R) \to C^{m+n}(X; R)$$

 $(\phi \smile \tau)(\sigma) = \phi(\sigma|_{[e_0, \dots, e_m]}) \tau(\sigma|_{[e_m, \dots, e_n]})$

$$\delta(\phi \smile \tau) = \delta\phi \smile \tau + (-1)^m \phi \smile \delta\tau$$

$$\delta \phi = 0, \, \delta \tau = 0 \text{ implies } \delta(\phi \smile \tau) = 0$$

 $\delta \phi \smile \tau = \delta(\phi \smile \tau) - (-1)^m \phi \smile \delta \tau$

$$R \text{ ring},$$

 $\phi \in C^m(X; R)$
 $\tau \in C^n(X; R)$

Define,
$$\smile: C^m(X; R) \times C^n(X; R) \to C^{m+n}(X; R)$$

 $(\phi \smile \tau)(\sigma) = \phi(\sigma|_{[e_0, \dots, e_m]}) \tau(\sigma|_{[e_m, \dots, e_n]})$

$$\delta(\phi \smile \tau) = \delta\phi \smile \tau + (-1)^m \phi \smile \delta\tau$$

$$\delta \phi = 0, \, \delta \tau = 0 \text{ implies } \delta(\phi \smile \tau) = 0$$

$$\delta \phi \smile \tau = \delta(\phi \smile \tau) - (-1)^m \phi \smile \delta \tau$$

$$\delta \tau = 0 \text{ implies } \delta \phi \smile \tau = \delta(\phi \smile \tau)$$

$$R \text{ ring},$$

 $\phi \in C^m(X; R)$
 $\tau \in C^n(X; R)$

Define,
$$\smile: C^m(X; R) \times C^n(X; R) \to C^{m+n}(X; R)$$

 $(\phi \smile \tau)(\sigma) = \phi(\sigma|_{[e_0, \dots, e_m]}) \tau(\sigma|_{[e_m, \dots, e_n]})$

$$\delta(\phi \smile \tau) = \delta\phi \smile \tau + (-1)^m \phi \smile \delta\tau$$

$$\delta\phi = 0, \, \delta\tau = 0 \text{ implies } \delta(\phi \smile \tau) = 0$$

$$\delta\phi \smile \tau = \delta(\phi \smile \tau) - (-1)^m \phi \smile \delta\tau$$

$$\delta\tau = 0 \text{ implies } \delta\phi \smile \tau = \delta(\phi \smile \tau)$$

$$\delta\phi \smile \tau = \delta(\phi \smile \tau) - (-1)^m \phi \smile \delta\tau$$

$$R \text{ ring},$$

 $\phi \in C^m(X;R)$
 $\tau \in C^n(X;R)$

Define,
$$\smile: C^m(X; R) \times C^n(X; R) \to C^{m+n}(X; R)$$

 $(\phi \smile \tau)(\sigma) = \phi(\sigma|_{[e_0, \dots, e_m]}) \tau(\sigma|_{[e_m, \dots, e_n]})$

$$\delta(\phi \smile \tau) = \delta\phi \smile \tau + (-1)^m \phi \smile \delta\tau$$

$$\delta\phi = 0, \, \delta\tau = 0 \text{ implies } \delta(\phi \smile \tau) = 0$$

$$\delta\phi \smile \tau = \delta(\phi \smile \tau) - (-1)^m \phi \smile \delta\tau$$

$$\delta\tau = 0 \text{ implies } \delta\phi \smile \tau = \delta(\phi \smile \tau)$$

$$\delta\phi \smile \tau = \delta(\phi \smile \tau) - (-1)^m \phi \smile \delta\tau$$

Defines, $\smile: H^m(X;R) \times H^n(X;R) \to H^{m+n}(X;R)$

R ring, $\phi \in C^m(X; R)$ $\tau \in C^n(X; R)$

Define,
$$\smile: C^m(X; R) \times C^n(X; R) \to C^{m+n}(X; R)$$

 $(\phi \smile \tau)(\sigma) = \phi(\sigma|_{[e_0, \dots, e_m]}) \tau(\sigma|_{[e_m, \dots, e_n]})$

$$\delta(\phi \smile \tau) = \delta\phi \smile \tau + (-1)^m \phi \smile \delta\tau$$

$$\delta\phi = 0, \, \delta\tau = 0 \text{ implies } \delta(\phi \smile \tau) = 0$$

$$\delta\phi \smile \tau = \delta(\phi \smile \tau) - (-1)^m \phi \smile \delta\tau$$

$$\delta\tau = 0 \text{ implies } \delta\phi \smile \tau = \delta(\phi \smile \tau)$$

$$\delta\phi \smile \tau = \delta(\phi \smile \tau) - (-1)^m \phi \smile \delta\tau$$

Defines, $\smile: H^m(X;R) \times H^n(X;R) \to H^{m+n}(X;R)$ Generally, \smile : $H^m(X,A;R) \times H^n(X,B;R) \rightarrow H^{m+n}(X,A\cup B;R)$

R ring, $\phi \in C^m(X; R)$ $\tau \in C^n(X; R)$

Define,
$$\smile: C^m(X; R) \times C^n(X; R) \to C^{m+n}(X; R)$$

 $(\phi \smile \tau)(\sigma) = \phi(\sigma|_{[e_0, \dots, e_m]}) \tau(\sigma|_{[e_m, \dots, e_n]})$

$$\delta(\phi \smile \tau) = \delta\phi \smile \tau + (-1)^m \phi \smile \delta\tau$$

$$\delta\phi = 0, \, \delta\tau = 0 \text{ implies } \delta(\phi \smile \tau) = 0$$

$$\delta\phi \smile \tau = \delta(\phi \smile \tau) - (-1)^m \phi \smile \delta\tau$$

$$\delta\tau = 0 \text{ implies } \delta\phi \smile \tau = \delta(\phi \smile \tau)$$

$$\delta\phi \smile \tau = \delta(\phi \smile \tau) - (-1)^m \phi \smile \delta\tau$$

Defines, $\smile: H^m(X; R) \times H^n(X; R) \to H^{m+n}(X; R)$ Generally, \smile : $H^m(X,A;R) \times H^n(X,B;R) \rightarrow H^{m+n}(X,A\cup B;R)$

Exercise. $f: X \to Y$, then $f^*(\alpha \smile \beta) = f^*(\alpha) \smile f^*(\beta)$

R ring, $\phi \in C^m(X;R)$ $\tau \in C^n(X;R)$

Define,
$$\smile: C^m(X; R) \times C^n(X; R) \to C^{m+n}(X; R)$$

 $(\phi \smile \tau)(\sigma) = \phi(\sigma|_{[e_0, \dots, e_m]}) \tau(\sigma|_{[e_m, \dots, e_n]})$

$$\delta(\phi \smile \tau) = \delta\phi \smile \tau + (-1)^m \phi \smile \delta\tau$$

$$\delta\phi = 0, \, \delta\tau = 0 \text{ implies } \delta(\phi \smile \tau) = 0$$

$$\delta\phi \smile \tau = \delta(\phi \smile \tau) - (-1)^m \phi \smile \delta\tau$$

$$\delta\tau = 0 \text{ implies } \delta\phi \smile \tau = \delta(\phi \smile \tau)$$

$$\delta\phi \smile \tau = \delta(\phi \smile \tau) - (-1)^m \phi \smile \delta\tau$$

Defines, $\smile: H^m(X;R) \times H^n(X;R) \to H^{m+n}(X;R)$

Generally,
$$\smile$$
: $H^m(X,A;R) \times H^n(X,B;R) \rightarrow H^{m+n}(X,A\cup B;R)$

Exercise. $f: X \to Y$, then $f^*(\alpha \smile \beta) = f^*(\alpha) \smile f^*(\beta)$

$$\times: H^m(X;R) \times H^n(Y;R) \to H^{m+n}(X \times Y;R)$$

R ring, $\phi \in C^m(X; R)$ $\tau \in C^n(X; R)$

Define,
$$\smile: C^m(X; R) \times C^n(X; R) \to C^{m+n}(X; R)$$

 $(\phi \smile \tau)(\sigma) = \phi(\sigma|_{[e_0, \dots, e_m]}) \tau(\sigma|_{[e_m, \dots, e_n]})$

$$\delta(\phi \smile \tau) = \delta\phi \smile \tau + (-1)^m \phi \smile \delta\tau$$

$$\delta\phi = 0, \, \delta\tau = 0 \text{ implies } \delta(\phi \smile \tau) = 0$$

$$\delta\phi \smile \tau = \delta(\phi \smile \tau) - (-1)^m \phi \smile \delta\tau$$

$$\delta\tau = 0 \text{ implies } \delta\phi \smile \tau = \delta(\phi \smile \tau)$$

$$\delta\phi \smile \tau = \delta(\phi \smile \tau) - (-1)^m \phi \smile \delta\tau$$

Defines, $\smile: H^m(X; R) \times H^n(X; R) \to H^{m+n}(X; R)$ Generally, \smile : $H^m(X,A;R) \times H^n(X,B;R) \rightarrow H^{m+n}(X,A\cup B;R)$

Exercise. $f: X \to Y$, then $f^*(\alpha \smile \beta) = f^*(\alpha) \smile f^*(\beta)$

 $\begin{array}{l} \times: H^m(X;R) \times H^n(Y;R) \to H^{m+n}(X \times Y;R) \\ x \times y = P_x^*(x) \smile p_y^*(y). \end{array}$

R ring, $\phi \in C^m(X;R)$ $\tau \in C^n(X;R)$

Define,
$$\smile: C^m(X; R) \times C^n(X; R) \to C^{m+n}(X; R)$$

 $(\phi \smile \tau)(\sigma) = \phi(\sigma|_{[e_0, \dots, e_m]}) \tau(\sigma|_{[e_m, \dots, e_n]})$

$$\delta(\phi \smile \tau) = \delta\phi \smile \tau + (-1)^m \phi \smile \delta\tau$$

$$\delta\phi = 0, \, \delta\tau = 0 \text{ implies } \delta(\phi \smile \tau) = 0$$

$$\delta\phi \smile \tau = \delta(\phi \smile \tau) - (-1)^m \phi \smile \delta\tau$$

$$\delta\tau = 0 \text{ implies } \delta\phi \smile \tau = \delta(\phi \smile \tau)$$

$$\delta\phi \smile \tau = \delta(\phi \smile \tau) - (-1)^m \phi \smile \delta\tau$$

Defines, $\smile: H^m(X;R) \times H^n(X;R) \to H^{m+n}(X;R)$ Generally, \smile : $H^m(X,A;R) \times H^n(X,B;R) \rightarrow H^{m+n}(X,A\cup B;R)$

Exercise. $f: X \to Y$, then $f^*(\alpha \smile \beta) = f^*(\alpha) \smile f^*(\beta)$

$$\begin{array}{l} \times: H^m(X;R) \times H^n(Y;R) \to H^{m+n}(X \times Y;R) \\ x \times y = P_x^*(x) \smile p_y^*(y). \end{array}$$

 $\times: H^m(X, A; R) \times H^n(Y, B; R) \to H^{m+n}(X \times Y, ??; R)$

R ring, $\phi \in C^m(X;R)$ $\tau \in C^n(X;R)$

Define,
$$\smile: C^m(X; R) \times C^n(X; R) \to C^{m+n}(X; R)$$

 $(\phi \smile \tau)(\sigma) = \phi(\sigma|_{[e_0, \dots, e_m]}) \tau(\sigma|_{[e_m, \dots, e_n]})$

$$\delta(\phi \smile \tau) = \delta\phi \smile \tau + (-1)^m \phi \smile \delta\tau$$

$$\delta\phi = 0, \, \delta\tau = 0 \text{ implies } \delta(\phi \smile \tau) = 0$$

$$\delta\phi \smile \tau = \delta(\phi \smile \tau) - (-1)^m \phi \smile \delta\tau$$

$$\delta\tau = 0 \text{ implies } \delta\phi \smile \tau = \delta(\phi \smile \tau)$$

$$\delta\phi \smile \tau = \delta(\phi \smile \tau) - (-1)^m \phi \smile \delta\tau$$

Defines, $\smile: H^m(X;R) \times H^n(X;R) \to H^{m+n}(X;R)$ Generally, \smile : $H^m(X, A; R) \times H^n(X, B; R) \rightarrow H^{m+n}(X, A \cup B; R)$

Exercise. $f: X \to Y$, then $f^*(\alpha \smile \beta) = f^*(\alpha) \smile f^*(\beta)$

$$\begin{array}{l} \times: H^m(X;R) \times H^n(Y;R) \to H^{m+n}(X \times Y;R) \\ x \times y = P_x^*(x) \smile p_y^*(y). \end{array}$$

 $\times: H^m(X, A; R) \times H^n(Y, B; R) \to H^{m+n}(X \times Y, A \times Y \cup X \times B; R)$

R ring, $\phi \in C^m(X; R)$ $\tau \in C^n(X; R)$

Define,
$$\smile: C^m(X; R) \times C^n(X; R) \to C^{m+n}(X; R)$$

 $(\phi \smile \tau)(\sigma) = \phi(\sigma|_{[e_0, \dots, e_m]}) \tau(\sigma|_{[e_m, \dots, e_n]})$

$$\delta(\phi \smile \tau) = \delta\phi \smile \tau + (-1)^m \phi \smile \delta\tau$$

$$\delta\phi = 0, \, \delta\tau = 0 \text{ implies } \delta(\phi \smile \tau) = 0$$

$$\delta\phi \smile \tau = \delta(\phi \smile \tau) - (-1)^m \phi \smile \delta\tau$$

$$\delta\tau = 0 \text{ implies } \delta\phi \smile \tau = \delta(\phi \smile \tau)$$

$$\delta\phi \smile \tau = \delta(\phi \smile \tau) - (-1)^m \phi \smile \delta\tau$$

Defines,

$$\smile: H^m(X;R) \times H^n(X;R) \to H^{m+n}(X;R)$$

Generally, \smile : $H^m(X,A;R) \times H^n(X,B;R) \rightarrow H^{m+n}(X,A\cup B;R)$

Exercise. $f: X \to Y$, then $f^*(\alpha \smile \beta) = f^*(\alpha) \smile f^*(\beta)$

$$\begin{array}{l} \times: H^m(X;R) \times H^n(Y;R) \to H^{m+n}(X \times Y;R) \\ x \times y = P_x^*(x) \smile p_y^*(y). \end{array}$$

$$\times: H^m(X,A;R) \times H^n(Y,B;R) \to H^{m+n}(X \times Y,A \times Y \cup X \times B;R)$$

Exercise. \smile is distributive over

+ and associative.

Cup products

R ring, $\phi \in C^m(X; R)$ $\tau \in C^n(X; R)$

Define, \smile : $C^m(X;R) \times C^n(X;R) \to C^{m+n}(X;R)$ $(\phi \smile \tau)(\sigma) = \phi(\sigma|_{[e_0,\dots,e_m]})\tau(\sigma|_{[e_m,\dots,e_n]})$

 $\delta(\phi \smile \tau) = \delta\phi \smile \tau + (-1)^m \phi \smile \delta\tau$

 $\delta\phi = 0, \, \delta\tau = 0 \text{ implies } \delta(\phi \smile \tau) = 0$ $\delta\phi \smile \tau = \delta(\phi \smile \tau) - (-1)^m \phi \smile \delta\tau$ $\delta\tau = 0 \text{ implies } \delta\phi \smile \tau = \delta(\phi \smile \tau)$ $\delta\phi \smile \tau = \delta(\phi \smile \tau) - (-1)^m \phi \smile \delta\tau$

Defines, $\smile: H^m(X;R) \times H^n(X;R) \to H^{m+n}(X;R)$ Generally, \smile : $H^m(X,A;R) \times H^n(X,B;R) \rightarrow H^{m+n}(X,A\cup B;R)$

Exercise. $f: X \to Y$, then $f^*(\alpha \smile \beta) = f^*(\alpha) \smile f^*(\beta)$

 $\begin{array}{l} \times: H^m(X;R) \times H^n(Y;R) \to H^{m+n}(X \times Y;R) \\ x \times y = P_x^*(x) \smile p_y^*(y). \end{array}$

 $\times: H^m(X,A;R) \times H^n(Y,B;R) \to H^{m+n}(X \times Y,A \times Y \cup X \times B;R)$

Exercise. \smile is distributive over

+ and associative.

 $H^*(X;R) := \bigoplus H^i(X;R)$ is a ring under \smile



 $\{\sigma_{\alpha}:\Delta_n\to X\}$

$$\{\sigma_{\alpha}: \Delta_n \to X\}$$

1. $\sigma_{\alpha}|_{\Delta \setminus \partial \Delta}$ injective.

$$\{\sigma_{\alpha}:\Delta_n\to X\}$$

1. $\sigma_{\alpha}|_{\Delta\setminus\partial\Delta}$ injective. If $\sigma_{\alpha}|_{\Delta\setminus\partial\Delta}(x_1) = y$ then there cannot be an x_2 such that $\sigma_{\beta}(x_2) = y$ for any β .

$$\{\sigma_{\alpha}: \Delta_n \to X\}$$

1. $\sigma_{\alpha}|_{\Delta\setminus\partial\Delta}$ injective. If $\sigma_{\alpha}|_{\Delta\setminus\partial\Delta}(x_1) = y$ then there cannot be an x_2 such that $\sigma_{\beta}(x_2) = y$ for any β .

The gluing happens only at the boundaries

$$\{\sigma_{\alpha}: \Delta_n \to X\}$$

- 1. $\sigma_{\alpha}|_{\Delta\setminus\partial\Delta}$ injective. If $\sigma_{\alpha}|_{\Delta\setminus\partial\Delta}(x_1) = y$ then there cannot be an x_2 such that $\sigma_{\beta}(x_2) = y$ for any β .

 The gluing happens only at the boundaries
- 2. σ_{α} restricted to a face of Δ_n is $\sigma_{\beta}: \Delta_{n-1} \to X$ for some X.

$$\{\sigma_{\alpha}: \Delta_n \to X\}$$

- 1. $\sigma_{\alpha}|_{\Delta\setminus\partial\Delta}$ injective. If $\sigma_{\alpha}|_{\Delta\setminus\partial\Delta}(x_1) = y$ then there cannot be an x_2 such that $\sigma_{\beta}(x_2) = y$ for any β . The gluing happens only at the boundaries
- 2. σ_{α} restricted to a face of Δ_n is $\sigma_{\beta}: \Delta_{n-1} \to X$ for some X. Linearity of the structure

$$\{\sigma_{\alpha}: \Delta_n \to X\}$$

- 1. $\sigma_{\alpha}|_{\Delta\setminus\partial\Delta}$ injective. If $\sigma_{\alpha}|_{\Delta\setminus\partial\Delta}(x_1) = y$ then there cannot be an x_2 such that $\sigma_{\beta}(x_2) = y$ for any β . The gluing happens only at the boundaries
- 2. σ_{α} restricted to a face of Δ_n is $\sigma_{\beta} : \Delta_{n-1} \to X$ for some X. Linearity of the structure
- 3. $A \subset X$ open in X equivalent to $\sigma_{\alpha}^{-1}(A)$ open for all α .

$$\{\sigma_{\alpha}: \Delta_n \to X\}$$

- 1. $\sigma_{\alpha}|_{\Delta\setminus\partial\Delta}$ injective. If $\sigma_{\alpha}|_{\Delta\setminus\partial\Delta}(x_1) = y$ then there cannot be an x_2 such that $\sigma_{\beta}(x_2) = y$ for any β .

 The gluing happens only at the boundaries
- 2. σ_{α} restricted to a face of Δ_n is $\sigma_{\beta} : \Delta_{n-1} \to X$ for some X. Linearity of the structure
- 3. $A \subset X$ open in X equivalent to $\sigma_{\alpha}^{-1}(A)$ open for all α . X is glued out of copies of Δ_n