Fourier Transform

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Preface

Fourier transform is a mathematical transformation, in the context of my exposition, that maps one function in time domain into another function in frequency domain. In order for a function to have its Fourier transform, the function needs to be Schwartz. However, in Physics, we extend the notion of function to more generalized case, called a generalized function. In a generalized function space, we allow more flexibility so that functions like Dirac delta function can have the Fourier transform even though such function is not Schwartz. In this article I will assume that every function that is discussed has its Fourier transform. I will call Fourier transform FT in short, and there are two types of FT: continuous and discrete. Discrete FT is of great importance in many modern data processing application as modern computers are digital, meaning they perform computation in discrete manner. In this article, I'll discuss a continous case first before moving onto the discrete case.

Contents

1 Prerequisite Mathematics

A few prerequisite mathematics will be discussed here first in anticipation of their requirements in the discussion of subsequent sections of Fourier transform. In this article, it is implied that all the functions discussed are generalized functions, $\mathbb{R} \mapsto \mathbb{C}$ (the domain is either t or f), and infinitely differentiable unless stated otherwise. Also, all the definite integrals and infinite sums are assumed to be convergent.

1.1 Dirac Delta Function

The Dirac Delta function, $\delta(t)$, has the following property:

$$\delta(t) = \begin{cases} \infty, & \text{if } t = 0 \\ 0, & \text{otherwise} \end{cases} \tag{1.1}$$

$$1 = \int_{-\infty}^{\infty} \delta(t) dt \tag{1.2}$$

(In my discussion of function, t variable represents time physically). It is easy to check that:

$$f(a) = \int_{-\infty}^{\infty} f(t)\delta(t) dt \tag{1.3}$$

$$\delta(t) = \delta(-t) \tag{1.4}$$

$$\delta(at) = \frac{\delta(t)}{|a|} \tag{1.5}$$

Also, the following integral is the Dirac Delta function as well:

$$\delta(t) = \int_{-\infty}^{\infty} e^{2\pi i f t} df \tag{1.6}$$

The above relations will be used throughout the discussion of this article.

1.2 Sinusoidal exponential decay function

Sinusoidal exponential decay function (I call it SED) is as follows:

$$x = x_0 \cos(2\pi f_0 t) e^{-t/T} \theta(t) \tag{1.7}$$

Where $\theta(t)$ is a Heaviside step function (the plot of such function is given in figure 1.1). You can have sine instead of cosine in the above expression. For our discussion, SED of choice is combination of sine and cosine exponential decays in complex form:

$$\begin{aligned} x &= x_0 \cos(2\pi f_0 t) e^{-t/T} \theta(t) + i x_0 \sin(2\pi f_0 t) e^{-t/T} \theta(t) \\ &= (\cos(2\pi f_0 t) + i \sin(2\pi f_0 t)) x_0 e^{-t/T} \theta(t) \\ &= x_0 e^{2\pi i f_0 t} e^{-t/T} \theta(t) \end{aligned} \tag{1.8}$$

The real part of (1.8) is simply (1.7).

1.3 Sampling function

The sampling function, denoted as $u(t)$, with the sampling interval Δt is as follows:

$$u(t) = \Delta t \sum_{n=-\infty}^{\infty} \delta(t - n\Delta t) \tag{1.9}$$

The function is periodic with the period being the sampling interval.

1.4 convolution

Convolution is a mathematical operation on two functions (say $x(t)$ and $y(t)$) that outputs another function. The symbol for the operator is $*$. The definition is as follows:

$$x(t) * y(t) = \int_{-\infty}^{\infty} x(\tau)y(t-\tau)d\tau \tag{1.10}$$

$(x * y)(t)$ notation is also used. Convolution operation is commutative, associative, and distributive.

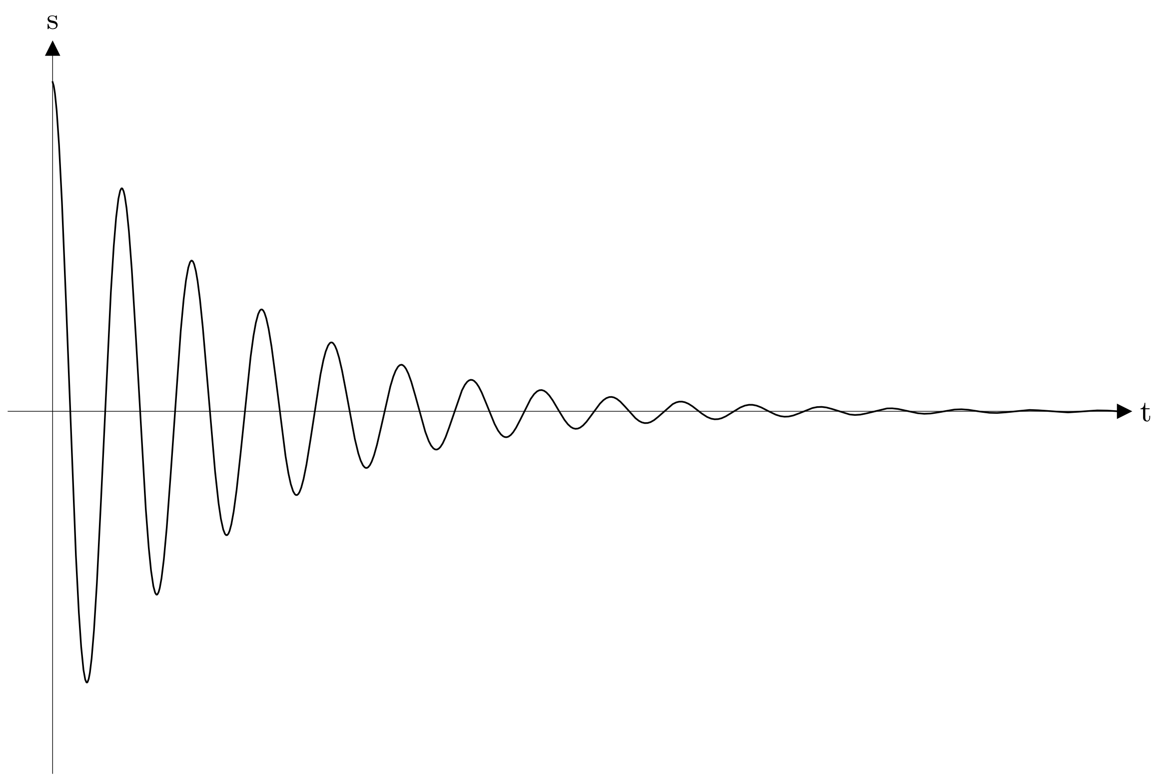

Figure 1.1: sed function plot, s vs. t

2 Continuous Fourier Transform

Continuous FT transforms a complex-valued function in t to another complex-valued function in f. Also, as we will see, the inverse FT of the function in f will recover the original function in t.

2.1 Definition of Fourier transform

Consider a function of t, $x(t)$. The continuous FT of x, $X(f)$, is as follows:

$$\begin{aligned} X(f) &= \mathcal{F}_t[x] \\ &= \int_{-\infty}^{\infty} x(t)e^{-2\pi ift}dt \end{aligned} \tag{2.1}$$

The domain of $X(f)$, f, is also in $\mathbb{R}$. (in my discussion t and f represent time and frequency respectively). $\mathcal{F}_t[x](f)$ notation can be used to explicitly indicate that x is transformed into f domain. It is easy to see that if a is a constant, then $\mathcal{F}_t[ax] = a\mathcal{F}_t[x]$. You can also perform inverse FT on X, which is:

$$\begin{aligned} \mathcal{F}_f[X] &= \int_{-\infty}^{\infty} X(f)e^{2\pi ift}df \\ &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} x(\tau)e^{-2\pi if\tau}d\tau\right) e^{2\pi ift}df \\ &= \int_{-\infty}^{\infty} x(\tau) \int_{-\infty}^{\infty} e^{2\pi if(t-\tau)}df d\tau \\ &= \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau \\ &= x(t) \end{aligned} \tag{2.2}$$

where I used (1.6) orthogonality relation (We will continue to use the notation $x(t)$ and $X(f)$ as an example of FT pair throughout the discussion). It is also easy to see that the inverse FT of one is the Dirac Delta function.

2.2 FT with the extra phase factor

Consider another function, $e^{2\pi if_0t}x(t)$. The FT of it is:

$$\begin{aligned} \mathcal{F}_t[e^{2\pi if_0t}x(t)] &= \int_{-\infty}^{\infty} e^{2\pi if_0t}x(t)e^{-2\pi ift}dt \\ &= \int_{-\infty}^{\infty} x(t)e^{-2\pi i(f-f_0)t}dt \\ &= \mathcal{F}_t[x](f-f_0) \\ &= X(f-f_0) \end{aligned} \tag{2.3}$$

It is simply the FT of x in $f - f_0$ in domain. By the same token, it is easy to see that $\mathcal{F}_t[x(t-\tau)] = e^{-2\pi if\tau}X(f)$.

2.3 FT of SED

Consider a SED function discussed before, $x = x_0 e^{2\pi i f_0 t} e^{-t/T}\theta(t)$. The FT of x is (with $1/T = r$) as follows:

$$\begin{aligned}
X(f) &= \mathcal{F}_t[x] \\
&= \mathcal{F}_t[x_0 e^{2\pi i f_0 t} e^{-t/T}\theta(t)] \\
&= s_0 \mathcal{F}_t[e^{-rt}\theta(t)](f - f_0) \\
&= \frac{s_0}{r + 2\pi i(f - f_0)} \\
&= s_0 \frac{r - 2\pi i(f - f_0)}{r^2 + 4\pi^2(f - f_0)^2} \\
&= s_0 \left[\frac{r}{r^2 + 4\pi^2(f - f_0)^2} + i\frac{2\pi(f_0 - f)}{r^2 + 4\pi^2(f - f_0)^2}\right] \qquad (2.4)
\end{aligned}$$

2.4 Poisson Summation Formula

Consider a function $x(t)$ and its FT pair $X(f)$. Then the Poisson Summation formula states that:

$$\sum_{n=-\infty}^{\infty} x(n) = \sum_{k=-\infty}^{\infty} X(k) \qquad (2.5)$$

For example, if $x(n) = e^{2\pi i a n}$, then we know that the FT of $x(n)$ is:

$$\begin{aligned}
X(k) &= \mathcal{F}_n[x](k) \\
&= \mathcal{F}_n[e^{2\pi i a n} \times 1](k) \\
&= \mathcal{F}_n[1](k - a) \\
&= \delta(k - a) \qquad (2.6)
\end{aligned}$$

Thus, according to the Poisson Summation formula, it follows that:

$$\sum_{n=-\infty}^{\infty} e^{2\pi i a n} = \sum_{k=-\infty}^{\infty} \delta(k - a) \qquad (2.7)$$

2.5 FT of the sampling function

Suppose $u(t)$ is the sampling function with the sampling interval Δt. Then the FT of u is:

$$\begin{aligned}
\mathcal{F}_t[u] &= \int_{-\infty}^{\infty} u(t) e^{-2\pi i f t} dt \\
&= \int_{-\infty}^{\infty} \Delta t \sum_{n=-\infty}^{\infty} \delta(t - n\Delta t) e^{-2\pi i f t} dt \\
&= \Delta t \sum_{n=-\infty}^{\infty} e^{-2\pi i f n \Delta t} \int_{-\infty}^{\infty} \delta(t - n\Delta t) dt \\
&= \Delta t \sum_{n=-\infty}^{\infty} e^{-2\pi i f n \Delta t} \qquad (2.8)
\end{aligned}$$

By applying (2.7), we can deduce that:

$$
\begin{aligned}
\mathcal{F}_t[u] &= \Delta t \sum_{n=-\infty}^{\infty} e^{-2\pi i f n \Delta t} \\
&= \Delta t \sum_{k=-\infty}^{\infty} \delta(k + f\Delta t) \\
&= \sum_{k=-\infty}^{\infty} \delta(f + \frac{k}{\Delta t})
\end{aligned} \tag{2.9}
$$

2.6 FT of product of u(t) and s(t)

Consider an arbitrary function $s(t)$. Let $x(t) = s(t)u(t)$. The FT of $x(t)$ is then:

$$
\begin{aligned}
\mathcal{F}_t[x] &= \int_{-\infty}^{\infty} s(t)u(t)e^{-2\pi i f t} dt \\
&= \int_{-\infty}^{\infty} s(t)\Delta t \sum_{n=-\infty}^{\infty} \delta(t - n\Delta t)e^{-2\pi i f t} dt \\
&= \sum_{n=-\infty}^{\infty} s(n\Delta t)e^{-2\pi i f n \Delta t}\Delta t \int_{-\infty}^{\infty} \delta(t - n\Delta t) dt \\
&= \sum_{n=-\infty}^{\infty} s(n\Delta t)e^{-2\pi i f n \Delta t}\Delta t
\end{aligned} \tag{2.10}
$$

If $\Delta t \to 0$, the above infinite sum becomes FT of $s(t)$.

2.7 FT of Convolution

Consider the convolution of two functions $x(t) * y(t)$. The FT of it is:

$$
\begin{aligned}
\mathcal{F}_t[(x*y)(t)] &= \int_{-\infty}^{\infty} (x*y)(t)e^{-2\pi i f t} dt \\
&= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} x(\tau)y(t-\tau) d\tau \right) e^{-2\pi i f t} dt \\
&= \int_{-\infty}^{\infty} x(\tau) \left(\int_{-\infty}^{\infty} y(t-\tau) e^{-2\pi i f t} dt \right) d\tau \\
&= \int_{-\infty}^{\infty} x(\tau) \left(e^{-2\pi i f \tau} \mathcal{F}_f[Y] \right) d\tau \\
&= \mathcal{F}_f[Y] \int_{-\infty}^{\infty} x(\tau) e^{-2\pi i f \tau} d\tau \\
&= \mathcal{F}_f[X]\mathcal{F}_f[Y] \\
&= X(f)Y(f)
\end{aligned} \tag{2.11}
$$

By the same token, $\mathcal{F}_t[x(t)y(t)] = X(f) * Y(f)$.

2.8 FT of $s(t)u(t)$ again

Consider $x(t) = s(t)u(t)$ discussed in section 2.6 again. We already showed that the FT of $x(t)$ is (2.10). We can also see that, by applying (2.11),