Fourier Transform

Sejin Nam
University of Hawaii at Manoa

Preface

Fourier transform is a mathematical transformation, in the context of my exposition, that maps one function in time domain into another function in frequency domain. There are two types of FT, Fourier transform in short: continuous and discrete. Discrete FT is of great importance in many modern data processing application as modern computers are digital, meaning they perform computation in discrete manner. In this article, I'll discuss a continous case first before moving onto the discrete case.

Contents

1   Prerequisite Mathematics

A few prerequisite mathematics will be discussed here first in anticipation of their requirements in the discussion of subsequent sections of Fourier transform. In this article, it is implied that all the functions discussed are $\mathbb{R} \mapsto \mathbb{C}$, infinitely differentiable, and absolutely integrable unless stated otherwise.

1.1   Dirac Delta Function

The Dirac Delta function, $\delta(t)$, has the following property:

$$\delta(t) = \begin{cases} \infty, & \text{if } t = 0 \\ 0, & \text{otherwise} \end{cases} \tag{1.1}$$

$$1 = \int_{-\infty}^{\infty} \delta(t)dt \tag{1.2}$$

(In my discussion of function, t variable represents time physically). The following integral is also the Dirac Delta function:

$$\delta(t) = \int_{-\infty}^{\infty} e^{2\pi i f t} df \tag{1.3}$$

The above relations will be used throughout the discussion of this article.

1.2   Sinusoidal exponential decay function

Sinusoidal exponential decay function (I call it SED) is as follows:

$$x = x_0 \cos{(2\pi f_0 t)} e^{-t/T} \theta(t) \tag{1.4}$$

Where $\theta(t)$ is a Heaviside step function (the plot of such function is given in figure 1.1). You can have sine instead of cosine in the above expression. For our discussion, SED of choice is combination of sine and cosine exponential decays in complex form:

$$\begin{aligned} x &= x_0 \cos{(2\pi f_0 t)} e^{-t/T} \theta(t) + i x_0 \sin{(2\pi f_0 t)} e^{-t/T} \theta(t) \\ &= (\cos{(2\pi f_0 t)} + i \sin{(2\pi f_0 t)}) x_0 e^{-t/T} \theta(t) \\ &= x_0 e^{2\pi i f_0 t} e^{-t/T} \theta(t) \end{aligned} \tag{1.5}$$
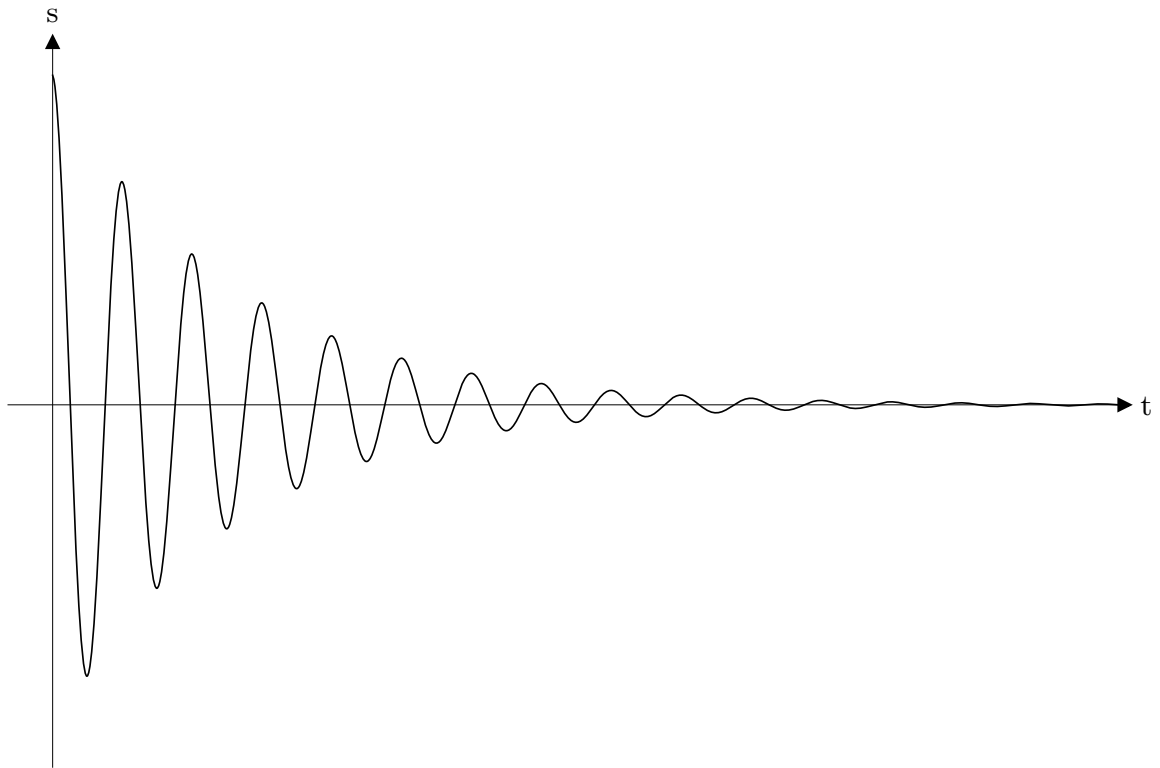
The real part of (1.5) is simply (1.4).

1.3   Sampling function

The sampling function, denoted as $u(t)$, with the sampling interval Δt is as follows:

$$u(t) = \Delta t \sum_{n=-\infty}^{\infty} \delta(t - n\Delta t) \tag{1.6}$$

The function is periodic with the period being the sampling interval.

Figure 1.1: sed function plot, s vs. t

2   Continuous Fourier Transform

2.1   Definition of Fourier transform

Consider a continuous function of t, $x(t)$. The continuous FT of x, $X(f)$, is as follows:

$$X(f) = \mathcal{F}_t[x]$$
$$= \int_{-\infty}^{\infty} x(t)e^{-2\pi i f t} dt \tag{2.1}$$

$X(f)$ is also continuous in f (in my discussion t and f represent time and frequency respectively). $\mathcal{F}_t[x](f)$ notation can be used to explicitly indicate that x is transformed into f domain. It is easy to see that if a is a constant, then $\mathcal{F}_t[ax] = a\mathcal{F}_t[x]$. You can also perform inverse FT on X, which is:

$$\mathcal{F}_f[X] = \int_{-\infty}^{\infty} X(f)e^{2\pi i f t} df$$
$$= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} x(\tau)e^{-2\pi i f \tau} d\tau \right) e^{2\pi i f t} df$$
$$= \int_{-\infty}^{\infty} x(\tau) \int_{-\infty}^{\infty} e^{2\pi i f (t-\tau)} df d\tau$$
$$= \int_{-\infty}^{\infty} x(\tau)\delta(t - \tau) d\tau$$
$$= x(t) \tag{2.2}$$

where I used (1.3) orthogonality relation (We will continue to use the notation $x(t)$ and $X(f)$ as an example of FT pair throughout the discussion). It is also easy to see that the inverse FT of one is the Dirac Delta function.

2.2   FT with the extra phase factor

Consider another function, $e^{2\pi i f_0 t}x(t)$. The FT of it is:

$$\mathcal{F}_t[e^{2\pi i f_0 t}x(t)] = \int_{-\infty}^{\infty} e^{2\pi i f_0 t}x(t)e^{-2\pi i f t} dt$$
$$= \int_{-\infty}^{\infty} x(t)e^{-2\pi i (f-f_0)t} dt$$
$$= \mathcal{F}_t[x](f - f_0)$$
$$= X(f - f_0) \tag{2.3}$$

It is simply the FT of x in $f - f_0$ in domain.

2.3 FT of SED

Consider a SED function discussed before, $x = x_0 e^{2\pi i f_0 t} e^{-t/T} \theta(t)$. The FT of x is (with $1/T = r$) as follows:

$$
\begin{aligned}
X(f) &= \mathcal{F}_t[x] \\
&= \mathcal{F}_t[x_0 e^{2\pi i f_0 t} e^{-t/T} \theta(t)] \\
&= s_0 \mathcal{F}_t[e^{-rt} \theta(t)](f - f_0) \\
&= \frac{s_0}{r + 2\pi i(f - f_0)} \\
&= s_0 \frac{r - 2\pi i(f - f_0)}{r^2 + 4\pi^2 (f - f_0)^2} \\
&= s_0 \left[\frac{r}{r^2 + 4\pi^2 (f - f_0)^2} + i \frac{2\pi(f_0 - f)}{r^2 + 4\pi^2 (f - f_0)^2} \right]
\end{aligned}
\tag{2.4}
$$

2.4 FT of the sampling function

Suppose $u(t)$ is the sampling function with the sampling interval Δt. Then the FT of u is:

$$
\begin{aligned}
\mathcal{F}_f[u] &= \int_{-\infty}^{\infty} u(t) e^{-2\pi i f t} dt \\
&= \int_{-\infty}^{\infty} \Delta t \sum_{n=-\infty}^{\infty} \delta(t - n\Delta t) e^{-2\pi i f t} dt \\
&= \Delta t \sum_{n=-\infty}^{\infty} e^{-2\pi i f n \Delta t} \int_{-\infty}^{\infty} \delta(t - n\Delta t) dt \\
&= \Delta t \sum_{n=-\infty}^{\infty} e^{-2\pi i f n \Delta t}
\end{aligned}
\tag{2.5}
$$