MATH 3311, FALL 2025: LECTURE 7, SEPTEMBER 10

Video: `https://youtu.be/7mq_T0_cEZI`

Dihedral groups

Definition 1. The **dihedral group** D_{2n} is the group consisting of the *rigid* symmetries of the regular n-gon: These consist of the rotations through multiples of $2\pi/n$, as well as reflections across medians. It is a finite group of order $2n$: There are n rotations (through each multiple of $2\pi/n$) and n reflections (across each of the medians).

Observation 1. More generally, for any n, we can take $\sigma \in D_{2n}$ to be rotation counterclocwise by $\frac{2\pi}{n}$ and τ to be any reflection (say across the median through vertex 1). We then find that the following things are true:

- The rotations are $e, \sigma, \ldots, \sigma^{n-1}$.
- The reflections are $\tau, \sigma \circ \tau, \ldots, \sigma^{n-1} \circ \tau$.
- We have $\sigma^n = e = \tau^2$, and $\tau \circ \sigma = \sigma^{n-1} \circ \tau$.

Note that $\sigma^{n-1} \circ \sigma = \sigma^n = e$, meaning that $\sigma^{n-1} = \sigma^{-1}$.

This information given by the last bullet point is sufficient to explain how to take the product of any two elements in the group, as we did in the previous example. See the lecture video for a longer breakdown of this.

Symmetric groups

Here is another important class of groups.

Definition 2. The **symmetric group on n letters**, denoted S_n, is the group of bijections from the set $\{1, \ldots, n\}$ to itself, under composition. In other words, in the notation from the homeworks, we have

$$S_n = \mathrm{Bij}\left(\{1, \ldots, n\}\right)$$

The group S_n is usually referred to as the **symmetric group on n letters**.

In other words, S_n is the group of permutations of n things. It has order $n!$.

Example 1. When $n = 1$, S_1 is just the trivial group.

Example 2. When $n = 2$, S_2 is a group of order 2 and so has to be isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

Example 3. When $n = 3$, S_3 is a group of order 6. What are some of its elements?

We have the permutation α that satifies:

$$\alpha : 1 \mapsto 2 \mapsto 3 \mapsto 1,$$

and the permutation β that fixes 1 and switches 2 and 3.

Note that these permutations are exactly how $\sigma, \tau \in D_6$ move the vertices of the equilateral triangle around! In particular, you can check that we have

$$\alpha^2 \circ \beta = \beta \circ \alpha.$$

This group is therefore non-abelian, and by Homework 2, it has to be isomorphic to D_6! In fact, we have an explicit isomorphism

$$\begin{aligned} D_6 &\xrightarrow{\simeq} S_3 \\ \sigma &\mapsto \alpha \\ \tau &\mapsto \beta \end{aligned}$$

. Once we know what happens to σ, τ, the outputs for all the other elements of D_6 will fall into place by the group homomorphism property (why?).

What's going on here is that we are basically 'forgetting' the geometric origins of the elements of D_6 and remembering only how they are permuting the vertices of the triangle. Of course the action on the vertices can be used to reconstruct the geometric interpretation, so we are not losing information.

Example 4. More generally, for any $n \geq 3$, D_{2n} permutes the n vertices of the regular n-gon, and in this way we can view every element $\gamma \in D_{2n}$ as a permutation $\rho(\gamma) \in S_n$ that tells us how γ permutes the vertices. In this way we get a group homomorphism

$$\rho : D_{2n} \to S_n.$$

This is in general injective but far from being an isomorphism. Indeed, the left hand side has only $2n$ elements while the right hand side has $n!$, usually a *much* larger number than $2n$. What this is saying is that not all permutations of the vertices of a regular n-gon arise from a rigid symmetry.

Question 1. Can you find a permutation of the vertices of the square that does *not* arise from an element of D_8?

Let us recap what is happening here: We have a group, in this case D_{2n}, and using its permutation of the vertices of the n-gon, we are able to relate it to the symmetric group S_n via a group homomorphism. This is a general and important phenomenon. Whenever a group can be viewed as permuting some set around, it gives rise to a group homomorphism to the relevant group of bijections of permutations. In fact, this is one of the *key* ways of understanding abstract groups: to make them more *concrete* and *tangible* by relating them with symmetric groups.

Here, this process is allowing us view D_{2n} *inside* S_n. This leads to the following definition:

Definition 3. A subset $H \subset G$ of a group G is a **subgroup** if the following conditions hold:

(1) (Closure under operation) If $h_1, h_2 \in H$, then $h_1 * h_2 \in H$.
(2) (Identity) $e \in H$.
(3) (Closure under inverse) If $h \in H$, then $h^{-1} \in H$

We denote this by writing $H \leq G$.

Remark 1. We can replace the condition $e \in H$ with the condition that H be *non-empty*. As soon as we have *some* element $h \in H$, then the last condition tells us $h^{-1} \in H$, and the first condition tells us that $h * h^{-1} = e \in H$.

Example 5. What we saw above is that D_{2n} can be realized as a subgroup $D_{2n} \leq S_n$. However, this depends on a *choice* of labeling of the vertices of the regular n-gon, because the realization is obtained by viewing elements of D_{2n} as permutation of the *labeled* vertices.