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Universal coefficient theorem of cohomology

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Theorem. $0 \to Ext(H_{n-1}(C), G) \to H^n(C; G) \to Hom(H_n(C), G) \to 0$
is natural and splits

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$Hom(C_n, G) \xrightarrow{i^*} Hom(Z_n, G) \to 0??$

$$0 \longrightarrow Z_{n+1} \longrightarrow C_{n+1} \longrightarrow B_n \longrightarrow 0$$

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$$
\begin{tikzcd}
0 \arrow[r] & Z_{n+1} \arrow[r] & C_{n+1} \arrow[r] \arrow[d, "\partial"] & B_n \arrow[r] & 0 \\
0 \arrow[r] & Z_n \arrow[r] & C_n \arrow[r] & B_{n-1} \arrow[r] & 0
\end{tikzcd}
$$

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0 \arrow[r] & Z_{n+1} \arrow[r] \arrow[d, "\partial"] & C_{n+1} \arrow[r] \arrow[d, "\partial"] & B_n \arrow[r] & 0 \\
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0 & \longrightarrow & Hom(B_{n-1},G) & \longrightarrow & Hom(C_n,G) & \longrightarrow & Hom(Z_n,G) & \longrightarrow & 0 \\
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$$\cdots \to Hom(Z_{n-1},G) \xrightarrow{i^*} Hom(B_{n-1},G) \to H^n(C;G) \to Hom(Z_n,G) \xrightarrow{i^*} Hom(B_n,G) \to \cdots$$

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\end{array}$$

$$\cdots \to Hom(Z_{n-1},G) \xrightarrow{i^*} Hom(B_{n-1},G) \to H^n(C;G) \to Hom(Z_n,G) \xrightarrow{i^*} Hom(B_n,G) \to \cdots$$

$$0 \to Hom(B_{n-1},G)/i^*(Hom(Z_{n-1},G)) \to H^n(C;G) \to Hom(Z_n/B_n,G) \to 0$$

$$\begin{tikzcd}
0 \arrow[r] & Hom(B_{n-1},G) \arrow[r] \arrow[d, "0"] & Hom(C_n,G) \arrow[r] \arrow[d, "\delta"] & Hom(Z_n,G) \arrow[r] \arrow[d, "0"] & 0 \\
0 \arrow[r] & Hom(B_n,G) \arrow[r] & Hom(C_{n+1},G) \arrow[r] & Hom(Z_{n+1},G) \arrow[r] & 0
\end{tikzcd}$$

$$\cdots \to Hom(Z_{n-1},G) \xrightarrow{i^*} Hom(B_{n-1},G) \to H^n(C;G) \to Hom(Z_n,G) \xrightarrow{i^*} Hom(B_n,G) \to \cdots$$

$$0 \to Hom(B_{n-1},G)/i^*(Hom(Z_{n-1},G)) \to H^n(C;G) \to Hom(Z_n/B_n,G) \to 0$$

$$0 \to B_n \xrightarrow{i} Z_n \to H_n \to 0$$

$$\begin{tikzcd}
0 \arrow[r] & Hom(B_{n-1},G) \arrow[r] \arrow[d, "0"] & Hom(C_n,G) \arrow[r] \arrow[d, "\delta"] & Hom(Z_n,G) \arrow[r] \arrow[d, "0"] & 0 \\
0 \arrow[r] & Hom(B_n,G) \arrow[r] & Hom(C_{n+1},G) \arrow[r] & Hom(Z_{n+1},G) \arrow[r] & 0
\end{tikzcd}$$

$$\cdots \to Hom(Z_{n-1},G) \xrightarrow{i^*} Hom(B_{n-1},G) \to H^n(C;G) \to Hom(Z_n,G) \xrightarrow{i^*} Hom(B_n,G) \to \cdots$$

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$$0 \to B_n \xrightarrow{i} Z_n \to H_n \to 0$$

$$0 \to Hom(H_n,G) \to Hom(Z_n,G) \xrightarrow{i^*} Hom(B_n,G) \to 0$$

$$\begin{tikzcd}
0 \arrow[r] & Hom(B_{n-1},G) \arrow[r] \arrow[d, "0"] & Hom(C_n,G) \arrow[r] \arrow[d, "\delta"] & Hom(Z_n,G) \arrow[r] \arrow[d, "0"] & 0 \\
0 \arrow[r] & Hom(B_n,G) \arrow[r] & Hom(C_{n+1},G) \arrow[r] & Hom(Z_{n+1},G) \arrow[r] & 0
\end{tikzcd}$$

$$\cdots \to Hom(Z_{n-1},G) \xrightarrow{i^*} Hom(B_{n-1},G) \to H^n(C;G) \to Hom(Z_n,G) \xrightarrow{i^*} Hom(B_n,G) \to \cdots$$

$$0 \to Hom(B_{n-1},G)/i^*(Hom(Z_{n-1},G)) \to H^n(C;G) \to Hom(Z_n/B_n,G) \to 0$$

$$0 \to B_n \xrightarrow{i} Z_n \to H_n \to 0$$

$$0 \to Hom(H_n,G) \to Hom(Z_n,G) \xrightarrow{i^*} Hom(B_n,G) \to 0$$

Theorem. $0 \to Ext(H_{n-1},G) \to H^n(C;G) \to Hom(H_n,G) \to 0$

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & Hom(B_{n-1},G) & \longrightarrow & Hom(C_n,G) & \longrightarrow & Hom(Z_n,G) & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle 0} & & \Big\downarrow{\scriptstyle \delta} & & \Big\downarrow{\scriptstyle 0} & & \\
0 & \longrightarrow & Hom(B_n,G) & \longrightarrow & Hom(C_{n+1},G) & \longrightarrow & Hom(Z_{n+1},G) & \longrightarrow & 0
\end{array}
$$

$$\cdots \to Hom(Z_{n-1},G) \xrightarrow{i^*} Hom(B_{n-1},G) \to H^n(C;G) \to Hom(Z_n,G) \xrightarrow{i^*} Hom(B_n,G) \to \cdots$$

$$0 \to Hom(B_{n-1},G)/i^*(Hom(Z_{n-1},G)) \to H^n(C;G) \to Hom(Z_n/B_n,G) \to 0$$

$$0 \to B_n \xrightarrow{i} Z_n \to H_n \to 0$$

$$0 \to Hom(H_n,G) \to Hom(Z_n,G) \xrightarrow{i^*} Hom(B_n,G) \to 0$$

Theorem. $0 \to Ext(H_{n-1},G) \to H^n(C;G) \to Hom(H_n,G) \to 0$
is a natural short exact sequence that splits.

$f : X \to Y$

$f : X \to Y$
$f_{\#} : C_n(X) \to C_n(Y)$

$$f : X \to Y$$
$$f_\# : C_n(X) \to C_n(Y)$$
$$f^\# : Hom(C_n(Y), G) \to Hom(C_n(X), G)$$

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$$f_{\#} : C_n(X) \to C_n(Y)$$
$$f^{\#} : Hom(C_n(Y), G) \to Hom(C_n(X), G)$$
$$f^{\#} : C^n(Y; G) \to C^n(X; G)$$

$f : X \to Y$
$f_\# : C_n(X) \to C_n(Y)$
$f^\# : Hom(C_n(Y), G) \to Hom(C_n(X), G)$
$f^\# : C^n(Y; G) \to C^n(X; G)$

$f_\# \circ \partial = \partial \circ f_\#$

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$f_\# : C_n(X) \to C_n(Y)$
$f^\# : Hom(C_n(Y), G) \to Hom(C_n(X), G)$
$f^\# : C^n(Y; G) \to C^n(X; G)$

$f_\# \circ \partial = \partial \circ f_\#$
$\partial^* \circ f^* = f^* \circ \partial^*$

$$f : X \to Y$$
$$f_{\#} : C_n(X) \to C_n(Y)$$
$$f^{\#} : Hom(C_n(Y), G) \to Hom(C_n(X), G)$$
$$f^{\#} : C^n(Y; G) \to C^n(X; G)$$

$$f_{\#} \circ \partial = \partial \circ f_{\#}$$
$$\partial^* \circ f^* = f^* \circ \partial^*$$
$$\delta \circ f^* = f^* \circ \delta$$

$$f : X \to Y$$
$$f_\# : C_n(X) \to C_n(Y)$$
$$f^\# : Hom(C_n(Y), G) \to Hom(C_n(X), G)$$
$$f^\# : C^n(Y; G) \to C^n(X; G)$$

$$f_\# \circ \partial = \partial \circ f_\#$$
$$\partial^* \circ f^* = f^* \circ \partial^*$$
$$\delta \circ f^* = f^* \circ \delta$$
$$f^* : H^n(Y; G) \to H^n(X; G)$$

$$f : X \to Y$$
$$f_{\#} : C_n(X) \to C_n(Y)$$
$$f^{\#} : Hom(C_n(Y), G) \to Hom(C_n(X), G)$$
$$f^{\#} : C^n(Y; G) \to C^n(X; G)$$

$$f_{\#} \circ \partial = \partial \circ f_{\#}$$
$$\partial^* \circ f^* = f^* \circ \partial^*$$
$$\delta \circ f^* = f^* \circ \delta$$
$$f^* : H^n(Y; G) \to H^n(X; G)$$

$$0 \to C_n(A) \to C_n(X) \to C_n(X, A) \to 0$$

.

$$0 \to C_n(A) \to C_n(X) \to C_n(X, A) \to 0$$

.

$$0 \to Hom(C_n(X, A), G) \to Hom(C_n(X), G) \to Hom(C_n(A), G) \to 0$$

.

$$0 \to C_n(A) \to C_n(X) \to C_n(X, A) \to 0$$

.

$$0 \to Hom(C_n(X, A), G) \to Hom(C_n(X), G) \to Hom(C_n(A), G) \to 0$$

.

$$\cdots \to H^n(X, A; G) \to H^n(X; G) \to H^n(A; G) \to H^{n+1}(X, A; G) \to \cdots$$

$$0 \to C_n(A) \to C_n(X) \to C_n(X, A) \to 0$$

.

$$0 \to Hom(C_n(X, A), G) \to Hom(C_n(X), G) \to Hom(C_n(A), G) \to 0$$

.

$$\cdots \to H^n(X, A; G) \to H^n(X; G) \to H^n(A; G) \to H^{n+1}(X, A; G) \to \cdots$$

Chain homotopy:

Chain homotopy:

$$f_\# - g_\# = \partial \circ P + P \circ \partial$$

Chain homotopy:

$$f_{\#} - g_{\#} = \partial \circ P + P \circ \partial$$
$$f^{\#} - g^{\#} = \partial^{*} \circ P^{*} + P^{*} \circ \partial^{*}$$

Chain homotopy:

$$f_{\#} - g_{\#} = \partial \circ P + P \circ \partial$$
$$f^{\#} - g^{\#} = \partial^{*} \circ P^{*} + P^{*} \circ \partial^{*}$$
$$f^{\#} - g^{\#} = P^{*} \circ \delta + \delta \circ P^{*}$$

Chain homotopy:

$$f_\# - g_\# = \partial \circ P + P \circ \partial$$
$$f^\# - g^\# = \partial^* \circ P^* + P^* \circ \partial^*$$
$$f^\# - g^\# = P^* \circ \delta + \delta \circ P^*$$
$$f^\# - g^\# = \delta \circ P^* + P^* \circ \delta$$

Chain homotopy:

$$f_\# - g_\# = \partial \circ P + P \circ \partial$$
$$f^\# - g^\# = \partial^* \circ P^* + P^* \circ \partial^*$$
$$f^\# - g^\# = P^* \circ \delta + \delta \circ P^*$$
$$f^\# - g^\# = \delta \circ P^* + P^* \circ \delta$$

Theorem. *f and g homotopic imply $f^* = g^*$*

Corollary. *X and Y homotopically equivalent implies $H^n(X) \cong H^n(Y)$ for all n.*

Excision:

Excision:

$A \subset X$

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$A \subset X$
(X, A)

Excision:

$A \subset X$
(X, A)

$H^n(X, A)$

Excision:

$B \subset A \subset X$
(X, A)

$H^n(X, A)$

Excision:

$B \subset A \subset X$
$(X \setminus B, A \setminus B) \xrightarrow{i} (X, A)$

$H^n(X, A)$

Excision:

$B \subset A \subset X$
$(X \setminus B, A \setminus B) \xrightarrow{i} (X, A)$

$H^n(X, A) \xrightarrow{i^*} H^n(X \setminus B, A \setminus B)$

Mayer-Vietoris:

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$X = Int\ A \cup Int\ B$

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$X = Int\ A \cup Int\ B$

$\cdots \to H^n(X) \to H^n(A) \oplus H^n(B) \to H^n(A \cap B) \to H^{n+1}(X) \to \cdots$

Cellular cohomology:

Cellular cohomology:
$H^i(X^n, X^{n-1})$

Cellular cohomology:

$H^i(X^n, X^{n-1})$ is $\oplus\mathbb{Z}$ if i = n and 0 otherwise.

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$H^n(X) \cong H^n(X^{n+1})$

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$H^i(X^n, X^{n-1})$ is $\oplus\mathbb{Z}$ if i = n and 0 otherwise.

$$H^n(X) \cong H^n(X^{n+1})$$

$$\cdots \to H^{n-1}(X^{n-1}, X^{n-2}) \to H^n(X^n, X^{n-1}) \to H^{n+1}(X^{n+1}, X^n) \to \cdots$$

Cellular cohomology:
$H^i(X^n, X^{n-1})$ is $\oplus\mathbb{Z}$ if i = n and 0 otherwise.

$H^n(X) \cong H^n(X^{n+1})$

$\cdots \to H^{n-1}(X^{n-1}, X^{n-2}) \to H^n(X^n, X^{n-1}) \to H^{n+1}(X^{n+1}, X^n) \to \cdots$
is the cellular chain complex whose homology is $H^n(X)$.

Cellular cohomology:
$H^i(X^n, X^{n-1})$ is $\oplus\mathbb{Z}$ if i = n and 0 otherwise.

$H^n(X) \cong H^n(X^{n+1})$

$\cdots \to H^{n-1}(X^{n-1}, X^{n-2}) \to H^n(X^n, X^{n-1}) \to H^{n+1}(X^{n+1}, X^n) \to \cdots$
is the cellular chain complex whose homology is $H^n(X)$.
Is dual to the cellular homology chain complex.

$H^n(X^{n+1})$

$$
\begin{tikzcd}
 & & H^n(X^n) \\
 & H^n(X^{n+1}) \arrow[ur] & \\
0 \arrow[ur] & &
\end{tikzcd}
$$

$$
\begin{tikzcd}[column sep=small]
 & & & H^{n+1}(X^{n+1}, X^n) \\
 & & H^n(X^n) \arrow[ur, "\delta_n"] & \\
 & H^n(X^{n+1}) \arrow[ur] & & \\
0 \arrow[ur] & & &
\end{tikzcd}
$$

$$\begin{tikzcd}
 & & & H^{n+1}(X^{n+1}, X^n) \\
 & & H^n(X^n) \arrow[ur, "\delta_n"] \arrow[dr] & \\
 & H^n(X^{n+1}) \arrow[ur] & & 0 \\
0 \arrow[ur] & & &
\end{tikzcd}$$

$$
\begin{tikzcd}
 & H^n(X^n, X^{n-1}) \arrow[dr] & & H^{n+1}(X^{n+1}, X^n) \\
 & & H^n(X^n) \arrow[ur, "\delta_n"] \arrow[dr] & \\
 & H^n(X^{n+1}) \arrow[ur] & & 0 \\
0 \arrow[ur] & & &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & H^n(X^n, X^{n-1}) \arrow[rr] \arrow[dr] & & H^{n+1}(X^{n+1}, X^n) \\
 & & H^n(X^n) \arrow[ur, "\delta_n"] \arrow[dr] & \\
 & H^n(X^{n+1}) \arrow[ur] & & 0 \\
0 \arrow[ur] & & &
\end{tikzcd}
$$

$$
\begin{tikzcd}
& H^{n-1}(X^{n-1}) \arrow[dr] & & & \\
H^{n-1}(X^{n-1}, X^{n-2}) \arrow[ur] & & H^n(X^n, X^{n-1}) \arrow[rr] \arrow[dr] & & H^{n+1}(X^{n+1}, X^n) \\
& & & H^n(X^n) \arrow[ur, "\delta_n"] \arrow[dr] & \\
& & H^n(X^{n+1}) \arrow[ur] & & 0 \\
& 0 \arrow[ur] & & &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & H^{n-1}(X^{n-1}) \arrow[dr] & & & \\
H^{n-1}(X^{n-1}, X^{n-2}) \arrow[ur] \arrow[rr] & & H^n(X^n, X^{n-1}) \arrow[rr] \arrow[dr] & & H^{n+1}(X^{n+1}, X^n) \\
 & & & H^n(X^n) \arrow[ur, "\delta_n"] \arrow[dr] & \\
 & & H^n(X^{n+1}) \arrow[ur] & & 0 \\
 & 0 \arrow[ur] & & &
\end{tikzcd}
$$