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$$
\begin{tikzcd}
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\cdots \arrow[r] & C'_2 \arrow[r] & C'_1 \arrow[r] & C'_0
\end{tikzcd}
$$

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$$\alpha_n - \beta_n = \partial'_{n+1} \circ D_n + D_{n-1} \circ \partial_n$$

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Assume, D_k defined for $k \xrightarrow{\partial_n} C_{n-1}(X) \to \cdots \to C_1(X) \xrightarrow{\partial_1} C_0(X)$

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & C_2 & \longrightarrow & C_1 & \longrightarrow & C_0 \\
 & & \downarrow{\scriptstyle \alpha_2,\beta_2} & & \downarrow{\scriptstyle \alpha_1,\beta_1} & & \downarrow{\scriptstyle \alpha_0,\beta_0} \\
\cdots & \longrightarrow & C_2' & \longrightarrow & C_1' & \longrightarrow & C_0'
\end{array}$$

$\alpha_n - \beta_n = \partial'_{n+1} \circ D_n + D_{n-1} \circ \partial_n$

Can we define D_n so that,
$\partial'_{n+1}(D_n(x)) = \alpha_n(x) - \beta_n(x) - D_{n-1}(\partial_n(x))$?

Assume, D_k defined for $k \xrightarrow{\partial_n \otimes Id} C_{n-1}(X) \otimes G \to \cdots \to C_1(X) \otimes G \xrightarrow{\partial_1 \otimes Id} C_0(X) \otimes G$
G abelian

$$
\begin{array}{ccccccc}
\cdots & \longrightarrow & C_2 & \longrightarrow & C_1 & \longrightarrow & C_0 \\
 & & \Big\downarrow{\scriptstyle \alpha_2,\beta_2} & & \Big\downarrow{\scriptstyle \alpha_1,\beta_1} & & \Big\downarrow{\scriptstyle \alpha_0,\beta_0} \\
\cdots & \longrightarrow & C_2' & \longrightarrow & C_1' & \longrightarrow & C_0'
\end{array}
$$

$\alpha_n - \beta_n = \partial'_{n+1} \circ D_n + D_{n-1} \circ \partial_n$

Can we define D_n so that,
$\partial'_{n+1}(D_n(x)) = \alpha_n(x) - \beta_n(x) - D_{n-1}(\partial_n(x))$?

Assume, D_k defined for k $\xrightarrow{\partial_n \otimes Id}$ $C_{n-1}(X) \otimes G$ $\to$
$\cdots \to C_1(X) \otimes G \xrightarrow{\partial_1 \otimes Id} C_0(X) \otimes G$
G abelian

$(\partial_n \otimes Id) \circ (\partial_{n-1} \otimes Id) = 0$

$$
\begin{array}{ccccccc}
\cdots & \longrightarrow & C_2 & \longrightarrow & C_1 & \longrightarrow & C_0 \\
 & & \Big\downarrow{\scriptstyle \alpha_2,\beta_2} & & \Big\downarrow{\scriptstyle \alpha_1,\beta_1} & & \Big\downarrow{\scriptstyle \alpha_0,\beta_0} \\
\cdots & \longrightarrow & C'_2 & \longrightarrow & C'_1 & \longrightarrow & C'_0
\end{array}
$$

$$\alpha_n - \beta_n = \partial'_{n+1} \circ D_n + D_{n-1} \circ \partial_n$$

Can we define D_n so that,
$\partial'_{n+1}(D_n(x)) = \alpha_n(x) - \beta_n(x) - D_{n-1}(\partial_n(x))$?

Assume, D_k defined for k $\xrightarrow{\partial_n \otimes Id}$ $C_{n-1}(X) \otimes G \to \cdots \to C_1(X) \otimes G \xrightarrow{\partial_1 \otimes Id} C_0(X) \otimes G$
G abelian

$$(\partial_n \otimes Id) \circ (\partial_{n-1} \otimes Id) = 0$$

$$Z_n(X; G) := ker\ (\partial_n \otimes Id)$$

$$\begin{array}{ccccccc}
\cdots \longrightarrow & C_2 & \longrightarrow & C_1 & \longrightarrow & C_0 \\
& \Big\downarrow{\scriptstyle \alpha_2,\beta_2} & & \Big\downarrow{\scriptstyle \alpha_1,\beta_1} & & \Big\downarrow{\scriptstyle \alpha_0,\beta_0} \\
\cdots \longrightarrow & C_2' & \longrightarrow & C_1' & \longrightarrow & C_0'
\end{array}$$

$\alpha_n - \beta_n = \partial'_{n+1} \circ D_n + D_{n-1} \circ \partial_n$

Can we define D_n so that,
$\partial'_{n+1}(D_n(x)) = \alpha_n(x) - \beta_n(x) - D_{n-1}(\partial_n(x))$?

Assume, D_k defined for k $\xrightarrow{\partial_n \otimes Id}$ $C_{n-1}(X) \otimes G$ $\rightarrow$
$\cdots \rightarrow C_1(X) \otimes G \xrightarrow{\partial_1 \otimes Id} C_0(X) \otimes G$
G abelian

$(\partial_n \otimes Id) \circ (\partial_{n-1} \otimes Id) = 0$

$Z_n(X;G) := ker\ (\partial_n \otimes Id)$

$B_n(X;G) := Im\ (\partial_{n+1} \otimes Id)$

$$\begin{tikzcd}
\cdots \arrow[r] & C_2 \arrow[r] \arrow[d, "\alpha_2{,}\beta_2"] & C_1 \arrow[r] \arrow[d, "\alpha_1{,}\beta_1"] & C_0 \arrow[d, "\alpha_0{,}\beta_0"] \\
\cdots \arrow[r] & C_2' \arrow[r] & C_1' \arrow[r] & C_0'
\end{tikzcd}$$

$\alpha_n - \beta_n = \partial'_{n+1} \circ D_n + D_{n-1} \circ \partial_n$

Can we define D_n so that,
$\partial'_{n+1}(D_n(x)) = \alpha_n(x) - \beta_n(x) - D_{n-1}(\partial_n(x))$?

Assume, D_k defined for k $\xrightarrow{\partial_n \otimes Id}$ $C_{n-1}(X) \otimes G \rightarrow$
$\cdots \rightarrow C_1(X) \otimes G \xrightarrow{\partial_1 \otimes Id} C_0(X) \otimes G$
G abelian

$(\partial_n \otimes Id) \circ (\partial_{n-1} \otimes Id) = 0$

$Z_n(X;G) := ker\ (\partial_n \otimes Id)$

$B_n(X;G) := Im\ (\partial_{n+1} \otimes Id)$

$H_n(X;G) := Z_n(X;G)/B_n(X;G)$

$$
\begin{array}{ccccccc}
\cdots & \longrightarrow & C_2 & \longrightarrow & C_1 & \longrightarrow & C_0 \\
 & & \Big\downarrow{\scriptstyle \alpha_2,\beta_2} & & \Big\downarrow{\scriptstyle \alpha_1,\beta_1} & & \Big\downarrow{\scriptstyle \alpha_0,\beta_0} \\
\cdots & \longrightarrow & C'_2 & \longrightarrow & C'_1 & \longrightarrow & C'_0
\end{array}
$$

$\alpha_n - \beta_n = \partial'_{n+1} \circ D_n + D_{n-1} \circ \partial_n$

Can we define D_n so that,
$\partial'_{n+1}(D_n(x)) = \alpha_n(x) - \beta_n(x) - D_{n-1}(\partial_n(x))$?

Assume, D_k defined for k $\xrightarrow{\partial_n \otimes Id} C_{n-1}(X) \otimes G \rightarrow \cdots \rightarrow C_1(X) \otimes G \xrightarrow{\partial_1 \otimes Id} C_0(X) \otimes G$
G abelian

$(\partial_n \otimes Id) \circ (\partial_{n-1} \otimes Id) = 0$

$Z_n(X; G) := ker\ (\partial_n \otimes Id)$

$B_n(X; G) := Im\ (\partial_{n+1} \otimes Id)$

$H_n(X; G) := Z_n(X; G)/B_n(X; G)$

$H_n(X) \otimes G \rightarrow H_n(X; G)$

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & C_2 & \longrightarrow & C_1 & \longrightarrow & C_0 \\
 & & \big\downarrow{\scriptstyle \alpha_2,\beta_2} & & \big\downarrow{\scriptstyle \alpha_1,\beta_1} & & \big\downarrow{\scriptstyle \alpha_0,\beta_0} \\
\cdots & \longrightarrow & C'_2 & \longrightarrow & C'_1 & \longrightarrow & C'_0
\end{array}$$

$\alpha_n - \beta_n = \partial'_{n+1} \circ D_n + D_{n-1} \circ \partial_n$

Can we define D_n so that,
$\partial'_{n+1}(D_n(x)) = \alpha_n(x) - \beta_n(x) - D_{n-1}(\partial_n(x))$?

Assume, D_k defined for k $\xrightarrow{\partial_n \otimes Id} C_{n-1}(X) \otimes G \to$
$\cdots \to C_1(X) \otimes G \xrightarrow{\partial_1 \otimes Id} C_0(X) \otimes G$
G abelian

$(\partial_n \otimes Id) \circ (\partial_{n-1} \otimes Id) = 0$

$Z_n(X;G) := ker\ (\partial_n \otimes Id)$

$B_n(X;G) := Im\ (\partial_{n+1} \otimes Id)$

$H_n(X;G) := Z_n(X;G)/B_n(X;G)$

$H_n(X) \otimes G \to H_n(X;G)\ [c] \otimes g \to [c \otimes g]$

$$0 \longrightarrow Z_{n+1} \longrightarrow C_{n+1} \longrightarrow B_n \longrightarrow 0$$

$$0 \longrightarrow Z_{n+1} \longrightarrow C_{n+1} \longrightarrow B_n \longrightarrow 0$$

$$0 \longrightarrow Z_n \longrightarrow C_n \longrightarrow B_{n-1} \longrightarrow 0$$

$$
\begin{tikzcd}
0 \arrow[r] & Z_{n+1} \arrow[r] & C_{n+1} \arrow[r] \arrow[d, "\partial"] & B_n \arrow[r] & 0 \\
0 \arrow[r] & Z_n \arrow[r] & C_n \arrow[r] & B_{n-1} \arrow[r] & 0
\end{tikzcd}
$$

$$
\begin{tikzcd}
0 \arrow[r] & Z_{n+1} \arrow[r] \arrow[d, "\partial"] & C_{n+1} \arrow[r] \arrow[d, "\partial"] & B_n \arrow[r] & 0 \\
0 \arrow[r] & Z_n \arrow[r] & C_n \arrow[r] & B_{n-1} \arrow[r] & 0
\end{tikzcd}
$$

$$\begin{tikzcd}
0 \arrow[r] & Z_{n+1} \arrow[r] \arrow[d, "\partial=0"] & C_{n+1} \arrow[r] \arrow[d, "\partial"] & B_n \arrow[r] & 0 \\
0 \arrow[r] & Z_n \arrow[r] & C_n \arrow[r] & B_{n-1} \arrow[r] & 0
\end{tikzcd}$$

$$
\begin{tikzcd}
0 \arrow[r] & Z_{n+1} \arrow[r] \arrow[d, "\partial=0"] & C_{n+1} \arrow[r] \arrow[d, "\partial"] & B_n \arrow[r] \arrow[d, "\partial"] & 0 \\
0 \arrow[r] & Z_n \arrow[r] & C_n \arrow[r] & B_{n-1} \arrow[r] & 0
\end{tikzcd}
$$

$$\begin{tikzcd}
0 \arrow[r] & Z_{n+1} \arrow[r] \arrow[d, "\partial=0"] & C_{n+1} \arrow[r] \arrow[d, "\partial"] & B_n \arrow[r] \arrow[d, "\partial=0"] & 0 \\
0 \arrow[r] & Z_n \arrow[r] & C_n \arrow[r] & B_{n-1} \arrow[r] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[r] & Z_{n+1} \arrow[r] \arrow[d, "\partial=0"] & C_{n+1} \arrow[r] \arrow[d, "\partial"] & B_n \arrow[r] \arrow[d, "\partial=0"] & 0 \\
0 \arrow[r] & Z_n \arrow[r] & C_n \arrow[r] & B_{n-1} \arrow[r] & 0
\end{tikzcd}$$

$$\cdots \to B_n \to Z_n \xrightarrow{j} H_n(C_*) \to B_{n-1} \to \cdots$$

$$b_n = \partial c_{n+1}$$

$$\begin{tikzcd}
0 \arrow[r] & Z_{n+1} \arrow[r] \arrow[d, "\partial=0"] & C_{n+1} \arrow[r] \arrow[d, "\partial"] & B_n \arrow[r] \arrow[d, "\partial=0"] & 0 \\
0 \arrow[r] & Z_n \arrow[r] & C_n \arrow[r] & B_{n-1} \arrow[r] & 0
\end{tikzcd}$$

$$\cdots \to B_n \to Z_n \xrightarrow{j} H_n(C_*) \to B_{n-1} \to \cdots$$

$b_n = \partial c_{n+1}$

$b_n \in B_n \to \partial c_{n+1} = b_n \in Z_n$

$$\begin{tikzcd}
0 \arrow[r] & Z_{n+1} \arrow[r] \arrow[d, "\partial=0"] & C_{n+1} \arrow[r] \arrow[d, "\partial"] & B_n \arrow[r] \arrow[d, "\partial=0"] & 0 \\
0 \arrow[r] & Z_n \arrow[r] & C_n \arrow[r] & B_{n-1} \arrow[r] & 0
\end{tikzcd}$$

$$\cdots \to B_n \to Z_n \xrightarrow{j} H_n(C_*) \to B_{n-1} \to \cdots$$

$b_n = \partial c_{n+1}$

$b_n \in B_n \to \partial c_{n+1} = b_n \in Z_n$

Therefore, $b_n \in B_n \to b_n \in Z_n$

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & Z_{n+1} & \longrightarrow & C_{n+1} & \longrightarrow & B_n & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \partial=0} & & \Big\downarrow{\scriptstyle \partial} & & \Big\downarrow{\scriptstyle \partial=0} & & \\
0 & \longrightarrow & Z_n & \longrightarrow & C_n & \longrightarrow & B_{n-1} & \longrightarrow & 0
\end{array}
$$

$$\cdots \to B_n \xrightarrow{i} Z_n \xrightarrow{j} H_n(C_*) \to B_{n-1} \to \cdots$$

$b_n = \partial c_{n+1}$

$b_n \in B_n \to \partial c_{n+1} = b_n \in Z_n$

Therefore, $b_n \in B_n \to b_n \in Z_n$

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Z_{n+1} & \longrightarrow & C_{n+1} & \longrightarrow & B_n & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \partial=0} & & \Big\downarrow{\scriptstyle \partial} & & \Big\downarrow{\scriptstyle \partial=0} & & \\
0 & \longrightarrow & Z_n & \longrightarrow & C_n & \longrightarrow & B_{n-1} & \longrightarrow & 0
\end{array}$$

$$\cdots \to B_n \xrightarrow{i} Z_n \xrightarrow{j} H_n(C_*) \to B_{n-1} \to \cdots$$

$b_n = \partial c_{n+1}$
$b_n \in B_n \to \partial c_{n+1} = b_n \in Z_n$

Therefore, $b_n \in B_n \to b_n \in Z_n$

$$Z_n \xrightarrow{j_*} H_n(C_*) \to ker\ i \to 0$$

$$
\begin{tikzcd}
0 \arrow[r] & Z_{n+1} \arrow[r] \arrow[d, "\partial=0"] & C_{n+1} \arrow[r] \arrow[d, "\partial"] & B_n \arrow[r] \arrow[d, "\partial=0"] & 0 \\
0 \arrow[r] & Z_n \arrow[r] & C_n \arrow[r] & B_{n-1} \arrow[r] & 0
\end{tikzcd}
$$

$$\cdots \to B_n \xrightarrow{i} Z_n \xrightarrow{j} H_n(C_*) \to B_{n-1} \to \cdots$$

$b_n = \partial c_{n+1}$
$b_n \in B_n \to \partial c_{n+1} = b_n \in Z_n$

Therefore, $b_n \in B_n \to b_n \in Z_n$

$$Z_n \xrightarrow{j_*} H_n(C_*) \to 0$$

$$\begin{tikzcd}
0 \arrow[r] & Z_{n+1} \arrow[r] \arrow[d, "\partial=0"] & C_{n+1} \arrow[r] \arrow[d, "\partial"] & B_n \arrow[r] \arrow[d, "\partial=0"] & 0 \\
0 \arrow[r] & Z_n \arrow[r] & C_n \arrow[r] & B_{n-1} \arrow[r] & 0
\end{tikzcd}$$

$$\cdots \to B_n \xrightarrow{i} Z_n \xrightarrow{j} H_n(C_*) \to B_{n-1} \to \cdots$$

$b_n = \partial c_{n+1}$

$b_n \in B_n \to \partial c_{n+1} = b_n \in Z_n$

Therefore, $b_n \in B_n \to b_n \in Z_n$

$$0 \to Z_n/ker\ j \xrightarrow{j_*} H_n(C_*) \to 0$$

$$
\begin{tikzcd}
0 \arrow[r] & Z_{n+1} \arrow[r] \arrow[d, "\partial=0"] & C_{n+1} \arrow[r] \arrow[d, "\partial"] & B_n \arrow[r] \arrow[d, "\partial=0"] & 0 \\
0 \arrow[r] & Z_n \arrow[r] & C_n \arrow[r] & B_{n-1} \arrow[r] & 0
\end{tikzcd}
$$

$$\cdots \to B_n \xrightarrow{i} Z_n \xrightarrow{j} H_n(C_*) \to B_{n-1} \to \cdots$$

$b_n = \partial c_{n+1}$
$b_n \in B_n \to \partial c_{n+1} = b_n \in Z_n$

Therefore, $b_n \in B_n \to b_n \in Z_n$

$$0 \to Z_n / i(B_n) \xrightarrow{j_*} H_n(C_*) \to 0$$

$$\begin{tikzcd}
0 \arrow[r] & Z_{n+1} \arrow[r] \arrow[d, "\partial=0"] & C_{n+1} \arrow[r] \arrow[d, "\partial"] & B_n \arrow[r] \arrow[d, "\partial=0"] & 0 \\
0 \arrow[r] & Z_n \arrow[r] & C_n \arrow[r] & B_{n-1} \arrow[r] & 0
\end{tikzcd}$$

$$\cdots \to B_n \xrightarrow{i} Z_n \xrightarrow{j} H_n(C_*) \to B_{n-1} \to \cdots$$

$b_n = \partial c_{n+1}$

$b_n \in B_n \to \partial c_{n+1} = b_n \in Z_n$

Therefore, $b_n \in B_n \to b_n \in Z_n$

$0 \to H_n(C_*) \xrightarrow{j_*} H_n(C_*) \to 0$

$$0 \longrightarrow Z_{n+1} \xrightarrow{j} C_{n+1} \xrightarrow{\partial} B_n \longrightarrow 0$$

$$0 \longrightarrow Z_{n+1} \xrightarrow{j} C_{n+1} \xrightarrow{\partial} B_n \longrightarrow 0$$

$$0 \longrightarrow Z_n \xrightarrow{j} C_n \xrightarrow{\partial} B_{n-1} \longrightarrow 0$$

$$
\begin{tikzcd}
0 \arrow[r] & Z_{n+1} \arrow[r, "j"] & C_{n+1} \arrow[r, "\partial"] \arrow[d, "\partial"] & B_n \arrow[r] & 0 \\
0 \arrow[r] & Z_n \arrow[r, "j"] & C_n \arrow[r, "\partial"] & B_{n-1} \arrow[r] & 0
\end{tikzcd}
$$

$$
\begin{tikzcd}
0 \arrow[r] & Z_{n+1} \arrow[r, "j"] \arrow[d, "\partial"] & C_{n+1} \arrow[r, "\partial"] \arrow[d, "\partial"] & B_n \arrow[r] & 0 \\
0 \arrow[r] & Z_n \arrow[r, "j"] & C_n \arrow[r, "\partial"] & B_{n-1} \arrow[r] & 0
\end{tikzcd}
$$

$$\begin{tikzcd}
0 \arrow[r] & Z_{n+1} \arrow[r, "j"] \arrow[d, "\partial=0"] & C_{n+1} \arrow[r, "\partial"] \arrow[d, "\partial"] & B_n \arrow[r] & 0 \\
0 \arrow[r] & Z_n \arrow[r, "j"] & C_n \arrow[r, "\partial"] & B_{n-1} \arrow[r] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[r] & Z_{n+1} \arrow[r, "j"] \arrow[d, "\partial=0"] & C_{n+1} \arrow[r, "\partial"] \arrow[d, "\partial"] & B_n \arrow[r] \arrow[d, "\partial"] & 0 \\
0 \arrow[r] & Z_n \arrow[r, "j"] & C_n \arrow[r, "\partial"] & B_{n-1} \arrow[r] & 0
\end{tikzcd}$$

$$
\begin{tikzcd}
0 \arrow[r] & Z_{n+1} \arrow[r, "j"] \arrow[d, "\partial=0"] & C_{n+1} \arrow[r, "\partial"] \arrow[d, "\partial"] & B_n \arrow[r] \arrow[d, "\partial=0"] & 0 \\
0 \arrow[r] & Z_n \arrow[r, "j"] & C_n \arrow[r, "\partial"] & B_{n-1} \arrow[r] & 0
\end{tikzcd}
$$

$$\begin{tikzcd}
0 \arrow[r] & Z_{n+1} \otimes G \arrow[r, "j\otimes Id"] \arrow[d, "\partial\otimes Id=0"] & C_{n+1} \otimes G \arrow[r, "\partial\otimes Id"] \arrow[d, "\partial\otimes Id"] & B_n \otimes G \arrow[r] \arrow[d, "\partial\otimes Id=0"] & 0 \\
0 \arrow[r] & Z_n \otimes G \arrow[r, "j\otimes Id"] & C_n \otimes G \arrow[r, "\partial\otimes Id"] & B_{n-1} \otimes G \arrow[r] & 0
\end{tikzcd}$$

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & Z_{n+1} \otimes G & \xrightarrow{j \otimes Id} & C_{n+1} \otimes G & \xrightarrow{\partial \otimes Id} & B_n \otimes G & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \partial \otimes Id = 0} & & \Big\downarrow{\scriptstyle \partial \otimes Id} & & \Big\downarrow{\scriptstyle \partial \otimes Id = 0} & & \\
0 & \longrightarrow & Z_n \otimes G & \xrightarrow{j \otimes Id} & C_n \otimes G & \xrightarrow{\partial \otimes Id} & B_{n-1} \otimes G & \longrightarrow & 0
\end{array}
$$

$$\cdots H_n(Z_* \otimes G)$$

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Z_{n+1}\otimes G & \xrightarrow{j\otimes Id} & C_{n+1}\otimes G & \xrightarrow{\partial\otimes Id} & B_n\otimes G & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \partial\otimes Id=0} & & \Big\downarrow{\scriptstyle \partial\otimes Id} & & \Big\downarrow{\scriptstyle \partial\otimes Id=0} & & \\
0 & \longrightarrow & Z_n\otimes G & \xrightarrow{j\otimes Id} & C_n\otimes G & \xrightarrow{\partial\otimes Id} & B_{n-1}\otimes G & \longrightarrow & 0
\end{array}$$

$$\cdots H_n(Z_*\otimes G)\to H_n(C_*\otimes G)$$

$$\begin{tikzcd}
0 \arrow[r] & Z_{n+1} \otimes G \arrow[r, "j\otimes Id"] \arrow[d, "\partial\otimes Id=0"] & C_{n+1} \otimes G \arrow[r, "\partial\otimes Id"] \arrow[d, "\partial\otimes Id"] & B_n \otimes G \arrow[r] \arrow[d, "\partial\otimes Id=0"] & 0 \\
0 \arrow[r] & Z_n \otimes G \arrow[r, "j\otimes Id"] & C_n \otimes G \arrow[r, "\partial\otimes Id"] & B_{n-1} \otimes G \arrow[r] & 0
\end{tikzcd}$$

$$\cdots H_n(Z_* \otimes G) \to H_n(C_* \otimes G) \to H_n(B_{*-1} \otimes G)$$

$$\begin{tikzcd}
0 \arrow[r] & Z_{n+1}\otimes G \arrow[r, "j\otimes Id"] \arrow[d, "\partial\otimes Id=0"] & C_{n+1}\otimes G \arrow[r, "\partial\otimes Id"] \arrow[d, "\partial\otimes Id"] & B_n\otimes G \arrow[r] \arrow[d, "\partial\otimes Id=0"] & 0 \\
0 \arrow[r] & Z_n\otimes G \arrow[r, "j\otimes Id"] & C_n\otimes G \arrow[r, "\partial\otimes Id"] & B_{n-1}\otimes G \arrow[r] & 0
\end{tikzcd}$$

$$\cdots H_n(Z_*\otimes G)\to H_n(C_*\otimes G)\to H_n(B_{*-1}\otimes G)\to H_{n-1}(Z_*\otimes G)\to\cdots$$

$$\begin{tikzcd}
0 \arrow[r] & Z_{n+1}\otimes G \arrow[r, "j\otimes Id"] \arrow[d, "\partial\otimes Id=0"] & C_{n+1}\otimes G \arrow[r, "\partial\otimes Id"] \arrow[d, "\partial\otimes Id"] & B_n\otimes G \arrow[r] \arrow[d, "\partial\otimes Id=0"] & 0 \\
0 \arrow[r] & Z_n\otimes G \arrow[r, "j\otimes Id"] & C_n\otimes G \arrow[r, "\partial\otimes Id"] & B_{n-1}\otimes G \arrow[r] & 0
\end{tikzcd}$$

$$\cdots Z_n\otimes G \to H_n(C_*\otimes G) \to H_n(B_{*-1}\otimes G) \to H_{n-1}(Z_*\otimes G) \to \cdots$$

$$\begin{tikzcd}
0 \arrow[r] & Z_{n+1} \otimes G \arrow[r, "j\otimes Id"] \arrow[d, "\partial\otimes Id=0"] & C_{n+1} \otimes G \arrow[r, "\partial\otimes Id"] \arrow[d, "\partial\otimes Id"] & B_n \otimes G \arrow[r] \arrow[d, "\partial\otimes Id=0"] & 0 \\
0 \arrow[r] & Z_n \otimes G \arrow[r, "j\otimes Id"] & C_n \otimes G \arrow[r, "\partial\otimes Id"] & B_{n-1} \otimes G \arrow[r] & 0
\end{tikzcd}$$

$$\cdots Z_n \otimes G \to H_n(C;G) \to H_n(B_{*-1} \otimes G) \to H_{n-1}(Z_* \otimes G) \to \cdots$$

$$\begin{tikzcd}
0 \arrow[r] & Z_{n+1} \otimes G \arrow[r, "j\otimes Id"] \arrow[d, "\partial\otimes Id=0"] & C_{n+1} \otimes G \arrow[r, "\partial\otimes Id"] \arrow[d, "\partial\otimes Id"] & B_n \otimes G \arrow[r] \arrow[d, "\partial\otimes Id=0"] & 0 \\
0 \arrow[r] & Z_n \otimes G \arrow[r, "j\otimes Id"] & C_n \otimes G \arrow[r, "\partial\otimes Id"] & B_{n-1} \otimes G \arrow[r] & 0
\end{tikzcd}$$

$$\cdots Z_n \otimes G \to H_n(C;G) \to B_{n-1} \otimes G \to H_{n-1}(Z_* \otimes G) \to \cdots$$

$$\begin{tikzcd}
0 \arrow[r] & Z_{n+1} \otimes G \arrow[r, "j\otimes Id"] \arrow[d, "\partial\otimes Id=0"] & C_{n+1} \otimes G \arrow[r, "\partial\otimes Id"] \arrow[d, "\partial\otimes Id"] & B_n \otimes G \arrow[r] \arrow[d, "\partial\otimes Id=0"] & 0 \\
0 \arrow[r] & Z_n \otimes G \arrow[r, "j\otimes Id"] & C_n \otimes G \arrow[r, "\partial\otimes Id"] & B_{n-1} \otimes G \arrow[r] & 0
\end{tikzcd}$$

$$\cdots Z_n \otimes G \to H_n(C;G) \to B_{n-1} \otimes G \to Z_{n-1} \otimes G \to \cdots$$

$$
\begin{tikzcd}
0 \arrow[r] & Z_{n+1} \otimes G \arrow[r, "j\otimes Id"] \arrow[d, "\partial\otimes Id=0"] & C_{n+1} \otimes G \arrow[r, "\partial\otimes Id"] \arrow[d, "\partial\otimes Id"] & B_n \otimes G \arrow[r] \arrow[d, "\partial\otimes Id=0"] & 0 \\
0 \arrow[r] & Z_n \otimes G \arrow[r, "j\otimes Id"] & C_n \otimes G \arrow[r, "\partial\otimes Id"] & B_{n-1} \otimes G \arrow[r] & 0
\end{tikzcd}
$$

$$\cdots Z_n \otimes G \to H_n(C;G) \to B_{n-1} \otimes G \xrightarrow{??} Z_{n-1} \otimes G \to \cdots$$

$$\begin{tikzcd}
0 \arrow[r] & Z_{n+1} \otimes G \arrow[r, "j\otimes Id"] \arrow[d, "\partial\otimes Id=0"] & C_{n+1} \otimes G \arrow[r, "\partial\otimes Id"] \arrow[d, "\partial\otimes Id"] & B_n \overset{b\otimes g}{\otimes} G \arrow[r] \arrow[d, "\partial\otimes Id=0"] & 0 \\
0 \arrow[r] & Z_n \otimes G \arrow[r, "j\otimes Id"] & C_n \otimes G \arrow[r, "\partial\otimes Id"] & B_{n-1} \otimes G \arrow[r] & 0
\end{tikzcd}$$

$$\cdots Z_n \otimes G \to H_n(C;G) \to B_{n-1} \otimes G \xrightarrow{??} Z_{n-1} \otimes G \to \cdots$$

$$
\begin{tikzcd}
 & & c\otimes g & b\otimes g=\partial c\otimes g & \\[-25pt]
0 \arrow[r] & Z_{n+1}\otimes G \arrow[r, "j\otimes Id"] \arrow[d, "\partial\otimes Id=0"] & C_{n+1}\otimes G \arrow[r, "\partial\otimes Id"] \arrow[d, "\partial\otimes Id"] & B_n\otimes G \arrow[r] \arrow[d, "\partial\otimes Id=0"] & 0 \\
0 \arrow[r] & Z_n\otimes G \arrow[r, "j\otimes Id"] & C_n\otimes G \arrow[r, "\partial\otimes Id"] & B_{n-1}\otimes G \arrow[r] & 0
\end{tikzcd}
$$

$$
\cdots Z_n\otimes G \to H_n(C;G) \to B_{n-1}\otimes G \xrightarrow{??} Z_{n-1}\otimes G \to \cdots
$$

$$
\begin{tikzcd}
0 \arrow[r] & Z_{n+1}\otimes G \arrow[r, "j\otimes Id"] \arrow[d, "\partial\otimes Id=0"] & \overset{c\otimes g}{C_{n+1}\otimes G} \arrow[r, "\partial\otimes Id"] \arrow[d, "\partial\otimes Id"] & \overset{b\otimes g=\partial c\otimes g}{B_n\otimes G} \arrow[r] \arrow[d, "\partial\otimes Id=0"] & 0 \\
0 \arrow[r] & Z_n\otimes G \arrow[r, "j\otimes Id"] & \overset{\partial c\otimes g}{C_n\otimes G} \arrow[r, "\partial\otimes Id"] & B_{n-1}\otimes G \arrow[r] & 0
\end{tikzcd}
$$

$$
\cdots Z_n\otimes G \to H_n(C;G) \to B_{n-1}\otimes G \xrightarrow{??} Z_{n-1}\otimes G \to \cdots
$$

$$\begin{tikzcd}
0 \arrow[r] & Z_{n+1}\otimes G \arrow[r, "j\otimes Id"] \arrow[d, "\partial\otimes Id=0"] & \overset{c\otimes g}{C_{n+1}\otimes G} \arrow[r, "\partial\otimes Id"] \arrow[d, "\partial\otimes Id"] & \overset{b\otimes g=\partial c\otimes g}{B_n\otimes G} \arrow[r] \arrow[d, "\partial\otimes Id=0"] & 0 \\
0 \arrow[r] & Z_n\otimes G \arrow[r, "j\otimes Id"] & \overset{\partial c\otimes g=b\otimes g}{C_n\otimes G} \arrow[r, "\partial\otimes Id"] & B_{n-1}\otimes G \arrow[r] & 0
\end{tikzcd}$$

$$\cdots Z_n\otimes G \to H_n(C;G) \to B_{n-1}\otimes G \xrightarrow{??} Z_{n-1}\otimes G \to \cdots$$

$$
\begin{tikzcd}
0 \arrow[r] & Z_{n+1} \otimes G \arrow[r, "j\otimes Id"] \arrow[d, "\partial\otimes Id=0"] & \overset{c\otimes g}{C_{n+1} \otimes G} \arrow[r, "\partial\otimes Id"] \arrow[d, "\partial\otimes Id"] & \overset{b\otimes g=\partial c\otimes g}{B_n \otimes G} \arrow[r] \arrow[d, "\partial\otimes Id=0"] & 0 \\
0 \arrow[r] & \overset{b\otimes g}{Z_n \otimes G} \arrow[r, "j\otimes Id"] & \overset{\partial c\otimes g=b\otimes g}{C_n \otimes G} \arrow[r, "\partial\otimes Id"] & B_{n-1} \otimes G \arrow[r] & 0
\end{tikzcd}
$$

$$
\cdots Z_n \otimes G \to H_n(C;G) \to B_{n-1} \otimes G \xrightarrow{??} Z_{n-1} \otimes G \to \cdots
$$

$$
\begin{tikzcd}
0 \arrow[r] & Z_{n+1}\otimes G \arrow[r, "j\otimes Id"] \arrow[d, "\partial\otimes Id=0"] & \overset{c\otimes g}{C_{n+1}\otimes G} \arrow[r, "\partial\otimes Id"] \arrow[d, "\partial\otimes Id"] & \overset{b\otimes g=\partial c\otimes g}{B_n\otimes G} \arrow[r] \arrow[d, "\partial\otimes Id=0"] & 0 \\
0 \arrow[r] & \overset{b\otimes g}{Z_n\otimes G} \arrow[r, "j\otimes Id"] & \overset{\partial c\otimes g=b\otimes g}{C_n\otimes G} \arrow[r, "\partial\otimes Id"] & B_{n-1}\otimes G \arrow[r] & 0
\end{tikzcd}
$$

$$
\cdots Z_n\otimes G \to H_n(C;G) \to B_{n-1}\otimes G \xrightarrow{i\otimes Id} Z_{n-1}\otimes G \to \cdots
$$

$$
\begin{tikzcd}
0 \arrow[r] & Z_{n+1}\otimes G \arrow[r, "j\otimes Id"] \arrow[d, "\partial\otimes Id=0"] & \overset{c\otimes g}{C_{n+1}\otimes G} \arrow[r, "\partial\otimes Id"] \arrow[d, "\partial\otimes Id"] & \overset{b\otimes g=\partial c\otimes g}{B_n\otimes G} \arrow[r] \arrow[d, "\partial\otimes Id=0"] & 0 \\
0 \arrow[r] & \overset{b\otimes g}{Z_n\otimes G} \arrow[r, "j\otimes Id"] & \overset{\partial c\otimes g=b\otimes g}{C_n\otimes G} \arrow[r, "\partial\otimes Id"] & B_{n-1}\otimes G \arrow[r] & 0
\end{tikzcd}
$$

$$
\cdots B_n\otimes G \xrightarrow{i\otimes Id} Z_n\otimes G \to H_n(C;G) \to B_{n-1}\otimes G \xrightarrow{i\otimes Id} Z_{n-1}\otimes G \to \cdots
$$

$$
\begin{tikzcd}
0 \arrow[r] & Z_{n+1}\otimes G \arrow[r, "j\otimes Id"] \arrow[d, "\partial\otimes Id=0"] & \overset{c\otimes g}{C_{n+1}\otimes G} \arrow[r, "\partial\otimes Id"] \arrow[d, "\partial\otimes Id"] & \overset{b\otimes g=\partial c\otimes g}{B_n\otimes G} \arrow[r] \arrow[d, "\partial\otimes Id=0"] & 0 \\
0 \arrow[r] & \overset{b\otimes g}{Z_n\otimes G} \arrow[r, "j\otimes Id"] & \overset{\partial c\otimes g=b\otimes g}{C_n\otimes G} \arrow[r, "\partial\otimes Id"] & B_{n-1}\otimes G \arrow[r] & 0
\end{tikzcd}
$$

$$\cdots B_n\otimes G \xrightarrow{i\otimes Id} Z_n\otimes G \to H_n(C;G) \to B_{n-1}\otimes G \xrightarrow{i\otimes Id} Z_{n-1}\otimes G \to \cdots$$

$$
\begin{tikzcd}
0 \arrow[r] & Z_{n+1}\otimes G \arrow[r, "j\otimes Id"] \arrow[d, "\partial\otimes Id=0"] & \overset{\color{blue}{c\otimes g}}{C_{n+1}\otimes G} \arrow[r, "\partial\otimes Id"] \arrow[d, "\partial\otimes Id"] & \overset{\color{blue}{b\otimes g=\partial c\otimes g}}{B_n\otimes G} \arrow[r] \arrow[d, "\partial\otimes Id=0"] & 0 \\
0 \arrow[r] & \overset{\color{blue}{b\otimes g}}{Z_n\otimes G} \arrow[r, "j\otimes Id"] & \overset{\color{blue}{\partial c\otimes g=b\otimes g}}{C_n\otimes G} \arrow[r, "\partial\otimes Id"] & B_{n-1}\otimes G \arrow[r] & 0
\end{tikzcd}
$$

$$
\cdots B_n\otimes G \xrightarrow{i\otimes Id} Z_n\otimes G \to H_n(C;G) \to B_{n-1}\otimes G \xrightarrow{i\otimes Id} Z_{n-1}\otimes G \to \cdots
$$

$$\begin{tikzcd}
0 \arrow[r] & Z_{n+1}\otimes G \arrow[r, "j\otimes Id"] \arrow[d, "\partial\otimes Id=0"] & \overset{c\otimes g}{C_{n+1}\otimes G} \arrow[r, "\partial\otimes Id"] \arrow[d, "\partial\otimes Id"] & \overset{b\otimes g=\partial c\otimes g}{B_n\otimes G} \arrow[r] \arrow[d, "\partial\otimes Id=0"] & 0 \\
0 \arrow[r] & \overset{b\otimes g}{Z_n\otimes G} \arrow[r, "j\otimes Id"] & \overset{\partial c\otimes g=b\otimes g}{C_n\otimes G} \arrow[r, "\partial\otimes Id"] & B_{n-1}\otimes G \arrow[r] & 0
\end{tikzcd}$$

$$\cdots B_n\otimes G \xrightarrow{i\otimes Id} Z_n\otimes G \to H_n(C;G) \to B_{n-1}\otimes G \xrightarrow{i\otimes Id} Z_{n-1}\otimes G \to \cdots$$

$$
\begin{tikzcd}
0 \arrow[r] & Z_{n+1}\otimes G \arrow[r, "j\otimes Id"] \arrow[d, "\partial\otimes Id=0"] & \overset{c\otimes g}{C_{n+1}\otimes G} \arrow[r, "\partial\otimes Id"] \arrow[d, "\partial\otimes Id"] & \overset{b\otimes g=\partial c\otimes g}{B_n\otimes G} \arrow[r] \arrow[d, "\partial\otimes Id=0"] & 0 \\
0 \arrow[r] & \overset{b\otimes g}{Z_n\otimes G} \arrow[r, "j\otimes Id"] & \overset{\partial c\otimes g=b\otimes g}{C_n\otimes G} \arrow[r, "\partial\otimes Id"] & B_{n-1}\otimes G \arrow[r] & 0
\end{tikzcd}
$$

$$
\cdots B_n\otimes G \xrightarrow{i\otimes Id} Z_n\otimes G \to H_n(C;G) \to B_{n-1}\otimes G \xrightarrow{i\otimes Id} Z_{n-1}\otimes G \to \cdots
$$

$$
\begin{tikzcd}
0 \arrow[r] & Z_{n+1}\otimes G \arrow[r, "j\otimes Id"] \arrow[d, "\partial\otimes Id=0"] & \overset{c\otimes g}{C_{n+1}\otimes G} \arrow[r, "\partial\otimes Id"] \arrow[d, "\partial\otimes Id"] & \overset{b\otimes g=\partial c\otimes g}{B_n\otimes G} \arrow[r] \arrow[d, "\partial\otimes Id=0"] & 0 \\
0 \arrow[r] & \overset{b\otimes g}{Z_n\otimes G} \arrow[r, "j\otimes Id"] & \overset{\partial c\otimes g=b\otimes g}{C_n\otimes G} \arrow[r, "\partial\otimes Id"] & B_{n-1}\otimes G \arrow[r] & 0
\end{tikzcd}
$$

$$\cdots B_n\otimes G \xrightarrow{i\otimes Id} Z_n\otimes G \to H_n(C;G) \to B_{n-1}\otimes G \xrightarrow{i\otimes Id} Z_{n-1}\otimes G \to \cdots$$

$$0 \to Ker\ (\partial\otimes Id)_* \to H_n(C;G) \to Im\ (\partial\otimes Id)_* \to 0$$

$$\begin{tikzcd}
0 \arrow[r] & Z_{n+1} \otimes G \arrow[r, "j\otimes Id"] \arrow[d, "\partial\otimes Id=0"] & \overset{c\otimes g}{C_{n+1} \otimes G} \arrow[r, "\partial\otimes Id"] \arrow[d, "\partial\otimes Id"] & \overset{b\otimes g=\partial c\otimes g}{B_n \otimes G} \arrow[r] \arrow[d, "\partial\otimes Id=0"] & 0 \\
0 \arrow[r] & \overset{b\otimes g}{Z_n \otimes G} \arrow[r, "j\otimes Id"] & \overset{\partial c\otimes g=b\otimes g}{C_n \otimes G} \arrow[r, "\partial\otimes Id"] & B_{n-1} \otimes G \arrow[r] & 0
\end{tikzcd}$$

$$\cdots B_n \otimes G \xrightarrow{i\otimes Id} Z_n \otimes G \to H_n(C;G) \to B_{n-1} \otimes G \xrightarrow{i\otimes Id} Z_{n-1} \otimes G \to \cdots$$

$$0 \to Ker\ (\partial \otimes Id)_* \to H_n(C;G) \to Ker\ (i \otimes Id) \to 0$$

$$
\begin{tikzcd}
0 \arrow[r] & Z_{n+1}\otimes G \arrow[r, "j\otimes Id"] \arrow[d, "\partial\otimes Id=0"] & \overset{c\otimes g}{C_{n+1}\otimes G} \arrow[r, "\partial\otimes Id"] \arrow[d, "\partial\otimes Id"] & \overset{b\otimes g=\partial c\otimes g}{B_n\otimes G} \arrow[r] \arrow[d, "\partial\otimes Id=0"] & 0 \\
0 \arrow[r] & \overset{b\otimes g}{Z_n\otimes G} \arrow[r, "j\otimes Id"] & \overset{\partial c\otimes g=b\otimes g}{C_n\otimes G} \arrow[r, "\partial\otimes Id"] & B_{n-1}\otimes G \arrow[r] & 0
\end{tikzcd}
$$

$$\cdots B_n\otimes G \xrightarrow{i\otimes Id} Z_n\otimes G \to H_n(C;G) \to B_{n-1}\otimes G \xrightarrow{i\otimes Id} Z_{n-1}\otimes G \to \cdots$$

$$0 \to Im\ (j\otimes Id)_* \to H_n(C;G) \to Ker\ (i\otimes Id) \to 0$$

$$
\begin{tikzcd}
0 \arrow[r] & Z_{n+1}\otimes G \arrow[r, "j\otimes Id"] \arrow[d, "\partial\otimes Id=0"] & \overset{c\otimes g}{C_{n+1}\otimes G} \arrow[r, "\partial\otimes Id"] \arrow[d, "\partial\otimes Id"] & \overset{b\otimes g=\partial c\otimes g}{B_n\otimes G} \arrow[r] \arrow[d, "\partial\otimes Id=0"] & 0 \\
0 \arrow[r] & \overset{b\otimes g}{Z_n\otimes G} \arrow[r, "j\otimes Id"] & \overset{\partial c\otimes g=b\otimes g}{C_n\otimes G} \arrow[r, "\partial\otimes Id"] & B_{n-1}\otimes G \arrow[r] & 0
\end{tikzcd}
$$

$$\cdots B_n\otimes G \xrightarrow{i\otimes Id} Z_n\otimes G \to H_n(C;G) \to B_{n-1}\otimes G \xrightarrow{i\otimes Id} Z_{n-1}\otimes G \to \cdots$$

$$0 \to Z_n\otimes G/Im\ (i\otimes Id) \to H_n(C;G) \to Ker\ (i\otimes Id) \to 0$$

$$
\begin{tikzcd}
0 \arrow[r] & Z_{n+1}\otimes G \arrow[r, "j\otimes Id"] \arrow[d, "\partial\otimes Id=0"] & \overset{c\otimes g}{C_{n+1}\otimes G} \arrow[r, "\partial\otimes Id"] \arrow[d, "\partial\otimes Id"] & \overset{b\otimes g=\partial c\otimes g}{B_n\otimes G} \arrow[r] \arrow[d, "\partial\otimes Id=0"] & 0 \\
0 \arrow[r] & \overset{b\otimes g}{Z_n\otimes G} \arrow[r, "j\otimes Id"] & \overset{\partial c\otimes g=b\otimes g}{C_n\otimes G} \arrow[r, "\partial\otimes Id"] & B_{n-1}\otimes G \arrow[r] & 0
\end{tikzcd}
$$

$$\cdots B_n\otimes G \xrightarrow{i\otimes Id} Z_n\otimes G \to H_n(C;G) \to B_{n-1}\otimes G \xrightarrow{i\otimes Id} Z_{n-1}\otimes G \to \cdots$$

$$0 \to (Z_n/i(B_n))\otimes G \to H_n(C;G) \to Ker\ (i\otimes Id) \to 0$$

because,

$$
\begin{tikzcd}
0 \arrow[r] & Z_{n+1}\otimes G \arrow[r, "j\otimes Id"] \arrow[d, "\partial\otimes Id=0"] & \overset{c\otimes g}{C_{n+1}\otimes G} \arrow[r, "\partial\otimes Id"] \arrow[d, "\partial\otimes Id"] & \overset{b\otimes g=\partial c\otimes g}{B_n\otimes G} \arrow[r] \arrow[d, "\partial\otimes Id=0"] & 0 \\
0 \arrow[r] & \overset{b\otimes g}{Z_n\otimes G} \arrow[r, "j\otimes Id"] & \overset{\partial c\otimes g=b\otimes g}{C_n\otimes G} \arrow[r, "\partial\otimes Id"] & B_{n-1}\otimes G \arrow[r] & 0
\end{tikzcd}
$$

$$\cdots B_n\otimes G \xrightarrow{i\otimes Id} Z_n\otimes G \to H_n(C;G) \to B_{n-1}\otimes G \xrightarrow{i\otimes Id} Z_{n-1}\otimes G \to \cdots$$

$$0 \to (Z_n/i(B_n))\otimes G \to H_n(C;G) \to Ker\ (i\otimes Id) \to 0$$

because,

$$0 \to B_n \xrightarrow{i} Z_n \to Z_n/B_n \to 0$$

$$\begin{tikzcd}
0 \arrow[r] & Z_{n+1}\otimes G \arrow[r, "j\otimes Id"] \arrow[d, "\partial\otimes Id=0"] & \overset{c\otimes g}{C_{n+1}\otimes G} \arrow[r, "\partial\otimes Id"] \arrow[d, "\partial\otimes Id"] & \overset{b\otimes g=\partial c\otimes g}{B_n\otimes G} \arrow[r] \arrow[d, "\partial\otimes Id=0"] & 0 \\
0 \arrow[r] & \overset{b\otimes g}{Z_n\otimes G} \arrow[r, "j\otimes Id"] & \overset{\partial c\otimes g=b\otimes g}{C_n\otimes G} \arrow[r, "\partial\otimes Id"] & B_{n-1}\otimes G \arrow[r] & 0
\end{tikzcd}$$

$$\cdots B_n\otimes G \xrightarrow{i\otimes Id} Z_n\otimes G \to H_n(C;G) \to B_{n-1}\otimes G \xrightarrow{i\otimes Id} Z_{n-1}\otimes G \to \cdots$$

$$0 \to (Z_n/i(B_n))\otimes G \to H_n(C;G) \to Ker\ (i\otimes Id) \to 0$$

because,

$$0 \to B_n \xrightarrow{i} Z_n \to Z_n/B_n \to 0$$

$$\implies B_n\otimes G \xrightarrow{i\otimes Id} Z_n\otimes G \to (Z_n/B_n)\otimes G \to 0$$

$$
\begin{tikzcd}
0 \arrow[r] & Z_{n+1} \otimes G \arrow[r, "j\otimes Id"] \arrow[d, "\partial\otimes Id=0"] & \overset{\textcolor{blue}{c\otimes g}}{C_{n+1} \otimes G} \arrow[r, "\partial\otimes Id"] \arrow[d, "\partial\otimes Id"] & \overset{\textcolor{blue}{b\otimes g=\partial c\otimes g}}{B_n \otimes G} \arrow[r] \arrow[d, "\partial\otimes Id=0"] & 0 \\
0 \arrow[r] & \overset{\textcolor{blue}{b\otimes g}}{Z_n \otimes G} \arrow[r, "j\otimes Id"] & \overset{\textcolor{blue}{\partial c\otimes g=b\otimes g}}{C_n \otimes G} \arrow[r, "\partial\otimes Id"] & B_{n-1} \otimes G \arrow[r] & 0
\end{tikzcd}
$$

$$\cdots B_n \otimes G \xrightarrow{i\otimes Id} Z_n \otimes G \to H_n(C;G) \to B_{n-1} \otimes G \xrightarrow{i\otimes Id} Z_{n-1} \otimes G \to \cdots$$

$$0 \to (Z_n/i(B_n)) \otimes G \to H_n(C;G) \to Ker\ (i \otimes Id) \to 0$$

because,

$$0 \to B_n \xrightarrow{i} Z_n \to Z_n/B_n \to 0$$

$$\implies B_n \otimes G \xrightarrow{i\otimes Id} Z_n \otimes G \to (Z_n/B_n) \otimes G \to 0$$

$$\implies Z_n \otimes G/Im\ (i \otimes Id) \cong (Z_n/B_n) \otimes G$$

$$\begin{tikzcd}
0 \arrow[r] & Z_{n+1}\otimes G \arrow[r, "j\otimes Id"] \arrow[d, "\partial\otimes Id=0"] & \overset{c\otimes g}{C_{n+1}\otimes G} \arrow[r, "\partial\otimes Id"] \arrow[d, "\partial\otimes Id"] & \overset{b\otimes g=\partial c\otimes g}{B_n\otimes G} \arrow[r] \arrow[d, "\partial\otimes Id=0"] & 0 \\
0 \arrow[r] & \overset{b\otimes g}{Z_n\otimes G} \arrow[r, "j\otimes Id"] & \overset{\partial c\otimes g=b\otimes g}{C_n\otimes G} \arrow[r, "\partial\otimes Id"] & B_{n-1}\otimes G \arrow[r] & 0
\end{tikzcd}$$

$$\cdots B_n\otimes G \xrightarrow{i\otimes Id} Z_n\otimes G \to H_n(C;G) \to B_{n-1}\otimes G \xrightarrow{i\otimes Id} Z_{n-1}\otimes G \to \cdots$$

$$0 \to (Z_n/i(B_n))\otimes G \to H_n(C;G) \to Ker\ (i\otimes Id) \to 0$$

because,

$$0 \to B_n \xrightarrow{i} Z_n \to Z_n/B_n \to 0$$

$$\implies\ B_n\otimes G \xrightarrow{i\otimes Id} Z_n\otimes G \to (Z_n/B_n)\otimes G \to 0$$

$$\implies\ Z_n\otimes G/Im\ (i\otimes Id) \cong (Z_n/B_n)\otimes G \cong H_n(C)\otimes G$$

$$
\begin{tikzcd}
0 \arrow[r] & Z_{n+1}\otimes G \arrow[r, "j\otimes Id"] \arrow[d, "\partial\otimes Id=0"] & \overset{\color{blue}c\otimes g}{C_{n+1}\otimes G} \arrow[r, "\partial\otimes Id"] \arrow[d, "\partial\otimes Id"] & \overset{\color{blue}b\otimes g=\partial c\otimes g}{B_n\otimes G} \arrow[r] \arrow[d, "\partial\otimes Id=0"] & 0 \\
0 \arrow[r] & \overset{\color{blue}b\otimes g}{Z_n\otimes G} \arrow[r, "j\otimes Id"] & \overset{\color{blue}\partial c\otimes g=b\otimes g}{C_n\otimes G} \arrow[r, "\partial\otimes Id"] & B_{n-1}\otimes G \arrow[r] & 0
\end{tikzcd}
$$

$$\cdots B_n\otimes G \xrightarrow{i\otimes Id} Z_n\otimes G \to H_n(C;G) \to B_{n-1}\otimes G \xrightarrow{i\otimes Id} Z_{n-1}\otimes G \to \cdots$$

$$0 \to Z_n\otimes G/Im\ (i\otimes Id) \to H_n(C;G) \to Ker\ (i\otimes Id) \to 0$$

because,

$$0 \to B_n \xrightarrow{i} Z_n \to Z_n/B_n \to 0$$

$$\implies B_n\otimes G \xrightarrow{i\otimes Id} Z_n\otimes G \to (Z_n/B_n)\otimes G \to 0$$

$$\implies Z_n\otimes G/Im\ (i\otimes Id) \cong (Z_n/B_n)\otimes G \cong H_n(C)\otimes G$$

$$
\begin{tikzcd}
0 \arrow[r] & Z_{n+1}\otimes G \arrow[r, "j\otimes Id"] \arrow[d, "\partial\otimes Id=0"] & \overset{c\otimes g}{C_{n+1}\otimes G} \arrow[r, "\partial\otimes Id"] \arrow[d, "\partial\otimes Id"] & \overset{b\otimes g=\partial c\otimes g}{B_n\otimes G} \arrow[r] \arrow[d, "\partial\otimes Id=0"] & 0 \\
0 \arrow[r] & \overset{b\otimes g}{Z_n\otimes G} \arrow[r, "j\otimes Id"] & \overset{\partial c\otimes g=b\otimes g}{C_n\otimes G} \arrow[r, "\partial\otimes Id"] & B_{n-1}\otimes G \arrow[r] & 0
\end{tikzcd}
$$

$$\cdots B_n\otimes G \xrightarrow{i\otimes Id} Z_n\otimes G \to H_n(C;G) \to B_{n-1}\otimes G \xrightarrow{i\otimes Id} Z_{n-1}\otimes G \to \cdots$$

$$0 \to (Z_n/i(B_n))\otimes G \to H_n(C;G) \to Ker\ (i\otimes Id) \to 0$$

because,

$$0 \to B_n \xrightarrow{i} Z_n \to Z_n/B_n \to 0$$

$$\implies B_n\otimes G \xrightarrow{i\otimes Id} Z_n\otimes G \to (Z_n/B_n)\otimes G \to 0$$

$$\implies Z_n\otimes G/Im\ (i\otimes Id) \cong (Z_n/B_n)\otimes G \cong H_n(C)\otimes G$$

$$\begin{tikzcd}
0 \arrow[r] & Z_{n+1}\otimes G \arrow[r, "j\otimes Id"] \arrow[d, "\partial\otimes Id=0"] & \overset{c\otimes g}{C_{n+1}\otimes G} \arrow[r, "\partial\otimes Id"] \arrow[d, "\partial\otimes Id"] & \overset{b\otimes g=\partial c\otimes g}{B_n\otimes G} \arrow[r] \arrow[d, "\partial\otimes Id=0"] & 0 \\
0 \arrow[r] & \overset{b\otimes g}{Z_n\otimes G} \arrow[r, "j\otimes Id"] & \overset{\partial c\otimes g=b\otimes g}{C_n\otimes G} \arrow[r, "\partial\otimes Id"] & B_{n-1}\otimes G \arrow[r] & 0
\end{tikzcd}$$

$$\cdots B_n\otimes G \xrightarrow{i\otimes Id} Z_n\otimes G \to H_n(C;G)\to B_{n-1}\otimes G \xrightarrow{i\otimes Id} Z_{n-1}\otimes G \to \cdots$$

$$0\to H_n(C)\otimes G\to H_n(C;G)\to Ker\ (i\otimes Id)\to 0$$

because,

$$0\to B_n \xrightarrow{i} Z_n \to Z_n/B_n \to 0$$

$$\implies B_n\otimes G \xrightarrow{i\otimes Id} Z_n\otimes G \to (Z_n/B_n)\otimes G\to 0$$

$$\implies Z_n\otimes G/Im\ (i\otimes Id)\cong (Z_n/B_n)\otimes G\cong H_n(C)\otimes G$$

$$
\begin{tikzcd}
0 \arrow[r] & Z_{n+1}\otimes G \arrow[r, "j\otimes Id"] \arrow[d, "\partial\otimes Id=0"] & \overset{\color{blue}{c\otimes g}}{C_{n+1}\otimes G} \arrow[r, "\partial\otimes Id"] \arrow[d, "\partial\otimes Id"] & \overset{\color{blue}{b\otimes g=\partial c\otimes g}}{B_n\otimes G} \arrow[r] \arrow[d, "\partial\otimes Id=0"] & 0 \\
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H_2(\mathbb{RP}^2; \mathbb{Z}/2) \arrow[r] \arrow[d, "f_*"] & Tor_1(H_1(\mathbb{RP}^2), \mathbb{Z}/2) \oplus (H_2(\mathbb{RP}^2) \otimes \mathbb{Z}/2) \arrow[d] \\
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To show f_* is an isomorphism:

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H_2(S^2;\mathbb{Z}/2) \arrow[r] & H_2(S^2, \mathbb{RP}^2 \setminus p;\mathbb{Z}/2) \arrow[r, "excision"] & H_2(U, U \setminus p)
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Example.

$$H_k(\mathbb{RP}^2) = \begin{cases} \mathbb{Z} & k = 0 \\ \mathbb{Z}/2 & k = 1 \\ 0 & otherwise \end{cases}$$

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because $\sigma : \Delta_k \to X$ lifts to
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$\tilde{X} \xrightarrow{p} X$ two sheeted covering

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$S^n \xrightarrow{\quad p \quad} \mathbb{RP}^n$

because $\sigma : \Delta_k \to X$ lifts to
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$$\begin{array}{ccc} C_k(S^n;\mathbb{Z}/2) & \xrightarrow{p_\#} & C_k(\mathbb{RP}^n;\mathbb{Z}/2) \\ \big\downarrow f_\# & & \big\downarrow f'_\# \\ C_k(S^n;\mathbb{Z}/2) & \xrightarrow{p_\#} & C_k(\mathbb{RP}^n;\mathbb{Z}/2) \end{array}$$

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$$\begin{tikzcd}
C_k(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\tau"] & C_k(S^n;\mathbb{Z}/2) \arrow[r, "p_\#"] \arrow[d, "f_\#"] & C_k(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[d, "f'_\#"] \\
C_k(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\tau"] & C_k(S^n;\mathbb{Z}/2) \arrow[r, "p_\#"] & C_k(\mathbb{RP}^n;\mathbb{Z}/2)
\end{tikzcd}$$

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$$\begin{array}{ccccc} C_k(\mathbb{RP}^n; \mathbb{Z}/2) & \xrightarrow{\tau} & C_k(S^n; \mathbb{Z}/2) & \xrightarrow{p_\#} & C_k(\mathbb{RP}^n; \mathbb{Z}/2) \\ \big\downarrow f'_\# & & \big\downarrow f_\# & & \big\downarrow f'_\# \\ C_k(\mathbb{RP}^n; \mathbb{Z}/2) & \xrightarrow{\tau} & C_k(S^n; \mathbb{Z}/2) & \xrightarrow{p_\#} & C_k(\mathbb{RP}^n; \mathbb{Z}/2) \end{array}$$

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$$\begin{array}{ccccccccc}
0 & \longrightarrow & C_k(\mathbb{RP}^n; \mathbb{Z}/2) & \xrightarrow{\tau} & C_k(S^n; \mathbb{Z}/2) & \xrightarrow{p_\#} & C_k(\mathbb{RP}^n; \mathbb{Z}/2) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f'_\#} & & \downarrow{\scriptstyle f_\#} & & \downarrow{\scriptstyle f'_\#} & & \\
0 & \longrightarrow & C_k(\mathbb{RP}^n; \mathbb{Z}/2) & \xrightarrow{\tau} & C_k(S^n; \mathbb{Z}/2) & \xrightarrow{p_\#} & C_k(\mathbb{RP}^n; \mathbb{Z}/2) & \longrightarrow & 0
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$$\begin{tikzcd}
0 \arrow[r] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\tau_*"] \arrow[d, "f'_*"] & H_n(S^n;\mathbb{Z}/2) \arrow[r, "p_*"] \arrow[d, "f_*"] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r] \arrow[d, "f'_*"] & H_{n-1}(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\tau_*"] \arrow[d, "f'_*"] & H_{n-1}(S^n;\mathbb{Z}/2) \arrow[d, "f_*"] \\
0 \arrow[r] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\tau_*"] & H_n(S^n;\mathbb{Z}/2) \arrow[r, "p_*"] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r] & H_{n-1}(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\tau_*"] & H_{n-1}(S^n;\mathbb{Z}/2)
\end{tikzcd}$$

$$
\begin{tikzcd}
0 \arrow[r] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\tau_*"] \arrow[d, "f'_*"] & H_n(S^n;\mathbb{Z}/2) \arrow[r, "p_*"] \arrow[d, "f_*"] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r] \arrow[d, "f'_*"] & H_{n-1}(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\tau_*"] \arrow[d, "f'_*"] & 0 \\
0 \arrow[r] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\tau_*"] & H_n(S^n;\mathbb{Z}/2) \arrow[r, "p_*"] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r] & H_{n-1}(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\tau_*"] & 0
\end{tikzcd}
$$

$$\begin{tikzcd}
0 \arrow[r] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\tau_*"] \arrow[d, "f'_*"] & H_n(S^n;\mathbb{Z}/2) \arrow[r, "p_*"] \arrow[d, "f_*"] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r] \arrow[d, "f'_*"] & H_{n-1}(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\tau_*"] \arrow[d, "f'_*"] & 0 \\
0 \arrow[r] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\tau_*"] & H_n(S^n;\mathbb{Z}/2) \arrow[r, "p_*"] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r] & H_{n-1}(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\tau_*"] & 0
\end{tikzcd}$$

θ is the (only) non-trivial deck transformation

$$\begin{tikzcd}
0 \arrow[r] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\tau_*"] \arrow[d, "f'_*"] & H_n(S^n;\mathbb{Z}/2) \arrow[r, "p_*"] \arrow[d, "f_*"] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r] \arrow[d, "f'_*"] & H_{n-1}(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\tau_*"] \arrow[d, "f'_*"] & 0 \\
0 \arrow[r] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\tau_*"] & H_n(S^n;\mathbb{Z}/2) \arrow[r, "p_*"] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r] & H_{n-1}(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\tau_*"] & 0
\end{tikzcd}$$

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$p_*([z]) = [p_\#(z)]$??

$$\begin{tikzcd}
0 \arrow[r] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\tau_*"] \arrow[d, "f'_*"] & H_n(S^n;\mathbb{Z}/2) \arrow[r, "p_*"] \arrow[d, "f_*"] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r] \arrow[d, "f'_*"] & H_{n-1}(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\tau_*"] \arrow[d, "f'_*"] & 0 \\
0 \arrow[r] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\tau_*"] & H_n(S^n;\mathbb{Z}/2) \arrow[r, "p_*"] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r] & H_{n-1}(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\tau_*"] & 0
\end{tikzcd}$$

θ is the (only) non-trivial deck transformation

$p_*([z]) = [p_\#(z)]??$

$\tau_*(p_*([z]))$

$$
\begin{tikzcd}
0 \arrow[r] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\tau_*"] \arrow[d, "f'_*"] & H_n(S^n;\mathbb{Z}/2) \arrow[r, "p_*"] \arrow[d, "f_*"] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r] \arrow[d, "f'_*"] & H_{n-1}(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\tau_*"] \arrow[d, "f'_*"] & 0 \\
0 \arrow[r] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\tau_*"] & H_n(S^n;\mathbb{Z}/2) \arrow[r, "p_*"] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r] & H_{n-1}(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\tau_*"] & 0
\end{tikzcd}
$$

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$\tau_*(p_*([z])) = \tau_*([p_\#(z)])$

$$\begin{tikzcd}
0 \arrow[r] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\tau_*"] \arrow[d, "f'_*"] & H_n(S^n;\mathbb{Z}/2) \arrow[r, "p_*"] \arrow[d, "f_*"] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r] \arrow[d, "f'_*"] & H_{n-1}(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\tau_*"] \arrow[d, "f'_*"] & 0 \\
0 \arrow[r] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\tau_*"] & H_n(S^n;\mathbb{Z}/2) \arrow[r, "p_*"] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r] & H_{n-1}(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\tau_*"] & 0
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$p_*([z]) = [p_\#(z)]??$

$\tau_*(p_*([z])) = \tau_*([p_\#(z)]) = [z + \theta_\#(z)]$

$$\begin{tikzcd}
0 \arrow[r] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\tau_*"] \arrow[d, "f'_*"] & H_n(S^n;\mathbb{Z}/2) \arrow[r, "p_*"] \arrow[d, "f_*"] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r] \arrow[d, "f'_*"] & H_{n-1}(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\tau_*"] \arrow[d, "f'_*"] & 0 \\
0 \arrow[r] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\tau_*"] & H_n(S^n;\mathbb{Z}/2) \arrow[r, "p_*"] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r] & H_{n-1}(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\tau_*"] & 0
\end{tikzcd}$$

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$\tau_*(p_*([z])) = \tau_*([p_\#(z)]) = [z + \theta_\#(z)] = [z] + [\theta_\#(z)]$

$$
\begin{tikzcd}
0 \arrow[r] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\tau_*"] \arrow[d, "f'_*"] & H_n(S^n;\mathbb{Z}/2) \arrow[r, "p_*"] \arrow[d, "f_*"] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r] \arrow[d, "f'_*"] & H_{n-1}(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\tau_*"] \arrow[d, "f'_*"] & 0 \\
0 \arrow[r] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\tau_*"] & H_n(S^n;\mathbb{Z}/2) \arrow[r, "p_*"] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r] & H_{n-1}(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\tau_*"] & 0
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$$

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0 \arrow[r] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\tau_*"] & H_n(S^n;\mathbb{Z}/2) \arrow[r, "p_*"] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r] & H_{n-1}(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\tau_*"] & 0
\end{tikzcd}$$

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$\tau_*(p_*([z])) = \tau_*([p_\#(z)]) = [z + \theta_\#(z)] = [z] + [\theta_\#(z)] = [z] + \theta_*[(z)] = 2[z]$

$$
\begin{tikzcd}
0 \arrow[r] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\tau_*"] \arrow[d, "f'_*"] & H_n(S^n;\mathbb{Z}/2) \arrow[r, "p_*"] \arrow[d, "f_*"] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r] \arrow[d, "f'_*"] & H_{n-1}(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\tau_*"] \arrow[d, "f'_*"] & 0 \\
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But τ_* is injective.

$$\begin{tikzcd}
0 \arrow[r] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\tau_*"] \arrow[d, "f'_*"] & H_n(S^n;\mathbb{Z}/2) \arrow[r, "p_*"] \arrow[d, "f_*"] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r] \arrow[d, "f'_*"] & H_{n-1}(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\tau_*"] \arrow[d, "f'_*"] & 0 \\
0 \arrow[r] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\tau_*"] & H_n(S^n;\mathbb{Z}/2) \arrow[r, "p_*"] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r] & H_{n-1}(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\tau_*"] & 0
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But τ_* is injective. Therefore, $p_* = 0$

$$
\begin{tikzcd}
0 \arrow[r] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\tau_*"] \arrow[d, "f'_*"] & H_n(S^n;\mathbb{Z}/2) \arrow[r, "0"] \arrow[d, "f_*"] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r] \arrow[d, "f'_*"] & H_{n-1}(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\tau_*"] \arrow[d, "f'_*"] & 0 \\
0 \arrow[r] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\tau_*"] & H_n(S^n;\mathbb{Z}/2) \arrow[r, "0"] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r] & H_{n-1}(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\tau_*"] & 0
\end{tikzcd}
$$

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0 \arrow[r] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\sim"] \arrow[d, "f'_*"] & H_n(S^n;\mathbb{Z}/2) \arrow[r, "0"] \arrow[d, "f_*"] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\sim"] \arrow[d, "f'_*"] & H_{n-1}(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\tau_*"] \arrow[d, "f'_*"] & 0 \\
0 \arrow[r] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\sim"] & H_n(S^n;\mathbb{Z}/2) \arrow[r, "0"] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\sim"] & H_{n-1}(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\tau_*"] & 0
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f has odd degree

$$
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0 \arrow[r] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\sim"] & H_n(S^n;\mathbb{Z}/2) \arrow[r, "0"] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\sim"] & H_{n-1}(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\tau_*"] & 0
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Theorem (Borsuk-Ulam)**.** *$f : S^n \to \mathbb{R}^n$, then $f(-x) = f(x)$ for some $x \in S^n$*

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Theorem (Borsuk-Ulam)**.** *$f : S^n \to \mathbb{R}^n$, then $f(-x) = f(x)$ for some $x \in S^n$*

Proof. Assume $f(x) - f(-x) \neq 0$ for any x

$\square$

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & H_n(\mathbb{RP}^n;\mathbb{Z}/2) & \xrightarrow{\sim} & H_n(S^n;\mathbb{Z}/2) & \xrightarrow{0} & H_n(\mathbb{RP}^n;\mathbb{Z}/2) & \xrightarrow{\sim} & H_{n-1}(\mathbb{RP}^n;\mathbb{Z}/2) & \xrightarrow{\tau_*} & 0 \\
 & & \downarrow{\scriptstyle f'_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f'_*} & & \downarrow{\scriptstyle f'_*} & & \\
0 & \longrightarrow & H_n(\mathbb{RP}^n;\mathbb{Z}/2) & \xrightarrow{\sim} & H_n(S^n;\mathbb{Z}/2) & \xrightarrow{0} & H_n(\mathbb{RP}^n;\mathbb{Z}/2) & \xrightarrow{\sim} & H_{n-1}(\mathbb{RP}^n;\mathbb{Z}/2) & \xrightarrow{\tau_*} & 0
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$h(x) = \frac{f(x)-f(-x)}{|f(x)-f(-x)|}$ is well defined and $h(-x) = -h(x)$. $\square$

$$\begin{tikzcd}
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0 \arrow[r] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\sim"] & H_n(S^n;\mathbb{Z}/2) \arrow[r, "0"] & H_n(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\sim"] & H_{n-1}(\mathbb{RP}^n;\mathbb{Z}/2) \arrow[r, "\tau_*"] & 0
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$S^n \xrightarrow{h} S^{n-1}$ $\square$

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & H_n(\mathbb{RP}^n;\mathbb{Z}/2) & \xrightarrow{\sim} & H_n(S^n;\mathbb{Z}/2) & \xrightarrow{0} & H_n(\mathbb{RP}^n;\mathbb{Z}/2) & \xrightarrow{\sim} & H_{n-1}(\mathbb{RP}^n;\mathbb{Z}/2) & \xrightarrow{\tau_*} & 0 \\
 & & \downarrow{\scriptstyle f'_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f'_*} & & \downarrow{\scriptstyle f'_*} & & \\
0 & \longrightarrow & H_n(\mathbb{RP}^n;\mathbb{Z}/2) & \xrightarrow{\sim} & H_n(S^n;\mathbb{Z}/2) & \xrightarrow{0} & H_n(\mathbb{RP}^n;\mathbb{Z}/2) & \xrightarrow{\sim} & H_{n-1}(\mathbb{RP}^n;\mathbb{Z}/2) & \xrightarrow{\tau_*} & 0
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$S^{n-1} \xrightarrow{i} S^n \xrightarrow{h} S^{n-1}$ has odd degree