Hamiltonian renormalisation VIII. $P(\Phi)_2$ quantum field theory

M. Rodriguez Zarate[1]*, T. Thiemann[1†]

[1] Inst. for Theor. Phys. III, FAU Erlangen – Nürnberg,
Staudtstr. 7, 91058 Erlangen, Germany

July 15, 2025

Abstract

In previous works in this series we focussed on Hamiltonian renormalisation of free field theories in all spacetime dimensions. In this paper we address the Hamiltonian renormalisation of the self-interacting scalar field in two spacetime dimensions with polynomial potential, called $P(\Phi)_2$. We consider only the finite volume case.

The $P(\Phi)_2$ theory is one of the few interacting QFT's that can be rigorously constructed non-perturbatively. We find that our Hamiltonian renormalisation flow finds this theory indeed as a fixed point.

1 Introduction

Constructing interacting quantum field theories (QFTs) rigorously in four and higher spacetime dimensions remains one of the most difficult challenges in theoretical and mathematical physics [1]. The difficulties come from the fact that quantum fields are operator valued distributions which means that products thereof as they appear typically in Hamiltonians are a priori ill-defined, being plagued by both short distance (UV) and large distance (IR) divergences. In the constructive QFT (CQFT) approach [2] one tames both types of divergences by introducing both UV (M) and IR cut-offs (R) to the effect that only a finite number of degrees survive at finite M, R. For instance, R could be a compactification radius and M a lattice spacing. Then at finite M, R one is in the safe realm of quantum mechanics. The problem is then how to remove the cut-offs. Usually one removes first M (continuum limit) and then R (thermodynamic limit). In this process the parameters (coupling constants) are taken to be cut-off dependent and they are tuned or renormalised in such a way that the limiting theory is well-defined when possible.

Non-perturbative renormalisation in CQFT (not to be confused with renormalisation in the perturbative approach to QFT) has a long tradition [3] and comes in both the functional integral language and the Hamiltonian language (see e.g. [4] and references therein). Focussing on UV cut-off removal, we consider quantum mechanical systems labelled by the cut-off M. If these quantum mechanical systems all descend form a well-defined continuum theory, then in the functional integral approach one obtains the theory at resolution M by integrating out all degrees of freedom referring to higher resolution while in the Hamiltonian approach one projects those out. This in particular implies that if one takes the quantum mechanical theory at resolution M' and integrates or projects out the degrees of freedom at resolutions between $M < M'$ and M' one obtains the quantum mechanical theory at resolution M. Vice versa, when this necessary set of *consistency conditions* is met, this typically also is sufficient to define a continuum theory.

The family of theories that one starts with, are constructed making various choices such as representations, factor orderings, discretisation errors, etc. and the afore mentioned consistency conditions are generically violated. However, one can define a sequence of such quantum mechanical theory families by defining a new theory at resolution M by integrating/projecting out the degrees of freedom between M and $M'(M)$ of the old theory at

*melissa.rodriguez@gravity.fau.de
†thomas.thiemann@gravity.fau.de

resolution $M'(M) > M$ where $M' : \; M \mapsto M'(M)$ is a fixed function on the set of resolution scales. Such a process is called a block spin transformation or coarse graining operation which typically leads to a renormalisation of the coupling constants. At a fixed point of this renormalisation flow of theories the consistency condition is enforced by construction and therefore fixed points qualify as continuum theories.

In previous parts of this series we have considered a Hamiltonian projection scheme [5, 4] which is motivated by the functional integral approach via Osterwalder-Schrader reconstruction. It was then applied to free QFT in Minkowski space [6, 7, 8, 9, 10] in any dimension and parametrised QFT [11] in 2d which shares some features with the free bosonic string. In all those cases the fixed point of the flow could be computed and was shown to coincide with the known continuum theory. In the present paper for the first time we expose the formalism to interacting quantum field theory (QFT), specifically to the self-interacting scalar quantum field theory in two spacetime dimensions with polynomial potential, called $P(\Phi)_2$ theory [12]. We focus on the UV part of the renormalisation and thus consider the cylinder spacetime $\mathbb{R} \times [0, R)$ with periodic boundary conditions where the circumference $2\pi \; R$ of the circle is fixed. Once the theory is constructed at finite R the thermodynamic limit $R \to \infty$ is taken by the methods described in [12] and references therein should one be interested on the spacetime $\mathbb{R}^2$.

This work is organised as follows:

In section 2 we introduce the classical and quantum $P(\Phi)_2$ model on the cylinder. Its quantum theory is well defined in the Fock representation selected by the free part of its Hamiltonian no matter what the polynomial degree of its potential is, even if it is not bounded from below, as long as the potential is normal ordered with respect to the same Fock structure. An elementary proof of this astonishing fact unique to two spacetime dimensions is provided in appendix A.

In section 3 we introduce the Hamiltonian renormalisation of this model. We pick the Dirichlet kernel [14] to define the block spin transformation mentioned above. An account of Dirichlet kernel renormalisation techniques, which are closely related to wavelet theory [13], may be found in appendix B. The Dirichlet kernel is a compromise between position locality and momentum decay properties, in particular it is smooth and therefore has advantages over the often chosen Schwarz kernel. The Schwarz kernel has better position locality properties but it has distributional derivatives which are potentially problematic with respect to treating the necessarily occurring field derivatives.

In section 4 we derive the flow of the model and show that its fixed point at finite resolution coincides with blocking from the known continuum QFT reviewed in section 2.

In section 5 we summarise and conclude.

2 Classical and quantum $P(\Phi)_2$ on the cylinder

In the first subsection we introduce the classical $P(\Phi)_2$ theory and in the second we show that its normal ordered self-interacting Hamiltonian is a densely defined, symmetric operator in the Fock representation selected by its free part.

2.1 Classical $P(\Phi)_2$ on the cylinder

We consider the cylinder spacetime manifold $\mathbb{R} \times [0, R)$ where R is any finite, fixed circumference of the circle. For convenience we pass from dimensionful coordinates $(c\; s, y)$ on that spacetime to dimensionless coordinates $X = (t = \frac{c\,s}{R}, x = \frac{y}{R})$ so that $x \in [0, 1)$ with endpoints identified. Then the classical action can be rewritten as

$$S[\Phi] = \int_{\mathbb{R}} dt \int_0^1 dx \; \{\frac{1}{2}[[\dot{\Phi}]^2 - [\Phi']^2 - p^2\Phi^2] - P(\Phi)\}(X) =: \int_{\mathbb{R}\times[0,1)} d^2 \; L(\Phi(x), \Phi'(x), \dot{\Phi}(X)), \tag{2.1}$$

where p is the mass, $\dot{(.)} = \partial_t(.)$, $(.)' = \partial_x(.)$ and

$$P(\Phi) = \sum_{k=0}^{N} g_k \; \Phi^k, \tag{2.2}$$

is any finite polynomial in the fields, i.e. a finite linear combination of powers $\Phi^k,\ k=0,..,N$ with real-valued, dimensionless coefficients g_k called coupling constants. We assume that whatever the value of g_2 is, we have $p^2=[M\ R]^2>0$ where M is the dimensional mass. This can always be achieved by redefining g_2 and is done in order to avoid the special treatment of zero modes. We take Φ dimension free and have dropped a constant pre-factor from the action that has the dimension of an action. The spacetime field Φ is subject to periodic boundary conditions $\Phi(t,0)=\Phi(t,1)$ for all $t\in\mathbb{R}$. Therefore in all that follows we never have to worry about boundary terms when integrating by parts w.r.t. x.

The Hamiltonian formulation is straightforward: The real valued time zero field $\phi(x):=\Phi(0,x)$ has the real valued conjugate momentum $\pi(x)=[\partial_t\Phi](0,x)$ which obey the canonical Poisson brackets

$$\{\phi(x),\phi(y)\}=\{\pi(x),\pi(y)\}=0,\ \ \{\pi(x),\phi(y)\}=\delta(x,y), \tag{2.3}$$

where $\delta(x,y)$ is the periodic delta distribution (see e.g. appendix B) and the Hamiltonian is obtained by the Legendre transformation

$$h[\phi,\pi]=\text{extr}_u\ \{\int_0^1\ dx\ [\pi(x)\ u(x)-L(\phi(x),\phi'(x),u(x))]\}=\int_0^1\ dx\ \{\frac{1}{2}[\pi^2+[\phi']^2+p^2\phi^2]+P(\phi)\}(x). \tag{2.4}$$

2.2 Quantum $P(\Phi)_2$ on the cylinder

The structure of the Hamiltonian (2.4) suggests the natural split

$$h=h_0+v,\ \ h_0=\int_0^1\ dx\ \{\frac{1}{2}[\pi^2+\phi\ \omega^2\cdot\phi\}(x),\ \ v=\int_0^1\ dx\ P(\phi(x)), \tag{2.5}$$

where

$$\omega^2:=p^2-\Delta,\ \ \Delta=\partial_x^2 \tag{2.6}$$

is a self-adjoint operator on the one particle Hilbert space

$$L=L_2([0,1),\ dx), \tag{2.7}$$

with pure point spectrum $\hat{\omega}_n^2=p^2+[2\pi i\ n]^2,\ n\in\mathbb{Z}$. The corresponding eigenfunctions are $e_n(x)=e^{2\pi\ i\ n\ x}$ which define an orthonormal basis of L. For obvious reasons, h_0,v are called the free part and interacting part respectively.

We define the unital Weyl $^*-$algebra $\mathfrak{A}$ in the usual way via its generating Weyl elements $w[f]=\exp(i<f,\phi>_L),\ w[g]=\exp(i<g,\pi>_L)$ for real valued $f,g\in L$ that are subject to the Weyl relations

$$w[g]\ w[f]\ w[-g]=e^{-i<g,f>_L}\ w[f],\ w[f]\ w[f']=w[f+f'],\ w[g]\ w[g']=w[g+g'],\ w[f]^*=w[-f],\ w[g]^*=w[-g]. \tag{2.8}$$

We define a cyclic Fock representation $(\rho,\mathcal{H},\Omega)$ of $\mathfrak{A}$ from the Fock state[1] ω_F

$$\omega_F(w[f]\ w[g]):=e^{\frac{i}{2}<f,g>_L}\ e^{-\frac{1}{4}[<f,\omega^{-1}\cdot f>_L+<g,\omega\cdot g>_L]}. \tag{2.9}$$

via the GNS construction. It is not difficult to show that (2.9) is unitarily equivalent to defining annihilation operator valued distributions corresponding to the algebra element

$$a(x):=\frac{1}{\sqrt{2}}[\omega^{1/2}\cdot\phi-i\omega^{-1/2}\cdot\pi](x), \tag{2.10}$$

so that $\rho(a(x))\Omega=0$ and everything else follows from the commutation relations of which the non-vanishing ones are

$$[a(x),a^*(y)]=\delta(x,y). \tag{2.11}$$

[1]Note that the symbol ω is reserved for both the covariance operator and the Fock state introduced in eq. (2.9); to distinguish the latter, it is denoted with a subscript F. In contrast, the letters w and W are used exclusively to denote Weyl elements.

The linear span $\mathcal{D}$ of the Fock vector states $\psi_{f_1,..,f_n} := \rho(< f_1, a >^*)..\rho(< f_n, a >^*)\Omega$ for $f_1, .., f_n \in L$ together with Ω is dense in the Fock representation space $\mathcal{H}$. The number n is called the particle number of the Fock vector where we assign $n = 0$ to the vacuum Ω void of particles. We will in fact consider $\mathcal{D}$ corresponding to $f_1, .., f_n \in L_0$ where L_0 is the span of the functions e_n. This $\mathcal{D}$ is still dense and has the advantage that finite products of functions in L_0 are still in L_0.

We use capital letters in order to denote the operator representatives of algebra elements, e.g. $A(x) = \rho(a(x)),\ A^\dagger(x) = \rho(a^*(x))$. We use this Fock structure in order to define both H_0 and V by their normal ordered symbols

$$\begin{aligned} H_0 &:= \rho(h_0) = \int_0^1 dx\ A^\dagger(x)\ [\omega \cdot A](x),\ V = \sum_{k=0}^N g_k V_k, \\ V_k &:= \rho(v_k) = 2^{-k/2} \sum_{l=0}^k \begin{pmatrix} k \\ l \end{pmatrix} V_k(l), \\ V_k(l) &:= \int_0^1 dx\ \{[\omega^{-1/2} \cdot A^\dagger](x)\}^{k-l}\ [\omega^{-1/2} \cdot A](x)\}^l. \end{aligned} \tag{2.12}$$

By construction, $\mathcal{D}$ is an invariant, dense domain of H_0. The astonishing fact, unique to two spacetime dimensions, when $m > 0$ and only when the spacetime is spatially compact, is that $\mathcal{D}$ is also a dense domain for V, albeit no longer an invariant one when $g_k \neq 0$ for at least one of $k = 2, .., N$. We give an elementary proof in appendix A.

3 Hamiltonian renormalisation of $P(\Phi)_2$

An account of the version of Hamiltonian renormalisation used below and employing the Dirichlet kernel can be found in appendix B, see [4, 5] for its motivation. Furthermore, in the following sections we adopt the notation introduced in appendix B: capital letters such as F, G denote functions in L; capital letters with a subscript M, such as F_M, G_M, refer to functions in L_M; and lowercase letters with a subscript M, such as f_M, g_M, denote functions in l_M. Elements of the abstract algebra are still written in lowercase, while their representatives (after GNS construction) are denoted by capital letters.

The starting point of the renormalisation scheme is to provide a family of theories $(\rho_M^{(0)},\ \mathcal{H}_M^{(0)},\ \Omega_M^{(0)},\ H_M^{(0)})$ consisting of a Hilbert space $\mathcal{H}_M^{(0)}$ with cyclic vector $\Omega_M^{(0)}$ that carries a representation $\rho_M^{(0)}$ of some $^*-$algebra $\mathfrak{A}_M$ and a Hamiltonian operator $H_M^{(0)}$ densely defined on some subspace $\mathcal{D}_M^{(0)}$ of $\mathcal{H}_M^{(0)}$. Equivalently, we consider a state $\omega_{M,F}^{(0)}$ on $\mathfrak{A}_M$ for which $(\rho_M^{(0)},\ \mathcal{H}_M^{(0)},\ \Omega_M^{(0)})$ are its GNS data. The objects $\mathfrak{A}_M^{(0)}$ and $H_M^{(0)}$ are to be thought of as discretised versions of the continuum objects $\mathfrak{A}$ and H. The label M is in general taken from a partially ordered and directed index set $\mathbb{O}$ that describes the (location, momentum, energy,..) resolution at which we probe the theory. In the present case the central tool is the Dirichlet kernel

$$P_M(x, y) = \sum_{n \in \mathbb{Z}_M} e_n(x - y),\ \ \mathbb{Z}_M = \{n \in \mathbb{Z};\ |n| \leq \frac{M-1}{2}\}, \tag{3.1}$$

which can be considered as a tamed version of the δ distribution on the circle. The index set is taken to be the set of odd naturals with ordering relation $M < M'$ iff $M'/M \in \mathbb{O}$. The Dirichlet kernel can be considered as an orthogonal projection P_M in L with image L_M. We define the Weyl algebra $\mathfrak{A}_M$ in analogy to (2.8) as the abstract $^*-$algebra generated by the Weyl elements $w_M[F_M],\ w_M[G_M]$ with real valued $F_M, G_M \in L_M$ subject to the relations

$$\begin{aligned} & w_M[G_M]\ w_M[F_M]\ w_M[-G_M] = e^{-i<G_M,F_M>_{L_M}}\ w_M[F_M],\ \ w_M[F_M]\ w_M[F'_M] = w_M[F_M + F'_M], \\ & w_M[G_M]\ w_M[G'_M] = w_M[G_M + G'_M],\ w_M[F_M]^* = w_M[-F_M],\ w_M[G_M]^* = w_M[-G_M], \end{aligned} \tag{3.2}$$

where the scalar product on the finite resolution 1-particle Hilbert space L_M coincides with the one on L. The state $\omega_{M,F}^{(0)}$ is chosen to be in analogy to (2.9) as the Fock state

$$\omega_{F,M}^{(0)}(w_M[F_M]\ w_M[G_M]) = e^{\frac{i}{2}<F_M,G_M>_{L_M}}\ e^{-\frac{1}{4}[<F_M,[\omega_M^{(0)}]^{-1}\cdot F_M>_{L_M} + <G_M,[\omega_M^{(0)}]\cdot G_M>_{L_M}]}, \tag{3.3}$$

which requires as a central input the definition of the kernel $\omega_M^{(0)}$ on L_M. Note that the unfortunate doubling of symbols ω for both an algebraic state and a kernel on the 1-particle Hilbert space is resolved by attaching an extra index "F" to the state to indicate "Fock". To motivate the choice of $\omega_M^{(0)}$ we follow the general prescription of appendix B and define h_M as a quantisation of

$$h_M[\phi_M, \pi_M] := h[\phi_M, \pi_M],\ \phi_M = P_M \cdot \phi,\ \pi_M = P_M \cdot \pi, \tag{3.4}$$

where h is the classical continuum Hamiltonian (2.5). For its free part we find

$$\begin{aligned} h_{0,M} &= \frac{1}{2} \int_0^1 dx\ [\pi_M(x)^2 + [\partial_x \phi_M(x)]^2 + p^2\ \phi_M(x)^2] \\ &= \frac{1}{2}[< \pi_M, \pi_M >_{L_M} + < \partial\phi_M, \partial\phi_M >_{L_M} + p^2 < \phi_M, \phi_M >_{L_M}]. \end{aligned} \tag{3.5}$$

Integrating the derivative term by parts (note that P_M preserves the space of periodic L_2 functions) we obtain

$$h_{0,M} = \frac{1}{2}[< \pi_M, \pi_M >_{L_M} - < \phi_M, \Delta_M \cdot \phi_M >_{L_M} + p^2\ < \phi_M, \phi_M >_{L_M}], \tag{3.6}$$

with the Laplacian

$$\Delta_M := \partial_M^2,\ \partial_M = P_M \cdot \partial \cdot P_M. \tag{3.7}$$

We have used the projection property $P_M \cdot P_M = P_M$ of the Dirichlet kernel to obtain (3.7). The specific form of $h_{0,M}$ suggests a Fock quantisation with annihilators

$$a_M(x) = 2^{-1/2}[[\omega_M]^{1/2} \cdot \phi_M - i\ [\omega_M]^{-1/2} \cdot \pi_M](x), \quad [\omega_M]^2 = p^2\ 1_M - \Delta_M, \tag{3.8}$$

which obey non-trivial commutation relations

$$[a_M(x), a_M(y)^*] = P_M(x, y), \tag{3.9}$$

if one defines the Poisson brackets between ϕ_M, π_M as the result of considering these as functions on the continuum phase space and using the continuum Poisson bracket

$$\{\pi_M(x), \phi_M(y)\} := \int du \int dv P_M(x, u)\ P_M(y, v)\ \{\pi(u), \phi(v)\} = P_M(x, y). \tag{3.10}$$

Then we see that (3.3) and (3.8) match provided that we interpret $w_M[F_M] = e^{i<F_M, \phi_M>_{L_M}}$, $w_M[G_M] = e^{i<G_M, \pi_M>_{L_M}}$.

We can finish the initialisation of the Hamiltonian renormalisation flow by defining

$$H_M^{(0)} = H_{0,M}^{(0)} + V_M^{(0)},\ H_{0,M}^{(0)} = \int_0^1 dx\ A_M^\dagger(x)\ [\omega_M^{(0)} \cdot A_M](x),\ V_M = \int_0^1 dx\ :V([[2\omega_M^{(0)}]^{-1/2} \cdot [A_M + A_M^\dagger]](x)) :_M, \tag{3.11}$$

where A_M is the operator representative of a_M and $: . :_M$ denotes normal ordering of the $A_M, A_M^\dagger$.

4 Hamiltonian renormalisation flow of $P(\Phi)_2$

In the first subsection we block the theory from the continuum in order to determine which fixed point family the renormalisation flow should find. In the second subsection, we use projector maps P_M -as defined in appendix B-, to project from L to the L_M subspaces and then compute the renormalisation flow. In the last subsection we discretise the fields, worked in the l_M spaces of square summable sequences and compute the renormalisation flow. In the two frameworks we show that the renormalisation flow of the initial family defined in the previous section is already at its fixed point. The reason for displaying the strictly equivalent flows in terms of projections and discretisations respectively is that the former is in the spirit of renormalisation schemes outside a lattice context while the former emphasises the traditional real space block spin interpretation of the renormalisation flow.

4.1 Blocking from the continuum

Blocking from the continuum means to define a state $\omega_{F,M}$ and a Hamiltonian H_M out of the continuum state ω_F and Hamiltonian H defined in section 2 via the formulas

$$\begin{aligned} \omega_{F,M}(w_M[F_M]\; w_M[G_M]) &:= \omega_F(w[F_M]\; w[G_M]), \\ < W_M[F_M]\Omega_M,\; H_M W_M[G_M] >_{\mathcal{H}_M} &:=< W[F_M]\Omega,\; H\, W[G_M]\Omega >_{\mathcal{H}} \end{aligned} \tag{4.1}$$

where $(\rho_M, \mathcal{H}_M, \Omega_M)$ and $(\rho, \mathcal{H}, \Omega)$ are the GNS data of $\omega_{F,M}$ and ω_F respectively and we have denoted the operator representatives by capital letters, i.e. $W_M[F_M] = \rho_M(w_M[F_M])$ and $W[F] = \rho(w[F])$. On the right hand side of (4.1) the elements $F_M, G_M \in L_M$ are to be considered as elements of the continuum 1-particle Hilbert space L by trivial embedding $F_M \mapsto F$.

We start with the first line of (4.1) and use (2.9)

$$\omega_F(w[F_M]\; w[G_M]) := e^{\frac{i}{2}\, <F_M,G_M>_L}\; e^{-\frac{1}{4}[<F_M,\omega^{-1}\cdot F_M>_L + <G_M,\omega\cdot G_M>_L]}. \tag{4.2}$$

We have explicitly with $\hat{\omega}_n = \sqrt{p^2 + [2\pi n]^2}$ using a resolution of identity with respect to the ONB e_n of L

$$\begin{aligned} &< F_M, \omega^{-1}\cdot F'_M >_L = \sum_{n\in\mathbb{Z}}\; < F_M, \omega^{-1}\, e_n >_L < e_n, F'_M >_L = \sum_{n\in\mathbb{Z}}\; \hat{\omega}_n^{-1}\; < F_M, e_n >_L < e_n, F'_M >_L \\ = &\sum_{n\in\mathbb{Z}_M} \hat{\omega}_n^{-1}\; < F_M, e_n >_{L_M} < e_n, F'_M >_{L_M} =:< F_M, \omega_M^{-1}\cdot F'_M >_{L_M}, \end{aligned} \tag{4.3}$$

where we used that F_M, F'_M are orthogonal to the $e_n,\; n \not\in \mathbb{Z}_M$. To interpret the operator ω_M on L_M we explicitly compute

$$\begin{aligned} \partial_M \cdot F_M &= \sum_{n\in\mathbb{Z}_M} e_n\; < e_n, \partial_M\cdot F_M >_{L_M} = \sum_{n\in\mathbb{Z}_M} e_n\; < e_n, \partial\cdot F_M >_{L_M} \\ = &\; -\sum_{n\in\mathbb{Z}_M} e_n\; < \partial\cdot e_n, F_M >_{L_M} = \sum_{n\in\mathbb{Z}_M} [2\pi i n]\; e_n\; < e_n, F_M >_{L_M}, \end{aligned} \tag{4.4}$$

hence

$$[p^2 - \Delta_M]\cdot F_M = \sum_{n\in\mathbb{Z}_M} \hat{\omega}_n^2\, e_n\; < e_n, F_M >_{L_M}. \tag{4.5}$$

It follows that $\omega_M = \sqrt{p^2 - \Delta_M}$ which is in fact identical to the natural choice $\omega_M^{(0)}$ of section 3 used to define the initial family of states. Concluding we find that $\omega_{F,M}$ coincides with $\omega_{F,M}^{(0)}$ defined in (3.3).

Considering the second line of (4.1) we use the elementary identity

$$\begin{aligned} &A(x)\; W[F] = W[F]\; W[F]^{-1}\; A(x)\; W[F] = W[F]\; (A(x) - i[< [2\omega]^{-1/2}\cdot F, A + A^\dagger >_L, A(x)]) \\ = &\; W[F]\; (A(x) + i([2\omega]^{-1/2}\cdot F)(x) 1_{\mathcal{H}_M}), \end{aligned} \tag{4.6}$$

to find for the free part

$$\begin{aligned} &< W[F_M]\Omega,\; H_0\, W[F'_M]\Omega >_{\mathcal{H}} = \int dx \int dy\, \omega(x,y)\;\; < A(x)\, W[F_M]\Omega,\; A(y)\, W[F'_M]\,\Omega >_{\mathcal{H}} \\ = &\int dx \int dy\, \omega(x,y)\, ([2\omega]^{-1/2}\cdot F_M)(x)\, ([2\omega]^{-1/2}\cdot F'_M)(y)\;\; < W[F_M]\Omega,\; W[F'_M]\,\Omega >_{\mathcal{H}} \\ = &\;\frac{1}{2}\; < \omega^{-1/2}\cdot F_M, \omega\cdot\omega^{-1/2}\cdot F'_M >_L < W_M[F_M]\Omega_M,\; W_M[F'_M]\Omega_M >_{\mathcal{H}_M} \\ = &\;\frac{1}{2}\; < F_M, F'_M >_{L_M} < W_M[F_M]\Omega_M,\; W_M[F'_M]\Omega_M >_{\mathcal{H}_M}, \end{aligned} \tag{4.7}$$

where $\omega(x,y) = \sum_{n\in\mathbb{Z}}\; \hat{\omega}_n e_n(x-y)$ is the integral kernel of ω and in the step before the last one we used that we already know that the GNS data of $\omega_{F,M}$ are those of $\omega_{F,M}^{(0)}$. Accordingly $H_M = H_M^{(0)}$ defined in (3.11). For the interacting part it suffices to consider a field monomial of order k

$$V_k = \int dx \ :\{([2\omega]^{-1/2} \cdot [A + A^\dagger])(x)\}^k := \sum_{l=0}^{k} \left(\begin{array}{c} k \\ l \end{array} \right) 2^{-k/2} \, V_k(l),$$
$$V_k(l) = \int dx \ ([\omega^{-1/2} \cdot A]^\dagger(x)])^{k-l} \ (\omega^{-1/2} \cdot A](x)])^l. \tag{4.8}$$

Using (4.6) it follows

$$< W[F_M]\Omega, \ V_k(l) \ W[F'_M]\Omega >_{\mathcal{H}} = i^{l-(k-l)} \int dx \ ([\omega^{-1/2} \cdot F_M](x))^{k-l} \ ([\omega^{-1/2} \cdot F'_M](x))^l \times$$
$$< W[F_M]\Omega, \ W[F'_M] \ \Omega >_{\mathcal{H}}$$
$$= i^{l-(k-l)} \int dx \ ([\omega_M^{-1/2} \cdot F_M](x))^{k-l} \ ([\omega_M^{-1/2} \cdot F'_M](x))^l < W_M[F_M]\Omega_M, \ W_M[F'_M] \ \Omega_M >_{\mathcal{H}_M}$$
$$=< W_M[F_M]\Omega_M, \ [V_k(l)]_M \ W_M[F'_M]\Omega_M >_{\mathcal{H}_M}, \tag{4.9}$$

where

$$[V_k(l)]_M = \int dx \ ([\omega_M^{-1/2} \cdot A_M]^\dagger(x)])^{k-l} \ (\omega_M^{-1/2} \cdot A_M](x)])^l. \tag{4.10}$$

It follows that the initial family defined in section 3 coincides with the family blocked from the continuum. The reason for this is simply the identity

$$\omega \cdot P_M = P_M \cdot \omega_M, \tag{4.11}$$

which is due to the fact that the derivative operator ∂ preserves the subspace L_M. This particular feature of the Dirichlet kernel is not shared by most other kernels such as the Schwarz kernel and in that case the initial family of section 3 does not coincide with the family blocked from the continuum of section 4.1 even for free theories, see [5, 6, 7, 8].

4.2 Renormalisation flow in terms of the projected fields

By definition, for $M < M'$ we consider the trivial embedding $L_M \to L_{M'}; \ F_M \mapsto F_{M'}$ which is possible because the L_M are nested subspaces of L. Then the analog of (4.1) is given by

$$\omega^{(n+1)}_{F,M}(w_M[F_M] \ w_M[G_M]) := \omega^{(n)}_{F,M'}(w_{M'}[F_M] \ w_{M'}[G_M]),$$
$$< W_M[F_M]\Omega^{(n+1)}_M, \ H^{(n+1)}_M \ W_M[F'_M]\Omega^{(n+1)}_M >_{\mathcal{H}^{(n+1)}_M} :=< W_{M'}[F_M]\Omega^{(n)}_{M'}, \ H^{(n)}_{M'} \ W_{M'}[F'_M]\Omega^{(n)}_{M'} >_{\mathcal{H}^{(n)}_{M'}}, \tag{4.12}$$

where $(\rho^{(n+1)}_M, \mathcal{H}^{(n+1)}_M, \Omega^{(n+1)}_M)$ and $(\rho^{(n)}_{M'}, \mathcal{H}^{(n)}_{M'}, \Omega^{(n)}_{M'})$ are the GNS data of $\omega^{(n+1)}_{F,M}$ and $\omega^{(n)}_{F,M'}$ respectively and we have denoted the operator representatives by capital letters, i.e. $W^{(n+1)}_M[F_M] = \rho^{(n+1)}_M(w_M[F_M])$ and $W_{M'}[F_{M'}] = \rho^{(n)}_{M'}(w_{M'}[F_{M'}])$. On the right hand side of (4.12) the elements F_M, G_M, F'_M are to be considered as elements of the continuum 1-particle Hilbert space $L_{M'}$ by trivial embedding $F_M \mapsto F_M$.

Going through literally the same steps as in section 4.1, we find that the flow is trivial: $\omega^{(n)}_{F,M} = \omega^{(0)}_{F,M} = \omega^*_{F,M} = \omega_{F,M}$ and $H^{(n)}_M = H^{(0)}_M = H^*_M = H_M$, that is, every sequence element of the family coincides with the initial element which is also the element blocked from the continuum and thus the fixed point element. This is again due to (for $M' > M$)

$$\omega_{M'} \cdot P_M = P_M \cdot \omega_M. \tag{4.13}$$

4.3 Renormalisation flow in terms of discretised fields

In order to test the version of Hamiltonian renormalisation method described in this series of papers less trivially for the solvable and self-interacting $P(\Phi)_2$ theory, one therefore should use an initial family which deviates from the natural choice of section 3. For example instead of the position non-local momentum ONB e_n of $L_M, \ n \in \mathbb{Z}_M$

one may use the position quasi-local ONB $\chi^M_m,\ m\in\mathbb{N}_M=\{0,1,..,M-1\}$ of L_M constructed in appendix B, see (B.7), (B.8). Indeed we can alternatively write (4.10) as

$$[V_k(l)]_M = M^{-k}\sum_{m_1,..,m_k\in\mathbb{N}_M} g_{M;m_1,m_2,..,m_k}\prod_{s=1}^{k-l}[\omega_M^{-1}\cdot A_M]^*(m_s)\prod_{s=k-l+1}^{k}[\omega_M^{-1}\cdot A_M](m_s), \tag{4.14}$$

where the "coupling constant" is given by

$$g_{M;m_1,m_2,..,m_k} = \int_0^1 dx\,\prod_{s=1}^{k}\chi^M_{m_s}(x) = \sum_{|n_1|,..,|n_k|\le(M-1)/2}\delta_{n_1+..+n_k=0}\prod_{s=1}^{k}e^{2\pi i n_k x^M_{m_s}}$$
$$= \sum_{|n_1|,..,|n_{k-1}|,|n_1+..+n_{k-1}|\le(M-1)/2}\ \prod_{s=1}^{k-1}e^{2\pi i n_k(x^M_{m_s}-x^M_{m_k})} \tag{4.15}$$

with $x^M_m=m/M$. If it was not for the constraint $|n_1+..+n_{k-1}|\le(M-1)/2$ this would collapse to

$$\prod_{s=1}^{k-1}\chi^M_{m_s}(x^M_{m_k}) = M^{k-1}\prod_{s=1}^{k-1}\delta_{m_s,m_k} \tag{4.16}$$

and we have

$$[\omega_M^{-1/2}\cdot A_M](m) = <\chi^M_m,\omega_M^{-1/2}\cdot A_M>_{L_M} = <\chi^M_m,\omega^{-1/2}A>_{L_M}, \tag{4.17}$$

because $\chi^M_m\in L_M$ and due to (4.11). Then (4.14) could be considered as the $k-$fold Riemann sum of the coupling constant times the displayed polynomial of discretised annihilation and creation operators as $\epsilon_M = x^M_{m+1}-x^M_m=M^{-1}$.

However, due to the constraint the coupling constant (4.15) is only quasi-local and it is for (4.15) that the flow is at its fixed point and not for (4.16). But one could start from the naive guess (4.16)

$$g^{(0)}_{M;m_1,m_2,..,m_k} = M^{k-1}\prod_{s=1}^{k-1}\delta_{m_s,m_k}, \tag{4.18}$$

and then run the renormalisation flow. Already after the first iterations step one finds that the form (4.18) is not preserved and that $g^{(1)}_{M;m_1,m_2,..,m_k}$ takes a more general form. Starting from this more general form one sees that (4.15) is a fixed point of the corresponding renormalisation flow equation.

In more detail, we make the general Ansatz

$$[V^{(r)}_k(l)]_M = M^{-k}\sum_{m_1,..,m_k\in\mathbb{N}_M} g^{(r)}_{M;m_1,m_2,..,m_k}\prod_{s=1}^{k-l}[\omega_M^{-1}\cdot A_M]^*(m_s)\prod_{s=k-l+1}^{k}[\omega_M^{-1}\cdot A_M](m_s), \tag{4.19}$$

for the r-th renormalisation step with initial condition (4.18). Then due to $\omega_{3M}^{-1}\ I_{M3M}=I_{M3M}\ \omega_{3M}^{-1}$ sandwiching $[V^{(r)}_k(l)]_{3M}$ between states of the form $w_{3M}[I_{M3M}f_M]\Omega_{3M}$ and requiring this to be $[V^{(r+1)}_k(l)]_M$ we see that we obtain

$$[V^{(r+1)}_k(l)]_M = (3M)^{-k}\sum_{m'_1,..,m'_k\in\mathbb{N}_{3M}} g^{(r)}_{3M;m'_1,m'_2,..,m'_k}\prod_{s=1}^{k-l}[I_{M3M}\omega_M^{-1}\cdot A_M]^*(m'_s)\prod_{s=k-l+1}^{k}[I_{M3M}\omega_M^{-1}\cdot A_M](m'_s), \tag{4.20}$$

which produces the flow equation

$$g^{(r+1)}_{M;m_1,..,m_k} = 3^{-k}\sum_{m'_1,..,m'_k}[\prod_{s=1}^{k}I_{M3M}(m'_s,m_s)]\ g^{(r)}_{3M;m'_1,..,m'_k} \tag{4.21}$$

It is easy to see that (4.21) has (4.15) as a fixed point because

$$\sum_{m'} I_{M3M}(m',m)\; \chi^{3M}_{m'}(x) = M^{-1} \sum_{m'} <\chi^{3M}_{m'}, \chi^M_m>\; \chi^{3M}_{m'}(x) = 3(P_{3M}\chi^M_m)(x) = 3\chi^M_m(x) \tag{4.22}$$

where completeness $\sum_m \chi^M_m(x)\chi^M_m(y) = M\; P_M(x,y)$ and $\chi^M_m \in L_M \subset L_{3M}$ was used.

We now use (4.16) as initial condition for the flow defined by (4.21). We need for $m' \in \mathbb{N}_{3M},\; m \in \mathbb{N}_M,\; n' \in \mathbb{Z}_{3M},\; n \in \mathbb{Z}_M$

$$<\chi^{3M}_{m'}, \chi^M_m> = \sum_{n,n'} e^{2\pi i\,[n'\,x^{3M}_{m'} - n\,x^M_m]} \;<e^{2\pi n' \cdot}, e^{2\pi i n \cdot} = \sum_n e^{2\pi i\, n[x^{3M}_{m'} - n\,x^M_m]} \tag{4.23}$$

since $\mathbb{Z}_M \subset \mathbb{Z}_{M'}$. Thus explicitly

$$\begin{aligned}
g^{(1)}_{M;m_1,..,m_k} &= \frac{1}{(3M)^k} \sum_{m' \in \mathbb{N}^k_{3M}} \prod_{s=1}^k <\chi^{3M}_{m'_s}, \chi^M_{m_s}>\; g^{(0)}_{3M;m'_1,..,m'_k} \\
&= \frac{1}{3M} \sum_{m' \in \mathbb{N}_{3M}} \prod_{s=1}^k <\chi^{3M}_{m'}, \chi^M_{m_s}> \\
&= \frac{1}{3M} \sum_{n \in \mathbb{Z}^k_M} \sum_{m' \in \mathbb{N}_{3M}} \prod_{s=1}^k e^{2\pi i\, n_s [x^{3M}_{m'} - x^M_{m_s}]} \\
&= \sum_{n \in \mathbb{Z}^k_M} [\prod_{s=1}^k e^{-2\pi i\, n_s\, x^M_{m_s}}]\; \delta_{n_1+..+n_k,0\;(\text{mod } 3M)}
\end{aligned} \tag{4.24}$$

where in the last step we took care of the fact that summing over $m' \in \mathbb{N}_{3M'}$ produces $3M$ when $n_1 + .. + n_k$ is an integer multiple of $3M$ and zero else. Comparing with (4.15) we see that the difference between the two expressions consists in the modulo $3M$ support of the Kronecker symbol because we can relabel $n_s \to -n_s$ as both sums are over the reflection invariant domain $\mathbb{Z}_M$.

At the next iteration step we find

$$\begin{aligned}
g^{(2)}_{M;m_1,..,m_k} &= \frac{1}{(3M)^k} \sum_{m' \in \mathbb{N}^k_{3M}} \prod_{s=1}^k <\chi^{3M}_{m'_s}, \chi^M_{m_s}>\; g^{(1)}_{3M;m'_1,..,m'_k} \\
&= \frac{1}{(3M)^k} \sum_{n' \in \mathbb{Z}^k_{3M}} \sum_{m' \in \mathbb{N}^k_{3M}} [\prod_{s=1}^k <\chi^{3M}_{m'_s}, \chi^M_{m_s}>\; e^{-2\pi i\, n'_s\, x^{3M}_{m'_s}}]\; \delta_{n'_1+..+n'_k,0\;(\text{mod } 3\,(3M))} \\
&= \frac{1}{(3M)^k} \sum_{n' \in \mathbb{Z}^k_{3M}} \sum_{n \in \mathbb{Z}^k_M} \sum_{m' \in \mathbb{N}^k_{3M}} [\prod_{s=1}^k e^{-2\pi i n_s x^M_m}\;\; e^{2\pi i\,(n_s - n'_s)\, x^{3M}_{m'_s}}]\; \delta_{n'_1+..+n'_k,0\;(\text{mod } 3^2\, M)} \\
&= \sum_{n' \in \mathbb{Z}^k_{3M}} \sum_{n \in \mathbb{Z}^k_M} [\prod_{s=1}^k e^{-2\pi i n_s x^M_m}\;\; \delta_{n_s,n'_s}]\; \delta_{n'_1+..+n'_k,0\;(\text{mod } 3^2\, M)} \\
&= \sum_{n \in \mathbb{Z}^k_M} [\prod_{s=1}^k e^{-2\pi i n_s x^M_m}]\; \delta_{n_1+..+n_k,0\;(\text{mod } 3^2\, M)}
\end{aligned} \tag{4.25}$$

where we used that $\delta_{n'-n,0\;(\text{mod}3M)} = \delta_{n'-n,0}$ as $|n'-n| \le \frac{3M-1+M-1}{2} < 2M < 3M$ i.e. the modulo operation can be dropped. We see that in the second iteration step the only change that happened compared to the first is that the modulo operation is now with respect to $3^2 M$ rather than $3^1 M$.

Iterating (formally one proceeds by induction) we find explicitly for $r \geq 1$

$$g^{(r)}_{M;m_1,..,m_k} = \sum_{n \in \mathbb{Z}_M^k} [\prod_{s=1}^k e^{-2\pi i n_s x_m^M}] \, \delta_{n_1+..+n_k,0 \text{ (mod } 3^r M)} \tag{4.26}$$

Now $|n_1+..+n_k| \leq k(M-1)/2$. For fixed resolution M, the number of steps r required such that the modulo operation can be dropped is when that estimate is lower than the smallest possible nonzero integer multiple of $3^r M$ i.e. $k(M-1)/2 < 3^r \, M$ i.e. $3^r > k\frac{M-1}{2M}$. Thus independently of M for a polynomial potential of degree k we need at most $r_k := 1 + [\frac{\ln(k/2)}{\ln(3)}]$ renormalisation steps until $g^{(r)}_{M;m_1,..,m_k} = g_{M;m_1,..,m_k}$ for all $r \geq r_k$ where [.] denotes the Gauss bracket. Note that this quick convergence of the flow would not occur had we not defined $\omega_M^2 = p^2 - \Delta_M, \; \Delta_M = \partial_M^2, \; \partial_M = I_M^\dagger \partial I_M$ but instead had replaced ∂_M by some of the more common discrete lattice derivatives (e.g. forward derivative $(\partial_M^+ f_M)(m) = M[f_M(m+1) - f_M(m)]$) because then the intertwining property $I_{M3M} \cdot \partial_M = \partial_{3M} \cdot I_{M3M}$ would not hold.

We end this section by illustrating graphically the degree of non-locality of the fixed point coupling (4.15) for the case $k=3$. Explicitly we consider

$$f_M(y_1, y_2) := M^{-2} \sum_{|n_1|,|n_2|,|n_1+n_2|<(M-1)/2} e^{2\,i(n_1\,y_1+n_2\,y_2)} \tag{4.27}$$

which yields

$$f(y_1 = \pi[x^M_{m_1} - x^M_{m_3}], y_2 = \pi[x^M_{m_2} - x^M_{m_3}]) = M^{-2} \; g_{M;m_1,m_2,m_3} \tag{4.28}$$

which we expect to be a quasi-local version of $\delta_{m_1,m_3} \; \delta_{m_2,m_3}$. The sums in (4.27) can be computed in closed form using the geometric series summation formula $\sum_{n=0}^{N-1} \; z^n = (z^N - 1)/(z-1)$. We subdivide the summation domain into the three sets

$$\begin{aligned} & \{n_1 = \{1,.., \frac{M-1}{2}\}, \; n_2 = \{-\frac{M-1}{2},.., \frac{M-1}{2} - n_1\}\} \\ \cup \quad & \{n_1 = \{-1,.., -\frac{M-1}{2}\}, \; n_2 = \{-\frac{M-1}{2} - n_1,.., \frac{M-1}{2}\}\} \\ \cup \quad & \{n_1 = \{0\}, \; n_2 = \{-\frac{M-1}{2},.., \frac{M-1}{2}\}\} \end{aligned} \tag{4.29}$$

One finds

$$f(y_1, y_2) = \frac{1}{2\;M^2}\{\frac{\cos(M(y_1-y_2))}{\sin(y_1)\sin(y_2)} + \frac{\cos(My_2)}{\sin(y_1)\sin(y_1-y_2)} - \frac{\cos(My_1)}{\sin(y_2)\sin(y_1-y_2)}\} \tag{4.30}$$

Despite its appearance, this function is everywhere smooth and bounded. Using $z_j = x_k - x_l, \; \epsilon_{jkl} = 1$ with $y_1 = x_1 - x_3, y_2 = x_2 - x_3$ one can write this in the manifestly permutation invariant form

$$f(z_1, z_2, z_3) = -\frac{1}{2\;M^2}\{\frac{\cos(M\;z_1)}{\sin(z_2)\sin(z_3)} + \frac{\cos(M\;z_2)}{\sin(z_3)\sin(z_1)} + \frac{\cos(M\;z_3)}{\sin(z_1)\sin(z_2)}\} \tag{4.31}$$

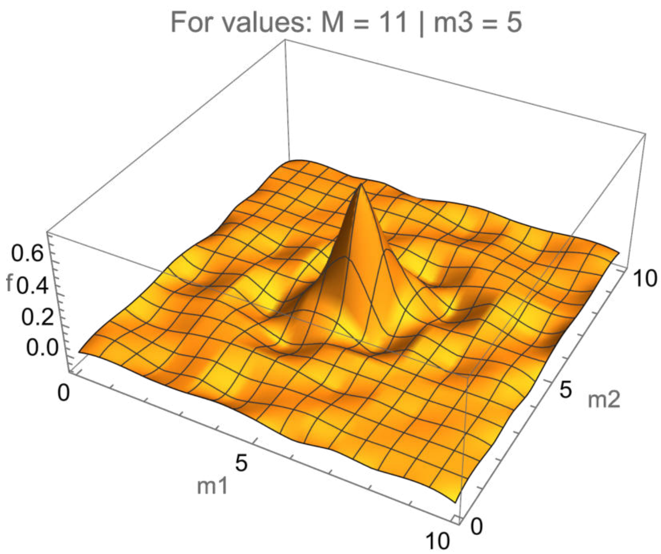
noticing the identity $z_1 + z_2 + z_3 = 0$ which displays the totally symmetric, smooth and bounded form (4.15) of this function. For any $z_j = 0 \; \Rightarrow z_k = -z_l = z, \; \epsilon_{jkl} = 1$ it reduces to

$$\tilde{f}(z) = \frac{1}{2\;M^2\sin^2(z)} \; \{M \; \sin(M\;z)\sin(z) + 1 - \cos(M\;z)\cos(z)\} \tag{4.32}$$

as one can confirm by using de l'Hospital's theorem. Using de l'Hospital's theorem one more time when taking $z \to 0$ one can explicitly check that the maximum of this function is given not by unity but by $f(0,0) = \tilde{f}(0) = \frac{3}{4} + \frac{1}{4M^2}$ corresponding to the fact that the constraints $|n_1 + n_2| \leq (M-1)/2$ on n_1, n_2 roughly delete one quarter of the unconstrained M^2 points. Away from the maximum the function is of order M^{-2} at a generic lattice point while it is of order M^{-1} on one of the coordinate axes and on the diagonal $y_1 = y_2$

(these are precisely the points $z_j = 0,\ j = 1, 2, 3$) but well away from the origin. This can be seen analytically noticing that $\cos(M\ z_j) = (-1)^{m_k - m_l},\ \epsilon_{jkl} = 1$ while $\sin(M\ z_j) = 0$. Accordingly if all $m_j - m_k$ are of the order $(M-1)/2$ the $\sin$ functions in the denominator of f are close to unity. If on the other hand say $m_2 = m_3$ corresponding to $z_1 = y_2 = 0, -z := z_3 = -z_2 = \pi(m_1 - m_2)/M =: m/M$ one finds $\tilde{f}(z) = \frac{1-(-1)^m \cos(m\pi/M)}{2\ M^2 \sin^2(m\pi/M)}$ which for m odd exceeds $\frac{1}{2\ M^2 \sin^2(m\pi/M)}$ and for large M decreases quasi continuously from approximatively $[2\pi^2]^{-1}$ to $[2M^2]^{-1}$ between $m = 1$ and $m = (M-1)/2$. It is of order $\pi/(2M)$ for $m \approx \sqrt{M/\pi}$ which is well away from the vicinity of the origin but for large M is outside of approximately the fraction $\sqrt{M}/M = 1/\sqrt{M}$ of the available points.

In figures 1, 2, 3 we illustrate this quasi-locality graphically for a few fixed resolutions $M = 11,\ 21,\ 51,\ 71,\ 111$ where we pick $m_3 = 5,\ 10,\ 25,\ 35,\ 55$ respectively while $m_1, m_2 \in \mathbb{N}_M$ take full range. We computed the quasi-local coupling at the exact lattice points and then let Mathematica interpolate between those values. One sees the effect that the maximum ≈ 0.75 becomes more pronounced and concentrated as the resolution increases while the function continues to display non-trivial oscillations in the vicinity of the maximum and along the coordinate axes and the diagonal.



(a)

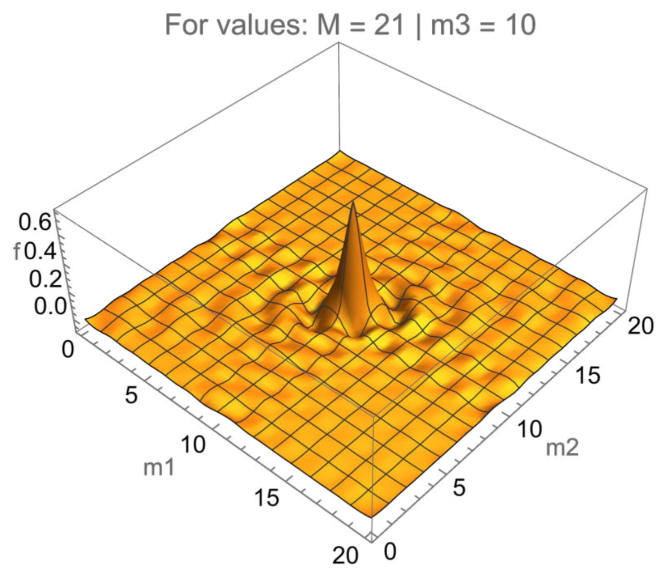
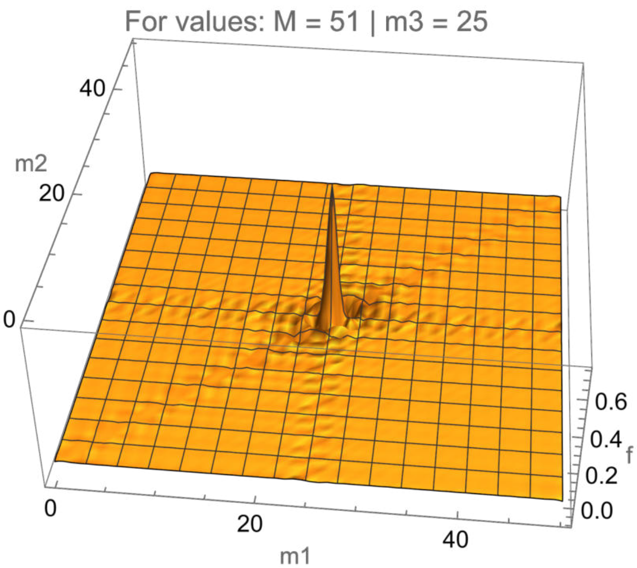
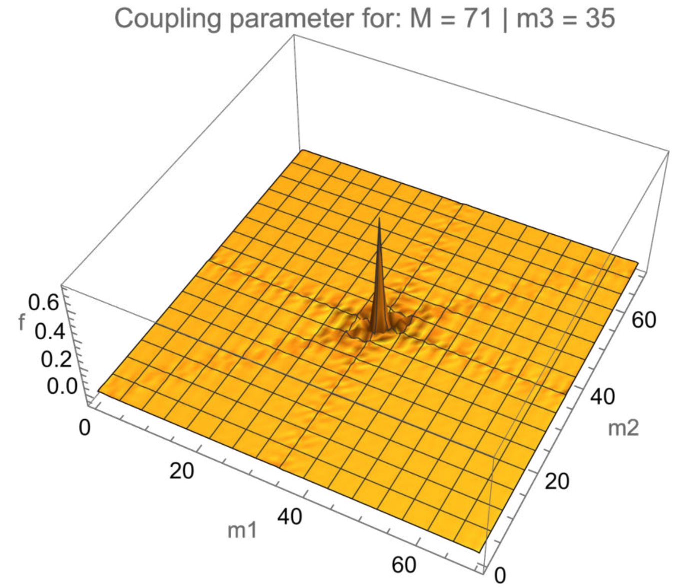


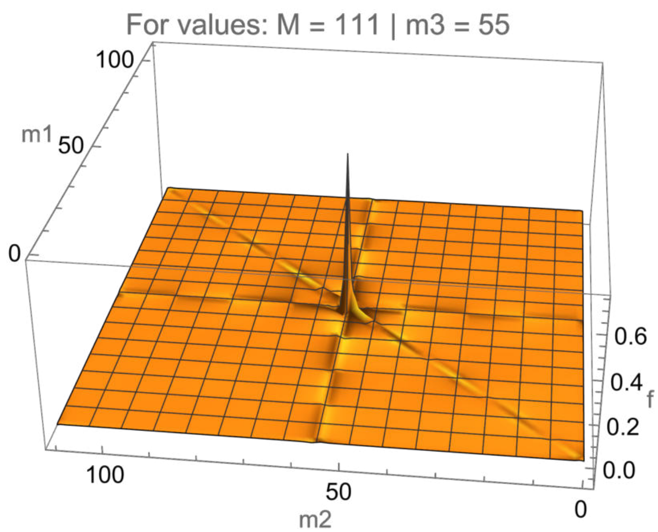
Figure 1: **Left:** Coupling for $M = 11$ and $m_3 = 5$. **Right:** Coupling for $M = 21$ and $m_3 = 10$.

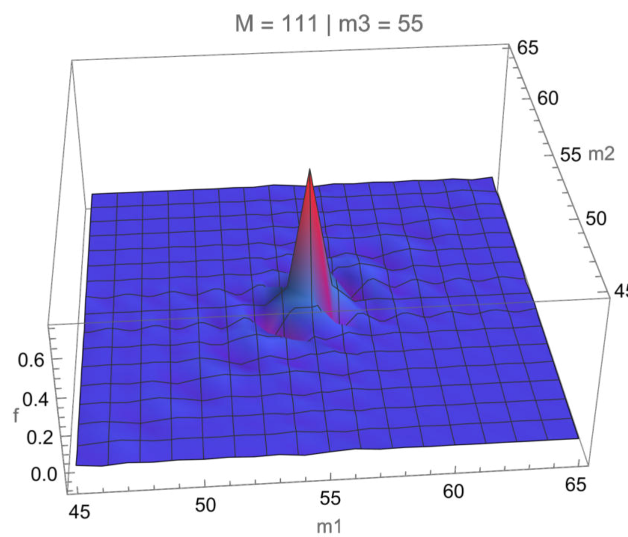




(a)

Figure 2: **Left:** Coupling for $M = 51$ and $m_3 = 25$. **Right:** Coupling for $M = 71$ and $m_3 = 35$.

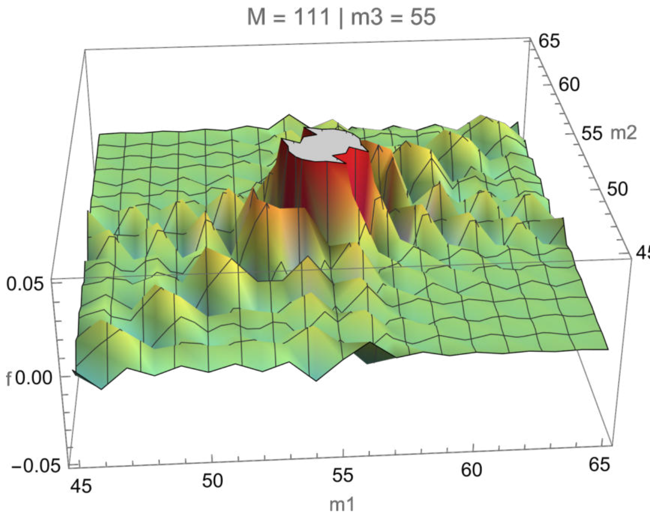




(a)

Figure 3: Coupling for $M = 111$ and $m_3 = 55$. **Left:** Full range. **Right:** Vicinity of maximum.

In figure 4 we zoom into the $\sqrt{M}$ vicinity of the maximum at $m_1 = m_2 = m_3 = 55$ for $M = 111$. We cut off the maximum peak at a convenient value in order not to suppress the values of the coupling in its vicinity.



(a)

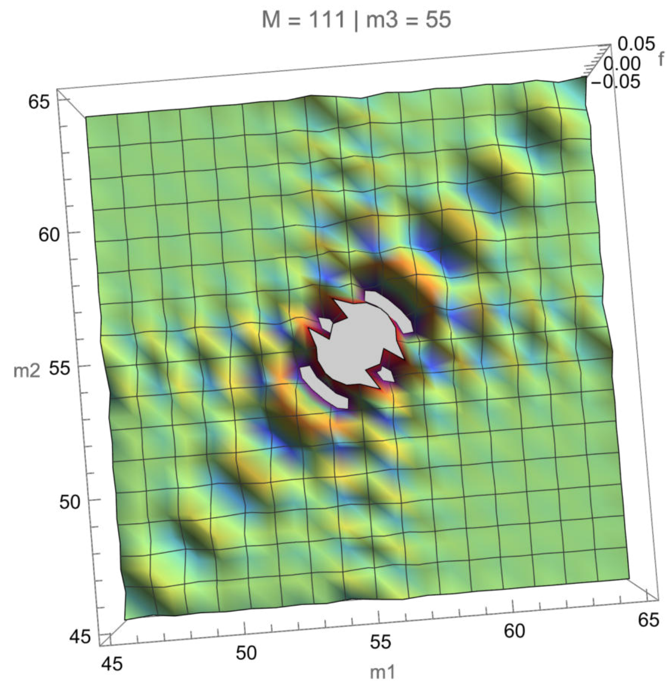


Figure 4: High resolution in the vicinity of the maximum for coupling for $M = 111$ and $m_3 = 55$.
Left: Side view. **Right:** Top view.

5 Conclusions

We have successfully applied the version of Hamiltonian renormalisation developed in the current sequence of papers for the first time to an interacting QFT in finite volume in 1+1 spacetime dimensions and showed that the renormalisation flow finds the known rigorous fixed point solution which is available in this case. The flow in fact stabilises rather quickly if one uses as coarse graining tools not the familiar position localised block spin transformations but rather smooth versions thereof which compromise between position and momentum locality and have better smoothness properties. This demonstrates that the tools developed in the course of this series, which so far has focussed on free QFT, are also successfully applicable in the interacting case and does find the correct continuum Hamiltonian **operator**.

Extrapolating to interacting QFT in higher, specifically four spacetime dimensions, the Hamiltonian $H = H_0 + V$ when quantised in the Fock representation adapted to H_0 is known to be no longer an operator but merely a **quadratic form**. Following the steps that we have carried out for the present model to any interacting QFT with Hamiltonian $H = H_0 + V$ on the D+1 manifold $\mathbb{R} \times T^D$ shows that the corresponding flow would indeed find that quadratic form as a fixed point in the Fock representation adapted to H_0. However, ideally renormalisation is designed to actually construct a continuum theory with H as an operator rather than merely a quadratic form which by Haag's theorem [1] requires to leave the realm of Fock representations adapted to H_0. To the best of our knowledge, the single known example where this actually worked in higher than 1+1 dimensions is Φ^4 theory in 2+1 dimensions [16]. In addition to making the coupling constants g_k to depend on M (which in the present model was not necessary) a new and non-trivial step successfully performed in that seminal work was to invent invertible "dressing transformations" T_M at resolution M on the cut-off Fock space $\mathcal{H}_M$ with modes confined to $n \in \mathbb{Z}_M$ such that in $H'_M := T_M^{-1}\, H_M T_M$ those terms are removed which prevent H_M from being an operator rather than a quadratic form as $M \to \infty$ (these are normal ordered monomials in field operator valued distributions with more than one creation operator). If that is the case then $\mathcal{D}$ is an invariant domain for H'_M where $\mathcal{D}$ is the span of Fock states in the free Fock space $\mathcal{H}$. Then H_M, defined on the new domain $\mathcal{D}'_M = T_M\mathcal{D}$ preserves $\mathcal{D}'_M$. The price to pay is that the norm of the vectors in $\mathcal{D}'_M$ with respect to the inner product $<.,.>$ on $\mathcal{H}$ diverges as $M \to \infty$. In [16] it was shown that one can define a new inner product $<.,.>'_M := \frac{<.,.>}{<T_M\Omega, T_M>}$ on $\mathcal{D}'_M$ where Ω is the Fock vacuum of H_0, such that one can take $M \to \infty$ and $H'_M \to H'$ becomes on operator densely defined on $\mathcal{D}'_M \to \mathcal{D}'$ whose Hilbert space completion with respect

$< .,. >'_M \to < .,. >'$ results in a new Hilbert space $\mathcal{H}'$ for the interacting theory. In order that this works, the norms of vectors $T_M\psi,\ \psi \in \mathcal{D}$ with respect to $< .,. >$ must all diverge at the same rate. We expect that similar mechanisms must be invoked in the present renormalisation scheme as well.

In a forthcoming work we will apply the version of Hamiltonian renormalisation developed in this series of papers to the self-interacting $U(1)^3$ model of the weak Newton constant realm of Euclidian signature vacuum quantum gravity in four spacetime dimensions [17]. This model has two known solutions, one in terms of operators [18] using a Narnhofer-Thirring type of representation [19] and one in terms of quadratic forms [20] in Fock representations [20] in the sense that the full algebra of quantum constraints (Gauss, spatial diffeomorphism and Hamiltonian) closes without anomaly. We will investigate the Hamiltonian renormalisation flow for both types of representations with the aim to gain new insights in how to renormalise full quantum gravity e.g. in the Loop Quantum Gravity (LQG) representation [21] in order to remove or reduce present quantisation ambiguities.

Acknowledgements
M.R.-Z. thanks Jonas Neuser for useful discussions on the topic and acknowledges the financial support provided by the Deutsche Akademische Austauschdienst e. V.

A Finiteness of $P(\Phi)_2$ QFT in Fock representations

As motivated in section 2 we pick the Fock representation selected by the free part of the Hamiltonian H_0. We want to show that the interaction part V is densely defined on $\mathcal{D}$, the linear span of Fock vectors with smearing functions in L_0, the L_2 functions with compact momentum support. Below we present an elementary proof that requires just first year's calculus knowledge.

To show this, consider a general Fock vector ψ. Then we have the elementary estimate

$$||V\psi||_{\mathcal{H}} \le \frac{|g_k|}{2^{k/2}} \sum_{l=0}^{k} \left(\begin{array}{c} k \\ l \end{array} \right) ||V_k(l)\psi||_{\mathcal{H}}, \tag{A.1}$$

and writing $\psi = \sum_{n=0}^N \psi_n$ where ψ_n is a Fock vector of particle number n we have further the estimate

$$||V_k(l)\psi||_{\mathcal{H}} \le \sum_{n=l}^{N} ||V_k(l)\psi_n||_{\mathcal{H}} \tag{A.2}$$

Here we used that $V_k(l)$ annihilates l particles and thus the r.h.s. of (A.2) vanishes for $N < l$. Let then $n \ge l$ and $\psi_0 = c\Omega,\ c =$const. or $\psi_n = \prod_{r=1} [< f_r, A >]^\dagger\ \Omega,\ n > 0$. Then for $l \ge 1$ we have

$$V_k(l)\psi_n = \sum_{I\in S_l} \int_0^1 dx\ [[\omega^{-1/2} \cdot A^*](x)]^{k-l} \prod_{i\in I} [\omega^{-1/2} \cdot f_i](x) \prod_{j\notin I} < f_j, A >^*\ \Omega \tag{A.3}$$

where S_l is the set of subsets of $\{1, .., n\}$ with l elements and for $l = 0$ we have

$$V_k(0)\psi_n = \int_0^1 dx\ [[\omega^{-1/2} \cdot A^*](x)]^k\ \psi_n \tag{A.4}$$

Using further elementary estimates it follows that V is densely defined on $\mathcal{D}$ if and only if objects of the form

$$\int_0^1 dx\ F_l(x)\ [[\omega^{-1/2} \cdot A^*](x)]^{k-l}\ \psi_{n-l} \tag{A.5}$$

are normalisable vectors in the Fock space where F_l is either the constant function equal to unity for $l = 0$ or a product of l functions in L_0 for $l > 0$. It is understood that $k, n \ge l$ and $k = 0, .., N$. For $k = 0$ and thus $l = 0$ there is nothing to show as (A.5) just equals $< 1, 1 >\ \psi_n$ which is finite as $1 \in L$ when space is compact. For

$k = 1$ (A.5) equals $< 1, A >^* \psi_n$ and $< 1, F_1 > \psi_{n-1}$ respectively which are both Fock vectors. The interesting terms therefore come form $k \geq 2$.

The norm squared of (A.5) is

$$\int_0^1 dx \int_0^1 dy\ F_l^*(x)\ F_l(y)\ < \psi_{n-l},\ [[\omega^{-1/2} \cdot A](x)]^{k-l}\ [[\omega^{-1/2} \cdot A^*](y)]^{k-l}\ \psi_{n-l} \tag{A.6}$$

When moving the annihilators to the right we get a sum of terms with $0 \leq r \leq k - l$ factors of

$$[(\omega^{-1/2} \cdot A)(x)\ ,\ (\omega^{-1/2} \cdot A^*)(y)] = \omega^{-1}(x, y) = \sum_{n \in \mathbb{Z}} \hat{\omega}_n^{-1}\ e_n(x - y) \tag{A.7}$$

which is the integral kernel of the operator ω^{-1} and another $k - l - r$ factors of functions in L_0 (for $k - l - r < 0$ that term vanishes). Altogether we see that V is densely defined on $\mathcal{D}$ if and only if integrals of the form

$$\int_0^1 dx \int_0^1 dy\ F_{k-r}^*(x)\ G_{k-r}(y)\ [\omega{-}1(x, y)]^r \tag{A.8}$$

converge with $2 \leq k \leq N$ and $0 \leq r \leq k$ with $F_{k-r}, G_{k-r} \in L_0$ when $k - r > 0$ and $F_0 = G_0 = 1$. Let $\hat{F}_n =< e_n, F >$ be the Fourier transform of $F \in L$, then (A.8) is either

$$\sum_{K \in \mathbb{Z}} \hat{F}_{k-r}^*(K)\ \hat{G}_{k-r}(K) \sum_{n_1,..,n_r \in \mathbb{Z}} \delta_{n_1+..+n_r,K} \prod_{s=1}^r \hat{\omega}_{n_s}^{-1} \tag{A.9}$$

for $k - r > 0$ or

$$\sum_{n_1,..,n_r \in \mathbb{Z}} \delta_{n_1+..+n_r,0} \prod_{s=1}^r \hat{\omega}_{n_s}^{-1} \tag{A.10}$$

for $k - r = 0$ where again compactness of space was important as the kernel is translation invariant. As functions in L_0 have compact momentum support, the sum over K in (A.9) is finite and it is sufficient to show that the sums

$$\sum_{n_1,..,n_r \in \mathbb{Z}} \delta_{n_1+..+n_r,K} \prod_{s=1}^r \hat{\omega}_{n_s} \tag{A.11}$$

converge for every $2 \leq r \leq N$ and any finite $K \in \mathbb{Z}$.

The intuitive reason why this is in fact the case is as follows: If it was not for the Kronecker symbol, the sums would decouple into a product of r sums each of which behaves as the $M \to \infty$ limit of $\zeta_M(1)$ which diverges as $\ln(M)$ and where $\zeta_M(z) = \sum_{n=1}^M n^{-z}$ is the M cut-off of the Riemann ζ function. However due to the Kronecker symbol we only get $r - 1$ sums over $n_1, .., n_{r-1}$ but still r factors, one depending on $|n_s|,\ s = 1, .., n_{r-1}$ and one depending on $|n_1 + .. + n_{r-1} - K|$ which roughly adds an additional factor of $\hat{\omega}_{n_s}^{1/(r-1)}$. Thus each of the $r - 1$ sums behave as $\zeta_M(1 + \frac{1}{r-1})$ which converges for every $r > 1$. Of course this heuristic argument is not rigorous.

We therefore rigorously estimate the multiple sum in (A.11) as follows: First we simplify

$$\begin{aligned} \sum_{n_1,..,n_r \in \mathbb{Z}} \prod_{s=1}^r \hat{\omega}_{n_s}^{-1}\ \delta_{n_1+..+n_r,K} &\leq \sum_{n_1,..,n_{r-1} \in \mathbb{Z} \cup \{0\}} \prod_{s=1}^r \hat{\omega}_{n_s}^{-1}\ \delta_{n_1+..+n_r,K} \\ = & \sum_{\sigma_1,..,\sigma_r=\pm 1} \sum_{n_1,..,n_r \in \mathbb{N}_0} \prod_{s=1}^r \hat{\omega}_{\sigma_s n_s}^{-1}\ \delta_{\sigma_1 n_1+..+\sigma_r n_r,K} \\ = & \sum_{\sigma_1,..,\sigma_r=\pm 1} \sum_{n_1,..,n_r \in \mathbb{N}_0} \prod_{s=1}^r \hat{\omega}_{n_s}^{-1}\ \delta_{\sigma_1 n_1+..+\sigma_r n_r,K} \\ = & \sum_{t=0}^r \left(\begin{array}{c} r \\ t \end{array} \right) \sum_{n_1,..,n_r \in \mathbb{N}_0} \prod_{s=1}^r \hat{\omega}_{n_s}^{-1} \times \\ & \{\theta(K-1)\ \delta_{[n_1+..+n_t]-[n_{t+1}+..+n_r+|K|],0} + \theta(-K-1)\ \delta_{[n_1+..+n_t+|K|]-[n_{t+1}+..+n_r],0} + \delta_{K,0}\ \delta_{[n_1+..+n_t]-[n_{t+1}+..+n_r],0}\} \end{aligned} \tag{A.12}$$

where we made use of reflection invariance $\hat{\omega}_n = \hat{\omega}_{-n}$ and that the product of the inverse eigenvalues is invariant under permutations of r summation variables $n_1, .., n_r$ so that the sum over $\sigma_1, ..\sigma_r$ can be reduced to a summation over the number $t = 0, 1, .., r$ of those $\sigma_1, ..\sigma_r$ which take the value $+$ which can be taken to be the $\sigma_1, .., \sigma_t$.

The first term vanishes when $t = 0$ and for $t = r$ the sum is constrained by $n_1, .., n_r \leq |K|$ as all summation variables are non negative. Likewise, the second term vanishes for $t = r$ and for $t = 0$ the sum is constrained by $n_1, .., n_r \leq |K|$. Finally the third term collapses to $n_1 = .. = n_r = 0$ for $t = 0, t = r$. Thus the terms with $t = 0, r$ are trivially finite and it is sufficient to consider only the terms $0 < t < r$. For those, in the second term we relabel $t \to r - t$ and switch the two groups $n_1, .., n_t$ and $n_{t+1}, .., n_r$. As $\theta(K-1) + \theta(-K-1) + \delta_{K,0} = 1$ we end up with the relevant contribution

$$
\begin{aligned}
S_r(K) &:= \sum_{t=1}^{r-1} \binom{r}{t} \sum_{n_1,..,n_r \in \mathbb{N}_0} \prod_{s=1}^{r} \hat{\omega}_{n_s}^{-1} \; \delta_{[n_1+..+n_t]-[n_{t+1}+..+n_r+|K|],0} \\
&= m^r \sum_{t=1}^{r-1} \binom{r}{t} \sum_{L=|K|}^{\infty} [\sum_{n_1,..,n_t \in \mathbb{N}_0} \prod_{s=1}^{t} w_{n_s}^{-1} \; \delta_{n_1+..+n_t,L}] \, [\sum_{n_{t+1},..,n_r \in \mathbb{N}_0} \prod_{s=t+1}^{r} w_{n_s}^{-1} \; \delta_{n_{t+1}..+n_r,L-|K|}]
\end{aligned}
\tag{A.13}
$$

where $\kappa := \frac{2\pi}{m}$ and $w_n = \sqrt{1 + [\kappa n]^2}$ and m is the mass parameter. Using the the abbreviation for $1 \leq t \leq r-1, \; M \geq 0$

$$
s_t(M) = \sum_{n_1,..,n_t \in \mathbb{N}_0} \prod_{s=1}^{t} w_{n_s}^{-1} \; \delta_{n_1+..+n_t,M} \tag{A.14}
$$

which for $t = 1$ just equals w_M^{-1} we have

$$
S_r(K) = m^r \sum_{t=1}^{r-1} \binom{r}{t} \sum_{L=|K|}^{\infty} s_t(L) \; s_{r-t}(L - |K|) \tag{A.15}
$$

The idea is now to show that $s_t(M)$ decays at least as $|M|^{-p}$ for some $p > 1/2$ no matter the value of t for then the sum over L converges and the sum over t is anyway finite. For $t = 1$ this is trivial, hence in what follows we consider $t \geq 2$.

To this end, we use the basic estimate

$$
1 + x^2 \geq c\,(1 + |x|)^2 \;\Leftrightarrow\; [|x| - \frac{c}{1-c}\, m]^2 + \frac{1-2c}{(1-c)^2} m^2 \geq 0 \tag{A.16}
$$

which holds for $0 < c \leq \frac{1}{2}$. We pick $c = \frac{1}{4}$ and thus

$$
w_n^{-1} \leq \frac{2}{1 + \kappa n} \tag{A.17}
$$

It follows

$$
\begin{aligned}
s_t(M) \leq 2^t & \sum_{n_1=0}^{M} [1 + \kappa n_1]^{-1} \sum_{n_2=0}^{M-n_1} [1 + \kappa n_2]^{-1} \; ... \\
& \sum_{n_{t-2}=0}^{M-[n_1+..+n_{t-3}]} [1 + \kappa n_{t-2}]^{-1} \sum_{n_{t-1}=0}^{M-[n_1+..+n_{t-2}]} [1 + \kappa n_{t-1}]^{-1} \; [1 + \kappa(M - [n_1 + .. + n_{t-1}])]^{-1}
\end{aligned}
\tag{A.18}
$$

To estimate the sum over n_{t-1} we use the abbreviation $l = M - [n_1 + .. + n_{t-2}]$ (which is defined to equal M when $t = 2$)

$$
\begin{aligned}
& \sum_{n=0}^{l} [1 + \kappa n]^{-1} \; [1 + \kappa(l - n)]^{-1} = [1 + \kappa\, l]^{-1} \sum_{n=0}^{l} ([1 + \kappa n]^{-1} + [1 + \kappa(l - n)]^{-1}) = \frac{2}{1 + \kappa l} \sum_{n=0}^{l} [1 + \kappa n]^{-1} \\
= \; & \frac{2}{1 + \kappa l} [1 + \sum_{n=1}^{l} [1 + \kappa n]^{-1}] = \frac{2}{1 + \kappa l} [1 + \sum_{n=1}^{l} \int_{n-1}^{n} dk \; [1 + \kappa n]^{-1}] \leq \frac{2}{1 + \kappa l} [1 + \sum_{n=1}^{l} \int_{n-1}^{n} dk \; [1 + \kappa k]^{-1}] \\
= \; & \frac{2}{1 + \kappa l} [1 + \int_{0}^{l} dk \; [1 + \kappa k]^{-1}] = \frac{2}{1 + \kappa l} \; [1 + \kappa^{-1} \; \ln(1 + \kappa l)]
\end{aligned}
\tag{A.19}
$$

Consider now any $0 < p < 1$ and note that $z =: 1 + \kappa l \geq 1$ then

$$\frac{\ln(z)}{z} \leq z^{-p}(1-p)^{-1}, \; z^{-1} \leq z^{-p} \tag{A.20}$$

Thus there exists a constant c_1 (depending on p_1, κ) such that (A.19) is bounded from above by $c_1 \; [1+\kappa l]^{-p_1}$ for any $0 < p_1 < 1$. It follows that

$$\begin{aligned} s_t(M) &\leq 2^t \; c_1 \sum_{n_1=0}^{M} [1+\kappa n_1]^{-1} \sum_{n_2=0}^{M-n_1} [1+\kappa n_2]^{-1} \; ... \\ &\sum_{n_{t-3}=0}^{M-[n_1+..+n_{t-4}]} [1+\kappa n_{t-3}]^{-1} \sum_{n_{t-2}=0}^{M-[n_1+..+n_{t-3}]} [1+\kappa n_{t-2}]^{-1} \; [1+\kappa(M-[n_1+..+n_{t-2}])]^{-p_1} \end{aligned} \tag{A.21}$$

To estimate the sum over n_{t-2} we use the abbreviation $l = M - [n_1 + .. + n_{t-3}]$ (which is defined to equal M when $t = 3$) and use again $z^{-1} < z^{-p_1}$ for $z \geq 1$ to find

$$\sum_{n=0}^{l} [1+\kappa n]^{-1} \; [1+\kappa(l-n)]^{-p_1} \leq \sum_{n=0}^{l} [1+\kappa n]^{-p_1} \; [1+\kappa(l-n)]^{-p_1} \leq [1+\kappa l]^{-p_1} \sum_{n=0}^{l} [[1+\kappa n]^{-1} + [1+\kappa(l-n)]^{-1}]^{p_1} \tag{A.22}$$

For any two numbers $a, b > 0$ and $p > 0$ we have

$$[a+b]^p = \begin{pmatrix} a^p \; [1+b/a]^p & a \geq b \\ b^p \; [1+a/b]^p & b \geq a \end{pmatrix} \leq \begin{pmatrix} (2a)^p & a \geq b \\ (2b)^p & b \geq a \end{pmatrix} \leq 2^p \; [a^p + b^p] \tag{A.23}$$

Therefore (A.22) can be further estimated by

$$\begin{aligned} & 2^{p_1} \; [1+\kappa l]^{-p_1} \sum_{n=0}^{l} \{[1+\kappa n]^{-p_1} + [1+\kappa(l-n)]^{-p_1}\} = 2^{1+p_1} \; [1+\kappa l]^{-p_1} \sum_{n=0}^{l} [1+\kappa n]^{-p_1} \\ = \; & \frac{2^{1+p_1}}{[1+\kappa l]^{p_1}} \; [1 + \sum_{n=1}^{l} [1+\kappa n]^{-p_1}] = \frac{2^{1+p_1}}{[1+\kappa l]^{p_1}} \; [1 + \sum_{n=1}^{l} \int_{n-1}^{n} dk \; [1+\kappa n]^{-p_1}] \\ \leq \; & \frac{2^{1+p_1}}{[1+\kappa l]^{p_1}} \; [1 + \sum_{n=1}^{l} \int_{n-1}^{n} dk \; [1+\kappa k]^{-p_1}] \\ = \; & \frac{2^{1+p_1}}{[1+\kappa l]^{p_1}} \; [1 + \int_{0}^{l} dk \; [1+\kappa k]^{-p_1}] = \frac{2^{1+p_1}}{[1+\kappa l]^{p_1}} \; [1 + \kappa^{-1} \; (1-p_1)^{-1} \; ([1+\kappa l]^{1-p_1} - 1)] \\ \leq \; & \frac{2^{1+p_1}}{[1+\kappa l]^{p_1}} \; [|1 - \frac{1}{\kappa(1-p_1)}| + \kappa^{-1} \; (1-p_1)^{-1} \; [1+\kappa l]^{1-p_1}] \end{aligned} \tag{A.24}$$

We choose p_1 so close to 1 that $p_2 := 2p_1 - 1$ still obeys $0 < p_2 < 1$ which is the case for $1/2 < p_1 < 1$. Then $p_2 < p_1$ so that also $[1+\kappa l]^{-p_1} < [1+\kappa l]^{-p_2}$. Thus there exists a constant c_2 (depending on p_1, κ) such that (A.24) is bounded from above by $c_2 \; [1+\kappa l]^{-p_2}$ for any $1/2 < p_1 < 1$ with $p_2 = 2p_1 - 1$. It follows that

$$\begin{aligned} s_t(M) &\leq 2^t \; c_1 \; c_2 \sum_{n_1=0}^{M} [1+\kappa n_1]^{-1} \sum_{n_2=0}^{M-n_1} [1+\kappa n_2]^{-1} \; ... \\ &\sum_{n_{t-4}=0}^{M-[n_1+..+n_{t-5}]} [1+\kappa n_{t-4}]^{-1} \sum_{n_{t-3}=0}^{M-[n_1+..+n_{t-4}]} [1+\kappa n_{t-3}]^{-1} \; [1+\kappa(M-[n_1+..+n_{t-3}])]^{-p_2} \end{aligned} \tag{A.25}$$

Comparing (A.21) and (A.25) we see that we can now iterate and define a sequence $p_u = 2p_{u-1} - 1, \; u = 1, .., t-1$ of powers provided we can keep them in the range $0 < p_u < 1$. The iteration is easily solved by

$$p_u = 2^{u-1} p_1 - (2^{u-1} - 1) \tag{A.26}$$

Pick $p_1 = 1 - 2^{-T}$ then

$$p_u = 1 - 2^{u-1-T} \tag{A.27}$$

which satisfies $0 < p_u < 1$ for $u = 1, .., t-1$ if $T \geq t$. As we have to perform $t-1$ summations in $s_t(M)$, for the choice $T = t$ we have $p_{t-1} = 3/4$ and find that for some constant c_t we have

$$s_t(M) \leq c_t \, [1 + \kappa M]^{-3/4} \tag{A.28}$$

Correspondingly

$$S_r(K) \leq m^r \, [\sum_{t=1}^{r-1} \left(\begin{array}{c} r \\ t \end{array} \right) \, c_t \, c_{r-t}] \sum_{L=|K|}^{\infty} ([1 + \kappa L] \, [1 + \kappa(L - |K|)])^{-3/4} \tag{A.29}$$

Let $\kappa' = \min(1, \kappa)$ so that $1 + \kappa M \geq \kappa'(1 + M)$ then for some constant $C_{r,m}$

$$S_r(K) \leq C_{r,m} \sum_{L=|K|}^{\infty} ([1 + L] \, [1 + L - |K|])^{-3/4} \leq C_{r,m} \, \zeta(\frac{3}{2}) \tag{A.30}$$

where we introduced $L - |K| = 0, 1, 2, ..$ as a new summation variable and estimated $1 + L + |K| \geq 1 + L$ so that the Riemann zeta $\zeta(z)$ appears which converges for $\Re(z) > 1$ (this fact can of course also proved elementarily using the above technique to estimate the sum by an integral). Note that the bound (A.30) is in fact independent of K and can certainly be optimised if needed.

B Renormalisation tools

More details on this section, in particular the relation to wavelet theory [13], can be found in [14].

We work on spacetimes diffeomorphic to $\mathbb{R} \times \sigma$. In a first step the spatial D-manifold σ is compactified to T^D. Therefore, all constructions that follow have to be done direction wise for each copy of S^1. On $X := S^1$, understood as $[0, 1)$ with endpoints identified, we consider the Hilbert space $L = L_2([0, 1), \, dx)$ with orthonormal basis

$$e_n(x) := e^{2\pi \, i \, n \, x}, \; n \in \mathbb{Z} \tag{B.1}$$

with respect to the inner product

$$< F, G >_L := \int_0^1 \, dx \, \overline{F(x)} \, G(x) \tag{B.2}$$

Let $\mathbb{O} \subset \mathbb{N}$ be the set of positive odd integers. We equip $\mathbb{O}$ with a partial order, namely

$$M < M' \;\; \Leftrightarrow \;\; \frac{M'}{M} \in \mathbb{N} \tag{B.3}$$

Note that this is not a linear order, i.e. not all elements of $\mathbb{O}$ are in relation, but $\mathbb{O}$ is directed, that is, for each $M, M' \in \mathbb{O}$ we find $M'' \in \mathbb{O}$ such that $M, M' < M''$ e.g. $M'' = MM'$. For each $M \in \mathbb{O}$, called a resolution scale, we introduce the subsets $\mathbb{N}_M \subset \mathbb{N}_0, \; \mathbb{Z}_M \subset \mathbb{Z}, X_M \subset X$ of respective cardinality M defined by

$$\mathbb{N}_M = \{0, 1, .., M-1\}, \; \mathbb{Z}_M = \{-\frac{M-1}{2}, -\frac{M-1}{2} + 1, .., \frac{M-1}{2}\}, \; X_M = \{x^M_m := \frac{m}{M}, \; m \in \mathbb{N}_M\} \tag{B.4}$$

It is easy to check that we have the lattice relation

$$X_M \subset X_{M'} \;\; \Leftrightarrow \;\; M < M' \tag{B.5}$$

The subspace $L_M \subset L$ is defined by

$$L_M := \text{span}(\{e_n, \; n \in \mathbb{Z}_M\}) \tag{B.6}$$

On L_M we use the same inner product as on L, hence the $e_n,\ n \in \mathbb{Z}_M$ provide an ONB for L_M. An alternative basis for L_M is defined by the functions

$$\chi_m^M(x) := \sum_{n\in\mathbb{Z}_M} e_n(x - x_m^M) \tag{B.7}$$

The motivation to introduce these functions is that in contrast to the plane waves e_n they are 1. spatially concentrated at $x = x_m^M$ and 2. real valued. This makes them useful for renormalisation purposes. In addition, in contrast to characteristic functions which have better spatial location properties, they are smooth. This is a crucial feature because quantum field theory involves products of derivatives of the fields and derivatives of characteristic functions yield δ distributions. More in general, renormalisation tools must make a compromise between localisation and smoothness.

The functions χ_m^M are still orthogonal but not orthonormal

$$< \chi_m^M,\ \chi_{\hat{m}}^M >_{L_M} = M\ \delta_{m,\hat{m}} \tag{B.8}$$

We choose not to normalise them in order to minimise the notational clutter in what follows. Let l_M be the space of square summable sequences $f_M = (f_{M,m})_{m\in\mathbb{N}_M}$ with M members and inner product given by

$$< f_M,\ g_M >_{l_M} := \frac{1}{M} \sum_{m\in\mathbb{N}_M} \overline{f_{M,m}}\ g_{M,m} \tag{B.9}$$

If we interpret $f_{M,m} = F(x_m^M)$ then (B.9) is a lattice approximant of $< F, G >_L$. We define

$$I_M:\ l_M \to L_M;\ (I_M \cdot f_M)(x) := < \chi_\cdot^M(x), f_M >_{l_M} = \frac{1}{M} \sum_{m\in\mathbb{N}_M} f_{M,m}\ \chi_m^M(x) \tag{B.10}$$

Its adjoint is defined by the requirement that

$$< I_M^* \cdot F_M,\ g_M >_{l_M} = < F_M,\ I_M \cdot g_M >_{L_M} \tag{B.11}$$

which demonstrates

$$I_M^*:\ L_M \to l_m,\ (I_M^* \cdot F_M)_m = < \chi_m^M, F_M >_{L_M} \tag{B.12}$$

One easily checks, using (B.8) that

$$I_M^* \cdot I_M = 1_{l_M},\ \ < I_M.,I_M. >_{L_M} = < .,. >_{l_M} \tag{B.13}$$

which shows that L_M, l_M are in 1-1 correspondence and that I_M is an isometry. Likewise

$$P_M := I_M \cdot I_M^* = 1_{L_M} \tag{B.14}$$

We can consider I_M also as a map $I_M:\ l_M \to L$ with image L_M and then $< I_M^*.,. >_{l_M} = < ., I_M >_L$ shows that $I_M^* : L \to l_M$ is given by the same formula (B.12) with $F \in L$ but now $P_M : L \to L_M$ is an orthogonal projection

$$P_M \cdot P_M = P_M,\ P_M^* = P_M \tag{B.15}$$

We have explicitly

$$(P_M \cdot F)(x) = \int_0^1 dy\ P_M(x,y)\ F(y),\ P_M(x,y) = \sum_{n\in\mathbb{Z}_M} e_n(x-y) \tag{B.16}$$

i.e. $P_M(x,y)$ is the M-cutoff of the δ distribution on X, i.e. modes $|n| > \frac{M-1}{2}$ are discarded.

Given a continuum function $F \in L$ we call $f_M = I_M^* \cdot F \in l_M$ or $F_M = P_M \cdot F \in L_M$ the discretisation of F at resolution M. In particular, if we have a Hamiltonian field theory on X with conjugate pair of fields (Φ, Π) i.e. the non-vanishing Poisson brackets are

$$\{\Pi(x), \Phi(y)\} = \delta_X(x,y) = \sum_{n\in\mathbb{Z}} e_n(x-y) \tag{B.17}$$

then their discretisations obey

$$\{\pi_{M,m},\phi_{M,\hat{m}}\} = M\ \delta_{m,\hat{m}},\quad \{\Pi_M(x),\Phi_M(y)\} = P_M(x,y) = \frac{\sin(M\pi(x-y))}{\sin(\pi(x-y))} \tag{B.18}$$

The latter formula is known as the Dirichlet kernel.

Given a functional $H[\Pi,\Phi]$ of the continuum fields we define its discretisation by

$$h_M[\pi_M,\phi_M] := H_M[\Pi_M,\Phi_M] = H[\Pi_M,\Phi_M] = (I'_M H)[\pi_M,\phi_M] \tag{B.19}$$

where I'_M denotes the pull-back by I_M. That is, in the continuum formula for H one substitutes $\Pi \to \Pi_M,\ \Phi \to \Phi_M$ in the formula for H upon which H is restricted to Π_M, Φ_M, i.e. H_M is that restriction, and then uses the identity $\Pi_M = I_M \cdot \pi_M,\ \Phi_M = I_M \cdot \phi_M$. In order for this to be well-defined it is important that I_M is sufficiently smooth as H typically depends of derivatives of Π, Φ. This is granted by our choice of I_M. In particular, as the derivative $\partial = \frac{\partial}{\partial x}$ preserves each of the spaces L_M we have a canonical discretisation of the derivative defined by

$$\partial_M := I_M^* \cdot \partial \cdot I_M \tag{B.20}$$

which obeys $\partial_M^n = I_M^* \cdot \partial^n \cdot I_M$ because $I_M \cdot I_M^* = P_M$ and $[\partial, P_M] = 0$.

Concerning quantisation, in the the continuum we define the Weyl algebra $\mathfrak{A}$ generated by the Weyl elements

$$W[F] = e^{-i<F,\Phi>_L},\ W[G] = e^{-i<G,\Pi>_L} \tag{B.21}$$

for real valued $F, G \in L$ (or a dense subspace thereof with additional properties such as smoothness and rapid momentum decrease of its Fourier modes $< e_n, F >_L,\ < e_n, G >_L$). That is, the non-trivial Weyl relations are

$$\begin{aligned} &W[G]\ W[F]\ W[-G] = e^{-i<G,F>_L}\ W[F],\ W[F]\ W[F'] = W[F+F'],\ W[G]\ W[G'] = W[G+G'] \\ &W[0] = 1_{\mathfrak{A}},\ W[F]^* = W[-F],\ W[G]^* = W[-G] \end{aligned} \tag{B.22}$$

Cyclic representations $(\rho, \mathcal{H}, \Omega)$ of $\mathfrak{A}$ with $\Omega \in \mathcal{H}$ a cyclic vector (i.e. $\mathcal{D} := \rho(\mathfrak{A})\mathcal{H}$ is dense) are generated from states (positive, linear, normalised functionals) ω on $\mathfrak{A}$ via the GNS construction [15]. The correspondence is given by

$$\omega(A) =< \Omega,\ \rho(A)\Omega >_{\mathcal{H}} \tag{B.23}$$

We may proceed analogously with the discretised objects. For each M we define the Weyl algebra $\mathfrak{A}_M$ generated by the Weyl elements

$$W_M[F_M] = e^{-i<F_M,\Phi_M>_{L_M}} = w_M[f_M] = e^{-i<f_M,\phi_M>_{l_M}},\ \ W_M[G_M] = e^{-i<G_M,\Pi_M>_{L_M}} = w_M[g_M] = e^{-i<g_M,\pi_M>_{l_M}} \tag{B.24}$$

where $F_M = I_M \cdot f_M,\ G_M = I_M \cdot g_M$ are real valued. Accordingly

$$\begin{aligned} &W_M[G_M]\ W_M[F_M]\ W_M[-G_M] = e^{-i<G_M,F_M>_{L_M}}\ W_M[F_M],\ W_M[F_M]\ W_M[F'_M] = W_M[F_M+F'_M], \quad \text{(B.25)} \\ &W_M[G_M]\ W_M[G'_M] = W_M[G_M+G'_M],\ W_M[0] = 1_{\mathfrak{A}_M},\ W_M[F_M]^* = W_M[-F_M],\ W_M[G_M]^* = W_M[-G_M] \end{aligned}$$

and completely analogous for ϕ_M, π_M if we substitute lower case letters for capital letters in (B.25). For each M we define a state ω_M on $\mathfrak{A}_M$ which gives rise to GNS data $(\rho_M, \mathcal{H}_M, \Omega_M)$ and the dense subspace $\mathcal{D}_M = \mathfrak{A}_M\Omega_M$. Note that $\mathfrak{A}_M$ is a subalgebra of $\mathfrak{A}_{M'}$ for $M < M'$ and that $\mathfrak{A}_M$ is a subalgebra of $\mathfrak{A}$. This follows from the identities

$$W_{M'}[F_M] = W_M[F_M],\ \ W_M[F_M] = W[F_M] \tag{B.26}$$

due to $P_{M'} \cdot P_M = P_M$ since $L_M \subset L_{M'}$ and $P_M \cdot P_M = P_M$ respectively.

The sole reason for discretisation is as follows: While finding states on $\mathfrak{A}$ is not difficult (e.g. Fock states) it is tremendously difficult to find such states which allow to define non-linear functionals of Π, Φ such as Hamiltonians densely on $\mathcal{D}$ due to UV singularities arising from the fact that Π, Φ are promoted to operator valued distributions whose product is a priori ill-defined. In the presence of the UV cut-off M this problem can be solved because e.g. $\Phi_M(x)^2$ is perfectly well-defined (Φ is smeared with the smooth kernel P_M). Suppose then that h_M or equivalently

H_M are somehow quantised on $\mathcal{D}_M$. We denote these quantisations by $\rho_M(h_M, c_M)$ or $\rho_M(H_M, c_M)$ respectively to emphasise that these operators are 1. densely defined on $\rho_M(\mathfrak{A}_M)\Omega_M$, 2. correspond to the classical symbol h_M of H_M respectively and 3. depend on a set of choices c_M for each M such as factor or normal ordering etc. It is therefore not at all clear whether the theories defined for each M in fact descend from a continuum theory. By "descendance" we mean that ω_M is the restriction of ω to $\mathfrak{A}_M$ and that $\rho_M(H_M, c_M)$ is the restriction of $\rho(H, c)$ to $\mathcal{D}_M$ as a quadratic form (i.e. in the sense of matrix elements). In formulas this means

$$\begin{aligned} \omega_M(A_M) = \omega(A_M), \\ < \rho_M(A_M)\Omega_M,\ \rho_M(H_M, c_M)\ \rho_M(B_M)\Omega_M >_{\mathcal{H}_M} =< \rho(A_M)\Omega,\ \rho(H, c)\ \rho(B_M)\Omega >_{\mathcal{H}}, \end{aligned} \tag{B.27}$$

for all $M \in \mathbb{O}$ and all $A_M, B_M \in \mathfrak{A}_M$. If they did, then we obtain the following identities for $M < M'$

$$\begin{aligned} \omega_{M'}(A_M) = \omega_M(A_M), \\ < \rho_M(A_M)\Omega_M,\ \rho_M(H_M, c_M)\ \rho_M(B_M)\Omega_M >_{\mathcal{H}_M} =< \rho_{M'}(A_M)\Omega_{M'},\ \rho_{M'}(H_{M'}, c_{M'})\ \rho_{M'}(B_M)\Omega_{M'} >_{\mathcal{H}_{M'}}, \end{aligned} \tag{B.28}$$

called consistency conditions. This follows from the fact that $A_{M'} := A_M, B_{M'} := B_M$ can be considered as elements of $\mathfrak{A}_M$ and then using (B.27). With some additional work [4] one can show that (B.28) are necessary and sufficient for $\omega, \rho(H)$ to exist (at least as a quadratic form).

In constructive quantum field theory (CQFT) [2] one proceeds as follows. One starts with an Ansatz of a family of discretised theories $(\omega_M^{(0)}, \rho_M^{(0)}(H_M, c_M^{(0)}))_{M\in\mathbb{O}}$. That Ansatz generically violates (B.28). We now define a renormalisation flow of states and quantisations by defining the sequence $(\omega_M^{(k)}, \rho_M^{(k)}(H_M, c_M^{(k)}))_{M\in\mathbb{O}}$ for $k \in \mathbb{N}_0$ via

$$\begin{aligned} & \omega_M^{(k+1)}(A_M) := \omega_{M'(M)}^{(n)}(A_M),\ < \rho_M^{(k+1)}(A_M)\Omega_M^{(k+1)},\ \rho_M^{(k+1)}(H_M, c_M^{(k+1)})\ \rho_M^{(k+1)}(B_M)\Omega_M^{(k+1)} >_{\mathcal{H}_M^{(k+1)}} \\ = & < \rho_{M'}^{(k)}(A_M)\Omega_{M'}^{(k)},\ \rho_{M'}^{(k)}(H_{M'}, c_{M'}^{(k)})\ \rho_{M'}^{(k)}(B_M)\Omega_{M'}^{(k)} >_{\mathcal{H}_{M'}^{(k)}} \end{aligned} \tag{B.29}$$

where $M' : \mathbb{O} \to \mathbb{O}$ is a fixed map with the property that $M'(M) > M,\ M'(M) \neq M$. The first relation defines a new state at the coarser resolution M as the restriction of the old state at the finer resolution $M'(M)$. This then defines also new GNS data $(\rho_M^{(k+1)}, \mathcal{H}_M^{(k+1)}, \Omega_M^{(k+1)})$ via the GNS construction. The second relation defines the matrix elements of an operator or quadratic form in that new representation and with new quantisation choices to be made at coarser resolution in terms of the restriction of the matrix elements of the old operator or quadratic form with old quantisation choices in the old representation at finer resolution. A fixed point family $(\omega_M^*, \rho_M^*(H_M, c_M^*))_{M\in\mathbb{O}}$ of the flow (B.29) solves (B.28) at least for $M' = M'(M)$ and all M and thus all $[M']^n(M),\ n \in \mathbb{N}_0$ and all M. This typically implies that (B.28) holds for all $M' < M$. In practice we will work with $M'(M) := 3\ M$

Note that for a general operator or quadratic form O defined densely on $\mathcal{D}$ it is not true that we find an element $a \in \mathfrak{A}$ such that $\rho(a) = O$ (e.g. unbounded operators) which is why the above statements cannot be made just in terms of the states ω. If one tried, one would need to use sequences or nets $a_n \in \mathfrak{A}$ whose limits lie outside of $\mathfrak{A}$. On the other hand, if one prefers to work with the Weyl elements $W_M[F_M]$ one may relate the spaces $l_M,\ L_M$ via the identities $w_M[f_M] = W_M[F_M],\ f_M = I_M^* \cdot F_M$. The $w_M[f_M],\ w_{M'}[f'_M]$ at resolution $M,\ M'$ respectively can be related via the *coarse graining* map $I_{MM'} := I_{M'}^* \cdot I_M;\ l_M \to l_{M'}$ such that $w_{M'}[I_{MM'} \cdot f_M] = w_M[f_M]$. This map obeys $I_{M_2M_3} \cdot I_{M_1M_2} = I_{M_1M_3}$ for $M_1 < M_2 < M_3$ because the image of I_M is L_M which is a subspace of $L_{M'}$ thus $I_{M_2M_3} \cdot I_{M_1M_2} = I_{M_3}^* \cdot P_{M_2} \cdot I_{M_1} = I_{M_3}^* \cdot I_{M_1}$. Then $W_{M'}[F_M] = w_{M'}[I_{M'}^* F_M] = w_{M'}[I_{M'}^* \cdot I_M \cdot f_M] = w_{M'}[I_{MM'} \cdot f_M]$ indeed. For the same reason $W_{M'}[F_M] = W_M[F_M]$ as L_M is embedded in $L_{M'}$ by the identity map. The renormalisation flow in terms of Weyl elements $w_M[f_M]$ and the coarse graining map $I_{M,M'}$ takes the form

$$\begin{aligned} \omega_M^{(k+1)}(w_M[f_M]) := \omega_{M'}^{(k)}(w_{M'}[I_{M,M'} f'_M]) \\ \langle w_M[f'_M]\Omega_M^{(k+1)}, H_M^{(k+1)}\ w_M[f_M]\Omega_M^{(k+1)}\rangle_{\mathcal{H}_M^{(k+1)}} := \langle w_{M'}[I_{M,M'} f'_M]\Omega_{M'}^{(k)}, H_{M'}^{(k)}\ w_{M'}[I_{M,M'} f_M]\Omega_{M'}^{(k)}\rangle_{\mathcal{H}_{M'}^{(k)}} . \end{aligned} \tag{B.30}$$

References

[1] R. Haag, "Local Quantum Physics", Springer Verlag, Berlin, 1984

[2] J. Glimm and A. Jaffe, "Quantum Physics", Springer Verlag, New York, 1987.

[3] K. G. Wilson. The renormalization group: Critical phenomena and the Kondo problem. Rev. Mod. Phys. **47** (1975) 773

[4] T. Thiemann. Canonical quantum gravity, constructive QFT and renormalisation. Front. in Phys. **8** (2020) 548232. arXiv:2003.13622 [gr-qc].

[5] T. Lang, K. Liegener, T. Thiemann. Hamiltonian Renormalisation I. Derivation from Osterwalder-Schrader Reconstruction. Class. Quant. Grav. **35** (2018) 245011. [arXiv:1711.05685]

[6] T. Lang, K. Liegener, T. Thiemann. Hamiltonian Renormalisation II. Renormalisation Flow of 1+1 dimensional free, scalar fields: Derivation. Class. Quant. Grav. **35** (2018) 245012. [arXiv:1711.06727]

[7] T. Lang, K. Liegener, T. Thiemann. Hamiltonian Renormalisation III. Renormalisation Flow of 1+1 dimensional free, scalar fields: Properties. Class. Quant. Grav. **35** (2018) 245013. [arXiv:1711.05688]

[8] T. Lang, K. Liegener, T. Thiemann. Hamiltonian Renormalisation IV. Renormalisation Flow of D+1 dimensional free scalar fields and Rotation Invariance. Class. Quant. Grav. **35** (2018) 245014, [arXiv:1711.05695]

[9] K. Liegener, T. Thiemann. Hamiltonian Renormalisation V. Free Vector Bosons. Front. Astron. Space Sci. **7** (2021) 547550. e-Print: 2003.13059 [gr-qc]

[10] T. Thiemann. Hamiltonian Renormalisation VII. Free Fermions and doubler free kernels. Phys. Rev. **D108** (2023) 12, 125007. e-Print: 2207.08291 [hep-th]

[11] E.-A. Zwicknagel. Hamiltonian renormalization. VI. Parametrized field theory on the cylinder. Phys. Rev. **D108** (2023) 12, 125006. e-Print: 2207.08290 [gr-qc]

[12] B. Simon, "The P(ϕ)2 Euclidean (Quantum) Field Theory", Princeton Unviersity Press, 1974

[13] I. Daubechies. Ten lectures of wavelets. Springer Verlag, Berlin, 1993.
A. Cohen, I. Daubechies, P.Vial. Wavelets on the interval and fast wavelet transforms. Appl. and Comp. Harm. Anlysis, Elsevier, 1993. [hal-01311753]

[14] T. Thiemann. Renormalization, wavelets, and the Dirichlet-Shannon kernels. Phys. Rev. D **108** (2023) 12, 125008. e-Print: 2207.08294 [hep-th]

[15] O. Bratteli, D. W. Robinson, "Operator Algebras and Quantum Statistical Mechanics", vol. 1,2, Springer Verlag, Berlin, 1997.

[16] J. Glimm, "Boson Fields with the : ϕ^4 : Interaction in Three Dimensions", Comm. Math. Phys. **10** (1968) 1-47.
J. Glimm, A. Jaffe, Positivity of the ϕ^4_3 Hamiltonian", Fortschr. Phys. **21** (1973) 327–376

[17] L. Smolin. The G(Newton) $\to 0$ limit of Euclidean quantum gravity. Class. Quant. Grav. **9** (1992) 883-894. e-Print: hep-th/9202076 [hep-th]

[18] Exact quantisation of U$(1)^3$ quantum gravity via exponentiation of the hypersurface deformation algebroid. Class. Quant. Grav. **40** (2023) 24, 245003. e-Print: 2207.08302 [gr-qc]

[19] H. Narnhofer, W.E. Thirring. Covariant QED without indefinite metric. Rev. Math. Phys. **4** (1992) spec01, 197-211

[20] T. Thiemann. Nonperturbative quantum gravity in Fock representations. Phys. Rev. **D 110** (2024) 12, 124023. e-Print: 2405.01212 [gr-qc]

[21] C. Rovelli, "Quantum Gravity", Cambridge University Press, Cambridge, 2004.
T. Thiemann, "Modern Canonical Quantum General Relativity", Cambridge University Press, Cambridge, 2007.
J. Pullin, R. Gambini, "A first course in Loop Quantum Gravity", Oxford University Press, New York, 2011
C. Rovelli, F. Vidotto, "Covariant Loop Quantum Gravity", Cambridge University Press, Cambridge, 2015
K. Giesel, H. Sahlmann, From Classical To Quantum Gravity: Introduction to Loop Quantum Gravity, PoS QGQGS2011 (2011) 002, [arXiv:1203.2733].

[22] K. Osterwalder, R. Schrader. Axioms for Euclidean Green's functions. Communications in Mathematical Physics **31**(2):83–112, 6 1973

[23] J. Glimm and A. Jaffe, "A $\lambda\phi^4$ quantum field theory without cutoffs. I", Phys. Rev. **176** (1968), 1945–1951. DOI: 10.1103/PhysRev.176.1945.
J. Glimm and A. Jaffe, "The $\lambda(\phi^4)_2$ quantum field theory without cutoffs: III. The physical vacuum", Acta Math. **125** (1970), 203–267. DOI: 10.1007/BF02392335.

J. Glimm and A. Jaffe, "The $\lambda(\phi^4)_2$ quantum field theory without cutoffs. II. The field operators and the approximate vacuum", Ann. of Math. **91** (1970), no. 2, 362–401. DOI: 10.2307/1970582.
J. Glimm and A. Jaffe, "The $\lambda(\phi^4)_2$ quantum field theory without cutoffs. IV. Perturbations of the Hamiltonian", J. Math. Phys. **13** (1972), no. 10, 1568–1584. DOI: 10.1063/1.1665879.