GENERALIZED FONTAINE-LAFFAILLE THEORY

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Abstract. We prove a version of Fontaine-Laffaille theory for arbitrary p-adic formal schemes. That is, we show that prismatic F-gauges over such formal schemes with Hodge-Tate weights in $\{0, \dots, p-2\}$ can be described in terms of modules over p-completed Hodge filtered derived de Rham cohomology with additional structure. This generalizes results of Terentiuk-Vologodsky-Xu, who considered the case of the formal scheme $\operatorname{Spf} W(\kappa)$ for a perfect field κ. We give applications to the classification of p-adic local systems over p-adic formal schemes. The proofs use descent to reduce to the case of semiperfect animated commutative $\mathbb{F}_p$-algebras, where we use frame theoretic methods appearing in the work of Gardner-Madapusi. The key computation is that of the syntomic cohomology of $\mathbb{F}_p/^{\mathbb{L}}p$.

Contents

1. Introduction

The goal of this article is to obtain a generalization of Fontaine-Laffaille theory over arbitrary p-adic formal schemes.

1.1. **Fontaine-Laffaille theory for smooth formal schemes.** We begin with the most concrete new consequence of our main result. Here is the setup: Let K be a complete discrete valuation field in mixed characteristic $(0,p)$ and with perfect residue field κ. Set $W = W(\kappa)$, let $\pi \in K$ be a uniformizer, and let $E(u) \in W[u]$ be the Eisenstein polynomial for π. This allows us to view $\mathscr{O}_K$ as the quotient $W[u]/(E(u))$. Let R be a p-completely smooth algebra over the ring of integers $\mathscr{O}_K$ of a finite extension $K/\mathbb{Q}_p$, and let $\tilde{R}$ be a smooth $W[u]$-algebra lifting R. Write S_R for the p-completed divided power envelope of $\tilde{R} \to R$: This admits a divided power filtration $\mathrm{Fil}^\bullet_{\mathrm{PD}} S_R$, where $\mathrm{Fil}^i_{\mathrm{PD}} S_R \subset S_R$ is the p-completed ideal generated by the elements $\frac{E(u)^j}{j!}$ for $j \geq i$. Assume also that $\tilde{R}$ admits a Frobenius lift $\varphi : \tilde{R} \to \tilde{R}$ extending the Frobenius lift $u \mapsto u^p$ on $W[u]$: Since S_R can also be viewed as the p-completed divided power envelope of $\tilde{R} \to R/pR$ compatible with the standard divided powers on $(p) \subset \tilde{R}$, φ extends uniquely to an endomorphism of S_R.

Definition 1.1.1 (Strongly divisible modules)**.** A **strongly divisible module** for R (with respect to $\tilde{R} \to R$) is a tuple $(\mathrm{Fil}^\bullet \mathcal{M}, \nabla_{\mathcal{M}}, \{\varphi_{\mathcal{M}}, i\}_{0 \leq i \leq p-1})$ where:

(1) $\mathrm{Fil}^\bullet \mathcal{M}$ is a filtered

le-breuil **Theorem A** (Fontaine-Laffaille-Breuil theory)**.**

nramified **Remark 1.1.2** (The unramified case)**.**

1.2. Reinterpretation in terms of the filtered de Rham stack.

Remark 1.2.1 (Filtered crystals via the filtered de Rham stack)**.**

1.3. Generalized Fontaine-Laffaille theory. To state our main technical result, which applies in quite a bit of generality, we need to introduce the cohomological stacks of Drinfeld and Bhatt-Lurie. The

_f-l_lift **Proposition 1.3.1.** *The natural restriction functor*

$$\mathrm{QCoh}_{[0,p-1]}(R^{\mathcal{N}}) \to \mathrm{QCoh}_{[0,p-1]}((R/^{\mathbb{L}}p)^{\mathcal{N}})$$

factors through a canonical lift

$$\mathrm{QCoh}_{[0,p-1]}(R^{\mathrm{dR},+}) \to \mathrm{QCoh}_{[0,p-1]}((R/^{\mathbb{L}}p)^{\mathcal{N}})$$

of the restriction functor

$$\mathrm{QCoh}_{[0,p-1]}(R^{\mathrm{dR},+}) \to \mathrm{QCoh}((R/^{\mathbb{L}}p)^{\triangle}).$$

Given this proposition, we can now make the following definition.

laffaille **Definition 1.3.2** (Fontaine-Laffaille complexes)**.** The stable ∞-category $\mathrm{DFL}(R)$ of **Fontaine-Laffaille complexes** over R is defined as the fiber product

$$\mathrm{DFL}(R) \overset{\mathrm{defn}}{=} \mathrm{QCoh}_{[0,p-1]}(R^{\mathrm{dR},+}) \times_{\mathrm{QCoh}_{[0,p-1]}((R/^{\mathbb{L}}p)^{\mathcal{N}})} \mathrm{QCoh}_{[0,p-1]}((R/^{\mathbb{L}}p)^{\mathrm{syn}}).$$

An object here is **nilpotent** if its image in $\mathrm{QCoh}_{[0,p-1]}((R/^{\mathbb{L}}p)^{\mathrm{syn}})$ is so. Within this ∞-category, for $[a,b] \subset [0,p-1]$, we have the ∞-subcategories $\mathrm{DFL}_{[a,b]}(R)$ (resp. $\mathrm{DFL}^{\mathrm{nilp}}_{[a,b]}(R)$) spanned by the objects (resp. nilpotent objects) with Hodge-Tate weights in $[a,b]$. Write $\mathrm{FL}(R)$ (and $\mathrm{FL}_{[a,b]}(R)$, etc.) for the subcategories spanned by the objects whose restrictions over $R^{\mathrm{dR},+}$ are vector bundles.

l_complex **Remark 1.3.3.** Effectively, giving an object of $\mathrm{DFL}(R)$ amounts to giving an F-gauge over $R/^{\mathbb{L}}p$ with Hodge-Tate weights in $[0,p-1]$ along with a lift of the underlying quasicoherent sheaf over $(R/^{\mathbb{L}}p)^{\mathcal{N}}$ to a sheaf over $R^{\mathrm{dR},+}$. An alternate persepctive is as follows: Restriction along the de Rham and Hodge-Tate embeddings of $(R/^{\mathbb{L}}p)^{\triangle}$ into $(R/^{\mathbb{L}}p)^{\mathcal{N}}$ combined with the functor from Proposition 1.3.1 yields two functors from $\mathrm{QCoh}_{[0,p-1]}(R^{\mathrm{dR},+})$ to $\mathrm{QCoh}((R/^{\mathbb{L}}p)^{\triangle})$, and giving an object in $\mathrm{DFL}(R)$ amounts to specifying an object in $\mathrm{QCoh}_{[0,p-1]}(R^{\mathrm{dR},+})$ along with an isomorphism between its images under these two functors.

ect_field **Example 1.3.4** (The case of $\mathrm{Spf}\, W(\kappa)$)**.** Suppose that $R = W(\kappa)$ where κ is a perfect field. Then there are canonical isomorphisms

$$W(\kappa)^{\mathrm{dR},+} \xrightarrow{\simeq} \mathbb{A}^1/\mathbb{G}_m \times \mathrm{Spf}\, W(\kappa)\ ;\ \kappa^{\mathcal{N}} \xrightarrow{\simeq} \mathcal{R}(\mathrm{Fil}^{\bullet}_p W(\kappa)).$$

Therefore, giving an object of $\mathrm{DFL}(W(\kappa))$ amounts to specifying an F-gauge over κ, along with a refinement of the underlying p-adically filtered complex of $W(\kappa)$-modules to a filtered complex over $W(\kappa)$ supported in degrees $[-(p-1),0]$. Restricting to the subcategory $\mathrm{FL}(W(\kappa))$, we recover (up to sign) the usual Fontaine-Laffaille category of finite free modules M, equipped with a filtration $\mathrm{Fil}^{\bullet} M$ by direct summands, with $\mathrm{gr}^{-i} M = 0$ for $i \notin [0,p-1]$, and an isomorphism

$$\sum_i p^{-i}\varphi^* \mathrm{Fil}^i M \xrightarrow{\simeq} M.$$

laffaille **Theorem B** (Generalized Fontaine-Laffaille theory)**.** *Let R be a p-complete animated commutative ring. Then the functors*

$$\mathrm{QCoh}_{[0,p-2]}(R^{\mathrm{syn}}) \to \mathrm{DFL}_{[0,p-2]}(R)\ ;\ \mathrm{QCoh}^{\mathrm{nilp}}_{[0,p-1]}(R^{\mathrm{syn}}) \to \mathrm{DFL}^{\mathrm{nilp}}_{[0,p-1]}(R)$$

induced by the lift in Proposition 1.3.1 are equivalences. Equivalently, the commutative squares of ∞-categories

$$\begin{tikzcd}
\mathrm{QCoh}_{[0,p-2]}(R^{\mathrm{syn}}) \arrow[r] \arrow[d] & \mathrm{QCoh}_{[0,p-2]}(R^{\mathrm{dR},+}) \arrow[d] \\
\mathrm{QCoh}_{[0,p-2]}((R/^{\mathbb{L}}p)^{\mathrm{syn}}) \arrow[r] & \mathrm{QCoh}_{[0,p-2]}((R/^{\mathbb{L}}p)^{\mathcal{N}})
\end{tikzcd}\ ;$$

$$\begin{tikzcd}
\mathrm{QCoh}^{\mathrm{nilp}}_{[0,p-1]}(R^{\mathrm{syn}}) \arrow[r] \arrow[d] & \mathrm{QCoh}_{[0,p-1]}(R^{\mathrm{dR},+}) \arrow[d] \\
\mathrm{QCoh}^{\mathrm{nilp}}_{[0,p-1]}((R/^{\mathbb{L}}p)^{\mathrm{syn}}) \arrow[r] & \mathrm{QCoh}_{[0,p-1]}((R/^{\mathbb{L}}p)^{\mathcal{N}})
\end{tikzcd}$$

are Cartesian.

Example 1.3.5 (Terentiuk-Vologodsky-Xu)**.** Specializing to the case of Example 1.3.4, we recover the main theorem of [3].

h_messing **Remark 1.3.6** (Grothendieck-Messing theory)**.** One can show that there are canonical equivalences of categories

$$\mathrm{QCoh}_{[0,1]}(R^{\mathrm{dR},+}) \xrightarrow{\simeq} \mathrm{QCoh}_{[0,1]}(\mathbb{A}^1/\mathbb{G}_m \times \mathrm{Spf}\, R) \times_{\mathrm{Mod}_R} \mathrm{QCoh}((R/^{\mathbb{L}}p)^{\mathbb{\Delta}});$$
$$\mathrm{QCoh}_{[0,1]}((R/^{\mathbb{L}}p)^{\mathcal{N}}) \xrightarrow{\simeq} \mathrm{QCoh}_{[0,1]}(\mathbb{A}^1/\mathbb{G}_m \times \mathrm{Spec}\, R/^{\mathbb{L}}p) \times_{\mathrm{Mod}_{R/^{\mathbb{L}}p}} \mathrm{QCoh}((R/^{\mathbb{L}}p)^{\mathbb{\Delta}}).$$

In particular, if we restrict to Hodge-Tate weights $\{0,1\}$, then we obtain canonical Cartesian squares (the first for $p > 2$)

$$\begin{tikzcd}
\mathrm{QCoh}_{[0,1]}(R^{\mathrm{syn}}) \arrow[r] \arrow[d] & \mathrm{QCoh}_{[0,1]}(\mathbb{A}^1/\mathbb{G}_m \times \mathrm{Spf}\, R) \arrow[d] \\
\mathrm{QCoh}_{[0,1]}((R/^{\mathbb{L}}p)^{\mathrm{syn}}) \arrow[r] & \mathrm{QCoh}_{[0,1]}(\mathbb{A}^1/\mathbb{G}_m \times \mathrm{Spec}(R/^{\mathbb{L}}p)) \times_{\mathrm{Mod}_{R/^{\mathbb{L}}p}} \mathrm{Mod}_R
\end{tikzcd};$$

$$\begin{tikzcd}
\mathrm{QCoh}^{\mathrm{nilp}}_{[0,1]}(R^{\mathrm{syn}}) \arrow[r] \arrow[d] & \mathrm{QCoh}_{[0,1]}(\mathbb{A}^1/\mathbb{G}_m \times \mathrm{Spf}\, R) \arrow[d] \\
\mathrm{QCoh}^{\mathrm{nilp}}_{[0,1]}((R/^{\mathbb{L}}p)^{\mathrm{syn}}) \arrow[r] & \mathrm{QCoh}_{[0,1]}(\mathbb{A}^1/\mathbb{G}_m \times \mathrm{Spec}(R/^{\mathbb{L}}p)) \times_{\mathrm{Mod}_{R/^{\mathbb{L}}p}} \mathrm{Mod}_R
\end{tikzcd}$$

This is an instance of the Grothendieck-Messing theory explained in [1, §8.7, 8.8]: Lifting an F-gauge with Hodge-Tate weights in $\{0,1\}$ from $R/^{\mathbb{L}}p$ to R amounts to lifting the Hodge filtration on its de Rham realization.

1.4. Application to syntomic cohomology.

1.5. Idea of proof.

The method of proof of the main Theorem B is essentially the same as the one used to establish the version of Grothendieck-Messing theory referenced in Remark 1.3.6. We begin by using derived descent for the map $R \to R/^{\mathbb{L}}p$ to reduce to the case of $\mathbb{F}_p$-algebras.

d_descent **Theorem C** (Derived mod-p descent)**.** *The natural functors*

$$\mathrm{DFL}(R) \to \mathrm{Tot}\left(\mathrm{DFL}\left(R \otimes \mathbb{F}_p^{\otimes_{\mathbb{Z}_p}(\bullet+1)}\right)\right)\ ;\ \mathrm{QCoh}(R^{\mathrm{syn}}) \to \mathrm{Tot}\left(\mathrm{QCoh}\left((R \otimes \mathbb{F}_p^{\otimes_{\mathbb{Z}_p}(\bullet+1)})^{\mathrm{syn}}\right)\right)$$

are equivalences. In particular, if Theorem B holds for all animated commutative $\mathbb{F}_p$-algebras, then it holds in general.

Suppose that R is an $\mathbb{F}_p$-algebra. Then we have a canonical section $\mathrm{Spec}\, R \to \mathrm{Spec}\, R/^{\mathbb{L}}p$, inducing sections $R^? \to (R/^{\mathbb{L}}p)^?$ for $? = \mathrm{syn}, \mathcal{N}, \mathbb{\Delta}$. Theorem B for $\mathbb{F}_p$-algebras is now a consequence of:

:inverted **Theorem D** (Inverted Fontaine-Laffaille theory)**.** *Suppose that R is an animated commutative $\mathbb{F}_p$-algebra. Then there are canonical Cartesian squares*

$$\begin{tikzcd} \mathrm{QCoh}_{[0,p-2]}((R/^{\mathbb{L}}p)^{\mathrm{syn}}) \arrow[r] \arrow[d] & \mathrm{QCoh}_{[0,p-2]}((R/^{\mathbb{L}}p)^{\mathcal{N}}) \arrow[d] \\ \mathrm{QCoh}_{[0,p-2]}(R^{\mathrm{syn}}) \arrow[r] & \mathrm{QCoh}_{[0,p-2]}(R^{\mathrm{dR},+}) \end{tikzcd} ;$$

$$\begin{tikzcd} \mathrm{QCoh}^{\mathrm{nilp}}_{[0,p-1]}((R/^{\mathbb{L}}p)^{\mathrm{syn}}) \arrow[r] \arrow[d] & \mathrm{QCoh}_{[0,p-1]}((R/^{\mathbb{L}}p)^{\mathcal{N}}) \arrow[d] \\ \mathrm{QCoh}^{\mathrm{nilp}}_{[0,p-1]}(R^{\mathrm{syn}}) \arrow[r] & \mathrm{QCoh}_{[0,p-1]}(R^{\mathrm{dR},+}) \end{tikzcd} .$$

Here the vertical functors on the left are the restrictions along the map $R^{\mathrm{syn}} \to (R/^{\mathbb{L}}p)^{\mathrm{syn}}$, while the ones on the right are left inverses to the functor from Proposition 1.3.1.

Remark 1.5.1. The technical reason for this inversion of roles is that the proof proceeds by using quasisyntomic descent to reduce to the case of semiperfect R. Here, the map $\Delta_{R/^{\mathbb{L}}p} \to \Delta_R$ is a surjective map of animated commutative rings, while the map in the other direction is not. This surjectivity is crucial for the deformation theoretic methods alluded to in the next remark.

Remark 1.5.2 (Frames and deformation theory)**.** The main observation now is that, for semiperfect R, all the ∞-categories involved in Theorem D can be described in terms of (animated higher) frames associated with the prismatic cohomology $\Delta_{R/^{\mathbb{L}}p}$ and Δ_R, viewed as animated commutative rings equipped with the Nygaard filtration and a filtered Frobenius map. Just as it was in [1], such a description is very helpful, because the category of frames admits several operations that are not evident on the level of the cohomological stacks. We combine these operations with the systematic use of filtered deformation theory to reduce the proof of the theorem to a very concrete nilpotence statement involving the prismatic cohomology of $\mathbb{F}_p/^{\mathbb{L}}p$, which we will now formulate.

ilpotence **Theorem E** (A nilpotence result)**.** *Let $K = \mathrm{fib}(\Delta_{\mathbb{F}_p/^{\mathbb{L}}p} \to \mathbb{Z}_p)$. Then there exists a canonical lift $K \to \mathrm{Fil}^{p-1}_{\mathcal{N}} \Delta_{\mathbb{F}_p/^{\mathbb{L}}p}$ and the endomorphism of K induced by the divided Frobenius map*

$$\varphi_{p-2} : \mathrm{Fil}^{p-2}_{\mathcal{N}} \Delta_{\mathbb{F}_p/^{\mathbb{L}}p} \to \Delta_{\mathbb{F}_p/^{\mathbb{L}}p}$$

is topologically locally nilpotent.

Remark 1.5.3 (Necessity of animated commutative rings)**.** Our proof uses derived descent along the map $R \to R/^{\mathbb{L}}p$ followed by quasisyntomic descent to reduce to the case of semiperfect animated commutative $\mathbb{F}_p$-algebras. In particular, even when $R = \mathbb{Z}_p$, the method requires proving the main theorem for the genuinely *non-discrete* animated commutative $\mathbb{F}_p$-algebras $\mathbb{F}_p^{\otimes_{\mathbb{Z}_p} m}$ for $m \geq 2$

Remark 1.5.4 ($(p-2)$-bounded stacks)**.** In fact the proof method applies to give similar Cartesian squares with $\mathrm{QCoh}_{[0,p-2]}$ replaced by quite general p-adic formal stacks over R^{syn} whose tangent complexes have Hodge-Tate weights bounded above by $p-2$ (or by $p-1$ with an additional nilpotence hypothesis) in a suitable sense. When $p = 2, 3$, this is covered by the study of 1-bounded stacks in [1, §8]. Since we do not have any applications in mind for such a general result, we refrain from elaborating on it here.

ed_frames

2. A lifting result for animated frames

In this section, modeled after [1, §5], we prove an abstract frame-theoretic lifting result that will later specialize to a proof of Theorem D.

eralities

2.1. Some generalities on filtered rings and modules.

ed_stacks

2.2. p-adic animated frames and their associated stacks.

Definition 2.2.1 (p-adic animated higher frames)**.** A **p-adic animated higher frame** or simply **frame** is a tuple $\underline{A}$

In the sequel, we will

nd_f_zips

2.3. Derived windows, F-zips and F-gauges.

ef_theory

2.4. Deformation theory and a lifting result.

up:frames **Setup 2.4.1.** We will fix a map of frames $\underline{B} \to \underline{A}$. We will stipulate that the map $B \to A$ is surjective, and that there exists an integer $m \geq 1$ and a factoring of the map of animated commutative rings $B \to B/\operatorname{Fil}^m B$ through a map $A \to B/\operatorname{Fil}^m B$.

Notation 2.4.2.

g_diagram **Remark 2.4.3.** We have a canonical commuting diagram of sheaves on $(R_A)_{\text{ét}}$

$$\begin{tikzcd} \mathcal{Q}^{(m)}_{\underline{B}} \arrow[r] \arrow[d] & \mathcal{Q}^{(m)}_{\operatorname{Fil}^\bullet B} \times_{\mathcal{Q}_B} \mathcal{Q}_A \arrow[d] \\ \mathcal{Q}^{(m)}_{\underline{A}} \arrow[r] & \mathcal{Q}^{(m)}_{\operatorname{Fil}^\bullet A} \end{tikzcd}$$

The right vertical arrow here factors through the projection onto $\mathcal{Q}^{(m)}_{\operatorname{Fil}^\bullet B}$.

_varphi_m **Construction 2.4.4.** Set $\operatorname{Fil}^\bullet K = \operatorname{fib}(\operatorname{Fil}^\bullet B \to \operatorname{Fil}^\bullet A)$. Then the factoring of $B \to B/\operatorname{Fil}^m B$ via A gives a splitting $\operatorname{Fil}^m K \simeq K \oplus (K/\operatorname{Fil}^m K)[-1]$. In particular, we obtain a semilinear endomorphism

$$\dot{\varphi}_m : K \to \operatorname{Fil}^m K \xrightarrow{\varphi_m|_{\operatorname{Fil}^m K}} K.$$

_operator **Construction 2.4.5.** Suppose that we have $\mathcal{M}$ in $\mathcal{Q}^{(m)}_{\underline{B}}(R_{B'})$. We get an operator

$$\psi_{\mathcal{M}} : \varphi^* \operatorname{gr}^{-m}_{\mathrm{Hdg}} M \to \operatorname{gr}^{-m}_{\mathrm{Hdg}} M$$

nilpotent **Definition 2.4.6.** We will say that $\mathcal{M}$ is **nilpotent** if the operator $\psi_{\mathcal{M}}$ is locally nilpotent. We will write

$$\mathcal{Q}^{(m),\mathrm{nilp}}_{\underline{B}} \subset \mathcal{Q}^{(m)}_{\underline{B}}$$

for the subsheaf of ∞-categories spanned by the nilpotent objects.

t_lifting **Proposition 2.4.7.** *The commuting square in Remark 2.4.3 restricts to a Cartesian square*

$$\begin{tikzcd} \mathcal{Q}^{(m),\mathrm{nilp}}_{\underline{B}} \arrow[r] \arrow[d] & \mathcal{Q}^{(m)}_{\operatorname{Fil}^\bullet B} \times_{\mathcal{Q}_B} \mathcal{Q}_A \arrow[d] \\ \mathcal{Q}^{(m),\mathrm{nilp}}_{\underline{A}} \arrow[r] & \mathcal{Q}^{(m)}_{\operatorname{Fil}^\bullet A} \end{tikzcd}$$

If the operator $\dot{\varphi}_m$ is topologically locally nilpotent, then the original square is also Cartesian.

ic_frames **Lemma 2.4.8.** *Suppose that we have a commuting diagram of frames*

$$\begin{tikzcd} \underline{B} \arrow[r] \arrow[d] & \underline{A} \arrow[d] \\ \underline{D} \arrow[r] & \underline{C} \end{tikzcd}$$

satisfying the following properties:

(1) *The horizontal arrows satisfy the hypotheses of Setup 2.4.1.*
(2) *The factorings of* $D \to D/\operatorname{Fil}^m D$ *through* C *is compatible with that of* $B \to B/\operatorname{Fil}^m B$ *through* A *in that we have a commuting diagram*

$$\begin{tikzcd} B \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & B/\operatorname{Fil}^m B \arrow[d] \\ D \arrow[r] & C \arrow[r] & D/\operatorname{Fil}^m D \end{tikzcd}$$

(3) *The vertical arrows are square-zero extensions. More precisely, the underlying maps* $\operatorname{Fil}^\bullet B \to \operatorname{Fil}^\bullet D$ *and* $\operatorname{Fil}^\bullet A \to \operatorname{Fil}^\bullet C$ *are square-zero extensions of filtered animated commutative rings.*
(4) *Proposition 2.4.7 holds with* $\underline{B} \to \underline{A}$ *replaced by* $\underline{D} \to \underline{C}$*.*

Then Proposition 2.4.7 holds for $\underline{B} \to \underline{A}$*.*

The bulk of the rest of this subsection will be devoted to the proof of Lemma 2.4.8. For now, let us see that the lemma implies the proposition.

Proof of Proposition 2.4.7 assuming Lemma 2.4.8. Applying the lemma to the squares

$$\begin{tikzcd} \tau_{\leq (k+1)}\underline{B} \arrow[r] \arrow[d] & \tau_{\leq (k+1)}\underline{A} \arrow[d] \\ \tau_{\leq k}\underline{B} \arrow[r] & \tau_{\leq k}\underline{A} \end{tikzcd}$$

for $k \geq 0$ and using nilcompleteness for the stacks involved, we reduce to the case where $\mathcal{R}(\operatorname{Fil}^\bullet B)$ and $\mathcal{R}(\operatorname{Fil}^\bullet A)$ are classical p-complete formal stacks (with bounded p-power torsion).

Now, the images of K in B/pB and $\pi_0(R_B)/p\pi_0(R_B)$ are locally nilpotent. Therefore, if we set[1]

$$\underline{B}_m = \pi_0(B/K^m \otimes_B \underline{B}),$$

then we have

$$\mathcal{Q}^{(m)}_{\underline{B}} = \varprojlim_m \mathcal{Q}^{(m)}_{\underline{B}_m} \ ; \ \mathcal{Q}^{(m)}_{\operatorname{Fil}^\bullet B} = \varprojlim_m \mathcal{Q}^{(m)}_{\operatorname{Fil}^\bullet B_m}.$$

Therefore, by applying the lemma to the squares

$$\begin{tikzcd} \underline{B}_{m+1} \arrow[r] \arrow[d] & \underline{A} \arrow[d, equal] \\ \underline{B}_m \arrow[r] & \underline{A}, \end{tikzcd}$$

we reduce to the case $B = B_1 = A$. Here, the result holds essentially trivially. □

3. The syntomification of $\mathbb{F}_p/^{\mathbb{L}}p$

F_p_mod_p

In this section, we will give an explicit description of the syntomification of $\mathbb{F}_p/^{\mathbb{L}}p$ and use it to prove Theorems E and D.

[1] We are implicitly using the following fact: If $I, J \subset B$ are δ-ideals in the classical δ-ring B (that is, we have $\delta(I) \subset I, \delta(J) \subset J$, or equivalently the δ-structure descends to both B/I and B/J), then IJ is once again a δ-ideal. This follows easily from the defining identites $\delta(x+y) = \delta(x) + \delta(y) + \sum_{i=1}^{p-1} \frac{1}{p}\binom{p}{i} x^i y^{p-i}$ and $\delta(xy) = x^p\delta(y) + y^p\delta(x) + p\delta(x)\delta(y)$.

f_field_p

3.1. **The syntomification of $\mathbb{F}_p$.**

d_p_mod_p

3.2. **The prismatization of $\mathbb{F}_p/^{\mathbb{L}}p$.**

t_functor **Remark 3.2.1** (The Witt scheme)**.** Let $\mathbb{Z}_p\{y\}^\wedge$ be the p-completed free δ-ring in the variable y. For any p-nilpotent C, we have canonical isomorphisms

$$\begin{aligned} W(C) &\simeq \mathrm{Map}_{\mathrm{CRing}^{p\text{-comp}}}(\mathbb{Z}_p[y]^\wedge, W(C)) \\ &\simeq \mathrm{Map}_{\mathrm{CRing}_\delta}(\mathbb{Z}_p\{y\}^\wedge, W(C)) \\ &\simeq \mathrm{Map}_{\mathrm{CRing}}(\mathbb{Z}_p\{y\}^\wedge, C). \end{aligned}$$

Here, the second isomorphism is from the defining property of $\mathbb{Z}_p\{y\}^\wedge$, while the last one is from the fact that $C \mapsto W(C)$ is the right adjoint to the forgetful functor from CRing_δ to $\mathrm{CRing}^{p\text{-comp}}$. Therefore, the p-adic Witt formal scheme W is canonically isomorphic to the affine formal scheme $\mathrm{Spf}\,\mathbb{Z}_p\{y\}^\wedge$.

_Fp_mod_p **Remark 3.2.2.** Since $\mathbb{F}_p/^{\mathbb{L}}p \simeq \mathbb{F}_p \otimes_{\mathbb{Z}} \mathbb{F}_p$ as $\mathbb{F}_p$-algebras (the $\mathbb{F}_p$-algebra structure on the right is via the first factor), for any animated commutative $\mathbb{F}_p$-algebra R, we have

$$\mathrm{Map}_{\mathrm{CRing}_{\mathbb{F}_p/}}(\mathbb{F}_p/^{\mathbb{L}}p, R) \simeq \mathrm{Map}_{\mathrm{CRing}}(\mathbb{F}_p, R).$$

If we have a surjective map of animated commutative rings $S \twoheadrightarrow R$ with fiber I, then we obtain a further isomorphism

$$\mathrm{Map}_{\mathrm{CRing}_{\mathbb{F}_p/}}(\mathbb{F}_p/^{\mathbb{L}}p, R) \simeq \mathrm{fib}_p(I \to S).$$

putations **Remark 3.2.3.** Suppose that A is a δ-ring. Then, for any $a \in A$, we have

$$\delta(pa) = (1 - p^{p-1})a^p + p\delta(a) = \varphi(a) - p^{p-1}a^p \equiv \varphi(a) \pmod p.$$

For $p > 2$,

$$\delta^2(pa) = \delta(\varphi(a) - p^{p-1}a^p) \equiv \varphi(\delta(a)) - \delta(p^{p-1}a^p) \equiv \varphi(\delta(a)) - p^{p-2}\varphi(a^p) \equiv \varphi(\delta(a)) \pmod p.$$

If $p = 2$, then $\delta(-x) = -x^2 - \delta(x)$, and so

$$\delta^2(2a) = \delta(\varphi(a) - 2a^2) \equiv \varphi(\delta(a)) + \delta(-2a^2) \equiv \varphi(\delta(a)) - \varphi(a)^2 \pmod 2.$$

_fp_mod_p **Proposition 3.2.4.**

Proof. By definition, $(\mathbb{F}_p/^{\mathbb{L}}p)^{\Delta}$ represents the functor

$$C \mapsto \mathrm{Map}_{\mathrm{CRing}_{\mathbb{F}_p/}}(\mathbb{F}_p/^{\mathbb{L}}p, W(C)/^{\mathbb{L}}p) \simeq \mathrm{fib}_p(W(C) \xrightarrow{p} W(C)).$$

We have used Remark 3.2.2 here. Therefore, we have a Cartesian diagram of p-adic formal stacks

$$\begin{tikzcd} (\mathbb{F}_p/^{\mathbb{L}}p)^{\Delta} \arrow[r] \arrow[d] & W \arrow[d] \\ \mathrm{Spf}\,\mathbb{Z}_p \arrow[r, "p"] & W. \end{tikzcd}$$

By Remark 3.2.1, one sees that the right vertical map is isomorphic to the map $\mathrm{Spf}\,\mathbb{Z}_p\{y\}^\wedge \to \mathrm{Spf}\,\mathbb{Z}_p\{z\}^\wedge$ given by the ring homomorphism

$$\mathbb{Z}_p\{z\}^\wedge \xrightarrow{\delta^i(z) \mapsto \delta^i(py)} \mathbb{Z}_p\{y\}^\wedge,$$

while the bottom horizontal arrow is given by the map $\mathrm{Spf}\,\mathbb{Z}_p \to \mathrm{Spf}\,\mathbb{Z}_p\{z\}^\wedge$ associated with $\mathbb{Z}_p\{z\}^\wedge \xrightarrow{\delta^i(z) \mapsto \delta^i(p)} \mathbb{Z}_p$. We have

$$(\mathbb{F}_p/^{\mathbb{L}}p)^{\Delta} \simeq \mathrm{Spf}\left(\mathbb{Z}_p \otimes^{\mathbb{L}}_{\mathbb{Z}_p\{z\}^\wedge} \mathbb{Z}_p\{y\}^\wedge\right).$$

$\square$

F_p_mod_p

3.3. The filtered prismatization of $\mathbb{F}_p/^{\mathbb{L}}p$.

ce_result

3.4. The key nilpotence result.

le_theory

3.5. Inverted Fontaine-Laffaille theory.

Proof of Theorem D. □

4. The filtered de Rham stack and filtered crystals

4.1. The filtered de Rham stack.

Definition 4.1.1 (The filtered de Rham stack)**.** For any derived p-complete R, the associated **filtered de Rham stack** $R^{\mathrm{dR},+}$ is given by

$$R^{\mathrm{dR},+} \overset{\mathrm{defn}}{=} R^{\mathcal{N}} \times_{\mathbb{Z}_p^{\mathcal{N}}} \mathbb{A}^1/\mathbb{G}_m.$$

Concretely, for any $C \in \mathrm{CRing}^{p\text{-nilp}}$, the fiber of $R^{\mathrm{dR},+}(C)$ over a point $(L \xrightarrow{t} C)$ of $(\mathbb{A}^1/\mathbb{G}_m)(C)$ is the mapping space

$$\mathrm{Map}_{\mathrm{CRing}}(R, (\mathbb{G}_a/\mathbf{V}(L^\vee)^\sharp)(C))$$

where

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Lemma 4.1.2 (Universal property of the divided power filtration)**.** *Suppose that $A \twoheadrightarrow R$ is a surjective map of derived p-complete animated commutative rings, and let $D \twoheadrightarrow R$ be its p-completed divided power envelope, equipped with its divided power filtration $\mathrm{Fil}^\bullet_{\mathrm{PD}} D$. Then, for any p-nilpotent C, and any section $(L \xrightarrow{t} C)$ of $(\mathbb{A}^1/\mathbb{G}_m)(C)$, there is a canonical isomorphism*

$$\mathrm{Map}_{\mathrm{Fun}([1],\mathrm{CRing})}((A \twoheadrightarrow R), (C \to (\mathbb{G}_a/\mathbf{V}(L^\vee)^\sharp)(C))) \xrightarrow{\simeq} \mathrm{Map}_{\mathrm{FilCRing}}(\mathrm{Fil}^\bullet_{\mathrm{PD}} D, \mathrm{Fil}^\bullet_L C).$$

iltration

Proposition 4.1.3 (The filtered de Rham stack and divided power envelopes)**.** *With the notation of Lemma 4.1.2, there is a canonical map of $\mathbb{A}^1/\mathbb{G}_m$-stacks*

$$\mathcal{R}(\mathrm{Fil}^\bullet_{\mathrm{PD}} D) \to R^{\mathrm{dR},+}.$$

Moreover, if A is a p-completed polynomial algebra over $\mathbb{Z}_p$, then the above map is a cover in the flat topology.

Proof. The first part of the proposition is immediate from Lemma 4.1.2. For the second, suppose that we have a section $(L \to C, R \to (\mathbb{G}_a/\mathbf{V}(L^\vee)^\sharp)(C))$ of $R^{\mathrm{dR},+}(C)$. By Lemma 4.1.2 again, it is enough to know that there is a faithfully flat map $C \to C'$ and a lift

$$(A \to R) \to (C' \to (\mathbb{G}_a/\mathbf{V}(L^\vee)^\sharp)(C'))$$

of the composition $R \to (\mathbb{G}_a/\mathbf{V}(L^\vee)^\sharp)(C) \to (\mathbb{G}_a/\mathbf{V}(L^\vee)^\sharp)(C')$. This is easily done since the map $\mathbb{G}_a \to \mathbb{G}_a/\mathbf{V}(L^\vee)^\sharp$ is surjective in the flat topology. □

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Corollary 4.1.4. *In the notation of Lemma 4.1.2, suppose that A is a p-completed polynomial algebra over $\mathbb{Z}_p$. For every $m \geq 0$, let $D^{(m)}$ be the p-completed divided power envelope of the map $A^{\otimes_{\mathbb{Z}_p}(m+1)} \to R^{\otimes_{\mathbb{Z}_p}(m+1)}$. Then there are canonical equivalences*

$$\mathrm{QCoh}(R^{\mathrm{dR},+}) \xrightarrow{\simeq} \mathrm{Tot}\ \mathrm{QCoh}(\mathcal{R}(\mathrm{Fil}^\bullet_{\mathrm{PD}} D^{(\bullet)})) \xrightarrow{\simeq} \mathrm{Tot}\ \mathrm{FilMod}^{p\text{-}comp}_{\mathrm{Fil}^\bullet_{\mathrm{PD}} D^{(\bullet)}}$$

5. Fontaine-Laffaille theory

5.1. Derived Fontaine-Laffaille complexes.

Proof of Proposition 1.3.1. □

5.2. Derived descent.

Remark 5.2.1 (F-gauges via Nygaard completions)**.**

Remark 5.2.2 (Fontaine-Laffaille complexes via Hodge completions)**.**

Proof of Theorem C. Using quasisyntomic descent, we can reduce to the case where R is semiperfectoid. By Remarks 5.2.1 and 5.2.2, it is enough to show that the functors

$$\mathrm{QCoh}(R^{\Delta}) \to \mathrm{Tot}\left(\mathrm{QCoh}((R \otimes \mathbb{F}_p^{(\bullet+1)})^{\Delta})\right) \; ; \; \mathrm{QCoh}(R^{\hat{\mathcal{N}}}) \to \mathrm{Tot}\left(\mathrm{QCoh}((R \otimes \mathbb{F}_p^{(\bullet+1)})^{\hat{\mathcal{N}}})\right)$$
$$\mathrm{QCoh}(R^{\mathrm{d\hat{R}},+}) \to \mathrm{Tot}\left(\mathrm{QCoh}((R \otimes \mathbb{F}_p^{(\bullet+1)})^{\mathrm{d\hat{R}},+})\right)$$

are equivalences. For the first functor, the argument from the proof of [1, Proposition 8.8.1] applies. For the other two, we will use the fact that the maps

$$R^{\mathcal{N}}_{(t=0)} \simeq R^{\hat{\mathcal{N}}}_{(t=0)} \to R^{\hat{\mathcal{N}}} \; ; \; R^{\mathrm{dR},+}_{(t=0)} \simeq R^{\mathrm{d\hat{R}},+}_{(t=0)} \to R^{\mathrm{d\hat{R}},+}$$

satisfy descent for QCoh. Since both $R^{\hat{\mathcal{N}}}$ and $R^{\mathrm{d\hat{R}},+}$ are complete along the $t = 0$ locus, this follows from [2, Corollary 3.1.6]. It is now enough to know that the maps

$$(R/^{\mathbb{L}}p)^{\mathcal{N}}_{(t=0)} \to R^{\mathcal{N}}_{(t=0)} \; ; \; (R/^{\mathbb{L}}p)^{\mathrm{dR},+}_{(t=0)} \to R^{\mathrm{dR},+}_{(t=0)}$$

satisfy descent for QCoh. In fact, it suffices to know that $(\mathbb{F}_p^{\mathcal{N}})_{(t=0)} \to (\mathbb{Z}_p^{\mathcal{N}})_{(t=0)}$ satisfies universal descent for QCoh—that is, it satisfies descent after arbitrary base-change. We can check this after base-change along the flat cover $\mathbb{G}_a \to (\mathbb{Z}_p^{\mathcal{N}})_{(t=0)}$, where we are now looking at the map $(\mathbb{G}_a \times \mathbb{G}_a^{\sharp}) \otimes \mathbb{F}_p \to \mathbb{G}_a$, which can of course be factored as

$$(\mathbb{G}_a \times \mathbb{G}_a^{\sharp}) \otimes \mathbb{F}_p \to \mathbb{G}_a \times \mathbb{G}_a^{\sharp} \to \mathbb{G}_a.$$

The second map here is faithfully flat and so has universal descent for QCoh, while the first satisfies universal descent for QCoh by [2, Corollary 3.1.6] again. □

Proof of Theorem B. □

6. The étale realization in the Fontaine-Laffaille range

6.1. Classification of p-adic local systems.

Proof of Theorem A. □

6.2. Relative syntomic cohomology of smooth proper schemes.

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