

# 1 Newton Polynomial Interpolation

## 1.1 Collocation

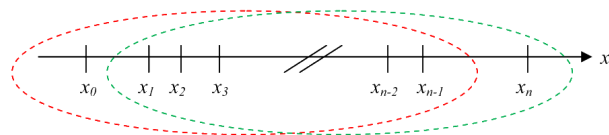
Collocation = All data points are hit by the approximating function (e.g., polynomial). The polynomial

$$y(x) = p(x) = c_0 + c_1x + c_2x^2 + \dots + c_mx^m \quad \text{with} \quad y(x_k) = p(x_k) = y_k \quad (k = 0, 1, \dots, n)$$

results in a linear system of  $n + 1$  equations.

## 1.2 Aitken-Neville Recursion Formula

Overlapping polynomial interpolation functions  $f_1 = p_{0,1,\dots,n-1}$ ,  $f_2 = p_{1,\dots,n}$  at the data points  $x_1, \dots, x_{n-1}$  can be combined.



$$p(x) = p_{0,1,\dots,n}(x) = \frac{(x - x_0) \cdot \overbrace{p_{1,2,\dots,n}(x)}^{\text{green}} - (x - x_n) \cdot \overbrace{p_{0,1,\dots,n-1}(x)}^{\text{red}}}{(x_n - x_0)}$$

## 1.3 Newton Polynomials

### Properties

- + Embedded (When a data point is added, not everything needs to be recalculated)
- + A single formula
- + Practical: Measurements are uniformly distributed.
- Runge's phenomenon (Oscillations at the edges)
- Computationally expensive

$$y_0 = a_0 \cdot \pi_0$$

↓

$$y_1 = a_0 \cdot \pi_0 + a_1 \cdot \pi_1$$

↓

↓

$$y_2 = a_0 \cdot \pi_0 + a_1 \cdot \pi_1 + a_2 \cdot \pi_2$$

⋮

↓

↘

↘

$$y_n = a_0 \cdot \pi_0(x_n) + a_1 \cdot \pi_1(x_n) + a_2 \cdot \pi_2(x_n) + \dots + a_n \pi_n(x_n)$$

$$\pi_0 = 1$$

$$\pi_1 = (x - x_0)$$

$$\pi_2 = (x - x_0)(x - x_1)$$

⋮

$$\pi_n = (x - x_0)(x - x_1) \dots (x - x_{n-1})$$

$$\pi_{n+1} = (x - x_0)(x - x_1) \dots (x - x_{n-1})(x - x_n)$$

Approximation of a “measurement data set”  
(Messreihe) with polynomials:

$$y(x) \approx p(x) = a_0\pi_0(x) + a_1\pi_1(x) + \dots + a_n\pi_n(x)$$

## 1.4 Divided Differences

Note:

$$a_{p-1} = y(x_0, x_1, x_2, \dots, x_{p-1}) \text{ belongs to } \pi_{p-1} = (x - x_0) \cdot (x - x_1) \cdot \dots \cdot (x - x_{p-2})$$

More elegant and faster resolution of Newton polynomials.

$$y(x_0, x_1, \dots, x_k) = \frac{y(x_1, x_2, \dots, x_k) - y(x_0, x_1, \dots, x_{k-1})}{(x_k - x_0)} \quad (k = 0, 1, \dots, n)$$

$$a(x_0, \dots, x_n) = \frac{a(x_1, \dots, x_n) - a(x_0, \dots, x_{n-1})}{x_n - x_0}$$

$$k = 0 \quad y(x_0)$$

$$k = 1 \quad \frac{y(x_1) - y(x_0)}{(x_1 - x_0)}$$

$$k = 2 \quad \frac{y(x_1, x_2) - y(x_0, x_1)}{(x_2 - x_0)}$$

$$k = 3 \quad \frac{y(x_1, x_2, x_3) - y(x_0, x_1, x_2)}{(x_3 - x_0)}$$

**Symmetry:**  $y(x_0, x_1) = y(x_1, x_0) \implies y(x_0, x_1, \dots, x_n)$  is independent of the order of x-values! (The last difference  $a_n$  matches, the other differences differ.)

**Example:**

	$x_k$	$y_k$
$x_0$	0	$\textcircled{1} a_0$
$x_1$	1	1 $\textcircled{0} a_1$
$x_2$	2	2 $\textcircled{\frac{1}{2}} a_2$
$x_3$	4	5 $\textcircled{-\frac{1}{12}} a_3$

$$y(x) \approx p(x) = a_0 \cdot \pi_0(x) + a_1 \cdot \pi_1(x) + a_2 \cdot \pi_2(x) + a_3 \pi_n(x)$$

$$= 1 \cdot \pi_0(x) + 0 \cdot \pi_1(x) + \frac{1}{2} \cdot \pi_2(x) - \frac{1}{12} \pi_n(x)$$

$$y(x) \approx p(x) = 1 + \frac{1}{2}(x - x_0)(x - x_1) - \frac{1}{12}(x - x_0)(x - x_1)(x - x_2)$$

$$= 1 + \frac{1}{2}(x - 0)(x - 1) - \frac{1}{12}(x - 0)(x - 1)(x - 2)$$

## 1.5 Collocation Error Formula

$$y(x) - p(x) = \frac{y^{(n+1)}(\xi)}{(n+1)!} (x - x_0)(x - x_1) \dots (x - x_{n-1})(x - x_n) = \frac{y^{(n+1)}(\xi)}{(n+1)!} \pi_{n+1}(x) \quad \xi \in (\min(x_i), \max(x_i))$$

Derivation:  $y = p + \frac{y^{(n+1)}(\xi)}{(n+1)!} \cdot \pi_{n+1} = y(x_0, x_1, \dots, x_n) + \frac{y^{(n+1)}(\xi)}{(n+1)!} \cdot \pi_{n+1} \implies y(x_0, x_1, \dots, x_{n+1}) = \frac{y^{(n+1)}(\xi)}{(n+1)!}$

The error term can be interpreted as a derivative. However, it is not clear where the derivative is evaluated in the interval  $[x_0, x_n]$ , as  $\xi$  is not known!

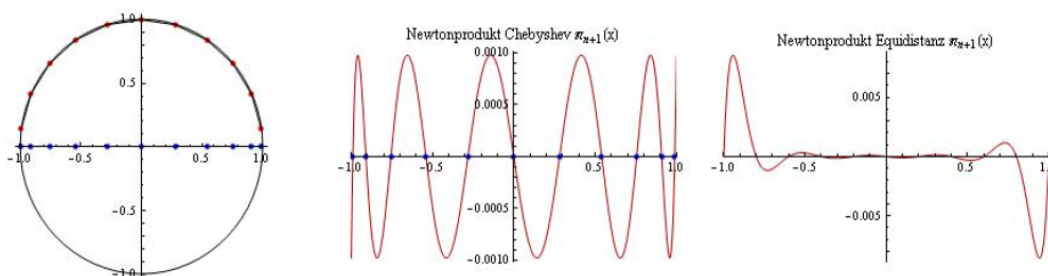
## 1.6 Runge Phenomenon and Chebyshev Arguments

**Runge Phenomenon:** Potential oscillations at the definition boundaries of polynomial collocations. Caused by high  $n$  and high  $\frac{y^{(n+1)}(x)}{(n+1)!}$ .

**Chebyshev Arguments:** Adjustment of measurement points through a different distribution (no longer uniformly distributed  $x_k$ , but uniformly distributed on the unit circle)

$$x_k = \cos\left(\frac{2k+1}{2(n+1)}\pi\right), \quad (k = 0, 1, \dots, n)$$

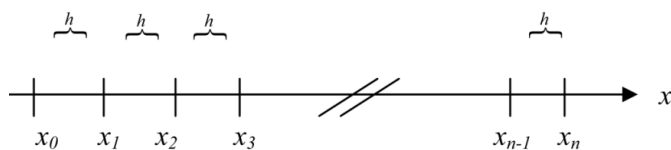
results in maximum amplitude  $\max_{-1 \leq x \leq 1} |\pi_{n+1}(x)| = \frac{1}{2^n}$ . With an affine transformation from the unit interval  $[-1, 1]$  to  $[a, b]: x \rightarrow a + \frac{b-a}{2}(x+1)$



### Properties

- + No Runge Phenomenon (oscillations at the edge)
- + A single formula
  - "Not embedded" (new measurements mean new polynomial calculation!)
  - Practically: Measurements are often only uniformly possible (limited edge resolution)
  - The range of interpolation is fixed (poor extrapolation outside the edge)

## 1.7 Uniformly Distributed Arguments



$$x_k = x_0 + kh \quad k = 0, 1, 2, \dots, n$$

$$y(x_0, x_1, \dots, x_k) = \frac{\Delta^k y_0}{h^k k!}$$

$$p(x) = y_0 + \frac{\Delta^1 y_0}{h^1 1!} \pi_1(x) + \frac{\Delta^2 y_0}{h^2 2!} \pi_2(x) + \dots + \frac{\Delta^n y_0}{h^n n!} \pi_n(x) = \sum_{k=0}^n \frac{\Delta^k y_0}{h^k k!} \pi_k(x)$$

$\Delta^0$	$y_0$	$y_1$	$y_2$	$y_3$	$y_4$
$\Delta^1$	$y_1 - y_0$	$y_2 - y_1$	$y_3 - y_2$	$y_4 - y_3$	
	$\Delta y_0$	$\Delta y_1$	$\Delta y_2$	$\Delta y_3$	
$\Delta^2$		$\Delta y_1 - \Delta y_0$	$\Delta y_2 - \Delta y_1$	$\Delta y_3 - \Delta y_2$	
		$\Delta^2 y_0$	$\Delta^2 y_1$	$\Delta^2 y_2$	

## 1.8 Taylor Series

$$f(x) = y_0 + \underbrace{\frac{y'(x_0)}{1!}(x-x_0) + \frac{y''(x_0)}{2!}(x-x_0)^2 + \dots + \frac{y^{(n)}(x_0)}{n!}(x-x_0)^n}_{\text{Taylor Polynom } p(x)} + \underbrace{\frac{y^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}}_{\text{Lagrange Fehlerterm}}$$

## 2 Hermite Interpolation (Osculation)

Extension of divided differences: Now, derivatives at data points are also possible as conditions.

From the definition of  $y(x)$ , we have  $y(x_0, x_0) = y'(x_0)$ . Generalized, this yields

$$y(\underbrace{x_0, \dots, x_n, x_{n+1}}_{(n+2)}) = \frac{y^{(n+1)}(\xi)}{(n+1)!}$$

Bsp. HS13/14

$$y(\underbrace{x_0, \dots, x_0}_{(n+1)}) = \lim_{\xi \rightarrow x_0} \frac{y^{(n)}(\xi)}{(n)!} = \frac{y^{(n)}(x_0)}{n!}$$

$$y(x_0, x_1, x_1) = \frac{y(x_1, x_1) - y(x_1, x_0)}{x_1 - x_0} = \frac{y'(x_1) - \frac{y(x_1) - y(x_0)}{x_1 - x_0}}{x_1 - x_0}$$

For Hermite interpolation, the same Newton tables are used, but with repetitions (see the example). The values marked in red are calculated as usual.

It is important that no gaps occur in the calculation of the divided differences (derivatives must be continuously available, e.g.,  $y', y'', y'''$  good;  $y', y''''$  bad)! Otherwise, the system of equations is unsolvable. In case of indeterminate results, a variable can be introduced, which can be determined at the end.

$x$	$y$
$x_0 = 2$	$y(x_0) = 1$
	$\frac{y^{(1)}(x_0)}{1!} = 1$
$x_0 = 2$	$y(x_0) = 1$
	$\frac{y^{(2)}(x_0)}{2!} = 0$
	$\frac{y^{(1)}(x_0)}{1!} = 1$
$x_0 = 2$	$y(x_0) = 1$
$x_1 = 4$	$y(x_1) = 2$
	$\frac{y^{(1)}(x_1)}{1!} = 0$
$x_1 = 4$	$y(x_1) = 2$
	$\frac{y^{(2)}(x_1)}{2!} = 0$
$x_1 = 4$	$y(x_1) = 2$
	$\frac{y^{(1)}(x_1)}{1!} = 0$

$x$	$y$
2	$\textcircled{1}^{a_0}$
	$\textcircled{1}^{a_1}$
2	1
	$\textcircled{0}^{a_2}$
	1
	$\textcircled{-\frac{1}{8}}^{a_3}$
2	1
	$-\frac{1}{4}$
	$\textcircled{\frac{1}{16}}^{a_4}$
4	2
	$\frac{1}{2}$
	$-\frac{1}{4}$
	$\frac{1}{16}$
4	2
	0
	$\frac{1}{8}$
4	2
	0
	$\textcircled{0}^{a_5}$

**Modified** Newton polynomials are used. In this case:

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$\pi_0 = 1$	$\pi_4 = (x - x_0)(x - x_0)(x - x_0)(x - x_1)$
$\pi_1 = (x - x_0)$	$\pi_5 = (x - x_0)(x - x_0)(x - x_0)(x - x_1)(x - x_1)$
$\pi_2 = (x - x_0)(x - x_0)$	$\pi_6 = (x - x_0)(x - x_0)(x - x_0)(x - x_1)(x - x_1)(x - x_1)$
$\pi_3 = (x - x_0)(x - x_0)(x - x_0)$	

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$$\begin{aligned}
 p_2(x) &= a_0 \cdot \pi_0 + a_1 \cdot \pi_1 + a_2 \cdot \pi_2 + a_3 \cdot \pi_3 + a_4 \cdot \pi_4 + a_5 \cdot \pi_5 \\
 &= a_0 \cdot 1 + a_1 \cdot (x - x_0) + a_2 \cdot (x - x_0)^2 + a_3 \cdot (x - x_0)^3 + a_4 \cdot (x - x_0)^3(x - x_1) + a_5 \cdot (x - x_0)^3(x - x_1)^2 \\
 &= 1 + (x - 2) - \frac{1}{8}(x - 2)^3 + \frac{1}{16}(x - 2)^3(x - 4)
 \end{aligned}$$

### 2.3 Error Formula

$$y(x) - p(x) = \frac{y^{(d)}(\xi)}{d!} (x - x_0)^{d_0} (x - x_1)^{d_1} \dots (x - x_n)^{d_n}$$

$$x, \xi \in \{\min x_i, \max x_i\}, \quad i = 0, 1, \dots, 2$$

$d$  is the total number of conditions, and  $d_i$  is the number of conditions per support point  $x_i$ . ( $d = \sum_{i=0}^n d_i$ )

### 2.4 Missing Derivatives

In case of missing derivative specifications, variables are introduced, and the tableau is computed with these. In the end, the missing variables are determined from the back.

**Example:**

Desired: 2nd order polynomial passing through points  $y(2) = 1$  and  $y(4) = 1$  with the derivative  $y'(4) = 1$ .

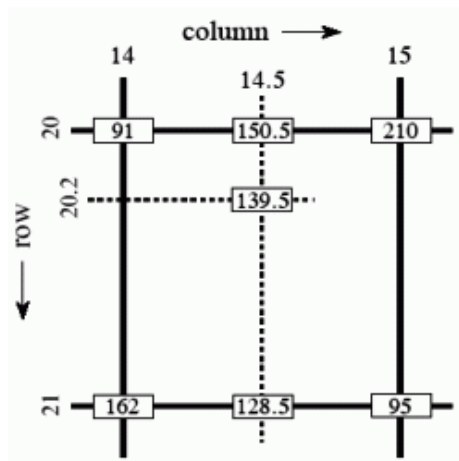
$x$	$y$
2	$\textcircled{1}^{a_0}$
	$\textcircled{\beta}^{a_1}$
2	1
	$\textcircled{-\frac{\beta}{2}}^{a_2}$
	0
4	1
	$\textcircled{\frac{1+\beta}{4}}^{a_3} \stackrel{!}{=} 0$
	$\frac{1}{2}$
4	1
	1

- Because the polynomial must have order  $d = 2$ , we have  $\frac{1+\beta}{4} = 0$ .  
From this, we get  $\beta = -1$ .

- The solution polynomial is:

$$\begin{aligned}
 p(x) &= a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + a_3(x - x_0)^2(x - x_1) \\
 &= 1 + \beta(x - x_0) - \frac{\beta}{2}(x - x_0)^2 + \frac{1+\beta}{4}(x - x_0)^2(x - x_1) \Big|_{\beta=-1} \\
 &= 1 - (x - 2) + \frac{1}{2}(x - 2)^2
 \end{aligned}$$

### 3 Multi-variate Polynomial Interpolation



- The basis of polynomials can be calculated through the *2-fold tensor product*  $\Rightarrow$  Bilinear Interpolation  
 $\{1, x, x^2\} \otimes \{1, y, y^2\} = \{1, x, y, x^2, y^2, xy, x^2y, xy^2, x^2y^2\}$ ;  
 This way, the number of equations  $n$  can be calculated:  
 $\mathbb{R}^n \otimes \mathbb{R}^k = \mathbb{R}^{nk}$

- Interpolate the desired order (linear, quadratic, ...) in the X-direction at position  $y_0$ :  $p(x, y_0)$
- Interpolate the desired order (linear, quadratic, ...) in the X-direction at position  $y_1$ :  $p(x, y_1)$
- Compute the tableau for divided differences (linear case here):

$y$	$z$
$y_0$	$p(x, y_0) = a_{y0}$
$y_1$	$p(x, y_1) \quad \frac{p(x, y_1) - p(x, y_0)}{y_1 - y_0} = a_{y1}$

- Formulate the polynomial:  
 $p(x, y) = a_{y0}\pi_0(y) + a_{y1}\pi_1(y) + a_{y2}\pi_2(y) + \dots$
- Expand the polynomial:  
 $p(x, y) = a_{0,0}\pi_0(x)\pi_0(y) + a_{1,0}\pi_1(x)\pi_0(y) + a_{0,1}\pi_0(x)\pi_1(y) + a_{1,1}\pi_1(x)\pi_1(y) + \dots$

#### 3.1 Example:

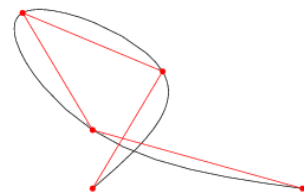
Given points  $(14, 20), (14, 21), (15, 20), (15, 21)$  with values  $\{91/255, 162/255, 210/255, 95/255\}$ . Linear interpolation is sought. This results in the following polynomial bases:  $(\{1, x\} \times \{1, y\} = \{1, x, y, xy\})$

1. Interpolation for  $y = y_0 = 20$ :  $p(x, y_0) = p(x_0, y_0)\pi_0(x) + \frac{p(x_1, y_0) - p(x_0, y_0)}{x_1 - x_0}\pi_1(x) = \frac{1}{255} \left( 91 \cdot 1 + \frac{210 - 91}{15 - 14} \cdot (x - 14) \right)$
2. The same for  $y = y_1 = 21$ :  $p(x, y_1) = \frac{1}{255} \left( 162 \cdot 1 + \frac{95 - 162}{15 - 14} \cdot (x - 14) \right)$
3. Then the final interpolation:  $p(x, y) = p(x, y_0)\pi_0(y) + \frac{p(x, y_1) - p(x, y_0)}{y_1 - y_0}\pi_1(y)$   
 $= \frac{1}{255} \left( \left( 91 \cdot 1 + \frac{210 - 91}{15 - 14} \cdot 1 \right) \cdot 1 + \frac{162 \cdot \frac{95 - 162}{15 - 14} \cdot (x - 14) - 91 \cdot 1 + \frac{210 - 91}{15 - 14} \cdot (x - 14)}{21 - 20} \cdot (y - 20) \right) = -215.98 + 15.0549x + 10.4902y - 0.7294xy$

bi-cubic und multivariate

## 4 Spline Interpolation

The idea of spline interpolation is not to interpolate the data with a high-degree polynomial but with several low-degree polynomials. This helps to avoid the tendency of high-degree polynomials to oscillate. At the "breakpoints," the transitions from one patch to the next, the derivatives of the neighboring patches must match up to a specified order of derivation.



### Properties

- + No Runge phenomenon (oscillations at the boundaries)
- + Polynomials have low degree
- + Computation effort lower than Newton [Spline:  $O(n)$  (due to tridiagonal band matrix), Newton:  $O(n^2)$ ]
- "Not embedded" (new measurements mean new polynomial calculation!)
- Polynomials must be composed for evaluation (post-processing effort is large)

### 4.1 One-Dimensional Splines

#### 4.1.1 Principle

For each *patch* (out of a total of  $n$ ), a polynomial of degree  $d$  (usually cubic,  $d = 3$ ) is computed. Conditions  $C^k$  ( $k = 1, \dots, d-1$ ) can be defined at the transitions at  $x_i$ .

**Degrees of Freedom**  $n$  patches with 1 polynomial each with  $(d+1)$  coefficients each give a total of  $n(d+1)$  degrees of freedom.

**Conditions**  $n$  patches with start and end  $= 2n$ ;  $d-1$  derivatives per inner point  $(n-1)(d-1)$  result in a total of  $n(d+1) - (d-1)$  conditions.

**Additional Conditions** When comparing degrees of freedom and conditions,  $d-1$  conditions need to be chosen. For the *natural spline*, the second derivatives are set to 0; for *clamped splines*, the first derivatives are specified.

#### 4.1.2 General Procedure (Cubic Splines)

The final interpolation has the following form (description of **one** patch):

$$S_i(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3$$

$$S'_i = b_i + 2c_i(x - x_i) + 3d_i(x - x_i)^2$$

$$S''_i = 2c_i + 6d_i(x - x_i)$$

Also needed (patch width)  $h_i = x_{i+1} - x_i = \Delta x_i$

1.  $a_i = y_i$  ( $i = 0, \dots, n-1$ )
2. For cubic splines, there are  $d-1 = 3-1 = 2$  additional conditions.

$$\begin{pmatrix} h_0 & 2(h_0+h_1) & h_1 & & & \\ & h_1 & 2(h_1+h_2) & h_2 & & \\ & & h_2 & 2(h_2+h_3) & h_3 & \\ & & & \ddots & \ddots & \ddots \\ & & & & h_{n-3} & 2(h_{n-3}+h_{n-2}) & h_{n-2} \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_{n-2} \\ c_{n-1} \end{pmatrix} = \begin{pmatrix} 3 \left( \frac{y_{i+1}-y_i}{h_i} - \frac{y_i-y_{i-1}}{h_{i-1}} \right) \end{pmatrix}_{i=1, \dots, n-2}$$

- **Natural Spline** (Boundary curvature):  $y_0'' = 0 = y_n''$  (minimizes the energy norm! ( $\min \int |f''(x)|^2 dx$ ))

(a)  $c_0 = c_n = 0$

- (b) Solve the system of equations for  $c_i$ :

$$\begin{pmatrix} 2(h_0+h_1) & h_1 & & & \\ h_1 & 2(h_1+h_2) & h_2 & & \\ & h_2 & 2(h_2+h_3) & h_3 & \\ & & \ddots & \ddots & \ddots \\ & & & h_{n-3} & 2(h_{n-3}+h_{n-2}) \\ & & & & h_{n-2} \\ & & & & 2(h_{n-2}+h_{n-1}) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_{n-2} \\ c_{n-1} \end{pmatrix} = \begin{pmatrix} 3 \left( \frac{y_{i+1}-y_i}{h_i} - \frac{y_i-y_{i-1}}{h_{i-1}} \right) \\ \vdots \\ 3 \left( \frac{y_{i+1}-y_i}{h_i} - \frac{y_i-y_{i-1}}{h_{i-1}} \right) \end{pmatrix}_{i=1, \dots, n-1}$$

- **Clamped Spline:**

(a)  $b_0 = y_0'; b_n = y_n'$

- (b) Solve the system of equations for  $c_i$ :

$$\begin{pmatrix} 2h_0 & h_0 & & & \\ h_0 & 2(h_0+h_1) & h_1 & & \\ & h_1 & 2(h_1+h_2) & h_2 & \\ & & h_2 & 2(h_2+h_3) & h_3 \\ & & & \ddots & \ddots \\ & & & h_{n-3} & 2(h_{n-3}+h_{n-2}) \\ & & & & h_{n-2} \\ & & & & 2h_{n-2} & 4h_{n-2}+3h_{n-1} \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_{n-2} \\ c_{n-1} \end{pmatrix} = \begin{pmatrix} 3 \left( \frac{a_1-a_0}{h_0} - y_0' \right) \\ \vdots \\ 3 \left( \frac{y_{i+1}-y_i}{h_i} - \frac{y_i-y_{i-1}}{h_{i-1}} \right) \\ \vdots \\ 9 \frac{y_n-a_{n-1}}{h_{n-1}} - 6 \frac{a_{n-1}-a_{n-2}}{h_{n-2}} - 3y_n' \end{pmatrix}_{i=1, \dots, n-2}$$

3.  $b_{i-1} = \frac{a_i-a_{i-1}}{h_{i-1}} - \frac{2c_{i-1}+c_i}{3}h_{i-1} \quad (i = 1, \dots, n-1)$

4.  $b_{n-1} = \frac{y_n-a_{n-1}}{h_{n-1}} - c_{n-1}h_{n-1} - d_{n-1}h_{n-1}^2 = \frac{y_n-y_{n-1}}{h_{n-1}} - \frac{2}{3}c_{n-1}h_{n-1}$

5.  $d_{i-1} = \frac{c_i-c_{i-1}}{3h_{i-1}} \quad (i = 1, \dots, n-1)$

6. Clamped Spline:  $d_{n-1} = \frac{y_n'-b_{n-1}-2c_{n-1}h_{n-1}}{3h_{n-1}^2}$       Natural Spline:  $d_{n-1} = -\frac{c_{n-1}}{3h_{n-1}}$

## Further Methods

In *Hermite cubic splines*,  $C^1$  functions are formed from patched cubic polynomials with prescribed first derivatives.

*Periodic splines* are  $C^2$  functions formed from patched cubic polynomials with periodicity ( $S'(x_0) = S'(x_n)$  and  $S''(x_0) = S''(x_n)$ ).

### 4.1.3 Error Estimation

For cubic splines with  $C^2$ , the following error estimate applies ( $H = \max h_i$  ( $i = 0, \dots, n-1$ )):

$$|y(x) - S(x)| \leq \max |y^{(4)}(x)| \frac{5}{384} H^4 \quad |y'(x) - S'(x)| \leq \max |y^{(4)}(x)| \frac{1}{24} H^3 \quad |y''(x) - S''(x)| \leq \max |y^{(4)}(x)| \frac{3}{8} H^2$$

## 4.2 Bernstein-Bézier Splines (B-B Splines)

### 4.2.1 Bernstein Polynomials

The Bernstein polynomials are defined in the interval  $t \in [0, 1]$  as

$$B_{i,n}(t) = \binom{n}{i} (1-t)^{n-i} t^i \quad t \in [0, 1] \quad i = 0, 1, \dots, n$$

with  $i$ : index and  $n$ : order of the spline.

Through an affine transformation into the interval  $t \in [a, b]$ , we have

$$B_{i,n}(u, a, b) = \frac{1}{(b-a)^n} \binom{n}{i} (b-u)^{n-i} (u-a)^i \quad u \in [a, b] \quad i = 0, 1, \dots, n$$

**Properties** Bernstein polynomials...

- form a linear basis for polynomials of order  $n$  (they can be used to build any polynomial of order  $n$ )
- have exactly one maximum at  $t = \frac{i}{n}$
- have a root at 0 (order  $i$ ) and at 1 (order  $n-1$ )
- are symmetric:  $B_{i,n}(t) = B_{n-i,n}(1-t)$
- are bounded between  $t \in [0, 1]$  and are restricted to  $[0, 1]$
- sum up to:  $\sum_{i=0}^n B_{i,n}(t) = 1$

$$\frac{d}{dt} B_{i,n}(t) = n(B_{i-1,n-1}(t) - B_{i,n-1}(t)) = -n\Delta B_{i-1,n-1}(t)$$

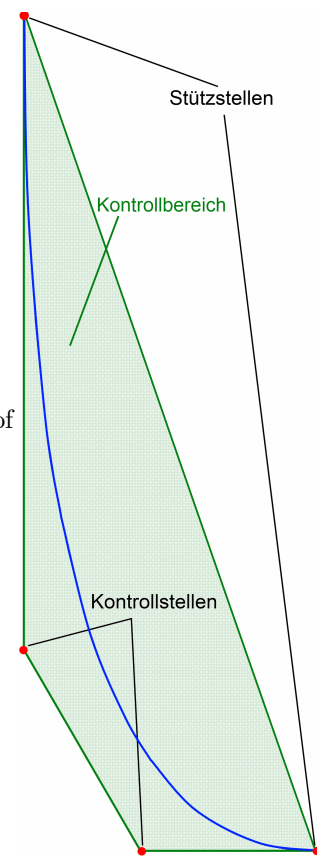
$$\frac{d^2}{dt^2} B_{i,n}(t) = n(n-1)(B_{i-2,n-2}(t) - 2B_{i-1,n-2}(t) + B_{i,n-2}(t)) = n(n-1)\Delta^2 B_{i-2,n-2}(t)$$

$$\frac{d^k}{dt^k} B_{i,n}(t) = (-1)^k n(n-1)\dots(n-k+1)\Delta^k B_{i-k,n-k}(t)$$

**Pascal's Triangle**

**Bernstein Polynomials**  $0 \leq t \leq 1$

$n = 0:$			1																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																															
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### 4.2.2 Simple Bézier Curves

A Bézier curve is defined by control points  $(\vec{P}_0, \vec{P}_1, \dots, \vec{P}_n \ (n \geq 2))$  in  $R^d$  and the Bernstein polynomials:

$$\vec{r}(t) = \sum_{i=0}^n \vec{P}_i B_{i,n}(t) \quad t \in [0, 1]$$

**Properties**

- Bézier curves always lie within the convex hull of the control points.

$$\vec{r}(0) = \vec{P}_0$$

$$\vec{r}(1) = \vec{P}_n$$

$$\vec{r}'(0) = n(\vec{P}_1 - \vec{P}_0)$$

$$\vec{r}'(1) = n(\vec{P}_n - \vec{P}_{n-1})$$

$$\vec{r}''(0) = n(n-1)(\vec{P}_2 - 2\vec{P}_1 + \vec{P}_0) \quad \vec{r}''(1) = n(n-1)(\vec{P}_n - 2\vec{P}_{n-1} + \vec{P}_{n-2})$$

- If  $C^k$  continuity is demanded at a point,  $k$  equations or  $k$  control points per point are necessary.



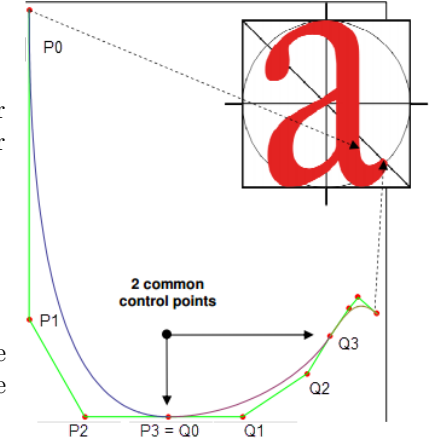
### 4.2.3 Casteljau Recurrence

The Casteljau recurrence is a similar idea to Neville-Aitken. It allows a point on a Bézier curve to be calculated as a linear combination of two points from Bézier curves of lower order.

$$\vec{r}_{\vec{P}_0, \vec{P}_1, \dots, \vec{P}_n}(t) = (1-t) \cdot \vec{r}_{\vec{P}_0, \vec{P}_1, \dots, \vec{P}_{n-1}}(t) + t \cdot \vec{r}_{\vec{P}_1, \vec{P}_2, \dots, \vec{P}_n}(t) \quad t \in [0, 1]$$

### 4.2.4 Composite Bézier Curves

The composite (simple) Bézier curves should satisfy the known conditions ( $C^k$ ). These are the equations to the point  $Q_0$  with control points  $Q_1, \dots, m-1$  after  $Q_0$ , as well as the control points  $P_1, \dots, n-1$  before  $Q_0$ .



Here,  $m = n = 3$

$$\begin{aligned} C^0 : & \quad \vec{P}_n = \vec{Q}_0 \\ C^1 : & \quad \vec{r}'_P(1) = \boxed{n(\vec{P}_n - \vec{P}_{n-1}) = m(\vec{Q}_1 - \vec{Q}_0)} = \vec{r}'_Q(0) \\ C^2 : & \quad \vec{r}''_P(1) = \boxed{n(n-1)(\vec{P}_n - 2\vec{P}_{n-1} + \vec{P}_{n-2}) = m(m-1)(\vec{Q}_2 - 2\vec{Q}_1 + \vec{Q}_0)} = \vec{r}''_Q(0) \\ C^k : & \quad \vec{r}^{(k)}_P(1) = \boxed{n(n-1) \dots (n-k+1)(\Delta^k \vec{P}_{n-k}) = m(m-1) \dots (m-k+1)(\Delta^k \vec{Q}_0)} = \vec{r}^{(k)}_Q(0) \end{aligned}$$

where  $m$  is the degree of  $\vec{r}_Q$ , and  $n$  is the degree of  $\vec{r}_P$ .

### 4.2.5 Composite Bézier Curves with $P_{i,j}$

When  $\vec{r}_j(t) = \sum_{i=0}^n \vec{P}_{i,j} B_{i,n}(t)$   $t \in [0, 1]$  is defined with the fixed points  $Q_j$  (degree:  $n$ ), the following formulas are obtained:

$$\begin{aligned} C^0 : & \quad \vec{P}_{n,j-1} = \vec{P}_{0,j} = \vec{Q}_j \\ C^1 : & \quad \vec{r}'_{j-1}(1) = \boxed{\vec{Q}_j - \vec{P}_{n-1,j-1} = \vec{P}_{1,j} - \vec{Q}_j} = \vec{r}'_j(0) \\ C^2 : & \quad \vec{r}''_{j-1}(1) = \boxed{\vec{Q}_j - 2\vec{P}_{n-1,j-1} + \vec{P}_{n-2,j-1} = \vec{P}_{2,j} - 2\vec{P}_{1,j} + \vec{Q}_j} = \vec{r}''_j(0) \\ C^k : & \quad \vec{r}^{(k)}_{j-1}(1) = \boxed{\Delta^k \vec{P}_{n-k,j-1} = \Delta^k \vec{P}_{0,j}} = \vec{r}^{(k)}_j(0) \end{aligned}$$

### 4.2.6 Bézier Surfaces as Tensor Splines

Bézier surfaces are formed in the same way as Bézier curves, using a tensor product of Bernstein polynomials. A surface in the interval  $[0, 1] \times [0, 1]$  is formed by

$$\vec{z}(s, t) = \sum_{i=0}^n \sum_{j=0}^m \vec{P}_{ij} B_{in}(s) B_{jm}(t) \quad s, t \in [0, 1]$$

Details and Formulas see Skript p. 19 – 23.

### 4.2.7 Comparison of Bézier and Newton Interpolation with Equally Distributed Arguments on the X-Axis

From the Bézier perspective:

- + Small error, no oscillation, no Runge phenomenon (with a low degree, e.g., 3)
- +  $O(n)$  (linear computation effort)
- Not embedded (new data requires recalculation of the interpolation)
- Post-processing (graphics, integration, derivatives) is more complex, as each curve segment must be considered individually.

## 5 Least-Squares Approximation

### 5.1 Linear Least-Squares

Goal: Approximation of data points by minimizing a function that indicates the deviation in  $y$  (residues). In this method, the data points are not precisely matched, and the degree  $m$  of the approximation polynomial is usually significantly smaller than the number of given data points  $N + 1$ .

From a set of basis functions  $\{g_0, g_1, \dots, g_m\}$  with  $m \ll N$  and the measurements  $(x_0, y_0), (x_1, y_1), \dots, (x_N, y_N)$ , the over-determined ( $m < N$ ) system of equations is set up and solved for the unknowns  $a_i$ .

$$\overbrace{\begin{pmatrix} g_0(x_0) & g_1(x_0) & \dots & g_m(x_0) \\ \vdots & \vdots & \ddots & \vdots \\ g_0(x_N) & g_1(x_N) & \dots & g_m(x_N) \end{pmatrix}}^{\text{Design Matrix } G} \cdot \begin{pmatrix} a_0 \\ \vdots \\ a_m \end{pmatrix} = \begin{pmatrix} y_0 \\ \vdots \\ y_N \end{pmatrix} \Leftrightarrow G \cdot a = y$$

Error function to be minimized:

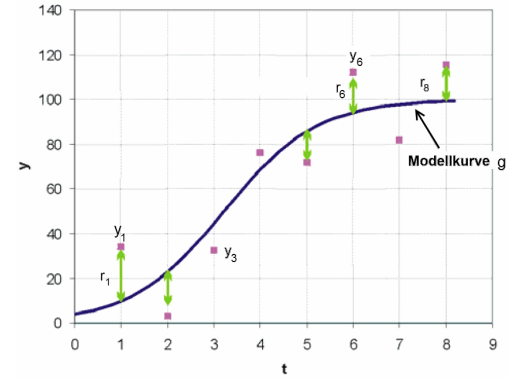
$$\underbrace{S}_{\text{Error}} = \sum_{i=0}^N \left( y_i - \underbrace{\sum_{j=0}^m a_j g_j}_{\text{Model}} \right)^2 = \sum_{i=0}^N r_i^2$$

Model function:

$$y = \sum_{j=0}^m a_j g_j \stackrel{\text{Poly!}}{=} \sum_{j=0}^m a_j x^j$$

Deviations (Residues):

$$r_i = y_i - \sum_{j=0}^m a_j g_j$$

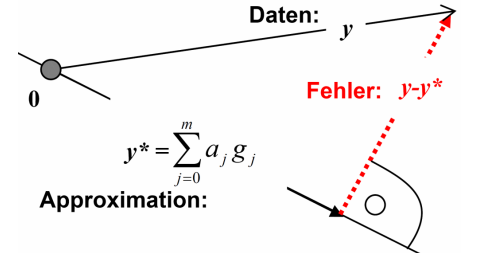


#### 5.1.1 Normal Equation

Minimizing the quadratic error with "normal matrix"  $G$  (design matrix):

$$\underbrace{G^T G}_{\text{Normal Matrix}} \cdot a = G^T y \Rightarrow a = (G^T G)^{-1} G^T y$$

$$\underbrace{\begin{pmatrix} G^T \\ (m+1) \times (N+1) \end{pmatrix}}_{(m+1) \times (m+1)} \cdot \underbrace{\begin{pmatrix} G \\ (N+1) \times (m+1) \end{pmatrix}}_{(m+1) \times (m+1)} \cdot \underbrace{a}_{(m+1) \times 1} = \underbrace{\begin{pmatrix} G^T \\ (m+1) \times (N+1) \end{pmatrix}}_{(m+1) \times (m+1)} \cdot \underbrace{y}_{(N+1) \times 1}$$



The symmetric  $(m+1) \times (m+1)$  matrix  $G^T \cdot G$  is positive definite if  $g_i$  are linearly independent. The new system of equations is regular and has a unique solution for the coefficients  $a_j$ .

$$\overbrace{\begin{pmatrix} \langle g_0 | g_0 \rangle & \langle g_1 | g_0 \rangle & \dots & \langle g_m | g_0 \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle g_0 | g_m \rangle & \langle g_1 | g_m \rangle & \dots & \langle g_m | g_m \rangle \end{pmatrix}}^{G^T \cdot G} \cdot \begin{pmatrix} a_0 \\ \vdots \\ a_m \end{pmatrix} = \begin{pmatrix} \langle y | g_0 \rangle \\ \vdots \\ \langle y | g_m \rangle \end{pmatrix}$$

#### 5.1.2 QR Decomposition

A square  $n \times n$  matrix  $Q$  is orthogonal if  $Q^T \cdot Q = Q \cdot Q^T = I$  or  $Q^T = Q^{-1}$ . Any matrix  $G$  with dimensions  $(N+1) \times (m+1)$  where  $N \geq m$  and rank  $m+1$  can be represented as the product of an orthogonal  $(N+1) \times (N+1)$  matrix  $Q$  and an upper triangular matrix  $R$  with dimensions  $(N+1) \times (m+1)$ .

The equation system  $Ga = y$  then becomes  $Ra = Q^T y$ , which is easier to solve<sup>Citation needed</sup>.

### 5.1.3 Singular Value Decomposition (SVD)

Any matrix  $\mathbf{G}$  with dimensions  $(N+1) \times (m+1)$  can be decomposed as follows:

$$\mathbf{G} = \mathbf{U}\mathbf{D}\mathbf{V}^T = \mathbf{U} \cdot \begin{bmatrix} d_{00} & 0 & \dots & 0 \\ 0 & d_{11} & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & \dots & 0 & d_{mm} \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & 0 \end{bmatrix} \cdot \mathbf{V}^T$$

$\mathbf{D}$   $((N+1) \times (m+1))$  is the central diagonal matrix with singular values,  $\mathbf{U}$   $((N+1) \times (N+1))$  and  $\mathbf{V}$   $((m+1) \times (m+1))$  are orthogonal and square matrices (meaning, among other things, that  $\mathbf{U}^T = \mathbf{U}^{-1}$ ).

The decomposition itself is not further explained here and can be looked up elsewhere. SVD can be used to solve equation systems (as here), compute eigenvectors and eigenvalues, and find inverses.

From  $\mathbf{G}\mathbf{a} = \mathbf{y}$  we get  $\mathbf{U}\mathbf{D}\mathbf{V}^T\mathbf{a} = \mathbf{y}$ . This can be solved to

$$\mathbf{a} = \mathbf{V}\mathbf{D}^{-1}\mathbf{U}^T \cdot \mathbf{y}$$

### 5.1.4 Monomials

With  $g = a_0 \underbrace{1}_{g_0} + a_1 \underbrace{x}_{g_1} + a_2 \underbrace{x^2}_{g_2} + \dots + a_m \underbrace{x^m}_{g_2}$  the Design Matrix becomes:

$$\mathbf{G} \underset{\text{When Monomials}}{=} \begin{bmatrix} 1 & x_0 & x_0^2 & \dots & x_0^m \\ 1 & x_1 & x_1^2 & \dots & x_1^m \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_N & x_N^2 & \dots & x_N^m \end{bmatrix}$$

### 5.1.5 Uniform Arguments (Orthogonality Properties)

With uniform elements  $x_i - x_j = (j - i)h$  for all  $i, j \rightarrow \{x_0 \dots x_N\} = \{x_0 + t \cdot h\}_{t=0 \dots N}$  and **orthogonal polynomials**,  $\mathbf{G}\mathbf{G}^T$  can be diagonalized, making the equations computationally simpler to solve.

$$p_{k,N}(t) = \sum_{i=0}^k (-1)^i \binom{k}{i} \binom{k+i}{i} \frac{t^{(i)}}{N^{(i)}} = 1 + \sum_{i=1}^k (-1)^i \binom{k}{i} \binom{k+i}{i} \frac{t(t-1)(t-2)\dots(t-i+1)}{N(N-1)(N-2)\dots(N-i+1)} \quad (k = 1, \dots, N)$$

with  $t = \frac{x-x_0}{h}$   $\binom{k}{i} = \frac{k!}{i!(k-i)!} = nCr(k, i)$

$p_{k,N}$  can now be substituted as  $g_k$  into the design matrix, and the product  $\mathbf{G}^T\mathbf{G}$  becomes a  $(m+1) \times (m+1)$  diagonal matrix. Finally,  $\mathbf{a}$  can be calculated using the known formula  $\mathbf{G}^T\mathbf{G}\mathbf{a} = \mathbf{G}^T\mathbf{y}$ .

**Example:**

$$\begin{aligned} \mathbf{x} &= \begin{bmatrix} 3 & 4 & 5 & 6 & 7 \end{bmatrix} \Rightarrow \mathbf{t} = \mathbf{x} - 3 = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \end{bmatrix} \\ \Rightarrow P_0(x) &= 1 \quad P_1(x) = 1 - \frac{x-3}{2} \quad P_2(x) = 1 - \frac{2 \cdot 3(x-3)}{4} + \frac{1 \cdot 3(x-3)(x-3-1)}{4 \cdot 3} = 1 - \frac{3x-9}{2} + \frac{1}{2}(x-4)(x-3)??? \\ \mathbf{G} &= \begin{bmatrix} P_0 & P_1 & P_2 \\ 1 & 1 & 1 \\ 1 & 1/2 & -1/2 \\ 1 & 0 & -1 \\ 1 & -1/2 & -1/2 \\ 1 & -1 & 1 \end{bmatrix} \Rightarrow \mathbf{G}^T\mathbf{G} = \begin{bmatrix} \|P_0\|^2 & 0 & 0 \\ 0 & \|P_1\|^2 & 0 \\ 0 & 0 & \|P_2\|^2 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & \frac{5}{2} & 0 \\ 0 & 0 & \frac{7}{2} \end{bmatrix} \end{aligned}$$

### 5.1.6 Chebyshev Orthogonal Polynomials

**Idea:** Approximation of a continuous polynomial using Chebyshev polynomials.

**Definition**

Chebyshev polynomials are defined as  $T_n(x) = \cos(n \arccos(x))$  with  $(n = 0, 1, \dots)$  and  $(-1 \leq x \leq 1)$ . This polynomial results in a clustering of data points in the boundary regions. The first few polynomials  $T_n$  and their expressions in terms of

$$\begin{array}{ll}
 T_0 = 1 & x^0 = 1 = T_0 \\
 T_1 = x & x^1 = x = T_1 \\
 T_2 = 2x^2 - 1 & x^2 = \frac{1}{2}T_2 + \frac{1}{2}T_0 \\
 T_3 = 4x^3 - 3x & x^3 = \frac{1}{4}T_3 + \frac{3}{4}T_1 \\
 T_4 = 8x^4 - 8x^2 + 1 & x^4 = \frac{1}{8}T_4 + \frac{1}{2}T_2 + \frac{3}{8}T_0 \\
 T_5 = 16x^5 - 20x^3 + 5x & x^5 = \frac{1}{16}T_5 + \frac{5}{16}T_3 + \frac{5}{8}T_1
 \end{array}$$

Additional Chebyshev polynomials can be computed using the recursion formula  $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$  ( $n \geq 2$ ) with initial conditions  $T_1(x) = x$ ,  $T_0(x) = 1$ .

**Properties:**

- The maximum amplitude of the Chebyshev polynomial is  $\frac{1}{2^n}$  (max amplitude of the error term).
- Amplitude:  $T_n(x) \in [-1, +1]$
- Roots:  $T_n(x) = 0 \Leftrightarrow x = \cos(\frac{2i+1}{2n}\pi)$   $i = 0, 1, \dots, n-1$  (These roots are the Chebyshev nodes)
- $T_n(x) = \pm 1 \Leftrightarrow x = \cos(\frac{i\pi}{n})$  ( $i = 0, 1, \dots, n$ )

**Recipe**

Chebyshev polynomials can only be used if  $x$  **with Chebyshev distances**  $x_i = \cos(\frac{2i+1}{2n}\pi)$  ( $i = 0, 1, \dots, n-1$ ) are used and not obtained with equidistant distances.

Goal: Approximate  $y(t) = p_N(t)$  (polynomial, defined in  $[a, b]$ ) with Chebyshev polynomials of degree  $m$ .

1. Normalize  $y(t)$  to the standard interval  $[-1, 1]$  (affine transformation)  $t = a + \frac{b-a}{2}(x+1)$ .
2. Compose  $y(x)$  from  $T_n$
3. Truncate  $y(x)$  to degree  $m$ :  $y_m(x)$
4. Back-transform:  $x = 2\frac{t-a}{b-a} - 1$
5. Substitute  $T_n$  into  $y_m$  (see table above)
6. Error estimate for truncate method:  
 $\max_t |y(t) - y_m(t)|$  (Truncated part)

**Example**

Wanted: Approximation  $y(t) = t^3$  with degree  $m = 2$  for interval  $(a, b) = (0, 1)$

1. Transformation with  $t = \frac{x+1}{2}$ :  
 $y(x) = \left(\frac{x+1}{2}\right)^3 = \frac{1}{8}(x^3 + 3x^2 + 3x + 1)$
2. Expand with table:  
 $y(x) = \frac{1}{8} \left( \frac{T_3(x)+3T_1(x)}{4} + 3\frac{T_2(x)+T_0}{2} + 3T_1(x) + T_0(x) \right)$   
 $= \frac{1}{32}T_3(x) + \frac{3}{16}T_2(x) + \frac{15}{32}T_1(x) + \frac{5}{16}T_0(x)$
3. Truncate to degree  $m = 2$ :  
 $y(x) \approx \frac{3}{16}T_2(x) + \frac{15}{32}T_1(x) + \frac{5}{16}T_0(x)$
4. Back-transform with  $x = 2t - 1$ :  
 $y(t) \approx \frac{3}{16}T_2(2t-1) + \frac{15}{32}T_1(2t-1) + \frac{5}{16}T_0(2t-1)$
5. Replace  $T_n(2t-1)$ :  
 $y(t) \approx \frac{3}{16}(2(2t-1)^2 - 1) + \frac{15}{32}(2t-1) + \frac{5}{16}$
6. Error estimate:  
 $\max_t \left| \frac{1}{32}T_3(2t-1) \right|$

The **Design Matrix** looks like this:

$$G \underset{\text{When Chebyshev}}{=} \begin{bmatrix} T_0(x_0) = 1 & T_1(x_0) = x_0 & \dots & T_m(x_0) \\ T_0(x_1) = 1 & T_1(x_1) = x_1 & \dots & T_m(x_1) \\ \vdots & \vdots & \ddots & \vdots \\ T_0(x_N) = 1 & T_1(x_N) = x_N & \dots & T_m(x_N) \end{bmatrix}_{N \times m}$$

With this,  $G^T G$  becomes:

$$G^T G \underset{\text{When Chebyshev}}{=} \begin{bmatrix} N+1 & 0 & \dots & 0 \\ 0 & \frac{N+1}{2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \frac{N+1}{2} \end{bmatrix}_{m \times m}$$

and  $(G^T G)^{-1} G^T$  becomes:

$$(G^T G)^{-1} G^T \underset{\text{When Chebyshev}}{=} \begin{bmatrix} \frac{1}{N+1} T_0(x_0) = \frac{1}{N+1} & \frac{1}{N+1} T_0(x_1) = \frac{1}{N+1} & \dots & \frac{1}{N+1} T_0(x_N) = \frac{1}{N+1} \\ \frac{2}{N+1} T_1(x_0) = \frac{2}{N+1} x_0 & \frac{2}{N+1} T_1(x_1) = \frac{2}{N+1} x_1 & \dots & \frac{2}{N+1} T_1(x_N) = \frac{2}{N+1} x_N \\ \vdots & \vdots & \ddots & \vdots \\ \frac{2}{N+1} T_m(x_0) & \frac{2}{N+1} T_m(x_1) & \dots & \frac{2}{N+1} T_m(x_N) \end{bmatrix}_{m \times N}$$

### 5.1.7 Discrete Least-Squares Chebyshev Approximation

Follows from  $a = (G^T G)^{-1} G^T y$  for  $x_i = \cos(\frac{2i+1}{2(N+1)}\pi)$  ( $i = 0, 1, \dots, N$ ):

$$p(x) = \sum_{j=0}^m a_j T_j(x) \quad \text{where} \quad a_j = \begin{cases} \frac{1}{N+1} \sum_{i=0}^N y(x_i) & j = 0 \\ \frac{2}{N+1} \sum_{i=0}^N T_j(x_i) y(x_i) & j > 0 \end{cases}$$

### 5.1.8 Continuous Least-Squares

For continuous functions that are **not polynomials** (!), the continuous version can also be used.

S: error function to be minimized, w: weight with  $(-1 \leq x \leq 1)$

$$S = \int_{-1}^1 (y(x) - p(x))^2 \cdot w(x) \cdot dx$$

### Continuous Least-Squares Chebyshev Approximation

The weight is particularly placed on the boundary (as is customary with Chebyshev). ( $w(x) = \frac{1}{\sqrt{1-x^2}}$ )

$$p(x) = \sum_{j=0}^m a_j T_j(x) \quad \text{where} \quad a_j = \begin{cases} \frac{1}{\pi} \int_{-1}^1 \frac{y(x)}{\sqrt{1-x^2}} dx & j = 0 \\ \frac{2}{\pi} \int_{-1}^1 \frac{y(x) T_j(x)}{\sqrt{1-x^2}} dx & j > 0 \end{cases}$$

### Continuous Least-Squares Legendre Approximation:

Here, the weight  $w(t) = 1$  is used.

Legendre Polynomials:

$$\begin{aligned} P_n(x) &= \frac{1}{2^n n!} \cdot \frac{d^n}{dx^n} (x^2 - 1)^n & P_0(x) &= 1 & P_1(x) &= x & P_2(x) &= \frac{1}{2}(3x^2 - 1) \\ P_3(x) &= \frac{1}{2}(5x^3 - 3x) & P_4(x) &= \frac{1}{8}(35x^4 - 30x^2 + 3) & P_5(x) &= \frac{1}{8}(63x^5 - 70x^3 + 15x) \\ 1 &= P_0(x) & x &= P_1(x) & x^2 &= \frac{2P_2(x) + P_0(x)}{3} & x^3 &= \frac{2P_3(x) + 3P_1(x)}{5} & x^4 &= \frac{8P_4(x) + 20P_2(x) + 7P_0(x)}{35} \end{aligned}$$

Coefficients:

$$p(x) = \sum_{j=0}^m a_j P_j(x) \quad \text{wobei} \quad a_j = \frac{2j+1}{2} \cdot \int_{-1}^1 y(x) P_j(x) \cdot dx \quad (j = 0, 1, \dots, m)$$

## 5.2 Multi-Variate Linear Least Squares Approximation

Multi-variate least square problems are solved exactly the same way as uni-variate least square problems (using the same methods)! The variable  $x$  is no longer a number but a vector with  $d$  dimensions!

**Basis Generation:** The bases are computed using the tensor product of uni-variate bases (multiply all with all in d-variables).

$$\sum_{j_1=0}^{m_1} \sum_{j_2=0}^{m_2} \dots \sum_{j_d=0}^{m_d} a_{j_1, j_2, \dots, j_d} \cdot g_{j_1}(x^{(1)}) g_{j_2}(x^{(2)}) \cdot \dots \cdot g_{j_d}(x^{(d)})$$

### Examples

Basis 3. Grad:  $\{1, x, y, x^2, 2xy, x^2, x^3, 3x^2y, 3xy^2, y^3\}$

Basis 4. Grad:  $\{1, x, y, x^2, 2xy, x^2, x^3, 3x^2y, 3xy^2, y^3, x^4, 4x^3y, 6x^2y^2, 4xy^2, y^4\}$

### 5.2.1 Matrix Condition Number

Given a linear system of equations in the form  $Ax = b$  with the solution  $x = A^{-1}b$ . The condition number  $\kappa(A)$  measures the maximum relative error of the solution  $x$  with respect to the relative error of the right-hand side  $b$ :

$$\kappa(A) = \begin{cases} \max \left( \frac{\|\Delta x\|/\|x\|}{\|\Delta b\|/\|b\|} \right) & \text{if } b \neq 0, \Delta b \neq 0, Ax = b \\ \infty & \text{otherwise} \end{cases} \quad \text{if the matrix } A \text{ is regular}$$

where  $\|\dots\|$  denotes a vector norm, usually the Euclidean norm  $\|\dots\|_2$  or the maximum norm  $\|\dots\|_\infty$ . If  $\kappa(A)$  is relatively small, the system/matrix is called "well-conditioned," otherwise, it is "ill-conditioned." For a regular matrix  $A$  and the Euclidean norm  $\|\dots\|_2$ :

- $\kappa(A) = \frac{|\sigma_{max}|}{|\sigma_{min}|}$ ,  $|\sigma_{max}|$  = absolute maximum singular value, and  $|\sigma_{min}|$  = absolute minimum singular value of  $A$
- If  $A$  is also a normal matrix:  $\kappa(A) = \frac{|\lambda_{max}|}{|\lambda_{min}|}$ ,  $|\lambda_{max}|$  = absolute maximum eigenvalue, and  $|\lambda_{min}|$  = absolute minimum eigenvalue

## 6 Error Propagation

### 6.1 Differential

The differential  $df$  denotes the linear part of the increment of a variable in a function.

$$\Delta f = f(x_0 + h) - f(x_0) \approx df = f'(x_0)dx = f'(x_0)h = f'(x_0)\Delta x$$

$\Delta f$  is the *absolute* error of  $f$  at  $x_0$ . The *relative* error is  $\frac{\Delta f}{f} \approx \frac{df(x_0)}{f(x_0)}$

Any  $n$  times differentiable function can be approximated with a **Taylor** series:

$$f(x_0 + h) = f(x_0) + hf'(x_0) + \frac{h^2}{2}f''(x_0) + \frac{h^3}{3!}f'''(x_0) + \dots + \frac{h^n}{n!}f^{(n)}(x_0) + R_n(x_0, h)$$

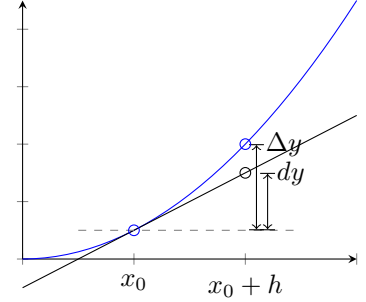
Where  $R_n(x_0, h)$  denotes the remainder term and converges to 0 with a speed of  $o(h^n)$  ("fast") as  $n \rightarrow \infty$ . There are, for example, the

$$\text{Lagrange Form: } R_n(x_0, h) = \frac{h^{n+1}}{(n+1)!} f^{(n+1)}(\underbrace{x_0 + \vartheta h}_{\xi}) \quad (\xi \in (x_0, x_0 + h), \vartheta \in (0, 1)) \quad \text{or the}$$

$$\text{Cauchy Form: } R_n(x_0, h) = \frac{h^n(1 - \vartheta)^n}{(n+1)!} f^{(n+1)}(\underbrace{x_0 + \vartheta h}_{\xi}) \quad (\xi \in (x_0, x_0 + h), \vartheta \in (0, 1))$$

With the Taylor series, the absolute error becomes

$$\Delta f = \frac{1}{1!}df(x_0) + \frac{1}{2!}d^2f(x_0) + \dots + \frac{1}{n!}d^nf(x_0) + R_n(x_0, h)$$



### 6.2 Multivariate Differentials, Taylor

In the multidimensional case, the differential looks like this:

$$\Delta f \approx df(\vec{x}) = \sum_{i=1}^n \frac{\partial f(\vec{x})}{\partial x_i} h_i = h_1 \frac{\partial f(\vec{x})}{\partial x_1} + \dots + h_n \frac{\partial f(\vec{x})}{\partial x_n} \quad \text{with } \vec{h} = [h_1, \dots, h_d]^T, \vec{x} = [x_1, \dots, x_d]^T$$

The vector field of partial derivatives is called the gradient field of  $f$  and is denoted by  $\text{grad}(f)$  or  $\vec{\nabla} f(\vec{x})$ . Thus,  $\Delta f \approx \vec{\nabla} f(\vec{x}) \cdot \vec{h}$ . In the multivariate case, the Taylor series for multi-indices  $\alpha = \{\alpha_1, \dots, \alpha_n\}$  and  $|\alpha| = \alpha_1 + \dots + \alpha_n$  is given by

$$f(\vec{x}_0 + \vec{h}) = f(\vec{x}_0) + \frac{1}{1!} \vec{\nabla} f(\vec{x}_0) \cdot \vec{h} + \sum_{|\alpha|=2}^N \frac{1}{\alpha!} \frac{\partial^{|\alpha|} f(\vec{x}_0)}{\partial x^\alpha} \cdot \vec{h}^\alpha + \sum_{|\alpha|=N+1} R_\alpha(\vec{x}_0, \vec{h}) \cdot \vec{h}^\alpha$$

or simplified for 2nd order

$$f(\vec{x}_0 + \vec{h}) = f(\vec{x}_0) + \vec{h} \nabla f(\vec{x}_0) + \underbrace{\left( \frac{h_1^2}{2!} \frac{\partial^2 f}{\partial x_1^2} + \frac{h_2^2}{2!} \frac{\partial^2 f}{\partial x_2^2} + \frac{h_1 h_2}{1!1!} \frac{\partial^2 f}{\partial x_1 \partial x_2} \right)}_{\text{for 2nd order}} + R_{(3,0)}(\vec{x}_0, \vec{h}) \cdot \vec{h}_1^3 + R_{(2,1)}(\vec{x}_0, \vec{h}) \cdot \vec{h}_1^2 \vec{h}_2 + R_{(1,2)}(\vec{x}_0, \vec{h}) \cdot \vec{h}_1 \vec{h}_2^2 + R_{(0,3)}(\vec{x}_0, \vec{h}) \cdot \vec{h}_2^3$$

### 6.3 Jacobian Matrix

The Jacobian matrix maps all first derivatives of a function. If the Jacobian matrix is square ( $m = n$ ), its determinant  $\det(J)$  can be interpreted as the transformed volume:

$$\underbrace{dy_1 dy_2 \dots dy_n}_{\text{„transformiertes Volumenelement“}} = \det(J_f(\vec{x})) \underbrace{dx_1 dx_2 \dots dx_n}_{\text{„Volumenelement“}}$$

$$J_f(\vec{x}) := \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\vec{x}) & \frac{\partial f_1}{\partial x_2}(\vec{x}) & \dots & \frac{\partial f_1}{\partial x_n}(\vec{x}) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(\vec{x}) & \frac{\partial f_m}{\partial x_2}(\vec{x}) & \dots & \frac{\partial f_m}{\partial x_n}(\vec{x}) \end{bmatrix}$$

The Jacobian determinant can also be a measure of error propagation. The calculation is done directly using the determinant rule, or

$$\frac{D(f_1, f_2, \dots, f_n)}{D(x_1, x_2, \dots, x_n)} = \det(J_f(\vec{x})) \quad \text{or} \quad \det(J_f) = \sqrt{\det \left( \underbrace{J_f^T(\vec{x}) J_f(\vec{x})}_{\text{diagonal matrix}} \right)}.$$

Also, for the transformation  $T$  and its inverse  $T^{-1}$ :  $\det(J_T) = \frac{1}{\det J_{T^{-1}}}$

## 7 Numerical Ordinary Differential Equations

### 7.1 Definitions

Ordinary Differential Equation (ODE):

$$y'(x) = f(x, y(x)) \quad \text{with} \quad y(x_0) = y_0$$

$f(x, y)$  is also referred to as the right-hand side (r.h.s.) function.

### 7.2 Explicit Methods

Forward time stepping is used without the need for solvers, based on the Taylor approximation. However, this approach becomes computationally intensive quickly, requiring higher derivatives in multiple dimensions.

$$y(x_0 + h) = y(x_0) + \frac{h}{1!}y'(x_0) + \frac{h^2}{2!}y''(x_0) + \dots + \frac{h^p}{p!}y^{(p)}(x_0) + \underbrace{\frac{h^{p+1}}{(p+1)!}y^{(p+1)}(\xi)}_{\text{remainder term}}$$

Further derivatives are derived using the chain rule:

$$y''(x) = \frac{\partial f(x, y)}{\partial x}1 + \frac{\partial f(x, y)}{\partial y}f(x, y) = \frac{\partial y'(x)}{\partial x}1 + \frac{\partial y'(x)}{\partial y}y'(x) \quad y'''(x) = \frac{\partial y''(x)}{\partial x}1 + \frac{\partial y''(x)}{\partial y}y'(x) \quad \dots$$

Derivatives up to the third degree are listed here (immense computational effort required):

$$\begin{aligned} y(x+h) = & y(x) + \frac{f(x, y)}{1!}h + \\ & \frac{1}{2!} \left( \frac{\partial f(x, y)}{\partial x}1 + \frac{\partial f(x, y)}{\partial y}f(x, y) \right) h^2 + \\ & \frac{1}{3!} \left( \frac{\partial^2 f(x, y)}{\partial x^2}1 + 2 \frac{\partial^2 f(x, y)}{\partial x \partial y}f(x, y) + \frac{\partial^2 f(x, y)}{\partial y^2}f(x, y)^2 + \left( \frac{\partial f(x, y)}{\partial y} \right)^2 f(x, y) + \frac{\partial f(x, y)}{\partial x} \frac{\partial f(x, y)}{\partial y} \right) h^3 + \dots + \\ & \frac{1}{4!}y^{(4)}(x)h^4 + \dots + \frac{1}{p!}y^{(p)}(x)h^p + \underbrace{\frac{1}{(p+1)!}y^{(p+1)}(\xi)h^{p+1}}_{\text{remainder term}} \end{aligned}$$

$$\text{Global error} = \max_{0 \leq i \leq k} |y_i - y(x_i)|$$

$$\text{Local error (slope)} = \tau_h(x_n) = \frac{y(x_n + h) - y(x_n)}{h} - \left( \frac{y'(x_n)}{1!} + \frac{y''(x_n)}{2!}h^1 + \dots + \frac{y^{(p)}(x_n)}{p!}h^{p-1} \right)$$

$$h\tau_h(x_n) = y(x_n + h) - y(x_n) - \left( \frac{y'(x_n)}{1!}h^1 + \frac{y''(x_n)}{2!}h^2 + \dots + \frac{y^{(p)}(x_n)}{p!}h^p \right)$$

Assuming that the initial value of the current step is exactly correct, i.e.,  $y_n = y(x_n)$ :

$$h\tau_h(x_n) = y(x_n + h) - \left( y(x_n) + \frac{y'(x_n)}{1!}h^1 + \frac{y''(x_n)}{2!}h^2 + \dots + \frac{y^{(p)}(x_n)}{p!}h^p \right) = y(x_n + h) - y_{n+1}$$

#### 7.2.1 Explicit Euler Method

Special case of explicit methods:

*Taylor with order  $p = 1$*

Relatively inaccurate approximation since the derivative remains unchanged in each step.

Global error:  $\max_{0 \leq i \leq k} |y_i - y(x_i)|$  with the number of steps  $k$

Local error:  $y(x_n + h) - y_{n+1}$  when the initial value of the current step is exactly correct

$$y_0 = y(x_0)$$

$$y(x_0 + h) \approx y_1 = y_0 + f(x_0, y_0)h \approx y(x_0) + y'(x_0)h$$

$$y(x_1 + h) \approx y_2 = y_1 + f(x_1, y_1)h \approx y(x_1) + y'(x_1)h$$

$\vdots$

$$y(x_{n-1} + h) \approx y_n = y_{n-1} + f(x_{n-1}, y_{n-1})h \approx y(x_{n-1}) + y'(x_{n-1})h$$



### 7.2.2 Explicit Taylor Methods of Higher Order

For  $p > 1$ , a better approximation is achieved, but the computational effort increases significantly. See Section 7.2 for calculation.

$$y_0 = y(x_0)$$
$$y_{k+1} = y_k + \frac{y'_k}{1!} h_k + \frac{y''_k}{2!} h_k^2 + \dots + \frac{y_k^{(p)}}{p!} h_k^p$$

### 7.2.3 Explicit Runge-Kutta (RK) Methods Skript S. 8

$s$  recursive stages  $k$  are defined, which, when summed, yield the Taylor approximation. Here,  $s = 4$ .

$$\begin{aligned}k_1 &= hf(x, y) \\k_2 &= hf(x + mh, y + mk_1) \\k_3 &= hf(x + nh, y + nk_2) \\k_4 &= hf(x + ph, y + pk_3)\end{aligned}$$

$$y(x + h) \approx y(x) + ak_1 + bk_2 + ck_3 + dk_4$$

In contrast to the script (p. 10), the  $h$  values are not included in the  $k$  definitions but appear in the final formula. With the correct boundary conditions (p. 11), this formula corresponds to the Taylor polynomial up to order 4.

By equating with Taylor, explicit RK methods define an overdetermined system of equations with 8 equations and 7 unknowns  $m, n, p$  and  $a, b, c, d$ .

	Solution Name	# Stages ( $s$ )	Solutions
There are different <b>solutions</b> , also listed here for order $s = 2$ and $s = 4$ :	Heun	2	$a = b = \frac{1}{2}, m = 1$
	Explicit Midpoint	2	$a = 0, b = 1, m =$
	Classical Runge-Kutta	4	$m = n = \frac{1}{2}, p = 1$

$$\text{Local error (slope)} = \tau_h(x_n) = \frac{y(x_n + h) - y(x_n)}{h} - \left( \frac{\tilde{k}_1 + 2\tilde{k}_2 + 2\tilde{k}_3 + \tilde{k}_4}{6h} \right)$$

The tilde superscript  $\tilde{k}$  indicates that the stage values  $k$  must be evaluated using exact values for  $x, y$ .

### 7.2.4 General Framework for Runge-Kutta Methods (Butcher Tableaus) Skript S. 13

Conditions (For Explicit Method):

$$\left. \begin{aligned}k_1 &= f(x + c_1 h, y + h \underbrace{\sum_{j=1}^s a_{1,j} k_j}_{s_1}) \\k_2 &= f(x + c_2 h, y + h \underbrace{\sum_{j=1}^s a_{2,j} k_j}_{s_2}) \\k_3 &= f(x + c_3 h, y + h \underbrace{\sum_{j=1}^s a_{3,j} k_j}_{s_3}) \\&\vdots \\k_s &= f(x + c_s h, y + h \underbrace{\sum_{j=1}^s a_{s,j} k_j}_{s_s}) \\y(x+h) &\approx y(x) + h \left( \sum_{j=1}^s b_j k_j \right)\end{aligned} \right\} s \text{ stages}$$

$$\begin{array}{c|cccccc}0 & 0 & 0 & \cdots & 0 & 0 \\c_2 & a_{2,1} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\c_s & a_{s,1} & a_{s,2} & \cdots & a_{s,s-1} & 0 \\ \hline & b_1 & b_2 & \cdots & b_{s-1} & b_s\end{array}$$

$a$  = Splitting in  $Y$   
 $b^T$  = Solutions given by the method  
 $c$  = Splitting in  $X$

FSAL (First Same As Last)  
 Special Form:

$$\begin{array}{c|cccccc}0 & 0 & 0 & \cdots & 0 & 0 \\c_2 & a_{2,1} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\c_{s-1} & a_{s-1,1} & a_{s-1,2} & \cdots & 0 & 0 \\1 & b_1 & b_2 & \cdots & b_{s-1} & 0 \\ \hline & b_1 & b_2 & \cdots & b_{s-1} & 0\end{array}$$

$$\bullet c_i = \sum_{j=1}^s a_{ij} = \sum_{j=1}^{i-1} a_{ij} \quad (i = 2, \dots, s)$$

$$\bullet \sum_{j=1}^s b_j = 1$$

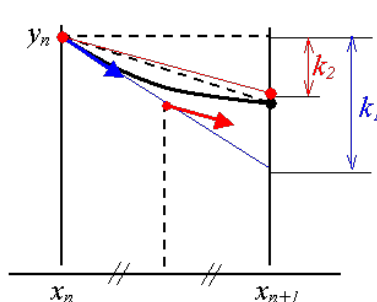
$$\bullet c_1 = 0$$

$$\bullet a_{1j} = 0 \quad 1 \leq j \leq s$$

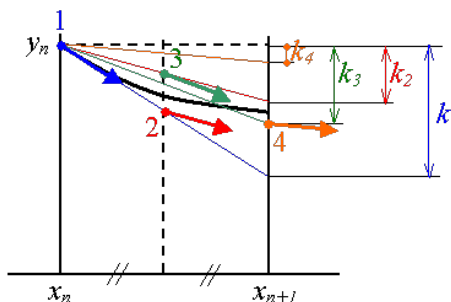
$$\bullet a_{ij} = 0 \quad j \geq i$$

$\begin{array}{c cc}0 & & \\1/2 & 1/2 & \\1/2 & 0 & 1/2 \\ \hline 1 & 0 & 0 & 1 \\ & 1/6 & 1/3 & 1/3 & 1/6\end{array}$	$\begin{array}{c c}0 & \\ \hline 1 & \\ \hline 0 & 1/2 & 1/2 \\ & 0 & 1\end{array}$	$\begin{array}{c cc}0 & & \\2/3 & 2/3 & \\ \hline 1/4 & 3/4 & \end{array}$
Classical Runge-Kutta of order 4 (cf. 1.10c)	Explicit Euler method (cf. 1.3A) Explicit Midpoint method (1.9b)	Yet another method (cf. 1.9c)

Heun Method (RK-2):  $b_1 = b_2 = \frac{1}{2}; c_2 = \frac{1}{2}; a_{2,1} = \frac{1}{2}$



Runge-Kutta 2<sup>ème</sup> ordre :  
 $k_2 = h f(x_n + h/2, y_n + k_1/2)$



Runge-Kutta 4<sup>ème</sup> ordre

### 7.2.5 Adaptive Explicit Methods Skript S. 17

Idea: Automatic adaptation of the step size  $h$ . This requires defining the maximum error of the approximation. The *Accuracy Goal* ( $ag$ ) specifies the minimum matching number of decimal places, while the *Precision Goal* ( $pg$ ) represents the number of significant figures in the result. The tolerance parameter is:

$$\varepsilon = \varepsilon_a + |y|\varepsilon_r = 10^{-ag} + |y|10^{-pg} \geq |e|.$$

#### Embedded Pairs of Explicit Runge-Kutta Methods

0	0	0	...	0	0
$c_2$	$a_{2,1}$	0	...	0	0
$\vdots$	$\vdots$	$\vdots$	$\ddots$	0	$\vdots$
$c_{s-1}$	$a_{s-1,1}$	$a_{s-1,2}$	...	0	0
$c_s$	$a_{s,1}$	$a_{s,2}$	...	$a_{s,s-1}$	0
	$b_1$	$b_2$	...	$b_{s-1}$	$b_s$
	$\hat{b}_1$	$\hat{b}_2$	...	$\hat{b}_{s-1}$	$\hat{b}_s$

The method introduces a second approximation, an *embedded order*  $\hat{p}$ , responsible for error calculation. The *primary order*  $p$  is required to calculate the step size. Usually,  $\hat{p} = p - 1$ .

The Butcher tableau is expanded by a row of  $b$  numbers.

Local error:

$$e_n = y(x+h) - \hat{y}(x+h) = h_n \sum_{j=1}^s (b_j - \hat{b}_j) k_j \Rightarrow \|e_n\| = \left\| h_n \sum_{j=1}^s (b_j - \hat{b}_j) k_j \right\|$$

If at the current step  $\frac{\|e_n\|}{\varepsilon} > 1$ , then the estimation of  $h_n$  was too optimistic, and the step must be repeated with a smaller step size (**Reject**).

Otherwise  $\frac{\|e_n\|}{\varepsilon} \leq 1$ , continue with the step size update (**Proceed**):

$$h_{n+1} = h_n \left( \frac{\varepsilon}{\|e_n\|} \right)^{\frac{1}{\tilde{p}}} = h_n \left( \frac{\|e_n\|}{\varepsilon} \right)^{-\frac{1}{\tilde{p}}}$$

$$\varepsilon = \varepsilon_a + \varepsilon_r |y_n| \quad \text{with} \quad \tilde{p} = \min(p, \hat{p}) + 1 \quad (\text{Order of the primary method})$$

#### Example

$x$	$y$	$h_n$	$\left(\frac{\ e_n\ }{\varepsilon}\right)^{-\frac{1}{\tilde{p}}}$	$h_{n+1}$	Status
10	$[1, -1]^T$	1	0.4	0.4	Reject
10	$[1, -1]^T$	0.4	1.5	0.6	Proceed
10.4	$[0.420958, -1.40852]^T$	0.6	0.5	0.3	Reject

### 7.2.6 Stability of Explicit Methods Skript S. 27

The global relative error must not diverge; it must be bounded. Since the analysis with the ODE under investigation is challenging due to the lack of knowledge about the solution, the general ODE is considered, the Dahlquist model:

$$y' = Ay \quad y(0) = 1 \quad \text{with} \quad A = \Re\{A\} + j\Im\{A\} \in \mathbb{C}$$

Its solution is  $y = e^{\Re\{A\}x} (\cos(\Im\{A\}x) + j \sin(\Im\{A\}x))$ , i.e., a vibration with exponential amplitude  $e^{\Re\{A\}x}$  and frequency  $\Im\{A\}$ .

With the method under investigation, a *linear stability function*  $F(z)$  can be calculated.

**Example** The examination is shown using the Heun method (RK-2):

$$k_1 = Ay_k$$

$$k_2 = A(y_k + 1hk_1) = A(y_k + Ah y_k) \quad \Rightarrow$$

$$y_0 = y(x_0) = y(0) = 1$$

$$y_{k+1} = y_k \underbrace{\left(1 + hA + (hA)^2 \frac{1}{2}\right)}_{F(hA)=F(z), \quad z=hA \in \mathbb{C}}$$

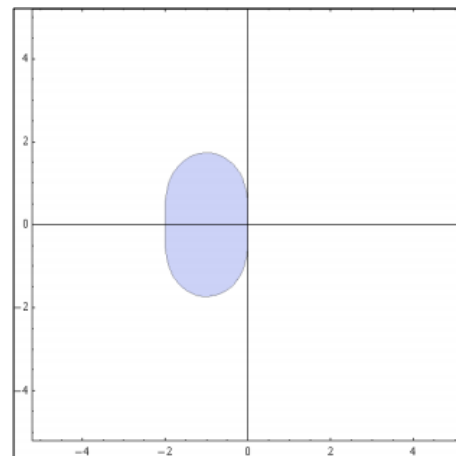
The linear stability function can now be substituted as follows:

$$y_{k+1} = \left( |F(z)| \exp(j \arg(F(z))) \right) y_k$$

Now three cases can be distinguished:

1.  $\Re\{A\} < 0$ : The amplitude of the ODE decreases exponentially. The stability condition requires that the approximated values  $y_k$  ( $k = 0, 1, \dots$ ) also decrease exponentially. Therefore, and from  $y_{k+1} \propto |F(z)|$ , it follows that  $|F(z)| < 1$ .
2.  $\Re\{A\} = 0$ : The amplitude of the ODE is constantly 1. The stability condition requires that the approximated values  $y_k$  ( $k = 0, 1, \dots$ ) also remain constant. Therefore, and from  $y_{k+1} \propto |F(z)|$ , it follows that  $|F(z)| = 1$ .
3.  $\Re\{A\} > 0$ : The amplitude of the ODE increases exponentially. The stability condition requires that the approximated values  $y_k$  ( $k = 0, 1, \dots$ ) also increase exponentially. Therefore, and from  $y_{k+1} \propto |F(z)|$ , it follows that  $|F(z)| > 1$ .

Cases 2 and 3 do not pose problems (conditions are always fulfilled); only the first case is critical.



Case 1 of the stability condition ( $\Re\{A\} < 0$ ):  
 $1 > |F(z)| = |1 + hA + \frac{1}{2}(hA)^2| \Rightarrow -2 < h\Re\{A\} < 0$ .  
 Cases 2 and 3 are not considered in the figure.

**Recursive Formula** S29 Recursive formulas exist for the stability polynomial  $F(z) = 1 + b_1k_1(z) + \dots + b_sk_s(z)$ :  
 $k_1(z) = z$ ,  $k_{j+1}(z) = z(1 + a_{j+1,1}k_1(z) + a_{j+1,2}k_2(z) + \dots + a_{j+1,j}k_j(z))$

**Weakness of Explicit Methods:** Stability!

### 7.2.7 Stiffness Skript S. 33

Definition: Stiffness = very challenging (according to Wikipedia). Rough idea: Explicit methods are forced to unacceptably small step sizes (computational complexity). This happens despite the actual smooth surfaces of the functions, which should intuitively not be very difficult to calculate.

If stability violations (stiffness) are detected, a switch to implicit methods (black-box solvers) should be made, and then, with non-stiffness detection, a switch back to explicit methods should be made.

Stiffness depends on:

1. Specific ODE
2. Specific step size  $h$
3. Specific method (e.g., RK-4)

**Stiffness Detection** Requirement:  $c_{s-1} = c_s = 1$ . Stiffness can be detected by testing  $\left| h \frac{\partial f(x,y)}{\partial y} \right|$  against the absolute limit of the stability region. Stiffness is present if  $|h\tilde{\lambda}|$  is outside or on the boundary of the stability region.  
 Example of explicit Runge-Kutta:

$$k_{s-1} = f\left(x + c_{s-1}h, y + h \underbrace{\sum_{j=1}^s a_{s-1,j}k_j}_{g_{s-1}}\right) \quad k_s = f\left(x + c_sh, y + h \underbrace{\sum_{j=1}^s a_{s,j}k_j}_{g_s}\right) \quad \Rightarrow \quad \tilde{\lambda} = \frac{\|k_s - k_{s-1}\|}{\|g_s - g_{s-1}\|}$$

$\tilde{\lambda}$  is an estimate for  $f_y = \frac{\partial}{\partial y}f(x, y)$ , e.g.,  $y' = x^4 - 25y^4 = f(x, y)$ , and takes on the role of  $\Re\{A\}$  in stability analysis.

## 7.3 Systems of ODEs and Higher-Order ODEs Skript S. 37

Systems of ODEs are written in vector notation and calculated as usual. That's all.

8    Differentiation Table

	$F(x)$	$F'(x)$
Addition	$f(x) \pm g(x)$	$f'(x) \pm g'(x)$
Linearity	$af(x)$	$af'(x)$
Product Rule	$f(x)g(x)$	$f'(x)g(x) + f(x)g'(x)$
Quotient Rule	$\frac{f(x)}{g(x)}$	$\frac{f'(x)g(x)-f(x)g'(x)}{(g(x))^2}$
Chain Rule	$f(g(x))$ $f^{-1}(x)$	$f'(g(x)) \cdot g'(x)$ $\frac{1}{f'(f^{-1}(x))}$
Basic functions	$x^n$ for any real $n$ $e^x$ $a^x$ ( $a > 0$ ) $\ln x$	$nx^{n-1}$ $e^x$ $(\ln a)a^x$ $\frac{1}{x}$
Trig functions	$\sin x$ $\cos x$ $\tan x$ $\arctan x = \tan^{-1} x$ $\arcsin x = \sin^{-1} x$	$\cos x$ $-\sin x$ $\frac{1}{\cos^2 x} = 1 + \tan^2 x$ $\frac{1}{1+x^2}$ $\frac{1}{\sqrt{1-x^2}}$
Hyperbolic Trig	$\sinh x$ $\cosh x$ $\tanh x$ $\sinh^{-1} x$ $\tanh^{-1} x$	$\cosh x$ $\sinh x$ $\frac{1}{\cosh^2 x}$ $\frac{1}{\sqrt{1+x^2}}$ $\frac{1}{1-x^2}$

9    Overview of polynomials

Name	Formula	Example	Reference
Newton	$\pi_n(x) = \prod_{i=1}^n (x - x_{i-1})$	$\pi_2 = (x - x_0)(x - x_1)$	1.3, S. 1
Bernstein	$B_{i,n}(x) = \binom{n}{i} (1 - x)^{n-i} x^i$	$B_{1,2} = 2x(1 - x)$	4.2.1, S. 8
Monomials	$x^n$	$x^2$	5.1.4, S. 11
Orthogonals	$p_{k,N}(t) = \sum_{i=0}^k (-1)^i \binom{k}{i} \binom{k+i}{i} \frac{t^{(i)}}{N^{(i)}}$	$P_1(x) = 1 - \frac{x-3}{2}$	5.1.5, S. 11
Chebyshev	$T_n(x) = \cos(n \arccos(x))$	$T_2 = 2x^2 - 1$	5.1.6, S. 12

## 10 Integration Table

$$\begin{array}{ll} \text{Linearity} & \int af(x) + bg(x) dx = a \int f(x) dx + b \int g(x) dx \\ \text{Substitution} & \int f(w(x))w'(x) dx = \int f(w) dw \\ \text{Integration by parts} & \int u(x)v'(x) dx = u(x)v(x) - \int u'(x)v(x) dx \end{array}$$

### Basic Functions

$$\begin{array}{ll} \int x^n dx = \frac{x^{n+1}}{n+1} + C & \int \frac{1}{x} dx = \ln |x| + C \\ \int e^{ax} dx = \frac{1}{a}e^x + C & \int a^x dx = \frac{a^x}{\ln a} + C \end{array}$$

### Trigonometric functions

$$\begin{array}{ll} \int \sin x dx = -\cos x + C & \int \cos x dx = \sin x + C \\ \int \frac{1}{\cos^2 x} dx = \tan x + C & \int \tan x dx = -\ln |\cos x| + C \\ \int \cot x dx = \ln |\sin x| + C & \end{array}$$

### Hyperbolic Trig functions

$$\begin{array}{ll} \int \sinh x dx = \cosh x + C & \int \cosh x dx = \sinh x + C \\ \int \tanh x dx = \ln(\cosh x) + C & \int \coth x dx = \ln |\sinh x| + C \end{array}$$

### Functions with $a^2 \pm x^2$

$$\begin{array}{ll} \int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C & \int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C \\ \int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \ln \left| \frac{x+a}{x-a} \right| + C & \\ \int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1}\left(\frac{x}{a}\right) + C & \int \frac{dx}{\sqrt{x^2 + a^2}} = \sinh^{-1}\left(\frac{x}{a}\right) + C \end{array}$$

### Inverse Functions

$$\begin{array}{ll} \int \ln x dx = x \ln x - x + C & \int \arcsin x dx = x \arcsin x + \sqrt{1-x^2} + C \\ \int \arctan x = x \arctan x - \frac{1}{2} \ln(1+x^2) + C & \end{array}$$

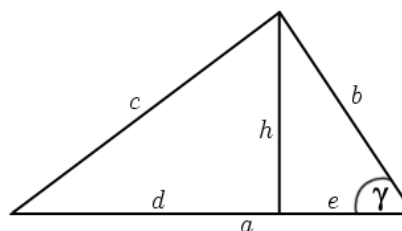
## 11 Idiotenseite

### 11.1 Dreiecksformeln

**Cosinussatz**  $c^2 = a^2 + b^2 - 2 \cdot a \cdot b \cdot \cos \gamma$

**Sinussatz**  $\frac{a}{\sin \alpha} = \frac{b}{\sin \beta} = \frac{c}{\sin \gamma} = 2r = \frac{u}{\pi}$

**Pythagoras**  $\sin^2(b) + \cos^2(b) = 1 \quad \tan(b) = \frac{\sin(b)}{\cos(b)}$



$$\sin \beta = \frac{b}{a} = \frac{\text{Gegenkathete}}{\text{Hypotenuse}}$$

$$\cos \beta = \frac{c}{a} = \frac{\text{Ankathete}}{\text{Hypotenuse}}$$

$$\tan \beta = \frac{c}{b} = \frac{\text{Gegenkathete}}{\text{Ankathete}}$$

$$\cot \beta = \frac{b}{c} = \frac{\text{Ankathete}}{\text{Gegenkathete}}$$

### 11.2 Funktionswerte für Winkelargumente

deg	rad	sin	cos	tan	deg	rad	sin	cos	deg	rad	sin	cos	deg	rad	sin	cos
0°	0	0	1	0	90°	$\frac{\pi}{2}$	1	0	180°	$\pi$	0	-1	270°	$\frac{3\pi}{2}$	-1	0
30°	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	120°	$\frac{2\pi}{3}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	210°	$\frac{7\pi}{6}$	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	300°	$\frac{5\pi}{3}$	$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
45°	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	135°	$\frac{3\pi}{4}$	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	225°	$\frac{5\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	315°	$\frac{7\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
60°	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	150°	$\frac{5\pi}{6}$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	240°	$\frac{4\pi}{3}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	330°	$\frac{11\pi}{6}$	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$

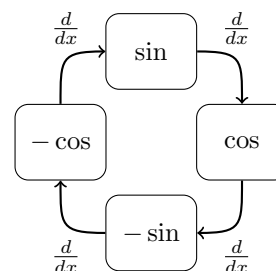
### 11.3 Periodizität

$$\cos(a + k \cdot 2\pi) = \cos(a) \quad \sin(a + k \cdot 2\pi) = \sin(a) \quad (k \in \mathbb{Z})$$

### 11.4 Quadrantenbeziehungen

$$\begin{aligned} \sin(-a) &= -\sin(a) & \cos(-a) &= \cos(a) \\ \sin(\pi - a) &= \sin(a) & \cos(\pi - a) &= -\cos(a) \\ \sin(\pi + a) &= -\sin(a) & \cos(\pi + a) &= -\cos(a) \\ \sin\left(\frac{\pi}{2} - a\right) &= \sin\left(\frac{\pi}{2} + a\right) = \cos(a) & \cos\left(\frac{\pi}{2} - a\right) &= -\cos\left(\frac{\pi}{2} + a\right) = \sin(a) \end{aligned}$$

### 11.5 Ableitungen



### 11.6 Additionstheoreme

$$\begin{aligned} \sin(a \pm b) &= \sin(a) \cdot \cos(b) \pm \cos(a) \cdot \sin(b) \\ \cos(a \pm b) &= \cos(a) \cdot \cos(b) \mp \sin(a) \cdot \sin(b) \\ \tan(a \pm b) &= \frac{\tan(a) \pm \tan(b)}{1 \mp \tan(a) \cdot \tan(b)} \end{aligned}$$

### 11.8 Produkte

$$\begin{aligned} \sin(a) \sin(b) &= \frac{1}{2}(\cos(a - b) - \cos(a + b)) \\ \cos(a) \cos(b) &= \frac{1}{2}(\cos(a - b) + \cos(a + b)) \\ \sin(a) \cos(b) &= \frac{1}{2}(\sin(a - b) + \sin(a + b)) \end{aligned}$$

### 11.9 Euler-Formeln

$$\begin{aligned} \sin(x) &= \frac{1}{2j}(e^{jx} - e^{-jx}) & \cos(x) &= \frac{1}{2}(e^{jx} + e^{-jx}) \\ e^{x+jy} &= e^x \cdot e^{jy} = e^x \cdot (\cos(y) + j \sin(y)) \\ e^{j\pi} &= e^{-j\pi} = -1 \end{aligned}$$

### 11.7 Doppel- und Halbwinkel

$$\begin{aligned} \sin(2a) &= 2 \sin(a) \cos(a) \\ \cos(2a) &= \cos^2(a) - \sin^2(a) = 2 \cos^2(a) - 1 = 1 - 2 \sin^2(a) \\ \cos^2\left(\frac{a}{2}\right) &= \frac{1 + \cos(a)}{2} & \sin^2\left(\frac{a}{2}\right) &= \frac{1 - \cos(a)}{2} \end{aligned}$$

### 11.10 Summe und Differenz

$$\begin{aligned} \sin(a) + \sin(b) &= 2 \cdot \sin\left(\frac{a+b}{2}\right) \cdot \cos\left(\frac{a-b}{2}\right) \\ \sin(a) - \sin(b) &= 2 \cdot \sin\left(\frac{a-b}{2}\right) \cdot \cos\left(\frac{a+b}{2}\right) \\ \cos(a) + \cos(b) &= 2 \cdot \cos\left(\frac{a+b}{2}\right) \cdot \cos\left(\frac{a-b}{2}\right) \\ \cos(a) - \cos(b) &= -2 \cdot \sin\left(\frac{a+b}{2}\right) \cdot \sin\left(\frac{a-b}{2}\right) \\ \tan(a) \pm \tan(b) &= \frac{\sin(a \pm b)}{\cos(a) \cos(b)} \end{aligned}$$