Fourier Transform

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Preface

Fourier transform is a mathematical transformation, in the context of my exposition, that maps one function in time domain into another function in frequency domain. In order for a function to have its Fourier transform, the function needs to be Schwartz. However, in Physics, we extend the notion of function to a more generalized case, called a generalized function. In a generalized function space, we allow more flexibility so that functions like Dirac delta function can have the Fourier transform even though such function is not Riemann integrable. In this article I will assume that every function that is discussed has its Fourier transform. I will call Fourier transform FT in short, and there are two types of FT: continuous and discrete. Discrete FT is of great importance in many modern data processing application as data can only be recorded in discrete and finite amount. Also, modern computers are digital, meaning they perform computation in discrete manner. In this article, I'll discuss a continous case first before moving onto the discrete case.

Contents

1 Prerequisite Mathematics

A few prerequisite mathematics will be discussed here first in anticipation of their requirements in the discussion of subsequent sections of Fourier transform. In this article, it is implied that all the functions discussed are generalized functions, $\mathbb{R} \mapsto \mathbb{C}$ (the domain is either t or f), and infinitely differentiable unless stated otherwise. Also, all the definite integrals and infinite sums are assumed to be convergent.

1.1 Dirac Delta Function

The Dirac Delta function, $\delta(t)$, has the following properties:

$$\delta(t) = \begin{cases} \infty, & \text{if } t = 0 \\ 0, & \text{otherwise} \end{cases} \tag{1.1}$$

$$1 = \int_{-\infty}^{\infty} \delta(t)dt \tag{1.2}$$

(In my discussion of function, t variable represents time physically). It is easy to check that:

$$f(a) = \int_{-\infty}^{\infty} f(t)\delta(t)dt \tag{1.3}$$

$$\delta(t) = \delta(-t) \tag{1.4}$$

$$\delta(at) = \frac{\delta(t)}{|a|} \tag{1.5}$$

Also, the following integral is the Dirac Delta function as well:

$$\delta(t) = \int_{-\infty}^{\infty} e^{2\pi i f t} df \tag{1.6}$$

with the understanding that the Dirac Delta function has meaning only when under an integral sign [1]. The above relations will be used throughout the discussion of this article.

1.2 Sampling function

The sampling function, denoted as $u(t)$, with the *sampling interval* Δt is as follows:

$$u(t) = \Delta t \sum_{n=-\infty}^{\infty} \delta(t - n\Delta t) \tag{1.7}$$

The function is periodic with the period being the sampling interval.

1.3 Convolution

Convolution is a mathematical operation on two functions (say $x(t)$ and $y(t)$) that outputs another function. The symbol for the operator is $*$. The definition is as follows:

$$x(t) * y(t) = \int_{-\infty}^{\infty} x(\tau)y(t - \tau)d\tau \tag{1.8}$$

$(x * y)(t)$ notation is also used. The term *convolution* refers to both the result function and to the process of computing it. The table 1.1 lists a few properties of Convolution [1].

Table 1.1: Properties of Convolution

Property	Mathematical Expression
Commutativity	$x * y = y * x$
Associativity	$x * (y * z) = (x * y) * z$
Distributivity	$x * (y + z) = x * y + x * z$
Multiplicative identity	$x * \delta = x$
Complex Conjugation	$\overline{x * y} = \overline{x} * \overline{y}$
Differentiation	$(x * y)' = x' * y = x * y'$
Integration	$\int_{\mathbb{R}} (x * t)(t)dt = \int_{\mathbb{R}} x(t)dt \int_{\mathbb{R}} y(t)dt$

1.4 Kronecker delta

Suppose i and j are integers. Then a Kronecker delta is defined as:

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{otherwise} \end{cases} \tag{1.9}$$

Now consider two integers n and p where $0 \leq n, p \leq N - 1$. Also consider the following sum:

$$\frac{1}{N} \sum_{k=0}^{N-1} e^{2\pi i k(n-p)/N} \tag{1.10}$$

If $n = p$, $e^{-2\pi i k(n-p)/N} = 1$, which means

$$\begin{aligned} \frac{1}{N} \sum_{k=0}^{N-1} e^{2\pi i k(n-p)/N} &= \frac{1}{N} \sum_{k=0}^{N-1} 1 \\ &= 1 \end{aligned} \tag{1.11}$$

If $n \neq p$, then $2\pi i(n-p)/N \neq 0$. Thus, it follows that

$$\begin{aligned} \frac{1}{N} \sum_{k=0}^{N-1} e^{2\pi i k(n-p)/N} &= \frac{1}{N} \sum_{k=0}^{N-1} (e^{2\pi i(n-p)/N})^k \\ &= \frac{1}{N} \frac{1 - e^{2\pi i(n-p)}}{1 - e^{2\pi i(n-p)/N}} \end{aligned} \tag{1.12}$$

Where I used the geometric series formula:

$$\sum_{k=0}^{N-1} a^k = \frac{1 - x^N}{1 - x} \tag{1.13}$$

with $a \neq 1$. Also, we know that $e^{2\pi i(n-p)} = 1$ because $n - p$ is an integer and $e^{2\pi i m} = 1$ for any integer m. Therefore, it follows that if $n \neq p$,

$$\begin{aligned} \frac{1}{N} \sum_{k=0}^{N-1} e^{2\pi i k(n-p)/N} &= \frac{1}{N} \frac{1 - e^{2\pi i(n-p)}}{1 - e^{2\pi i(n-p)/N}} \\ &= 0 \end{aligned} \tag{1.14}$$

Thus, the above sum satisfies the Kronecker delta condition. In other words,

$$\delta_{np} = \frac{1}{N} \sum_{k=0}^{N-1} e^{2\pi i k(n-p)/N} \tag{1.15}$$

1.5 Sinusoidal exponential decay function

Sinusoidal exponential decay function (I call it SED) is as follows:

$$x = x_0 \cos{(2\pi f_0 t)} e^{-t/T} \theta(t) \tag{1.16}$$

Where $\theta(t)$ is a Heaviside step function (the plot of such function is given in figure 1.1). You can have sine instead of cosine in the above expression. For our discussion, SED of choice is combination of sine and cosine exponential decays in complex form:

$$\begin{aligned} x &= x_0 \cos{(2\pi f_0 t)} e^{-t/T} \theta(t) + i x_0 \sin{(2\pi f_0 t)} e^{-t/T} \theta(t) \\ &= (\cos{(2\pi f_0 t)} + i \sin{(2\pi f_0 t)}) x_0 e^{-t/T} \theta(t) \\ &= x_0 e^{2\pi i f_0 t} e^{-t/T} \theta(t) \end{aligned} \tag{1.17}$$

The real part of (1.17) is simply (1.16).

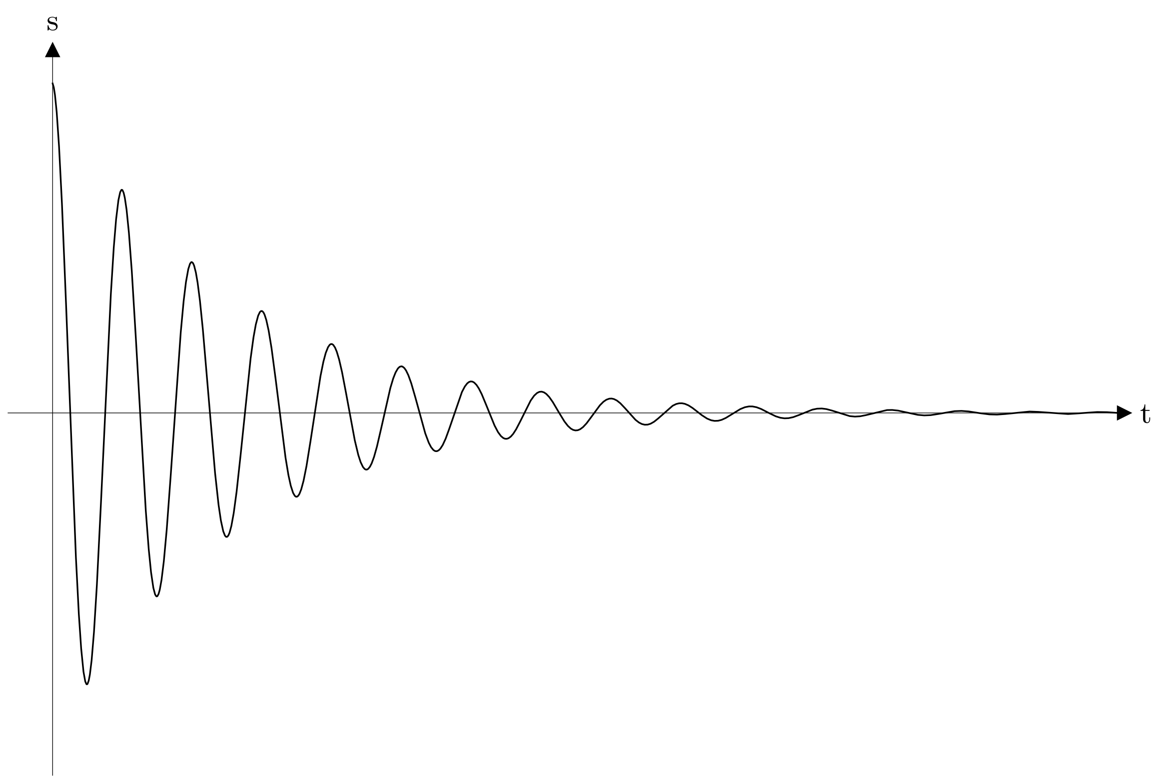

Figure 1.1: sed function plot, s vs. t

2 Continuous Fourier Transform

Continuous FT transforms a complex-valued function in t to another complex-valued function in f. Also, as we will see, the inverse FT of the function in f will recover the original function in t.

2.1 Definition of Fourier transform

Consider a function of t, $x(t)$. The continuous FT of x, $X(f)$, is as follows:

$$\begin{aligned} X(f) &= \mathcal{F}_t[x] \\ &= \int_{-\infty}^{\infty} x(t)e^{-2\pi i f t} dt \end{aligned} \tag{2.1}$$

It is sometimes called continuous-time Fourier transform (CTFT) to emphasize that t variable represents physical continuous time in an application. The domain of $X(f)$, f, is also in $\mathbb{R}$. (in my discussion t and f represent time and frequency respectively). $\mathcal{F}_t[x](f)$ notation can be used to explicitly indicate that x is transformed into f domain. It is easy to see that if a is a constant, then $\mathcal{F}_t[ax] = a\mathcal{F}_t[x]$. You can also perform inverse FT on X, which is:

$$\begin{aligned} \mathcal{F}_f^{-1}[X] &= \int_{-\infty}^{\infty} X(f)e^{2\pi i f t} df \\ &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} x(\tau)e^{-2\pi i f \tau} d\tau \right) e^{2\pi i f t} df \\ &= \int_{-\infty}^{\infty} x(\tau) \int_{-\infty}^{\infty} e^{2\pi i f (t-\tau)} df d\tau \\ &= \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau) d\tau \\ &= x(t) \end{aligned} \tag{2.2}$$

where I used (1.6) orthogonality relation .We will continue to use the notation $x(t)$ and $X(f)$ as an example of FT pair throughout the discussion (they are called 'Fourier pair'). It is also easy to see that the inverse FT of one is the Dirac Delta function [2].

2.2 FT with the extra phase factor

Consider another function, $e^{2\pi i f_0 t}x(t)$. The FT of it is:

$$\begin{aligned} \mathcal{F}_t[e^{2\pi i f_0 t}x(t)] &= \int_{-\infty}^{\infty} e^{2\pi i f_0 t}x(t)e^{-2\pi i f t} dt \\ &= \int_{-\infty}^{\infty} x(t)e^{-2\pi i (f-f_0) t} dt \\ &= \mathcal{F}_t[x](f-f_0) \\ &= X(f-f_0) \end{aligned} \tag{2.3}$$

It is simply the FT of x in $f - f_0$ in domain. By the same token, it is easy to see that $\mathcal{F}_t[x(t-\tau)] = e^{-2\pi i f \tau}X(f)$. A list of FT properties is in the table 2.1 [1]:

Table 2.1: Properties of Fourier transform

Property	Mathematical Expression
Linearity	$\mathcal{F}_t[ax+by] = aX + bY$
Shift in time	$\mathcal{F}_t[x(t-\tau)] = e^{-2\pi i f \tau} X(f)$
Phase factor	$\mathcal{F}_t[e^{2\pi i f_0 t} x(t)] = X(f - f_0)$
Scaling in time domain	$\mathcal{F}_t[x(at)] = \frac{1}{a} X(f/a)$
Complex conjugation	$\overline{\mathcal{F}_t[x]} = \overline{X(-f)}$
FT of Convolution	$\mathcal{F}_t[x * y] = X(f)Y(f)$
FT of product	$\mathcal{F}_t[xy] = X(f) * Y(f)$

2.3 Poisson Summation Formula

Consider a Fourier pair $x(t)$ and $X(f)$. Then the Poisson Summation formula states that [3]:

$$\sum_{n=-\infty}^{\infty} x(n) = \sum_{k=-\infty}^{\infty} X(k) \tag{2.4}$$

For example, if $x(n) = e^{2\pi i a n}$, then we know that the FT of $x(n)$ is:

$$\begin{aligned} X(k) &= \mathcal{F}_n[x](k) \\ &= \mathcal{F}_n[e^{2\pi i a n} \times 1](k) \\ &= \mathcal{F}_n[1](k-a) \\ &= \delta(k-a) \end{aligned} \tag{2.5}$$

Thus, according to the Poisson Summation formula, it follows that:

$$\sum_{n=-\infty}^{\infty} e^{2\pi i a n} = \sum_{k=-\infty}^{\infty} \delta(k-a) \tag{2.6}$$

2.4 FT of the sampling function

Suppose $u(t)$ is the sampling function with the sampling interval Δt. Then the FT of u is:

$$\begin{aligned} \mathcal{F}_t[u] &= \int_{-\infty}^{\infty} u(t) e^{-2\pi i f t} dt \\ &= \int_{-\infty}^{\infty} \Delta t \sum_{n=-\infty}^{\infty} \delta(t - n\Delta t) e^{-2\pi i f t} dt \\ &= \Delta t \sum_{n=-\infty}^{\infty} e^{-2\pi i f n \Delta t} \int_{-\infty}^{\infty} \delta(t - n\Delta t) dt \\ &= \Delta t \sum_{n=-\infty}^{\infty} e^{-2\pi i f n \Delta t} \end{aligned} \tag{2.7}$$

By applying (2.6), we can deduce that:

$$
\begin{aligned}
\mathcal{F}_t[u] &= \Delta t \sum_{n=-\infty}^{\infty} e^{-2\pi i f n \Delta t} \\
&= \Delta t \sum_{k=-\infty}^{\infty} \delta(k + f\Delta t) \\
&= \sum_{k=-\infty}^{\infty} \delta(f + k f_s)
\end{aligned} \tag{2.8}
$$

Where $f_s = 1/\Delta t$ is called a *sampling rate*.

2.5 FT of product of $u(t)$ and $s(t)$

Consider an arbitrary function $s(t)$. Let $x(t) = s(t)u(t)$. The FT of $x(t)$ is then:

$$
\begin{aligned}
\mathcal{F}_t[x] &= \int_{-\infty}^{\infty} s(t)u(t)e^{-2\pi i f t} dt \\
&= \int_{-\infty}^{\infty} s(t)\Delta t \sum_{n=-\infty}^{\infty} \delta(t - n\Delta t) e^{-2\pi i f t} dt \\
&= \sum_{n=-\infty}^{\infty} s(n\Delta t) e^{-2\pi i f n \Delta t} \Delta t \int_{-\infty}^{\infty} \delta(t - n\Delta t) dt \\
&= \sum_{n=-\infty}^{\infty} s(n\Delta t) e^{-2\pi i f n \Delta t} \Delta t
\end{aligned} \tag{2.9}
$$

If $\Delta t \to 0$, the above infinite sum becomes FT of $s(t)$.

2.6 FT of Convolution

Consider the convolution of two functions $x(t) * y(t)$. The FT of it is:

$$
\begin{aligned}
\mathcal{F}_t[(x * y)(t)] &= \int_{-\infty}^{\infty} (x * y)(t) e^{-2\pi i f t} dt \\
&= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} x(\tau) y(t - \tau) d\tau \right) e^{-2\pi i f t} dt \\
&= \int_{-\infty}^{\infty} x(\tau) \left(\int_{-\infty}^{\infty} y(t - \tau) e^{-2\pi i f t} dt \right) d\tau \\
&= \int_{-\infty}^{\infty} x(\tau) \left(e^{-2\pi i f \tau} \mathcal{F}_f[Y] \right) d\tau \\
&= \mathcal{F}_f[Y] \int_{-\infty}^{\infty} x(\tau) e^{-2\pi i f \tau} d\tau \\
&= \mathcal{F}_f[X]\mathcal{F}_f[Y] \\
&= X(f)Y(f)
\end{aligned} \tag{2.10}
$$

by the same token,

$$
\mathcal{F}_t[x(t)y(t)] = X(f) * Y(f) \tag{2.11}
$$

Here you can see convolutions can be used to make FT easier [2].

2.7 FT of $s(t)u(t)$ again

Consider $x(t) = s(t)u(t)$ discussed in section 2.5 again. We already showed that $X(f)$ is (2.9). We can also see that, by applying (2.11),

$$\begin{aligned}
\mathcal{F}_t[su] &= S(f) * U(f) \\
&= \int_{-\infty}^{\infty} S(\xi)U(f-\xi)d\xi \\
&= \int_{-\infty}^{\infty} S(\xi) \left(\sum_{k=-\infty}^{\infty} \delta(f + kf_s - \xi) \right) d\xi \\
&= \sum_{k=-\infty}^{\infty} S(f + kf_s) \int_{-\infty}^{\infty} \delta(f + kf_s - \xi) d\xi \\
&= \sum_{k=-\infty}^{\infty} S(f + kf_s) \qquad (2.12)
\end{aligned}$$

In other words,

$$\sum_{n=-\infty}^{\infty} s(n\Delta t)e^{-2\pi i f n\Delta t}\Delta t = \sum_{k=-\infty}^{\infty} S(f + kf_s) \qquad (2.13)$$

2.8 FT of SED

We will now consider a specific function, in this case, SED function discussed in section 1, $x = x_0 e^{2\pi i f_0 t} e^{-t/T}\theta(t)$. The FT of x is (with $1/T = r$) as follows:

$$\begin{aligned}
X(f) &= \mathcal{F}_t[x] \\
&= \mathcal{F}_t[x_0 e^{2\pi i f_0 t} e^{-t/T}\theta(t)] \\
&= x_0 \mathcal{F}_t[e^{-rt}\theta(t)](f - f_0) \\
&= \frac{x_0}{r + 2\pi i(f - f_0)} \\
&= x_0 \frac{r - 2\pi i(f - f_0)}{r^2 + 4\pi^2(f - f_0)^2} \\
&= x_0 \left[\frac{r}{r^2 + 4\pi^2(f - f_0)^2} + i\frac{2\pi(f_0 - f)}{r^2 + 4\pi^2(f - f_0)^2} \right] \qquad (2.14)
\end{aligned}$$

3 Discrete Fourier transform

Unlike continuous FT, discrete FT, or DFT in short, converts a finite sequence $\{x_n\}_{n=0}^{N-1}$, instead of continuous function, into an element of another finite sequence $\{X_n\}_{n=0}^{N-1}$. As you can see from the above sequences, there are N elements of both x and X respectively. From now on it is implied that all the sequences without explicit subscripts and superscripts run from $n = 0$ to $n = N - 1$.

3.1 Definition of DFT

Consider a sequence $\{x_n\}$. Then DFT of x_n is:

$$\begin{aligned} X_k &= \mathcal{F}[x_n] \\ &= \sum_{n=0}^{N-1} x_n \, e^{-2\pi ikn/N} \end{aligned} \tag{3.1}$$

The inverse DFT of X_k is:

$$\begin{aligned} \mathcal{F}^{-1}[X_k] &= \frac{1}{N} \sum_{k=0}^{N-1} X_k e^{2\pi ink/N} \\ &= \frac{1}{N} \sum_{k=0}^{N-1} \sum_{p=0}^{N-1} x_p \, e^{-2\pi ikp/N} e^{2\pi ink/N} \\ &= \sum_{p=0}^{N-1} x_p \frac{1}{N} \sum_{k=0}^{N-1} e^{2\pi ik(n-p)/N} \\ &= \sum_{p=0}^{N-1} x_p \delta_{np} \\ &= x_n \end{aligned} \tag{3.2}$$

Where I used the orthogonal relation (1.15). There is another version of DFT called discrete-time Fourier transform (DTFT in short). The only difference is that the sum runs from negative infinity to positive infinity. The example is (2.9), which is DTFT of $\Delta t \cdot s(n\Delta t)$.

References

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[3] Elias M. Stein and Rami Shakarchi. *Fourier Analysis: An Introduction*. 1st ed. 41 William Street, Princeton, New Jersey 08540, USA: Princeton University Press, 2003. ISBN: 9780691113845.

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