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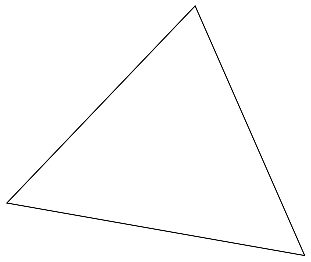
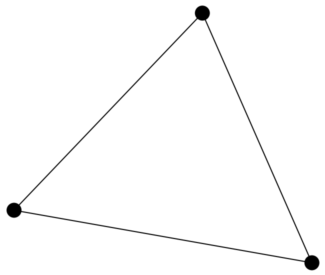
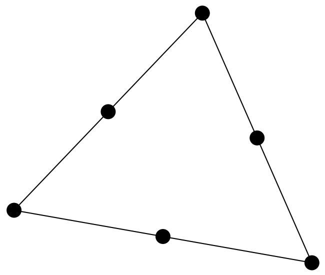
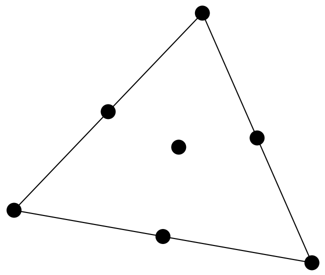
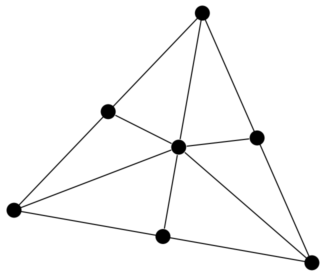
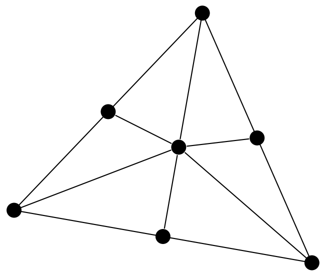


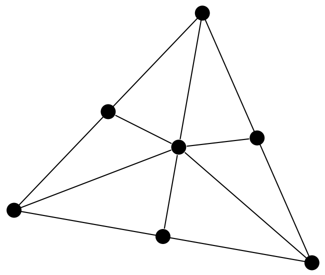


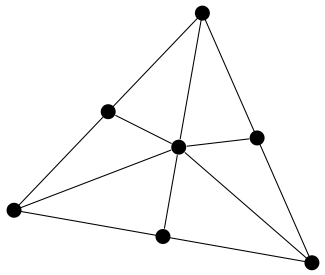


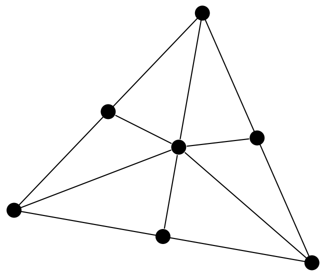


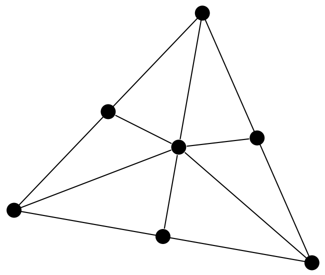


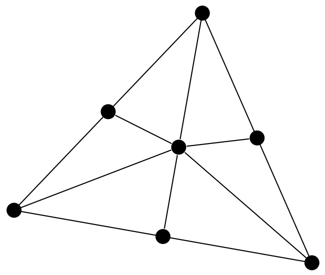


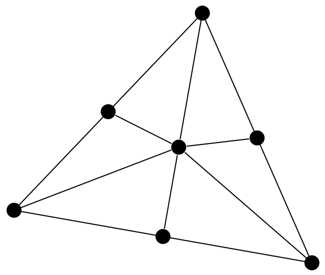




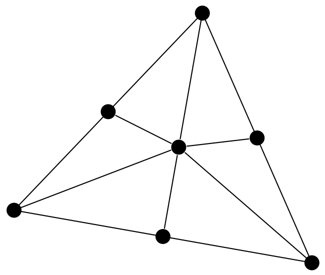




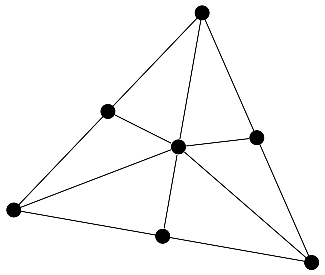




$$S(\lambda) = \hat{b}(S(\partial\lambda))$$



$$S(\lambda) = \hat{b}(S(\partial\lambda)) \text{ where } \hat{b}(\lambda \restriction_{[v_0, v_1, \ldots, v_k]}) = \lambda \restriction_{[b, v_0, v_1, \ldots, v_k]}$$



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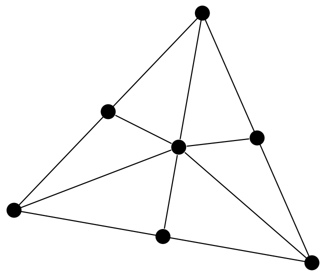

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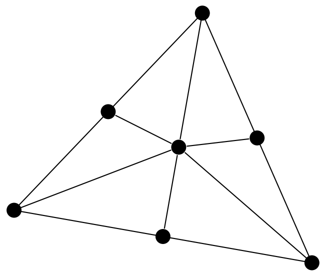

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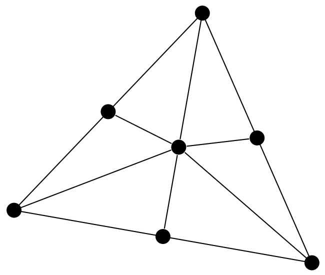

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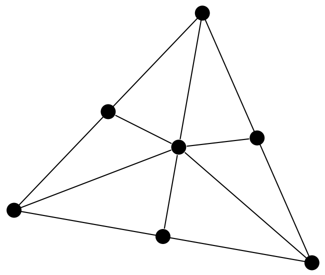

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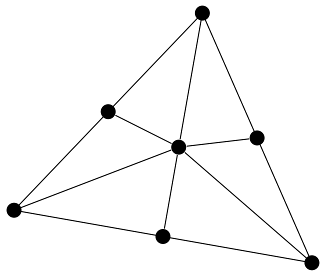

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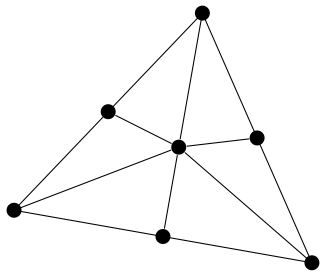

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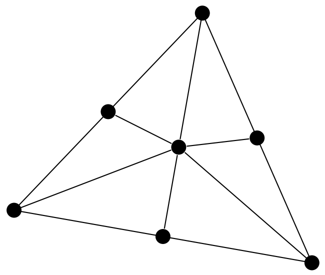

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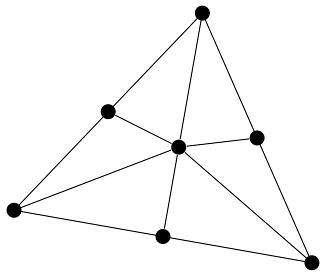
Extend to non-linear: $S(\sigma) = \sigma_{\#}(S(Id_{\Delta_n})$...



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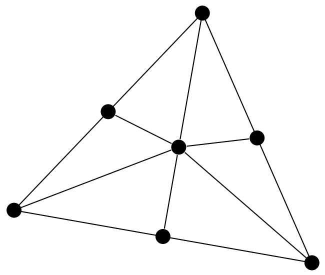

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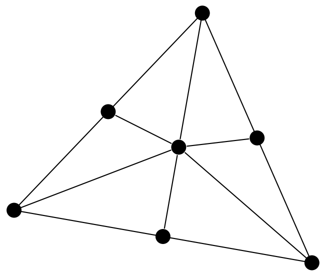

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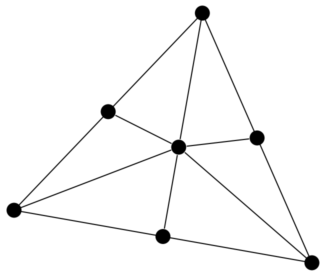

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$\partial_n \circ S_n = S_{n-1} \circ \partial_n$
$S'_n(\sigma) := \sigma_\#(S_n(Id_{\Delta_n}))$

$K \subset \mathbb{R}^k$ convex
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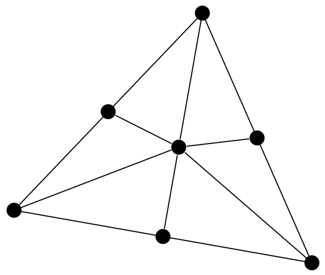

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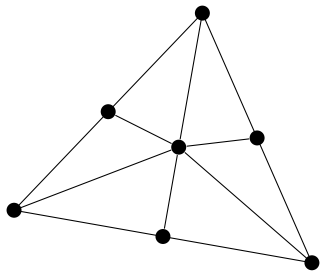

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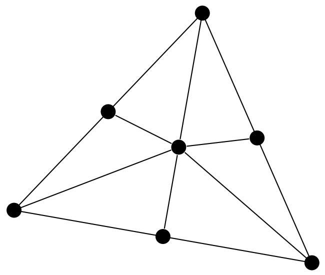

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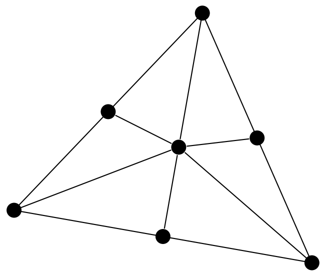

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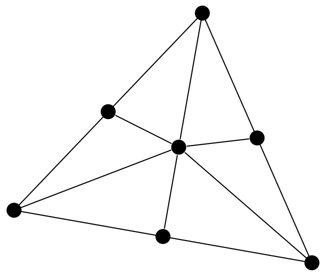

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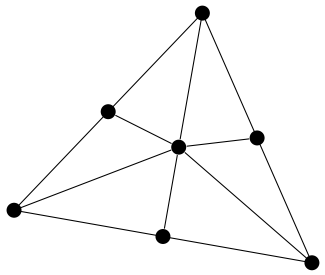

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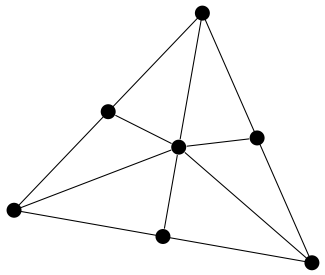

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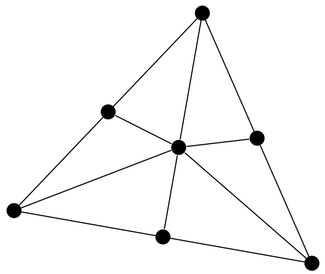
$$



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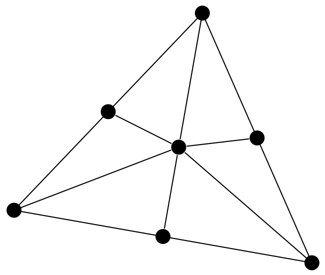

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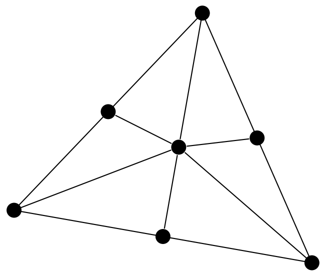
\end{aligned}$$

$$= S'_{n-1}(\partial\sigma)$$



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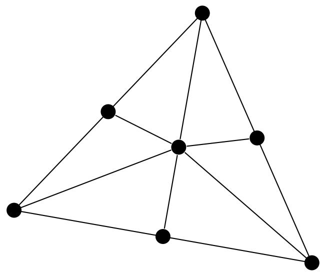

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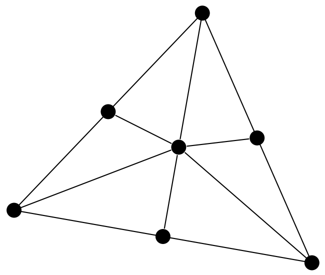

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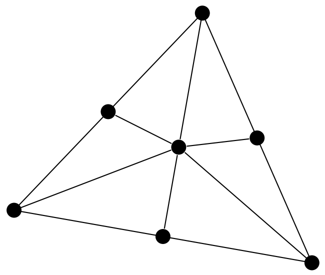

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\\
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&= \Sigma_i(-1)^i S'_{n-1}(\sigma \circ F_i) \\
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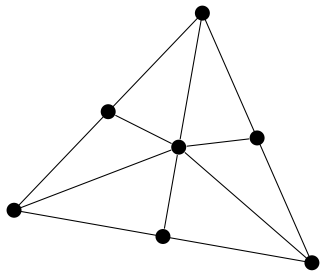

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&= \Sigma_i(-1)^i\sigma_\# S_{n-1}F_i i \\
&= \Sigma_i(-1)^i\sigma_\#((F_i)_\#(S_{n-1}Id_{\Delta_{n-1}})) \\
&= \Sigma_i(-1)^i(\sigma \circ F_i)_\# S_{n-1}Id_{\Delta_{n-1}} \\
&= \Sigma_i(-1)^i S'_{n-1}(\sigma \circ F_i) \\
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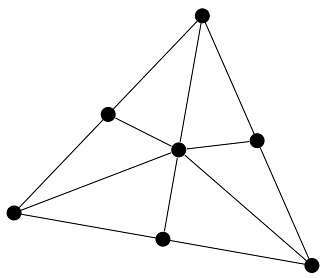
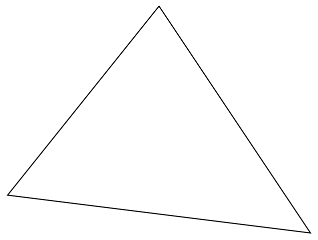
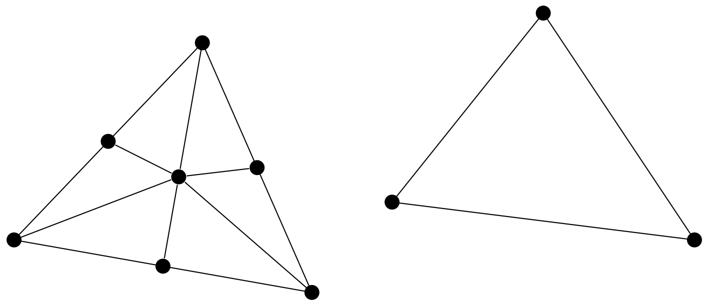
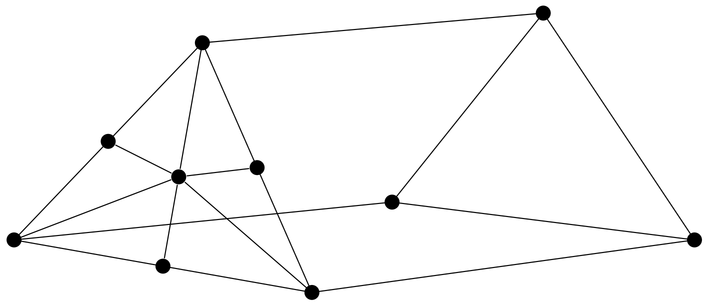
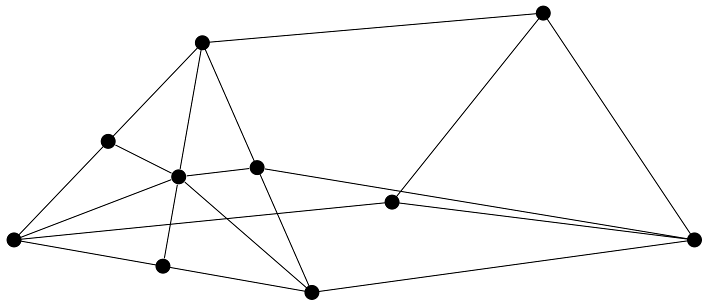
\end{aligned}$$

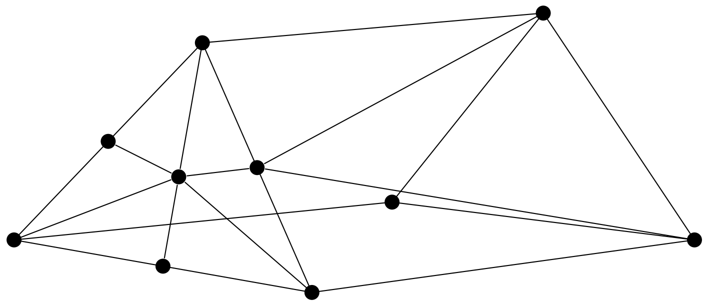


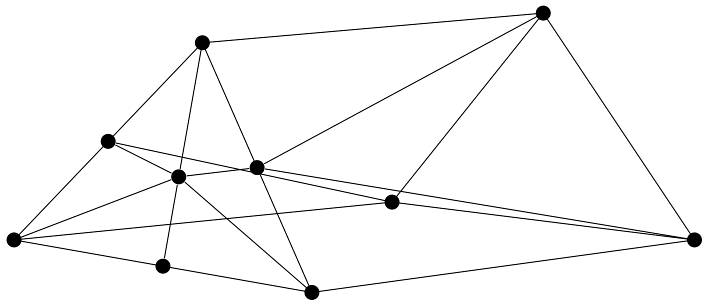


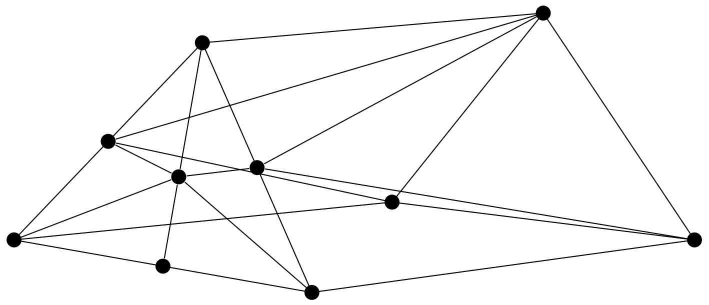


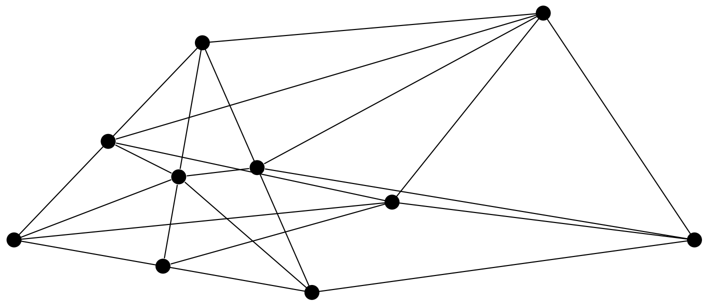


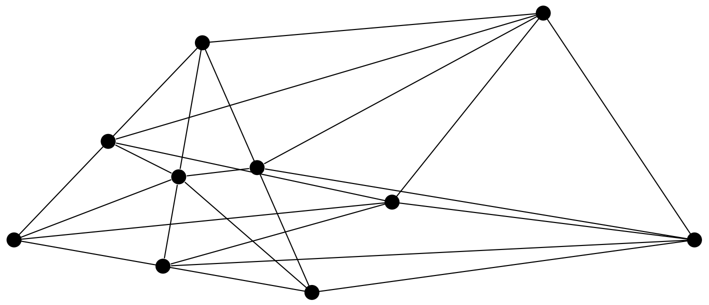


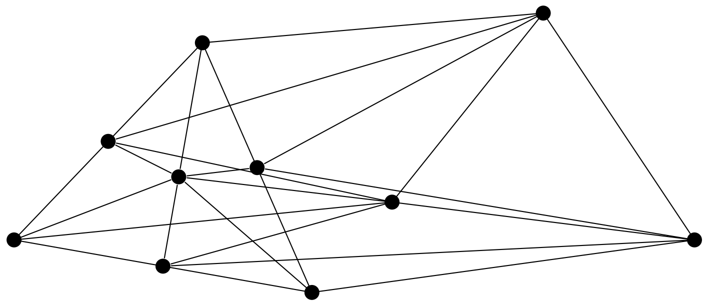


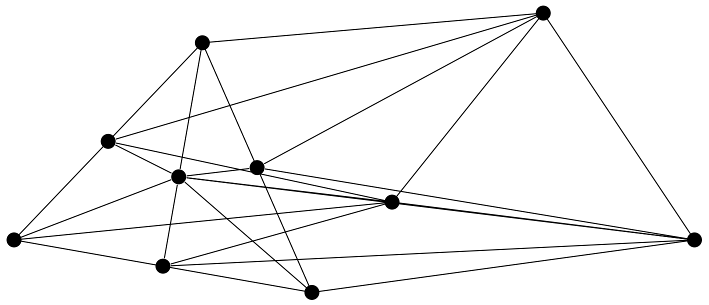


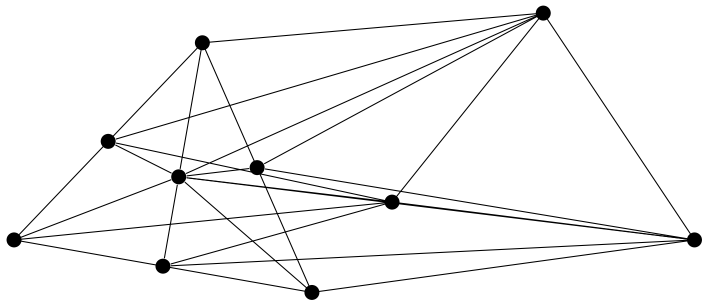


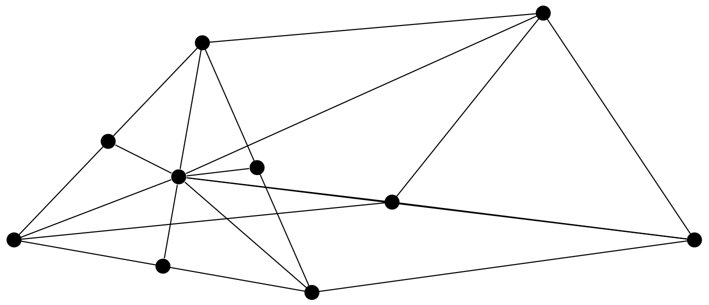


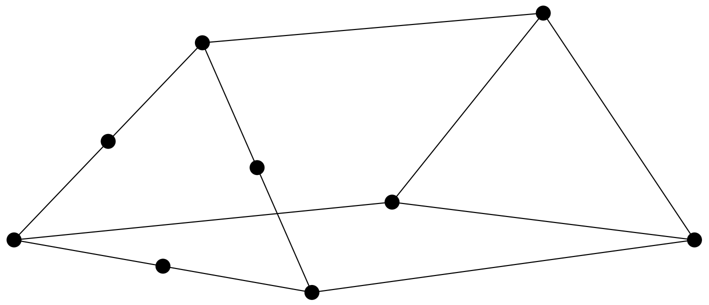


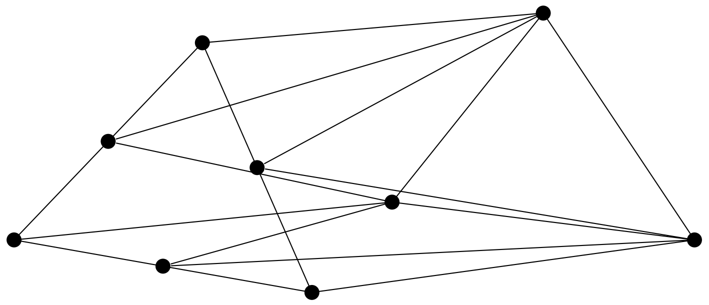


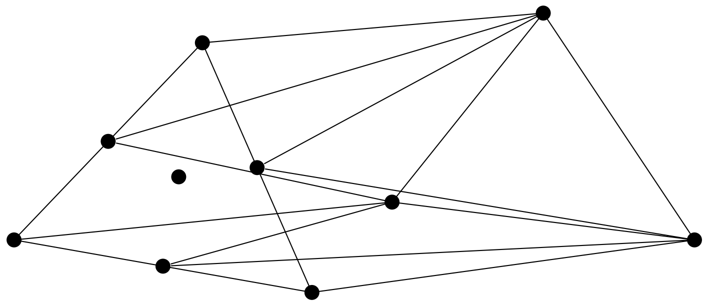


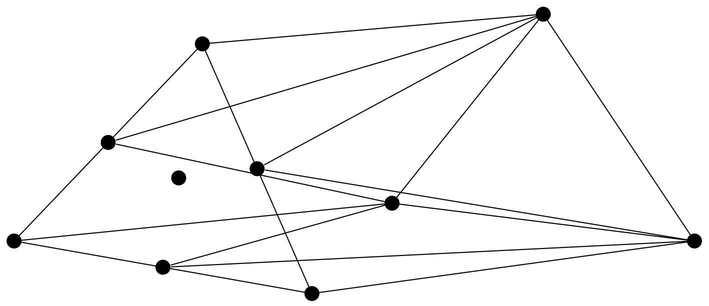


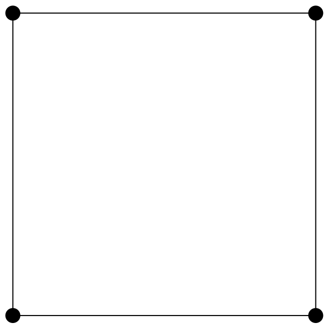


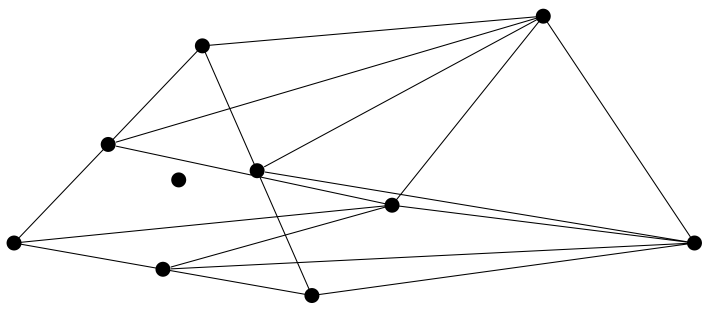


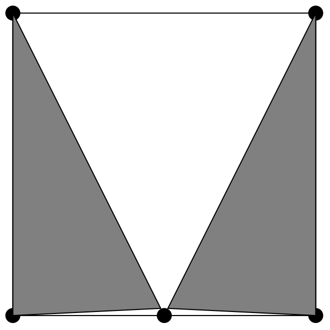


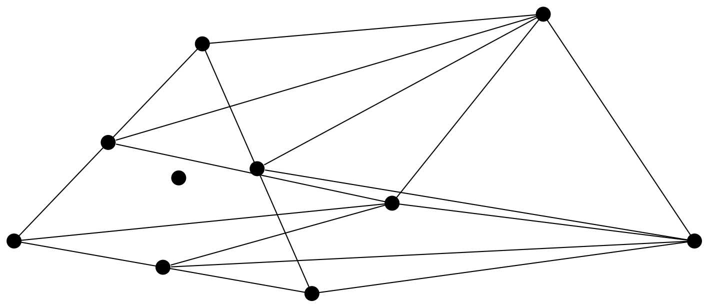


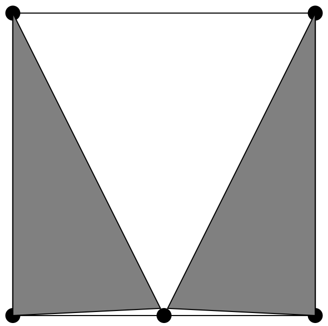


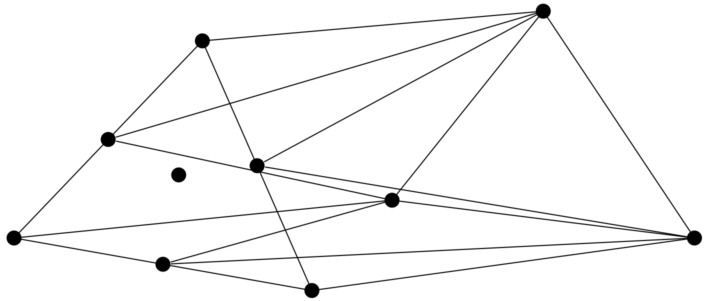


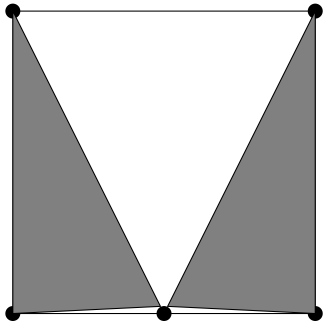




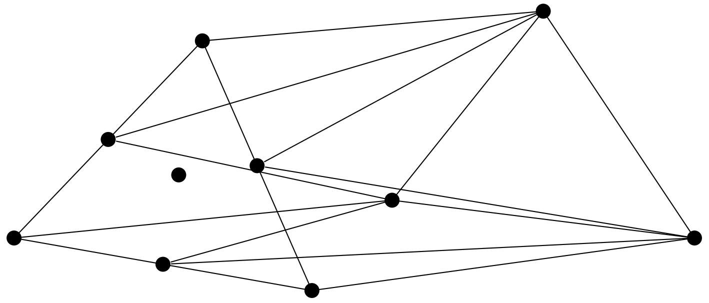


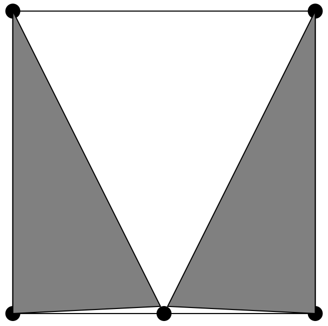




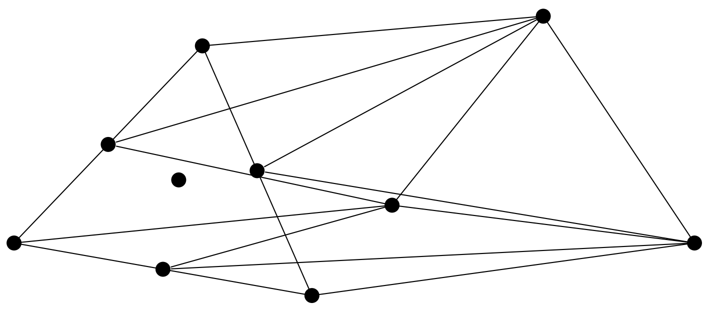


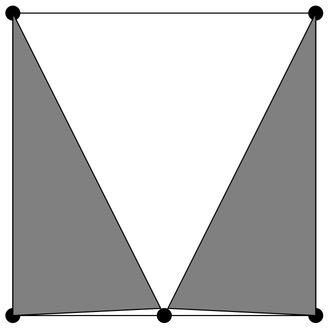
$$T(\lambda) := -\hat{b}(T\partial\lambda)+$$





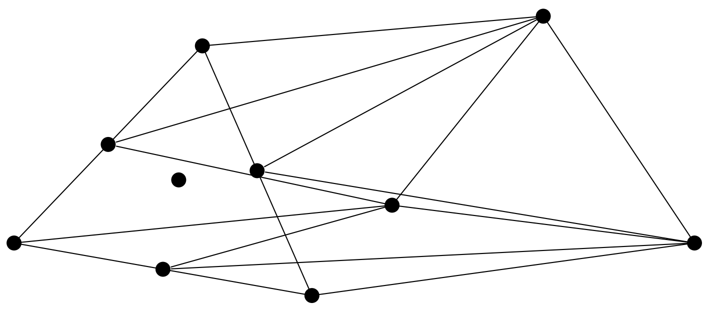
$$T(\lambda) := -\hat{b}(T\partial\lambda) + \hat{b}(\lambda)$$

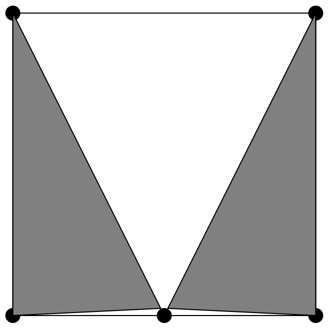




$$T(\lambda) := -\hat{b}(T\partial\lambda) + \hat{b}(\lambda)$$

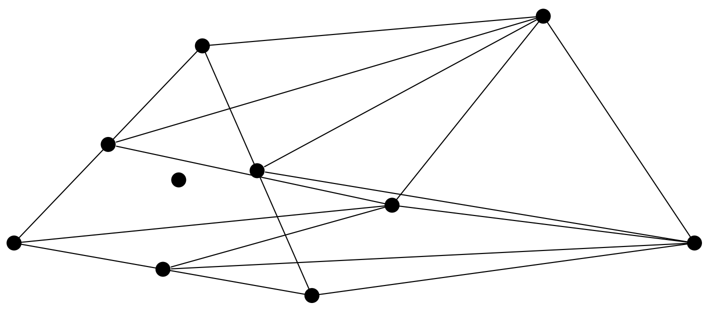
$$\partial T(\lambda) \;=\; \partial\hat{b}(\lambda) - \partial\hat{b}(T\partial\lambda)$$

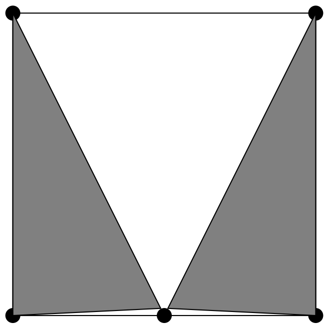




$$T(\lambda) := -\hat{b}(T\partial\lambda) + \hat{b}(\lambda)$$

$$\begin{aligned} \partial T(\lambda) &= \partial\hat{b}(\lambda) - \partial\hat{b}(T\partial\lambda) \\ &= \lambda - \hat{b}(\partial\lambda) - \partial\hat{b}(T\partial\lambda) \end{aligned}$$





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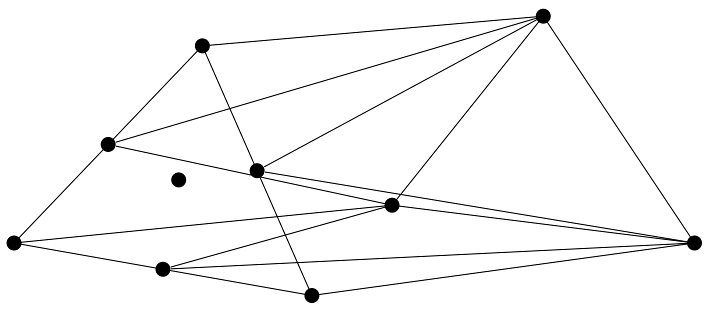
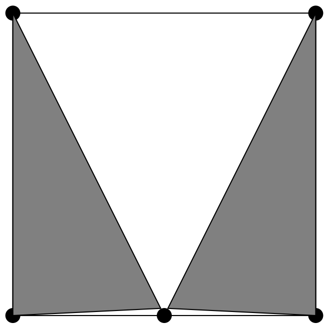
&= \lambda - \hat{b}(\partial\lambda) - T\partial\lambda + \hat{b}(\partial T\partial\lambda)
\end{aligned}$$





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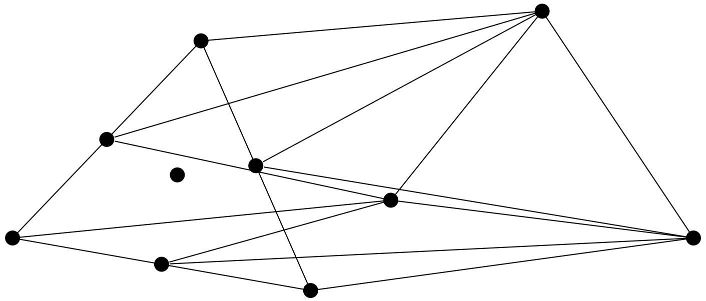
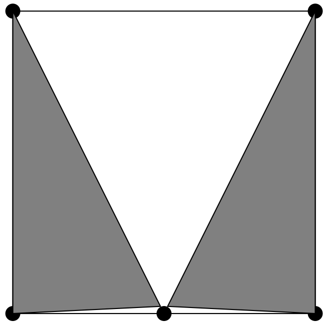
&= \lambda - \hat{b}(\partial\lambda) - T\partial\lambda + \hat{b}(\partial\lambda - S(\partial\lambda) - T(\partial\partial\lambda)
\end{aligned}$$





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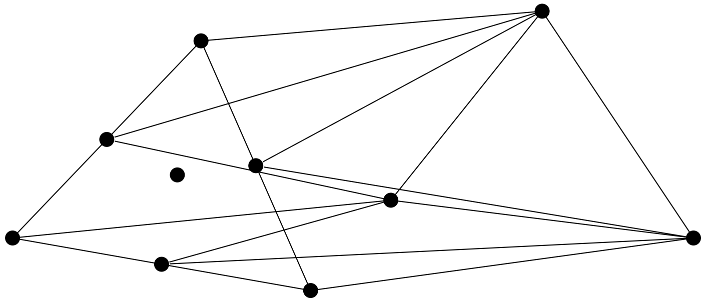

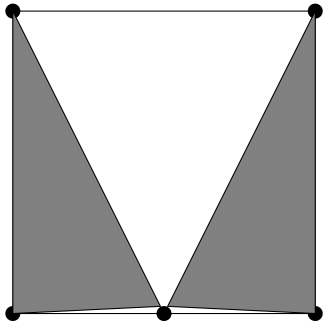

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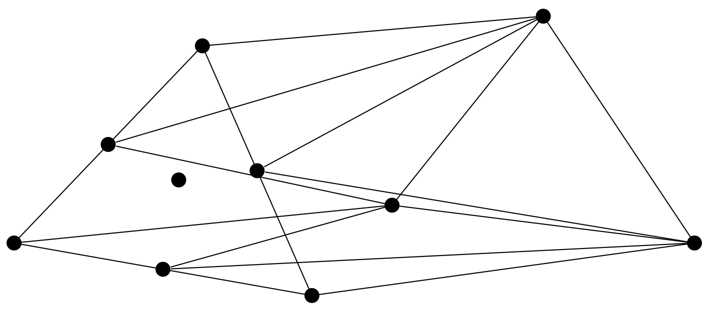
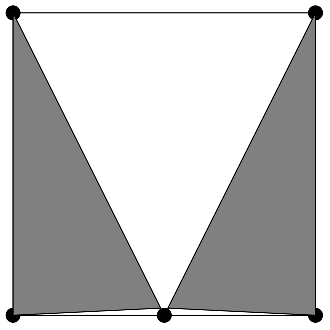
\end{aligned}$$





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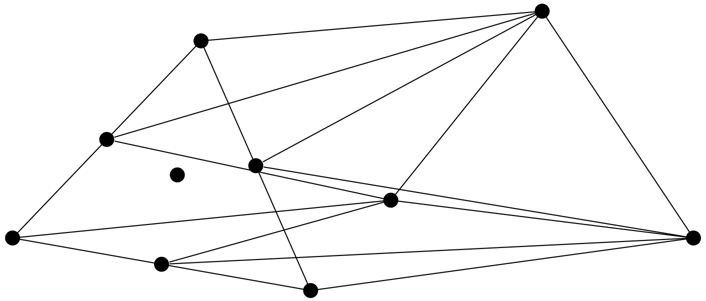
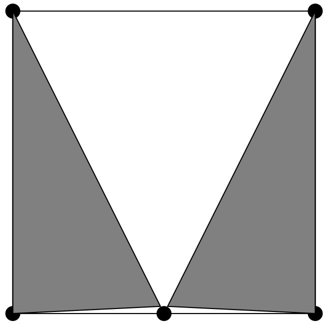
\end{aligned}$$





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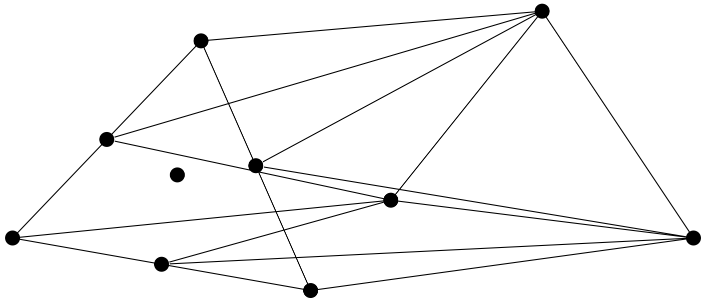

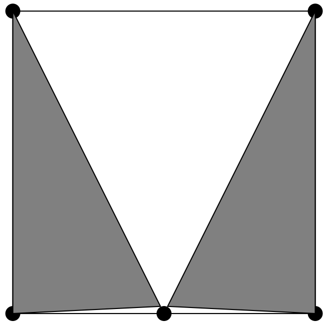

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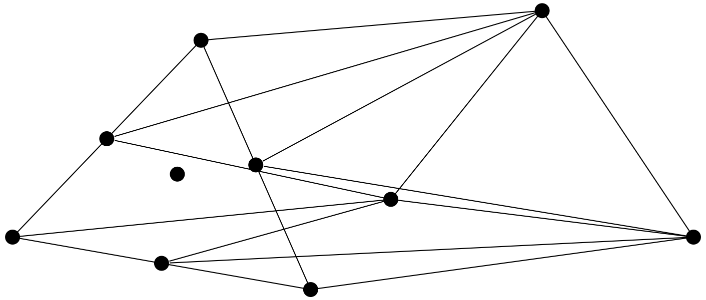
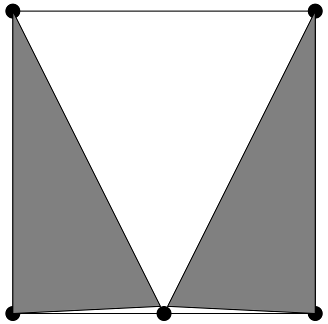
\end{aligned}$$





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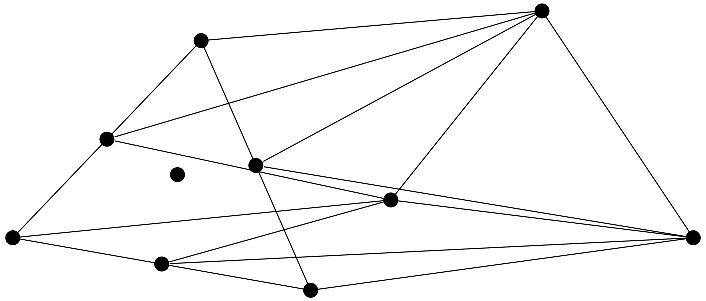
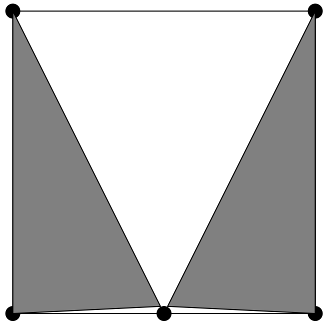
&= \lambda - \cancel{\hat{b}(\partial\lambda)} - T\partial\lambda + \cancel{\hat{b}(\partial\lambda)} - \underbrace{\hat{b}(S(\partial\lambda))}_{S(\lambda)}
\end{aligned}$$





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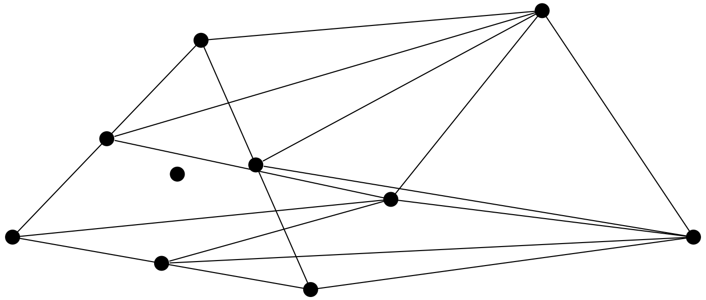
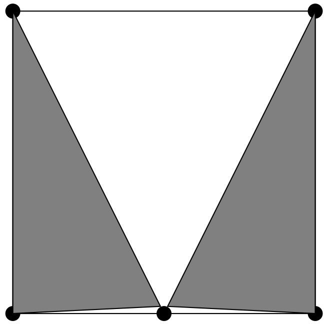
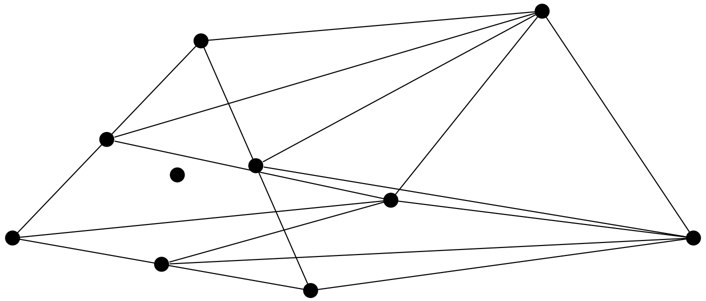
&= \lambda - \cancel{\hat{b}(\partial\lambda)} - T\partial\lambda + \cancel{\hat{b}(\partial\lambda)} - \underbrace{\hat{b}(S(\partial\lambda))}_{S(\lambda)}
\end{aligned}$$

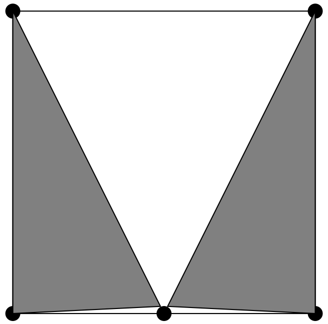
Extend to non-linear: $T(\sigma) = \sigma_{\#}(T(Id_{\Delta_n})...$





$$\partial T(\lambda) = \lambda - S\lambda - T(\partial\lambda)$$





$$\partial T(\lambda) = \lambda - S\lambda - T(\partial\lambda)$$
$$T'(\sigma) := \sigma_{\#}(T(Id_{\Delta_n}))$$

$K \subset \mathbb{R}^k$ convex

$\lambda : \Delta_n \to K$ linear (linear singular n-simplex)

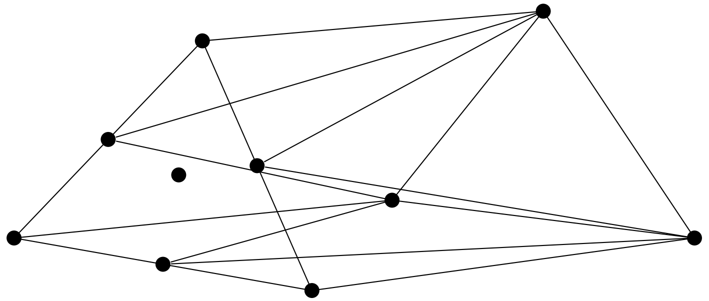
denoted $[v_0, v_1, \ldots, v_n]$ where $\lambda(e_i) = v_i$

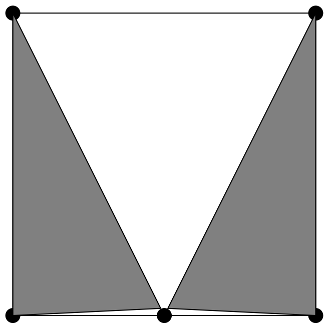
$\sigma = \sigma \circ Id_{\Delta_n} = \sigma_{\#}(Id_{\Delta_n})$

$F_i : \text{span}\{e_0, \ldots, e_{n-1}\} \to \text{span}\{e_0, \ldots, \hat{e}_i, \ldots, e_n\}$

$\sigma_i = \sigma \circ F_i = \sigma_{\#}(F_i)$

$\partial\sigma = \Sigma_i(-1)^i\sigma_{\#}(F_i)$





$$\partial T(\lambda) = \lambda - S\lambda - T(\partial\lambda)$$
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$$\partial T'(\sigma)$$

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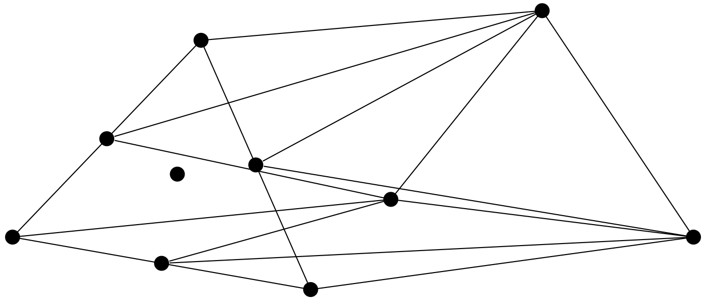

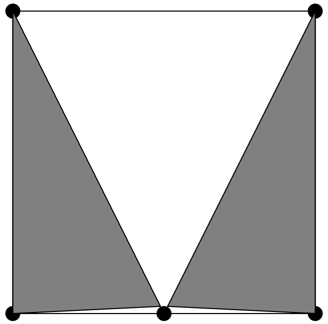

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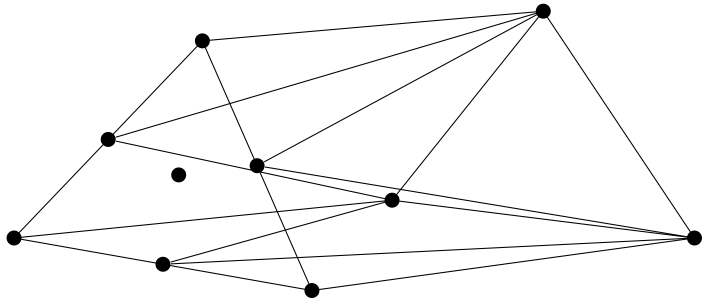

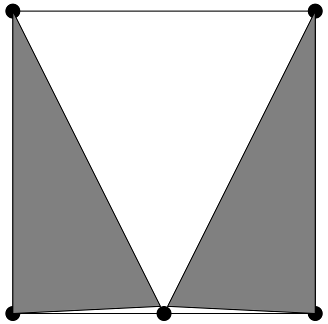

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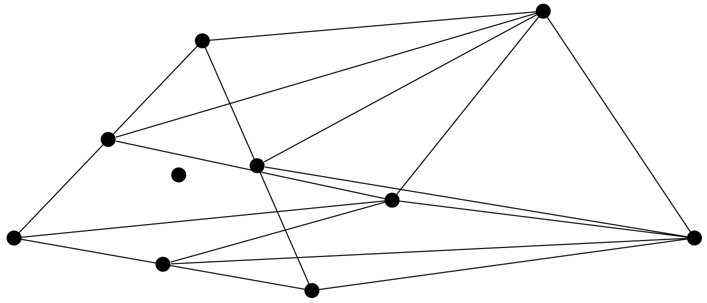

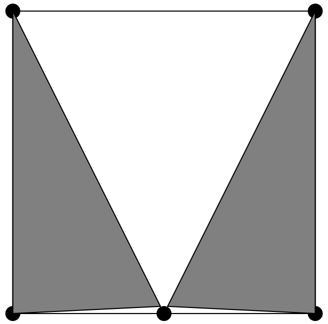

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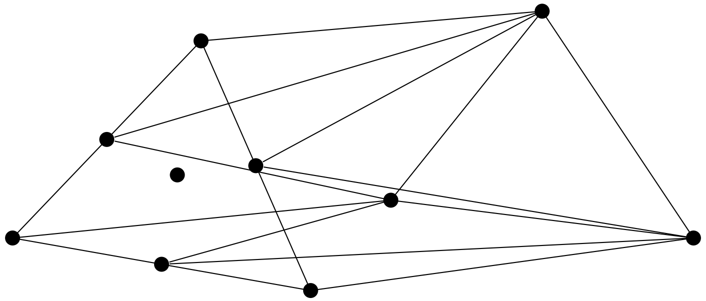

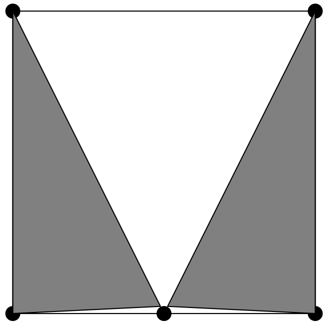

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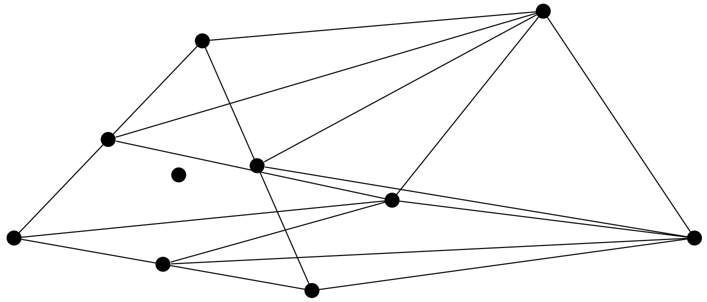

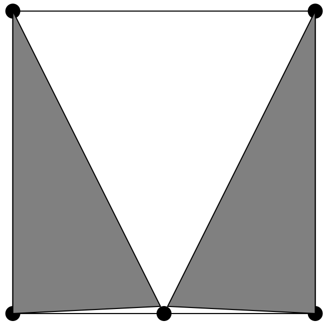

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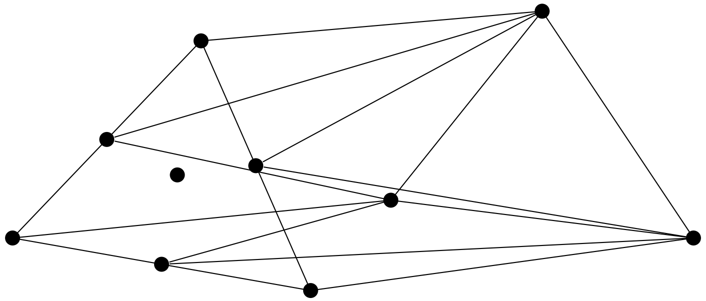

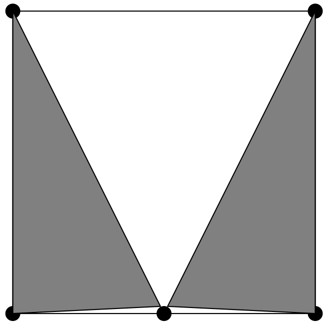

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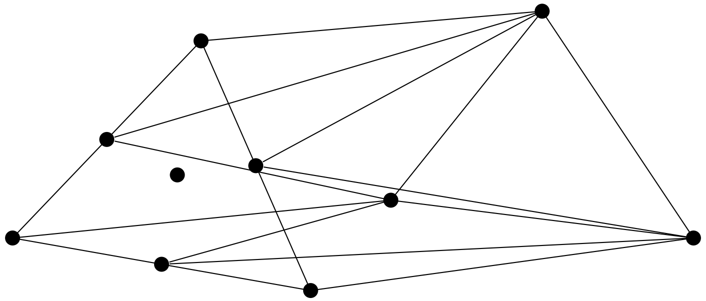

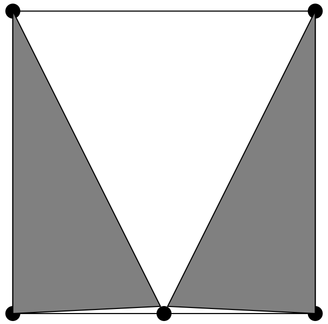

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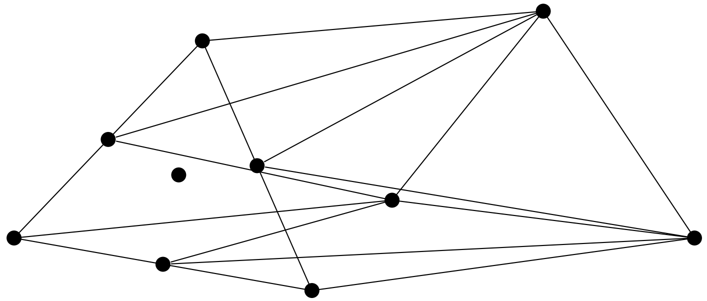

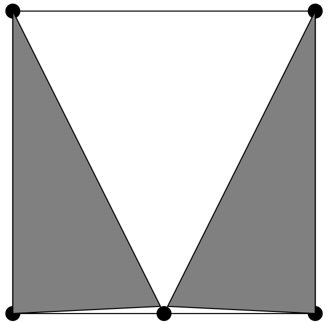

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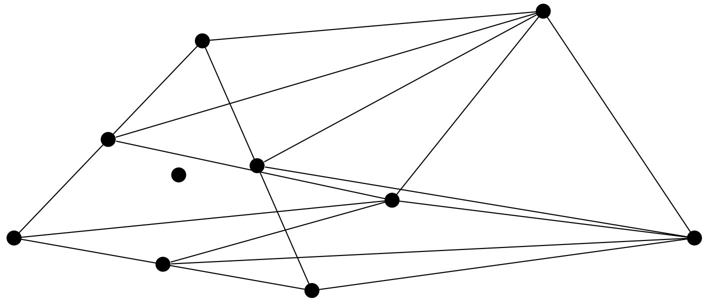

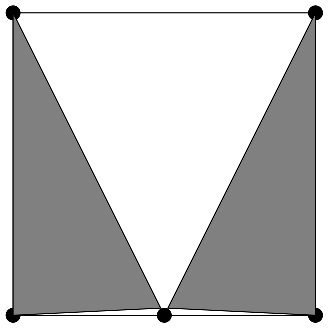

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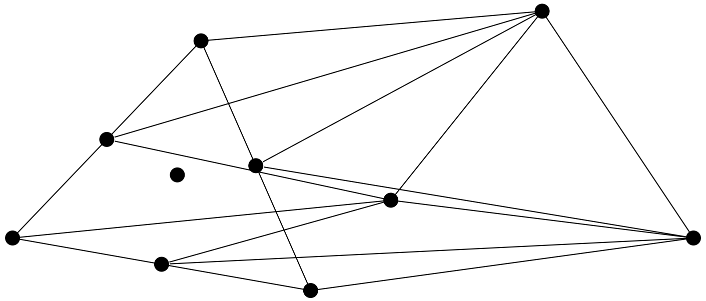

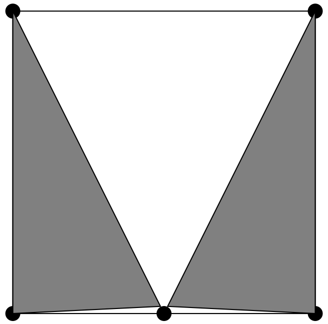

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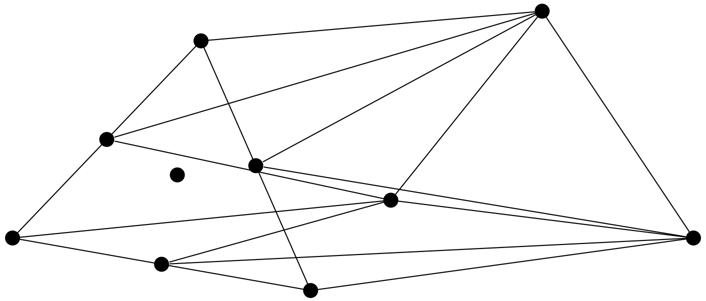

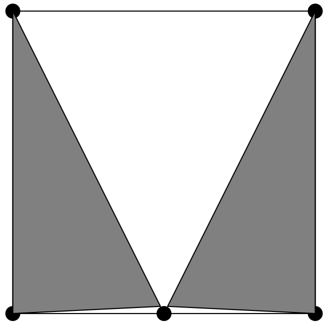

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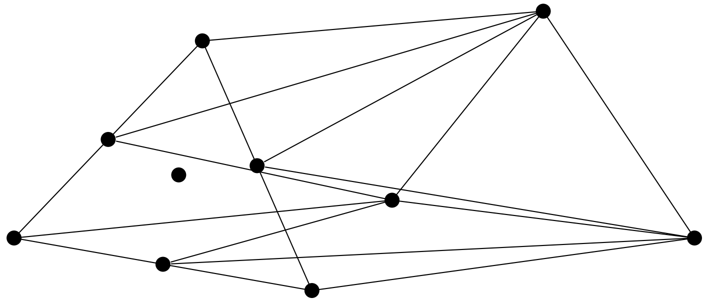

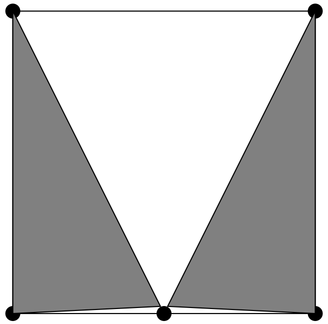

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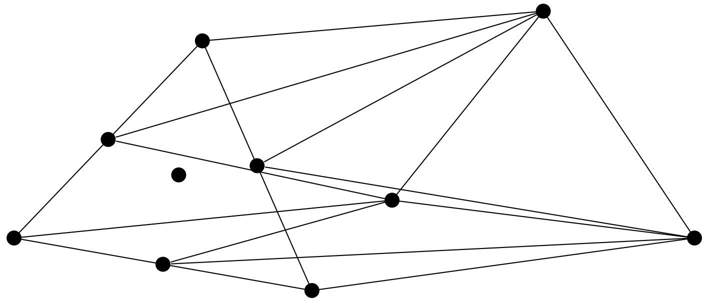
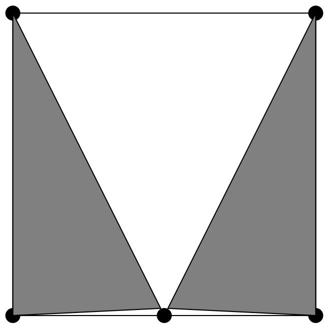
$$
\begin{aligned}
&= \sigma - S(\sigma) - \Sigma_i(-1)^i T'_{n-1}(\sigma \circ F_i) \\
&= \sigma - S(\sigma) - \Sigma_i(-1)^i T'_{n-1}\sigma_i \\
&= \sigma - S(\sigma) - T'_{n-1}(\Sigma_i(-1)^i \sigma_i) \\
&= \sigma - S(\sigma) - T'_{n-1}(\partial\sigma)
\end{aligned}
$$



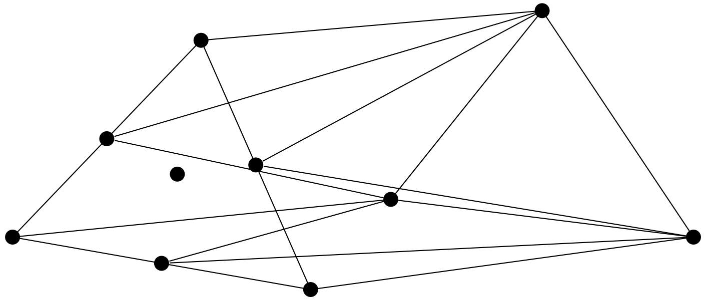


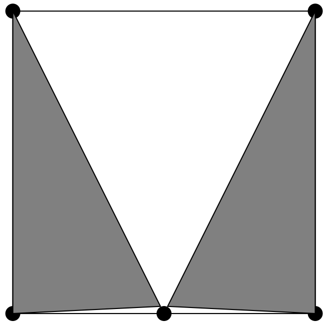
$K \subset \mathbb{R}^k$ convex
$\lambda : \Delta_n \to K$ linear (linear singular n-simplex)
denoted $[v_0, v_1, \ldots, v_n]$ where $\lambda(e_i) = v_i$
$\sigma = \sigma \circ Id_{\Delta_n} = \sigma_{\#}(Id_{\Delta_n})$
$F_i : \text{span}\{e_0, \ldots, e_{n-1}\} \to \text{span}\{e_0, \ldots, \hat{e}_i, \ldots, e_n\}$
$\sigma_i = \sigma \circ F_i = \sigma_{\#}(F_i)$
$\partial\sigma = \Sigma_i(-1)^i \sigma_{\#}(F_i)$

$\partial T(\lambda) = \lambda - S\lambda - T(\partial\lambda)$
$T'(\sigma) := \sigma_{\#}(T(Id_{\Delta_n}))$

$$\begin{aligned}
\partial T'(\sigma) &= \partial\sigma_{\#}(T_n(Id_{\Delta_n})) \\
&= \sigma_{\#}(\partial T_n(Id_{\Delta_n})) \\
&= \sigma_{\#}(Id_{\Delta_n} - S(Id_{\Delta_n}) - T_{n-1}\partial(Id_{\Delta_n})) \\
&= \sigma_{\#}(Id_{\Delta_n}) - \sigma_{\#}(S(Id_{\Delta_n})) - \sigma_{\#}(T_{n-1}\partial(Id_{\Delta_n})) \\
&= \sigma - S(\sigma) - \sigma_{\#}(T_{n-1}\Sigma_i(-1)^i Id_{\#}(F_i)) \\
&= \sigma - S(\sigma) - \sigma_{\#}(T_{n-1}\Sigma_i(-1)^i F_i) \\
&= \sigma - S(\sigma) - \sigma_{\#}(\Sigma_i(-1)^i T_{n-1}F_i) \\
&= \sigma - S(\sigma) - \Sigma_i(-1)^i \sigma_{\#} T_{n-1} F_i
\end{aligned}$$

$$\begin{aligned}
&= \sigma - S(\sigma) - \Sigma_i(-1)^i (\sigma \circ F_i)_{\#} T_{n-1} Id_{\Delta_{n-1}} \\
&= \sigma - S(\sigma) - \Sigma_i(-1)^i T'_{n-1}(\sigma \circ F_i) \\
&= \sigma - S(\sigma) - \Sigma_i(-1)^i T'_{n-1}\sigma_i \\
&= \sigma - S(\sigma) - T'_{n-1}(\Sigma_i(-1)^i \sigma_i) \\
&= \sigma - S(\sigma) - T'_{n-1}(\partial\sigma)
\end{aligned}$$





$K \subset \mathbb{R}^k$ convex
$\lambda : \Delta_n \to K$ linear (linear singular n-simplex)
denoted $[v_0, v_1, \ldots, v_n]$ where $\lambda(e_i) = v_i$
$\sigma = \sigma \circ Id_{\Delta_n} = \sigma_\#(Id_{\Delta_n})$
$F_i : \text{span}\{e_0, \ldots, e_{n-1}\} \to \text{span}\{e_0, \ldots, \hat{e}_i, \ldots, e_n\}$
$\sigma_i = \sigma \circ F_i = \sigma_\#(F_i)$
$\partial\sigma = \Sigma_i(-1)^i \sigma_\#(F_i)$

$$\partial T(\lambda) = \lambda - S\lambda - T(\partial\lambda)$$
$$T'(\sigma) := \sigma_\#(T(Id_{\Delta_n}))$$

$$\begin{aligned}
\partial T'(\sigma) &= \partial\sigma_\#(T_n(Id_{\Delta_n})) \\
&= \sigma_\#(\partial T_n(Id_{\Delta_n})) \\
&= \sigma_\#(Id_{\Delta_n} - S(Id_{\Delta_n}) - T_{n-1}\partial(Id_{\Delta_n})) \\
&= \sigma_\#(Id_{\Delta_n}) - \sigma_\#(S(Id_{\Delta_n})) - \sigma_\#(T_{n-1}\partial(Id_{\Delta_n})) \\
&= \sigma - S(\sigma) - \sigma_\#(T_{n-1}\Sigma_i(-1)^i Id_\#(F_i)) \\
&= \sigma - S(\sigma) - \sigma_\#(T_{n-1}\Sigma_i(-1)^i F_i) \\
&= \sigma - S(\sigma) - \sigma_\#(\Sigma_i(-1)^i T_{n-1}F_i) \\
&= \sigma - S(\sigma) - \Sigma_i(-1)^i \sigma_\# T_{n-1}F_i \\
&= \sigma - S(\sigma) - \Sigma_i(-1)^i \sigma_\#((F_i)_\#(T_{n-1}Id_{\Delta_{n-1}})) \\
&= \sigma - S(\sigma) - \Sigma_i(-1)^i (\sigma \circ F_i)_\# T_{n-1}Id_{\Delta_{n-1}} \\
&= \sigma - S(\sigma) - \Sigma_i(-1)^i T'_{n-1}(\sigma \circ F_i) \\
&= \sigma - S(\sigma) - \Sigma_i(-1)^i T'_{n-1}\sigma_i \\
&= \sigma - S(\sigma) - T'_{n-1}(\Sigma_i(-1)^i \sigma_i) \\
&= \sigma - S(\sigma) - T'_{n-1}(\partial\sigma)
\end{aligned}$$

$$\sigma - S\sigma = \partial T\sigma + T\partial\sigma$$

$$\sigma - S\sigma = \partial T\sigma + T\partial\sigma$$
$$S\sigma - S^2\sigma = \partial TS\sigma + T\partial(S\sigma)$$

$$\sigma - S\sigma = \partial T\sigma + T\partial\sigma$$
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$$\sigma - S^n\sigma = \partial(\Sigma_{i=0}^n TS^i)\sigma + (\Sigma_{i=0}^n TS^i)\partial(\sigma)$$

$$\sigma - S\sigma = \partial T\sigma + T\partial\sigma$$
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$$\sigma - S^n\sigma = \partial\underbrace{(\Sigma_{i=0}^{n} TS^i)}_{D_n}\sigma + \underbrace{(\Sigma_{i=0}^{n} TS^i)}_{D_n}\partial(\sigma)$$

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$$\sigma - S\sigma = \partial T\sigma + T\partial\sigma$$

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$$D(\sigma) := D_{m(\sigma)}(\sigma)$$

$$\sigma - S\sigma = \partial T\sigma + T\partial\sigma$$
$$S\sigma - S^2\sigma = \partial TS\sigma + TS\partial(\sigma)$$
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$$\boxed{\sigma - S^n\sigma = \partial(\underbrace{\Sigma_{i=0}^n TS^i}_{D_n})\sigma + (\underbrace{\Sigma_{i=0}^n TS^i}_{D_n})\partial(\sigma)}$$

$D(\sigma) := D_{m(\sigma)}(\sigma)$ where $m(\sigma)$ is the smallest m such that $S_m(\sigma) \in C_n^{\mathfrak{U}}(X)$

$$\partial D(\sigma)$$

$$\sigma - S\sigma = \partial T\sigma + T\partial\sigma$$
$$S\sigma - S^2\sigma = \partial TS\sigma + TS\partial(\sigma)$$
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$D(\sigma) := D_{m(\sigma)}(\sigma)$ where $m(\sigma)$ is the smallest m such that $S_m(\sigma) \in C_n^{\mathfrak{U}}(X)$

$$\partial D(\sigma) = \partial D_{m(\sigma)}(\sigma)$$

$$\sigma - S\sigma = \partial T\sigma + T\partial\sigma$$
$$S\sigma - S^2\sigma = \partial TS\sigma + TS\partial(\sigma)$$
$$S^2\sigma - S^3\sigma = \partial TS^2\sigma + TS^2\partial(\sigma)$$
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$$\begin{aligned} \partial D(\sigma) &= \partial D_{m(\sigma)}(\sigma) \\ &= Id(\sigma) - S^{m(\sigma)}(\sigma) - D_{m(\sigma)}(\partial\sigma) \end{aligned}$$

$$\sigma - S\sigma = \partial T\sigma + T\partial\sigma$$
$$S\sigma - S^2\sigma = \partial TS\sigma + TS\partial(\sigma)$$
$$S^2\sigma - S^3\sigma = \partial TS^2\sigma + TS^2\partial(\sigma)$$
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$$\begin{aligned} \partial D(\sigma) &= \partial D_{m(\sigma)}(\sigma) \\ &= Id(\sigma) - S^{m(\sigma)}(\sigma) - \underbrace{D_{m(\sigma)}}_{\neq D}(\partial\sigma) \end{aligned}$$

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$$\begin{aligned}\partial D(\sigma) &= \partial D_{m(\sigma)}(\sigma)\\ &= Id(\sigma) - S^{m(\sigma)}(\sigma) - \underbrace{D_{m(\sigma)}}_{\neq D}(\partial\sigma)\end{aligned}$$

$$\boxed{D_{m(\sigma)}(\tau) - D_{m(\tau)}(\tau)}$$

$$\sigma - S\sigma = \partial T\sigma + T\partial\sigma$$
$$S\sigma - S^2\sigma = \partial TS\sigma + TS\partial(\sigma)$$
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$$\boxed{D_{m(\sigma)}(\tau) - D_{m(\tau)}(\tau) = \Sigma_{i=m(\tau)}^{m(\sigma)} TS^i(\tau)}$$

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$$\boxed{D_{m(\sigma)}(\tau) = \Sigma_{i=m(\tau)}^{m(\sigma)} TS^i(\tau) + D_{m(\tau)}(\tau)}$$

$D(\sigma) := D_{m(\sigma)}(\sigma)$ where $m(\sigma)$ is the smallest m such that $S_m(\sigma) \in C_n^{\mathfrak{U}}(X)$

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$$\begin{aligned} \partial D(\sigma) &= \partial D_{m(\sigma)}(\sigma) \\ &= Id(\sigma) - S^{m(\sigma)}(\sigma) - (\alpha + D)(\partial\sigma) \end{aligned}$$

$$\boxed{D_{m(\sigma)}(\tau) = \underbrace{\Sigma_{i=m(\tau)}^{m(\sigma)} TS^i}_{\alpha}(\tau) + D(\tau)}$$

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$$\begin{aligned} \partial D(\sigma) &= \partial D_{m(\sigma)}(\sigma) \\ &= Id(\sigma) - S^{m(\sigma)}(\sigma) - \alpha(\partial\sigma) - D(\partial\sigma) \end{aligned}$$

$$\boxed{D_{m(\sigma)}(\tau) = \underbrace{\Sigma_{i=m(\tau)}^{m(\sigma)} TS^i}_{\alpha}(\tau) + D(\tau)}$$

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$$\boxed{D_{m(\sigma)}(\tau) = \underbrace{\Sigma_{i=m(\tau)}^{m(\sigma)} TS^i}_{\alpha}(\tau) + D(\tau)}$$

$D(\sigma) := D_{m(\sigma)}(\sigma)$ where $m(\sigma)$ is the smallest m such that $S_m(\sigma) \in C_n^{\mathfrak{U}}(X)$

$$\begin{aligned} \partial D(\sigma) &= \partial D_{m(\sigma)}(\sigma) \\ &= Id(\sigma) - \rho(\sigma) - D(\partial\sigma) \end{aligned}$$

where

$$\rho(\sigma) := S^{m(\sigma)}(\sigma) + \alpha(\partial\sigma)$$

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$$S\sigma - S^2\sigma = \partial TS\sigma + TS\partial(\sigma)$$
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where

$$\begin{aligned} \rho(\sigma) &:= S^{m(\sigma)}(\sigma) + \alpha(\partial\sigma) \\ &= S^{m(\sigma)}(\sigma) + D_{m(\sigma)}(\partial\sigma) - D(\partial\sigma) \end{aligned}$$

$$\boxed{D_{m(\sigma)}(\tau) = \underbrace{\Sigma_{i=m(\tau)}^{m(\sigma)} TS^i}_{\alpha}(\tau) + D(\tau)}$$

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$D(\sigma) := D_{m(\sigma)}(\sigma)$ where $m(\sigma)$ is the smallest m such that $S_m(\sigma) \in C_n^{\mathfrak{U}}(X)$

$$\begin{aligned} \partial D(\sigma) &= \partial D_{m(\sigma)}(\sigma) \\ &= Id(\sigma) - \rho(\sigma) - D(\partial\sigma) \end{aligned}$$

where

$$\begin{aligned} \rho(\sigma) &:= S^{m(\sigma)}(\sigma) + \alpha(\partial\sigma) \\ &= S^{m(\sigma)}(\sigma) + D_{m(\sigma)}(\partial\sigma) - D(\partial\sigma) \\ &= \sigma - \partial D(\sigma) - D(\partial\sigma) \end{aligned}$$

$$\boxed{D_{m(\sigma)}(\tau) = \underbrace{\Sigma_{i=m(\tau)}^{m(\sigma)} TS^i}_{\alpha}(\tau) + D(\tau)}$$

$$\sigma - S\sigma = \partial T\sigma + T\partial\sigma$$
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Proposition. $\partial : C_n^{\mathfrak{U}}(X) \to C_{n-1}^{\mathfrak{U}}(X)$ *well defined*

$$\sigma - S\sigma = \partial T\sigma + T\partial\sigma$$
$$S\sigma - S^2\sigma = \partial TS\sigma + TS\partial(\sigma)$$
$$S^2\sigma - S^3\sigma = \partial TS^2\sigma + TS^2\partial(\sigma)$$
$$\sigma - S^3\sigma = \partial(T + TS + TS^2)\sigma + (T + TS + TS^2)\partial(\sigma)$$

$$\boxed{\sigma - S^n\sigma = \partial(\underbrace{\Sigma_{i=0}^{n} TS^i}_{D_n})\sigma + (\underbrace{\Sigma_{i=0}^{n} TS^i}_{D_n})\partial(\sigma)}$$

Proposition. $\partial : C_n^{\mathfrak{U}}(X) \to C_{n-1}^{\mathfrak{U}}(X)$ *well defined*

$H_n^{\mathfrak{U}}(X) := ker\ \partial / Im\ \partial$

$$\sigma - S\sigma = \partial T\sigma + T\partial\sigma$$
$$S\sigma - S^2\sigma = \partial TS\sigma + TS\partial(\sigma)$$
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j is an isomorphism (2nd isomorphism theorem)

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i_* is an isomorphism by the last proposition

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Proof. $B = X \setminus Z$ … (Exercise!) □