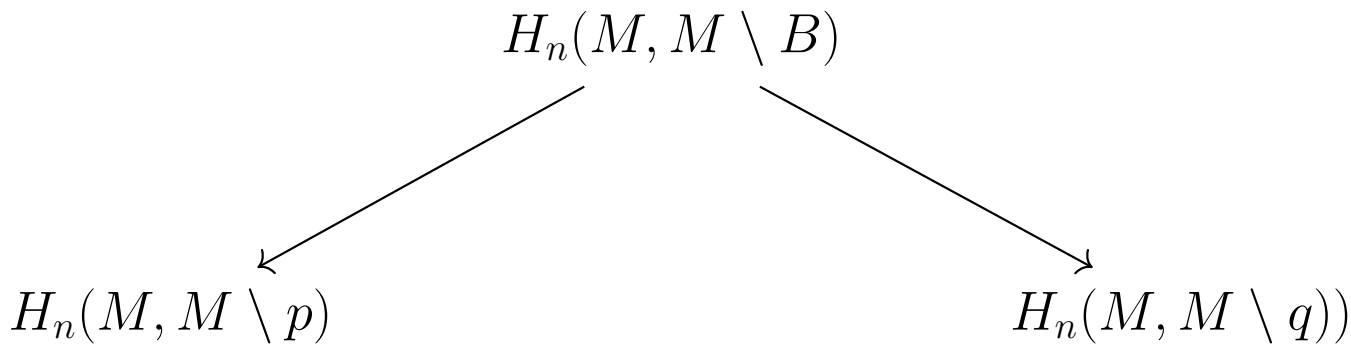
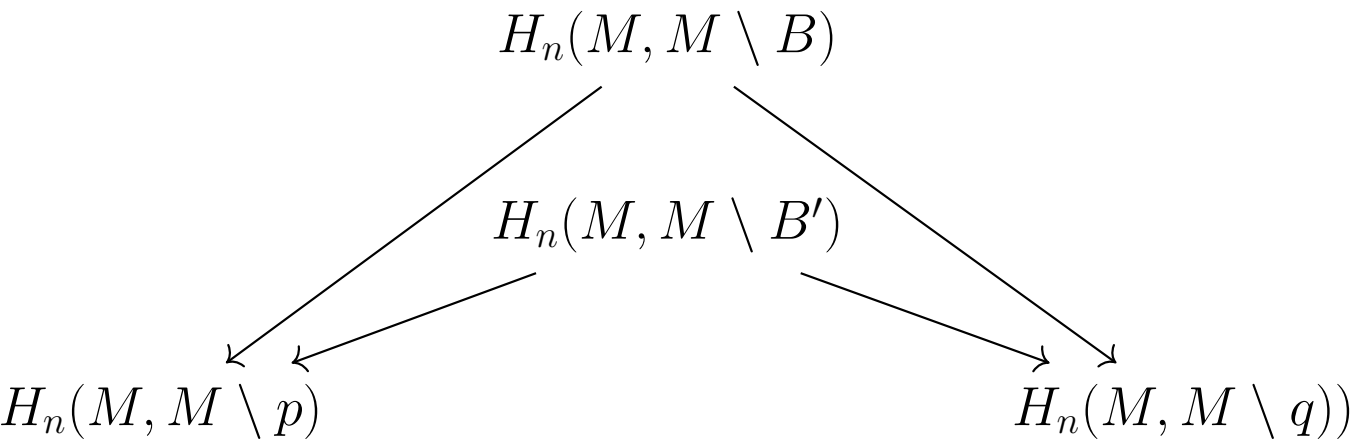


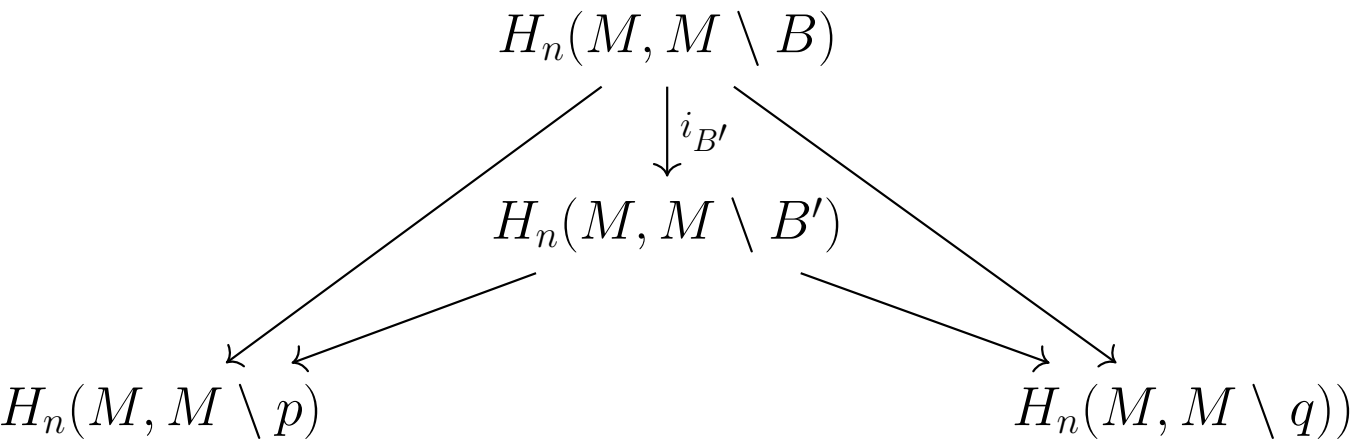
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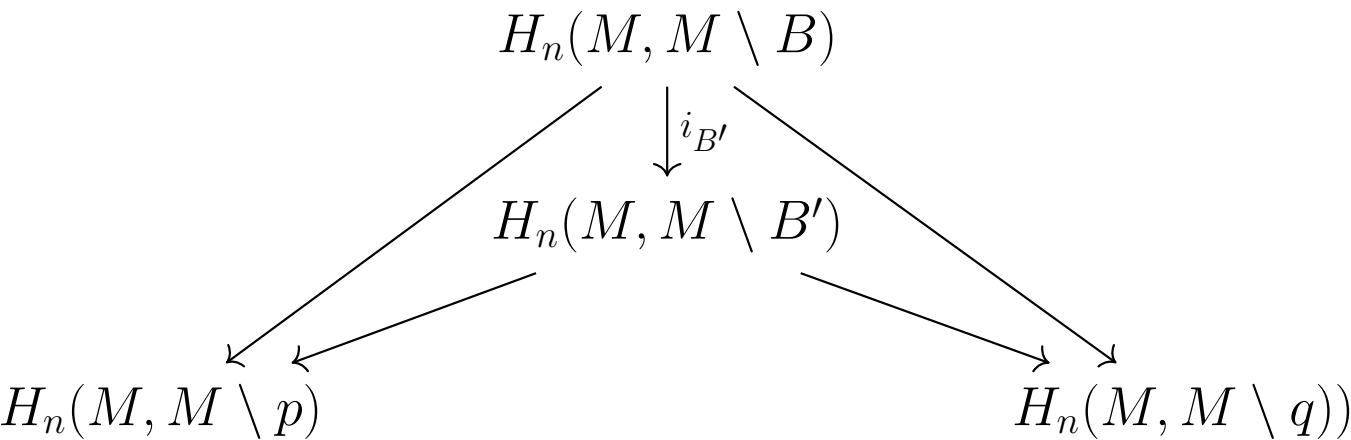
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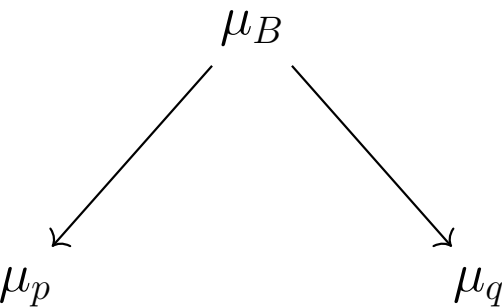
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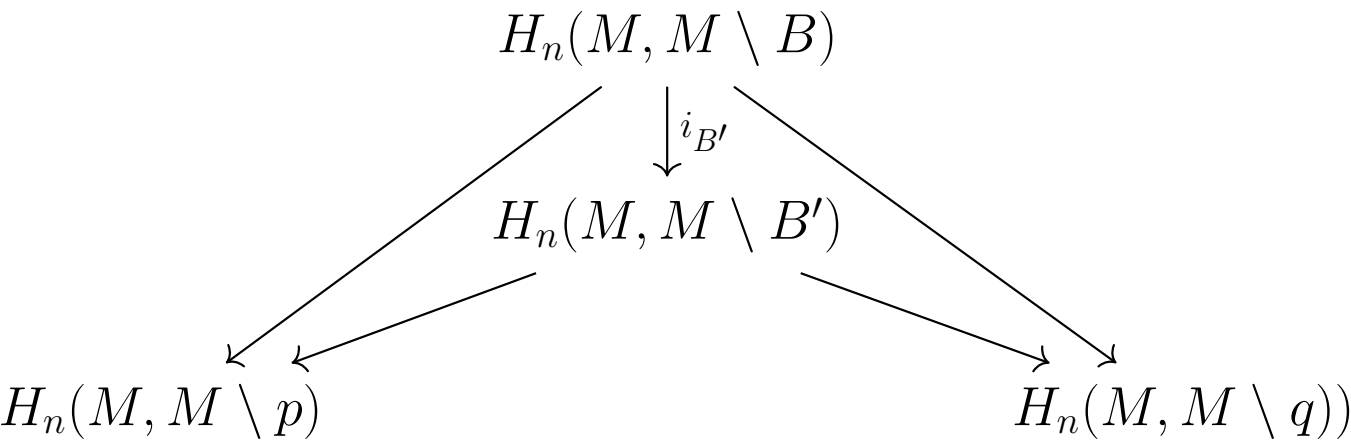
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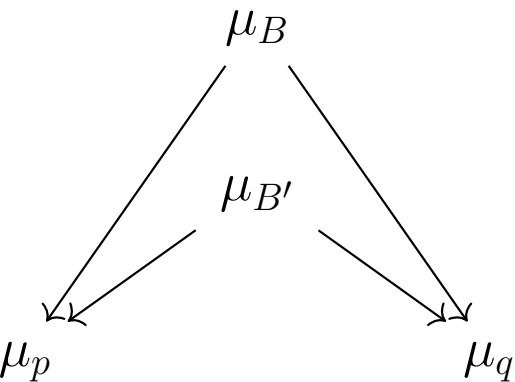
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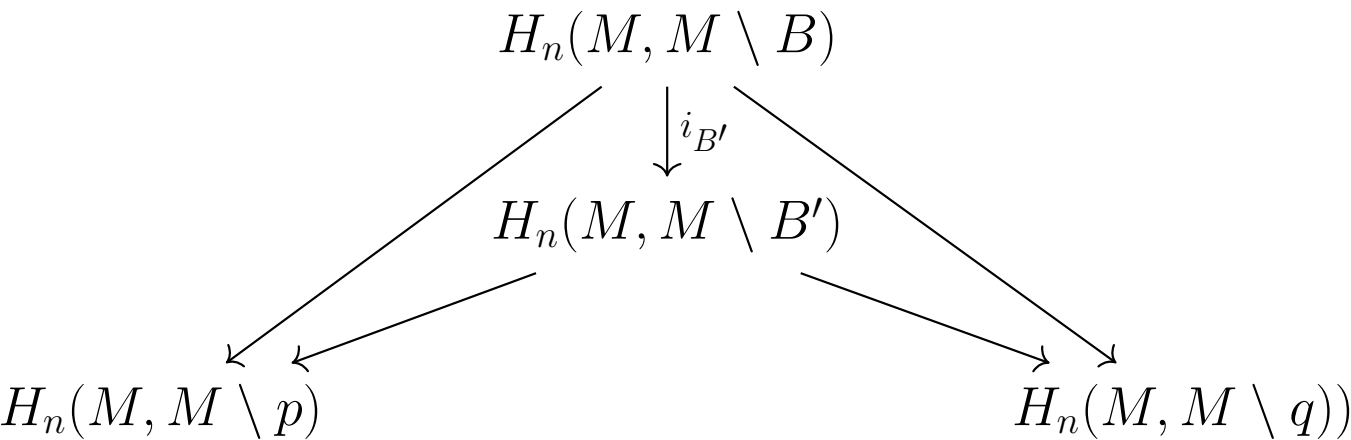
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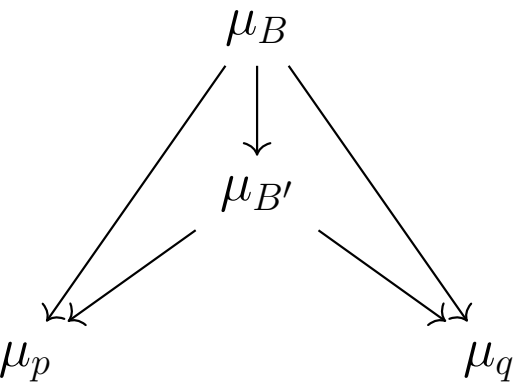
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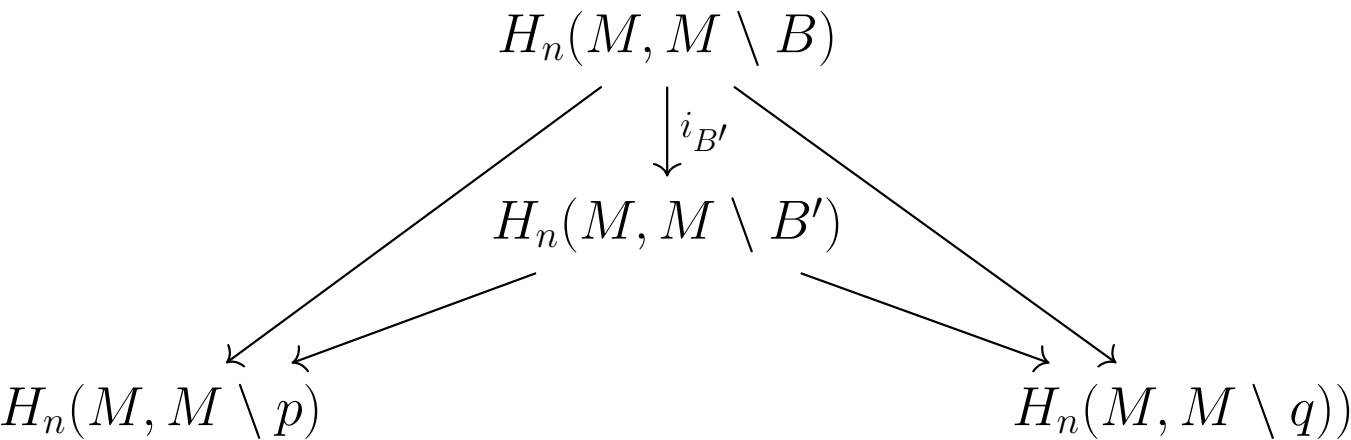


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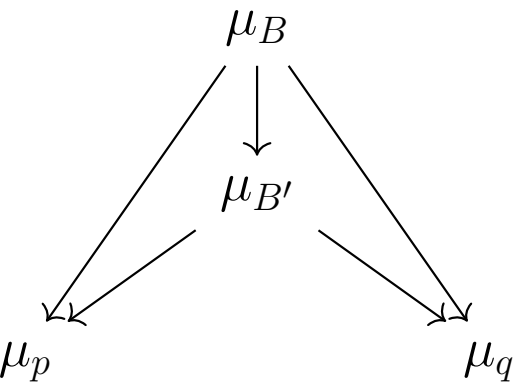


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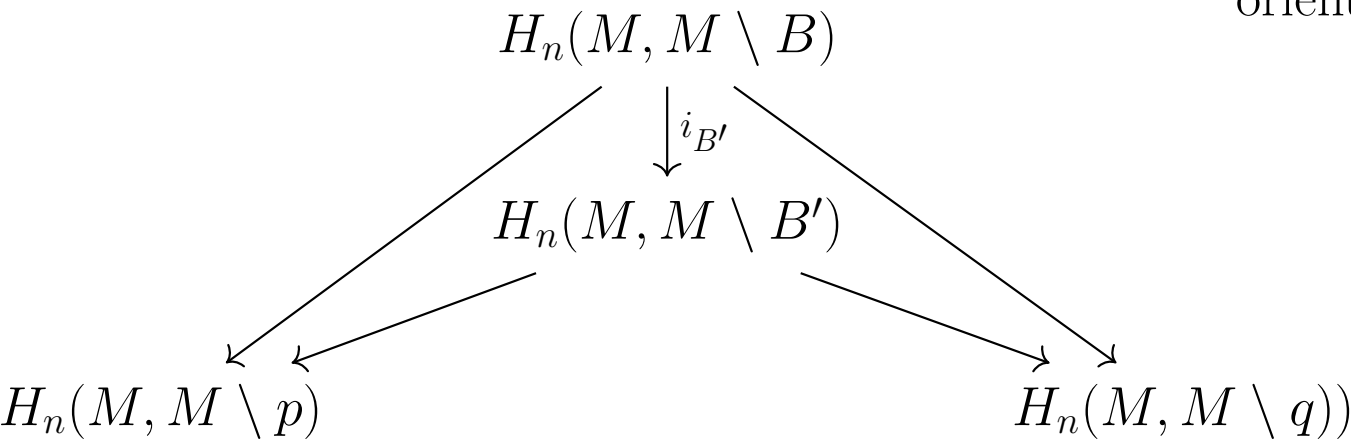
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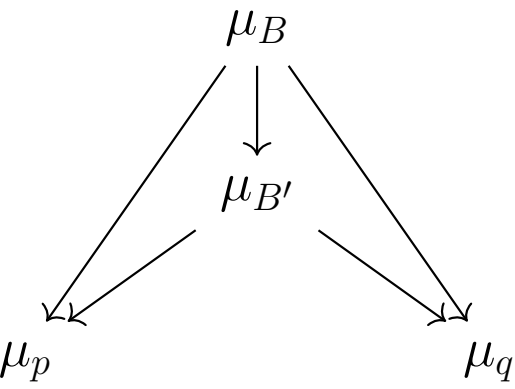
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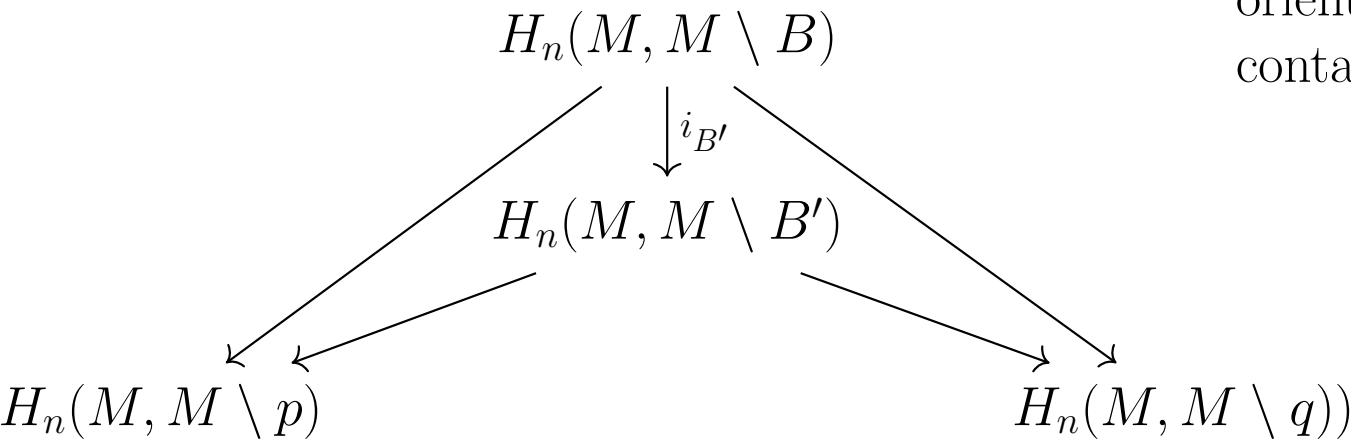


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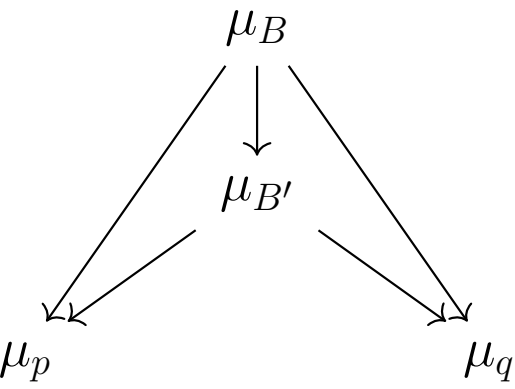


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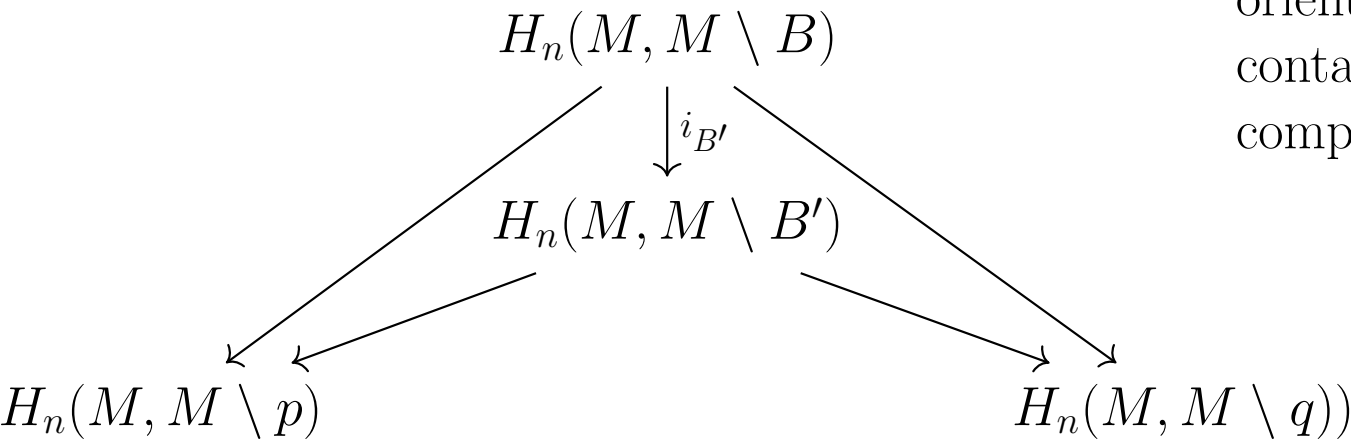


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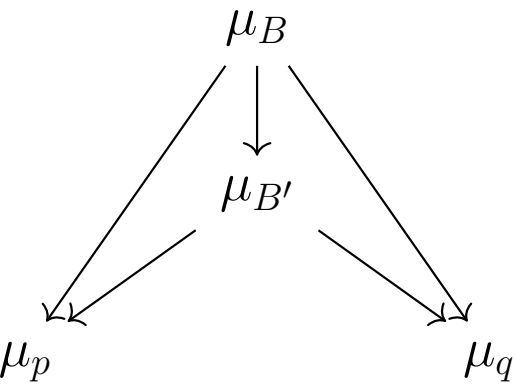
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$\Gamma_{\mathbb{Z}}(M) := \{s : M \rightarrow M_{\mathbb{Z}} \mid \pi \circ s = Id, s \text{ continuous}\}$   
(called, space of sections)

$\Gamma_{\mathbb{Z}}(M)$  is an abelian group (0 section assigns the 0 homology class to each point)

**Example.**  $\alpha \rightarrow \alpha_x$

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**Example.**  $s \in \Gamma_{\mathbb{Z}}(\mathbb{R}^n)$   
 $s = ms_0$ , *where  $s_0$  is the orientation section and  $m \in \mathbb{Z}$ . For any ball  $B$*

Question:  
 $A$  compact.

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