

## Exercise sheet 5

1. Define  $\mathbb{RP}^n = (\mathbb{R}^{n+1} \setminus 0) / \sim$  where  $(x_0, x_1, \dots, x_n) \sim (y_0, y_1, \dots, y_n)$  if and only if there is an  $\alpha \in \mathbb{R}$  such that  $(x_0, x_1, \dots, x_n) = \alpha(y_0, y_1, \dots, y_n)$ . Show that it is a CW-complex for each  $n$  and compute its homology.
2. Define  $\mathbb{CP}^n = (\mathbb{C}^{n+1} \setminus 0) / \sim$  where  $(x_0, x_1, \dots, x_n) \sim (y_0, y_1, \dots, y_n)$  if and only if there is an  $\alpha \in \mathbb{C}$  such that  $(x_0, x_1, \dots, x_n) = \alpha(y_0, y_1, \dots, y_n)$ . Show that it is a CW-complex for each  $n$  and compute its homology.
3. Exercises 7 and 8 from section 2.2 of Hatcher's Algebraic Topology (page 155).
4. Prove that  $H_n(X_n)$  is always free, where  $X_n$  denotes the  $n$  skeleton of a CW-complex  $X$ .
5. Compute  $H_k(S^n; G)$  for any abelian group,  $G$ .
6. Prove that if  $f : S^n \rightarrow S^n$  has degree  $d$ , then the induced map  $f : H_k(S^n; G) \rightarrow H_k(S^n; G)$  is multiplication by  $d$ .
7. Let  $X$  is a topological space such that  $H_k(X)$  is always finitely generated, then prove that if  $F$  is a field, then the  $\chi(X) = \sum_i (-1)^i \dim H_i(X; F)$ .
8. Prove the following properties of  $Tor_1$ :
  - (a)  $Tor_1(A, B) = Tor_1(B, A)$
  - (b)  $Tor(A, G) = 0$  if  $G$  is free
  - (c) If  $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$  is a short exact sequence then, the following sequence is exact:
 
$$0 \rightarrow Tor_1(A, G) \rightarrow Tor_1(B, G) \rightarrow Tor_1(C, G) \rightarrow A \otimes G \rightarrow B \otimes G \rightarrow C \otimes G \rightarrow 0$$