Mayer-Vietoris sequence

$$\cdots \to H_{n+1}(X) \xrightarrow{\partial} H_n(A \cap B) \xrightarrow{i_*} H_n(A) \oplus H_n(B) \xrightarrow{j_*} H_n^{\{A,B\}}(X) \xrightarrow{\partial} H_{n-1}(A \cap B) \to \cdots$$

is exact

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If $X = Int\ A \cup Int\ B$,

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Recall Van-Kampen!

Jordan curve theorem

Theorem. *$f : S^1 \to S^2$ continuous and homeomorphism onto its image, then $S^2 \setminus f(S^1)$ has two components*

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$$\tilde{H}_0(S^2 \setminus f(S^1))$$

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$$\cdots \to \tilde{H}_1(U) \oplus \tilde{H}_1(V) \to \tilde{H}_1(U \cup V) \to \tilde{H}_0(U \cap V) \to \tilde{H}_0(U) \oplus \tilde{H}_0(V) \to \cdots$$

Claim: $H_i(S^2 \setminus f(I)) = 0$, $\square$

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Claim: $H_i(S^2 \setminus f(I)) = 0$, i.e. $\tilde{H}_i(U) = \tilde{H}_i(V) = 0$

$\tilde{H}_1(U \cup V) = \tilde{H}_0(U \cap V)$

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Proof of claim (i.e. $H_i(S^2 \setminus f([0,1])) = 0$):
$I = I_1 \cup I_2$

Proof of claim (i.e. $H_i(S^2 \setminus f([0,1])) = 0$*):*

$I = I_1 \cup I_2$,

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$[x] \to ([i_U(x)], [-i_V(x)])$

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$[x] \to ([i_U(x)], [-i_V(x)])$

$[x] = [0]$ if and only if $[i_U(x)] = 0$ AND $[i_V(x)] = 0$

Proof of claim (i.e. $H_i(S^2 \setminus f([0,1])) = 0$):
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Lemma (Five lemma).

$$\begin{array}{ccccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E \\ \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & D' & \longrightarrow & E' \end{array}$$

rows exact

Lemma (Five lemma).

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"] & B \arrow[r] \arrow[d, "g"] & C \arrow[r] \arrow[d, "h"] & D \arrow[r] \arrow[d, "i"] & E \arrow[d, "j"] \\ A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & D' \arrow[r] & E' \end{tikzcd}$$

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$$\begin{tikzcd}
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\end{tikzcd}$$
rows exact and f, g, i, j isomorphisms, diagram commutes $\implies$ h isomorphism

Proof. Exercise! $\square$

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$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_i & \longrightarrow & B_i & \longrightarrow & C_i & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A'_i & \longrightarrow & B'_i & \longrightarrow & C'_i & \longrightarrow & 0 \end{array}$$

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$$\begin{tikzcd} \cdots \arrow[r] & H_i(A) \arrow[r] \arrow[d] & H_i(B) \arrow[r] \arrow[d] & H_i(C) \arrow[r] \arrow[d] & H_{i-1}(A) \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & H_i(A') \arrow[r] & H_i(B') \arrow[r] & H_i(C') \arrow[r] & H_{i-1}(A) \arrow[r] & \cdots \end{tikzcd}$$

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and

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Proof. Exercise! □