Contents

1 Theory

1.1 Terms and Classification

1.1.1 Order

As in ordinary differential equations, the order is the highest derivative of the unknown function appearing in the differential equation.

PDE 1st Order: $F\left(x_1, \ldots, x_n, u, \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n}\right)$

Using the substitution $\frac{\partial u}{\partial x_i} \to p_i$, it can be expressed as $F(x_1, \ldots, x_n, u, p_1, \ldots, p_n)$.

PDE 2nd Order: $F\left(x_1, \ldots, x_n, u, \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n}, \frac{\partial^2 u}{\partial x_1^2}, \ldots, \frac{\partial^2 u}{\partial x_n^2}\right)$

Using the substitution $\frac{\partial u}{\partial x_i} \to p_i$, $\frac{\partial^2 u}{\partial x_i \partial x_j} \to t_{ij}$, it can be expressed as $F(x_1, \ldots, x_n, u, p_1, \ldots, p_n, t_{11}, t_{12}, \ldots, t_{n,n-1}, t_{nn})$.

Examples:

$$\frac{\partial^2 u}{\partial x_1 \partial x_2} = 0 \quad \Rightarrow \quad F(t_{12}) = t_{12} = 0 \quad \Leftrightarrow \quad F\left(\frac{\partial^2 u}{\partial x_1 \partial x_2}\right) = 0$$

$$\frac{\partial u}{\partial x_1} = \frac{\partial u}{\partial x_2} \quad \Rightarrow \quad F(p_1, p_2) = p_1 - p_2 = 0 \quad \Leftrightarrow \quad F\left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}\right) = 0$$

$$x_1 \frac{\partial u}{\partial x_1} + x_2 \frac{\partial u}{\partial x_2} = \frac{\partial u}{\partial x_3} \quad \Rightarrow \quad F(x_1, x_2, p_1, p_2, p_3) = x_1 p_1 + x_2 p_2 - p_3 \quad \Leftrightarrow \quad F\left(x_1, x_2, \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \frac{\partial u}{\partial x_3}\right) = x_1 \frac{\partial u}{\partial x_1} + x_2 \frac{\partial u}{\partial x_2} - \frac{\partial u}{\partial x_3} = 0$$

1.1.2 Laplace Operator

Cartesian: $\Delta u(x, y, z) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$

Polar: $\Delta u(r, \varphi) = \frac{1}{r} \frac{\partial u}{\partial r} r \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \varphi^2}$

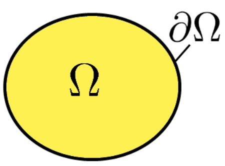
Cylindrical: $\Delta u(r, \varphi, z) = \frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial u}{\partial r}\right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \varphi^2} + \frac{\partial^2 u}{\partial z^2}$

Spherical: $\Delta u(r, \theta, \varphi) = \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial u}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 u}{\partial \varphi^2}$

1.1.3 Conversion to Lower Order System

Given:	$F\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial x \partial y}, \frac{\partial^2 u}{\partial y^2}\right) = 0.$
Substitution:	$p = \frac{\partial u}{\partial x}, \qquad q = \frac{\partial u}{\partial y}$
For second derivatives:	$\frac{\partial^2 u}{\partial x^2} = \frac{\partial p}{\partial x}, \quad \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial p}{\partial y} = \frac{\partial q}{\partial x}, \quad \frac{\partial^2 u}{\partial y^2} = \frac{\partial q}{\partial y}$
System of 1st order equations	$p = \frac{\partial u}{\partial x}, \quad q = \frac{\partial u}{\partial y}, \quad \frac{\partial p}{\partial y} = \frac{\partial q}{\partial x}$

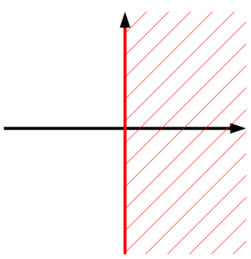
1.1.4 Domain of a PDE (Ω)



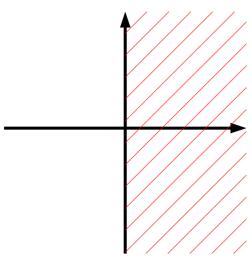
$\mathring{\Omega}$	Interior points
$\partial\Omega$	Boundary
$\bar{\Omega}$	Domain Ω and boundary $\partial\Omega$

The domain of a PDE **must** be open; only then is the partial derivative defined everywhere. The domain is open if, around every point in the domain Ω, a small ball can be drawn that also lies within the domain Ω.

Not a domain:



Domain:



Solution of a PDE:

Given: Domain Ω, PDE, boundary values on $\partial\Omega$

Solution: Function u: $\bar{\Omega} \to \mathbb{R}$, PDE in Ω and boundary values on $\partial\Omega$

'well-posed' when the information uniquely determines the solution

1.1.5 Classification of a PDE

Order:	Highest occurring partial derivative		
Type:	Linear:	Linear in $u, x_1, \ldots, x_n, \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n}, \frac{\partial^2 u}{\partial x_1^2}, \frac{\partial^2 u}{\partial x_1 \partial x_2}, \ldots, \frac{\partial^2 u}{\partial x_n^2}$	(“No products of us”)
	Quasilinear:	Linear in $\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n}, \frac{\partial^2 u}{\partial x_1^2}, \frac{\partial^2 u}{\partial x_1 \partial x_2}, \ldots, \frac{\partial^2 u}{\partial x_n^2}$	(“No products of derivatives of u”)
	Nonlinear:	Everything else	

1.2 Method of Characteristics

Important: Characteristics **cannot** be used as initial conditions, otherwise the characteristic becomes the solution (instead of obtaining a surface, you get a curve).
Important: The characteristic must pass through the boundary only once.
Useful for Quasilinear PDEs of 1st order. If separation is possible, this (simpler) method should be used.
Initial condition:

$$a(x,y,u) \cdot \frac{\partial u}{\partial x} + b(x,y,u) \cdot \frac{\partial u}{\partial y} - c(x,y,u) = 0$$

Characteristic:

$$\frac{d}{dt}\begin{bmatrix} x(t) \\ y(t) \\ u(t) \end{bmatrix} = \begin{bmatrix} a(x,y,u) \\ b(x,y,u) \\ c(x,y,u) \end{bmatrix}$$

Region: $\Omega\{\ldots | x > 0, \text{all } y\}$ Boundary condition: $u(0, y_0) = g(y_0)$

Vector notation: $\begin{bmatrix} a(x,y,u) \\ b(x,y,u) \\ c(x,y,u) \end{bmatrix} \underbrace{\begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & -1 \end{bmatrix}}_{\vec{n}:\ \text{Normal to surface}} = 0$

Tangents: $\vec{t}_x = \begin{bmatrix} 1 \\ 0 \\ \frac{\partial u}{\partial x} \end{bmatrix}$ $\vec{t}_y = \begin{bmatrix} 0 \\ 1 \\ \frac{\partial u}{\partial y} \end{bmatrix}$ $\vec{n} \bullet \vec{t_x} = 0$ $\vec{n} \bullet \vec{t_y} = 0$ $\vec{t_x} \bullet \vec{t_y} = \vec{n}$

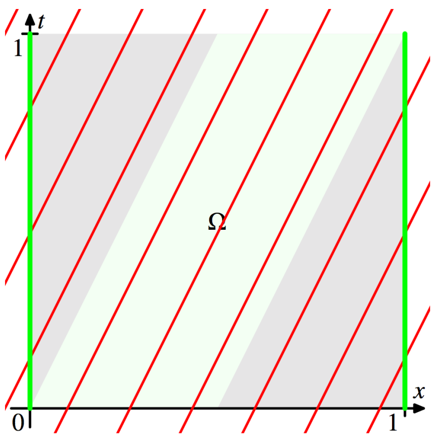
Solution approach: For each initial point $\begin{bmatrix} 0 \\ y_0 \\ g(y_0) \end{bmatrix}$ find a characteristic, then solve for x, y.

Boundary Conditions A solution function $u(x, y)$ must be covered by characteristics. The solution is now determined by the boundary values.

For the depicted region Ω, different cases are possible:

1. Boundary values on the *left* and *right* are given. A region in the middle is undetermined.
2. Boundary values on the *upper* and *lower* boundaries are given. A part of the region is over-determined.
3. Boundary values on the *left* and *lower* boundaries are given. The function is uniquely determined (but not necessarily differentiable everywhere).

The solution is not determinable for all boundary values.
If two characteristics meet → Singularity



Example:

1. PDE with boundary conditions and domain: $\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = 3, \quad u(0,y) = g(y) = \sin(y) \Rightarrow u(0, y_0) = g(y_0) = \sin(y_0)$

   Terms in matrix form: $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

2. Calculate characteristics from PDE → ODE: $\frac{d}{dt}\begin{bmatrix}x(t)\\y(t)\\u(t)\end{bmatrix} = \begin{bmatrix}1\\2\\3\end{bmatrix}$

3. Solve ODEs (for standard ODEs, see 7.5.8 on page 32.): $\begin{bmatrix}x\\y\\u\end{bmatrix} = \begin{bmatrix}1t+x_0\\2t+y_0\\3t+u_0\end{bmatrix}$

4. Substitute initial conditions: $\begin{bmatrix}x\\y\\u\end{bmatrix} = \begin{bmatrix}1t+x_0\\2t+y_0\\3t+u_0\end{bmatrix}\Bigg|_{t=0} = \begin{bmatrix}x_0\\y_0\\u_0\end{bmatrix} = \begin{bmatrix}0\\y_0\\\sin(y_0)\end{bmatrix}$
   Solution of ODE is: $\begin{bmatrix}x\\y\\u\end{bmatrix} = \begin{bmatrix}1\\2\\3\end{bmatrix}\cdot t + \begin{bmatrix}0\\y_0\\\sin(y_0)\end{bmatrix}$

5. Eliminate all variables except u, x, y: $u = 3x + \sin(y - 2x)$

6. Verification: Derive the result ($u = 3x + \sin(y - 2x)$) and substitute it into the original PDE $\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = 3$ to check if it is satisfied.

1.3 Method of Separation

Choosing a suitable coordinate system is important.

1. **Approach** (Highest derivative decisive):
   - For PDEs of 1st Order: $U(x, y) = X(x) + Y(y)$
   - For PDEs of 2nd Order: $U(x, y) = X(x) \cdot Y(y)$

2. **Substitution:** Substitute the approach into the PDE.

3. **Separation:** Each side of the PDE should only have one variable. The two resulting ordinary differential equations are coupled by a constant k (fixing the variable). If oscillation is expected, $k < 0$ and $k = -\lambda^2$.

4. **Solving the ODEs:** Obtain a family of solutions

5. **Constructing the General Solution:** (Linear combination of solutions), adhere to boundary conditions!

Example 1: PDE: $\frac{1}{x}\frac{\partial u}{\partial x}+\frac{1}{y}\frac{\partial u}{\partial y}=\frac{1}{y^2}$

1. Approach:

   $u(x,y)=X(x)+Y(y)$ (1st Order)

2. Substitution:

   $\frac{\partial u}{\partial x}=X'(x) \qquad \frac{\partial u}{\partial y}=Y'(y) \quad \Rightarrow \quad \frac{X'(x)}{x}+\frac{Y'(y)}{y}=\frac{1}{y^2}$

3. Separation:

   $\frac{X'(x)}{x}=k=\frac{1}{y^2}-\frac{Y'(y)}{y}$

4. Solving ODEs:

   $X'(x)=k\cdot x \quad \Rightarrow \quad X(x)=\frac{1}{2}kx^2+C_x$
   $Y'(y)=\frac{1}{y}-ky \quad \Rightarrow \quad Y(y)=\ln(y)-\frac{1}{2}ky^2+C_y$

5. Linear Combination:

   $u(x,y)=\frac{1}{2}kx^2-\frac{1}{2}ky^2+ln(y)+C$

Example 2: PDE: $x^2\frac{\partial^2 u}{\partial x^2}+x\frac{\partial u}{\partial x}+y^2\frac{\partial^2 u}{\partial y^2}+y\frac{\partial u}{\partial y}=0$
Boundary conditions: $\Omega=[1,2]\times[1,2] \qquad u=0$ on $\partial\Omega$

1. Approach:

   $u(x,y)=X(x)\cdot Y(y)$ (2nd Order)

2. Substitution:

   $x^2X''(x)Y(y)+xX'(x)Y(y)+y^2X(x)Y''(y)+yX(x)Y'(y)=0$

3. Separation: Division by $X(x)Y(y)$

   $\frac{x^2X''(x)}{X(x)}+\frac{xX'(x)}{X(x)}+\frac{y^2Y''(y)}{Y(y)}+\frac{yY'(y)}{Y(y)}=0 \quad \Rightarrow \quad \frac{x^2X''(x)}{X(x)}+\frac{xX'(x)}{X(x)}=k=-\frac{y^2Y''(y)}{Y(y)}-\frac{yY'(y)}{Y(y)}$

4. Solving ODEs:

   $\frac{x^2X''(x)}{X(x)}+\frac{xX'(x)}{X(x)}=k \quad \Rightarrow \quad x^2X''(x)+xX'(x)-kX(x)=0$ with $X(1)=X(2)=0$
   $\frac{y^2Y''(y)}{Y(y)}-\frac{yY'(y)}{Y(y)}=-k \quad \Rightarrow \quad y^2Y''(y)+yY'(y)+kY(y)=0$ with $Y(1)=Y(2)=0$

   Solution of ODEs not done here.

Example 3: PDE: $\frac{\partial^2 u}{\partial t^2}=\frac{\partial^2 u}{\partial x^2} \qquad u(t=0,x)=0$
Boundary conditions: $x=[0,\pi]$
$\frac{\partial u}{\partial t}(t=0,x)=\sin^3(x)=\frac{3}{4}\sin(x)-\frac{1}{4}\sin(3x)$

1. Approach:

   $u(t,x)=T(t)\cdot X(x)$ (2nd Order)

2. Substitution:

   $T''(t)\cdot X(x)=X''(x)\cdot T(t)$

3. Separation:

   $\frac{X''(x)}{X(x)}=-\mu^2=\frac{T''(t)}{T(t)}$

4. Solving ODEs:

   $X(x)=\sin(\mu x) \qquad T(t)=\sin(\mu t)$
   $X(x)=\cos(\mu x) \qquad T(t)=\cos(\mu t)$

5. Linear Combination:

   The boundary conditions $x=0$ and $x=\pi$ can only be satisfied with $\sin(\mu x)$ and a positive, integer μ. The $cos(nx)$ terms are eliminated.

   $u(t,x)=\sum_{n=1}^{\infty}a_n\sin(nx)\sin(nt)+\sum_{n=1}^{\infty}b_n\sin(nx)\cos(nt)$

   The coefficients a_n and b_n must be determined using the initial conditions at time $t=0$:

   $u(0,x)=\sum_{n=1}^{\infty}b_n\sin(nx)=0 \quad \Rightarrow \quad b_n=0$

   $\frac{\partial u}{\partial t}(\pi,x)=\sum_{n=1}^{\infty}a_n n\sin(nx)=\sin^3(x)=\frac{3}{4}\sin(x)-\frac{1}{4}\sin(3x) \quad \Rightarrow \quad a_1=\frac{3}{4} \quad a_3=-\frac{1}{12} \quad a_k=0$ for $k\neq 1,3$

   $u(t,x)=\frac{3}{4}\sin(x)\sin(t)-\frac{1}{12}\sin(3x)\sin(3t)$

1.4 Hamilton-Jacobi Theory

The Hamilton-Jacobi Theory starts with a total energy $H(x_i,p_i)$ depending on position and momentum. To achieve this, a function $S(x_i,t)$ must be found for which

$$\frac{\partial S}{\partial t}=H\left(x_i,p_i\right)=H\left(x_i,\frac{\partial S}{\partial x_i}\right) \qquad \text{with} \quad p_i=\frac{\partial S}{\partial x_i}$$

This can usually be solved by integration, introducing integration constants P_i. The *trajectory parameters* Q_i are

$$Q_i=\frac{\partial S}{\partial P_i}$$

and the trajectory has the form

$$x_i(t,Q_i,P_i)$$

1.5 Transformations

- The transition from functions to Fourier series transforms a partial differential equation into a family of ordinary differential equations for the individual Fourier coefficients.
- Integral transformations can transform a partial differential equation into a family of partial differential equations with fewer variables or even ordinary differential equations.
- Integral transformations and their inverses can provide formulas for solutions to certain partial differential equations, answering the question of well-posedness for given boundary conditions.

Domain	Transformation
$[0, \infty)$	Laplace transformation
$\mathbb{R}$	Fourier transformation
$[-\pi, \pi]$	Fourier series

1.5.1 Fourier Series

$$\boxed{u(t,x) = \frac{a_0(t)}{2} + \sum_{k=1}^{\infty} a_k(t)\cos(kx) + b_k(t)\sin(kx)}$$

1.5.2 Example: Vibrating String

$$\boxed{\partial_t^2 u = \partial_x^2 u}$$

1. Insert Fourier analysis approach into PDE:

$$\partial_t^2(t,x) = \frac{a_0''(t)}{2} + \sum_{k=1}^{\infty} a_k''(t)\cos(kx) + b_k''(t)\sin(kx) \qquad \partial_x^2(t,x) = -\sum_{k=1}^{\infty} a_k(t)k^2\cos(kx) + b_k(t)k^2\sin(kx)$$

$$\partial_t^2(t,x) = \partial_x^2(t,x) \qquad \Longleftrightarrow \qquad \frac{a_0''(t)}{2} + \sum_{k=1}^{\infty} a_k''(t)\cos(kx) + b_k''(t)\sin(kx) = -\sum_{k=1}^{\infty} a_k(t)k^2\cos(kx) + b_k(t)k^2\sin(kx)$$

$$\boxed{\Rightarrow \quad \frac{a_0''(t)}{2} + \sum_{k=1}^{\infty} \left(a_k''(t) + a_k(t)k^2\right)\cos(kx) + \left(b_k''(t) + b_k(t)k^2\right)\sin(kx) = 0}$$

2. This equation is only solvable if all coefficients vanish (Fourier theory):

   $a_0''(t) = 0 \qquad a_k''(t) = -k^2 a_k(t) \qquad b_k''(t) = -k^2 b_k(t)$

3. The Fourier transformation has transformed the PDE into a system of ODEs, the solutions of which are well-known:

   $a_0(t) = m_0(t) + c_0 \qquad a_k(t) = A_k^a\cos(kt) + B_k^a\sin(kt) \qquad b_k(t) = A_k^b\cos(kt) + B_k^b\sin(kt)$

Initial Conditions: The differential equations for the coefficients $a_k(t)$ and $b_k(t)$ can only be completely solved when initial or boundary conditions are given.

- Initial conditions for wave equation:
  $u(0,x) = f(x) \qquad \frac{\partial u}{\partial t} = g(x)$
- The functions f and g can also be represented as Fourier series:
  $f(x) = \frac{a_0^f}{2} + \sum\limits_{k=1}^{\infty} a_k^f\cos(kx) + b_k^f\sin(kx)$

  $g(x) = \frac{a_0^g}{2} + \sum\limits_{k=1}^{\infty} a_k^g\cos(kx) + b_k^g\sin(kx)$
- Together with the approach for $u(t,x)$, the equations (for $t = 0$) are obtained:
  $\frac{a_0(0)}{2} + \sum\limits_{k=1}^{\infty} a_k(0)\cos(kx) + b_k(0)\sin(kx) = \frac{a_0^f}{2} + \sum\limits_{k=1}^{\infty} a_k^f\cos(kx) + b_k^f\sin(kx)$

  $\frac{a_0'(0)}{2} + \sum\limits_{k=1}^{\infty} a_k'(0)\cos(kx) + b_k'(0)\sin(kx) = \frac{a_0^g}{2} + \sum\limits_{k=1}^{\infty} a_k^g\cos(kx) + b_k^g\sin(kx)$

- Coefficient comparison yields:
  $a_k(0) = a_k^f \qquad a'_k(0) = a_k^g \qquad b_k(0) = b_k^f \qquad b'_k(0) = b_k^g$
- The complete solution is thus:
  $u(t,x) = \frac{a_0^g(t)+a_0^f}{2} + \sum\limits_{k=1}^{\infty} \left(a_k^f \cos(kt) + \frac{1}{k}a_k^g \sin(kt)\right)\cos(kx) + \left(b_k^f \cos(kt) + \frac{1}{k}b_k^g \sin(kt)\right)\sin(kx)$

1.5.3 Inhomogeneous Wave Equation

The method can also be generalized to the inhomogeneous wave equation. The perturbation term is also expanded as a Fourier series.
$\partial_t^2 u - \partial_x^2 u = f \qquad \Rightarrow \qquad f(t,x) = \frac{a_0^f(t)}{2} + \sum\limits_{k=1}^{\infty} a_k^f(t)\cos(kx) + b_k^f \sin(kx)$

1.5.4 Laplace Transform

$\boxed{F(t) = \int\limits_0^\infty f(t)\mathrm{e}^{-st}dt}$ See also later in the summary! (Chapter 6.1 on page 27)

Solution of an ODE:

$\dot{x}(t) + px(t) = f(t) \qquad f(t) = q$

$\dot{x}(t) + px(t) = f(t) \circ\!\!-\!\!\bullet\; sX(s) - x(0) + pX(s) = F(s) \qquad f(t) \circ\!\!-\!\!\bullet\; F(s) = \frac{q}{s}$

$\Rightarrow X(s) = \frac{F(s)+x(0)}{s+p} = \frac{q+x(0)}{s(s+p)}\Big|_{x(0)=0} \bullet\!\!-\!\!\circ\; x(t) = \frac{q}{p}(1 - \mathrm{e}^{-pt})$

Solution of a PDE:

$\frac{\partial u}{\partial t} + x\frac{\partial u}{\partial x} = x \qquad t \geq 0, \quad x \geq 0 \qquad u(x,0) = 0, \quad u(0,t) = 0 \qquad x,t > 0$

Transformation: $\frac{\partial u}{\partial t} + x\frac{\partial u}{\partial x} = x \circ\!\!-\!\!\bullet\; sU(s,x) - u(x,0) + x\frac{\partial U(s,x)}{\partial x} = \frac{x}{s} \qquad \Rightarrow \qquad U(s,x) = \frac{x}{s(s+1)}$

$U(s,x) \bullet\!\!-\!\!\circ\; x(1 - \mathrm{e}^{-t})$

1.5.5 Example: Heat Conduction

At time $t = 0$, the rod has temperatures of -1 at $x = -\frac{\pi}{2}$ and 1 at $x = \frac{\pi}{2}$ → stationary state. At time $t = 0$, the reservoirs are removed, and the rod is left to itself. In particular, no heat can be dissipated through the ends.

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad \text{or in general:} \qquad \frac{\partial u}{\partial t} = a^2\frac{\partial^2 u}{\partial x^2}$$

"Triangle function"

$$d(x) = \begin{cases} -2 - \dfrac{2x}{\pi} & -\dfrac{\pi}{2} \leq x \\ \dfrac{2x}{\pi} & -\dfrac{\pi}{2} \leq x \leq \dfrac{\pi}{2} \\ 2 - \dfrac{2x}{\pi} & x \leq \dfrac{\pi}{2} \end{cases} \qquad = \qquad \sum_{n=0}^{\infty} \frac{8(-1)^n}{\pi^2(2n+1)^2}\sin((2n+1)x)$$

$\hat{u}(t,k)$ Fourier-Sine-Coefficients / $\mathcal{L}u$ Laplace Transformation.
Initial conditions are odd → solution of the differential equation for all times odd.
Boundary conditions: $\partial_x u(t, -\frac{\pi}{2}) = \partial_x u(t, \frac{\pi}{2}) = 0$ → Reflection at $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ → extend to a 2π-periodic function on $\mathbb{R}$

$$\partial_t \hat{u}(t,k) = -k^2 \hat{u}(t,k)$$

Now this equation can be Laplace-transformed:

$$\begin{aligned} s\mathcal{L}\hat{u}(s,k) - \hat{u}(0,k) &= -k^2 \mathcal{L}\hat{u}(s,k) \\ (s+k^2)\mathcal{L}\hat{u}(s,k) &= \hat{u}(0,k) \\ \mathcal{L}\hat{u}(s,k) &= \frac{\hat{u}(0,k)}{s+k^2} \end{aligned}$$

Inverse transformation yields:

$$\hat{u}(t,k) = \hat{u}(0,k)e^{-k^2 t}$$

Now, only the Fourier coefficients need to be determined, which can be obtained from the triangle function:

$$\hat{u}(0, 2n+1) = \frac{8(-1)^n}{\pi^2(2n+1)^2}$$

and thus, the final solution is obtained by summing the Fourier series:

$$u(t,x) = \sum_{n=0}^{\infty} \frac{8(-1)^n}{\pi^2(2n+1)^2} e^{-(2n+1)^2 t} \sin((2n+1)x)$$

1.6 PDE 2nd Order

Linear partial differential equations of second order have the form: $\boxed{\sum_{i,j=1}^{n} a_{ij}\partial_i\partial_j u + \sum_{i=1}^{n} b_i\partial_i u + cu = f}$

1.6.1 Classification

Classification only for PDEs of second order!
Eigenvalue calculation: (e.g., for $\partial_x^2 u + 2\partial_x\partial_y u + \partial_y^2 u = 0$)

1. Form a symmetric matrix and subtract λ on the diagonal. For example: $A = \begin{pmatrix} \partial_x^2 & \partial_x\partial_y \\ \partial_y\partial_x & \partial_y^2 \end{pmatrix}$
   In diagonal matrices, the eigenvalues correspond to the diagonal entries.

2. Set determinant equal to 0: $\det(\mathbf{A}-\lambda\mathbf{I}) = 0 \quad \Rightarrow \quad \lambda_i$

3. Solve the equation

Alternatively (if, for example, a very complicated PDE needs to be classified), the signs of the eigenvalues can also be determined via trace and determinant:

1. See left (Eigenvalue calculation): Form matrix A

2. Calculate determinant and try to read from the table: $\det A = a_{11}a_{22} - a_{12}a_{21} = \lambda_1\lambda_2$

3. Calculate trace and try to read from the table: $\mathrm{tr}(A) = a_{11} + a_{22} = \lambda_1 + \lambda_2$

Class	Number of Eigenvalues			det(A)	Example
	Positive	Negative	Zero(=0)	for n=2	
Hyperbolic	n-1	1	0	det < 0	Wave Equation: $\frac{\partial^2 u}{\partial t^2} = \Delta u$
Parabolic	n-1	0	1	det $= 0$	Heat Equation: $\frac{\partial u}{\partial t} = \Delta u$
Elliptic	n	0	0	det > 0	Potential: $\Delta u = f$
Ultrahyperbolic	>1	>1	0	-	-

1.7 Elliptic PDEs

$\Delta u = f \qquad \omega = \{(x,y)|y \geq 0\}, \quad u(x,y) = ay$
Theorem: If Ω is bounded and connected, then the solution u is always unique.

Proof: Assume: $u = u_1 - u_2$
Substitute: $\Delta u_1 - \Delta u_2 = f - f = 0$
$(u_1 - u_2)|_{\partial\omega} = g - g = 0$
$\Delta u = 0 \qquad u|_{\partial\Omega} = 0$
If $u = 0$ is a solution, then there is only one solution.

1.7.1 Maximum Principle

If $\Delta u = 0$, then u is *harmonic*, and the extrema (maxima and minima of the function) are located on the boundary $\partial\Omega$.

1.7.2 Example (Exercise Solutions)

An elliptic PDE like $\Delta u = c$ has only one solution with the given Dirichlet boundary values. Remember: The reason was the maximum principle. If there were a second solution $\bar{v}(r,\phi)$ with the same boundary values, $v - \bar{v}$ would be a solution to the equation $\Delta(v - \bar{v}) = 0$, a harmonic function. The boundary values of $v - \bar{v}$ are 0. Since a harmonic function attains the maximum on the boundary, $v - \bar{v} = 0$ is the solution, so it is unique.

1.7.3 Green's Function

An elliptic PDE is solved by inverting Δ. This inversion is done using Green's function, which is the inverse function of Δ
Δ: Laplace operator.

$$u(x) = \int_\Omega \sigma(x,\xi) f(\xi) d\xi + \int_\Omega h(x,\xi) f(\xi) d\xi \qquad \sigma(x,\xi) = \begin{cases} \frac{1}{2}\lvert x-\xi\rvert & n=1 \\ \frac{1}{2\pi}\log\lvert x-\xi\rvert & n=2 \\ -\frac{1}{4\pi}\frac{1}{\lvert x-\xi\rvert} & n=3 \\ \frac{1}{(2-n)\mu(S^{n-1})}\lvert x-\xi\rvert^{2-n} & n\geq 3 \end{cases}$$

Green's function: $G(x,\xi) = \sigma(x,\xi) + h(x,\xi)$
Theorem: If Ω is a domain where the Dirichlet problem is uniquely solvable, then there exists a function $G(x,\xi)$, which, as a function of x, solves the equation $\Delta G(x,\xi) = \delta(x-\xi)$ with homogeneous boundary conditions. Solution: $u(x) = \int_\Omega G(x,\xi) f(\xi) d\xi + \int_{\partial\Omega} g(\xi) \cdot \underset{\xi}{\text{grad}}\, G(x,\xi) d\eta \qquad \eta:$ normal vector of ∂G

1.7.4 Mean Value Property of Harmonic Functions

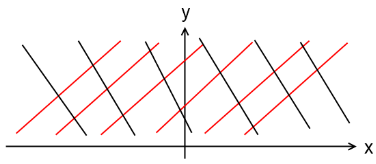
$$\Delta h = 0 \qquad \text{Mean Value Property:} \qquad h(x) = \begin{cases} \frac{h(x+\delta)+h(x-\delta)}{2} & n=1 \\ \frac{1}{2\pi r}\int_{S_r^1} h(x+\xi)d\xi & n=2 \\ \frac{1}{4\pi r^2}\int_{S_r^2} h(x+\xi)d\xi & n=3 \end{cases}$$

1.8 Hyperbolic PDEs

PDE: $\frac{\partial^2 u}{\partial t^2} - a^2\frac{\partial^2 u}{\partial x^2} = 0 \qquad \Omega = \{(x,t) \mid t>0\} \qquad u_0 = u(x_0, 0)$

Trick: $(\partial_t + a\partial_x)(\partial_t - a\partial_x)u = (\partial_t^2 - a^2\partial_x^2)u = 0 \qquad$ (for constant velocity a)

Two possible solutions: $\underbrace{(\partial_t + a\partial_x)u = 0}_{\boxed{\text{Wave to the right}}} \quad \underbrace{(\partial_t - a\partial_x)u = 0}_{\boxed{\text{Wave to the left}}}$



Solution using characteristics: $\frac{\partial}{\partial s}\begin{Bmatrix} x(s) \\ t(s) \\ u(s) \end{Bmatrix} = \begin{Bmatrix} \pm a \\ 1 \\ 0 \end{Bmatrix} \begin{matrix} \Rightarrow & x = \pm as + x_0 \\ \Rightarrow & t = s + t_0 = s \\ \Rightarrow & u = u_0 \end{matrix} \qquad (t_0 = 0)$

$x = \pm at + x_0 \quad \Rightarrow \quad x_0 = x \mp at \quad \Rightarrow \quad u(x,t) = u_0(x \mp at)$

General solution from linear combination: $\boxed{u(x,t) = u_+(x+at) + u_-(x-at)}$

$\Rightarrow$ **Two** initial conditions are needed to determine u_+ **and** u_-.

e.g.: $u(x,0) = u_0(x) \qquad \frac{\partial u}{\partial t}(x,0) = g_0(x)$

1.8.1 Strips/Characteristics

PDE: $a\partial_x^2 u + 2b\partial_x\partial_y u + c\partial_y^2 u + d\partial_x u + e\partial_y u + fu = g$ Symbol matrix: $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$

Along the curve $t \mapsto (x(t), y(t))$, the initial values / partial derivatives are $\left.\begin{aligned} u(x(t),y(t)) &= u(t) \\ \partial_x u(x(t),y(t)) &= p(t) \\ \partial_y u(x(t),y(t)) &= q(t) \end{aligned}\right\}$

Characteristics satisfy the ODE:

$$a\dot{y}(t)^2 - 2b\dot{x}(t)\dot{y}(t) + c\dot{x}(t)^2 = 0$$

Characteristic strip additionally satisfies: $a\dot{p}(t)\dot{y}(t) - h\dot{x}(t)\dot{y}(t) + c\dot{x}(t)\dot{q}(t) = 0$

2 Numerics

2.1 Discretization

2.1.1 1st Derivative

$$g'(x) \approx \frac{g(x+\Delta x) - g(x)}{\Delta x} \qquad \text{or} \qquad \boxed{g'(x) \approx \frac{g(x+\Delta x) - g(x-\Delta x)}{2\Delta x}} \qquad \text{(Central Difference: Better Quality)}$$

2.1.2 2nd Derivative

$\boxed{g''(x) \approx \frac{g(x-\Delta x) - 2g(x) + g(x+\Delta x)}{\Delta x^2}}$ is the same for the second (better quality) version.

2.2 FDM

TIP: Derive the equation system by hand for non-zero initial conditions, reduces the chance of errors.

2.2.1 Basic Equation: $-u''(x) = f(x)$

$A^{(n)}\tilde{u}^{(n)} = f^{(n)}$

$$A^{(n)} = \frac{1}{\Delta x^2}\operatorname{tridiag}_{n-1}(-1, 2, -1) = \frac{1}{\Delta x^2}\begin{bmatrix} 2 & -1 & 0 & \dots \\ -1 & 2 & -1 & \dots \\ 0 & -1 & 2 & \dots \\ 0 & 0 & -1 & \dots \\ \dots \end{bmatrix} \qquad \text{(an } (n-1)\times(n-1) \text{ matrix)}$$

Boundary: $u(0) = a \qquad u(n) = b \qquad A^{(n)}\tilde{u}^{(n)} = \begin{bmatrix} f(x_1^{(n)}) + \dfrac{a}{\Delta x^2} \\ f(x_2^{(n)}) \\ \vdots \\ f(x_{(n-1)}^{(n)}) + \dfrac{b}{\Delta x^2} \end{bmatrix}$

2.2.2 Basic Equation: $T''(x) - hT(x) = T_A$

$-T'' + hT(x) = hT_A$

$$A^{(n)} = \frac{1}{\Delta x^2}\operatorname{tridiag}_{n-1}(-1, 2+h\Delta x^2, -1) = \frac{1}{\Delta x^2}\begin{bmatrix} 2+h\Delta x^2 & -1 & 0 & \dots \\ -1 & 2+h\Delta x^2 & -1 & \dots \\ 0 & -1 & 2+h\Delta x^2 & \dots \\ 0 & 0 & -1 & \dots \\ \dots \end{bmatrix}$$

2.2.3 Example Homework 7

Given: $u''(x) = 4(u(x) - x)$ with boundary values $u(0) = 0$ and $u(1) = 2$ with $\Delta x = 1/3$.
To find: $\tilde{u}(\frac{1}{3})$ and $\tilde{u}(\frac{2}{3})$
Solution: Substitute the derivative of u'' into the equation, see above, giving the general equation $\frac{u(x-\Delta x) - 2u(x) + u(x+\Delta x)}{\Delta x^2} = 4(u(x) - x)$, for points P1 and P2, this yields:
P1: $\frac{0 - 2\tilde{u}(\frac{1}{3}) + \tilde{u}(\frac{2}{3})}{(\frac{1}{3})^2} = 4(\tilde{u}(\frac{1}{3}) - \frac{1}{3})$ P2: $\frac{\tilde{u}(\frac{1}{3}) - 2\tilde{u}(\frac{2}{3}) + 2}{(\frac{1}{3})^2} = 4(\tilde{u}(\frac{2}{3}) - \frac{2}{3})$
Now solve the system of equations to get $\tilde{u}(\frac{1}{3})$ and $\tilde{u}(\frac{2}{3})$

2.3 Convergence

A model is convergent if, as $n \to \infty$, the approximation $\tilde{u}$ and u coincide.

$$\boxed{||v||_{\Delta x} = \sqrt{\Delta x(v_1^2 + v_2^2 + \dots v_{n-1}^2)} = \sqrt{\Delta x}||v||}$$

It converges if: $\lim_{n\to\infty} ||\tilde{u}^{(n)} - u^{(n)}||_{1/n} = \sqrt{\frac{1}{n}}||\tilde{u}^{(n)} - u^{(n)}|| = \sqrt{\frac{1}{n}}\sqrt{(\tilde{u}_1 - u_1)^2 + \dots + (\tilde{u}_{n-1} - u_{n-1})} = 0$

2.4 Consistency

A model is consistent when the model aligns with reality through simplification.

2.4.1 Residue

Exact: $A^{(n)} \cdot \tilde{u}^{(n)} - f^{(n)} = 0$
Residue: $A^{(n)} \cdot (u^{(n)} - \tilde{u}^{(n)}) = r^{(n)}$
An approximation method is consistent if $\boxed{\lim_{n\to\infty} ||r^{(n)}||_{1/n} = 0}$ holds.

Consistency is a necessary but not sufficient condition for the convergence of a method.

2.4.2 Taylor

$g(x) = \sum_{k=0}^{n} \frac{1}{k!} g^{(k)}(x_0)(x-x_0)^k + \frac{1}{(n+1)!} g^{(n+1)}(\xi)(x-x_0)^{n+1}$ = Taylor Approximation Polynomial + Lagrange Remainder
$\xi \longmapsto [x_0 < \xi < x]$

Alternative $f(x_0+h) = f(x_0) + f'(x_0)\frac{h}{1!} + ... + f^{(n)}(x_0)\frac{h^n}{n!} + R_n(h)$ where $h = x - x_0$

2.4.3 Forward/Backward Difference

$g'(x) - \frac{g(x+\Delta x)+g(x)}{\Delta x} = O(\Delta x) = \frac{g''(\xi)}{2}\Delta x \Rightarrow$ 1st Order

2.4.4 Central Difference

$g'(x) - \frac{g(x+\Delta x)+g(x-\Delta x)}{2\Delta x} = O(\Delta x^2) = \frac{g'''(\xi_1)+g'''(\xi_2)}{12}\Delta x^2 \Rightarrow$ 2nd Order

2.4.5 2nd Derivative

$g''(x) - \frac{g(x+\Delta x)-2g(x)+g(x-\Delta x)}{2\Delta x} = O(\Delta x^2) = \frac{g''''(\xi_1)+g''''(\xi_2)}{24}\Delta x^2 \Rightarrow$ 2nd Order

Global consistency error for 2nd Derivative: $||r^{(n)}||_{1/n} \leq \frac{1}{12} \max_{\xi\in[0,1]} |f''(\xi)| \cdot \Delta x^2$

2.5 Stability

The stability of a matrix can be determined through its norm $||A||_*$.

The following applies: $||A||_* = \max_{||x||_*=1} ||A \cdot x||_* \qquad ||A \cdot x||_* \leq ||A||\ ||x||_*$

An approximation method is stable if, independent of the constant C, the following holds: $\boxed{||{A^{(n)}}^{-1}||_{1/n} \leq C}$

Determining $||A||$ is generally not easy, so $||A||$ is often determined through the diagonalization of A.

$$y = A \cdot x \qquad \Rightarrow \qquad \tilde{y} = D \cdot \tilde{x} \qquad \text{with} \qquad D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

$TAT^T = D$, where T is the transformation matrix from x coordinate system to $\tilde{x}$ coordinate system. T is orthogonal. The diagonal elements $\lambda_1, \dots, \lambda_n$ are also called eigenvalues.

It follows: $\boxed{||A|| = \max_k |\lambda_k|}$ and $\boxed{||A^{-1}|| = \{\min_k |\lambda_k|\}^{-1}}$
Determining eigenvalues: $\boxed{\det(A - \lambda I) = |A - \lambda I| = 0} \qquad \Rightarrow \qquad \lambda_1, \dots, \lambda_n$
Determining eigenvectors (for each λ_i): $(A - \lambda_i I) \cdot v_i = 0 \qquad \Rightarrow \qquad v_1, \dots, v_n$

2.6 FDM for Elliptic PDEs (Poisson: $-\Delta u = f$)

Equation: $-\Delta u(x,y) = f(x,y)$ $\qquad -\Delta u(x,y) = -\left(\frac{g(x+\Delta x,y)-2g(x,y)+g(x-\Delta x,y)}{2\Delta x} + \frac{g(x,y+\Delta y)-2g(x,y)+g(x,y-\Delta y)}{2\Delta y}\right)$

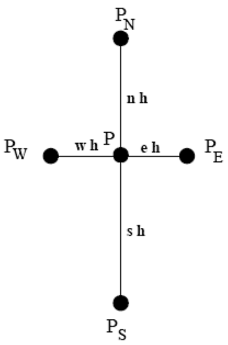
$h = \Delta x = \Delta y \Rightarrow \boxed{-\frac{1}{h^2}(\tilde{u}_{j,k+1} + \tilde{u}_{j+1,k} + \tilde{u}_{j,k-1} + \tilde{u}_{j-1,k} - 4\tilde{u}_{j,k}) = f_{j,k}}$

$$B\tilde{u} = f \Rightarrow B = \begin{bmatrix} T & D & 0 & \dots \\ D & T & D & \dots \\ 0 & D & T & \dots \\ 0 & 0 & D & \dots \\ \dots \end{bmatrix} \text{ where } T = \frac{1}{h^2}\begin{bmatrix} 4 & -1 & 0 & \dots \\ -1 & 4 & -1 & \dots \\ 0 & -1 & 4 & \dots \\ 0 & 0 & -1 & \dots \\ \dots \end{bmatrix} \text{ and } D = \frac{1}{h^2}\begin{bmatrix} -1 & 0 & 0 & \dots \\ 0 & -1 & & \dots \\ 0 & 0 & -1 & \dots \\ \dots \end{bmatrix}$$

$$\tilde{u} = \begin{bmatrix} \tilde{u}_{1,1} \\ \tilde{u}_{2,1} \\ \vdots \\ \tilde{u}_{1,2} \\ \tilde{u}_{2,2} \\ \vdots \end{bmatrix} \quad f = \begin{bmatrix} f_{1,1} \\ f_{2,1} \\ \vdots \\ f_{1,2} \\ f_{2,2} \\ \vdots \end{bmatrix} + \frac{1}{h^2}\begin{bmatrix} u(0,0)+u(1,0)+u(0,1) \\ u(2,0) \\ \vdots \\ u(0,2) \\ 0 \\ \vdots \end{bmatrix}$$

Boundary conditions must be incorporated into f if $u(x,y) \neq 0$ on $\partial\Omega$

2.6.1 Irregular Grid (for the Boundary)



$$-\frac{2}{h^2}\cdot\left(\frac{u(x+e\cdot h,y)-u(x,y)}{e(e+w)} + \frac{u(x-w\cdot h,y)-u(x,y)}{w(e+w)} + \frac{u(x,y+n\cdot h)-u(x,y)}{n(n+s)} + \frac{u(x,y-s\cdot h)-u(x,y)}{s(n+s)}\right) = f(x,y)$$

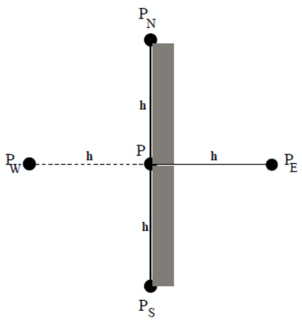
or

$$-\frac{2}{h^2}\cdot\left(\frac{u(P_E)-u(P)}{e(e+w)} + \frac{u(P_W)-u(P)}{w(e+w)} + \frac{u(P_N)-u(P)}{n(n+s)} + \frac{u(P_S)-u(P)}{s(n+s)}\right) = f(x,y)$$

If $\Delta x, \Delta y$ are constant ($w\cdot h = e\cdot h$, $n\cdot h = s\cdot h$) and $h=1$:

$$-\left(\frac{u(P_E)+u(P_W)-2u(P)}{\Delta x^2} + \frac{u(P_N)+u(P_S)-2u(P)}{\Delta y^2}\right) = f(x,y)$$

2.6.2 Neumann Boundary



For Neumann boundary conditions, the boundary points must also be calculated. In the figure, P, P_N, and P_S are on the boundary, P_E is inside, and P_W is outside of Ω. The Neumann boundary condition in P is given: $\boxed{\frac{\partial u}{\partial n}(P) = g(P)}$

From the derivative $u_x(P) = \frac{\partial u}{\partial n}(P) = \frac{u(P_E)-u(P_W)}{2h}$, the outside point $u(P_W)$ can be calculated: $u(P_W) = u(P_E) - 2h\cdot u_x(P)$. Thus, it holds:

$$\boxed{\frac{2u(P_E)+u(P_N)+u(P_S)-4u(P)-2h\cdot u_x(P)}{h^2}}$$

If P_W and P_E are swapped, the sign is reversed: $u(P_W) = u(P_E) + 2h\cdot u_x(P)$.

Mirror Method:
If $u_x(x,y) = 0$, it is also referred to as the mirror method. Points P_W and P_E then have the same value ($P_W = P_E$).

2.7 FDM for Parabolic PDEs

Heat conduction equation: $\boxed{u_t(x,t) = u_{xx}(x,t)}$ $\qquad f(0) = f(1) = 0 \qquad \bar{\Omega} = [0,1]\times[0,\infty]$

Boundary conditions: $u(x,0) = f(x) \qquad u(0,t) = u(1,t) = 0 \qquad x\in(0,1) \qquad t\in[0,\infty)$

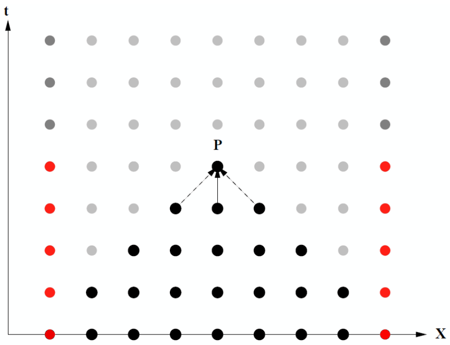
2.7.1 Explicit Method (Richardson Method)

$$\boxed{\frac{\tilde{u}(x,t+\Delta t)-\tilde{u}(x,t)}{\Delta t}=\frac{\tilde{u}(x+\Delta x,t)-2\tilde{u}(x,y)+\tilde{u}(x-\Delta x,t)}{\Delta x^2}} \qquad \Delta x=\tfrac{1}{n} \qquad \Delta t=\tfrac{r}{n^2} \qquad \boxed{r=\frac{\Delta t}{\Delta x^2}}$$

Idea: From positions k, calculate $k+1$: $\tilde{u}_{j,k+1}=r\tilde{u}_{j-1,k}+(1-2r)\tilde{u}_{j,k}+r\tilde{u}_{j+1,k}$
Discretization of t: $k, k+1, \dots$
Discretization of x: $j, j+1, \dots$

- Initialization, boundary condition: $\tilde{u}_{j,0}=f(j/n) \qquad \tilde{u}_{0,k}=\tilde{u}_{n,k}=0$
- Approximation matrix: $C^{(n)}=\text{tridiag}_{n-1}(r,1-2r,r)=\begin{bmatrix}1-2r & r & 0 & 0 & \cdots\\ r & 1-2r & r & 0 & \cdots\\ 0 & r & 1-2r & r & \cdots\\ 0 & 0 & r & 1-2r & \cdots\\ \vdots & \vdots & \vdots & \vdots & \ddots\end{bmatrix}$
- Calculate one step: $\tilde{u}^{(k+1)}=C^{(n)}\tilde{u}^{(k)}$
- Calculate k steps: $\tilde{u}^{(k)}=\left\{C^{(n)}\right\}^k\tilde{u}^{(0)}$

Convergence Behavior:



The method is stable if: $||C^{(n)}||<1 \qquad \Rightarrow \qquad r<\frac{1}{2}$

This requires very small time steps, making the method impractical due to high computational capacity needs.

The reason for the poor convergence behavior can be visualized geometrically. The calculation of the value at node P includes the values of all nodes shaded in black. Only the 0-th level of the boundary values is considered.

To calculate the method with C^k for k steps, the red values (left and right) must be set to 0 (Boundary Condition). For the boundary vector $\tilde{u}^{(0)}$, only the lowest black 7 points are filled in.

2.7.2 Implicit Method

In contrast to the explicit method, which uses values from the previous time step, here a system of equations is solved globally.

$$\boxed{\frac{\tilde{u}(x,t)-\tilde{u}(x,t-\Delta t)}{\Delta t}=\frac{\tilde{u}(x+\Delta x,t)-2\tilde{u}(x,y)+\tilde{u}(x-\Delta x,t)}{\Delta x^2}} \qquad \Delta x=\tfrac{1}{n} \qquad \Delta t=\tfrac{r}{n^2} \qquad \boxed{r=\frac{\Delta t}{\Delta x^2}}$$

$\tilde{u}_{j,k}=-r\tilde{u}_{j-1,k+1}+(1+2r)\tilde{u}_{j,k+1}-r\tilde{u}_{j+1,k+1}$
Idea: Calculate derivatives using backward differences

- Initialization, boundary condition: $\tilde{u}_{j,0}=f(j/n) \qquad \tilde{u}_{0,k}=\tilde{u}_{n,k}=0$
- Approximation matrix: $E^{(n)}=\text{tridiag}_{n-1}(-r,1+2r,-r)=\begin{bmatrix}1+2r & -r & 0 & 0 & \cdots\\ -r & 1+2r & -r & 0 & \cdots\\ 0 & -r & 1+2r & -r & \cdots\\ 0 & 0 & -r & 1+2r & \cdots\\ \vdots & \vdots & \vdots & \vdots & \ddots\end{bmatrix}$
- Equation: $\tilde{u}^{(k)}=E^{(n)}\cdot\tilde{u}^{(k+1)}$
- Calculate one step: $\tilde{u}^{(k+1)}=\left\{E^{(n)}\right\}^{-1}\tilde{u}^{(k)}$
- Calculate k steps: $\tilde{u}^{(k)}=\left\{E^{(n)}\right\}^{-k}\tilde{u}^{(0)}$

Advantage: The implicit method is always stable, regardless of the time resolution Δt
Disadvantage: Requires expensive matrix inversion.

2.7.3 Crank-Nicolson Method (Mixed Method)

The idea of the Crank-Nicolson method is to average the two approximations

$$\boxed{\frac{\tilde{u}(x,t+\Delta t)-\tilde{u}(x,t)}{\Delta t}=\frac{\tilde{u}(x+\Delta x,t)-2\tilde{u}(x,t)+\tilde{u}(x-\Delta x,t)}{\Delta x^2}}$$

$$\boxed{\frac{\tilde{u}(x,t+\Delta t)-\tilde{u}(x,t)}{\Delta t}=\frac{\tilde{u}(x+\Delta x,t+\Delta t)-2\tilde{u}(x,t+\Delta t)+\tilde{u}(x-\Delta x,t+\Delta t)}{\Delta x^2}}$$

With this idea, the continuous problem is transformed into the following discrete problem:

$-r\tilde{u}_{j-1,k+1}+(2+2r)\tilde{u}_{j,k+1}-r\tilde{u}_{j+1,k+1}=r\tilde{u}_{j-1,k}+(2-2r)\tilde{u}_{j,k}+r\tilde{u}_{j+1,k}$

As with the other methods: $\Delta x=\frac{1}{n}$ $\quad \Delta t=\frac{r}{n^2}$ $\quad \boxed{r=\frac{\Delta t}{\Delta x^2}}$

- Initialization, boundary condition: $\tilde{u}_{j,0}=f(j/n)$ $\quad \tilde{u}_{0,k}=\tilde{u}_{n,k}=0$
- Approximation matrices:

$$F^{(n)}=E^{(n)}+I=\text{tridiag}_{n-1}(-r,2+2r,-r)=\begin{bmatrix}2+2r & -r & 0 & 0 & \cdots\\ -r & 2+2r & -r & 0 & \cdots\\ 0 & -r & 2+2r & -r & \cdots\\ 0 & 0 & -r & 2+2r & \cdots\\ \vdots & \vdots & \vdots & \vdots & \ddots\end{bmatrix}$$

$$G^{(n)}=C^{(n)}+I=\text{tridiag}_{n-1}(\ r,\ 2-2r,\ r\ )=\begin{bmatrix}2-2r & r & 0 & 0 & \cdots\\ r & 2-2r & r & 0 & \cdots\\ 0 & r & 2-2r & r & \cdots\\ 0 & 0 & r & 2-2r & \cdots\\ \vdots & \vdots & \vdots & \vdots & \ddots\end{bmatrix}$$

- Equation: $F^{(n)}\cdot\tilde{u}^{(k+1)}=G^{(n)}\cdot\tilde{u}^{(k)}$
- Calculate one step: $\tilde{u}^{(k+1)}=\left\{F^{(n)}\right\}^{-1}\cdot G^{(n)}\cdot\tilde{u}^{(k)}$
- Calculate k steps: $\tilde{u}^{(k)}=\left(\left\{F^{(n)}\right\}^{-1}\cdot G^{(n)}\right)^k\cdot\tilde{u}^{(0)}$

2.8 FDM for Hyperbolic PDEs

$u_{tt}=u_{xx}\rightarrow$ homogeneous
$u_{tt}-u_{xx}=v(x,t)\rightarrow$ inhomogeneous

Initial conditions:
$u(x,0)=f(x)\quad u_t(x,0)=g(x)$

2.8.1 Leap-Frog Scheme

$\tilde{u}_{j,k+1}=r^2\tilde{u}_{j-1,k}+2(1-r^2)\tilde{u}_{j,k}+r^2\tilde{u}_{j+1,k}-\tilde{u}_{j,k-1}$ $\quad r=\frac{\Delta t}{\Delta x}$

$\tilde{u}_{j,0}=f(j\Delta x)$ $\quad \tilde{u}_{j,1}=f(j\Delta x)+g(j\Delta x)\Delta t+f''(j\Delta x)\frac{\Delta t^2}{2}$

2.8.2 Transport Equation

$$u_x(x,t)+u_t(x,t)=0 \qquad u(x,0)=f(x)\longrightarrow u(x,t)=f(x-t)$$

Downwind Scheme

$$\frac{\tilde{u}(x,t+\Delta t)-\tilde{u}(x,t)}{\Delta t}+\frac{\tilde{u}(x+\Delta x,t)-\tilde{u}(x,t)}{\Delta x}=0 \qquad \tilde{u}_{j,k+1}=(1+r)\tilde{u}_{j,k}-r\tilde{u}_{j+1,k} \quad r=\frac{\Delta t}{\Delta x} \quad \text{Mostly Divergent}$$

Upwind Scheme

$$\frac{\tilde{u}(x,t+\Delta t)-\tilde{u}(x,t)}{\Delta t}+\frac{\tilde{u}(x,t)-\tilde{u}(x-\Delta x,t)}{\Delta x}=0 \qquad \tilde{u}_{j,k+1}=(1-r)\tilde{u}_{j,k}+r\tilde{u}_{j-1,k} \quad \text{Convergent for} \quad r=\frac{\Delta t}{\Delta x}\le 1$$

Centered Scheme

$$\frac{\tilde{u}(x,t+\Delta t)-\tilde{u}(x,t)}{\Delta t}+\frac{\tilde{u}(x+\Delta x,t)-\tilde{u}(x-\Delta x,t)}{2\Delta x}=0 \qquad \tilde{u}_{j,k+1}=-\frac{r}{2}\tilde{u}_{j+1,k}+\tilde{u}_{j,k}+\frac{r}{2}\tilde{u}_{j-1,k} \quad r=\frac{\Delta t}{\Delta x}$$

Lax-Wendroff Scheme

$$\tilde{u}_{j,k+1} = A\tilde{u}_{j+1,k} + B\tilde{u}_{j,k} + C\tilde{u}_{j-1,k} \quad r = \frac{\Delta t}{\Delta x} \quad A = \frac{r^2 - r}{2} \quad B = 1 - r^2 \quad C = \frac{r^2 + r}{2}$$

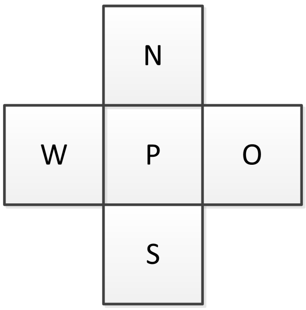
2.9 FVM (Finite Volume Method, Voronoi's Method)

$\Delta u = 0$ in Ω
$u(x, y) = f(x, y)$ on $\partial\Omega$

Gauss's theorem states: $\boxed{\oint_{\Gamma} \Delta u(x, y)dxdy = \int_{\Gamma} \text{div grad } u(x, y)dxdy = \oint_{\partial\Gamma} \text{grad } u(x, y)d\vec{n}}$

Where the boundary normal vector $\vec{n}$ is always directed perpendicular to the outer region Γ.

$\Rightarrow \oint_{\Gamma} \Delta u(x, y)dxdy = \oint_{\partial\Gamma} \text{grad } u(x, y)d\vec{n} = 0$

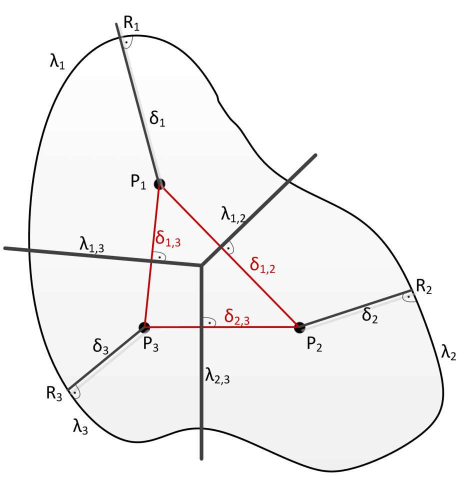


$$\frac{u(P_E)-u(P_P)}{h} \cdot h + \frac{u(P_N)-u(P_P)}{h} \cdot h + \frac{u(P_W)-u(P_P)}{h} \cdot h + \frac{u(P_S)-u(P_P)}{h} \cdot h \approx 0$$

$$\Rightarrow \tilde{u}(P_E) + \tilde{u}(P_N) + \tilde{u}(P_W) + \tilde{u}(P_S) - 4 \cdot \tilde{u}(P_P) = 0$$

Advantages:

- Allows computation with flow quantities and balances, thus avoiding the Laplace operator (Δ) and the associated complex mathematics.
- Can handle complicated geometries.



Procedure for Calculation:

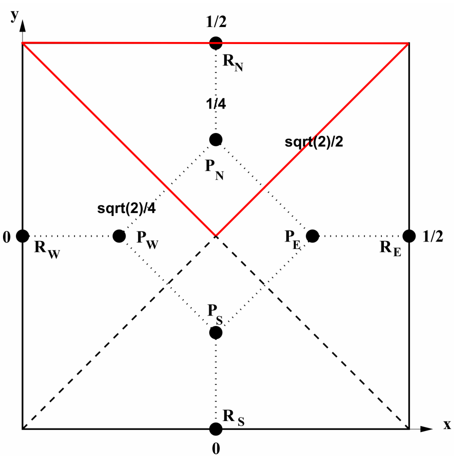
1. Choose points $P_1, \ldots, P_n$.
2. Divide the region into small subregions, e.g. by bisecting perpendiculars.
3. Discretize the boundary.

For the P_i-cell: $\sum_j \frac{u(P_{i,j}) - u(P_i)}{\delta_{i,j}} \cdot \lambda_{i,j} = 0$

For the P_1-cell: $\frac{\tilde{u}(P_2)-\tilde{u}(P_1)}{\delta_{1,2}} \cdot \lambda_{1,2} + \frac{\tilde{u}(P_3)-\tilde{u}(P_1)}{\delta_{1,3}} \cdot \lambda_{1,3} + \frac{\tilde{u}(R_1)-\tilde{u}(P_1)}{\delta_1} \cdot \lambda_1 = 0$

For the P_2-cell: $\frac{\tilde{u}(P_1)-\tilde{u}(P_2)}{\delta_{1,2}} \cdot \lambda_{1,2} + \frac{\tilde{u}(P_3)-\tilde{u}(P_2)}{\delta_{2,3}} \cdot \lambda_{2,3} + \frac{\tilde{u}(R_2)-\tilde{u}(P_2)}{\delta_2} \cdot \lambda_2 = 0$

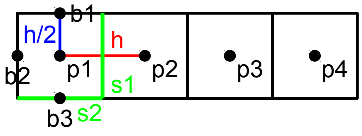
For the P_3-cell: $\frac{\tilde{u}(P_2)-\tilde{u}(P_3)}{\delta_{2,3}} \cdot \lambda_{2,3} + \frac{\tilde{u}(P_1)-\tilde{u}(P_3)}{\delta_{1,3}} \cdot \lambda_{1,3} + \frac{\tilde{u}(R_3)-\tilde{u}(P_3)}{\delta_3} \cdot \lambda_3 = 0$



$$\frac{\tilde{u}(P_E)-\tilde{u}(P_N)}{1/4\cdot\sqrt{2}} \cdot \frac{\sqrt{2}}{2} + \frac{\tilde{u}(P_W)-\tilde{u}(P_N)}{1/4\cdot\sqrt{2}} \cdot \frac{\sqrt{2}}{2} + \frac{\tilde{u}(R_N)-\tilde{u}(P_N)}{1/4} \cdot 1 = 0$$

$$(\tilde{u}_E - \tilde{u}_N) \cdot 2 + (\tilde{u}_W - \tilde{u}_N) \cdot 2 + (1/2 - \tilde{u}_N) \cdot 4 = 0$$

$$0 \cdot \tilde{u}_S + 2 \cdot \tilde{u}_E + 2 \cdot \tilde{u}_W - 8 \cdot \tilde{u}_N + 2 = 0$$



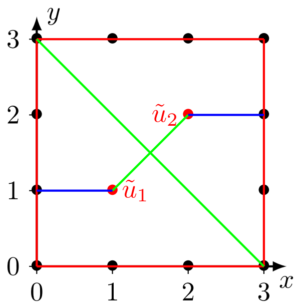
u1: $\frac{u(b_1)-u(p_1)}{h/2}\cdot h+\frac{u(b_2)-u(p_1)}{h/2}\cdot h+\frac{u(b_3)-u(p_1)}{h/2}\cdot s_2+\frac{u(p_2)-u(p_1)}{h}\cdot s_1$

Caution for quick calculations: $\frac{u(b_2)-u(p_1)}{h/2}\cdot h=(u(b_2)-u(p_1))\cdot 2$

Example with $\Delta u(x,y)\neq 0$ and two voronoi points:
The function $u(x,y)$ is defined on the square $\Omega=[0,3]\times[0,3]$. The function $u(x,y)$ satisfies

$$\Delta u(x,y)+4=0$$

in Ω and $u(x,y)=0$ on the boundary of Ω. Determine approximate values for $u(1,1)$ and $u(2,2)$. Use finite volumes à la Voronoi with Voronoi-points $(1,1)$ and $(2,2)$



$$\frac{1}{2}\int_0^3\int_0^3 -4\,dxdy=-18$$

Function integrated over the voronoi cell example 68 in the script

$$\frac{0-\tilde{u}_1}{1}6+\frac{\tilde{u}_2-\tilde{u}_1}{\sqrt{2}}3\sqrt{2}=-18$$

$$\frac{0-\tilde{u}_2}{1}6+\frac{\tilde{u}_1-\tilde{u}_2}{\sqrt{2}}3\sqrt{2}=-18$$

$$A=\begin{bmatrix}-9 & 3\\ 3 & -9\end{bmatrix}\begin{pmatrix}\tilde{u}_1\\ \tilde{u}_2\end{pmatrix}=\begin{pmatrix}-18\\ -18\end{pmatrix}$$

3 FEM (Finite Element Method)

The vector space $\mathbb{V}$ has infinitely many dimensions. If we choose n independent functions $v_1, \ldots, v_n$, then the functions $a_1 \cdot v_1(x) + \ldots + a_n \cdot v_n(x)$ span an n-dimensional subspace $\mathbb{V}^{(n)}$ of $\mathbb{V}$. The following applies:

$$\boxed{\tilde{u}^{(n)} = a_1 \cdot v_1(x) + \ldots + a_n \cdot v_n(x)}$$

3.1 The Method of Ritz

Ritz Matrix: $R^{(n)} = \begin{bmatrix} R_{1,1} & R_{1,2} & \cdots \\ R_{2,1} & R_{2,2} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}$ with $R_{j,k}^{(n)} = \int_0^1 v_j'(x) \cdot v_k'(x) dx$ and $j, k = 1, \ldots, n$

Ritz Vector: $r^{(n)} = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \end{bmatrix}$ with $r_k^{(n)} = \int_0^1 f(x) \cdot v_k(x) dx$ and $k = 1, \ldots, n$

Solution according to Ritz: $R^{(n)} \cdot a = r^{(n)} \quad \Rightarrow \quad a = \left\{R^{(n)}\right\}^{-1} \cdot r^{(n)}$

3.2 The Method of Galerkin

Galerkin Matrix: $G^{(n)} = \begin{bmatrix} G_{1,1} & G_{1,2} & \cdots \\ G_{2,1} & G_{2,2} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}$ with $G_{j,k}^{(n)} = \int_0^1 \underbrace{(v_j''(x))}_{v_j \text{ in the form of the DE!}} \cdot v_k(x) dx$ and $j, k = 1, \ldots, n$

Galerkin Vector: $g^{(n)} = \begin{bmatrix} g_1 \\ g_2 \\ \vdots \end{bmatrix}$ with $g_k^{(n)} = \int_0^1 f(x) \cdot v_k(x) dx$ and $k = 1, \ldots, n$

Solution according to Galerkin: $G^{(n)} \cdot a + g^{(n)} = 0 \quad \Rightarrow \quad a = -\left\{G^{(n)}\right\}^{-1} \cdot g^{(n)}$ according to Ritz $G^{(n)} = -R^{(n)}$ $\quad g^{(n)} = r^{(n)}$

The above matrix is only valid for the PDE $-u''(x) = f(x)$ with the ansatz $\tilde{u}(x) = a_1 \cdot v_1(x) + a_2 \cdot v_2(x)$. Otherwise, a system of equations for $v_k = v_1$ and v_2 must be established (example for ODE: $u''(x) + u(x) + x = 0$):

$$\int_0^1 (a_1 \cdot v_1''(x) + a_2 \cdot v_2''(x) + a_1 \cdot v_1(x) + a_2 \cdot v_2(x) + x) \cdot v_k(x) dx = 0 \rightarrow G_{j,k}^{(n)} = \int_0^1 (v_j''(x) + v_j(x)) \cdot v_k(x) dx$$

3.3 Weighted Residuals

Weighting functions: $\{w_1(x), \ldots, w_n(x)\}$

Matrix (weighted residuals): $M^{(n)} = \begin{bmatrix} M_{1,1} & M_{1,2} & \cdots \\ M_{2,1} & M_{2,2} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}$ with $M_{j,k}^{(n)} = \int_0^1 v_j''(x) \cdot w_k(x) dx$

Vector (weighted residuals): $m^{(n)} = \begin{bmatrix} m_1 \\ m_2 \\ \vdots \end{bmatrix}$ with $m_k^{(n)} = \int_0^1 f(x) \cdot w_k(x) dx$

Solution of the weighted residuals: $M^{(n)} \cdot a + m^{(n)} = 0 \quad \Rightarrow \quad a = -\left\{M^{(n)}\right\}^{-1} \cdot m^{(n)}$

3.4 Point Collocation

In the context of point collocation (individual points must match between the true result and the approximation), n support points are chosen in the interval $[0, 1]$.

$\begin{bmatrix} v_1''(x_1) & v_2''(x_1) & \cdots \\ v_1''(x_2) & v_2''(x_2) & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} \cdot \begin{bmatrix} a_1 \\ a_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} -f(x_1) \\ -f(x_2) \\ \vdots \end{bmatrix}$ Solve the system of equations for a.

The above matrix is only valid for the PDE $-u''(x) = f(x)$ with the ansatz $\tilde{u}(x) = a_1 \cdot v_1(x) + a_2 \cdot v_2(x)$. Otherwise, the DE must be formulated with the trial functions, and substituted at both points to determine a_1 and a_2:
DE: $u''(x) + u(x) = -x \Rightarrow$ Equation at Point 1: $a_1 \cdot v_1''(x_1) + a_2 \cdot v_2''(x_1) + a_1 \cdot v(x_1) + a_2 \cdot v(x_1) = -x_1$

3.5 Domain Collocation

In contrast to point collocation, entire domains (intervals I_k) rather than individual points must match. For $-u''(x) = f(x)$, this system of equations is set up.

$$\begin{bmatrix} \int_{I_1} v_1'' & \int_{I_1} v_2'' & \cdots \\ \int_{I_2} v_1'' & \int_{I_2} v_2'' & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} \cdot \begin{bmatrix} a_1 \\ a_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} -\int_{I_1} f(x) \\ -\int_{I_2} f(x) \\ \vdots \end{bmatrix}$$

Solve the system of equations for a.

3.6 Gauss's Method (MSE)

Gauss Matrix: $Q^{(n)} = \begin{bmatrix} Q_{1,1} & Q_{1,2} & \cdots \\ Q_{2,1} & Q_{2,2} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}$ with $Q_{j,k}^{(n)} = \int_0^1 v_j''(x) \cdot v_k''(x) dx$

Gauss Vector: $q^{(n)} = \begin{bmatrix} q_1 \\ q_2 \\ \vdots \end{bmatrix}$ with $q_k^{(n)} = \int_0^1 f(x) \cdot v_k''(x) dx$

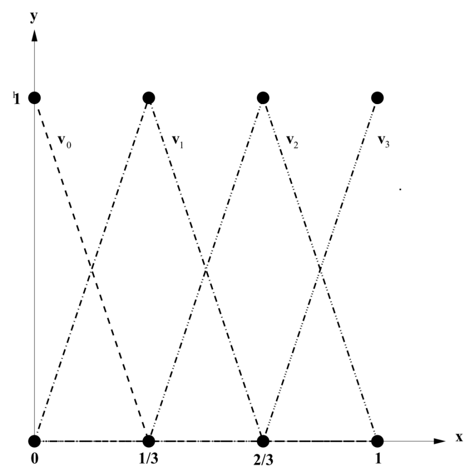
Solution according to Gauss: $Q^{(n)} \cdot a + q^{(n)} = 0 \quad \Rightarrow \quad a = -\left\{Q^{(n)}\right\}^{-1} \cdot q^{(n)}$

3.7 Finite Elements

The discussed methods assume the choice of a set $v_1(x), \ldots, v_n(x)$ of basis functions. In FEM, local supports (basis functions) are used, which are non-zero only on a small interval. The advantage of this approach is that in a given region, only one support influences the approximation function. The disadvantage lies in the high number of required supports.
IMPORTANT: All methods are presented with a discretization of $h = 1/3$.

3.7.1 Node Variables

First, on the interval $[0, 1]$, n nodes, usually uniformly distributed, are introduced.



This divides the interval $[0, 1]$ into subintervals (meshes).
For $n = 3$: $I_1 = [0, 1/3]$ $I_1 = [1/3, 2/3]$ $I_1 = [2/3, 1]$
Next, each node x_k is assigned an approximation variable (node variable).
Approximation: $\tilde{u}(0) = a_0$ $\tilde{u}(1/3) = a_1$ $\tilde{u}(2/3) = a_2$ $\tilde{u}(1) = a_3$

Additional conditions:
$$\begin{array}{llll} v_0(0) = 1 & v_0(1/3) = 0 & v_0(2/3) = 0 & v_0(1) = 0 \\ v_1(0) = 0 & v_1(1/3) = 1 & v_1(2/3) = 0 & v_1(1) = 0 \\ v_2(0) = 0 & v_2(1/3) = 0 & v_2(2/3) = 1 & v_2(1) = 0 \\ v_3(0) = 0 & v_3(1/3) = 0 & v_3(2/3) = 0 & v_3(1) = 1 \end{array}$$

3.7.2 Shape Functions

The local basis functions are intended to be composed of sections of simpler functions, such as polynomials, defined only on a single mesh.
Two possible shape functions are, for example, $l_1(x) = 1 - x$ and $l_2(x) = x$.

$t \in$	$[0, 1/3]$	$[1/3, 2/3]$	$[2/3, 1]$
$v_0 =$	$1 - 3x$	0	0
$v_1 =$	$3x$	$2 - 3x$	0
$v_2 =$	0	$-1 + 3x$	$3 - 3x$
$v_3 =$	0	0	$-2 + 3x$

$\Longrightarrow$

$t \in$	$[0, 1/3]$	$[1/3, 2/3]$	$[2/3, 1]$
$v_0 =$	$l_1(3x)$	0	0
$v_1 =$	$l_2(3x)$	$l_1(3x - 1)$	0
$v_2 =$	0	$l_2(3x - 1)$	$l_1(3x - 2)$
$v_3 =$	0	0	$l_2(3x - 2)$

3.7.3 Element Matrices

In principle, the approximation variable can be determined by any method. Because in a linear approximation function, the second derivative is trivial ($= 0$), the choice of the Ritz method is enforced.
The integrals are evaluated element-wise:

$$\int_0^1 = \int_0^{1/3} + \int_{1/3}^{2/3} + \int_{2/3}^1$$

This approach calculates the Ritz matrix for each mesh individually and then sums them up to form the global Ritz matrix:

$$R^{(4)} = R^{(4,1)} + R^{(4,2)} + R^{(4,3)} = \begin{bmatrix} * & * & 0 & 0 \\ * & * & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & * & * & 0 \\ 0 & * & * & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{bmatrix}$$

The 2×2 matrices marked with $*$ are called mesh matrices:

$$M^{(4,1)} = M^{(4,2)} = M^{(4,3)} = \begin{bmatrix} * & * \\ * & * \end{bmatrix} = 3 \cdot \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \qquad \Rightarrow \qquad \boxed{M = \frac{1}{h} \cdot \underbrace{\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}}_{\mathbf{E}\text{: Element matrix}} = \frac{1}{h} \cdot \mathbf{E}}$$

The element matrix is now inserted into the corresponding Ritz matrix and superimposed. For the quantization of $h = 1/3$, we get:

$$R^{(4)} = \begin{bmatrix} -3 & 3 & 0 & 0 \\ 3 & -3-3 & 3 & 0 \\ 0 & 3 & -3-3 & 3 \\ 0 & 0 & 3 & -3 \end{bmatrix} = \begin{bmatrix} -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -3 \end{bmatrix}$$

The Ritz vector must be calculated by integration:

$$r^{(4)} = \begin{bmatrix} \int_0^1 f(x) \cdot v_0(x) dx \\ \int_0^1 f(x) \cdot v_1(x) dx \\ \int_0^1 f(x) \cdot v_2(x) dx \\ \int_0^1 f(x) \cdot v_3(x) dx \end{bmatrix}$$

The corresponding Ritz equation system is: $\boxed{R^{(4)} \cdot a + r^{(4)} = 0}$ $\quad \Rightarrow \quad$ $a = -\left\{R^{(4)}\right\}^{-1} \cdot r^{(4)}$

Initial conditions: The initial conditions a_0 and a_n can directly be substituted.

$a_0 = 10 \qquad a_3 = 20$

$$\begin{bmatrix} 3 & -6 & 3 & 0 \\ 0 & 3 & -6 & 3 \end{bmatrix} \cdot \begin{bmatrix} 10 \\ a_1 \\ a_2 \\ 20 \end{bmatrix} + r^{(4)} = 0 \qquad \Rightarrow \qquad \begin{bmatrix} -6 & 3 \\ 3 & -6 \end{bmatrix} \cdot \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} 30 \\ 60 \end{bmatrix} + r^{(4)} = 0$$

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} -6 & 3 \\ 3 & -6 \end{bmatrix}^{-1} \cdot \begin{bmatrix} -\left(r_1^{(4)} + 3 \cdot a_0\right) \\ -\left(r_2^{(4)} + 3 \cdot a_3\right) \end{bmatrix}$$

3.7.4 Finite Element Manual Calculation

Problem statement: $u''(x) + f(x) = 0 \qquad f(x) = 20 \qquad u(0) = 10 \qquad u(1) = 20$

The approximation should be performed on the **NON**-uniform intervals: $[0, 1/6]$, $[1/6, 1/2]$, $[1/2, 1]$

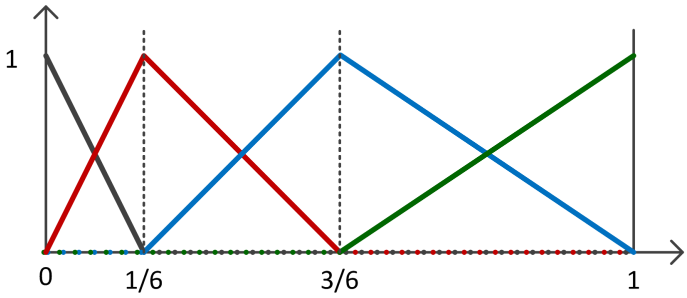
The corresponding element matrices E are:

$$\frac{1}{1/6} \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -6 & 6 \\ 6 & -6 \end{bmatrix} \qquad \frac{1}{1/2-1/6} \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix} \qquad \frac{1}{1-1/2} \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}$$

The Ritz Vector and the Ritz Matrix are:

$$R^n = \begin{bmatrix} -6 & 6 & 0 & 0 \\ 6 & -9 & 3 & 0 \\ 0 & 3 & -5 & 2 \\ 0 & 0 & 2 & -2 \end{bmatrix}$$

$$r^n = \begin{bmatrix} \int\limits_0^{1/6} f(x)\cdot(1-6x)dx \\ \int\limits_0^{1/6} f(x)\cdot(6x) + \int\limits_{1/6}^{3/6} f(x)\cdot(3/2-3x)dx \\ \int\limits_{1/6}^{3/6} f(x)\cdot(3x-1/2) + \int\limits_{3/6}^{1} f(x)\cdot(2-2x)dx \\ \int\limits_{1/2}^{1} f(x)\cdot(2x-1)dx \end{bmatrix} = \begin{bmatrix} 5/3 \\ 5 \\ 25/3 \\ 5 \end{bmatrix}$$



$$R^n \cdot a + r^n = 0 \qquad \Rightarrow \qquad \begin{bmatrix} -6 & 6 & 0 & 0 \\ 6 & -9 & 3 & 0 \\ 0 & 3 & -5 & 2 \\ 0 & 0 & 2 & -2 \end{bmatrix} \cdot \begin{bmatrix} 10 \\ a_1 \\ a_2 \\ 20 \end{bmatrix} + \begin{bmatrix} 5/3 \\ 5 \\ 25/3 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \Rightarrow \qquad \begin{bmatrix} 6 & -9 & 3 & 0 \\ 0 & 3 & -5 & 2 \end{bmatrix} \cdot \begin{bmatrix} 10 \\ a_1 \\ a_2 \\ 20 \end{bmatrix} + \begin{bmatrix} 5/3 \\ 5 \\ 25/3 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \qquad \begin{bmatrix} -9 & 3 \\ 3 & -5 \end{bmatrix} \cdot \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} 6\cdot 10 \\ 2\cdot 20 \end{bmatrix} + \begin{bmatrix} 5 \\ 25/3 \end{bmatrix} = 0 \qquad \Rightarrow \qquad \begin{bmatrix} -9 & 3 \\ 3 & -5 \end{bmatrix} \cdot \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} -65 \\ -145/3 \end{bmatrix} \qquad \Rightarrow \qquad \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 235/18 \\ 35/2 \end{bmatrix}$$

$$\Rightarrow \qquad \tilde{u}(x) = 10\cdot v_0(x) + \tfrac{235}{18} v_1(x) + \tfrac{35}{2}\cdot v_2(x) + 20\cdot v_3(x) =$$

3.7.5 h-Strategy

The basic idea of the h-strategy is to refine the resolution. In other words, the mesh width h is reduced. To cover the entire range, more meshes are required.

3.7.6 p-Strategy

With the p-strategy, the mesh remains unchanged. The trial functions should now be composed of higher-order polynomials, and new nodes and node variables are introduced.

Problem statement: $u''(x) + f(x) = 0 \qquad u(0) = a_0 \qquad u(1) = a_6$

The approximation should apply to the equally spaced intervals: $[0, 1/3], \quad [1/3, 2/3], \quad [2/3, 1]$

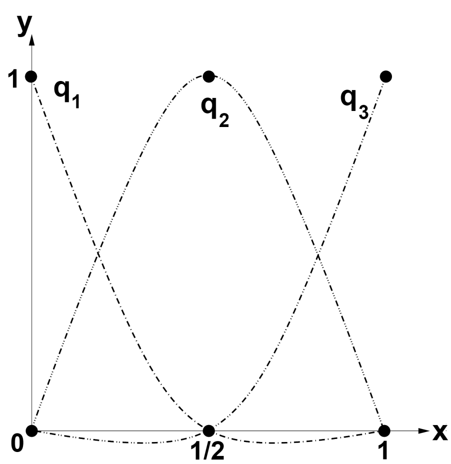
Shape functions:

$q_1(x) = (1-x)\cdot(1-2x)$
$q_2(x) = 4x\cdot(1-x)$
$q_3(x) = -x\cdot(1-2x)$

Element matrix:

$$\boxed{E = \frac{1}{3}\begin{bmatrix} -7 & 8 & -1 \\ 8 & -16 & 8 \\ -1 & 8 & -7 \end{bmatrix}}$$

$x \in$	$[0, 1/3]$	$[1/3, 2/3]$	$[2/3, 1]$
$v_0 =$	$q_1(3x)$	0	0
$v_1 =$	$q_2(3x)$	0	0
$v_2 =$	$q_3(3x)$	$q_1(3x-1)$	0
$v_3 =$	0	$q_2(3x-1)$	0
$v_4 =$	0	$q_3(3x-1)$	$q_1(3x-2)$
$v_5 =$	0	0	$q_2(3x-2)$
$v_6 =$	0	0	$q_3(3x-2)$



$$\underbrace{\begin{bmatrix} -7 & 8 & -1 & 0 & 0 & 0 & 0 \\ 8 & -16 & 8 & 0 & 0 & 0 & 0 \\ -1 & 8 & -14 & 8 & -1 & 0 & 0 \\ 0 & 0 & 8 & -16 & 8 & 0 & 0 \\ 0 & 0 & -1 & 8 & -14 & 8 & -1 \\ 0 & 0 & 0 & 0 & 8 & -16 & 8 \\ 0 & 0 & 0 & 0 & -1 & 8 & -7 \end{bmatrix}}_{\text{Ritz Matrix } R^{(8)} \text{ for } h=1/3} \cdot \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} + \underbrace{\begin{bmatrix} \int_0^1 f(x)\cdot v_0(x)dx \\ \int_0^1 f(x)\cdot v_1(x)dx \\ \int_0^1 f(x)\cdot v_2(x)dx \\ \int_0^1 f(x)\cdot v_3(x)dx \\ \int_0^1 f(x)\cdot v_4(x)dx \\ \int_0^1 f(x)\cdot v_5(x)dx \\ \int_0^1 f(x)\cdot v_6(x)dx \end{bmatrix}}_{\text{Ritz Vector } r^{(8)} \text{ for } h=1/3} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Advantage of the p-Strategy over the h-Strategy: In both strategies, the dimension of the system matrices increases. However, there is a justified hope that the increase required to achieve comparable accuracy is much smaller with the p-strategy than with the h-strategy.

3.8 Conformity and Completeness

If the approximate solution must now be differentiable twice, the approach of the last section is no longer considered conforming.

To ensure single differentiability at the nodes, new basis functions must be found.

$\tilde{u}(x) = a_0 v_0(x) + a_1 v_1(x) + a_2 v_2(x) + a_3 v_3(x) + \tilde{a}_0 \tilde{v}_0(x) + \tilde{a}_1 \tilde{v}_1(x) + \tilde{a}_2 \tilde{v}_2(x) + \tilde{a}_3 \tilde{v}_3(x)$

Two basis functions ensure the correct value at the nodes. Two additional basis functions are needed to ensure the first derivative (slope) at the transition nodes; they ensure the completeness of the basis functions. (Without the two additional basis functions, only a slope of zero would be possible at the transition nodes.)

3.9 Hermite Polynomials of Third Order

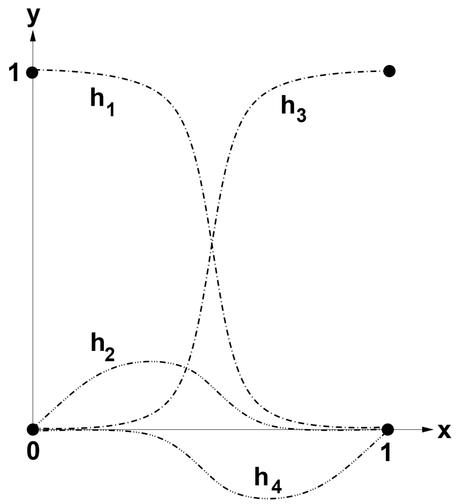
Agreement up to the 1st derivative at the nodes

Problem statement: $u''(x) + f(x) = 0 \qquad u(0) = a_0 \qquad u'(0) = \tilde{a}_0 \qquad u(1) = a_3 \qquad u'(1) = \tilde{a}_3$

The approximation should apply to the equally spaced intervals: $[0, 1/3], \quad [1/3, 2/3], \quad [2/3, 1]$

Shape functions:
$h_1(x) = 2x^3 - 3x^2 + 1$
$h_2(x) = x^3 - 2x^2 + x$
$h_3(x) = -2x^3 + 3x^2$
$h_4(x) = x^3 - x^2$

$x \in$	$[0, 1/3]$	$[1/3, 2/3]$	$[2/3, 1]$
$v_0 =$	$h_1(3x)$	0	0
$\tilde{v}_0 =$	$\frac{1}{3}h_2(3x)$	0	0
$v_1 =$	$h_3(3x)$	$h_1(3x-1)$	0
$\tilde{v}_1 =$	$\frac{1}{3}h_4(3x)$	$\frac{1}{3}h_2(3x-1)$	0
$v_2 =$	0	$h_3(3x-1)$	$h_1(3x-2)$
$\tilde{v}_2 =$	0	$\frac{1}{3}h_4(3x-1)$	$\frac{1}{3}h_2(3x-2)$
$v_3 =$	0	0	$h_3(3x-2)$
$\tilde{v}_3 =$	0	0	$\frac{1}{3}h_2(3x-1)$



$$E = \frac{1}{30}\begin{bmatrix} -36 & -3 & 36 & -3 \\ -3 & -4 & 3 & 1 \\ 36 & 3 & -36 & 3 \\ -3 & 1 & 3 & -4 \end{bmatrix} \quad \Rightarrow \quad \boxed{M = \frac{1}{30 \cdot h}\begin{bmatrix} -36 & -3 \cdot h & 36 & -3 \cdot h \\ -3 \cdot h & -4 \cdot h^2 & 3 \cdot h & 1 \cdot h^2 \\ 36 & 3 \cdot h & -36 & 3 \cdot h \\ -3 \cdot h & 1 \cdot h^2 & 3 \cdot h & -4 \cdot h^2 \end{bmatrix}}$$

$$\underbrace{\frac{3}{30}\begin{bmatrix} -36 & -1 & 36 & -1 & 0 & 0 & 0 & 0 \\ -1 & -4/9 & 1 & 1/9 & 0 & 0 & 0 & 0 \\ 36 & 1 & -72 & 0 & 36 & -1 & 0 & 0 \\ -1 & 1/9 & 0 & -8/9 & 1 & 1/9 & 0 & 0 \\ 0 & 0 & 36 & 1 & -72 & 0 & 36 & -1 \\ 0 & 0 & -1 & 1/9 & 0 & -8/9 & 1 & 1/9 \\ 0 & 0 & 0 & 0 & 36 & 1 & -36 & 1 \\ 0 & 0 & 0 & 0 & -1 & 1/9 & 1 & -4/9 \end{bmatrix}}_{\text{Ritz Matrix } R^{(8)} \text{ for } h=1/3} \cdot \begin{bmatrix} a_0 \\ \tilde{a}_0 \\ a_1 \\ \tilde{a}_1 \\ a_2 \\ \tilde{a}_2 \\ a_3 \\ \tilde{a}_3 \end{bmatrix} + \underbrace{\begin{bmatrix} \int_0^1 f(x) \cdot v_0(x) dx \\ \int_0^1 f(x) \cdot \tilde{v}_0(x) dx \\ \int_0^1 f(x) \cdot v_1(x) dx \\ \int_0^1 f(x) \cdot \tilde{v}_1(x) dx \\ \int_0^1 f(x) \cdot v_2(x) dx \\ \int_0^1 f(x) \cdot \tilde{v}_2(x) dx \\ \int_0^1 f(x) \cdot v_3(x) dx \\ \int_0^1 f(x) \cdot \tilde{v}_3(x) dx \end{bmatrix}}_{\text{Ritz Vector } r^{(8)} \text{ for } h=1/3} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

4 Fourier Series

Complex:

$$\boxed{f(t) = \sum_{k=-\infty}^{\infty} c_k \cdot e^{jk\omega_f t}} = \sum_{k=0}^{\infty} \left(c_k \cdot e^{jk\omega_f t} + \overline{c_k} \cdot e^{-jk\omega_f t}\right) \qquad \boxed{c_k = \overline{c_{-k}} = \frac{1}{T}\int_0^T f(t) \cdot e^{-jk\omega_f t} dt}$$

Real:

$$\boxed{f(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left[a_k \cos(k\omega_f t) + b_k \sin(k\omega_f t)\right]} = \frac{A_0}{2} + \sum_{k=1}^{\infty} A_k \cos(k\omega_f t + \varphi_k) \quad k \in \mathbb{Z}$$

$$\boxed{a_0 = \frac{2}{T}\int_0^T f(t)dt, \quad a_k = \frac{2}{T}\int_0^T f(t)\cos(k\omega_f t)dt, \quad b_k = \frac{2}{T}\int_0^T f(t)\sin(k\omega_f t)dt} \qquad \boxed{\omega_f = \frac{2\pi}{T} = 2\pi f}$$

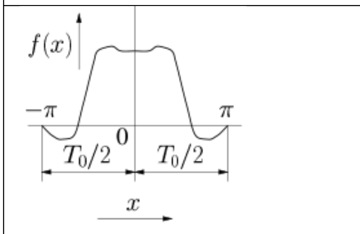
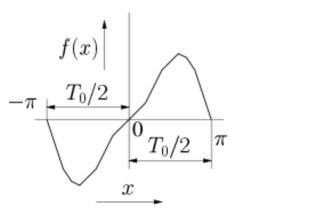
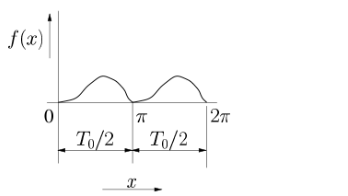
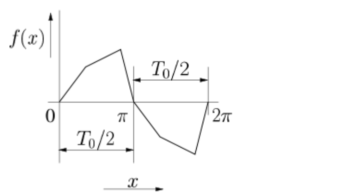
a_0, c_0, A_0 are *constants*, ω_f is the *fundamental frequency*, a_k and b_k are the *real coefficients*, c_k is the *complex coefficient*, A_k is the *amplitude*, and φ_k is the *phase*.

$a_k = c_k + \bar{c_k} = 2\,\mathrm{Re}(c_k) = A_k \cos(\varphi_k)$	$b_k = j(c_k + \bar{c_k}) = -2\,\mathrm{Im}(c_k) = -A_k \sin(\varphi_k)$
$c_k = \frac{a_k - jb_k}{2} = \frac{A_k}{2} e^{j\varphi_k}$	$c_{-k} = \overline{c_k} = \frac{a_k + jb_k}{2} = \frac{A_k}{2} e^{-j\varphi_k}$
$A_k = 2\lvert c_k\rvert = \sqrt{a_k^2 + b_k^2}$	

Calculation of φ_k from a_k and b_k

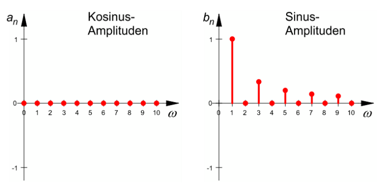
$a_k > 0:$	$\varphi_k = -\arctan(\frac{b_k}{a_k})$	$a_k < 0:$	$\varphi_k = -\arctan(\frac{b_k}{a_k}) + \pi$
$a_k = 0; b_k > 0:$	$\varphi_k = -\frac{\pi}{2}$	$a_k = 0; b_k < 0:$	$\varphi_k = \frac{\pi}{2}$
$a_k = b_k = 0:$	$\varphi_k =$ not defined		$\varphi_k = arg(c_k)$

4.1 Symmetry

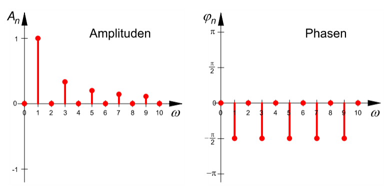
even function	**odd function**	**first half period**	**second half period**
			
$f(-t) = f(t)$ $b_k = 0$ $a_k = \frac{4}{T}\int_0^{\frac{T}{2}} f(t) \cdot \cos(k\omega_f t)dt$	$f(-t) = -f(t)$ $a_k = 0$ $b_k = \frac{4}{T}\int_0^{\frac{T}{2}} f(t) \cdot \sin(k\omega_f t)dt$	$f(t) = f(t+\pi)$ $a_{2k+1} = 0$ $b_{2k+1} = 0$	$f(t) = -f(t+\pi)$ $a_{2k} = 0$ $b_{2k} = 0$

4.2 Spectra

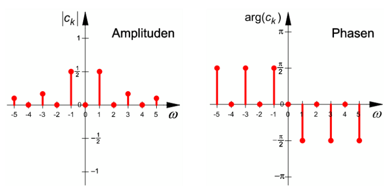
Cosine-Sine Amplitude Spectrum



Single-sided Amplitude/Phase Spectrum



Two-sided Amplitude/Phase Spectrum



The one-sided and two-sided spectra differ only in the amplitude diagram. The phase diagram for positive k is identical. The amplitude values are distributed evenly between positive and negative k.

5 Fourier Transform

$$\boxed{f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}d\omega} = \frac{1}{2\pi}\int_{-\infty}^{\infty}[R(\omega)\cos(\omega t) + X(\omega)\sin(\omega t)]d\omega + \frac{j}{2\pi}\int_{-\infty}^{\infty}[R(\omega)\sin(\omega t) - X(\omega)\cos(\omega t)]d\omega$$

$$\boxed{F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt} = R(\omega) - jX(\omega) \quad R(\omega) = \int_{-\infty}^{\infty} f(t)\cos(\omega t)dt \quad \text{and} \quad X(\omega) = \int_{-\infty}^{\infty} f(t)\sin(\omega t)dt$$

$\sigma(t) \circ\!\!-\!\!\bullet \frac{1}{j\omega} + \pi\cdot\delta(\omega)$ $\qquad$ $\frac{1}{\pi\cdot t} \circ\!\!-\!\!\bullet -j\cdot sgn(\omega)$

$1 \circ\!\!-\!\!\bullet 2\pi\cdot\delta(t) \underbrace{\longleftrightarrow}_{Careful} \delta(\omega) \circ\!\!-\!\!\bullet 1$ $\qquad$ $sgn(t) \circ\!\!-\!\!\bullet \frac{2}{j\omega}$

5.1 Properties

Fourier integral exists if $\int_{-\infty}^{\infty}|f(t)|dt < \infty$

Linearity	$\alpha\cdot f(t) + \beta\cdot g(t)$	$\circ\!\!-\!\!\bullet$	$\alpha\cdot F(\omega) + \beta\cdot G(\omega)$
Time reversal (reflection at the y-axis)	$f(-t)$	$\circ\!\!-\!\!\bullet$	$F(-\omega) = F^*(\omega)$
Similarity	$f(\alpha t)$	$\circ\!\!-\!\!\bullet$	$\frac{1}{\|\alpha\|}F\left(\frac{\omega}{\alpha}\right) \quad \alpha\in\mathbb{R}\setminus\{0\}$
Time-domain shift	$f(t\pm t_0)$	$\circ\!\!-\!\!\bullet$	$F(\omega)e^{\pm j\omega t_0}$
Frequency-domain shift	$f(t)e^{\pm j\omega_0 t}$	$\circ\!\!-\!\!\bullet$	$F(\omega\mp\omega_0)$
Time-domain differentiation	$\frac{\partial^n f(t)}{\partial t^n}$	$\circ\!\!-\!\!\bullet$	$(j\omega)^n F(\omega)$
Time-domain integration	$\int_{-\infty}^{t} f(\tau)d\tau$	$\circ\!\!-\!\!\bullet$	$\frac{F(\omega)}{j\omega} + \pi F(0)\delta(\omega)$
Frequency-domain differentiation	$t^n f(t)$	$\circ\!\!-\!\!\bullet$	$j^n\frac{\partial F(\omega)}{\partial\omega^n}$
Time-domain convolution	$f(t) * g(t)$	$\circ\!\!-\!\!\bullet$	$F(\omega)\cdot G(\omega)$
Frequency-domain convolution	$f(t)\cdot g(t)$	$\circ\!\!-\!\!\bullet$	$\frac{1}{2\pi}F(\omega) * G(j\omega)$
Exchange theorem (Duality)	$f(t)$	$\circ\!\!-\!\!\bullet$	$F(\omega)$
	$F(t)$	$\circ\!\!-\!\!\bullet$	$2\pi\cdot f(-\omega)$
Modulation	$\cos(\alpha t)\cdot f(t)$	$\circ\!\!-\!\!\bullet$	$\frac{1}{2}\cdot[F(\omega-\alpha) + F(\omega+\alpha)]$
	$\sin(\alpha t)\cdot f(t)$	$\circ\!\!-\!\!\bullet$	$\frac{1}{2j}\cdot[F(\omega-\alpha) - F(\omega+\alpha)]$
Parseval's Theorem	$\int_{-\infty}^{\infty} f(t)g^*(t)dt$	$=$	$\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)G^*(\omega)d\omega$
Bessel's Theorem	$\int_{-\infty}^{\infty}\|f(t)\|^2 dt$	$=$	$\frac{1}{2\pi}\int_{-\infty}^{\infty}\|F(\omega)\|^2 d\omega$
Initial values	$f(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)d\omega$		$F(0) = \int_{-\infty}^{\infty} f(t)dt$
Infinite sequence of δ-impulses	$\sum_{n=-\infty}^{\infty}\delta(t - n\cdot t_0)$	$\circ\!\!-\!\!\bullet$	$\sum_{n=-\infty}^{\infty}\frac{2\pi}{t_0}\delta(\omega - n\cdot\frac{2\pi}{t_0})$

6 Laplace Transformation

$$\boxed{F(s) = \int_0^\infty f(t) e^{-st} dt} \qquad s = \sigma + j\omega$$

- Applicable only for **causal** systems $\boxed{t \geq 0}$
- Integrable over the interval $(0, \infty)$
- Growth less than that of an exponential function $\boxed{\sigma > 0}$
- σ is the damping factor: $e^{-s} = e^{-\sigma} \cdot e^{-j\omega}$
- Fourier-transform $F(\omega)$ can be expressed through Laplace transformation $F(s)$.
- Fourier $\longleftrightarrow$ Laplace transformations only if there is a pole $(\sigma > 0)$, i.e., to the left of the $j\omega$ axis and causal!
- $f(0+)$: Corresponds to the initial condition at a time greater than 0 (causal).

6.1 Properties

Linearity	$\alpha \cdot f(t) + \beta \cdot g(t)$	$\circ\!\!-\!\!\bullet$	$\alpha \cdot F(s) + \beta \cdot G(s)$
Time-domain shift	$f(t \pm t_0)$	$\circ\!\!-\!\!\bullet$	$F(s)e^{\pm t_0 s}$
Damping (Frequency-domain shift)	$f(t)e^{\mp \alpha t}$	$\circ\!\!-\!\!\bullet$	$F(s \pm \alpha)$
Similarity	$f(\alpha t)$	$\circ\!\!-\!\!\bullet$	$\frac{1}{\alpha} F\left(\frac{s}{\alpha}\right) \quad 0 < \alpha \in \mathbb{R}$
Time-domain convolution	$f(t) * g(t)$	$\circ\!\!-\!\!\bullet$	$F(s) \cdot G(s)$
Frequency-domain convolution	$f(t) \cdot g(t)$	$\circ\!\!-\!\!\bullet$	$\frac{1}{2\pi j} F(s) * G(s)$
Time-domain differentiation	$f'(t)$	$\circ\!\!-\!\!\bullet$	$sF(s) - f(0+)$
	$f''(t)$	$\circ\!\!-\!\!\bullet$	$s^2 F(s) - sf(0+) - f'(0+)$
	$f^{(n)}(t)$	$\circ\!\!-\!\!\bullet$	$s^n F(s) - s^{n-1} f(0+) - s^{n-2} f'(0+) - \ldots - sf^{(n-2)}(0+) - f^{(n-1)}(0+)$
Frequency-domain differentiation	$(-t)^n f(t)$	$\circ\!\!-\!\!\bullet$	$F^{(n)}(s)$
Integration	$\int_0^t f(\tau) d\tau$	$\circ\!\!-\!\!\bullet$	$\frac{F(s)}{s}$
Initial value	$\lim_{t \to 0} f(t)$ (must exist)	$=$	$\lim_{s \to \infty} sF(s)$
Final value	$\lim_{t \to \infty} f(t)$ (must exist)	$=$	$\lim_{s \to 0} sF(s)$

6.2 From Laplace to Fourier

$s \to j\omega$ This can only be done if the pole $(\sigma > 0)$ is to the left of the $j\omega$-axis and the system is causal.

6.3 Inverse Transformation (Complex Integration)

$$f(t) = \int_{x-j\infty}^{x+j\infty} F(s) \cdot e^{st} \cdot ds$$

6.4 Procedure for Inverse Transformation

1. Approach Attempt to make the numerator have the same factor as the denominator and then simplify (corrections!)
2. Approach Partial fraction decomposition

6.5 Inverse Transformation via Table

σ = Step function. When 1 is transformed, σ should be taken (so in the frequency domain, it is $\frac{1}{s}$).

$\sigma(t)$	$\circ\!\!-\!\!\bullet$	$\frac{1}{s}$	$\sigma(t)\cdot t^2\cdot e^{\alpha\,t}$	$\circ\!\!-\!\!\bullet$	$\frac{2}{(s-\alpha)^3}$
$\sigma(t)\cdot t$	$\circ\!\!-\!\!\bullet$	$\frac{1}{s^2}$	$\sigma(t)\cdot t^n\cdot e^{\alpha\,t}$	$\circ\!\!-\!\!\bullet$	$\frac{n!}{(s-\alpha)^{n+1}}$
$\sigma(t)\cdot t^2$	$\circ\!\!-\!\!\bullet$	$\frac{2}{s^3}$	$\sigma(t)\cdot\sin(\omega\,t)$	$\circ\!\!-\!\!\bullet$	$\frac{\omega}{s^2+\omega^2}$
$\sigma(t)\cdot t^n$	$\circ\!\!-\!\!\bullet$	$\frac{n!}{s^{n+1}}$	$\sigma(t)\cdot\cos(\omega\,t)$	$\circ\!\!-\!\!\bullet$	$\frac{s}{s^2+\omega^2}$
$\sigma(t)\cdot e^{\alpha\,t}$	$\circ\!\!-\!\!\bullet$	$\frac{1}{s-\alpha}$	$\delta(t)$	$\circ\!\!-\!\!\bullet$	$1(s)$
$\sigma(t)\cdot t\cdot e^{\alpha\,t}$	$\circ\!\!-\!\!\bullet$	$\frac{1}{(s-\alpha)^2}$	$\delta(t-a)$	$\circ\!\!-\!\!\bullet$	$e^{-a\,s}$

7 Math Fundamentals

7.1 Partial Fraction Decomposition

$f(x) = \frac{x^2+20x+149}{x^3+4x^2-11x-30} \Rightarrow$ Factorize denominator using Horner's scheme, binomial, etc. $\Rightarrow x^3+4x^2-11x-30 = (x+2)(x^2+2x-15) = (x+2)(x+5)(x-3)$

Approach:

$$f(x) = \frac{x^2+20x+149}{x^3+4x^2-11x-30} = \frac{A}{x-3} + \frac{B}{x+2} + \frac{C}{x+5} = \frac{A(x+2)(x+5)+B(x-3)(x+5)+C(x-3)(x+2)}{(x-3)(x+2)(x+5)}$$

Set up a system of equations with arbitrary values of x_i (preferably choose poles or 0,1,-1):

$$\begin{aligned} x_1 = 3 &: \ -9+60+149 = A\cdot 5\cdot 8 \ \Rightarrow A = 5 \\ x_2 = -2 &: \ -4-40+149 = B(-5)\cdot 3 \ \Rightarrow B = -7 \\ x_3 = -5 &: \ -25-100+149 = C(-8)(-3) \Rightarrow C = 1 \end{aligned} \quad \Rightarrow f(x) = \frac{5}{x-3} - \frac{7}{x+2} + \frac{1}{x+5}$$

Additional approaches for other types of terms:

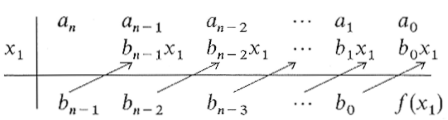
$$f(x) = \frac{5x^2-37x+54}{x^3-6x^2+9x} = \frac{A}{x} + \frac{B}{x-3} + \frac{C}{(x-3)^2} = \frac{A(x-3)^2+Bx(x-3)+Cx}{x(x-3)^2}$$

$$f(x) = \frac{1.5x}{x^3-6x^2+12x-8} = \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{C}{(x-2)^3} = \frac{A(x-2)^2+B(x-2)+C}{(x-2)^3}$$

$$f(x) = \frac{x^2-1}{x^3+2x^2-2x-12} = \frac{A}{x-2} + \frac{Bx+C}{x^2+4x+6} = \frac{A(x^2+4x+6)+(Bx+C)(x-2)}{(x-2)(x^2+4x+6)}$$

7.1.1 Horner's Scheme

- Arrows $\Rightarrow$ Multiplication
- Numbers in each column are added



$x_1 \Rightarrow$ Root (must be guessed!!)
Top row = polynomial to be factored

Example:
$f(x) = x^3 - 67x - 126$

	1	0	-67	-126
$x_1 = -2$		-2	4	$+126$
	1	-2	-63	$0 = f(-2)$
	$\uparrow$	$\uparrow$	$\uparrow$	
	b_2	b_1	b_0	

$\Rightarrow f(x) = (x-x_1)(b_2x^2+b_1x+b_0) = (x+2)(x^2-2x-63)$

7.2 Trigonometry

$\sin^2(b) + \cos^2(b) = 1 \qquad \tan(b) = \frac{\sin(b)}{\cos(b)} \qquad \cosh(b)^2 - \sinh(b)^2 = 1 \qquad \tanh(b) = \frac{\sinh(b)}{\cosh(b)}$

7.2.1 Function Values for Angle Arguments

deg	rad	sin	cos	tan
0	0	0	1	0
30	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
45	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
60	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$

deg	rad	sin	cos
90	$\frac{\pi}{2}$	1	0
120	$\frac{2\pi}{3}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$
135	$\frac{3\pi}{4}$	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$
150	$\frac{5\pi}{6}$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$

deg	rad	sin	cos
180	π	0	-1
210	$\frac{7\pi}{6}$	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$
225	$\frac{5\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$
240	$\frac{4\pi}{3}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$

deg	rad	sin	cos
270	$\frac{3\pi}{2}$	-1	0
300	$\frac{5\pi}{3}$	$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
315	$\frac{7\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
330	$\frac{11\pi}{6}$	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$

7.2.2 Quadrant Relations

$\sin(-a) = -\sin(a) \qquad \cos(-a) = \cos(a)$
$\sin(\pi - a) = \sin(a) \qquad \cos(\pi - a) = -\cos(a)$
$\sin(\pi + a) = -\sin(a) \qquad \cos(\pi + a) = -\cos(a)$
$\sin\left(\frac{\pi}{2} - a\right) = \sin\left(\frac{\pi}{2} + a\right) = \cos(a) \qquad \cos\left(\frac{\pi}{2} - a\right) = -\cos\left(\frac{\pi}{2} + a\right) = \sin(a)$

Addition Theorems $\sin(a \pm b) = \sin(a) \cdot \cos(b) \pm \cos(a) \cdot \sin(b) \qquad \cos(a \pm b) = \cos(a) \cdot \cos(b) \mp \sin(a) \cdot \sin(b)$
$\tan(a \pm b) = \dfrac{\tan(a) \pm \tan(b)}{1 \mp \tan(a) \cdot \tan(b)}$

7.2.3 Double and Half Angles

$\sin(2a) = 2\sin(a)\cos(a) \qquad \cos(2a) = \cos^2(a) - \sin^2(a) = 2\cos^2(a) - 1 = 1 - 2\sin^2(a)$
$\cos^2\left(\frac{a}{2}\right) = \frac{1+\cos(a)}{2} \qquad \sin^2\left(\dfrac{a}{2}\right) = \frac{1-\cos(a)}{2}$

7.2.4 Products

$\sin(a)\sin(b) = \frac{1}{2}(\cos(a-b) - \cos(a+b))$
$\cos(a)\cos(b) = \frac{1}{2}(\cos(a-b) + \cos(a+b))$
$\sin(a)\cos(b) = \frac{1}{2}(\sin(a-b) + \sin(a+b))$

7.2.5 Hyperbolic

$\sinh(z) = \frac{1}{2}\left(e^z - e^{-z}\right) \qquad \cosh(z) = \frac{1}{2}\left(e^z + e^{-z}\right)$

7.2.6 Sum and Difference

$\sin(a) + \sin(b) = 2 \cdot \sin\left(\frac{a+b}{2}\right) \cdot \cos\left(\frac{a-b}{2}\right)$
$\sin(a) - \sin(b) = 2 \cdot \sin\left(\frac{a-b}{2}\right) \cdot \cos\left(\frac{a+b}{2}\right)$
$\cos(a) + \cos(b) = 2 \cdot \cos\left(\frac{a+b}{2}\right) \cdot \cos\left(\frac{a-b}{2}\right)$
$\cos(a) - \cos(b) = -2 \cdot \sin\left(\frac{a+b}{2}\right) \cdot \sin\left(\frac{a-b}{2}\right)$
$\tan(a) \pm \tan(b) = \dfrac{\sin(a \pm b)}{\cos(a)\cos(b)}$

7.2.7 Euler

$sin(z) = \frac{e^{jz} - e^{-jz}}{2j} \qquad cos(z) = \frac{e^{jz} + e^{-jz}}{2} \qquad e^{j\varphi} = cjs(\varphi) = cos(\varphi) + jsin(\varphi)$

7.2.8 Complex

Magnitude:	$\lvert z \rvert = \sqrt{Re(z)^2 + Im(z)^2} = \sqrt{z \cdot \bar{z}}$	
Complex conjugate:	$z = z_1 + jz_2$	$\bar{z} = z^* = z_1 - jz_2$

7.3 Taylor Polynomial

$f(x_0 + h) = f(x_0) + f'(x_0)h + \frac{f''(x_0)}{2}h^2 + \frac{f'''(x_0)}{3!}h^3 + \ldots + \frac{f^{(n)}(x_0)}{n!}h^n + R_n(x_0, h)$

7.4 Integral Calculus

Integration	$A = \int\limits_a^b f(t)dt = [F(t)]_a^b = F(b) - F(a)$
Linearity	$\int f(\alpha x + \beta)dx = \frac{1}{\alpha} \cdot F(\alpha x + \beta) + C$
Partial Integration	$\int\limits_a^b \underset{\Uparrow}{u'(x)} \cdot \underset{\Downarrow}{v(x)}dx = \Big[u(x) \cdot v(x)\Big]_a^b - \int\limits_a^b u(x) \cdot v'(x)dx$

Substitution (Rationalization)	$t=\tan\frac{x}{2},\quad dx=\frac{2dt}{1+t^2}\quad \sin x=\frac{2t}{1+t^2}\quad \cos x=\frac{1-t^2}{1+t^2}\quad \int R(\sin(x)\cos(x))dx$
General Substitution	$\int_a^b f(x)dx=\int_{g^{-1}(a)}^{g^{-1}(b)} f(g(t))\cdot g'(t)dt\quad t=g^{-1}(x)\quad \boxed{\text{x=g(t)}}\quad dx=g'(t)\cdot dt$
Logarithmic Integration	$\int\frac{f'(x)}{f(x)}dx=\ln\lvert f(x)\rvert+C\quad (f(x)\neq 1)$
Special Form of the Integrand	$\int f'(x)\cdot(f(x))^\alpha dx=f(x)^{\alpha+1}\cdot\frac{1}{\alpha+1}+C\quad(\alpha\neq -1)$
Differentiation	$\int_a^b f'(t)dt=f(b)-f(a)\quad \frac{d}{dx}\int_1^x f(t)dt=f(x)$

7.4.1 Some Indefinite Integrals

$\int \mathrm{d}x = x + C$
$\int x^\alpha \mathrm{d}x = \frac{x^{\alpha+1}}{\alpha+1}+C,\ x\in\mathbb{R}^+,\ \alpha\in\mathbb{R}\backslash\{-1\}$
$\int \frac{1}{x}\mathrm{d}x=\ln|x|+C,\ x\neq 0$
$\int e^x\mathrm{d}x=e^x+C$
$\int a^x\mathrm{d}x=\frac{a^x}{\ln(a)}+C,\ a\in\mathbb{R}^+\backslash\{1\}$
$\int \sin x\mathrm{d}x=-\cos x+C$
$\int \cos x\mathrm{d}x=\sin x+C$
$\int\frac{\mathrm{d}x}{\sin^2 x}=-\cot(x)+C,\ x\neq k\pi$ with $k\in\mathbb{Z}$
$\int\frac{\mathrm{d}x}{\cos^2 x}=\tan(x)+C,\ x\neq\frac{\pi}{2}+k\pi$ with $k\in\mathbb{Z}$
$\int\sinh(x)\mathrm{d}x=\cosh(x)+C$
$\int\cosh(x)\mathrm{d}x=\sinh(x)+C$
$\int\frac{\mathrm{d}x}{\sinh^2 x}=-\coth(x)+C,\ x\neq 0$
$\int\frac{\mathrm{d}x}{\cosh^2 x}=\tanh(x)+C,\ x\neq 0$
$\int\frac{\mathrm{d}x}{ax+b}=\frac{1}{a}\ln|ax+b|+C,\ a\neq 0,\ x\neq-\frac{b}{a}$
$\int\frac{\mathrm{d}x}{a^2x^2+b^2}=\frac{1}{ab}\arctan(\frac{a}{b}x)+C,\ a\neq 0,\ x\neq-\frac{b}{a}\ x\neq-\frac{b}{a}$
$\int\frac{\mathrm{d}x}{a^2x^2-b^2}=\frac{1}{2ab}\ln|\frac{ax-b}{ax+b}|+C,\ a\neq 0,\ x\neq-\frac{b}{a}\ x\neq-\frac{b}{a}$
$\int\sqrt{a^2x^2+b^2}\mathrm{d}x=\frac{x}{2}\sqrt{a^2x^2+b^2}+\frac{b^2}{2a}\ln(ax+\sqrt{a^2x^2+b^2})+C,\ a\neq 0,\ b\neq 0$
$\int\sqrt{a^2x^2-b^2}\mathrm{d}x=\frac{x}{2}\sqrt{a^2x^2-b^2}+\frac{b^2}{2a}\ln|ax+\sqrt{a^2x^2-b^2}|+C,\ a\neq 0,\ b\neq 0,\ a^2x^2\geq b^2$
$\int\sqrt{b^2-a^2x^2}\mathrm{d}x=\frac{x}{2}\sqrt{b^2-a^2x^2}+\frac{b^2}{2a}\arcsin(\frac{a}{b}x)+C),\ a\neq 0,\ b\neq 0,\ a^2x^2\geq b^2$
$\int\frac{\mathrm{d}x}{\sqrt{a^2x^2+b^2}}=\frac{1}{a}\ln(ax+\sqrt{a^2x^2+b^2})+C,\ a\neq 0,\ b\neq 0$
$\int\frac{\mathrm{d}x}{\sqrt{a^2x^2-b^2}}=\frac{1}{a}\ln(ax+\sqrt{a^2x^2-b^2})+C,\ a\neq 0,\ b\neq 0\ a^2x^2>b^2$
$\int\frac{\mathrm{d}x}{\sqrt{b^2-a^2x^2}}=\frac{1}{a}\arcsin(\frac{a}{b}x)+C,\ a\neq 0,\ b\neq 0,\ a^2x^2<b^2$

The integrals $\int\frac{dx}{X}$, $\int\sqrt{X}\ dx$, $\int\frac{dx}{\sqrt{X}}$ with $X=ax^2+2bx+c$, $a\neq 0$, are transformed by the transformation $X=a\left(x+\frac{b}{a}\right)^2+\left(c-\frac{b^2}{a}\right)$ and the substitution $t=x+\frac{b}{a}$ into the integrals 15. to 22.

$\int\frac{x\ \mathrm{d}x}{X}=\frac{1}{2a}\ln|X|-\frac{b}{a}\int\frac{\mathrm{d}x}{X},\ a\neq 0,\ X=ax^2+2bx+c$
$\int\sin^2(ax)\ \mathrm{d}x=\frac{x}{2}-\frac{1}{4a}\cdot\sin(2ax)+C,\ a\neq 0$
$\int\cos^2(ax)\ \mathrm{d}x=\frac{x}{2}+\frac{1}{4a}\cdot\sin(2ax)+C,\ a\neq 0$
$\int\sin^n(ax)\ \mathrm{d}x=\frac{\sin^{n-1}(ax)\cdot\cos(ax)}{na}+\frac{n-1}{n}\int\sin^{n-2}(ax)\mathrm{d}x,\ n\in\mathbb{N},\ a\neq 0$
$\int\cos^n(ax)\ \mathrm{d}x=\frac{cos^{n-1}(ax)\cdot\sin(ax)}{na}+\frac{n-1}{n}\int\cos^{n-2}(ax)\mathrm{d}x,\ n\in\mathbb{N},\ a\neq 0$
$\int\frac{dx}{\sin(ax)}=\frac{1}{a}\ln|\tan(\frac{ax}{2})|+C,\ a\neq 0,\ x\neq k\frac{\pi}{a}$ with $k\in\mathbb{Z}$
$\int\frac{dx}{cos(ax)}=\frac{1}{a}\ln|\tan(\frac{ax}{2}+\frac{\pi}{4})|+C,\ a\neq 0,\ x\neq\frac{\pi}{2a}+k\frac{\pi}{a}$ with $k\in\mathbb{Z}$
$\int\tan(ax)\mathrm{d}x=-\frac{1}{a}\ln|\cos(ax)|+C,\ a\neq 0,\ x\neq\frac{\pi}{2a}+k\frac{\pi}{a}$ with $k\in\mathbb{Z}$
$\int\cot(ax)\mathrm{d}x=\frac{1}{a}\ln|\sin(ax)|+C,\ a\neq 0,\ x\neq k\frac{\pi}{a}$ with $k\in\mathbb{Z}$
$\int x^n\sin(ax)\mathrm{d}x=-\frac{x^n}{a}\cos(ax)+\frac{n}{a}\int x^{n-1}\cos(ax)\mathrm{d}x,\ n\in\mathbb{N},\ a\neq 0$
$\int x^n\cos(ax)\mathrm{d}x=\frac{x^n}{a}\sin(ax)-\frac{n}{a}\int x^{n-1}\sin(ax)\mathrm{d}x,\ n\in\mathbb{N},\ a\neq 0$
$\int x^ne^{ax}\mathrm{d}x=\frac{1}{a}x^ne^{ax}-\frac{n}{a}\int x^{n-1}e^{ax}\mathrm{d}x,\ n\in\mathbb{N},\ a\neq 0$
$\int e^{ax}\sin(bx)\mathrm{d}x=\frac{e^{ax}}{a^2+b^2}(a\cdot\sin(bx)-b\cdot\cos(bx))+C,\ a\neq 0,\ b\neq 0$
$\int e^{ax}\cos(bx)\mathrm{d}x=\frac{e^{ax}}{a^2+b^2}(a\cdot\cos(bx)+b\cdot\sin(bx))+C,\ a\neq 0,\ b\neq 0$
$\int\ln(x)\mathrm{d}x=x(\ln(x)-1)+C,\ x\in\mathbb{R}^+$
$\int x^\alpha\cdot\ln(x)\mathrm{d}x=\frac{x^{\alpha+1}}{(\alpha+1)^2}[(\alpha+1)\ln(x)-1]+C,\ x\in\mathbb{R}^+,\ \alpha\in\mathbb{R}\backslash\{-1\}$

7.5 Differential Equations

7.5.1 Linear Differential Equations of 1st Order

Form: $y' + f(x)y = g(x)$ **Procedure:** $y = y_H + y_p$ $y_H = k \cdot e^{-\int f(x)dx}$ where $k = y_0$
$y_p = k \cdot e^{-\int f(x)dx}$ where $k = \int (g(x) \cdot e^{\int f(x)dx})dx$

7.5.2 Linear Differential Equation of 2nd Order with Constant Coefficients

Form: $y'' + a_1 \cdot y' + a_0 \cdot y = f(x)$
Forcing Term: $f(x)$
Homogeneous Differential Equation: $f(x) = 0$
Inhomogeneous Differential Equation: $f(x) \neq 0$

7.5.3 General Solution of a Homogeneous ODE: Y_H

Characteristic Polynomial $\underline{\lambda^2 + a_1 \cdot \lambda + a_0 = 0}$ of $\underline{y'' + a_1 \cdot y' + a_0 \cdot y = 0}$ $(\lambda_{1,2} = -\frac{a_1}{2} \pm \frac{\sqrt{a_1^2 - 4a_0}}{2})$

If $\lambda_1 \neq \lambda_2$ and $\lambda_{1,2} \in R$:	$Y_H = Ae^{\lambda_1 x} + Be^{\lambda_2 x}$
If $\lambda_1 = \lambda_2$ and $\lambda_{1,2} \in R$:	$Y_H = e^{\lambda_1 x}(A + B \cdot x)$
If $\lambda_{1,2} = -\frac{a_1}{2} \pm j\alpha$:	$Y_H = e^{-\frac{1}{2}a_1 x}(A\cos(\alpha x) + B\sin(\alpha x))$

7.5.4 General Solution of an Inhomogeneous ODE: $y = Y_H + y_P$

Basic Solution Method of an Inhomogeneous ODE: y_P
Homogeneous ODE: $y'' + a_1 \cdot y' + a_0 \cdot y = 0$ for which ($g(x) = Y_H$ homogeneous solution) $g(x_0) = 0$ and $g'(x_0) = 1$ holds, is:

$$y_P(x) = \int_{x_o}^{x} g(x + x_0 - t) \cdot f(t)dt$$

the particular solution of $y'' + a_1 \cdot y' + a_0 \cdot y = f(x)$

The Approach of an Inhomogeneous ODE in the Form of The Forcing Term: y_P

$f(x) = p_n(x)$ ($p_n(x)$ and $q_n(x)$ are polynomials of the same degree)

Case a: $a_0 \neq 0$:	$y_P = q_n(x)$
Case b: $a_0 = 0, a_1 \neq 0$:	$y_P = x \cdot q_n(x)$
Case c: $a_0 = a_1 = 0$:	$y_P = x^2 \cdot q_n(x)$

$f(x) = e^{bx} \cdot p_n(x)$

Case a: b not a root of the characteristic polynomial:	$y_P = e^{bx} \cdot q_n(x)$
Case b: b a simple root of the characteristic polynomial:	$y_P = e^{bx} \cdot x \cdot q_n(x)$
Case c: b a double root of the characteristic polynomial:	$y_P = e^{bx} \cdot x^2 \cdot q_n(x)$

$f(x) = e^{cx} \cdot (p_n(x)\cos(bx) + q_n(x)\sin(bx))$

Case a: $c + jb$ not a solution of the characteristic equation:	$y_P = e^{cx} \cdot (r_n(x)\cos(bx) + s_n(x)\sin(bx))$
Case b: $c + jb$ a solution of the characteristic equation:	$y_P = e^{cx} \cdot x \cdot (r_n(x)\cos(bx) + s_n(x)\sin(bx))$

Superposition Principle
$f(x) = c_1 f_1(x) + c_2 f_2(x)$

y_1 is a specific solution of the ODE	$y'' + a_1 \cdot y' + a_0 \cdot y = c_1 f_1(x)$
y_2 is a specific solution of the ODE	$y'' + a_1 \cdot y' + a_0 \cdot y = c_2 f_2(x)$
then:	$y_P = c_1 y_1 + c_2 y_2$

7.5.5 Linear Differential Equation of nth Order with Constant Coefficients

Form: $y^{(n)} + a_{n-1} \cdot y^{(n-1)} + \ldots + a_0 \cdot y = f(x) \Leftrightarrow \sum_{k=0}^{n} a_k y^{(k)} = f(x)$

Homogeneous Solutions

Case a: r real solutions λ:	$y_1 = e^{\lambda x}$, $y_2 = xe^{\lambda x}$, $\ldots$,$y_r = x^{r-1}e^{\lambda x}$	Strong damping/creeper case
Case b: k complex solutions $\lambda = \alpha + j\beta$:	$y_1 = e^{\alpha x}\cos(\beta x)$, $y_3 = e^{\alpha x}x^1\cos(\beta x)$, ... (odd)	Weak damping /
	$y_2 = e^{\alpha x}\sin(\beta x)$, $y_4 = e^{\alpha x}x^1\sin(\beta x)$, ... (even)	Oscillation case

Degrees of freedom $(A, B, C, ...)$ and additional x^n should not be forgotten!!!

Most General Solution of the Particular Part:

$$\underbrace{\sum_{k=0}^{n} a_k y^{(k)}}_{f(y,y',y'',\ldots)} = \underbrace{e^{\alpha x}(p_{m1}(x)\cos(\beta x) + q_{m2}(x)\sin(\beta x))}_{\text{Forcing term}}$$

Distinguish solutions of the characteristic polynomial (λ): with m = max(m1, m2)

Case a: $\alpha + j\beta \neq \lambda$, then	$y_P = e^{\alpha x}(r_m(x)\cos(\beta x) + s_m(x)\sin(\beta x))$
Case b: $\alpha + j\beta$ is a u-fold solution of λ, then	$y_P = e^{\alpha x}x^u(r_m(x)\cos(\beta x) + s_m(x)\sin(\beta x))$
	u-fold resonance

Basic Solution Method

$$\begin{pmatrix} g(x_0) = & 0 & = & c_1g_1(x_0) + c_2g_2(x_0) + \ldots + c_n(x_0) \\ g'(x_0) = & 0 & = & c_1g_1'(x_0) + c_2g_2'(x_0) + \ldots + c_ng_n'(x_0) \\ \vdots & \vdots & & \\ g^{(n-1)}(x_0) = & 1 & = & c_1g_1^{(n-1)}(x_0) + c_2g_2^{(n-1)}(x_0) + \ldots + c_ng_n^{(n-1)}(x_0) \end{pmatrix}$$

Results in $c_1, \ldots, c_n$ for
$y_P(x) = \int_{x_0}^{x} g(x + x_0 - t)f(t)dt$

7.5.6 Linear Differential Equation Systems of First Order with Constant Coefficients

Form: $\dot{x} = ax + by + f(t) \leftrightarrow y = \frac{1}{b}(\dot{x} - ax - f(t))$
$\dot{y} = cx + dy + g(t)$

The general solution results from the ODE: $\ddot{x} - (a + d)\dot{x} + (ad - bc)x = \dot{f}(t) - d \cdot f(t) + b \cdot g(t)$

$\ddot{x}, \dot{x}, \dot{y}$ are each differentiated with respect to t!

7.5.7 Solving ODEs with Laplace Transformation

To solve an ODE with Laplace (causal!), the equation must first be transformed into the Laplace domain. After that, the equation can be solved algebraically. The result must then be transformed back into the original domain through inverse transformation.

Note:
- $H(s) = \frac{1}{p(s)}$ where $p(s)$ represents the characteristic polynomial

Stability

A system is stable if the root of the characteristic polynomial $p(s)$ lies in the left half-plane:

$$Re[p(s)] < 0$$

7.5.8 Common DEs

DE	Solution	DE	Solution
$\frac{dx}{dt} = 0$	C	$\frac{dx}{dt} = 1$	$t + C$
$\frac{dx}{dt} = y$	$t \cdot y + C$	$\frac{dx}{dt} = kx$	Ce^{kt}
$\frac{du}{dt} = \sin(t)$	$C - \cos(t)$	$\frac{d^2x}{dt^2} = k^2x$	$A * \cosh(kt) + B * \sinh(kt) = \frac{1}{2}((A + B)e^{kt} + (A - B)e^{-kt})$
$\frac{d^2x(t)}{dt^2} = -\omega^2x(t)$	$A\cos(\omega t) + B\sin(\omega t)$		

7.6 Differential Calculus

$f'(x_0) = \lim\limits_{\Delta x \to 0} \frac{f(x_0+\Delta)x - f(x_0)}{\Delta x}$

Chain rule:	$f\big(g(x)\big)'$	$=$	$g'(x) \cdot f'\big(g(x)\big)$ or $\dfrac{df(g(x))}{dx} = f'(g(x)) \cdot g'(x)$
Product rule:	$(f(x) \cdot g(x))'$	$=$	$f'(x) \cdot g(x) + f(x) \cdot g'(x)$
Quotient rule:	$\dfrac{f(x)}{g(x)}$	$=$	$\dfrac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$

7.7 Miscellaneous

7.7.1 Quadratic Formula

$ax^2 + bx + c = 0 \quad \Rightarrow \quad x_{1,2} = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

7.7.2 Determinants

$\det\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$

7.7.3 Matrix Inversion

$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \Rightarrow \quad A^{-1} = \frac{1}{det(A)} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$

7.7.4 Eigenvalues/ Eigenvectors

Eigenvalue: $\det(A - \lambda I) \quad \Rightarrow \quad \lambda i$
Eigenvector: $(A - \lambda_i I)v = 0 \quad \Rightarrow \quad v_i \qquad$ (For each λ_i)
Definition: $A \cdot \underline{v} = \lambda \cdot \underline{v}$

7.7.5 TI-89

Equation for multiple values
$(3x + y^2) \mid x = 1$ and $y = 2 \rightarrow Result$
Matrix Editor

- APPS / Data/Matrix Editor
- New
- Type: Matrix
- Define Variable, Row, Column
- Enter values

Delete saved variables

- Explorer: 2nd / VAR-LINK
- Select variable
- Delete: DEL
- Confirm deletion: ENTER