Fourier Transform

Sejin Nam
University of Hawaii at Manoa

Contents

1 Prerequisite Mathematics

A few prerequisite mathematics will be discussed here in anticipation of their requirements in the subsequent sections after this section.

1.1 Dirac Delta Function

Dirac Delta function, $\delta(t)$, has the following property:

$$\delta(t) = \begin{cases} \infty, & \text{if } t = 0 \\ 0, & \text{otherwise} \end{cases} \tag{1.1}$$

$$1 = \int_{-\infty}^{\infty} \delta(t) dt \tag{1.2}$$

(In my discussion of function, t variable represents time physically). The following integral is also a Dirac Delta function:

$$\delta(t) = \int_{-\infty}^{\infty} e^{2\pi i f t} df \tag{1.3}$$

The above relations will be used throughout the discussion of this article.

2 Fourier Transform

Fourier transform is a mathematical transformation, in the context of my discussion, that maps one function in time domain into another function in frequency domain. There are two types of FT, Fourier transform in short: continuous and discrete. Discrete FT is of great importance in many modern data processing application as modern computers are digital, meaning they perform computation in discrete manner. I'll discuss a continous case first before moving onto the discrete case.

2.1 Continuous Fourier Transform

Consider a continuous function of t, $x(t)$. The continuous FT of x is as follows:

$$\begin{aligned} X(f) &= \mathcal{F}_t[x] \\ &= \int_{-\infty}^{\infty} x(t) e^{-2\pi i f t} dt \end{aligned} \tag{2.1}$$

$X(f)$ is also continuous in f (in my discussion f represents frequency physically). You can also perform inverse FT on X, which is:

$$\begin{aligned} \mathcal{F}_f[X] &= \int_{-\infty}^{\infty} X(f) e^{2\pi i f t} df \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(\tau) e^{-2\pi i f \tau} d\tau e^{2\pi i f t} df \\ &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} x(\tau) e^{-2\pi i f \tau} d\tau \right) e^{2\pi i f t} df \\ &= \int_{-\infty}^{\infty} x(\tau) \int_{-\infty}^{\infty} e^{2\pi i f (t-\tau)} df d\tau \\ &= \int_{-\infty}^{\infty} x(\tau) \delta(t-\tau) d\tau \\ &= x(t) \end{aligned} \tag{2.2}$$

where I used (**??**) orthogonality relation.

$$s = s_0 \cos{(2\pi f_0 t)} e^{-t/T} \theta(t) \tag{2.3}$$

Where $\theta(t)$ is a Heaviside step function. You can have sine instead of cosine in the above expression. For our discussion, SED of choice is combination of sine and cosine exponential decays in complex form:

$$s = s_0 e^{2\pi i f_0 t} e^{-t/T} \theta(t) \tag{2.4}$$

The real part or the imaginary part can represent a real physical signal. The FT of SED is (with $1/T = r$):

$$\begin{aligned} S(f) &= \mathcal{F}_t[s] \\ &= \mathcal{F}_t[s_0 e^{2\pi i f_0 t} e^{-t/T} \theta(t)] \\ &= s_0 \mathcal{F}_t[e^{-rt}\theta(t)](f - f_0) \\ &= \frac{s_0}{r + 2\pi i (f - f_0)} \\ &= s_0 \frac{r - 2\pi i (f - f_0)}{r^2 + 4\pi^2 (f - f_0)^2} \\ &= s_0 \left[\frac{r}{r^2 + 4\pi^2 (f - f_0)^2} + i \frac{2\pi (f - f_0)}{r^2 + 4\pi^2 (f - f_0)^2} \right] \end{aligned} \tag{2.5}$$