A SYNTOMIC CHARACTERIZATION OF INTEGRAL CANONICAL MODELS OF SHIMURA VARIETIES

KEERTHI MADAPUSI AND ALEX YOUCIS

Abstract. We give a characterization—and in many cases, a new construction—of integral canonical models of Shimura varieties that uses the notion of an aperture appearing in work of the first author with Gardner on some conjectures of Drinfeld. This applies to Shimura varieties of pre-abelian type at primes of hyperspecial level, recovering and extending previous work of Kisin, Kim-Madapusi and Imai-Kato-Youcis, but also to exceptional Shimura varieties for large enough primes. The characterization in the exceptional case is *a priori* different from the one recently shown by Bakker-Shankar-Tsimerman, and recovers many of their results, such as the existence of prime-to-p Hecke operators and the non-emptiness of the μ-ordinary stratum. In fact, we show the non-emptiness of *all* possible Newton strata. Under a crystalline good reduction condition that holds when the Shimura datum is of pre-abelian type, we also prove that the syntomic realization map is surjective. An important ingredient in the proofs is a generalization of Tate's full faithfulness theorem for p-divisible groups to the context of apertures. This leads to a mapping property for the integral canonical model that characterizes maps into it from all normal, flat and excellent schemes over $\mathbb{Z}_{(p)}$.

1. Introduction

The purpose of this article is to lay the groundwork for applying the results of [24] to global questions. In doing so, we give a primarily p-adic characterization of integral canonical models of Shimura varieties at unramified primes, including for those of exceptional type considered by Bakker-Shankar-Tsimerman [4]. This characterization is *a priori* different from the ℓ-adic ones given by these authors and others, and is closely related to the ones introduced by Pappas [70] and Imai-Kato-Youcis [38]. For Shimura varieties of abelian type, this method actually gives a new *construction* that avoids the use of twisting abelian varieties and also extends to Shimura data of *pre*-abelian type.[1]

1.1. **The main results.** The key observation animating this paper is that apertures, the mixed characteristic analogues of the local shtukas of Genestier-Lafforgue,[1] introduced by Drinfeld [22] and studied in the work of the first author with Gardner [24], have very strong connections with the integral theory of Shimura varieties, and in fact characterize the 'good' models completely. This connection with Shimura varieties was already expressed by Drinfeld as an 'expectation' in [22]. Significant progress towards the realization of this expectation was made in [38], where the authors actually showed that the prismatic realization on an integral model of a Shimura variety of abelian type is sufficient to pin down the model at *finite* level, without considering the whole prime-to-p Hecke tower.

[1] Alex: Is this really the 'mixed-characterization version of local shtukas'?

Our primary aim here is to situate this result in what we believe to be the right context, and to substantially broaden the kinds of Shimura data it applies to by exploiting the geometry made available now by the theory of [24].

Setup 1.1.1. We will fix a prime p. An **unramified Shimura datum** is a triple $(G, \mathcal{G}, X)$ where (G, X) is a Shimura datum and $\mathcal{G}$ is a reductive model for G over $\mathbb{Z}_p$. We will also set $K_p = \mathcal{G}(\mathbb{Z}_p)$, and only consider level subgroups $K \subset G(\mathbb{A}_f)$ of the form K_pK^p for some neat compact open subgroup $K^p \subset G(\mathbb{A}_f^p)$. We will call the tuple $(G, \mathcal{G}, X, K)$ and **unramified Shimura tuple**. We will also fix a

[1] Pre-abelian Shimura data have a simple type-theoretic characterization: the adjoint datum is of type $A, B, C, D^{\mathbb{R}}$, or $D^{\mathbb{H}}$. The first non-trivial example of one that is not of abelian type is a type $D^{\mathbb{H}}$ Shimura datum with simply connected derived subgroup.

place $v|p$ of the reflex field E of (G,X). Over the associated Shimura variety Sh_K, we have a canonical $\mathcal{G}^c(\mathbb{Z}_p)$-local system $\mathbf{Et}_{K,p}$, where $\mathcal{G}^c$ is the so-called cuspidal quotient of $\mathcal{G}$ (see § 6.1). For $\ell \neq p$, we also have K_ℓ^c-local systems $\mathbf{Et}_{K,\ell}$, where K_ℓ^c is the image of K_ℓ in $G^c(\mathbb{Q}_\ell)$.

Remark 1.1.2. The Shimura datum gives an E_v-rational conjugacy class of cocharacters $\{\mu_v\}$. We in fact have a representative $\mu_v : \mathbb{G}_{m,\mathscr{O}_{E_v}} \to \mathcal{G}_{\mathscr{O}_{E_v}}$ for $\{\mu_v\}$. We will view μ_v as a cocharacter of $\mathcal{G}^c_{\mathscr{O}_{E_v}}$. To this we can attach the smooth algebraic formal stacks $\mathrm{BT}_n^{\mathcal{G}^c,-\mu_v}$ over $\mathscr{O}_{E_v}$ constructed in [24]. We can also consider their inverse limit $\mathrm{BT}_\infty^{\mathcal{G}^c,-\mu_v}$. These stacks generalize the formal stacks of (truncated) polarized p-divisible groups of height $2g$ in the case of the Siegel Shimura data, and give a group theoretic way of defining 'p-divisible groups with $\mathcal{G}^c$-structure' without actually requiring one to work with p-divisible groups. Moreover, they can be constructed from data that is not necessarily of 'Hodge type', where there might not be any connection with p-divisible groups at all.

Remark 1.1.3. For any p-adic formal scheme $\mathfrak{X}$, there is a canonical étale realization functor

$$T_{\text{ét}} : \mathrm{BT}_\infty^{\mathcal{G}^c,-\mu_v}(\mathfrak{X}) \to \mathrm{Loc}_{\mathcal{G}^c(\mathbb{Z}_p)}(\mathfrak{X}_\eta)$$

where the right hand side is the groupoid of $\mathcal{G}^c(\mathbb{Z}_p)$-local systems over the adic generic fiber of $\mathfrak{X}$. When (G,X) is the Siegel Shimura datum, then what we have here is simply the functor taking a polarized p-divisible group to its Tate module viewed as a symplectic space.

Definition 1.1.4 (Crystalline integral canonical models)**.** An integral model $\mathscr{S}_K$ for Sh_K over $\mathscr{O}_{E,(v)}$ with v-adic formal completion $\widehat{\mathscr{S}_K}$ is a **crystalline integral canonical model** (Cr-**ICM** for short) if the following conditions hold:

(1) (Pointwise criterion) For any mixed characteristic $(0,p)$ complete discrete valuation field F over $\mathscr{O}_{E_v}$ with perfect residue field, and $x \in \mathrm{Sh}_K(F)^2$, the following are equivalent:
  (a) $x \in \mathscr{S}_K(\mathscr{O}_F)$;
  (b) $\mathbf{Et}_{K,p,x}$ is crystalline;
  (c) $\mathbf{Et}_{K,p,x}$ is potentially crystalline.

(2) (Serre-Tate property) There exists a formally étale map of p-adic formal stacks over Spf $\mathscr{O}_{E,v}$

$$\varpi : \widehat{\mathscr{S}_K} \to \mathrm{BT}_\infty^{\mathcal{G}^c,-\mu_v}$$

such that the associated étale realization over the adic generic fiber is isomorphic to the restriction of $\mathbf{Et}_{K,p}$.

_canonical **Definition 1.1.5** (Étale integral canonical models)**.** An integral model $\mathscr{S}_K$ for Sh_K over $\mathscr{O}_{E,(v)}$ is an **étale integral canonical model** (or Ét-**ICM** for short) if it satisfies the same Serre-Tate property as a Cr-ICM, but the pointwise condition is now replaced by: For any mixed characteristic $(0,p)$ complete discrete valuation field F over $\mathscr{O}_{E_v}$ and every $x \in \mathrm{Sh}_K(F)$, the following conditions are equivalent:

(1) $x \in \mathscr{S}_K(\mathscr{O}_F)$.
(2) $\mathbf{Et}_{K,\ell,x}$ is an unramified K_ℓ^c-local system for all $\ell \neq p$;
(3) $\mathbf{Et}_{K,\ell,x}$ is a potentially unramified K_ℓ^c-local system for all $\ell \neq p$;

We will call $\mathscr{S}_K$ an **integral canonical model**—or ICM—if it is either a crystalline or an étale integral canonical model.

hodge_type **Remark 1.1.6.** When (G,X) is of Hodge type, one can view $\mathrm{BT}_\infty^{\mathcal{G}^c,-\mu_v}$ as parameterizing polarized p-divisible groups with certain additional structure, and the map ϖ carries a polarized abelian variety with additional structure to the corresponding p-divisible group version. In this case, condition (1) in both definitions is just saying that $\mathscr{S}_K(\mathscr{O}_F)$ consists of the locus in $\mathrm{Sh}_K(F)$ parameterizing abelian varieties with good reduction. In particular, an Ét-ICM is the same as a Cr-ICM in this case. This

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[2]Here and elsewhere, we will write $\mathrm{Sh}_K(F)$ for what should properly be written as $(\mathrm{Sh}_K \otimes_E E_v)(F)$.

last assertion holds in general for Shimura varieties of pre-abelian type, and we expect that this is true in general as well.

We can now state the main results of this paper.

Theorem A (Uniqueness of integral canonical models, Corollaries 6.4.4 and 6.4.7)**.** *An integral canonical model for $\mathscr{S}_K$, if it exists, is determined uniquely up to isomorphism. Moreover, it is functorial in the tuple $(G, \mathcal{G}, X, K)$.*

Theorem B (Néronian property for proper canonical models, Corollary 6.9.2)**.** *Suppose that $\mathscr{S}_K$ is a proper integral model for* Sh_K*. Then $\mathscr{S}_K$ is a* Cr-*ICM for* Sh_K *if and only if it is an* Ét-*ICM.*[3] *Moreover, in this case, if $p > 3$, then it is a* Néron model *for* Sh_K *in the sense of* [11, Definition 1]*: For any smooth separated $\mathscr{O}_{E,(v)}$-scheme S, every map $S[1/p] \to \mathrm{Sh}_K$ extends to a map $S \to \mathscr{S}_K$.*

Theorem C (Insensitivity to central extensions, Theorem 6.7.2)**.** *Let $(G, \mathcal{G}, X, K) \to (\overline{G}, \overline{\mathcal{G}}, \overline{X}, \overline{K})$ be a map of unramified Shimura tuples, where $\mathcal{G} \to \overline{\mathcal{G}}$ is a surjection with central kernel. Then* Sh_K *admits a* Cr-*ICM over $\mathscr{O}_{E,(v)}$ if and only if* $\mathrm{Sh}_{\overline{K}}$ *does so over $\mathscr{O}_{\overline{E},(\overline{v})}$ for the place $\overline{v} \mid p$ of the reflex field $\overline{E} = E(\overline{G}, \overline{X})$ lying under v.*[4]

Remark 1.1.7 (Insensitivity to prime-to-p level)**.** We can apply Theorem C—as well as the weaker version for étale integral canonical models alluded to in Footnote 4—to the case where the map $\mathcal{G} \to \overline{\mathcal{G}}$ is the identity. Here, it tells us that the existence of ICMs is insensitive to the choice of the prime-to-p level subgroup. That is if Sh_K admits an ICM over $\mathscr{O}_{E,(v)}$ for some choice of neat subgroup $K^p \subset G(\mathbb{A}_f^p)$, then it does so for all such subgroups. In the sequel, we will simply say that the triple $(G, \mathcal{G}, X)$ **admits an ICM over** $\mathscr{O}_{E,(v)}$: This means that Sh_K admits an ICM over $\mathscr{O}_{E,(v)}$ for some (hence every) choice of $K^p \subset G(\mathbb{A}_f^p)$.

Theorem D (Integral canonical models from reduction of structure group, Theorem 6.7.1)**.** *Suppose that $(G, \mathcal{G}, X) \to (G^\sharp, \mathcal{G}^\sharp, X^\sharp)$ is a map of unramified Shimura data with $\mathcal{G} \to \mathcal{G}^\sharp$ a closed immersion of reductive group schemes. If $(G^\sharp, \mathcal{G}^\sharp, X^\sharp)$ admits an ICM (for the place $v^\sharp \mid p$ of the reflex field $E^\sharp \subset E$ lying under v), then so does $(G, \mathcal{G}, X)$.*

The next result is originally due to Imai-Kato-Youcis [38, Theorem 3.34] when $p > 2$ and the Shimura variety is of abelian type.

Theorem E (Pre-abelian type canonical models, Theorem 6.8.3)**.** *Suppose that $(G, \mathcal{G}, X)$ is of pre-abelian type. Then a* Cr-*ICM exists over $\mathscr{O}_{E,(v)}$ and this is also an* Ét-*ICM. In fact, if $(G, \mathcal{G}, X)$ is of abelian type, then the models constructed by Kisin* [44] *(and Kim-Madapusi* [42] *for $p = 2$) are integral canonical models in the sense used here.*

Finally, we have the following version of the theorem of Bakker-Shankar-Tsimerman [4]:

Theorem F (Exceptional integral canonical models)**.** *For any unramified Shimura datum $(G, \mathcal{G}, X)$, and all p large enough, an* Ét-*ICM exists over $\mathscr{O}_{E,(v)}$ for any $v|p$. If $G/Z(G)$ is anisotropic, then this model is proper and so is a* Cr-*ICM.*

Let us now record some remarks about the definitions and results above.

Remark 1.1.8 (Comparison with existing notions of canonicity)**.** This notion of canonicity is *different* from the ones appearing in [68], [44] or [4], which only use ℓ-adic realizations for $\ell \neq p$ to characterize the model instead. Also, the characterizations appearing in the first two of these, whose origins are in work of Milne, are not of models at any finite level, but of the whole prime-to-p Hecke tower. As such, they are slightly tricky to formulate and check. The notion of a Cr-ICM here was introduced

[3] The content of this assertion is knowing that $\mathbf{Et}_{K,\ell}$ extends to a local system over a proper Cr-ICM.

[4] One has a weaker version for Ét-canonical models. See the referenced result in the body of the paper.

by Imai-Kato-Youcis [38], and this in turn has its antecedents in the work of Pappas [70] and Pappas-Rapoport [71]. The important observation that this should give a characterization at *finite level* is found in [38].

Remark 1.1.9. Theorem A is a consequence of a more refined statement that gives a complete description of the points of a Cr-ICM valued in *any* flat normal $\mathscr{O}_{E,(v)}$-scheme of finite type. See Theorem G below.

Remark 1.1.10. The Néronian property appearing in Theorem B appears to be a new observation even for Hodge and PEL type Shimura varieties. It is a special case of a more general statement for not necessarily proper Shimura varieties; see Theorem H below. We expect that the bound in both results can be improved to $p > 2$: the larger bound we obtain here is an artifact of the method of proof.

Remark 1.1.11. A version of Theorem C appears in the work of Moonen [68] for his modified version of Milne's canonicity condition. The result here is quite a bit more general and subsumes for instance the construction in [44] of integral models of abelian type from those of Hodge type.

Remark 1.1.12. Theorem D is already known by work of Pappas–Rapoport and Daniels–van Hoften–Kim–Zhang in the context of Pappas-Rapoport canonical models, even for quasi-parahoric level structures; see [17, Theorem 4.1.8].

orem:kisin **Remark 1.1.13** (Comparison with the work of Kisin)**.** The first general construction of integral canonical models of abelian type is due to Kisin [44].[5] For Shimura varieties of Hodge type, what one will find in this paper is essentially a streamlined version of the arguments in *loc. cit.*, making use of the geometry of the stacks $\mathrm{BT}_n^{\mathcal{G},-\mu_v}$ to compress some of the p-adic Hodge theory used there. The real innovation here shows up most strikingly in the transition from models of Hodge type to those of pre-abelian type. For his constructions of models of abelian type, Kisin uses an integral extension of Deligne's construction of canonical models, and shows that this can be understood moduli-theoretically in terms of a 'twisting' operation on polarized abelian varieties. The argument one will find here is much softer, does not use any global moduli interpretation and works also for pre-abelian Shimura data. A philosophical explanation for this is that, before the introduction of $(\mathcal{G},-\mu_v)$-apertures, there was no systematic analogue of 'p-divisible groups with $\mathcal{G}$-structure' in the pre-abelian (or even *abelian*) regime. This is of course the key tool here.

m:lovering **Remark 1.1.14** (Comparison with the work of Lovering)**.** Since the characterization given here is at finite level, it works without any additional argument to give integral canonical models of pre-abelian type over $\mathscr{O}_E[1/N]$ where N is the set of primes where K is unramified. It also immediately gives a functorial way of obtaining filtered F-crystals lifting automorphic vector bundles (see Remark **??**). In particular, it recovers Lovering's results from [57, 56] and extends them to the pre-abelian context.

ato_youcis **Remark 1.1.15** (Comparison with Imai-Kato-Youcis)**.** The proof we give for Theorem E is different from that of Imai-Kato-Youcis and works also when $p = 2$. We do make use of intermediate results from [36, 37, 38]. However, the work of Imai–Kato–Youcis assumes the existence of integral models constructed by Kisin, while this article gives a construction from scratch and also works in the pre-abelian situation.

Remark 1.1.16. Given Theorem E, the content of Theorem F is of course that it applies to the so-called exceptional Shimura data that are not of pre-abelian type and hence cannot be related to the moduli of abelian varieties in any reasonably direct way. Note that the properness of the integral canonical models associated with anisotropic pre-abelian Shimura data follows from [64, Theorem 1].

[5]There is earlier work of Vasiu [87]. Note, however, that despite the title of that paper, no actual construction of models of *pre*-abelian type appears within it.

comparison **Remark 1.1.17** (Comparison with Bakker–Shankar–Tsimerman)**.** The idea that one should be able to provably characterize integral canonical models of *all* Shimura varieties, including those not of abelian type, first appears in the already mentioned recent work of Bakker–Shankar–Tsimerman. Their version of the mapping property applies to smooth inputs admitting a log smooth compactification, and uses the prime-to-p étale realizations. The one in Theorem F primarily uses the p-adic or syntomic realization instead, and as we will see below yields a mapping property for all flat normal schemes of finite type over $\mathscr{O}_{E,(v)}$.

However, the results of Bakker–Shankar–Tsimerman extend well beyond the ambit of Shimura varieties, and apply to integral models of *non-minuscule* variations of Hodge structure. This is in some sense the main philosophical difference between these works: Here, we make full use of the minusculeness (though in the sense of G-bundles, not just vector bundles) of the family of realizations over a Shimura variety, which yields stronger results about the global geometry of the integral models (see for instance Remark 1.3.9 below). But these methods say nothing about what happens beyond the minuscule regime.

The properties of the integral models we now need to establish Theorem F are:

(1) The *pointwise* crystallinity of $\mathbf{Et}_p$: this is one of the key results from the work of Pila-Shankar-Tsimerman-Esnault-Groechnig on the André-Oort conjecture [73], and is required to check condition (2) for canonicity.
(2) A Neròn-Ogg-Shafarevich type good reduction criterion that combines work of Klevdal–Patrikis [49] with a short argument appearing in [73]. This goes into checking condition (1).
(3) Versality of the Kodaira-Spencer map, which holds generically and therefore for large enough primes. This is logically weaker than the ampleness condition for the Griffiths bundle imposed by the authors of [4].

In summary, the theorem depends quite seriously on results from [73] and [49], but is logically independent of [4]. Moreover, because of point (3), it shows canonicity for an *a priori* larger set of primes.

Remark 1.1.18 (Crystalline canonicity of exceptional models?)**.** We do not know if one can upgrade the conclusion of Theorem F to give the existence of a Cr-ICM in general. The issue is that we do not know yet if the potentially crystalline points of the log smooth compactifications considered by Bakker–Shankar–Tsimerman in [4] all lie in the good reduction locus. This would follow from a crystalline (or semi-stable) compatibility result at the boundary along the lines of what is shown for the interior in [72] and [35].

orem:hecke **Remark 1.1.19** (Hecke correspondences and semisimplicity)**.** Remark 1.1.7, Theorem A, and Theorem F together give us the existence of ℓ-Hecke correspondences on integral models for large enough primes $p \neq \ell$. This observation already appears in [4], whose authors make the remarkable additional observation that this is sufficient to extend Tate's argument for semisimplicity to the exceptional context.

1.2. **Mapping properties of canonical models.** Now, we arrive at the precise mapping property that in particular implies Theorem A. It takes an especially simple form for crystalline ICMs:

g_property **Theorem G** (Mapping property of integral canonical models, Theorem 6.4.3)**.** *A* Cr-*ICM (resp. an* Ét-*ICM) has the following mapping property: Suppose that we have a normal flat excellent algebraic space $\mathcal{X}$ over $\mathscr{O}_{E,(v)}$ equipped with:*

(1) A map $f : X \overset{\text{defn}}{=} \mathcal{X}[1/p] \to \mathrm{Sh}_K$ of E-schemes (resp. such that, for all $\ell \neq p$, the restriction of $\mathbf{Et}_{K,\ell}$ over X extends to a local system over $\mathcal{X}$);
(2) A map
$$\eta : \widehat{\mathcal{X}} \to \mathrm{BT}_{\infty}^{\mathcal{G}^c, -\mu_v}$$

such that the associated $\mathcal{G}^c(\mathbb{Z}_p)$-local system over the adic generic fiber is isomorphic to the restriction of $\mathbf{Et}_{K,p}$.

Then f extends uniquely to a map $\tilde{f} : \mathcal{X} \to \mathscr{S}_K$ of $\mathscr{O}_{E,(v)}$-schemes along with an isomorphism $\varpi \circ \tilde{f} \simeq \eta$.

er_product **Remark 1.2.1** (A fiber product interpretation)**.** The[2] above mapping property for integral canonical models can be interpreted as saying that we have a fiber product diagram

[2] **Alex:** Alex: read this again, it seems slightly wrong?

$$\begin{array}{ccc} \widehat{\mathscr{S}}_{K,\eta} & \longrightarrow & \mathrm{BT}^{\mathcal{G}^c,-\mu_v}_{\infty,\eta} \\ \downarrow & & \downarrow \\ \mathrm{Sh}^{\mathrm{an}}_K & \longrightarrow & \mathrm{Loc}^{\mathrm{an}}_{\mathcal{G}^c(\mathbb{Z}_p)} \end{array}$$

of stacks on normal adic spaces over E_v. Here the bottom right is the adic moduli of $\mathcal{G}^c(\mathbb{Z}_p)$-local systems, and the bottom arrow classifies the analytic restriction of $\mathbf{Et}_{K,p}$.

_PR_models **Remark 1.2.2** (Mapping property for Pappas-Rapoport models)**.** In fact, the method of proof of Theorem G can be combined with some full faithfulness results of Pappas–Rapoport [71] to establish a more general mapping property for integral models for *parahoric* levels satisfying the axioms from *loc. cit.*; see Theorem 6.17.19.

For primes larger than 3, we have a simpler mapping property for integral canonical models with respect to smooth $\mathscr{O}_{E,(v)}$-schemes. The Nèronian property for proper models in Theorem B is a consequence of this.

d_neronian **Theorem H** (Pointwise mapping property)**.** *Suppose that $p > 3$, and that $\mathcal{X}$ is a smooth separated $\mathscr{O}_{E,(v)}$-scheme whose generic fiber X is equipped with a map $f : X \to \mathrm{Sh}_K$. Suppose that, for all finite extensions F/E_v and all $x \in \mathcal{X}(\mathscr{O}_F)$, $\mathbf{Et}_{K,p,x}$ is crystalline. Then f extends to a map $\mathcal{X} \to \mathscr{S}_K$.*

Remark 1.2.3. We can view this theorem as saying that, for $p > 3$, the formal algebraic space $\widehat{\mathscr{S}_K}$ is a Néron model for the crystalline locus in $\mathrm{Sh}^{\mathrm{an}}_K$, and that $\mathscr{S}_K$ is obtained by gluing this Néron model to Sh_K along the crystalline locus as in [2].

1.3. **Applications.**

Definition 1.3.1. We will say that two unramified Shimura data $(G, \mathcal{G}, X)$ and $(G', \mathcal{G}', X')$ are in the same **central equivalence class** if the associated adjoint unramified Shimura data are isomorphic.

Remark 1.3.2. Theorem C says that the existence of crystalline ICMs is actually a property that depends only on the central equivalence class.

Consider the following technical assumption, which holds for all unramified Shimura data of pre-abelian type and also for arbitrary Shimura data with $G/Z(G)$ anisotropic and p sufficiently large:

tification **Assumption 1.3.3.** There exists $(G', \mathcal{G}', X')$ in the central equivalence class of $(G, \mathcal{G}, X)$ and a closed immersion $(G', \mathcal{G}', X') \hookrightarrow (G^\sharp, \mathcal{G}^\sharp, X^\sharp)$ into another unramified Shimura datum such that, for all places $v^\sharp | p$ of its reflex field $E^\sharp$, $\mathrm{Sh}_{K^\sharp}$ admits a Cr-ICM $\mathscr{S}_{K^\sharp}$ over $\mathscr{O}_{E^\sharp,v^\sharp}$ equipped with a *smooth* compactification $\mathscr{S}_{K^\sharp} \hookrightarrow \overline{\mathscr{S}}_{K^\sharp}$.

The next result can be viewed as an algebraization theorem for apertures.

surjective **Theorem I** (Surjectivity of the syntomic realization)**.** *Suppose that $(G, \mathcal{G}, X)$ satisfies Assumption 1.3.3. Then it admits integral canonical models $\mathscr{S}_K$ over $\mathscr{O}_{E,(v)}$, and, for all $n \geq 1$, the smooth map of formal Artin stacks*

$$\widehat{\mathscr{S}_K} \to \mathrm{BT}_n^{\mathcal{G}^c_{\mathbb{Z}_p},-\mu_v}$$

is surjective.

rjectivity **Remark 1.3.4.** We have the following consequence that is classical and certainly known to experts, but we were not able to find it in the literature: Given a principally polarized 1-truncated Barsotti-Tate group G_1 over an algebraically closed field κ,[6] there exists a principally polarized abelian variety A over κ such that $A[p] \simeq G_1$ as principally polarized truncated BT group schemes.

Remark 1.3.5. Here is a sketch of a classical proof of the fact from the previous remark conveyed to us by Ching-Li Chai: One can show that every principally polarized 1-truncated BT group scheme can be lifted to a principally polarized p-divisible group, so it is enough to know that every such p-divisible group can be realized from an abelian variety. However, this property is an isogeny invariant, so it is enough to know that every Newton stratum of the Siegel modular variety is non-empty, and this is known; see for instance [85, p. 98].

A similar argument works for Shimura varieties for Hodge type, but one has to work quite a bit harder by combining the results of [46] and [45]. The proof given here, on the other hand, is entirely geometric and works uniformly for all Shimura varieties of pre-abelian type.

Next, we show some non-emptiness results.

ton_strata **Theorem J** (Non-emptiness of Newton strata)**.** *Let $\mathscr{S}_K$ be an integral canonical or an Ét-ICM. Then every point $x \in \mathscr{S}_K(\kappa)$ with κ an algebraic closed field over $k(v)$ yields a canonical element $[b_x] \in B(G^c, \{-\mu_v\})$, and the resulting map $\mathscr{S}_K(\kappa) \to B(G^c, \{-\mu_v\})$ is surjective.*

Here, $B(G^c, \{-\mu_v\})$ is the set defined by Kottwitz [50] and studied by Rapoport-Richartz [75].

nemptiness **Remark 1.3.6.** We actually give two different proofs of Theorem J. The first applies to integral canonical models satisfying Assumption 1.3.3, and uses Theorem I and a theorem of Wintenberger [91]. The second uses results of [46] (which also ultimately rely on the aforementioned result of Wintenberger) and applies to all integral canonical models. In fact, the latter proof only uses the following weaker property (in addition to the existence of the universal aperture): Every CM point of Sh_K valued in a finite extension F/E extends to a point of $\mathscr{S}_K(\mathscr{O}_{F,(w)})$ for $w|v$.

edahl-oort **Theorem K** (Non-emptiness of Ekedahl-Oort strata)**.** *Suppose that $\mathscr{S}_K$ is an integral canonical model. Then there exists a smooth map of stacks*

$$\mathscr{S}_{K,k(v)} \to \mathcal{G}^c\text{-}\mathrm{zip}_{-\mu_v}.$$

Here, the right hand side is the stack of $\mathcal{G}^c$-zips of type $-\mu_v$ defined in [74]*. Moreover:*

(1) The μ-ordinary stratum in $\mathscr{S}_{K,k(v)}$ is non-empty and dense.

(2) If $\mathscr{S}_K$ satisfies the hypotheses of Theorem I, then this map is also surjective: That is, all Ekedahl-Oort strata of the special fiber are non-empty.

Remark 1.3.7. Theorem K is essentially an immediate consequence of Theorem I, and the fact—shown in [24]—that there is a natural smooth surjective map of stacks $\mathrm{BT}_1^{\mathcal{G}^c,-\mu_v} \otimes \mathbb{F}_p \to \mathcal{G}^c\text{-}\mathrm{zip}_{-\mu_v}$.

ral_period **Remark 1.3.8** (Integral period morphisms)**.** The mapping criterion from Theorem G can be applied to certain moduli spaces to obtain integral extensions of period or Torelli maps to Shimura varieties, modeled on the constructions appearing for instance in [63]. In *loc. cit.* one had to use stronger regularity hypotheses on the moduli spaces, and there is also the small technical headache that at that time a mapping property was known only for the full prime-to-p Hecke tower. The additional flexibility provided by the methods of this paper make for easier formulation and proof of such results. This is explored briefly in § 6.17.

plications **Remark 1.3.9** (Other applications)**.** The existence of integral canonical models has some important consequences for the geometry and arithmetic of Shimura varieties, which will be explored in subsequent work:

[6] See the discussion in [24, §11.6] regarding some subtleties when $p = 2$.

(1) (Isogenies, p-Hecke correspondences and Igusa stacks) The first application is to the construction of p-Hecke correspondences on crystalline ICMs, and will be worked out in forthcoming work of the first author with Si Ying Lee [53]. This involves a new notion of isogeny for $(\mathcal{G}^c, -\mu_v)$-apertures that uses the Vinberg monoid and is, even in the Siegel modular case, a slightly different notion than the usual one given by isogenies for (polarized) p-divisible groups. Via this, one also obtains a general treatment of Rapoport-Zink spaces associated with all minuscule local Shimura data, including the exceptional ones, as well as a *formal* construction of the Igusa stack and the associated Cartesian diagram as in [92, 16] for all crystalline ICMs (including the exceptional cases, once those can be shown to exist). This specializes in particular to a 'pure thought' construction of a map from the relevant affine Deligne-Lusztig varieties to the mod-p fiber of the integral model. In turn, this has applications to the count of mod-p points on the integral models, and thence to the Langlands-Rapoport conjectures, along with the accompanying applications to the computation of the zeta function of Shimura varieties.
(2) (Langlands-Kottwitz–Scholze method) In [79], Scholze constructs certain functions using deformation spaces of p-divisible groups, and shows that these functions allow one to extend to deeper levels the Langlands–Kottwitz method for computing traces of the Frobenius–Hecke action at hyperspecial level for certain Shimura varieties of PEL type. Among other things, this method allowed Scholze to reprove the local Langlands conjecture for GL_n in simpler terms than that in [33]. In forthcoming work of the second author, it is shown that one can formulate such functions in complete generality, replacing deformation spaces of p-divisible groups with deformation spaces of apertures. It is further shown that the analogue of the Langlands–Kottwitz–Scholze method holds for any integral canonical model as defined here.
(3) (The structure of μ-ordinary loci and central leaves) Another consequence that is somewhat new even for Shimura varieties of Hodge type is a global view of the μ-ordinary loci of integral models, which will be explained in [62]. The corresponding locus of $\mathrm{BT}_\infty^{\mathcal{G}^c,-\mu_v}$ can be shown to be an iterated biextension of classifying stacks of p-divisible groups equipped with a canonical lift of Frobenius. This gives a global (in the sense of geometry) explanation of the fact that complete local rings at μ-ordinary points have the structure of *cascades* as defined by Moonen [69] (see also the work of Shankar-Zhou [81]), as well as of the existence of canonical lifts. In fact, we will make use of this structure in this paper to prove one direction of Theorem C. The same methods also work to systematically study central leaves in Shimura varieties.
(4) (Special cycles on Shimura varieties) In an upcoming revision of [61] by the first author, the current *ad hoc* crystalline construction of special cycles on integral models will be replaced by a more robust treatment that goes through the world of apertures, with substantially simpler and conceptually clearer arguments.

m:toroidal **Remark 1.3.10** (Extension to toroidal compactifications)**.** In ongoing work, joint with Teruhisa Koshikawa,[3] we construct logarithmic analogues of the stacks $\mathrm{BT}_n^{\mathcal{G},\mu}$ as smooth 0-dimensional formal Artin stacks fibered over the p-adic completion of Olsson's $\mathcal{L}og$ stack of fine log structures. We expect that one can use these analogues to give a definition of integral canonical models for the toroidal compactifications of Shimura varieties, with similarly robust mapping properties. This should also have the pleasant consequence of not having to specify any conditions for the set of $\mathscr{O}_F$-points, since we will be dealing with *proper* algebraic spaces. One can see the work of Inoue [39] as a first step in this direction.

[3] Alex: Knowing him, I can't imagine he'll care, but we should confirm with him it's OK to say this before we post.

1.4. **Tate full faithfulness.** A key ingredient in our proofs of the canonicity results, and especially for establishing the mapping property in Theorem G, is a generalization of Tate's full faithfulness theorem for p-divisible groups. This holds in the following setting: $\mathcal{G}$ will be a smooth affine group scheme over $\mathbb{Z}_p$ and μ will be a cocharacter of $\mathcal{G}$ defined over an unramified ring of integers $\hat{\mathcal{O}}$. We will assume that μ is 1-*bounded* in the sense of Lau [52, Definition 6.3.1]: when $\mathcal{G}$ is reductive, this is the same as being minuscule. Then we have the associated formal algebraic stacks $\mathrm{BT}_n^{\mathcal{G},\mu}$ constructed in [24, §9], and their inverse limit $\mathrm{BT}_\infty^{\mathcal{G},\mu}$.

rothm:tate **Theorem L** (Tate full faithfulness, Theorem 4.1.5)**.** *If $\mathfrak{X}$ is a normal p-completely flat formal scheme over $\hat{\mathcal{O}}$, the étale realization functor from $\mathrm{BT}_\infty^{\mathcal{G},\mu}(\mathfrak{X})$ to $\mathcal{G}(\mathbb{Z}_p)$-local systems over $\mathfrak{X}_\eta$ is fully faithful.*

Remark 1.4.1. The above result should be viewed as a prismatic analogue of [71, Theorem 2.7.7]. See Remark 2.7.9 in *loc. cit.*

The argument for Theorem L is roughly along the lines of Tate's original proof for p-divisible groups, by reducing to the case of a mixed characteristic DVR. For this, we need the full faithfulness results of Bhatt–Scholze [10] and Guo-Reinecke [31], and, crucially, because this argument involves not necessarily p-complete rings and uses the algebraic Hartog's lemma for normal rings, we need to show that $\mathrm{BT}_n^{\mathcal{G},\mu}$ can be algebraized into a stack with *affine* diagonal. This is the content of the next result.

braization **Theorem M** (Algebraization, Theorem 4.1.3)**.** *There is a unique smooth 0-dimensional algebraic stack $\mathrm{BT}_n^{\mathcal{G},\mu,\mathrm{alg}}$ over $\mathcal{O}$ with generic fiber $\mathrm{Loc}_{\mathcal{G}(\mathbb{Z}/p^n\mathbb{Z})}$ and with p-adic formal completion $\mathrm{BT}_n^{\mathcal{G},\mu}$. Moreover, the stack has* affine *diagonal.*

Remark 1.4.2. When $\mathcal{G} = \mathrm{GL}_h$ and $\mathcal{O} = \mathbb{Z}_p$, what we have here is simply the algebraic stack over $\mathbb{Z}_p$ classifying n-truncated Barsotti-Tate group schemes of height h and some fixed dimension. In this case, Theorem L recovers Tate's original result for p-divisible groups. See [24, Theorem 11.1.4, Proposition 11.9.2].

Remark 1.4.3. Somewhat suprisingly to at least one of us, while the existence of the algebraic stack is a straightforward application of Artin-Lurie representability, the affineness of the diagonal turns out to require some finer analysis, which is the content of § 4.2. Note that, in the context of truncated Barsotti-Tate groups, this affineness statement is the trivial assertion that the scheme of isomorphisms between two such group schemes is affine.

1.5. Further remarks on the proofs.

onventions

1.6. Notational conventions.

- We uniformly suppress notation indicating what ideal a completion (e.g., of a scheme or ring) is with respect to, as it will always be clear from context. The vast majority of the time this completion is the p-adic one for the relevant prime p.
- [4]

[4] **Alex:** Add something about our definitions/conventions for $B(G)$.

d_drinfeld

2. Prismatic F-gauges

2.1. The stacks of Bhatt-Lurie and Drinfeld. Canonically associated with any derived p-adic formal scheme $\mathfrak{X}$ are three derived p-adic formal stacks[7] $\mathfrak{X}^{\Delta}, \mathfrak{X}^{\mathcal{N}}, \mathfrak{X}^{\mathrm{syn}}$ (shortened to R^{Δ}, $R^{\mathcal{N}}$, and R^{syn} when $\mathfrak{X} = \mathrm{Spf}(R)$), the **prismatization**, **filtered prismatization** and **syntomification**, respectively. We will review what we'll need to know of these cohomological stacks for this paper. The reader can find more details in [21], [5], [24, §6] and [65, §3]. The first two named sources explain how to construct these stacks as classical formal prestacks, and the third extends the constructions—following Bhatt-Lurie [8]—to the animated framework.

matization **Remark 2.1.1** (Prisms and the prismatization)**.** The prismatization $\mathfrak{X}^{\Delta}$ parameterizes *Cartier-Witt divisors* on $\mathfrak{X}$. These are related to the prismatic site for $\mathfrak{X}$. Indeed, suppose that we have an object $(A, I, \mathrm{Spf}(\overline{A}) \to \mathfrak{X})$ in the absolute prismatic site for $\mathfrak{X}$. Endow A with the (p, I)-adic topology. As in [8, Construction 3.10], we find a canonical map $\iota_{(A,I)} : \mathrm{Spf}\, A \to \mathfrak{X}^{\Delta}$ taking a map $\mathrm{Spec}(B) \to \mathrm{Spf}(A)$,

[7] This means that they are étale (and in fact fpqc) sheaves on p-nilpotent derived affine schemes, i.e. derived schemes of the form $\mathrm{Spec}\, A$ where A is a p-nilpotent animated commutative ring.

where B is a p-nilpotent ring, to the Cartier-Witt divisor $I \otimes_A W(B) \to W(B)$ where $A \to W(B)$ is the unique δ-map lifting $A \to B$, and where we equip this Cariter–Witt divisor with structure map

$$\mathrm{Spec}(W(B)/^{\mathbb{L}}(I \otimes_A W(B))) \to \mathrm{Spf}(\overline{A}) \to \mathfrak{X},$$

Remark 2.1.2 (Relationship between the stacks)**.** The stack $\mathfrak{X}^{\mathcal{N}}$ is a *filtered* formal stack, meaning that it is fibered naturally over the formal Artin stack $\mathbb{A}^1/\mathbb{G}_m$ parameterizing line bundles L over p-complete rings C equipped with a cosection $L \to C$. The pre-image of the open point

$$\mathrm{Spf}\,\mathbb{Z}_p \simeq \mathbb{G}_m/\mathbb{G}_m \subset \mathbb{A}^1/\mathbb{G}_m$$

is canonically isomorphic to $\mathfrak{X}^{\mathbb{\Delta}}$, and is called the **de Rham locus** of $\mathfrak{X}^{\mathcal{N}}$ and its inclusion is denote $j_{\mathrm{dR}}\colon \mathfrak{X}^{\mathbb{\Delta}} \to \mathfrak{X}^{\mathcal{N}}$. There is also another open immersion $j_{\mathrm{HT}} : \mathfrak{X}^{\mathbb{\Delta}} \to \mathfrak{X}^{\mathcal{N}}$ called the **Hodge-Tate locus** which is physically disjoint from the de Rham locus. The stack $\mathfrak{X}^{\mathrm{syn}}$ is the coequalizer of these two open immersions and is therefore equipped with a canonical open immersion $j_{\mathbb{\Delta}} : \mathfrak{X}^{\mathbb{\Delta}} \to \mathfrak{X}^{\mathrm{syn}}$.

cohomology **Remark 2.1.3** (Nygaard filtered absolute prismatic cohomology)**.** Bhatt-Lurie show that the values of these stacks on quasiregular semiperfectoid (qrsp) inputs can be described in terms of Nygaard filtered absolute prismatic cohomology. For such rings R, their **absolute prismatic cohomology** is a classical p-complete ring $\mathbb{\Delta}_R$, which is in fact a δ-ring equipped with a canonical prism structure $(\mathbb{\Delta}_R, I_R)$: this is also the *initial* prism for R. In [7, §5.5], Bhatt and Lurie construct the **Nygaard filtration** $\mathrm{Fil}^\bullet_{\mathcal{N}}\,\mathbb{\Delta}_R$ on absolute prismatic cohomology. When R is qrsp, we have

$$\mathrm{Fil}^i_{\mathcal{N}}\,\mathbb{\Delta}_R = \{x \in \mathbb{\Delta}_R :\ \varphi(x) \in \mathrm{Fil}^i_{I_R}\,\mathbb{\Delta}_R\},$$

where $\mathrm{Fil}^\bullet_{I_R}\,\mathbb{\Delta}_R$ is the I_R-adic filtration. The Nygaard filtered prismatization $R^{\mathcal{N}}$ is now canonically isomorphic to the *formal Rees stack* $\mathcal{R}(\mathrm{Fil}^\bullet_{\mathcal{N}}\,\mathbb{\Delta}_R)$ associated with this filtration.[5] The open immersion j_{dR} amounts to 'forgetting' the filtration (that is, restriction to the open substack $\mathbb{G}_m/\mathbb{G}_m \subset \mathbb{A}^1/\mathbb{G}_m$), while j_{HT} arises from the filtered Frobenius lift

$$\mathrm{Fil}^\bullet_{\mathcal{N}}\,\mathbb{\Delta}_R \to \mathrm{Fil}^\bullet_{I_R}\,\mathbb{\Delta}_R.$$

This description can be extended to all *semiperfectoid* rings; see [24, Theorem 6.11.7].[6]

rham_point **Remark 2.1.4** (The (filtered) de Rham point)**.** There are canonical maps

$$x^{\mathcal{N}}_{\mathrm{dR}} : \mathbb{A}^1/\mathbb{G}_m \times \mathfrak{X} \to \mathfrak{X}^{\mathcal{N}}\ ;\ x_{\mathrm{dR}} : \mathfrak{X} \to \mathfrak{X}^{\mathbb{\Delta}}$$

with the second being the restriction of the first over $\mathbb{G}_m/\mathbb{G}_m$. In terms of the 'affine' description from Remark 2.1.3 for semiperfectoid rings, the first map corresponds to the canonical map of filtered rings from $\mathrm{Fil}^\bullet_{\mathcal{N}}\,\mathbb{\Delta}_R$ to R equipped with its trivial descending filtration supported in graded degree 0.[7]

Remark 2.1.5 (Functoriality and étale descent)**.** Each of the assignments $\mathfrak{X} \mapsto \mathfrak{X}^?$, for $? = \mathbb{\Delta}, \mathcal{N}, \mathrm{syn}$, is functorial in the derved formal scheme $\mathfrak{X}$, and preserves products: That is, we have canonical isomorphisms $(\mathfrak{X} \times \mathfrak{Y})^? \simeq \mathfrak{X}^? \times \mathfrak{Y}$. Furthermore, if $\mathfrak{Y} \to \mathfrak{X}$ is a p-adically étale and faithfully flat map[8], then $\mathfrak{Y}^? \to \mathfrak{X}^?$ is an étale cover of derived formal stacks (see [24, Proposition 6.12.3]).

Definition 2.1.6. A map $\mathfrak{Y} \to \mathfrak{X}$ of derived p-adic formal schemes is **quasisyntomic** if it is p-completely flat (that is, $\mathfrak{Y} \otimes^{\mathbb{L}} \mathbb{F}_p \to \mathfrak{X} \otimes^{\mathbb{L}} \mathbb{F}_p$ is flat), and if $\mathbb{L}_{\mathfrak{Y}/\mathfrak{X}}$ has p-complete Tor amplitude $[-1, 0]$: that is, $\mathbb{L}_{\mathfrak{Y}/\mathfrak{X}}/^{\mathbb{L}}p$ has Tor amplitude $[-1, 0]$ as a quasi-coherent sheaf on $\mathfrak{Y} \otimes^{\mathbb{L}} \mathbb{F}_p$. The map $\mathfrak{Y} \to \mathfrak{X}$ is a **quasisyntomic cover** if it is quasisyntomic and $\mathfrak{Y} \otimes^{\mathbb{L}} \mathbb{F}_p \to \mathfrak{X} \otimes^{\mathbb{L}} \mathbb{F}_p$ is an fpqc cover.

nt_descent **Proposition 2.1.7.** *If $\mathfrak{Y} \to \mathfrak{X}$ is a quasisyntomic cover, then $\mathfrak{Y}^{\mathcal{N}} \to \mathfrak{X}^{\mathcal{N}}$ is surjective in the p-completely flat topology. In fact, if $\mathfrak{Y} = \mathrm{Spf}\,S$ and $\mathfrak{X} = \mathrm{Spf}\,R$ with R and S semiperfectoid rings, then the map $\mathfrak{Y}^{\mathcal{N}} \to \mathfrak{X}^{\mathcal{N}}$ is faithfully flat.*

Proof. See [24, Proposition 6.12.3 and Corollary 6.12.8].[8] ☐

[8]Recall this means that the induced map $\mathfrak{Y} \otimes^{\mathbb{L}} \mathbb{F}_p \to \mathfrak{X} \otimes^{\mathbb{L}} \mathbb{F}_p$ is étale and faithfully flat.

[5] **Alex:** Perhaps we can give more setup here and a reference as the formal Rees stack is maybe not universally known, and quite important. For example, I think later we use the notation σ and τ and don't define it. I am open to a better reference, but maybe [38, §1.1.2]?

[6] **Alex:** Perhaps it could be worth remarking that one can describe the arbitrary situation as a colimit of semiperfectoid cases?

[7] **Alex:** Not strictly necessary, but I think it could be worth spelling this out a bit more, or giving a reference. For example, what you are describing is what happens on filtered rings, but you mean that this map is the associated map on formal Rees stacks of this map of filtered rings.

[8] **Alex:** You had Corollary 6.12.5, but that seems wrong?

tomic_qrsp **Remark 2.1.8.** If $\mathfrak{X} = \operatorname{Spf} R$ is affine and the structure map $\mathfrak{X} \to \operatorname{Spf}(\mathbb{Z}_p)$ is quasisyntomic, there exists an affine quasisyntomic cover $\operatorname{Spf} R_\infty = \mathfrak{X}_\infty \to \mathfrak{X}$ such that $R_\infty^{\otimes_R m}$ is qrsp and p-torsion free for all m (see [9, Lemmas 4.28 and 4.30]). Therefore, in many cases, Proposition 2.1.7 reduces questions about the stacks $\mathfrak{X}^?$ to ones about the corresponding stacks associated with (the formal spectra of) qrsp and p-torsion free rings.

d_f_gauges

2.2. **F-gauges and prismatic F-crystals.** Let $\mathfrak{X}$ be a derived p-adic formal scheme.

Definition 2.2.1 ((Prismatic) F-gauges)**.** A **vector bundle in prismatic F-gauges** or simply **vector bundle F-gauge** over $\mathfrak{X}$ is a vector bundle over $\mathfrak{X}^{\mathrm{syn}}$. We will write $\operatorname{Vect}(\mathfrak{X}^{\mathrm{syn}})$ for the category of such objects.[9]

f-crystals **Remark 2.2.2** (Prismatic F-crystals)**.** Recall from [10] the notion of *prismatic crystals* and *prismatic F-crystals* over $\mathfrak{X}$: These are objects over the absolute prismatic site $\mathfrak{X}_{\Delta}$ (shortened to R_{Δ} when $\mathfrak{X} = \operatorname{Spf}(R)$). To begin, we have the *structure sheaf* $\mathscr{O}_{\Delta}$ and a generalized Cartier divisor $\mathcal{I}_{\Delta} \to \mathscr{O}_{\Delta}$ given by the assignment

$$(\mathcal{I}_{\Delta} \to \mathscr{O}_{\Delta}) : (A, I, \operatorname{Spf}(\overline{A}) \to \mathfrak{X}) \mapsto (I \to A).$$

A **prismatic crystal** (in vector bundles) over $\mathfrak{X}$ is a vector bundle $\mathcal{E}$ over $(\mathfrak{X}_{\Delta}, \mathscr{O}_{\Delta})$, and a **prismatic F-crystal** is a pair $(\mathcal{E}, \varphi_{\mathcal{E}})$ where $\mathcal{E}$ is a prismatic crystal and

$$\varphi_{\mathcal{E}} : \varphi^*\mathcal{E}[1/\mathcal{I}_{\Delta}] \xrightarrow{\simeq} \mathcal{E}[1/\mathcal{I}_{\Delta}]$$

is an $\mathscr{O}_{\Delta}[1/\mathcal{I}_{\Delta}]$-linear isomorphism of sheaves on $\mathfrak{X}_{\Delta}$. Write $\operatorname{Vect}^{\varphi}(\mathfrak{X}_{\Delta}, \mathscr{O}_{\Delta})$ for the category of prismatic F-crystals over $\mathfrak{X}$.

f-crystals **Remark 2.2.3.** Concretely, a prismatic crystal is an assignment

$$(A, I, \operatorname{Spf}(\overline{A}) \to \mathfrak{X}) \mapsto \mathcal{E}(A, I, \operatorname{Spf}(\overline{A}) \to \mathfrak{X})$$

with the value being a finite locally free A-module, and satisfying the usual crystal property: For all maps $(A, I, \operatorname{Spf}(\overline{A}) \to \mathfrak{X}) \to (B, J, \operatorname{Spf}(\overline{B}) \to \mathfrak{X})$ in $\mathfrak{X}_{\Delta}$, we have isomorphisms[9]

$$B \otimes_A \mathcal{E}(A, I, \operatorname{Spf}(\overline{A}) \to \mathfrak{X}) \xrightarrow{\simeq} \mathcal{E}(B, J, \operatorname{Spf}(\overline{B}) \to \mathfrak{X})$$

satisfying the expected compatibility relations. Endowing this with the structure of a prismatic F-crystal now amounts to giving isomorphisms

$$A \otimes_{\varphi, A} \mathcal{E}(A, I, R \to \overline{A})[1/I] \xrightarrow{\simeq} \mathcal{E}(A, I, R \to \overline{A})[1/I]$$

that are compatible with the isomorphisms coming from the crystal property.

[9] **Alex:** Maybe the natural map is an isomorphism? The natural map being the one induced by functoriality and B-linearity.

-prismatic **Remark 2.2.4** (F-gauges and prismatic F-crystals)**.** By Remark 2.1.1, any vector bundle on $\mathfrak{X}^{\Delta}$ yields a prismatic crystal over $\mathfrak{X}$. If the vector bundle arises from a vector bundle F-gauge via pullback along $j_{\Delta} : \mathfrak{X}^{\Delta} \to \mathfrak{X}^{\mathrm{syn}}$, then $\mathcal{E}$ has a canonical structure of a prismatic F-crystal over $\mathfrak{X}$; see [5, Remark 6.3.4] and [38, Construction 1.21]. This gives rise to a natural functor $\operatorname{Vect}(\mathfrak{X}^{\mathrm{syn}}) \to \operatorname{Vect}^{\varphi}(\mathfrak{X}_{\Delta}, \mathscr{O}_{\Delta})$.

_prismatic **Proposition 2.2.5.** *Suppose that $\mathfrak{X}$ is p-quasisyntomic and flat over $\mathbb{Z}_p$. Then the natural functor*

$$\operatorname{Vect}(\mathfrak{X}^{\mathrm{syn}}) \to \operatorname{Vect}^{\varphi}(\mathfrak{X}_{\Delta}, \mathscr{O}_{\Delta})$$

is fully faithful

Proof. See [30, Corollary 3.53]. ☐

[9] In general, $\operatorname{Vect}(\mathfrak{X}^{\mathrm{syn}})$ is actually an ∞-category, but when $\mathfrak{X}$ is p-quasisyntomic and flat over $\mathbb{Z}_p$, Remark 2.1.8 combined with Remark 2.1.3 tells us that this is in fact a classical category.

ealization

2.3. **Analytic prismatic F-crystals and crystalline local systems.** We now recall some generalizations due to Du-Liu-Moon-Shimizu [23] and Guo-Reinecke [31] of the classification of Bhatt-Scholze [10] of crystalline Galois representations. For more details, see the discussion in [36, §2.3]. In this subsection, K will denote a complete mixed characteristic $(0, p)$ discrete valuation field with perfect residue field.

Remark 2.3.1 (Analytic prismatic F-crystals)**.** We will take $\mathrm{Vect}^{\mathrm{an},\varphi}(\mathfrak{X}_{\Delta}, \mathscr{O}_{\Delta})$ to be the category of *analytic prismatic F-crystals* defined in [31, §3]. Concretely, this amounts to giving essentially the same data as that of a prismatic F-crystal, explained in Remark 2.2.3, except that $\mathcal{E}(A, I, \mathrm{Spf}(\overline{A}) \to \mathfrak{X})$ is now a vector bundle over $\mathrm{Spec}(A)\backslash V(p, I)$. Lemma 3.4 of [31] ensures that this is a sensible notion.

ealization **Remark 2.3.2** (The étale realization)**.** Via [5, Constructions 6.3.1, 6.3.2] (see also [31, Construction 3.9]), we find a canonical functor

$$T_{\text{ét}} : \mathrm{Vect}^{\mathrm{an},\varphi}(\mathfrak{X}_{\Delta}, \mathscr{O}_{\Delta}) \to \mathrm{Loc}_{\mathbb{Z}_p}(\mathfrak{X}_{\eta})$$

where the right hand side is the category of proétale $\mathbb{Z}_p$-local systems on the rigid generic fiber of $\mathfrak{X}$. Since both sides satisfy quasisyntomic descent (see [31, Lemma 3.6] for descent for analytic prismatic F-crystals), it suffices to specify the equivalence for semiperfectoid $\mathrm{Spf}(R) \to \mathfrak{X}$ with R semiperfectoid. This is given as

$$\Gamma((\mathrm{Spf}\, R)_{\eta}, T_{\text{ét}}(\mathcal{E})) = \left(\mathcal{E}(\Delta_R, I_R, \mathrm{Spf}(\overline{\Delta}_R) \to \mathrm{Spf}(R) \to \mathfrak{X})[1/I_R]^{\wedge}\right)^{\varphi_{\mathcal{E}}=\mathrm{id}}.$$

Here, we are viewing $\mathcal{E}(\Delta_R, I_R, \mathrm{Spf}(\overline{\Delta}_R) \to \mathrm{Spf}(R) \to \mathfrak{X})[1/I_R]$ as a finite locally free module over $\Delta_R[1/I_R]$, and $(-)^{\wedge}$ denotes the p-adic completion.

Remark 2.3.3 (Relationship between various types of local systems)**.** In the sequel we will frequently make use of the following natural equivalences:

(1) Let X be a locally of finite type K-scheme, and $X^{\mathrm{an}} \overset{\mathrm{defn}}{=} X \times_{\mathrm{Spec}(K)} \mathrm{Spa}(K)$ its analytification. Then, there is a (bi-)exact monoidal equivalence $\mathrm{Loc}_{\mathbb{Z}_p}(X) \xrightarrow{\sim} \mathrm{Loc}_{\mathbb{Z}_p}(X^{\mathrm{an}})$ (see [36, §2.1.3] and the references therein).

(2) If (A, A^+) is a Huber pair, there is a (bi-)exact monoidal equivalence

$$\mathrm{Loc}_{\mathbb{Z}_p}(\mathrm{Spec}(A)) \xrightarrow{\sim} \mathrm{Loc}_{\mathbb{Z}_p}(\mathrm{Spa}(A, A^+)).$$

See [36, §2.1.4] and the references therein.

We will often glibly use the above identifications without further comment. Moreover, by (2) (and the independence of base points for connected spaces) there is no ambiguity in the notation $\mathrm{Loc}_{\mathbb{Z}_p}(A)$ for a Huber pair (A, A^+), and we use this notational shortcut often.

n_analytic **Proposition 2.3.4.** *Suppose that $\mathfrak{X}$ is p-quasisyntomic and flat over $\mathbb{Z}_p$. Then the natural functor $\mathrm{Vect}^{\varphi}(\mathfrak{X}_{\Delta}, \mathscr{O}_{\Delta}) \to \mathrm{Vect}^{\mathrm{an},\varphi}(\mathfrak{X}_{\Delta}, \mathscr{O}_{\Delta})$ is fully faithful.*

Proof. See [31, Proposition 3.7]. ☐

cal_system **Remark 2.3.5** (Crystalline local systems)**.** Suppose that $\mathfrak{X}$ is a *base formal $\mathscr{O}_K$-scheme* as defined in [36, §1.1] and the references therein. This implies in particular that R is p-quasisyntomic and p-torsion free. Then within $\mathrm{Loc}_{\mathbb{Z}_p}(\mathfrak{X}_{\eta})$, we have the full subcategory $\mathrm{Loc}^{\mathrm{crys}}_{\mathbb{Z}_p}(\mathfrak{X})$ of *crystalline* $\mathbb{Z}_p$-local systems; see [23, §2.2], [31, §2.4], [36, §2.3.3]. When $\mathfrak{X} = \mathrm{Spf}(\mathscr{O}_K)$, this recovers the classical notion of a Galois stable $\mathbb{Z}_p$-lattice in a crystalline $\mathbb{Q}_p$-representation of the absolute Galois group of K. Recent work of Guo-Yang [32, Theorem 1.1] shows that, when $\mathfrak{X}$ is smooth over $\mathscr{O}_K$, a $\mathbb{Z}_p$-local system over $\mathfrak{X}$ is crystalline if and only if its restriction over every classical point of $\mathfrak{X}_{\eta}$ is crystalline.

o_reinecke **Theorem 2.3.6.** *Suppose that $\mathfrak{X}$ is p-quasisyntomic and flat over $\mathbb{Z}_p$. Then:*

(1) *The étale realization functor* $T_{\text{ét}} : \mathrm{Vect}^{\mathrm{an},\varphi}(\mathfrak{X}_{\Delta}, \mathscr{O}_{\Delta}) \to \mathrm{Loc}_{\mathbb{Z}_p}(\mathfrak{X}_\eta)$ *is faithful. In particular, its restriction to* $\mathrm{Vect}(\mathfrak{X}^{\mathrm{syn}})$ *is also faithful.*

(2) *Suppose that* $\mathfrak{X}$ *is smooth over* $\mathscr{O}_K$*; or, more generally, that* $\mathfrak{X}$ *is a base formal* $\mathscr{O}_K$*-scheme. Then the functor induces a bi-exact symmetric monoidal equivalence*

$$\mathrm{Vect}^{\mathrm{an},\varphi}(\mathfrak{X}_{\Delta}, \mathscr{O}_{\Delta}) \xrightarrow{\simeq} \mathrm{Loc}^{\mathrm{crys}}_{\mathbb{Z}_p}(\mathfrak{X}_\eta).$$

In particular, its restriction to $\mathrm{Vect}(\mathfrak{X}^{\mathrm{syn}})$ *is fully faithful.*

Proof. This is essentially a translation of results of Guo-Reinecke [31] and Du-Liu-Moon-Shimizu [23]).

The first result follows from Propositions 2.2.5 and 2.3.4 combined with [31, Proposition 3.7] and the proof of [31, Lemma 4.1]. Note that the last result only claims to prove faithfulness for R p-completely smooth over $\mathscr{O}_K$. However, the proof actually shows that the functor $T_{\text{ét}}$ is faithful for any p-torsion free qrsp algebra[10], and so implies what we need by quasisyntomic descent.

The second assertion follows from [31, Theorem 4.5], [23, Theorem 3.29] and [36, Proposition 2.22]. □

oth_affine **Definition 2.3.7** (Local systems for smooth affine group schemes)**.** Suppose that $\mathcal{G}$ is a smooth affine group scheme over $\mathbb{Z}_p$ and that X is a locally Noetherian scheme or adic space. A $\mathcal{G}(\mathbb{Z}_p)$**-local system** over X is a $\mathcal{G}(\mathbb{Z}_p)$-torsor on the proétale site of X.[11] We will write $\mathrm{Loc}_{\mathcal{G}(\mathbb{Z}_p)}(X)$ for the groupoid of such local systems. If X is affine/affinoid with global sections R, we will also write $\mathrm{Loc}_{\mathcal{G}(\mathbb{Z}_p)}(R)$ instead. We will also use analogous notation for the variants where $\mathcal{G}(\mathbb{Z}_p)$ is replaced by $\mathcal{G}(\mathbb{Z}/p^n\mathbb{Z})$. See [36, §2.1] for a more detailed discussion.

_tannakian **Remark 2.3.8** (Tannakian perspective on local systems)**.** Giving a $\mathcal{G}(\mathbb{Z}_p)$-local system $\mathcal{P}$ over X is equivalent to giving an exact symmetric monoidal functor $V \mapsto (V)_{\mathcal{P}}$ from the category $\mathrm{Rep}_{\mathbb{Z}_p}(\mathcal{G})$ of algebraic representations $\mathcal{G} \to \mathrm{GL}(V)$ defined over $\mathbb{Z}_p$ to the category $\mathrm{Loc}_{\mathbb{Z}_p}(X)$. See [36, Propositions 2.3 and 2.8]. In fact, the cited result shows that giving a $\mathcal{G}(\mathbb{Z}/p^n\mathbb{Z})$-local system $\mathcal{P}_n$ over X is the same as giving an exact symmetric monoidal functor $\mathrm{Rep}_{\mathbb{Z}_p}(\mathcal{G}) \to \mathrm{Loc}_{\mathbb{Z}/p^n\mathbb{Z}}(X)$.

lline_GZ_p **Definition 2.3.9** (Crystalline $\mathcal{G}(\mathbb{Z}_p)$-local systems)**.** Let $\mathfrak{X}$ be a base formal $\mathscr{O}_K$-scheme. A $\mathcal{G}(\mathbb{Z}_p)$-local system $\mathcal{P}$ on $\mathfrak{X}_\eta$ is **crystalline** if, for every representation V in $\mathrm{Rep}_{\mathbb{Z}_p}(\mathcal{G})$, the local system $(V)_{\mathcal{P}}$ is crystalline. Write $\mathrm{Loc}^{\mathrm{crys}}_{\mathcal{G}(\mathbb{Z}_p)}(\mathfrak{X}_\eta)$ for the full subcategory of $\mathrm{Loc}_{\mathcal{G}(\mathbb{Z}_p)}(\mathfrak{X}_\eta)$ of crystalline $\mathcal{G}(\mathbb{Z}_p)$-local systems.

lline_GZ_p **Remark 2.3.10.** By [36, Proposition 2.20], to check that a $\mathcal{G}(\mathbb{Z}_p)$-local system $\mathcal{P}$ is crystalline, it is sufficient to check that $(V)_{\mathcal{P}}$ is crystalline for a single faithful representation V.

_tannakian **Remark 2.3.11** ($\mathcal{G}$-bundles in analytic prismatic F-crystals)**.** Suppose that $\mathfrak{X}$ is a base formal $\mathscr{O}_K$-scheme. Combining Remark 2.3.8 with Theorem 2.3.6, we see that the category $\mathrm{Loc}^{\mathrm{crys}}_{\mathcal{G}(\mathbb{Z}_p)}(\mathfrak{X}_\eta)$ is equivalent to the category of exact symmetric monoidal functors

$$\omega : \mathrm{Rep}_{\mathbb{Z}_p}(\mathcal{G}) \to \mathrm{Vect}^{\mathrm{an},\varphi}(\mathfrak{X}_{\Delta}, \mathscr{O}_{\Delta}).$$

o_reinecke **Corollary 2.3.12.** *Suppose that* $\mathfrak{X}$ *is* p*-quasisyntomic and flat over* $\mathbb{Z}_p$*.*

(1) *There is a natural faithful functor*

$$T_{\text{ét}} : (B\mathcal{G})(\mathfrak{X}^{\mathrm{syn}}) \to \mathrm{Loc}_{\mathcal{G}(\mathbb{Z}_p)}(\mathfrak{X}_\eta)$$

(2) *If* $\mathfrak{X}$ *is a base* $\mathscr{O}_K$*-formal scheme, then the functor is also full and takes its values in* $\mathrm{Loc}^{\mathrm{crys}}_{\mathcal{G}(\mathbb{Z}_p)}(\mathfrak{X}_\eta)$*.*

[10] The argument in this generality is actually needed to apply this lemma to the proof of Theorem 4.5 in *loc. cit.*

[11] Concretely, this amounts to giving a compatible inverse system $\mathcal{P} = \{\mathcal{P}_n\}_{n\geq 1}$ where $\mathcal{P}_n \to X$ is a finite étale torsor for the finite group $\mathcal{G}(\mathbb{Z}/p^n\mathbb{Z})$ (viewed as a locally constant sheaf), and the maps $\mathcal{P}_{n+1} \to \mathcal{P}_n$ are equivariant for the natural surjections $\mathcal{G}(\mathbb{Z}/p^{n+1}\mathbb{Z}) \to \mathcal{G}(\mathbb{Z}/p^n\mathbb{Z})$.

Proof. Under these hypotheses, $\mathfrak{X}^{\mathrm{syn}}$ is a classical formal stack. Therefore, we find from [60, Corollary 9.3.7.3] that the groupoid $(B\mathcal{G})(\mathfrak{X}^{\mathrm{syn}})$ is equivalent to the groupoid of exact symmetric monoidal functors

$$\mathrm{Rep}_{\mathbb{Z}_p}(\mathcal{G}) \to \mathrm{Vect}(\mathfrak{X}^{\mathrm{syn}}).$$

The corollary now follows from Theorem 2.3.6 and Remark 2.3.8. □

ble_groups

2.4. F-gauges and p-divisible groups.

Definition 2.4.1 (Hodge-Tate weights)**.** Let $\mathfrak{X}$ be a p-adic formal scheme. Every vector bundle F-gauge $\mathcal{V}$ over $\mathfrak{X}$ yields a graded finite locally free R-module via pullback along the composition

$$B\mathbb{G}_m \times \mathfrak{X} \to \mathbb{A}^1/\mathbb{G}_m \times \mathfrak{X} \xrightarrow{x^{\mathcal{N}}_{\mathrm{dR}}} \mathfrak{X}^{\mathcal{N}} \to \mathfrak{X}^{\mathrm{syn}}.$$

The **Hodge-Tate weights** of $\mathcal{V}$ are the graded degrees in which this graded module is non-zero.

The following result is essentially due to Anschütz-Le Bras [3] over quasisyntomic bases; the general statement here is shown in [24]. See Theorem 11.1.4 and Proposition 11.8.2 of *loc. cit.*

:dieudonne

Theorem 2.4.2 (Prismatic Dieudonné theory)**.** *Let $\mathfrak{X}$ be a derived p-adic formal scheme and let $\mathcal{BT}_p(\mathfrak{X})$ be the category of p-divisible groups over $\mathfrak{X}$, and let $\mathrm{Vect}_{[0,1]}(\mathfrak{X}^{\mathrm{syn}})$ be the category of vector bundle F-gauges over $\mathfrak{X}$ with Hodge-Tate weights in $\{0,1\}$. Then:*

(1) There is a canonical equivalence of categories

$$\mathcal{G} : \mathrm{Vect}_{[0,1]}(\mathfrak{X}^{\mathrm{syn}}) \xrightarrow{\simeq} \mathcal{BT}_p(\mathfrak{X})$$

compatible with Cartier duality.

(2) There is a natural equivalence $T_p \circ \mathcal{G} \simeq T_{\mathrm{\acute{e}t}}$, where $T_p(H)$ is the Tate module of a p-divisible group H.

ec:gmudisp

3. $(\mathcal{G}, \mu)$-APERTURES

The purpose of this section is to recall the results of [24] and to record some complements that will be useful in the sequel.

in_results

3.1. The main representability result.

This subsection is a quick review of [24, §9].

setup:g_mu

Setup 3.1.1. $\mathcal{G}$ will be a smooth connected affine group scheme over $\mathbb{Z}_p$, $\hat{\mathcal{O}}$ the ring of integers in a finite unramified extension of $\mathbb{Q}_p$, and $\mu : \mathbb{G}_{m,\hat{\mathcal{O}}} \to \mathcal{G}_{\hat{\mathcal{O}}}$ a **1-bounded** cocharacter, whose weights on $\mathrm{Lie}(G)_{\hat{\mathcal{O}}}$ via the adjoint action are bounded above by 1. Let $\mathcal{P}^-_\mu \subset \mathcal{G}$ be the smooth subgroup scheme whose Lie algebra is identified with the sum of the weight-i spaces in $\mathrm{Lie}(G)_{\hat{\mathcal{O}}}$ for $i \le 0$. When $\mathcal{G}$ is reductive, μ is minuscule and $\mathcal{P}^-_\mu$ is a parabolic subgroup associated with μ.

torsor_Qmu

Remark 3.1.2. Associated with μ is the map of classifying stacks $B\mathbb{G}_{m,\hat{\mathcal{O}}} \to B\mathcal{G}_{\hat{\mathcal{O}}}$. This classifies a $\mathcal{G}$-torsor $\mathcal{Q}_\mu$ over $B\mathbb{G}_{m,\hat{\mathcal{O}}}$.

Definition 3.1.3. For $R \in \mathrm{CRing}^{p\text{-comp}}_{\hat{\mathcal{O}}/}$, an n**-truncated** $(\mathcal{G}, \mu)$**-aperture** over R is a $\mathcal{G}$-torsor $\mathfrak{Q}$ over $R^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z}$[12] satisfying the following condition: For every geometric point $R \to \kappa$ of $\mathrm{Spf}\, R$, the restriction of $(x^{\mathcal{N}}_{dR})^*\mathfrak{Q}$ (see Remark 2.1.4) over $B\mathbb{G}_m \times \mathrm{Spec}\,\kappa$ is isomorphic to that of $\mathcal{Q}_\mu$. These organize into an ∞-groupoid $\mathrm{BT}^{\mathcal{G},\mu}_n(R)$, and the assignment $R \mapsto \mathrm{BT}^{\mathcal{G},\mu}_n(R)$ is a derived p-adic formal prestack. We will also set[10]

$$\mathrm{BT}^{\mathcal{G},\mu}_\infty = \varprojlim_n \mathrm{BT}^{\mathcal{G},\mu}_n.$$

Objects in $\mathrm{BT}^{\mathcal{G},\mu}_\infty(R)$ will be called $(\mathcal{G}, \mu)$**-apertures** over R. They classify $\mathcal{G}$-torsors on R^{syn} that are bounded by μ in the sense explained above.

[10] **Alex:** In theory you could make the convention that $p^\infty = 0$ so that you don't have to differentiate here/elsewhere.

[12]By this, we mean a map of p-adic formal stacks $R^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z} \to B\mathcal{G}$.

ass_needed **Remark 3.1.4.** Note that the isomorphism class of $\mathcal{Q}_\mu$ depends only on the conjugacy class of μ. This implies that the stacks $\mathrm{BT}_n^{\mathcal{G},\mu}$ also only depend on this conjugacy class and not on the particular choice of representative.

The next result is [24, Theorem 9.3.2].

thm:gmm **Theorem 3.1.5.** *The formal prestack* $\mathrm{BT}_n^{\mathcal{G},\mu}$ *is represented by a zero-dimensional connected quasi-compact smooth p-adic formal Artin stack over* $\hat{\mathcal{O}}$ *with affine diagonal. Moreover, for any pro-nilpotent divided power thickening* $R' \twoheadrightarrow R$ *in* $\mathrm{CRing}^{p-\mathrm{comp}}_{\hat{\mathcal{O}}/}$*, we have a Cartesian* Grothendieck-Messing *square*[11]

ing_square (3.1.5.1)
$$\begin{array}{ccc} \mathrm{BT}_n^{\mathcal{G},\mu}(R') & \longrightarrow & B\mathcal{P}_\mu^-(R'/^{\mathbb{L}}p^n) \\ \downarrow & & \downarrow \\ \mathrm{BT}_n^{\mathcal{G},\mu}(R) & \longrightarrow & B\mathcal{P}_\mu^-(R/^{\mathbb{L}}p^n)\times_{B\mathcal{G}(R/^{\mathbb{L}}p^n)} B\mathcal{G}(R'/^{\mathbb{L}}p^n) \end{array}.$$

The transition maps $\mathrm{BT}_{n+1}^{\mathcal{G},\mu} \to \mathrm{BT}_n^{\mathcal{G},\mu}$ *are smooth and surjective.*

[11] **Alex:** Is this not misleading as it hides the dependence on the PD structure.
What's misleading about it?

tion_rings **Remark 3.1.6** (Deformation rings)**.** For any point $x \in \mathrm{BT}_\infty^{\mathcal{G},\mu}(\kappa)$ valued in a perfect field κ, the deformation functor for $\mathrm{BT}_\infty^{\mathcal{G},\mu}$ at x takes discrete values and is represented by a complete local formally smooth $W(\kappa)$-algebra of relative dimension $\dim\mathcal{G} - \dim\mathcal{P}_\mu^-$. This is explained in [24, Lemma 10.2.5]. It can also be seen from the results of Ito [40], combined with *a priori* knowledge of the formal smoothness of $\mathrm{BT}_\infty^{\mathcal{G},\mu}$; see [38, Proposition 3.32]. One can give this complete local $W(\kappa)$-algebra explicit coordinates

assicality **Remark 3.1.7** (Classicality)**.** There is a somewhat interesting phenomenon here. When R is a discrete p-complete ring—even though R^{syn} is in general only a derived stack that is covered flat locally by the spectra of animated, not necessarily discrete, rings—$\mathrm{BT}_n^{\mathcal{G},\mu}(R)$ is still just a 1-groupoid (or a 1-truncated ∞-groupoid). When R/pR satisfies a mild finiteness condition, this is now explained by [65, Remark 9.9.11], which tells us that $\mathrm{BT}_n^{\mathcal{G},\mu}(R)$ can be computed using only the classical truncation of R^{syn}.

nd_windows

3.2. **Frames and windows.**

In [24, §5 and §10], we find a general discussion of frames and windows, which allow us to give more concrete descriptions of the category of $(\mathcal{G},\mu)$-apertures in certain situations. Here, we will isolate a particular instance that will prove very useful to us.

Definition 3.2.1 (Breuil-Kisin frames)**.** Let R be a p-complete ring. A **Breuil-Kisin frame** for R is a prism $\underline{A} = (A, I')$ such that A is p-complete and p-torsion free, $I' \subset A$ is an ideal and $R = A/I'$.

[12] **Alex:** Should we say this is fancy way of saying $(\mathrm{A}_{\inf} f(R), (\tilde{\xi}))$?

e_algebras **Example 3.2.2** (Frames for base $\mathscr{O}_K$-algebras)**.** Suppose that K is a complete discrete valuation field with perfect residue field κ, and set $K_0 = W(\kappa)[1/p] \subset K$. Fix a monic Eisenstein polynomial $E(u) \in W(\kappa)[u]$ such that $E(\pi) = 0$ for some uniformizer $\pi \in \mathscr{O}_K$. Suppose that R is a base $\mathscr{O}_K$-algebra in the sense of [36, §1.1]: Then $R = \mathscr{O}_K \otimes_{W(\kappa)} R_0$ where R_0 is a flat $W(\kappa)$-algebra that admits a Frobenius lift and hence a δ-structure. The prism $(\mathfrak{S}_R, (E(u)))$ where $\mathfrak{S}_R = R_0[\![u]\!]$ equipped with the δ-R_0-algebra structure satisfying $\delta(u) = 0$ is now a Breuil-Kisin frame for R.

[13] **Alex:** Are all the I vs. I' correct in this example? Also we didn't say what $\pm$ means. Maybe we should also give some context about this, as it is confusing as we talked about coming from the 'classic' story (e.g., as in Bhargav's notes). Also maybe give reference for Rees stack?

toid_frame **Example 3.2.3** (Frames for perfectoid rings)**.** If R is a perfectoid ring, then $\underline{\Delta}_R = (\Delta_R, \mathrm{Fil}^1_{\mathcal{N}}\Delta_R)$ is a Breuil-Kisin frame for R.[12]

ting_frame **Example 3.2.4** (Frames via Frobenius liftings)**.** Suppose that R is a flat $\hat{\mathcal{O}}$-algebra equipped with a Frobenius lift $\varphi : R \to R$. We can then view $(R, (p))$ as a Breuil-Kisin frame $\underline{R}$ for R/pR.

s_liftings **Remark 3.2.5** (Liftings of étale covers)**.** Suppose that $\underline{A}$ is a Breuil-Kisin frame for R. Then any p-completely étale map $R \to \tilde{R}$ lifts uniquely to a (p, I')-completely étale map $A \to \tilde{A}$ of δ-rings, and $(\tilde{A}, I'\tilde{A})$ is a Breuil-Kisin frame for $\tilde{R}$. See [24, Proposition 5.4.23].

s_bk_frame **Remark 3.2.6** (The Rees stack associated with a Breuil-Kisin frame)**.** Suppose[13] that $\underline{A}$ is a Breuil-Kisin frame for R. Set $I = \varphi^* I' \subset A$. We then have the (p, I')-complete formal Rees stack $\mathcal{R}(\mathrm{Fil}^\bullet_{I'} A)$ associated with the I'-adic filtration on A, and the Frobenius lift on A extends to a map of Rees stacks $\mathcal{R}(\mathrm{Fil}^\bullet_I A) \to \mathcal{R}(\mathrm{Fil}^\bullet_{I'} A)$. We obtain two maps $\tau, \sigma : \operatorname{Spf} A \to \mathcal{R}(\mathrm{Fil}^\bullet_{I'} A)$, where τ is the pullback of $\mathbb{G}_m/\mathbb{G}_m \to \mathbb{A}^1/\mathbb{G}_m$ and σ is obtained from the filtered Frobenius lift and the isomorphism

$$\operatorname{Spf} A \xrightarrow{\simeq} \mathcal{R}(\mathrm{Fil}^\bullet_{I,\pm} A).$$

Note also that we have a canonical map $x_{\underline{A}} : \mathbb{A}^1/\mathbb{G}_m \times \operatorname{Spf} R \to \mathcal{R}(\mathrm{Fil}^\bullet_{I'} A)$ associated with the map $\mathrm{Fil}^\bullet_{I'} A \to \mathrm{Fil}^\bullet_{\mathrm{triv}} R$ of filtered rings. Here, $\mathrm{Fil}^\bullet_{\mathrm{triv}} R$ is the trivial filtration on R supported in graded degree 0.

_G-torsors **Definition 3.2.7** ($\mathcal{G}$-torsors over the Rees stack bounded by μ)**.** A $\mathcal{G}$-torsor $\mathcal{Q}_{\mathrm{Rees}}$ over $\mathcal{R}(\mathrm{Fil}^\bullet_{I'} A)$ is **bounded by** μ if, for any map $R \to \kappa$ to an algebraically closed field κ, the restriction of $x^*_{\underline{A}} \mathcal{Q}_{\mathrm{Rees}}$ along the composition

$$B\mathbb{G}_m \times \operatorname{Spec} \kappa \to B\mathbb{G}_m \times \operatorname{Spf} R \to \mathbb{A}^1/\mathbb{G}_m \times \operatorname{Spf} R$$

is isomorphic to $\mathcal{Q}_\mu$.

oundedness **Remark 3.2.8** (Consequences of 1-boundedness)**.** Since we have assumed that μ is 1-bounded, a $\mathcal{G}$-torsor $\mathcal{Q}_{\mathrm{Rees}}$ over $\mathcal{R}(\mathrm{Fil}^\bullet_{I'} A)$ bounded by μ is actually determined up to isomorphism by the following data:

- The $\mathcal{G}$-torsor $\mathcal{Q} \overset{\mathrm{defn}}{=} \tau^* \mathcal{Q}_{\mathrm{Rees}}$ over $\operatorname{Spf} A$;
- The $\mathcal{G}$-torsor $x^*_{\underline{A}} \mathcal{Q}_{\mathrm{Rees}}$ over $\mathbb{A}^1/\mathbb{G}_m \times \operatorname{Spf} R$, which in turn is equivalent to giving a reduction of structure $Q^- \subset Q$ to a $\mathcal{P}^-_\mu$-torsor for the $\mathcal{G}$-torsor Q over $\operatorname{Spf} R$ obtained via restriction from $\mathcal{Q}$.

This can be deduced for instance from [24, Proposition 4.12.3].

ifications **Remark 3.2.9** (Description in terms of modifications)**.** As a particular consequence of the description in Remark 3.2.8, we find that we can associate with the pair $(\mathcal{Q}, Q^- \subset Q)$ the $\mathcal{G}$-torsor $\sigma^* \mathcal{Q}_{\mathrm{Rees}}$ over $\operatorname{Spf} A$. This can be obtained directly from the pair. For simplicity, we will restrict ourselves to the case where $I' = (E)$ is principal with a distinguished choice of generator E. Consider the sheaf of groups

$$H_\mu : \tilde{R} \mapsto \mathcal{G}(\tilde{A}) \times_{\mathcal{G}(\tilde{R})} \mathcal{P}^-_\mu(\tilde{R}) \subset \mathcal{G}(\tilde{A}) \overset{\mathrm{defn}}{=} (L^+\mathcal{G})(\tilde{R})$$

on the p-completely étale site of R. and note that conjugation by $\mu(E) \in \mathcal{G}(A[1/E])$ yields a map $\mathrm{int}(\mu(E)) : H_\mu \to L^+\mathcal{G}$. We can view $\mathcal{Q}$ as an $L^+\mathcal{G}$-torsor, and Q^- as yielding a reduction of structure group to an H_μ-torsor $\mathcal{Q}^- \subset \mathcal{Q}$. Pushforward of $\mathcal{Q}^-$ along $\mathrm{int}(\mu(E))$ now yields a modification $\mathcal{Q}'$ of $\mathcal{Q}$ and a further pushforward along the endomorphism of $L^+\mathcal{G}$ induced by the Frobenius lift φ on A now yields the $\mathcal{G}$-torsor $\sigma^* \mathcal{Q}_{\mathrm{Rees}}$.

_bk_frames **Definition 3.2.10** ($(\mathcal{G}, \mu)$-windows over Breuil-Kisin frames)**.** Suppose that $\underline{A}$ is a Breuil-Kisin frame for R. A $(\mathcal{G}, \mu)$**-window** over $\underline{A}$ is a $\mathcal{G}$-torsor $\mathcal{Q}$ over $\mathcal{R}(\mathrm{Fil}^\bullet_{I'} A)$ bounded by μ and equipped with an isomorphism $\alpha : \sigma^* \mathcal{Q} \xrightarrow{\simeq} \tau^* \mathcal{Q}$. Write $\mathrm{Wind}^{\mathcal{G},\mu}_{\underline{A},\infty}(R)$ for the groupoid of $(\mathcal{G}, \mu)$-windows over $\underline{A}$.

:gl_n_case **Example 3.2.11** (The case of $\mathcal{G} = \mathrm{GL}_n$)**.** Suppose that $\mathcal{G} = \mathrm{GL}_n$ and μ_d is the cocharacter splitting the 2-step filtration $0 \subset \mathbb{Z}_p^d \times \{0\} \subset \mathbb{Z}_p^d \times \mathbb{Z}_p^{n-d} = \mathbb{Z}_p^n$. Then, by [24, Proposition 5.6.8, Corollary 5.7.4] and the argument from [65, Proposition 9.5.9], there is a canonical equivalence of groupoids

$$\bigsqcup_{d \le n} \mathrm{Wind}^{\mathrm{GL}_n, \mu_d}_{\underline{A},\infty}(R) \xrightarrow{\sim} \mathrm{BK}_{\underline{A},\infty}(R)$$

where the right hand side is the groupoid of triples $(\mathcal{M}, \varphi_{\mathcal{M}}, V_{\mathcal{M}})$ where:

- $\mathcal{M}$ is a finite locally free A-module;

- $\varphi_{\mathcal{M}} : \varphi^*\mathcal{M} \to \mathcal{M}$ and $V_{\mathcal{M}} : I' \otimes_A \mathcal{M} \to \varphi^*\mathcal{M}$ such that the maps
$$V_{\mathcal{M}} \circ (1 \otimes \varphi_{\mathcal{M}}) : I' \otimes_A \varphi^*\mathcal{M} \to \varphi^*\mathcal{M}\ ;\ \varphi_{\mathcal{M}} \circ V_{\mathcal{M}} : I' \otimes_A \mathcal{M} \to \mathcal{M}$$
are the maps induced by the inclusion $I' \subset A$;

It is easy to see that in fact the datum of $V_{\mathcal{M}}$ is redundant and can be replaced with the *condition* that $\varphi_{\mathcal{M}}$ is injective with cokernel a finite locally free module over R. In the case of Example 3.2.2, this is the category of *Breuil windows* over $\mathfrak{S}_R$ introduced in [88].

_alternate **Remark 3.2.12** (Alternate description via modifications)**.** The groupoid $\mathrm{Wind}^{\mathcal{G},\mu}_{\underline{A},\infty}(R)$ can also be described as follows using Remark 3.2.9 (after having chosen a generator $E \in I'$): An object here is a triple $(\mathcal{Q}, Q^-, \alpha)$ where $\mathcal{Q}$ is a $\mathcal{G}$-torsor over $\mathrm{Spf}\, A$, $Q^- \subset Q$ is a reduction of structure group of $Q \stackrel{\text{defn}}{=} \mathcal{Q}|_{\mathrm{Spf}\, R}$ to a $\mathcal{P}^-_\mu$-torsor, and $\alpha : \varphi^*\mathcal{Q}' \xrightarrow{\simeq} \mathcal{Q}$ is an isomorphism of $\mathcal{G}$-torsors, where $\mathcal{Q}'$ is the modification of $\mathcal{Q}$ along Q^-. More explicitly, the sheaf of groupoids
$$\tilde{R} \mapsto \mathrm{Wind}^{\mathcal{G},\mu}_{\underline{\tilde{A}},\infty}(\tilde{R})$$
on the p-completely étale site over R is the quotient $[L^+\mathcal{G}/H_\mu]$ where H_μ acts on $L^+\mathcal{G}$ via the right action $g \cdot h = \tau(h)^{-1} g \varphi(\mathrm{int}(\mu(E))(h))$, where $\tau : H_\mu \to L^+\mathcal{G}$ is the tautological map.

sentations **Remark 3.2.13** (Modifications in terms of representations)**.** Suppose that we have $\Lambda \in \mathrm{Rep}_{\mathbb{Z}_p}(\mathcal{G})$ and let $\mathcal{V}$ be its twist by $\mathcal{Q}$. Then the associated twist $\mathcal{V}'$ by $\mathcal{Q}'$ can be described as follows: Let $V = R \otimes_A \mathcal{V}$. The reduction of structure Q^- yields a filtration $\mathrm{Fil}^\bullet V$. Working étale locally on R if necessary we can assume that $\mathrm{Fil}^\bullet V$ is split by a cocharacter of (a twist of) $\mathcal{G}_R$ that is geometrically conjugate to μ: This gives us a splitting $V = \oplus_i V^i$ where $\mathrm{Fil}^i V = \oplus_{j \geq i} V^j$. Lift this to a cocharacter of (a twist of) $\mathcal{G}_A$ and consider the associated splitting $\mathcal{V} = \bigoplus_i \mathcal{V}^i$. Then we have
$$\mathcal{V}' = \bigoplus_i E^{-i}\mathcal{V}^i = \sum_i E^{-i}\,\mathrm{Fil}^i \mathcal{V} \subset \mathcal{V}[E^{-1}],$$
where $\mathrm{Fil}^i \mathcal{V} = \oplus_{j \geq i} \mathcal{V}^i$. In particular, the datum of the isomorphism α yielding a window gives an isomorphism
$$\varphi^*\left(\sum_i E^{-i}\,\mathrm{Fil}^i \mathcal{V}\right) \xrightarrow{\simeq} \mathcal{V}.$$

pn_tensors **Setup 3.2.14** (Tensor packages)**.** It will be convenient to make the following choices. Let $\mathcal{G} \to \mathrm{GL}(\Lambda)$ be a faithful representation. We choose finitely many tensors $\{s_\alpha\} \subset \Lambda^\otimes$ such that $\mathcal{G}$ is their pointwise stabilizer: This is always possible by [13, Theorem 1.1].

r_packages **Remark 3.2.15** (Explicit description using tensor packages)**.** Using Remark 3.2.9, we see that giving $\mathfrak{Q}$ in $\mathrm{Wind}^{\mathcal{G},\mu}_{\underline{A},\infty}(R)$ is equivalent to the groupoid of triples $(\mathcal{Q}, Q^-, \xi)$, where:

- $\mathcal{Q}$ is a $\mathcal{G}$-torsor over A;
- $Q^- \subset Q \stackrel{\text{defn}}{=} \mathcal{Q}|_{\mathrm{Spec}\, R}$ is a reduction of structure group to a $\mathcal{P}^-_\mu$-torsor;
- ξ is an isomorphism $\varphi^*\mathcal{Q}' \xrightarrow{\simeq} \mathcal{Q}$ of $\mathcal{G}$-torsors over A, where $\mathcal{Q}'$ is the modification of $\mathcal{Q}$ along Q^-.

This can be made even more concrete using the tensor package from Setup 3.2.14.

(1) Giving $\mathcal{Q}$ is equivalent to specifying a finite locally free A-module $\mathcal{L}$[14] and tensors $\{s_{\alpha,\mathcal{L}}\} \subset \mathcal{L}^\otimes$ such that, for some p-completely étale and faithfully flat map $R \to \widetilde{R}$ with corresponding lift $A \to \widetilde{A}$, there is an isomorphism
$$\eta : \widetilde{A} \otimes_{\mathbb{Z}_p} \Lambda \to A' \otimes_A \mathcal{L}$$
carrying $\{1 \otimes s_\alpha\}$ to $\{1 \otimes s_{\alpha,\mathcal{L}}\}$.

(2) The reduction of structure group $Q^- \subset Q$ amounts to giving a filtration $\mathrm{Fil}^\bullet L$ on $L \stackrel{\text{defn}}{=} R \otimes_A \mathcal{L}$ such that for some $R \to R'$ as above the isomorphism η can be chosen so that the induced filtration on $R' \otimes_{\mathbb{Z}_p} \Lambda$ is split by μ.

[14] **Alex:** Mostly agnostic, but I do wonder if $\mathcal{L}$ (I assume for lattice) is the best choice for an arbitrary rank vector bundle.

(3) If we set $\mathcal{L}^m \subset \mathcal{L}$ to be the pre-image of $\mathrm{Fil}^m L$, the modification $\mathcal{Q}'$ now corresponds to the A-module

$$\mathcal{L}' = \sum_m E^{-m}\mathcal{L}^m \subset \mathcal{L}[1/E]$$

along with the same collection of tensors $\{s_{\alpha,\mathcal{L}}\}$, now viewed inside $\mathcal{L}'^{,\otimes}$.

(4) The isomorphism ξ corresponds to an isomorphism of A-modules

$$\varphi^*\mathcal{L}' = \sum_m \varphi(E)^{-m}\varphi^*\mathcal{L}^m \xrightarrow{\simeq} \mathcal{L}$$

carrying $\{\varphi^* s_{\alpha,\mathcal{L}}\}$ to $\{s_{\alpha,\mathcal{L}}\}$.

matization **Proposition 3.2.16** (Mapping the Rees stack to the filtered prismatization)**.** *There is a canonical map of formal stacks $\iota^{\mathcal{N}}_{\underline{A}} : \mathcal{R}(\mathrm{Fil}^\bullet_{I'} A) \to R^{\mathcal{N}}$ such that*

$$\iota^{\mathcal{N}}_{\underline{A}} \circ \tau = j_{\mathrm{dR}} \circ \iota_{(A,I)} \ ; \ \iota^{\mathcal{N}}_{\underline{A}} \circ \sigma = j_{\mathrm{HT}} \circ \iota_{(A,I)} \ ; \ \iota^{\mathcal{N}}_{\underline{A}} \circ x_{\underline{A}} = x^{\mathcal{N}}_{\mathrm{dR}}.$$

Proof. See [24, Example 6.10.5]. □

to_windows **Corollary 3.2.17** (Apertures to windows)**.** *Suppose that R is a p-complete $\hat{\mathcal{O}}$-algebra. Then there is a canonical functor*

$$\mathrm{BT}^{\mathcal{G},\mu}_\infty(R) \to \mathrm{Wind}^{\mathcal{G},\mu}_{\underline{A},\infty}(R).$$

The next result is a special case of [24, Lemma 9.2.3].

toid_rings **Proposition 3.2.18** (Description for perfectoid rings)**.** *Suppose that R is perfectoid. Then the functor*

$$\mathrm{BT}^{\mathcal{G},\mu}_\infty(R) \to \mathrm{Wind}^{\mathcal{G},\mu}_{\underline{\Delta_R},\infty}(R)$$

is an equivalence.

perfectoid **Remark 3.2.19** (Quotient description for perfectoid rings)**.** Let $\xi \in \Delta_R$ be a generator for $\mathrm{Fil}^1_{\mathcal{N}} \Delta_R$. Another way of phrasing the proposition, using Remark 3.2.12, is that the restriction of the stack $\mathrm{BT}^{\mathcal{G},\mu}_\infty$ to perfectoid R-algebras is the quotient $[L^+\mathcal{G}/H_\mu]$ (computed in the p-completely étale topology) where H_μ acts on $L^+\mathcal{G}$ via the right action $g \cdot h = \tau(h)^{-1} g \varphi(\mathrm{int}(\mu(\xi))(h))$. Here, $\tau \colon H_\mu \to L^+\mathcal{G}$ is the tautological map. In fact a similar description holds for the n-truncated versions as well: One has to now look at the étale sheafification of the assignment $R \mapsto \mathcal{G}(\Delta_R/p^n\Delta_R)/H^{(n)}_\mu(R)$, where

$$H^{(n)}_\mu(R) = \mathcal{G}(\Delta_R/p^n\Delta_R) \times_{\mathcal{G}(R/^{\mathbb{L}}p^n)} \mathcal{P}^-_\mu(R/^{\mathbb{L}}p^n).$$

When R is p-torsion free, we have

_free_desc (3.2.19.1) $$H^{(n)}_\mu(R) = \mathcal{G}(\Delta_R/p^n\Delta_R) \times_{\mathcal{G}(R/p^nR)} \mathcal{P}^-_\mu(R/p^nR) \subset \mathcal{G}(\Delta/p^n\Delta_R).$$

3.3. **Isocrystals with G-structure.**

Here, we recall the relationship between apertures and F-isocrystals with G-structure.

newton_map **Construction 3.3.1** (Apertures to isocrystals)**.** Let R be a perfect $\mathbb{F}_p$-algebra. Then by Remark 3.2.19 we have

$$\mathrm{BT}^{\mathcal{G},\mu}_\infty(R) = [L^+\mathcal{G}/H_\mu](R),$$

where H_μ is the étale sheaf on perfect $\mathbb{F}_p$-algebras given by $H_\mu = L^+\mathcal{G} \times_{\mathcal{G}} \mathcal{P}^-_\mu$. The action is via

$$\begin{aligned} L^+\mathcal{G} \times H_\mu &\to L^+\mathcal{G} \\ (g,h) &\mapsto h^{-1} g \varphi(\mu(p) h \mu(p)^{-1}). \end{aligned}$$

Set $G = \mathcal{G}_{\mathbb{Q}_p}$ and let $LG : R \mapsto G(W(R)[1/p])$ be the Witt vector loop group associated with G. Then we also have the étale quotient

$$\mathrm{Isoc}_G \overset{\mathrm{defn}}{=} [LG/_{\mathrm{Ad}_\varphi} LG]$$

associated with the φ-twisted adjoint action

$$\begin{aligned}\mathrm{Ad}_\varphi\colon LG \times LG &\to LG\\ (g,h) &\mapsto h^{-1} g\varphi(h).\end{aligned}$$

The map $L^+\mathcal{G} \xrightarrow{g \mapsto g\varphi(\mu(p))} LG$ now descends to a functorial map

$$\mathrm{BT}_\infty^{\mathcal{G},\mu}(R) \to \mathrm{Isoc}_G(R)$$

for perfect $\mathbb{F}_p$-algebras R.[15]

Remark 3.3.2. When $R = \kappa$ is an algebraically closed field, then the quotient description of $\mathrm{BT}_\infty^{\mathcal{G},\mu}(\kappa)$ appearing above shows that we have

$$\mathrm{BT}_\infty^{\mathcal{G},\mu}(\kappa) = [\mathcal{G}(W(\kappa))/H_\mu(\kappa)] \xrightarrow[\simeq]{gH_\mu(\kappa) \mapsto g\varphi(\mu(p))} [\mathcal{G}(W(\kappa))\varphi(\mu(p))\mathcal{G}(W(\kappa))/\mathcal{G}/_{\mathrm{Ad}_\varphi}\mathcal{G}(W(\kappa))].$$

The right hand side is the set $C(\mathcal{G},\{\varphi(\mu)\})$ from . The set of isomorphism classes in $\mathrm{Isoc}_G(\kappa)$ can be identified with Kottwitz's set $B(G)$, within which we have the *μ-admissible* subset $B(G,\{\mu\})$, which depends only on the Galois orbit of the conjugacy class $\{\mu\}$ of μ.

ttwitz_map **Lemma 3.3.3.** *For an algebraically closed field κ, the map $\mathrm{BT}_\infty^{\mathcal{G},\mu}(\kappa) \to B(G)$ induced from Construction 3.3.1 and then taking isomorphism classes has image $B(G,\{\mu\}) = B(G,\{\varphi(\mu)\})$.*

Proof. That the image is inside $B(G,\{\mu\})$ is just a restatement of [75, Theorem 4.2], which says that the image of $C(G,\{\varphi(\mu)\})$ in $B(G)$ lands inside $B(G,\{\varphi(\mu)\})$. To show that every element $[b] \in B(G,\varphi(\mu))$ is obtained in this way, we need a result of Wintenberger [91, Corollaire 3] (see also the more general non-emptiness result of Gashi [25, Theorem 5.2]): This tells us that there exists $g \in G(W(\kappa)[1/p])$ and $h_1, h_2 \in G(W(\kappa))$ such that $g^{-1}b\varphi(g) = h_1\varphi(\mu(p))h_2$. After replacing g with $g\varphi^{-1}(h_2)^{-1}$, we find that the class of $g^{-1}b\varphi(g) \in \mathcal{G}(W(\kappa))\varphi(\mu(p))$ is in the image of $\mathrm{BT}_\infty^{\mathcal{G},\mu}(\kappa)$, as desired. □

ealization

3.4. The de Rham realization.

Construction 3.4.1 (The Hodge-filtered de Rham realization)**.** Suppose that we have $\mathfrak{Q} \in \mathrm{BT}_\infty^{\mathcal{G},\mu}(R)$. Then pulling back along the filtered de Rham point $x_{\mathrm{dR}}^{\mathcal{N}}$ from Remark 2.1.4 gives us a $\mathcal{G}$-bundle $\mathrm{Fil}^\bullet_{\mathrm{Hdg}}\,T_{\mathrm{dR}}(\mathfrak{Q})$ over $\mathbb{A}^1/\mathbb{G}_m \times \mathrm{Spf}\,R$.

ed_de_rham **Remark 3.4.2.** It follows from the definition of $\mathrm{BT}_\infty^{\mathcal{G},\mu}(R)$ and [24, Remark 4.9.7] that $\mathrm{Fil}^\bullet_{\mathrm{Hdg}}\,T_{\mathrm{dR}}(\mathfrak{Q})$ is étale locally on $\mathrm{Spf}\,R$ isomorphic to the pullback of the $\mathcal{G}$-torsor $\mathcal{Q}_\mu$ over $B\mathbb{G}_m \times \mathrm{Spf}\,\hat{\mathcal{O}}$ from Remark 3.1.2. Moreover, *loc. cit.* also shows that giving such a $\mathcal{G}$-bundle over $\mathbb{A}^1/\mathbb{G}_m \times \mathrm{Spf}\,R$ is equivalent to giving a $\mathcal{P}_\mu^-$-torsor over R. That is, we have constructed a canonical map $\mathrm{BT}_\infty^{\mathcal{G},\mu} \to B\mathcal{P}_\mu^-$. This is precisely the one showing up in the deformation theory explained in Theorem 3.1.5.

ality_bt_n **Remark 3.4.3** (A period map and versality)**.** Here is a reformulation of the deformation theory in Theorem 3.1.5 that will be useful. Let Gr_μ be the Grassmannian scheme over $\hat{\mathcal{O}}$ associated with μ: It depends only on the conjugacy class of μ and can be presented as the fppf quotient $\mathcal{G}_{\hat{\mathcal{O}}}/\mathcal{P}_\mu^-$ for our choice of representative μ. We can view the classifying stack $B\mathcal{P}_\mu^-$ from this perspective as the fppf quotient $\mathrm{Gr}_\mu/\mathcal{G}_{\hat{\mathcal{O}}}$. Now, $\mathrm{Fil}^\bullet_{\mathrm{Hdg}}\,T_{\mathrm{dR}}(\mathfrak{Q})$ can be viewed as arising from a canonical map $\mathrm{BT}_\infty^{\mathcal{G},\mu} \to \mathrm{Gr}_\mu/\mathcal{G}_{\hat{\mathcal{O}}}$. Then Grothendieck-Messing theory tells us that the resulting map on cotangent complexes (all p-complete and over $\hat{\mathcal{O}}$)

$$\mathbb{L}^\wedge_{\mathrm{Gr}_\mu/\mathcal{G}_{\hat{\mathcal{O}}}}|_{\mathrm{BT}_\infty^{\mathcal{G},\mu}} \to \mathbb{L}^\wedge_{\mathrm{BT}_\infty^{\mathcal{G},\mu}}$$

factors through an isomorphism

$$\mathbb{L}^\wedge_{(\mathrm{Gr}_\mu/\mathcal{G}_{\hat{\mathcal{O}}})/B\mathcal{G}_{\hat{\mathcal{O}}}}|_{\mathrm{BT}_\infty^{\mathcal{G},\mu}} \xrightarrow{\sim} \mathbb{L}^\wedge_{\mathrm{BT}_\infty^{\mathcal{G},\mu}}.$$

sub:Fzips

3.5. F-zips and de Rham F-zips.

[15] **Alex:** This is quite nice, but I think it would be nice to say that the composition (broke up into pieces as this is a margin note) $\mathrm{Rep}_{\mathbb{Z}_p}(\mathcal{G}) \to \mathrm{Vect}(R^{\mathrm{syn}})$ then to $\mathrm{Vect}^\varphi(R_{\Delta}) = \mathrm{Vect}^\varphi(R_{\mathrm{crys}})$ which equals $\mathrm{Vect}^\varphi(W(R))$, you get an F-crystal with $\mathcal{G}$-structure. This functor you have written is just the group-theoretic way of writing this the associated F-isocrystal with G-structure. In fact, this is already discussed in Construction 3.6.2.

reference

socrystals

3.6. **Filtered F-crystals.** Here, we review some material on the crystalline realizations of apertures. More details from a Tannakian perspective can be found in [37].[16]

[16] Alex: I don't know if it's the Tannakian perspective—I guess you just mean the case of perfect complexes which recover the discussion here by Tannakian considerations. Also, throughout are we assuming $p > 2$?

emiperfect

Remark 3.6.1 (Apertures over semiperfectoid rings)**.** For any semiperfect (animated commutative) $\mathbb{F}_p$-algebra R, we have a canonical comparison isomorphism $\mathbb{\Delta}_R \xrightarrow{\simeq} A_{\mathrm{crys}}(R)$ (see for instance [24, Remark 6.9.5]). Hence a $\mathcal{G}$-bundle over $R^{\mathbb{\Delta}} = \operatorname{Spf} \mathbb{\Delta}_R$ is the same as one over $A_{\mathrm{crys}}(R)$. Suppose now that we have $\mathfrak{Q} \in \mathrm{BT}^{\mathcal{G},\mu}_{\infty}(R)$: Then just as in [5, Remark 6.3.4], the $\mathcal{G}$-bundle $T_{\mathrm{crys}}(\mathfrak{Q})$ in crystals associated with $j^*_{\mathbb{\Delta}}\mathfrak{Q}$ is equipped with an isomorphism $\varphi^* T_{\mathrm{crys}}(\mathfrak{Q})[p^{-1}] \xrightarrow{\simeq} T_{\mathrm{crys}}(\mathfrak{Q})[p^{-1}]$ of $\mathcal{G}$-bundles over $A_{\mathrm{crys}}(R)$.

ealization

Construction 3.6.2 (Crystalline realization)**.** By quasisyntomic descent, Remark 3.6.1 tells us that for any $\mathbb{F}_p$-algebra R, every $\mathfrak{Q} \in \mathrm{BT}^{\mathcal{G},\mu}_{\infty}(R)$ yields a $\mathcal{G}$-bundle in F-crystals over R, which we denote by $T_{\mathrm{crys}}(\mathfrak{Q})$. See [37, §1.1.1] for an explanation in the context of vector bundle F-gauges.

Remark 3.6.3 (F-isocrystals over perfect rings)**.** Suppose that R is a perfect $\mathbb{F}_p$-algebra. Then, $\mathrm{Isoc}_G(R)$ is the groupoid of pairs $(\mathcal{E}, \varphi_{\mathcal{E}})$, where $\mathcal{E}$ is a G-torsor over $W(R)[1/p]$ and $\varphi_{\mathcal{E}} : \varphi^*\mathcal{E} \xrightarrow{\simeq} \mathcal{E}$ is an isomorphism of G-bundles. The functor $\mathrm{BT}^{\mathcal{G},\mu}_{\infty}(R) \to \mathrm{Isoc}_G(R)$ from Construction 3.3.1 can now be understood as follows. Given $\mathfrak{Q} \in \mathrm{BT}^{\mathcal{G},\mu}_{\infty}(R)$, one considers the G-bundle in F-isocrystals underlying the $\mathcal{G}$-bundle in F-crystals $T_{\mathrm{crys}}(\mathfrak{Q})$: This corresponds exactly to a G-bundle over $W(R)[1/p]$ equipped with an isomorphism from its Frobenius twist, and hence an object of $\mathrm{Isoc}_G(R)$.

nd_de_rham

Remark 3.6.4 (Crystals and the de Rham realization)**.** Suppose that R is a p-complete $\hat{\mathcal{O}}$-algebra. Then every crystal in $\mathcal{G}$-bundles over R/pR yields a $\mathcal{G}$-bundle over R; here, we are using the fact that $R \to R/pR$ is a pro-divided power thickening. In particular, if we have $\mathfrak{Q} \in \mathrm{BT}^{\mathcal{G},\mu}_{\infty}(R)$ reducing to $\mathfrak{Q}_0 \in \mathrm{BT}^{\mathcal{G},\mu}_{\infty}(R/pR)$, then we obtain a $\mathcal{G}$-bundle over R associated with $T_{\mathrm{crys}}(\mathfrak{Q}_0)$. This $\mathcal{G}$-bundle is in fact canonically isomorphic to $T_{\mathrm{dR}}(\mathfrak{Q})$; see [37, Theorem 1.19] and its proof.

nius_lifts

Remark 3.6.5 (Frobenius lifts and filtrations)**.** Suppose that R is a flat $\hat{\mathcal{O}}$-algebra equipped with a Frobenius lift $\varphi : R \to R$. As in Example 3.2.4, this gives us a Breuil-Kisin frame $\underline{R}$ for R, and we obtain a functor $\Phi_{\underline{R}} : \mathrm{BT}^{\mathcal{G},\mu}_{\infty}(R/pR) \to \mathrm{Wind}^{\mathcal{G},\mu}_{\underline{R},\infty}(R/pR)$. Suppose that we have $\mathfrak{Q}_0 \in \mathrm{BT}^{\mathcal{G},\mu}_{\infty}(R/pR)$. One finds that in the notation of Remark 3.2.12, $\Phi_{\underline{R}}(\mathfrak{Q}_0)$ corresponds to the tuple

$$(T_{\mathrm{crys}}(\mathfrak{Q}_0), \mathrm{Fil}^{\bullet}_{\mathrm{Hdg}} T_{\mathrm{dR}}(\mathfrak{Q}_0), \alpha)$$

with α an isomorphism

$$\varphi^* T_{\mathrm{crys}}(\mathfrak{Q}_0)' \xrightarrow{\simeq} T_{\mathrm{crys}}(\mathfrak{Q}_0),$$

where $T_{\mathrm{crys}}(\mathfrak{Q}_0)'$ is the modification along $\mathrm{Fil}^{\bullet}_{\mathrm{Hdg}} T_{\mathrm{dR}}(\mathfrak{Q}_0)$.

visibility

Remark 3.6.6 (Strong divisibility)**.** In the situation of Remark 3.6.5, suppose that $\mathfrak{Q}_0$ lifts to $\mathfrak{Q} \in \mathrm{BT}^{\mathcal{G},\mu}_{\infty}(R)$. Then we have a lift $\mathrm{Fil}^{\bullet}_{\mathrm{Hdg}} T_{\mathrm{dR}}(\mathfrak{Q})$ of the filtered bundle $\mathrm{Fil}^{\bullet}_{\mathrm{Hdg}} T_{\mathrm{dR}}(\mathfrak{Q})$. For $V \in \mathrm{Rep}_{\mathbb{Z}_p}(\mathcal{G})$, let $\mathrm{Fil}^{\bullet}_{\mathrm{Hdg}} V_{\mathrm{twist}}$ be the filtered finite locally free R-module obtained by twisting V by $\mathrm{Fil}^{\bullet}_{\mathrm{Hdg}} T_{\mathrm{dR}}(\mathfrak{Q})$.[17] Then Remark 3.2.13 tells us that we have a functorial-in-V isomorphism

$$\varphi^*\left(\sum_i p^{-i}\,\mathrm{Fil}^i_{\mathrm{Hdg}} V_{\mathrm{twist}}\right) \xrightarrow{\sim} V_{\mathrm{twist}}$$

of finite locally free R-modules.

[17] Alex: I'm not sure I like this terminology, does V_{twist} mean the filtered bundle you get by evaluating our filtered fiber functor on V? This is confusing because we also call the thing appearing in the source of the below isomorphism a twist. Maybe we can use notation similar to that in Remark 3.2.13.

_F-crystal

Construction 3.6.7 (Filtered F-crystals)**.** Let κ be a perfect field, and suppose that $\operatorname{Spf} R$ is a base formal $W(\kappa)$-scheme. Suppose that we have $\mathfrak{Q} \in \mathrm{BT}^{\mathcal{G},\mu}_{\infty}(R)$ lifting $\mathfrak{Q}_0 \in \mathrm{BT}^{\mathcal{G},\mu}_{\infty}(R/pR)$. Then $T_{\mathrm{dR}}(\mathfrak{Q})$ is equipped with a topologically nilpotent integrable connection ∇, which completely determines the $\mathcal{G}$-bundle in crystals $T_{\mathrm{crys}}(\mathfrak{Q}_0)$. Now, if R is equipped with a Frobenius lift $\varphi : R \to R$, then the F-crystal structure on $T_{\mathrm{crys}}(\mathfrak{Q}_0)$ gives us an isomorphism

$$\varphi^* T_{\mathrm{dR}}(\mathfrak{Q})[p^{-1}] \xrightarrow{\simeq} T_{\mathrm{dR}}(\mathfrak{Q})[p^{-1}]$$

of $\mathcal{G}$-bundles over $R[p^{-1}]$ that is parallel for ∇. For any representation V, this gives us an isomorphism $\varphi^* V_{\mathrm{twist}}[p^{-1}] \xrightarrow{\simeq} V_{\mathrm{twist}}[p^{-1}]$, which restricts to the integral isomorphism explained in Remark 3.6.6. Moreover, the filtration $\mathrm{Fil}^\bullet_{\mathrm{Hdg}}\, T_{\mathrm{dR}}(\mathfrak{Q})$ satisfies Griffiths transversality (in the sense that the associated filtration on the twist of any representation of $\mathcal{G}$ by $T_{\mathrm{dR}}(\mathfrak{Q})$ satisfies Griffiths transversality); see [37, Theorem 2.10]. In sum, in the language of [37, §2.1.2], for any base formal $W(\kappa)$-scheme $\mathfrak{X}$, and $\mathfrak{Q} \in \mathrm{BT}^{\mathcal{G},\mu}_\infty(\mathfrak{X})$, we can associate with $\mathfrak{Q}$ an exact and monoidal functor from $\mathrm{Rep}_{\mathbb{Z}_p}(\mathcal{G})$ to the category of *strongly divisible filtered F-crystals* over $\mathfrak{X}$.

ealization

3.7. **The étale realization and p-adic comparison.** Here, we recall the construction of the étale realization of apertures and some p-adic comparison results for it that make precise its relationship with the objects defined in the previous subsection.

_apertures

Construction 3.7.1 (The étale realization for apertures)**.** Let R be a derived p-complete animated commutative ring. Suppose that we have $\mathfrak{Q}$ in $B\mathcal{G}(R^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z})$. We will attach to it an object $T_{\mathrm{\acute{e}t}}(\mathfrak{Q})$ that specializes to the construction given in Corollary 2.3.12 when R is p-quasisyntomic and p-torsion free. The point is that, for every object $(A, I, R \to \overline{A})$ in R_{Δ}, Remark 2.1.1 tells us that $j^*_{\Delta}\mathfrak{Q} \in B\mathcal{G}(R^{\Delta} \otimes \mathbb{Z}/p^n\mathbb{Z})$ yields a $\mathcal{G}$-torsor $\mathcal{P}_{\mathfrak{Q}}(A, I, R \to \overline{A})$ over $A/^{\mathbb{L}}p^n$. Furthermore, just as in [5, Remark 6.3.4], the fact that we have the underlying syntomic torsor $\mathfrak{Q}$ implies that there is a natural isomorphism of $\mathcal{G}$-torsors

$$\varphi_{\mathcal{P}} \colon \varphi^*\mathcal{P}_{\mathfrak{Q}}(A, I, R \to \overline{A})[1/I] \xrightarrow{\simeq} \mathcal{P}_{\mathfrak{Q}}(A, I, R \to \overline{A})[1/I]$$

over $(A/^{\mathbb{L}}p^n)[1/I]$. Now, just as in Constructions 6.3.1 and 6.3.2 of *loc. cit.*, we obtain an object $T_{\mathrm{\acute{e}t}}(\mathfrak{Q})$ in $\mathrm{Loc}_{\mathcal{G}(\mathbb{Z}/p^n\mathbb{Z})}(R[1/p])$ characterized by the property that for p-torsion free perfectoid R-algebras T, we have

$$\Gamma(\operatorname{Spec} T[1/p], T_{\mathrm{\acute{e}t}}(\mathfrak{Q})) = \mathcal{P}_{\mathfrak{Q}}(\Delta_T, I_T, R \to T)[1/I_T]^{\varphi_{\mathcal{P}}=\mathrm{id}}.$$

Taking the limit over n gives us a canonical realization functor

$$T_{\mathrm{\acute{e}t}} : B\mathcal{G}(R^{\mathrm{syn}}) \to \mathrm{Loc}_{\mathcal{G}(\mathbb{Z}_p)}(R[1/p]).$$

This construction globalizes to a general derived p-adic formal scheme $\mathfrak{X}$ in the obvious way.

socrystals

Remark 3.7.2 (Filtered F-isocrystals and crystalline comparison)**.** Let κ be a perfect field over $\hat{\mathcal{O}}$ of characteristic p and let $\mathfrak{X}$ be a base formal $W(\kappa)$-scheme. Suppose that we have $\mathfrak{Q} \in \mathrm{BT}^{\mathcal{G},\mu}_\infty(\mathfrak{X})$. For any $V \in \mathrm{Rep}_{\mathbb{Z}_p}(\mathcal{G})$, we can now associate two filtered F-isocrystals[13] over the special fiber $\mathfrak{X}_\kappa$:

- The filtered F-isocrystal associated with the filtered F-crystal $\mathrm{Fil}^\bullet_{\mathrm{Hdg}}\, V_{\mathrm{twist}}$ from Construction 3.6.7;
- The filtered F-isocrystal associated with the crystalline $\mathbb{Z}_p$-local system $(V)_{T_{\mathrm{\acute{e}t}}(\mathfrak{Q})}$.

By [37, Theorem 2.10], these two filtered F-isocrystals are canonically isomorphic.[18]

[18] Alex: We actually probably have something stronger as we're in the minuscule situation—you should be able to upgrade this to crystals.

comparison

Remark 3.7.3 (Filtered bundles and de Rham comparison)**.** As usual, let $\mathscr{O}_K$ be a complete DVR over $\hat{\mathcal{O}}$ with perfect residue field, and let $\mathfrak{X}$ be a base formal $\mathscr{O}_K$-scheme. Suppose that we are given $\mathfrak{Q} \in \mathrm{BT}^{\mathcal{G},\mu}_\infty(\mathfrak{X})$ lifting $\mathsf{Q} \in \mathrm{Loc}^{\mathrm{crys}}_{\mathcal{G}(\mathbb{Z}_p)}(\mathfrak{X}_\eta)$. We can now associate two filtered G-bundles over $\mathfrak{X}_\eta$ equipped with integrable connections:

- The generic fiber of the Hodge-filtered de Rham realization $\mathrm{Fil}^\bullet_{\mathrm{Hdg}}\, T_{\mathrm{dR}}(\mathfrak{Q})$;
- Since Q is crystalline, and therefore in particular de Rham, we have the filtered G-bundle $\mathrm{Fil}^\bullet D_{\mathrm{dR}}(\mathsf{Q})$ associating with each $V \in \mathrm{Rep}_{\mathbb{Q}_p}(G)$, the filtered vector bundle $\mathrm{Fil}^\bullet D_{\mathrm{dR}}((V)_{\mathsf{Q}})$ from [54, Theorem 3.7(iv)].

comparison

Lemma 3.7.4. *The two filtered bundles above with integrable connections are canonically isomorphic.*

[13] See [36, p. 28] for this notion

Proof. When $W(\kappa) = \mathscr{O}_K$, this is immediate from Remark 3.7.2. The proof of Imai-Kato-Youcis cited there works also in this context, if one ignores F-structures. Indeed, everything after the first paragraph in *loc. cit.* works exactly the same, replacing $\mathbb{D}_{\mathrm{crys}}[1/p]$ and D_{crys} with $\mathrm{Fil}^\bullet_{\mathrm{Hdg}} T_{\mathrm{dR}}(\mathfrak{Q})[1/p]$ and $\mathrm{Fil}^\bullet D_{\mathrm{dR}}(\mathsf{Q})$, respectively. ☐

perfectoid

3.8. **Étale realization over perfectoid rings.** In this subsection, we will see that the étale realization on $\mathrm{BT}_n^{\mathcal{G},\mu}$ is faithful for base formal $\mathscr{O}_K$-schemes.

perfectoid **Remark 3.8.1** ($\mathcal{G}(\mathbb{Z}/p^n\mathbb{Z})$-torsors over perfectoid rings)**.** Suppose that R is perfectoid and p-torsion free. By Remark 2.3.8, giving a $\mathcal{G}(\mathbb{Z}/p^n\mathbb{Z})$-local system over $R[1/p]$ is equivalent to giving an exact symmetric monoidal functor

$$\mathrm{Rep}_{\mathbb{Z}_p}(\mathcal{G}) \to \mathrm{Loc}_{\mathbb{Z}/p^n\mathbb{Z}}(R[1/p]).$$

Now, tilting combined with Katz's Riemann-Hilbert equivalence (see [10, Proposition 3.6]) tells us that the right hand side is equivalent to the category of pairs $(\mathcal{V},\eta)$, where $\mathcal{V}$ is a finite locally free module over $\Delta_R/p^n\Delta_R[1/\xi]$ and $\eta : \varphi^*\mathcal{V} \xrightarrow{\simeq} \mathcal{V}$ is an isomorphism of $\Delta_R/p^n\Delta_R[\xi^{-1}]$-modules. Therefore, we conclude that giving a $\mathcal{G}(\mathbb{Z}/p^n\mathbb{Z})$-local system over $R[1/p]$ is equivalent to giving a $\mathcal{G}$-torsor $\mathcal{Q}$ over $(\Delta_R/p^n\Delta_R)[1/\xi]$ along with an isomorphism $\varphi_{\mathcal{Q}} : \varphi^*\mathcal{Q} \xrightarrow{\simeq} \mathcal{Q}$.

al_systems **Remark 3.8.2** (Quotient description of the image of the étale realization)**.** In terms of the description of the target from Remark 3.8.1, the étale realization functor $\mathrm{BT}_n^{\mathcal{G},\mu}(R) \to \mathrm{Loc}_{\mathcal{G}(\mathbb{Z}/p^n\mathbb{Z})}(R[1/p])$ factors through the fully faithful sub-groupoid spanned by pairs $(\mathcal{Q},\varphi_{\mathcal{Q}})$ where $\mathcal{Q}$ can be trivialized over $(\Delta_{\tilde{R}}/p^n\Delta_{\tilde{R}})[1/\xi]$ for some p-completely faithfully flat and étale map $R \to \tilde{R}$ of perfectoid rings. This sub-groupoid is the evaluation at R of the étale sheafification of the functor on p-torsion free perfectoid rings given by

$$R \mapsto [\mathcal{G}(\Delta_R/p^n\Delta_R[1/\xi])/_{\mathrm{Ad}_\varphi}(\mathcal{G}(\Delta_R/p^n\Delta_R[1/\xi]))].$$

Here, the right hand side is the quotient by the φ-semilinear adjoint action

$$\begin{aligned} \mathcal{G}(\Delta_R/p^n\Delta_R[1/\xi]) \times \mathcal{G}(\Delta_R/p^n\Delta_R[1/\xi]) &\to \mathcal{G}(\Delta_R/p^n\Delta_R[1/\xi]) \\ (g,h) &\mapsto h^{-1}g\varphi(h). \end{aligned}$$

From this perspective, the étale realization map from Construction 3.7.1 can be viewed as the étale sheafification of the map

$$[\mathcal{G}(\Delta_R/p^n\Delta_R)/H_\mu^{(n)}(R)] \to [\mathcal{G}(\Delta_R/p^n\Delta_R[1/\xi])/_{\mathrm{Ad}_\varphi}(\mathcal{G}(\Delta_R/p^n\Delta_R[1/\xi]))]$$

induced by

$$\mathcal{G}(\Delta_R/p^n\Delta_R) \xrightarrow{g\mapsto g\varphi(\mu(\xi))} \mathcal{G}(\Delta_R/p^n\Delta_R[1/\xi])$$

and the map of groups

:hmu_to_LG (3.8.2.1) $$H_\mu^{(n)}(R) = \mathcal{G}(\Delta_R/p^n\Delta_R) \times_{\mathcal{G}(R/p^nR)} \mathcal{P}_\mu^-(R/p^nR) \subset \mathcal{G}(\Delta/p^n\Delta_R) \to \mathcal{G}(\Delta_R/p^n\Delta_R[1/\xi])$$

d_faithful **Proposition 3.8.3.** *Suppose that R is perfectoid and p-torsion free. Then the étale realization functor*

$$\mathrm{BT}_n^{\mathcal{G},\mu}(R) \to \mathrm{Loc}_{\mathcal{G}(\mathbb{Z}/p^n\mathbb{Z})}(R[1/p])$$

is faithful.

Proof. Given Remark 3.8.2 and the description of $H_\mu^{(n)}(R)$ in (3.2.19.1), this reduces to the easy assertion that the map (3.8.2.1) is injective. ☐

h_faithful **Corollary 3.8.4.** *Suppose that $\mathfrak{X}$ is a base formal $\mathscr{O}_K$-scheme. Then the étale realization functor*

$$\mathrm{BT}_n^{\mathcal{G},\mu}(\mathfrak{X}) \to \mathrm{Loc}_{\mathcal{G}(\mathbb{Z}/p^n\mathbb{Z})}(\mathfrak{X}_\eta)$$

is faithful.

Proof. This is because $\mathfrak{X}$ admits a quasisyntomic cover $\{\mathrm{Spf}(R_i) \to \mathfrak{X}\}$ with R_i perfectoid and p-torsion free for all i; see [36, Lemma 1.15]. ☐

nary_locus

3.9. **The μ-ordinary locus.** Suppose that $\mathcal{G}$ is reductive. We'll now look at the μ-ordinary locus of $\mathrm{BT}_n^{\mathcal{G},\mu}$, which is more accurately an étale map $\mathrm{BT}_n^{\mathcal{G},\mu,\mathrm{ord}} \to \mathrm{BT}_n^{\mathcal{G},\mu}$.

Setup 3.9.1. To begin, choose a maximal torus $\mathcal{T} \subset \mathcal{G}$ contained in a Borel subgroup of $\mathcal{G}$. We can assume that μ has been chosen within its conjugacy class to factor through $\mathcal{T}_{\hat{\mathcal{O}}}$.[14] The sum of the Galois conjugates of μ now gives a cocharacter ν of $\mathcal{T}$ that is defined over $\mathbb{Z}_p$. Let $\mathcal{P}_\nu^+ \subset \mathcal{G}$ be the parabolic subgroup whose Lie algebra is the sum of the non-negative eigenspaces for ν.

nu_central **Lemma 3.9.2.** *Let $\mathcal{M}_\nu \subset \mathcal{G}$ and $\mathcal{M}_\mu \subset \mathcal{G}_{\hat{\mathcal{O}}}$ be the centralizers of ν and μ, respectively. Then we have the inclusion $\mathcal{M}_{\nu,\hat{\mathcal{O}}} \subset \mathcal{M}_\mu$.*

Proof. It is enough to know that μ is central in $M_{\nu,\hat{\mathcal{O}}}$. By [81, Lemma 2.7], this would follow if the average $\overline{\mu}$ of the Galois conjugates of μ is central in $M_{\nu,\hat{\mathcal{O}}}$. But this is clear, since ν is an integer multiple of $\overline{\mu}$. ☐

Definition 3.9.3 (The μ-ordinary locus)**.** The **μ-ordinary locus** of $\mathrm{BT}_n^{\mathcal{G},\mu}$ is the formal algebraic stack

$$\mathrm{BT}_n^{\mathcal{G},\mu,\mathrm{ord}} \overset{\mathrm{defn}}{=} \mathrm{BT}_n^{\mathcal{P}_\nu^+,\mu}.$$

nary_local **Proposition 3.9.4** (Local structure of ordinary locus)**.** *The map $\mathrm{BT}_n^{\mathcal{G},\mu,\mathrm{ord}} \to \mathrm{BT}_n^{\mathcal{G},\mu}$ is étale with dense image. Moreover, for every $x \in \mathrm{BT}_\infty^{\mathcal{G},\mu,\mathrm{ord}}(\kappa)$ with κ perfect, the universal deformation space for $\mathrm{BT}_\infty^{\mathcal{G},\mu}$ at the image of x has the structure of a* cascade *over $W(\kappa)$ in the sense of* [69].

Proof. The first assertion follows from [24, Proposition 9.5.1]. The second will be shown in [62]. ☐

ion_points **Remark 3.9.5** (Cascades and torsion points)**.** Since it has made an appearance in the proposition above, let us say a few words about the notion of a cascade. In fact, for each integer $n \geq 1$, Moonen defines the notion of an **n-cascade** in inductive fashion, and a cascade is just an n-cascade for some n. Without recalling the precise definition, let us just say that a 2-cascade in the context of the theorem is simply a p-divisible formal group, while a 3-cascade is a biextension of p-divisible formal groups, and so on. In particular, every cascade Γ over $W(\kappa)$ has an associated **p^n-torsion** formal subscheme $\Gamma[p^n] \subset \Gamma$, which is finite flat and generically étale over $W(\kappa)$. Fix this discussion.

4. Tate's full faithfulness for apertures and its consequences

nsequences

In this section, we prove a generalization Tate's full faithfulness theorem for p-divisible groups to the context of apertures. We also derive some corollaries to this that will be of importance for the main theorems of this paper.

ubsec:tate

4.1. **Algebraization and Tate's full faithfulness.** Let $\mathcal{O}$ be a mixed characteristic $(0,p)$ discrete valuation ring with completion $\hat{\mathcal{O}}$. We will now define an algebraic variant of $\mathrm{BT}_n^{\mathcal{G},\mu}$ over $\mathcal{O}$ by gluing it with the stack of $\mathcal{G}(\mathbb{Z}/p^n\mathbb{Z})$-local systems over E, and we will use it to prove the aperture analogue of Tate's full faithfulness theorem for p-divisible groups by reducing it ultimately to Corollary 2.3.12.

Construction 4.1.1 (Algebraic BT)**.** For any animated commutative $\mathcal{O}$-algebra R with derived p-adic completion $\hat{R}$, set

$$\mathrm{BT}_n^{\mathcal{G},\mu,\mathrm{alg}}(R) \overset{\mathrm{defn}}{=} \mathrm{BT}_n^{\mathcal{G},\mu}(\hat{R}) \times_{\mathrm{Loc}_{\mathcal{G}(\mathbb{Z}/p^n\mathbb{Z})}(\hat{R}[1/p])} \mathrm{Loc}_{\mathcal{G}(\mathbb{Z}/p^n\mathbb{Z})}(R[1/p]).$$

Also, set

$$\mathrm{BT}_\infty^{\mathcal{G},\mu,\mathrm{alg}}(R) = \varprojlim_n \mathrm{BT}_n^{\mathcal{G},\mu,\mathrm{alg}}(R).$$

[14]By [51, Lemma 1.1.3], μ can be conjugated to a cocharacter factoring through $\mathcal{T}_{\hat{\mathcal{O}}[1/p]}$ dominant with respect to the choice of Borel. This will necessarily be a cocharacter of $\mathcal{T}_{\mathcal{O}}$.

Remark 4.1.2. For any R, we obtain a canonical étale realization map

$$T_{\text{ét}} : \mathrm{BT}_n^{\mathcal{G},\mu,\mathrm{alg}}(R) \to \mathrm{Loc}_{\mathcal{G}(\mathbb{Z}/p^n\mathbb{Z})}(R[1/p])$$

obtained as the composition

$$\begin{aligned}\mathrm{BT}_n^{\mathcal{G},\mu,\mathrm{alg}}(R) &= \mathrm{BT}_n^{\mathcal{G},\mu}(\hat{R}) \times_{\mathrm{Loc}_{\mathcal{G}(\mathbb{Z}/p^n\mathbb{Z})}(\hat{R}[1/p])} \mathrm{Loc}_{\mathcal{G}(\mathbb{Z}/p^n\mathbb{Z})}(R[1/p]) \\ &\xrightarrow{T_{\text{ét}}\times \mathrm{id}} \mathrm{Loc}_{\mathcal{G}(\mathbb{Z}/p^n\mathbb{Z})}(\hat{R}[1/p]) \times_{\mathrm{Loc}_{\mathcal{G}(\mathbb{Z}/p^n\mathbb{Z})}(\hat{R}[1/p])} \mathrm{Loc}_{\mathcal{G}(\mathbb{Z}/p^n\mathbb{Z})}(R[1/p]) \\ &= \mathrm{Loc}_{\mathcal{G}(\mathbb{Z}/p^n\mathbb{Z})}(R[1/p]).\end{aligned}$$

Here, we are using the realization functor from Construction 3.7.1 for $\hat{R}$. This globalizes: For any Deligne-Mumford stack $\mathcal{X}$ over $\mathcal{O}$ with generic fiber X, we obtain a functor

$$T_{\text{ét}} : \mathrm{BT}_n^{\mathcal{G},\mu,\mathrm{alg}}(\mathcal{X}) \to \mathrm{Loc}_{\mathcal{G}(\mathbb{Z}/p^n\mathbb{Z})}(X)$$

braization **Theorem 4.1.3** (Algebraic representability and affineness of diagonal)**.** *The functor* $\mathrm{BT}_n^{\mathcal{G},\mu,\mathrm{alg}}$ *is represented by a smooth* 0*-dimensional Artin stack over* $\mathcal{O}$ *with* affine *diagonal.*

The rest of the subsection will be devoted to the proof of this theorem, with the completion of the proof of the affineness of the diagonal postponed to the next subsection. For now, let us record an important consequence.

eta_normal **Definition 4.1.4** (η-normality)**.** Suppose that $\mathcal{X}$ is a Deligne-Mumford stack over $\mathcal{O}$. We will say that $\mathcal{X}$ is **η-normal**[15] if it admits an étale cover by affine schemes of the form $\operatorname{Spec} R$ with R p-torsion free and integrally closed in $R[1/p]$. Note that, if $\mathcal{X}$ is normal then it is automatically η-normal.

ithfulness **Theorem 4.1.5** (Tate full faithfulness for apertures)**.** *Let* $\mathcal{X}$ *be a Noetherian,* η*-normal Deligne-Mumford stack over* $\mathcal{O}$ *with generic fiber* X*. Then the functor* $T_{\text{ét}} : \mathrm{BT}_\infty^{\mathcal{G},\mu,\mathrm{alg}}\mathcal{X}) \to \mathrm{Loc}_{\mathcal{G}(\mathbb{Z}_p)}(X)$ *is fully faithful.*

Proof. By étale descent, we reduce to the case where $\mathcal{X} = \operatorname{Spec} R$ is affine. We essentially follow (a part of) the proof by Tate of his theorem for p-divisible groups [84]. By Theorem 4.1.3, for any pair of objects

$$(\mathfrak{Q}_1, \mathfrak{Q}_2) \in \mathrm{BT}_\infty^{\mathcal{G},\mu,\mathrm{alg}}(R) \times \mathrm{BT}_\infty^{\mathcal{G},\mu,\mathrm{alg}}(R)$$

the scheme of isomorphisms from $\mathfrak{Q}_1$ to $\mathfrak{Q}_2$ is represented by an affine scheme $\mathcal{I} \to \operatorname{Spec} R$. The theorem comes down to knowing that every $R[1/p]$-point of $\mathcal{I}$ extends to an R-valued point. Reference?

For this, by the η-normality of R, it is enough to know that an $R[1/p]$-valued point of $\mathcal{I}$ extends over the localization at every height 1 prime of R that is minimal over pR. This reduces us to the case where R is a mixed characteristic discrete valuation ring with maximal ideal $\mathfrak{m}$. Now, let κ be the residue field of R. For any purely inseparable finite extension κ'/κ, there exists a finite flat quasisyntomic map of discrete valuation rings $R \to R'$ such that $R'/\mathfrak{m}R' \simeq \kappa'$[16] Using this, one sees that there exists an ind-finite flat quasisyntomic cover $R \to R_\infty$ where R_∞ is a discrete valuation ring with perfect residue field. Via flat descent, it is now enough to show:

(1) $\mathcal{I}(R_\infty) \to \mathcal{I}(R_\infty[1/p])$ is a bijection.
(2) $\mathcal{I}(R_\infty^{\otimes_R m}) \to \mathcal{I}(R_\infty^{\otimes_R m}[1/p])$ is injective.

By the definition of $\mathrm{BT}_n^{\mathcal{G},\mu,\mathrm{alg}}$, to prove assertion (1) (resp. (2)), we can replace R_∞ (resp. $R_\infty^{\otimes_R m}$) with its (derived) p-completion. Note that in assertion (2) the algebras involved are quasisyntomic over R_∞. Therefore, both assertions follow from Theorem 2.3.6. $\square$

[15] See [1, Appendix A]

[16] If $\kappa' = \kappa(a^{1/p})$ for some $a \in \kappa \backslash \kappa^p$, then we can take $R' = R[X]/(X^p - \pi X - \tilde{a})$, for some uniformizer $\pi \in R$ and some lift $\tilde{a} \in R$ of a. The general case follows by induction on $[\kappa' : \kappa]$.

ithfulness **Corollary 4.1.6.** *Suppose that $\mathcal{X}$ is a normal flat Noetherian algebraic space over $\mathcal{O}$ with $\mathsf{Q} \in \mathrm{Loc}_{\mathcal{G}(\mathbb{Z}_p)}(\mathcal{X}_\eta)$. Then there exists a (unique) $\mathfrak{Q} \in \mathrm{BT}_\infty^{\mathcal{G},\mu,\mathrm{alg}}(\mathcal{X})$ with $T_{\text{ét}}(\mathfrak{Q}) \simeq \mathsf{Q}$ if and only if there is a smooth cover $\mathcal{X}' \to \mathcal{X}$ by a scheme $\mathcal{X}'$ and $\mathfrak{Q}' \in \mathrm{BT}_\infty^{\mathcal{G},\mu,\mathrm{alg}}(\mathcal{X}')$ with $T_{\text{ét}}(\mathfrak{Q}') \simeq \mathsf{Q}|_{\mathcal{X}'[1/p]}$.*

Proof. The only if direction is obvious. For the if direction, since $\mathcal{X}' \to \mathcal{X}$ is smooth each k-fold self-fiber product $(\mathcal{X}')^{\times_{\mathcal{X}} k}$ is still a normal flat Noetherian $\mathcal{O}$-scheme. Therefore, Theorem 4.1.5 tells us that the tautological descent datum for $\mathsf{Q}|_{\mathcal{X}'_\eta}$ yields a descent datum for $\mathfrak{Q}'$, which is effective. □

Remark 4.1.7. Combining the theorem with Theorem 2.4.2, we obtain a proof of Tate's seminal full faithfulness theorem for p-divisible groups [84, Theorem 4]. Unwinding everything, one sees that the proof obtained here is not very different from that of Tate's, except that his final reduction is to the case where there exists a homomorphism of p-divisible groups over a complete valuation ring inducing a given isomorphism of Tate modules. Since we do not have any direct way of defining a map between apertures that is not an isomorphism, we end up appealing to the faithfulness assertion of Theorem 2.3.6 instead.

Let us now begin preparations for the proof of Theorem 4.1.3.

esentation **Lemma 4.1.8** (Completed finite presentation)**.** *Suppose that $\{R_i\}_{i\in I}$ is an inductive system of derived p-complete animated commutative rings with colimit R, and let $\hat{R}$ be the derived p-completion of R. Then the natural map*

$$\varinjlim_i \mathrm{BT}_n^{\mathcal{G},\mu}(R_i) \to \mathrm{BT}_n^{\mathcal{G},\mu}(\hat{R})$$

is an equivalence.

Proof. If the inductive system is one of $\mathbb{Z}/p^m\mathbb{Z}$-algebras, then the conclusion holds because $\mathrm{BT}_n^{\mathcal{G},\mu}$ is a finitely presented formal Artin stack.

For the general case, set $\overline{R}_i \overset{\text{defn}}{=} R_i/^{\mathbb{L}}p^2$ and $\overline{R} = R/^{\mathbb{L}}p^2$[17], and note that by Grothendieck-Messing theory we have a Cartesian square

$$\begin{array}{ccc} \mathrm{BT}_n^{\mathcal{G},\mu}(\hat{R}) & \longrightarrow & B\mathcal{P}_\mu^-(R/^{\mathbb{L}}p^n) \\ \downarrow & & \downarrow \\ \mathrm{BT}_n^{\mathcal{G},\mu}(\overline{R}) & \longrightarrow & B\mathcal{P}_\mu^-(\overline{R}/^{\mathbb{L}}p^n) \times_{B\mathcal{G}(\overline{R}/^{\mathbb{L}}p^n)} B\mathcal{G}(R/^{\mathbb{L}}p^n) \end{array}$$

and similarly with $\hat{R}$ replaced by R_i. We can now conclude using the previous paragraph, and the finite presentation of the stacks $B\mathcal{P}_\mu^-$ and $B\mathcal{G}$.[18] □

esentation **Lemma 4.1.9** (Algebraic finite presentation)**.** *The prestack $\mathrm{BT}_n^{\mathcal{G},\mu,\mathrm{alg}}$ over $\mathcal{O}$ is locally of finite presentation.*

Proof. We need to know that the natural map

$$\begin{array}{c} \varinjlim_i \left[\mathrm{BT}_n^{\mathcal{G},\mu}(\hat{R}_i) \times_{\mathrm{Loc}_{\mathcal{G}(\mathbb{Z}/p^n\mathbb{Z})}(\hat{R}_i[1/p])} \mathrm{Loc}_{\mathcal{G}(\mathbb{Z}/p^n\mathbb{Z})}(R_i[1/p])\right] \\ \downarrow \\ \mathrm{BT}_n^{\mathcal{G},\mu}(\hat{R}) \times_{\mathrm{Loc}_{\mathcal{G}(\mathbb{Z}/p^n\mathbb{Z})}(\hat{R}[1/p])} \mathrm{Loc}_{\mathcal{G}(\mathbb{Z}/p^n\mathbb{Z})}(R[1/p]) \end{array}$$

---

[17] The choice of p^2 here is to ensure that $R \to \overline{R}$ is an inverse limit of *nilpotent* divided power extensions even when $p = 2$: For p odd, one could also choose to work with $R/^{\mathbb{L}}p$ instead.

[18] We are using the fact that filtered colimits of ∞-groupoids commute with finite limits; see [59, Proposition 5.3.3.3,].

is an isomorphism for any inductive system $\{R_i\}$ of animated commutative $\mathcal{O}$-algebras with colimit R.

Now, since $\mathrm{Loc}_{\mathcal{G}(\mathbb{Z}/p^n\mathbb{Z})}$ is a finite Deligne-Mumford stack, it is in particular finitely presented, and the natural map

esentation

$$\varinjlim_i \mathrm{Loc}_{\mathcal{G}(\mathbb{Z}/p^n\mathbb{Z})}(R_i[1/p]) \to \mathrm{Loc}_{\mathcal{G}(\mathbb{Z}/p^n\mathbb{Z})}(\varinjlim_i R_i[1/p]); \tag{4.1.9.1}$$

is an isomorphism. Therefore, it is enough to verify that the following commutative square is Cartesian

$$\begin{tikzcd}
\varinjlim_i \mathrm{BT}_n^{\mathcal{G},\mu}(\hat{R}_i) \arrow[r] \arrow[d] & \mathrm{BT}_n^{\mathcal{G},\mu}(\hat{R}) \arrow[d] \\
\mathrm{Loc}_{\mathcal{G}(\mathbb{Z}/p^n\mathbb{Z})}(\varinjlim_i \hat{R}_i[1/p]) \arrow[r] & \mathrm{Loc}_{\mathcal{G}(\mathbb{Z}/p^n\mathbb{Z})}(\hat{R}[1/p]).
\end{tikzcd}$$

The ring $\varinjlim_i \hat{R}_i$ is p-Henselian with p-adic completion $\hat{R}$. Therefore, by [12, Corollary 2.1.20], the bottom arrow is an isomorphism. The top arrow is an isomorphism by Lemma 4.1.8. $\square$

The next result is immediate from the definitions, the étaleness[19] of $\mathrm{Loc}_{\mathcal{G}(\mathbb{Z}/p^n\mathbb{Z})}$ and Theorem 3.1.5.

ion_theory

Lemma 4.1.10 (Algebraic deformation theory). *For every nilpotent divided power extension $(R' \twoheadrightarrow R, \gamma)$ of animated commutative $\mathcal{O}$-algebras, there is a canonical Cartesian diagram*

$$\begin{tikzcd}
\mathrm{BT}_n^{\mathcal{G},\mu,\mathrm{alg}}(R') \arrow[r] \arrow[d] & B\mathcal{P}_\mu^-(R'/^{\mathbb{L}}p^n) \arrow[d] \\
\mathrm{BT}_n^{\mathcal{G},\mu,\mathrm{alg}}(R) \arrow[r] & B\mathcal{P}_\mu^-(R/^{\mathbb{L}}p^n) \times_{B\mathcal{G}(R/^{\mathbb{L}}p^n)} B\mathcal{G}(R'/^{\mathbb{L}}p^n)
\end{tikzcd}$$

Proof of Theorem 4.1.3. The proof will be via Artin-Lurie representability [58, Theorem 7.1.6]. It is important for this that we allow animated inputs, though with this caveat the reader will note that the argument itself—except for the parts establishing finite presentation and affineness of the diagonal—is quite formal. In what follows, we will use without comment the fact that $\mathrm{BT}_n^{\mathcal{G},\mu} \otimes \mathbb{Z}/p^m\mathbb{Z}$ (for $m \geq 1$) and $\mathrm{Loc}_{\mathcal{G}(\mathbb{Z}/p^n\mathbb{Z})}$ are both locally finitely presented algebraic stacks over $\mathcal{O}$, and so satisfy the conditions of *loc. cit.*.

Condition (1) of *loc. cit.* on local finite presentation is verified by Lemma 4.1.9.

Conditions (2: being an étale sheaf), (3: integrability) , (5: infinitesimal cohesiveness) and (6: nilcompleteness) of [58, Theorem 7.1.6] all involve behavior with respect to limits and are easily verified from the definitions.

For condition (4) on the existence of a cotangent complex, note that, for any $\mathcal{O}$-algebra R, and a map $x : \mathrm{Spf}\, \hat{R} \to \mathrm{BT}_n^{\mathcal{G},\mu}$, the cotangent complex $\mathbb{L}_{\mathrm{BT}_n^{\mathcal{G},\mu}/\mathscr{O}_{E_v},x}$ is a perfect complex over $\hat{R}$ with Tor amplitude in $[0,1]$ and cohomology killed by p^n. In particular, it algebraizes canonically to a perfect complex over R.[20] If x lifts to a point $y \in \mathrm{BT}_n^{\mathcal{G},\mu,\mathrm{alg}}(R)$, then Lemma 4.1.10 shows that the assignment $y \mapsto \mathbb{L}_{\mathrm{BT}_n^{\mathcal{G},\mu}/\mathscr{O}_{E_v},x}$ is the desired cotangent complex for $\mathrm{BT}_n^{\mathcal{G},\mu,\mathrm{alg}}$ over $\mathcal{O}$.[21] Note that this is still a perfect complex over $\mathrm{BT}_n^{\mathcal{G},\mu,\mathrm{alg}}$ with Tor amplitude in $[0,1]$.

Condition (7) is verified by seeing that $\mathrm{BT}_n^{\mathcal{G},\mu,\mathrm{alg}}(R)$ is 1-truncated for any discrete $\mathcal{O}$-algebra R: Indeed, this is true for all the spaces involved in the fiber product defining it.

Thus we have verified all the conditions of Lurie's theorem, and so can conclude using this, the description of the cotangent complex, and the quasi-compactness of both $\mathrm{BT}_n^{\mathcal{G},\mu}$ and $\mathrm{Loc}_{\mathcal{G}(\mathbb{Z}/p^n\mathbb{Z})}$ that $\mathrm{BT}_n^{\mathcal{G},\mu,\mathrm{alg}}$ is a smooth 0-dimensional Artin stack over $\mathcal{O}$.

We are now tasked with showing that the diagonal Δ[22] is affine when $\mathcal{G}$ is reductive. For this, note that $\Delta[1/p]$ is affine (in fact, finite étale) and the p-adic completion $\widehat{\Delta}$ is also affine by Theorem 3.1.5.

[19] **Alex:** Maybe *formal étaleness* would be better.

[20] **Alex:** Maybe it would be worth explaining this a bit more, e.g., citing [6, Lemma 8.2] or maybe https://arxiv.org/pdf/2411.1 to save some teeth gnashing.

[21] **Alex:** It might be clearer to say the assignment $(x', x) \mapsto \mathbb{L}_{\mathrm{BT}_n^{\mathcal{G},\mu}/\mathscr{O}_{E_v}}|_x$ is the cotangent complex, where (x', x) is a point of $\mathrm{BT}_n^{\mathcal{G},\mu,\mathrm{alg}}$?

[22] **Alex:** Maybe just for readability we remind the reader we currently know that Δ is is a locally of finite type algebraic space over $\mathcal{O}$.

Moreover, it follows from Proposition 3.8.3 that the diagonal is a *separated* algebraic space: Indeed, the cited result shows that the map $\Delta(R) \to \Delta(F)$ is injective for any perfectoid valuation ring R of mixed characteristic with fraction field F. Since every mixed characteristic DVR is dominated by a perfectoid valuation ring, we see that the injectivity also holds for such DVRs. For DVRs of equal characteristic, this injectivity is immediate from the already noted facts about $\Delta[1/p]$ and $\widehat{\Delta}$.

Choose a smooth cover

$$\operatorname{Spec} R \to \mathrm{BT}_n^{\mathcal{G},\mu,\mathrm{alg}} \times \mathrm{BT}_n^{\mathcal{G},\mu,\mathrm{alg}}$$

by a smooth affine scheme over $\mathcal{O}$. Let B be the finite étale $R[1/p]$-algebra representing the pullback of $\Delta[1/p]$, and let S be thetopologically of finite type $\hat{R}$-algebra representing the pullback of $\widehat{\Delta}$. Then the criterion of Achinger-Youcis from [2, Proposition 3.6] tells us that $\Delta|_{\operatorname{Spec} R}$ is affine if and only if the natural map $B \to S[1/p]$ has dense image. This follows from Proposition 4.2.4 below. □

e_diagonal

4.2. **Analytic properties of the diagonal.**

The purpose of this subsection is to prove Proposition 4.2.4, and so to complete the proof of the affineness of the diagonal of $\mathrm{BT}_n^{\mathcal{G},\mu,\mathrm{alg}}$.

p:diagonal

Setup 4.2.1. Suppose that we have a smooth covering $\operatorname{Spec} R \to \mathrm{BT}_n^{\mathcal{G},\mu,\mathrm{alg}} \times \mathrm{BT}_n^{\mathcal{G},\mu,\mathrm{alg}}$ as in the proof of Theorem 4.1.3. This classifies a pair of n-truncated apertures $(\mathfrak{Q}_1, \mathfrak{Q}_2)$ over the p-completion $\hat{R}$ and a pair of $\mathcal{G}(\mathbb{Z}/p^n\mathbb{Z})$-local systems $(\mathsf{Q}_1, \mathsf{Q}_2)$ over $R[1/p]$ whose restrictions over $\hat{R}[1/p]$ are isomorphic to $(T_{\text{ét}}(\mathfrak{Q}_1), T_{\text{ét}}(\mathfrak{Q}_2))$. Write $\Delta_R \to \operatorname{Spec} R$ for the restriction of the diagonal, and let $\hat{\Delta}_R \to \operatorname{Spf} \hat{R}$ be the associated formal algebraic space. As we observed in the proof of Theorem 4.1.3, Δ_R is a finitely presented *separated* algebraic space over R.

tup_affine

Remark 4.2.2. By definition, $\hat{\Delta}_R$ parameterizes isomorphisms between the n-truncated apertures $\mathfrak{Q}_1$ and $\mathfrak{Q}_2$. The diagonal of $\mathrm{BT}_n^{\mathcal{G},\mu}$ is an affine, finitely presented map, and so $\hat{\Delta}_R$ is represented by topologically of finite type $\hat{R}$-algebra S. Similarly, $\Delta_R[1/p]$ is the finite étale scheme over $R[1/p]$ parameterizing isomorphisms between the $\mathcal{G}(\mathbb{Z}/p^n\mathbb{Z})$-local systems Q_1 and Q_2, and so is represented by a finite étale $R[1/p]$-algebra B.

setup_adic

Remark 4.2.3. Suppose now that R is of finite type over $\mathcal{O}$.[23] We can then associate with Δ_R the generic fiber $X \overset{\text{defn}}{=} (\widehat{\Delta}_R)_\eta$ of the formal scheme $\hat{\Delta}_R$: This is a finitely presented affinoid adic space over $Y \overset{\text{defn}}{=} \operatorname{Spa}(\hat{R}[1/p], \hat{R})$. We can also consider the affinoid adic space X' over Y obtained by taking the analytification of the finite étale $\hat{R}[1/p]$-scheme $\hat{R}[1/p] \otimes_R \Delta_R$.[24] Concretely, we have $X = \operatorname{Spa}(S[1/p], S)$ and $X' = \operatorname{Spa}(\hat{B}, \hat{B}^+)$, where $\hat{B} = B \otimes_{R[1/p]} \hat{R}[1/p]$, and $\hat{B}^+ \subset \hat{B}$ is the integral closure of $\hat{R}$.[25] We then obtain a diagram of adic spaces

$$\begin{tikzcd} X \arrow[r] \arrow[dr] & X' \arrow[d] \\ & Y \end{tikzcd}$$

where the vertical arrow is finite [34, (1.4.2)], and where the top horizontal map is an open immersion [34, Proposition 1.9.6]. For the latter, which follows from the separatedness of Δ_R, see also [2, Proposition 2.16].

Note that the map $X \to X'$ from Remark 4.2.3 yields maps of $R[1/p]$-algebras $B \to \hat{B} \to S[1/p]$. Our goal is to prove:

ional_open

Proposition 4.2.4. *The map $B \to S[1/p]$ has dense image.*

al_diamond

Remark 4.2.5 (The diagonal as an analytic diamond)**.** Suppose that T is a perfectoid R-algebra. Unwinding Remark 3.2.19 in the n-truncated case, and after étale localization on $\operatorname{Spf} T$ if necessary, we can assume that the restriction of $(\mathfrak{Q}_1, \mathfrak{Q}_2)$ over T arises from a pair (x_1, x_2) belonging to the set

[23] **Alex:** Was this not already our setup, or were you assuming they cover was not a smooth cover?

[24] **Alex:** Because it may be confusing for people not familiar with this sort of maneuver, maybe we should emphasize this is analytification relative to R.

[25] **Alex:** Nothing is technically wrong as written, but I just want to comment that $\widehat{R}$ may not be integrally closed in $\widehat{R}[1/p]$—this is fine as it makes sense to write $\operatorname{Spa}(A, S)$ where S is not integrally closed (it just only 'rembembers' this integral closure).

$\mathcal{G}(\mathbb{A}_T/p^n\mathbb{A}_T) \times \mathcal{G}(\mathbb{A}_T/p^n\mathbb{A}_T)$. Then, for any p-torsion free perfectoid T-algebra T' that is integrally closed in $T'[1/p]$, we have canonical identifications

$$\begin{aligned} X(\mathrm{Spa}(T'[1/p], T')) &= \{h \in H_\mu^{(n)}(T') :\ x_2 = h^{-1}x_1\varphi(\mu(\xi)h\mu(\xi)^{-1})\} \\ &= \{h \in H_\mu^{(n)}(T') :\ h = x_1\varphi(\mu(\xi)h\mu(\xi)^{-1})x_2^{-1}\}, \end{aligned}$$

and

$$\begin{aligned} X'(\mathrm{Spa}(T'[1/p], T')) &= \{h \in \mathcal{G}(\mathbb{A}_{T'}/p^n\mathbb{A}_{T'}[1/\xi]) :\ x_2 = h^{-1}x_1\varphi(\mu(\xi)h\mu(\xi)^{-1})\} \\ &= \{h \in \mathcal{G}(\mathbb{A}_{T'}/p^n\mathbb{A}_{T'}[1/\xi]) :\ h = x_1\varphi(\mu(\xi)h\mu(\xi)^{-1})x_2^{-1}\} \end{aligned}$$

In other words, we have concretely described the diamonds $X_T^\diamond$, $X_T^{',\diamond}$ underlying $\mathrm{Spa}(T[1/p], T) \times_Y X$ and $\mathrm{Spa}(T[1/p], T) \times_Y X'$.

earization **Remark 4.2.6** (Linearization)**.** Keep the setup from the previous remark. Choose a faithful representation $\mathcal{G} \hookrightarrow \mathrm{GL}(\Lambda)$, and set $M = \mathrm{End}(\Lambda) \otimes_{\mathbb{Z}_p} \mathbb{A}_T/p^n\mathbb{A}_T$: This is equipped with a canonical φ-semilinear bijection $1 \otimes \varphi$, which we will denote simply by φ. Write $\Phi : M[\xi^{-1}] \xrightarrow{\simeq} M[\xi^{-1}]$ for the operator $m \mapsto x_1\varphi(\mu(\xi))\varphi(m)\varphi(\mu(\xi))^{-1}x_2^{-1}$, and define functors $Z^\diamond, Z^{',\diamond}$ on affinoid perfectoid spaces over T by

$$\begin{aligned} Z^\diamond(\mathrm{Spa}(T'[1/p], T')) &= \{m \in \mathbb{A}_{T'} \otimes_{\mathbb{A}_T} M :\ m = \Phi(m) \in (\mathbb{A}_{T'} \otimes_{\mathbb{A}_T} M)[\xi^{-1}]\} \\ Z^{',\diamond}(\mathrm{Spa}(T'[1/p], T')) &= \{m \in (\mathbb{A}_{T'} \otimes_{\mathbb{A}_T} M)[\xi^{-1}] :\ m = \Phi(m) \in (\mathbb{A}_{T'} \otimes_{\mathbb{A}_T} M)[\xi^{-1}]\} \end{aligned}$$

We then have a canonical isomorphism $X_T^\diamond \xrightarrow{\simeq} Z^\diamond \times_{Z',\diamond} X_T^{',\diamond}$ of diamonds over $\mathrm{Spd}(T[1/p], T) = \mathrm{Spa}(T^\flat[1/\varpi], T^\flat)$, where $\varpi \in T^\flat$ is the pseudouniformizer obtained by taking the image of ξ.

earization **Lemma 4.2.7.** *$Z^\diamond$ and $Z^{',\diamond}$ are represented by affinoid perfectoid spaces $Z = \mathrm{Spa}(D, D^+)$ and $Z' = \mathrm{Spa}(D', D^{',+})$ over $\mathrm{Spa}(T[1/p], T)$. Moreover, the map $Z \to Z'$ is a rational open embedding, and the map $D \to D'$ has dense image.*

Proof. Via the usual dévissage, we reduce to the case where $n = 1$. Choose $k \geq 0$ such that $\Phi(M) \subset \varpi^{-k}M$ and set $\tilde{\Phi} = \varpi^k\Phi$.

By Artin-Schreier theory, the functor $C \mapsto (C \otimes_{T^\flat[\varpi^{-1}]} M[\varpi^{-1}])^{\Phi=1}$ is represented by a finite étale group scheme over $T^\flat[\varpi^{-1}]$ locally isomorphic to $\mathbb{F}_p^{\mathrm{rank}_{T^\flat} M}$. Therefore, there exists $r \geq 0$ such that, for all affinoid perfectoids $\mathrm{Spa}(T'[1/p], T')$ over T, we have

me_diamond
$$\begin{aligned} Z^{',\diamond}(\mathrm{Spa}(T'[1/p], T')) &= \{m \in \varpi^{-r}(T^{',\flat} \otimes_{T^\flat} M) :\ \Phi(m) = m\} \\ &\xrightarrow[\simeq]{m \mapsto \varpi^r m} \{n \in T^{',\flat} \otimes_{T^\flat} M :\ \Phi(n) = \varpi^{pr}n\} \\ &= \{n \in T^{',\flat} \otimes_{T^\flat} M :\ \tilde{\Phi}(n) = \varpi^{k+pr}n\}. \end{aligned} \tag{4.2.7.1}$$

Moreover, via this identification, we have

:z_diamond
$$Z^{',\diamond}(\mathrm{Spa}(T'[1/p], T')) \simeq Z^{',\diamond}(\mathrm{Spa}(T'[1/p], T')) \cap \varpi^r(T^{',\flat} \otimes_{T^\flat} M) \subset T^{',\flat} \otimes_{T^\flat} M. \tag{4.2.7.2}$$

Now, (4.2.7.1) tells us that $Z^{',\diamond}$ is represented by an affinoid perfectoid over $\mathrm{Spa}(T[1/p], T)$: it is a closed subsheaf of the perfectoid closed unit disk

$$\mathbf{D}_{\mathrm{perf}}(M) : (T'[1/p], T') \mapsto T^{',\flat} \otimes_{T^\flat} M$$

associated with M. Furthermore, (4.2.7.2) tells us that $Z^\diamond$ is a rational open in $Z^{',\diamond}$, obtained as the pre-image of the rational open immersion $\mathbf{D}_{\mathrm{perf}}(M) \xrightarrow{\varpi^r} \mathbf{D}_{\mathrm{perf}}(M)$. From this, the lemma follows. $\square$

Proof of Proposition 4.2.4. Note that $\hat{R}$ is p-completely smooth over $\mathcal{O}$ and so admits a p-completely flat map $\hat{R} \to \hat{R}_\infty$ where $\hat{R}_\infty$ is perfectoid and is the p-completed filtered colimit of finite flat $\hat{R}$-algebras. Lemma 4.2.7 and Remark 4.2.6 together tell us that

$$B\hat{\otimes}\hat{R}_\infty \to (S\hat{\otimes}\hat{R}_\infty)[1/p] \tag{4.2.7.3}$$

has dense image. Indeed, it suffices to show that if Q denotes the cokernel of the map of R-modules $B \to S[1/p]/S$ then $Q = 0$. Let us note that Q is p^∞-torsion, and as $R \to \hat{R}_\infty$ is p-completely faithfully flat, we see that Q is zero if and only if $Q \otimes_R \hat{R}_\infty$ is zero. But, this follows from the density of the image in (4.2.7.3). □

4.3. **An extension property.** The following proposition will be used to prove Theorem G.

Proposition 4.3.1. *Suppose that we have two η-normal algebraic spaces $\mathcal{X}_1, \mathcal{X}_2$ over $\mathcal{O}$ with $\mathfrak{Q}_i \in \mathrm{BT}_\infty^{\mathcal{G},\mu,\mathrm{alg}}(X_i)$ lifting $\mathsf{Q}_i \in \mathrm{Loc}_{\mathcal{G}(\mathbb{Z}_p)}(X_i)$ where $X_i \overset{\mathrm{defn}}{=} \mathcal{X}_i[1/p]$, and suppose that the following holds:*

1. *$\mathcal{X}_2$ is of finite type over $\mathcal{O}$ and $\mathcal{X}_1$ is excellent.*
2. *The formal classifying map for $\mathfrak{Q}_2$*
$$\widehat{\mathcal{X}}_2 \to \mathrm{BT}_\infty^{\mathcal{G},\mu}$$
is formally unramified.
3. *There is a map $f : X_1 \to X_2$ and an isomorphism $\zeta : \mathsf{Q}_2|_{X_1} \xrightarrow{\simeq} \mathsf{Q}_1$.*
4. *For every mixed characteristic $(0,p)$ complete discrete valuation field F over $\mathcal{O}$ with perfect residue field, the map $X_1(F) \to X_2(F)$ maps $\mathcal{X}_1(\mathscr{O}_F)$ into $\mathcal{X}_2(\mathscr{O}_F)$.*

Then f extends uniquely to a map $\mathcal{X}_1 \to \mathcal{X}_2$ and the isomorphism ζ lifts uniquely to an isomorphism $\mathfrak{Q}_1|_{\mathcal{X}_1} \xrightarrow{\simeq} \mathfrak{Q}_2$.

Proof. The proof we are about to see is essentially a generalization of an argument of Pappas from [70, Theorem 7.1.7] (see also [38, Theorem 3.13]).

Let $\mathcal{Y} \to \mathcal{X}_1 \times_{\mathrm{Spec}\,\mathcal{O}} \mathcal{X}_2$ be constructed as follows: Take the Zariski closure $\mathcal{Y}'$ of the graph

$$X_1 \xrightarrow{\mathrm{id}\times f} X_1 \times_{\mathrm{Spec}\,E} X_2 \hookrightarrow \mathcal{X}_1 \times_{\mathrm{Spec}\,\mathcal{O}} \mathcal{X}_2.$$

Since $\mathcal{X}_2$ is of finite type over $\mathcal{O}$, $\mathcal{Y}'$ is of finite type over $\mathcal{X}_1$. Now take $\mathcal{Y}$ to be the η-normalization of $\mathcal{Y}'$[19]: This is of finite type over $\mathcal{X}_1$ because of the excellence hypothesis [82, Lemma 035S].

The restrictions of $\mathfrak{Q}_1$ and $\mathfrak{Q}_2$ yield two objects in $\mathrm{BT}_\infty^{\mathcal{G},\mu}(\mathcal{Y})$ equipped with an isomorphism of the underlying local systems in $\mathrm{Loc}_{\mathcal{G}(\mathbb{Z}_p)}(\mathcal{Y}[1/p])$. Therefore, by Theorem 4.1.5, we see that there is an isomorphism $\mathfrak{Q}_1|_{\mathcal{Y}} \xrightarrow{\simeq} \mathfrak{Q}_2|_{\mathcal{Y}}$. In other words, the classifying maps

$$\mathcal{Y} \to \mathcal{X}_1 \to \mathrm{BT}_\infty^{\mathcal{G},\mu,\mathrm{alg}}\ ;\ \mathcal{Y} \to \mathcal{X}_2 \to \mathrm{BT}_\infty^{\mathcal{G},\mu,\mathrm{alg}}$$

are isomorphic.

Suppose that we have a closed geometric mod-p point $y \in \mathcal{Y}(\kappa)$ mapping to $x_i \in \mathcal{X}_i(\kappa)$. We claim that the map of complete local Noetherian rings $\widehat{\mathscr{O}}_{\mathcal{X}_1,x_1} \to \widehat{\mathscr{O}}_{\mathcal{Y},y}$ is finite.

Write $z \in \mathrm{BT}_\infty^{\mathcal{G},\mu}(W(\kappa))$ for the point corresponding to $\mathfrak{Q}_{1,x_1} \simeq \mathfrak{Q}_{2,x_2}$. Let $\widehat{\mathscr{O}}_y$ (resp. $\widehat{\mathscr{O}}_{x_1}$, $\widehat{\mathscr{O}}_{x_2}$, $\widehat{\mathscr{O}}_z$) be the deformation rings over $W(\kappa)$ for $\mathcal{Y}$ (resp. $\mathcal{X}_1$, $\mathcal{X}_1$, $\mathrm{BT}_\infty^{\mathcal{G},\mu,\mathrm{alg}}$) at their respective κ-points. We now have factorings

$$\widehat{\mathscr{O}}_z \to \widehat{\mathscr{O}}_{x_1} \to \widehat{\mathscr{O}}_y\ ;\ \widehat{\mathscr{O}}_z \to \widehat{\mathscr{O}}_{x_2} \to \widehat{\mathscr{O}}_y$$

of the map $\widehat{\mathscr{O}}_z \to \widehat{\mathscr{O}}_y$. By our formal unramifiedness hypothesis, the first map in the second factoring is surjective. By construction, the combined map

$$\widehat{\mathscr{O}}_{x_1}\hat{\otimes}_{W(\kappa)}\widehat{\mathscr{O}}_{x_2} \to \widehat{\mathscr{O}}_y,$$

[19]More specifically, take $\mathcal{Y}$ to be relative spectrum over $\mathcal{Y}'$ of the quasi-coherent $\mathcal{O}_{\mathcal{Y}'}$-algebra given by integral closure of $\mathcal{O}_{\mathcal{Y}'}$ in $j_*\mathcal{O}_{\mathcal{Y}'_\eta}$. where $j\colon \mathcal{Y}'_\eta \to \mathcal{Y}'$ is the open inclusion.

whose image is now the same as that of the second factor $\widehat{\mathscr{O}}_{x_2}$, is finite (since $\mathcal{Y} \to \mathcal{X}_1 \times_{\mathrm{Spf}\,\mathcal{O}} \mathcal{X}_2$ is finite). We conclude that the map $\widehat{\mathscr{O}}_{x_1} \to \widehat{\mathscr{O}}_y$ is already finite.

What we have seen above shows that the map $\mathcal{Y} \to \mathcal{X}_1$ is quasi-finite; see for instance Ch. IV, Propositions 2 and 3 of [76]. Since it is also an isomorphism after inverting p, it follows from Zariski's Main Theorem and the η-normality of $\mathcal{X}_1$ that it is in fact an open immersion; see for instance the proof of [63, Corollary 5.15].

We now claim that every mod-p geometric point $x_1 \in \mathcal{X}_1(\kappa)$ is inside the open $\mathcal{Y}$. Indeed, by the flatness of $\mathcal{X}_1$, we can find a mixed characteristic complete DVR $\mathscr{O}_F$ with residue field κ such that x_1 lifts to $\mathcal{X}_1(\mathscr{O}_F)$. But hypothesis (4) tells us that this lift lies in $\mathcal{Y}(\mathscr{O}_F)$.

We have now shown $\mathcal{Y} \xrightarrow{\simeq} \mathcal{X}_1$, which implies that $\mathcal{Y}$ is the graph of a unique extension $\mathcal{X}_1 \to \mathcal{X}_2$ with the desired properties. $\square$

ng_central 4.4. **Descending along central covers.** In this subsection, we will consider the problem of descending apertures along central extensions of smooth affine group schemes.

ral_covers **Lemma 4.4.1** (Behavior under central covers)**.** *Suppose that we have a central extension $\mathcal{G} \to \overline{\mathcal{G}}$ of smooth affine $\mathbb{Z}_p$-groups. Then the map*

$$\mathrm{BT}_n^{\tilde{\mathcal{G}},\mu,\mathrm{alg}} \to \mathrm{BT}_n^{\mathcal{G},\mu,\mathrm{alg}}$$

is an étale surjection.

Proof. Our hypothesis implies that we have $\mathcal{P}_\mu^- = \mathcal{G} \times_{\overline{\mathcal{G}}} \overline{\mathcal{P}}_\mu^-$. Therefore, the deformation theory explained in Theorem 3.1.5 tells us that the map in question is étale after taking v-adic completions, and is étale after inverting p by definition. To see that it is also surjective, we can check on algebraically closed points. In characteristic 0, the surjectivity is clear, while in characteristic p it follows from the quotient description given in Remark 3.2.19. $\square$

over_gerbe **Remark 4.4.2.** In fact, the description obtained from Remark 3.2.19 says more. For every algebraically closed field κ, the map from $\mathrm{BT}_n^{\mathcal{G},\mu}(\kappa)$ to $\mathrm{BT}_n^{\overline{\mathcal{G}},\overline{\mu}}(\kappa)$ is isomorphic to

$$[\mathcal{G}(W_n(\kappa))/H_\mu^{(n)}(\kappa)] \to [\overline{\mathcal{G}}(W_n(\kappa))/H_{\overline{\mu}}^{(n)}(\kappa)].$$

If $\mathcal{Z} = \ker(\mathcal{G} \to \overline{\mathcal{G}})$, then one finds that this map is a torsor under the groupoid $[\mathcal{Z}(W_n(\kappa))/\mathrm{Ad}_\varphi \mathcal{Z}(W_n(\kappa))]$, where the notation is as in Construction 3.3.1 below: That is, $\mathcal{Z}(W_n(\kappa))$ is acting on itself via σ-conjugation.

If $\mathcal{Z}_{\text{ét}}$ is the quotient of $\mathcal{Z}$ by its connected part $\mathcal{Z}^\circ$, then Lang's lemma tells us that this groupoid is an extension of

$$[\mathcal{Z}_{\text{ét}}(\kappa)/\mathrm{Ad}_\varphi \mathcal{Z}_{\text{ét}}(\kappa)] \simeq R\Gamma_{\text{ét}}(\mathrm{Spec}\,\mathbb{F}_p, \mathcal{Z}_{\text{ét}})[1]$$

by the classifying stack $B\underline{\mathcal{Z}^\circ(\mathbb{Z}/p^n\mathbb{Z})} \simeq \mathrm{Loc}_{\mathcal{Z}^\circ(\mathbb{Z}/p^n\mathbb{Z})}$. That is, $\mathrm{BT}_n^{\mathcal{G},\mu} \to \mathrm{BT}_n^{\overline{\mathcal{G}},\overline{\mu}}$ factors as a gerbe banded by $\mathcal{Z}^\circ(\mathbb{Z}/p^n\mathbb{Z})$ followed by a torsor for the finite groupoid $R\Gamma_{\text{ét}}(\mathrm{Spec}\,\mathbb{F}_p, \mathcal{Z}_{\text{ét}})[1]$. Another way of saying this is that it factors as a gerbe banded by $\mathcal{Z}(\mathbb{Z}/p^n\mathbb{Z})$ followed by a torsor for $H^1_{\text{ét}}(\mathrm{Spec}\,\mathbb{F}_p, \mathcal{Z}_{\text{ét}})$.

_algebraic **Remark 4.4.3.** The description at the end of Remark 4.4.2 is valid for the map of algebraic stacks $\mathrm{BT}_n^{\mathcal{G},\mu,\mathrm{alg}} \to \mathrm{BT}_n^{\overline{\mathcal{G}},\overline{\mu},\mathrm{alg}}$. To see this, just note that the map $\mathrm{Loc}_{\mathcal{G}(\mathbb{Z}/p^n\mathbb{Z})} \to \mathrm{Loc}_{\overline{\mathcal{G}}(\mathbb{Z}/p^n\mathbb{Z})}$ factors as

$$\mathrm{Loc}_{\mathcal{G}(\mathbb{Z}/p^n\mathbb{Z})} \to \mathrm{Loc}_{\mathcal{G}(\mathbb{Z}/p^n\mathbb{Z})/\mathcal{Z}(\mathbb{Z}/p^n\mathbb{Z})} \to \mathrm{Loc}_{\overline{\mathcal{G}}(\mathbb{Z}/p^n\mathbb{Z})}.$$

The first map is a gerbe banded by $\mathcal{Z}(\mathbb{Z}/p^n\mathbb{Z})$, while the second is a torsor under

$$\overline{\mathcal{G}}(\mathbb{Z}/p^n\mathbb{Z})/\mathrm{im}\ \mathcal{G}(\mathbb{Z}/p^n\mathbb{Z}) \simeq H^1_{\text{ét}}(\mathrm{Spec}\,\mathbb{F}_p, \mathcal{Z}_{\text{ét}}).$$

The isomorphism here follows from the connectedness of $\mathcal{G}$ and $\mathcal{Z}^\circ$, and Lang's lemma.

hfulness_2 **Proposition 4.4.4** (Descending along central covers)**.** *Suppose that $\mathcal{X}$ is a normal flat Noetherian algebraic space over $\mathcal{O}$ with $\mathsf{Q} \in \mathrm{Loc}_{\mathcal{G}(\mathbb{Z}_p)}(\mathcal{X}[1/p])$. Suppose that there exist:*

(1) A central extension $\tilde{\mathcal{G}} \to \mathcal{G}$ of reductive $\mathbb{Z}_p$-groups;
(2) A finite extension $\tilde{E}/E$, a place $\tilde{v}|v$ of $\tilde{E}$ and a lift $\tilde{\mu} : \mathbb{G}_{m,\tilde{\mathcal{O}}} \to \tilde{\mathcal{G}}_{\tilde{\mathcal{O}}}$ of μ, where $\tilde{\mathcal{O}} = \mathscr{O}_{\tilde{E}_{\tilde{v}}}$.
(3) An étale cover $\mathcal{X}' \to \mathcal{X} \otimes \mathscr{O}_{\tilde{E},(\tilde{v})}$ with a lift $\tilde{\mathsf{Q}} \in \mathrm{Loc}_{\tilde{\mathcal{G}}}(\mathcal{X}'[1/p])$ of the restriction of Q;
(4) A lift $\tilde{\mathfrak{Q}} \in \mathrm{BT}_{\infty}^{\tilde{\mathcal{G}},\tilde{\mu},\mathrm{alg}}(\mathcal{X}')$ of $\tilde{\mathsf{Q}}$.

Then Q lifts to $\mathfrak{Q} \in \mathrm{BT}_{\infty}^{\mathcal{G},\mu,\mathrm{alg}}(\mathcal{X})$, and the p-completed classifying map $\widehat{\mathcal{X}} \to \mathrm{BT}_{\infty}^{\mathcal{G},\mu}$ for $\mathfrak{Q}$ is formally étale if and only if the corresponding one $\widehat{\mathcal{X}}'_{\tilde{v}} \to \mathrm{BT}_{\infty}^{\tilde{\mathcal{G}},\tilde{\mu}}$ is so.

Proof. The first part of the conclusion is immediate from Corollary 4.1.6, while the second is a consequence of Lemma 4.4.1. □

ral_covers

4.5. **Lifting along finite central covers.** The purpose of this subsection is to give some pointwise criteria for lifting apertures along finite central covers of group schemes. The prime 2 poses a bit of difficulty, so the reader willing to assume $p > 2$ can ignore the additional hypotheses imposed to avoid this assumption.

Definition 4.5.1. Let $\mathcal{O}$ be a p-torsion free, absolutely unramified DVR with perfect residue field k. Fix an algebraic closure κ for k. Suppose that $\mathcal{X}$ is an algebraic space of finite type over a complete DVR with generic fiber X and that $f : Y \to X$ is a finite étale cover. Let $\hat{\mathcal{O}}$ be the p-adic completion of $\mathcal{O}$. Given an extension of complete discrete valuation fields $\hat{\mathcal{O}}[1/p] \subset K$ and a point $x \in \mathcal{X}(\mathscr{O}_K)$, we will say that f is **unramified at** x—or, equivalently, that x is f**-unramified**—if the finite étale K-algebra corresponding to $Y \times_X \operatorname{Spec} K$ is a product of unramified extensions of K.

on_torsors

Lemma 4.5.2. *Suppose that $\mathcal{X}$ is a smooth algebraic space over $\mathcal{O}$ with generic fiber X, and suppose that $f : Y \to X$ is a finite Galois cover with abelian Galois group Δ satisfying the following property:*

- *There is a dense open subspace $U \subset \mathcal{X} \otimes k$ such that, for every $x \in U(\kappa)$, there exists an f-unramified lift $\tilde{x} \in \mathcal{X}(W(\kappa))$ of x.*

If $p = 2$, impose the following additional condition:

- *For every $x \in U(\kappa)$ as above, there exists a finite extension $K/W(\kappa)[1/p]$ with uniformizer π such that every lift of x to $\mathcal{X}(\mathscr{O}_K/\pi^2\mathscr{O}_K)$ admits a further lift to an f-unramified point $\tilde{x} \in \mathcal{X}(\mathscr{O}_K)$.*

Then Y extends to a Galois cover $\mathcal{Y} \to \mathcal{X}$.

Proof. Let $F : \mathcal{Y} \to \mathcal{X}$ be the normalization of $\mathcal{X}$ in Y. We want to show that F is finite étale. Since $\mathcal{O} \to \hat{\mathcal{O}}$ is faithfully flat, F is finite étale if and only if its base-change over $\widehat{\mathcal{O}}$ is so. Moreover, $\mathcal{Y} \otimes_{\mathcal{O}} \hat{\mathcal{O}}$ is the normalization of $\mathcal{X} \otimes_{\mathcal{O}} \hat{\mathcal{O}}$ in $Y \otimes_{\mathcal{O}[1/p]} \hat{\mathcal{O}}[1/p]$: this follows because $\hat{\mathcal{O}}$ is a filtered colimit of smooth $\mathcal{O}$-algebras [82, Section 07GB]. Therefore, without loss of generality, we can assume that $\mathcal{O} = \hat{\mathcal{O}} = W(\kappa)$ is p-adically complete and that $k = \kappa$ is algebraically closed.

By dévissage, we can reduce to the case where $\Delta = \mathbb{Z}/\ell\mathbb{Z}$ for some prime ℓ: This reduction only needs the observation that, if we have a factoring $\mathcal{Y} \to \mathcal{X}' \to \mathcal{X}$ with the second map finite étale, then the pre-image of U in $\mathcal{X}'$ satisfies the same properties as U but now with respect to $\mathcal{Y} \to \mathcal{X}'$.

Given our first hypothesis, it follows from Lemma 4.5.3 below that, for every generic point $\eta \to \mathcal{Y} \otimes \kappa$, the local ring $\mathscr{O}_{\mathcal{Y},\eta}$ is a DVR with maximal ideal generated by p.

By purity of branch locus [29, Corollaire 3.3], it suffices to know that, for any such η, lying above a generic point $\eta' \to \mathcal{X} \otimes \kappa$, the map $\mathscr{O}_{\mathcal{X},\eta'} \to \mathscr{O}_{\mathcal{Y},\eta}$ of DVRs, both admitting p as a uniformizer, is étale. If the corresponding finite extension of residue fields $k(\eta)/k(\eta')$ is separable, and in particular, if $\ell \neq p$, then this is immediate.

Therefore, we can assume that $\ell = p$ and that the extension of residue fields is purely inseparable of degree p. In this case, any generator of Δ will act as the identity on the residue field $k(\eta)$. If $p > 2$, this is impossible: See for instance the proof of [68, Proposition 3.22].

So we are left now with the case where $p = 2$ with a purely inseparable extension of residue fields. We then have

$$\mathscr{O}_{\mathcal{Y},\eta} = \mathscr{O}_{\mathcal{X},\eta'}[x]/(x^2 - t)$$

where $t \in \mathscr{O}^{\times}_{\mathcal{X},\eta'}$ maps to an element in $k(\eta')^{\times} \backslash (k(\eta')^{\times})^2$. We will see in this situation that there exists an open subscheme $V \subset \mathcal{X} \otimes \kappa$ such that, for all $x \in \mathcal{X}(\kappa)$, and for any finite extension $K/W(\kappa)[1/p]$ with uniformizer π, there are lifts $x_1 \in \mathcal{X}(\mathscr{O}_K/\pi^2\mathscr{O}_K)$ such that every further lift $\tilde{x} \in \mathcal{X}(\mathscr{O}_K)$ of x_1 fails to be f-unramified. This of course contradicts our prevailing hypothesis when $p = 2$.

For this, localizing on $\mathcal{X}$ if necessary, we can assume that t extends to a global section of $\mathscr{O}^{\times}_{\mathcal{X}}$ and this section acquires a square-root over $\mathcal{Y}$. We can also assume that the differential form dt is nowhere vanishing (see for instance the argument in [32, Proposition 6.14]). This means that, for any $x \in \mathcal{X}(\kappa)$ and any finite extension $K/W(\kappa)[1/p]$ with uniformizer π, there are lifts $x_1 \in \mathcal{X}(\mathscr{O}_K/\pi^2\mathscr{O}_K)$ such that $t(x_1) \in (\mathscr{O}_K/\pi^2\mathscr{O}_K)^{\times}$ is not a square. But then, for any lift $\tilde{x} \in \mathcal{X}(\mathscr{O}_K)$ of x_1, $t(\tilde{x}) \in \mathscr{O}^{\times}_K$ is not a square. This implies that $\tilde{x}$ cannot be f-unramified. □

educedness **Lemma 4.5.3.** *Let $\mathcal{O}$ be a complete DVR with uniformizer π and algebraically closed residue field k. Suppose that $\mathcal{Y}$ is an integral scheme, flat and of finite type over $\mathcal{O}$ such that, for every generic point η of the special fiber $\mathcal{Y}_k$, the local ring $\mathscr{O}_{\mathcal{Y},\eta}$ is a DVR. Suppose also that the set*

$$\mathrm{im}(\mathcal{Y}(\mathcal{O}) \to \mathcal{Y}(k))$$

is set-theoretically Zariski dense in $\mathcal{Y}_k$. Then, $\mathcal{Y}_k$ is generically reduced. Equivalently, for any generic point η for $\mathcal{Y}_k$, π is a uniformizer in $\mathscr{O}_{\mathcal{Y},\eta}$

Proof. This is essentially [32, Lemma 6.6], but, since the hypotheses there are a bit stronger, we recall the proof here. Let $\varpi \in \mathscr{O}_{\mathcal{Y},\eta}$ be a uniformizer, and suppose that we have $\pi = u\varpi^m$ for some $u \in \mathscr{O}^{\times}_{\mathcal{Y},\eta}$. By localizing on $\mathcal{Y}$, we can assume that $\mathcal{Y} = \operatorname{Spec} A$ is affine and that we have $\varpi \in A$ and $u \in A^{\times}$. Furthermore, by our density hypothesis, we can also assume that we have a section $x : A \to \mathcal{O}$ of the $\mathcal{O}$-algebra A. This shows $\pi = x(u)x(\varpi)^m \in \mathcal{O}$. Clearly, this is possible only if $m = 1$. □

on_torsors **Remark 4.5.4.** Given $\mathcal{X}$ and $Y \to X$ as in Lemma 4.5.2, the hypotheses there hold in the following situation: There is an open dense subspace $U \subset \mathcal{X} \otimes k$ such that, for every $x \in U(\kappa)$, there exists an ideal $J_x \subset \widehat{\mathscr{O}}_{\mathcal{X},x}$ with the following properties:

(1) $R_x = \widehat{\mathscr{O}}_{\mathcal{X},x}/J_x$ is reduced and finite flat over $W(\kappa)$;
(2) For every finite extension $K/W(\kappa)[1/p]$ and every map $R_x \to \mathscr{O}_K$ of $W(\kappa)$-algebras, the corresponding point $\tilde{x} \in \mathcal{X}(\mathscr{O}_K)$ is f-unramified.
(3) There is a section $R_x \to W(\kappa)$;
(4) $J_x + (p) = \mathfrak{m}_x^2 + (p)$, where $\mathfrak{m}_x \subset \widehat{\mathscr{O}}_{\mathcal{X},x}$ is the maximal ideal;

Indeed, conditions (2) and (3) ensure that every $x \in U(\kappa)$ admits a $W(\kappa)$-valued f-unramified lift.

_extending **Lemma 4.5.5.** *Suppose that $\mathcal{X}$ is a smooth algebraic space over $\mathcal{O}$ with generic fiber X and that $\tilde{\mathcal{X}} \to \mathcal{X}$ is a gerbe banded by a finite abelian group Δ. Suppose that there is a section $s : X \to \tilde{\mathcal{X}}$ and an open dense subset $U \subset \mathcal{X} \otimes k$ with the following property:*

- *For every $x \in U(\kappa)$, there exists a lift $\tilde{x} \in \mathcal{X}(W(\kappa))$ of x such that the section $s \circ \tilde{x}[1/p]$: $\operatorname{Spec} W(\kappa)[1/p] \to \tilde{\mathcal{X}}$ extends over $\operatorname{Spec} W(\kappa)$.*

If $p = 2$, impose the following additional condition:

- *For every $x \in U(\kappa)$ as above, there exists a finite extension $K/W(\kappa)[1/p]$ with uniformizer π such that every lift of x to $\mathcal{X}(\mathscr{O}_K/\pi^2\mathscr{O}_K)$ admits a further lift $\tilde{x} \in \mathcal{X}(\mathscr{O}_K)$ such that $s \circ \tilde{x}[1/p]$ extends over $\operatorname{Spec} \mathscr{O}_K$.*

Then s extends to a section $\mathcal{X} \to \tilde{\mathcal{X}}$.

Proof. To begin, if we have two extensions of s over $\mathcal{X}$, then the space of isomorphisms between them is a Δ-torsor over $\mathcal{X}$, admitting a canonical trivialization over X, and so is itself canonically trivial (see for instance [82, Lemma 0BQG]). In particular, to prove the existence of such an extension, we can work étale locally and assume that the gerbe is trivial. The section s now corresponds to a Δ-torsor $Y \to X$, and the extension property amounts to the concrete assertion that the normalization of $\mathcal{X}$ in Y is a finite Galois cover $\mathcal{Y} \to \mathcal{X}$. This follows from our hypotheses and Lemma 4.5.2. $\square$

tral_cover **Setup 4.5.6.** We will now return to the notation from §4.1. Suppose that we have the following data:

(1) A flat normal algebraic space $\mathcal{X}$ over $\mathcal{O}$ with generic fiber X;
(2) A surjective map $\mathcal{G} \to \overline{\mathcal{G}}$ of smooth affine $\mathbb{Z}_p$-groups with finite flat kernel $\mathcal{Z}$;
(3) An aperture $\overline{\mathfrak{Q}} \in \mathrm{BT}_\infty^{\overline{\mathcal{G}},\overline{\mu},\mathrm{alg}}(\mathcal{X})$ lifting $\overline{\mathsf{Q}} \in \mathrm{Loc}_{\overline{\mathcal{G}}(\mathbb{Z}_p)}(X)$;
(4) A finite Galois cover $Y \to X$ with abelian Galois group Δ such that $\overline{\mathsf{Q}}$ lifts to $\mathsf{Q} \in \mathrm{Loc}_{\mathcal{G}(\mathbb{Z}_p)}(Y)$.

Let $\mathcal{Y} \to \mathcal{X}$ be the normalization of $\mathcal{X}$ in Y. We will be interested in the following questions:

(1) Does Q lift to $\mathfrak{Q} \in \mathrm{BT}_\infty^{\mathcal{G},\mu,\mathrm{alg}}(\mathcal{X})$?
(2) Is $\mathcal{Y} \to \mathcal{X}$ once again a Galois cover?

Definition 4.5.7. With the setup above, given an extension of complete discrete valuation fields $E_v \subset K$ and a point $x \in \mathcal{X}(\mathscr{O}_K)$, we will say that x is (f, Q)**-apertile** if f is unramified at x and if the following additional condition holds:

- Suppose that $Y \times_{X,x[1/p]} \operatorname{Spec} E = \operatorname{Spec} A_x[1/p]$, where A_x is a product of rings of integers in unramified extensions of K. Then the restriction of Q over $A_x[1/p]$ lifts to
$$\mathfrak{Q}_x \in \mathrm{BT}_\infty^{\mathcal{G},\mu,\mathrm{alg}}(A_x).$$

tral_cover **Proposition 4.5.8** (Lifting along finite central covers)**.** *Fix an algebraic closure κ for $k(v)$. Suppose that we are in the situation of Setup 4.5.6, and suppose that the following conditions hold:*

(1) $\mathcal{X}$ is a smooth algebraic space over $\mathcal{O}$;
(2) There exists a dense open subspace $U \subset \mathcal{X} \otimes k(v)$ such that, for all $x \in U(\kappa)$, there exists an (f, Q)-apertile lift $\tilde{x} \in \mathcal{X}(W(\kappa))$.
(3) If $p = 2$, suppose also that for $x \in U(\kappa)$ as above, there exists a finite extension $K/W(\kappa)[1/p]$ with uniformizer π such that every lift of x to $\mathcal{X}(\mathscr{O}_K/\pi^2\mathscr{O}_K)$ admits a further lift to an (f, Q)-apertile point of $\mathcal{X}(\mathscr{O}_K)$.

Then Q lifts to $\mathfrak{Q} \in \mathrm{BT}_\infty^{\mathcal{G},\mu,\mathrm{alg}}(\mathcal{Y})$ and $\mathcal{Y} \to \mathcal{X}$ is a finite Galois cover.

Proof. That $\mathcal{Y} \to \mathcal{X}$ is a finite Galois cover is immediate from our hypotheses and Lemma 4.5.2. In particular, $\mathcal{Y}$ is smooth over $\mathcal{O}$.

Since $\mathcal{Z} = \ker(\mathcal{G} \to \overline{\mathcal{G}})$ is finite flat, the map $\mathcal{Z}(\mathbb{Z}/p^m\mathbb{Z}) \to \mathcal{Z}(\mathbb{Z}/p^n\mathbb{Z})$ is an isomorphism for all $m \geq n$ large enough. Therefore, the description from Remark 4.4.3 shows that the map
$$\mathrm{BT}_\infty^{\mathcal{G},\mu,\mathrm{alg}} \to \mathrm{BT}_\infty^{\overline{\mathcal{G}},\mu,\mathrm{alg}} \times_{\mathrm{BT}_n^{\overline{\mathcal{G}},\overline{\mu},\mathrm{alg}}} \mathrm{BT}_n^{\mathcal{G},\mu,\mathrm{alg}}$$
is an isomorphism for n sufficiently large. In particular, the map
$$\tilde{\mathcal{Y}} \overset{\mathrm{defn}}{=} \mathrm{BT}_\infty^{\mathcal{G},\mu,\mathrm{alg}} \times_{\mathrm{BT}_\infty^{\overline{\mathcal{G}},\overline{\mu},\mathrm{alg}}} \mathcal{Y} \to \mathcal{Y}$$
is a finite étale stack factoring as
$$\tilde{\mathcal{Y}} \to \tilde{\mathcal{Y}}_1 \to \mathcal{Y}$$
where the first map is a gerbe banded by the finite abelian group $\Delta_1 \overset{\mathrm{defn}}{=} \mathcal{Z}(\mathbb{Z}/p^n\mathbb{Z})$ and the second is a torsor for the finite abelian group $\Delta_2 \overset{\mathrm{defn}}{=} H^1_{\text{ét}}(\operatorname{Spec} \mathbb{F}_p, \mathcal{Z})$.

The existence of Q gives us a section of $\tilde{\mathcal{Y}}$ over Y, and we want to show that this section extends to one over $\mathcal{Y}$. For the associated section of $\tilde{\mathcal{Y}}_1$, this is immediate. Therefore, we have to show that the trivialization of the Δ_1-gerbe $\mathcal{Y} \times_{\tilde{\mathcal{Y}}_1} \tilde{\mathcal{Y}} \to \mathcal{Y}$ over the generic fiber given by Q extends over $\mathcal{Y}$. This now follows from Lemma 4.5.5. $\square$

nt_lifting **Remark 4.5.9** (An ℓ-adic variant)**.** The argument used above also applies to ℓ-adic local systems for $\ell \neq p$. Namely, suppose that $\mathcal{H} \to \overline{\mathcal{H}}$ is a finite central cover of smooth affine group schemes over $\mathbb{Z}_\ell$. Suppose that we have:

(1) A smooth algebraic space $\mathcal{X}$ over $\mathcal{O}$ with generic fiber X;
(2) A local system $\overline{\mathsf{T}} \in \mathrm{Loc}_{\overline{\mathcal{H}}(\mathbb{Z}_\ell)}(\mathcal{X})$;
(3) A finite Galois cover $f : Y \to X$ with abelian Galois group Δ such that $\overline{\mathsf{T}}|_Y$ lifts to $\mathsf{T} \in \mathrm{Loc}_{\mathcal{H}(\mathbb{Z}_\ell)}(Y)$.

Let us say that, for $K/W(\kappa)[1/p]$ a finite extension, a point $\tilde{x} \in \mathcal{X}(\mathscr{O}_K)$ is (f, T)**-unramified** if it is f-unramified, and if the restriction of T over the fiber $Y_{\tilde{x}}$ is also an unramified $\mathcal{H}(\mathbb{Z}_\ell)$-local system. Suppose that there is a dense open subspace $U \subset \mathcal{X} \otimes k(v)$ such that, for all $x \in U(\kappa)$, there exists an (f, T)-unramified lift $\tilde{x} \in \mathcal{X}(\mathscr{O}_{E_v})$. If $p = 2$, then suppose also that for $x \in U(\kappa)$ as above, there exists a finite extension $K/W(\kappa)[1/p]$ with uniformizer π such that every lift of x to $\mathcal{X}(\mathscr{O}_K/\pi^2\mathscr{O}_K)$ admits a further lift to an (f, T)-unramified point of $\mathcal{X}(\mathscr{O}_K)$. Then:

(1) f extends to a finite Galois cover $\mathcal{Y} \to \mathcal{X}$;
(2) T extends to a $\mathcal{H}(\mathbb{Z}_\ell)$-local system over $\mathcal{Y}$.

4.6. **A criterion for discreteness.** The next result will be used in § 6.7.

s_a_scheme **Lemma 4.6.1.** *Suppose that we have a finite étale map $\mathcal{X}' \to \mathcal{X}$ of normal Deligne-Mumford stacks flat over $\mathcal{O}$ with the following properties:*

(1) $\mathcal{X} = \mathcal{X}'/H$ for a finite group H;
(2) $\mathcal{X}'$ and $\mathcal{X}[1/p]$ are separated algebraic spaces;
(3) There exists a map $\mathcal{X} \to \mathrm{BT}_\infty^{\mathcal{G},\mu,\mathrm{alg}}$ whose v-adic formal completion is formally étale.

Then in fact H acts freely on $\mathcal{X}'$. Therefore, $\mathcal{X}$ is also a separated algebraic space, and, if $\mathcal{X}'$ is a quasi-projective scheme, then $\mathcal{X}$ is also a scheme.

Proof. Choose a closed point $x \in \mathcal{X}(\kappa)$ valued in a perfect field κ, and suppose that it has a lift $x' \in \mathcal{X}'(\kappa)$. Let $R_{x'}$ be the complete local ring of $\mathcal{X}'$ at x', and set $R_x = R_{x'}^{H_x}$, where $H_x \leq H$ is the decomposition group of x. We want to show that H_x is trivial. Since the action is free after inverting p, this is equivalent to showing that the map $R_x \to R_{x'}$ of normal $\mathbb{Z}_p$-flat local rings is an isomorphism. Moreover, the freeness also shows that the map

$$\operatorname{Spec} R_{x'}[1/p] \to \mathcal{X}[1/p] \to \mathrm{Loc}_{\mathcal{G}(\mathbb{Z}_p)}$$

factors through $\operatorname{Spec} R_x[1/p]$.

By our hypotheses, we have a $(\mathcal{G}, \mu)$-aperture $\mathfrak{Q}_{x'}$ over x' and $R_{x'}$ is its universal deformation ring. Giving a map $y : R_x \to W(\kappa)$ is equivalent to giving a deformation $\mathfrak{Q}_y$ of $\mathfrak{Q}_{x'}$ over $W(\kappa)$, but by Proposition 5.1.3 such a deformation is determined completely by the étale realization $T_{\mathrm{\acute{e}t}}(\mathfrak{Q}_y)$, which in turn depends only on the image of y in $\operatorname{Spec} R_x$.

This shows that $P_y = \ker(y)$ is the only prime lying above $P_y \cap R_x$. In turn, this implies that, for all $h \in H_x$, we have $h(P_y) = P_y$. Now, if $f \in R_{x'}$ is such that $y(f) = a$, then we have $f - a \in P_y$, and hence $h(f) - a \in h(P_y) = P_y$. So we find that $f - h(f)$ belongs to P_y for all $y : R_{x'} \to W(\kappa)$, and therefore must be identically zero (recall that $R_{x'}$ is formally smooth over $W(\kappa)$). In other words, we have $R_x = R_{x'}$, as desired. ☐

_apertures
5. Criteria for the existence of apertures

The purpose of this section is to develop criteria for proving the existence of apertures when $\mathcal{G}$ is reductive. These are mainly of a 'pointwise' nature, and will be employed in the next section to prove the canonicity of integral models. We will maintain the notation from the previous section. In particular, we will fix a smooth affine $\mathbb{Z}_p$-group scheme $\mathcal{G}$ and a 1-bounded cocharacter μ defined over an unramified ring of integers $\mathcal{O}$. Most of our results here will carry the additional assumption that $\mathcal{G}$ is in fact reductive.

5.1. **Apertures over p-completely smooth $\mathscr{O}_K$-algebras.** Let K be a complete discrete valuation with perfect residue field κ and fix a uniformizer $\pi \in K$. Our goal here is to characterize the image of the fully faithful étale realization functor for certain base $\mathscr{O}_K$-schemes.

Definition 5.1.1 (The type of a crystalline local system)**.** Suppose that G is reductive. Suppose that we have $\mathsf{Q} \in \mathrm{Loc}^{\mathrm{crys}}_{\mathcal{G}(\mathbb{Z}_p)}(K)$: this is equivalent to saying $(V)_{\mathsf{Q}}$ is in $\mathrm{Loc}^{\mathrm{crys}}_{\mathbb{Q}_p}(K)$ for all $V \in \mathrm{Rep}_{\mathbb{Q}_p}(\mathcal{G})$. In particular, the associated Galois representation is de Rham, and so via Fontaine's D_{dR} functor we obtain a filtered K-vector space $\mathrm{Fil}^\bullet D_{\mathrm{dR}}((V)_{\mathsf{Q}})$. By [44, Lemma (1.4.5)], there exists a cocharacter of G_K that splits these filtrations simultaneously for all V. In particular, this gives us a well-defined geometric conjugacy class of cocharacters of G. This conjugacy class is the **type** of Q. If λ is a representative of the conjugacy class defined over K, then we will abuse terminology and say that Q has type λ.

Notation 5.1.2. For a cocharacter λ of G_K, $\mathrm{Loc}^{\mathrm{crys},\lambda}_{\mathcal{G}(\mathbb{Z}_p)}(K) \subset \mathrm{Loc}^{\mathrm{crys}}_{\mathcal{G}(\mathbb{Z}_p)}(K)$ be the subgroupoid spanned by the local systems Q of type λ.

The two following results characterize the image of the fully faithful étale realization functor on $\mathrm{BT}^{\mathcal{G},\mu}_\infty(R)$ in some cases.

Proposition 5.1.3. *Suppose that $\mathcal{G}$ is reductive. Then the fully faithful functor $T_{\text{ét}} : \mathrm{BT}^{\mathcal{G},\mu}_\infty(\mathscr{O}_K) \to \mathrm{Loc}_{\mathcal{G}(\mathbb{Z}_p)}(K)$ has essential image $\mathrm{Loc}^{\mathrm{crys},\mu}_{\mathcal{G}(\mathbb{Z}_p)}(K)$.*

Theorem 5.1.4. *Let e be the absolute ramification index of K. Suppose $2p < e-1$, that R is a p-completely smooth $\mathscr{O}_K$-algebra and that $\mathcal{G}$ is reductive. Then a local system $\mathsf{Q} \in \mathrm{Loc}_{\mathcal{G}(\mathbb{Z}_p)}(R[1/p])$ lifts to $\mathrm{BT}^{\mathcal{G},\mu}_\infty(R)$ if and only if, for every finite extension K'/K and every $\mathscr{O}_K$-algebra map $x : R \to \mathscr{O}_{K'}$, the local system Q_x over K' is in $\mathrm{Loc}^{\mathrm{crys},\mu}_{\mathcal{G}(\mathbb{Z}_p)}(K')$.*

We will need a bit of preparation.

Lemma 5.1.5. *The étale realization functor $T_{\text{ét}} : \mathrm{BT}^{\mathcal{G},\mu}_\infty(\mathscr{O}_K) \to \mathrm{Loc}^{\mathrm{crys}}_{\mathcal{G}(\mathbb{Z}_p)}(K)$ lands in $\mathrm{Loc}^{\mathrm{crys},\mu}_{\mathcal{G}(\mathbb{Z}_p)}(K)$*

Proof. Given $\mathfrak{Q} \in \mathrm{BT}^{\mathcal{G},\mu}_\infty(\mathscr{O}_K)$ lifting $\mathsf{Q} \in \mathrm{Loc}_{\mathcal{G}(\mathbb{Z}_p)}(K)$, Lemma 3.7.4 shows that $\mathrm{Fil}^\bullet D_{\mathrm{dR}}(\mathsf{Q})$ is canonically isomorphic to $\mathrm{Fil}^\bullet_{\mathrm{Hdg}} T_{\mathrm{dR}}(\mathfrak{Q})$. This, combined with the definition of $\mathrm{BT}^{\mathcal{G},\mu}_\infty$, implies that Q has type μ. □

Remark 5.1.6 (Crystallinity from pointwise crystallinity)**.** As explained in Remark 2.3.5, the work of Guo-Yang [32] tells us that our pointwise condition in Theorem 5.1.4 in fact implies that Q is in $\mathrm{Loc}^{\mathrm{crys}}_{\mathcal{G}(\mathbb{Z}_p)}(R[1/p])$.

Remark 5.1.7 (Reduction to a question of algebraicity)**.** Given the previous remark, Theorem 2.3.6 tells us that the tensor functor $\mathrm{Rep}_{\mathbb{Z}_p}(\mathcal{G}) \to \mathrm{Loc}_{\mathbb{Z}_p}(R[1/p])$ associated with Q factors through an exact symmetric monoidal functor

$$\mathrm{Rep}_{\mathbb{Z}_p}(\mathcal{G}) \to \mathrm{Vect}^{\mathrm{an},\varphi}(R_{\Delta}, \mathscr{O}_{\Delta}).$$

By [38, Propositions 1.28, 1.39], the theorem now amounts to showing that this functor factors through the fully faithful subcategory $\mathrm{Vect}^{\varphi}(R_{\Delta}, \mathscr{O}_{\Delta})$. Indeed, the cited results would then imply that Q arises from a $\mathcal{G}$-torsor $\mathfrak{Q}$ over R^{syn}, and our pointwise assumption on the type, combined with Lemma 5.1.5, implies that $\mathfrak{Q}$ must be in $\mathrm{BT}^{\mathcal{G},\mu}_\infty(R)$.

To establish the desired factoring, it is now sufficient to show that the analytic prismatic F-crystal associated with $(\Lambda)_{\mathsf{Q}}$ is in fact a prismatic F-crystal. Here, Λ can be *any* faithful representation of $\mathcal{G}$.

_criterion **Remark 5.1.8** (Reduction to a question of finite freeness)**.** Working étale locally on $\mathrm{Spf}\, R$, we can assume that R is a base $\mathscr{O}_K$-algebra admitting a Breuil-Kisin frame $\underline{\mathfrak{S}}_R$ as in Example 3.2.2. The analytic prismatic F-crystal associated with $(\Lambda)_{\mathsf{Q}}$ gives a vector bundle $\mathcal{V}^{\mathrm{an}}$ over $\mathrm{Spec}\,\mathfrak{S}_R\backslash V(p,(E(u)))$. Knowing that we in fact have a prismatic F-crystal amounts to knowing that the reflexive $\mathfrak{S}_R$-module

$$\mathcal{L} = H^0(\mathrm{Spec}\,\mathfrak{S}_R\backslash V(p,(E(u))), \mathcal{L}^{\mathrm{an}})$$

is in fact finite locally free.

Proof of Proposition 5.1.3. In this case, $\mathfrak{S}_{\mathscr{O}_K}$ is a regular local Noetherian ring of dimension 2, and so every reflexive finitely generated module over it is finite free. □

e_of_torus **Lemma 5.1.9.** *Suppose that $\mathcal{G} = \mathcal{T}$ is a torus over $\mathbb{Z}_p$. Then Theorem 5.1.4 holds.*

Proof. We can reduce to the situation in Remark 5.1.8, and so need to check that the reflexive $\mathfrak{S}_R$-module $\mathcal{L}$ associated with a faithful representation Λ is finite locally free. Since this can be checked after a faithfully flat base-change, we can assume without loss of generality that $\mathcal{T}$ is split over $\mathfrak{S}_R$. In particular, $\mathcal{L}^{\mathrm{an}}$ is now a direct sum of line bundles, and this implies that $\mathcal{L}$ is finite locally free; see [28, Théorème XI.3.18]. □

lete_local **Lemma 5.1.10** (Reduction to the complete local case)**.** *Suppose that e is arbitrary and that the following equivalent conditions hold for every maximal ideal $\mathfrak{m} \subset R$:*

(1) The restriction of Q over $\mathrm{Spec}\,\hat{R}_{\mathfrak{m}}[1/p]$ lifts to $\mathrm{BT}_{\infty}^{\mathcal{G},\mu}(\hat{R}_{\mathfrak{m}})$;

(2) The analytic prismatic F-crystal over $\hat{R}_{\mathfrak{m}}$ associated with $(\Lambda)_{\mathsf{Q}}$ is in fact a prismatic F-crystal;

Then Q lifts to $\mathrm{BT}_{\infty}^{\mathcal{G},\mu}(R)$.

Proof. As in Remark 5.1.8, we can assume that R is a base $\mathscr{O}_K$-algebra of the form $R_0 \otimes_{W(\kappa)} \mathscr{O}_K$, and are therefore reduced to checking that the $\mathfrak{S}_R$-module $\mathcal{L}$ is finite locally free. For this, it is enough to check that, for every maximal ideal $\mathfrak{m} \subset R$, lifting to a maximal ideal $\mathfrak{n} \subset \mathfrak{S}_R$, the completion of $\mathcal{L}$ at $\mathfrak{n}$ is finite locally free. However, the completion $(\mathfrak{S}_R)^{\wedge}_{\mathfrak{n}}$ underlies a Breuil-Kisin frame for $\hat{R}_{\mathfrak{m}} = \hat{R}_{0,\mathfrak{m}_0} \otimes_{W(\kappa)} \mathscr{O}_K$, where $\mathfrak{m}_0 = \mathfrak{m} \cap R_0$. Therefore, our equivalent hypotheses tell us that $\hat{\mathcal{L}}_{\mathfrak{n}}$ is obtained from the evaluation of a prismatic F-crystal over $\hat{R}_{\mathfrak{m}}$, and is thus finite free, as desired. □

Notation 5.1.11. Suppose now that R is a base $\mathscr{O}_K$-algebra and that T is the completion of R at a maximal ideal. Let $\underline{\mathfrak{S}}_T$ be the Breuil-Kisin frame for T obtained as a completion of $\mathfrak{S}_R$ as in the proof above, and set $\mathcal{L}_T = T \otimes_R \mathcal{L}$.

f_aperture **Lemma 5.1.12.** *Suppose that μ acts on Λ with weights in the interval $[a,b]$ with $|b-a| = r$. Then, if $er < p-1$, $\mathcal{L}_T$ is finite free over T. In particular, if $\mathcal{G}$ is adjoint, then Theorem 5.1.4 holds.*

Proof. The first assertion follows from the argument in [55, Proposition 4.3]. The second now follows from the fact that we can choose Λ to be the adjoint representation, which has μ-weights $\{-1,0,1\}$. □

_criterion **Remark 5.1.13.** Suppose that κ is algebraically closed. Then one sees that a $\mathcal{G}$-torsor over $U \overset{\mathrm{defn}}{=} \mathrm{Spec}\,\mathfrak{S}_T\backslash V(p,E(u))$ extends to a $\mathcal{G}$-torsor over $\mathfrak{S}_T$ if and only if it is actually trivial. This is because every $\mathcal{G}$-torsor over $\mathfrak{S}_T$ is trivial, since $\mathfrak{S}_T$ is now a strictly Henselian local ring, and also, the restriction map for $\mathcal{G}$-torsors from $\mathrm{Spec}\,\mathfrak{S}_R$ to U is fully faithful,

Proof of Theorem 5.1.4. Without loss of generality, we can assume that κ is algebraically closed and that R is a base $\mathscr{O}_K$-algebra with Breuil-Kisin frame $\underline{\mathfrak{S}}_R$. We only have to show that for every completion T of R, the $\mathcal{G}$-torsor $\mathcal{Q}_T^{\mathrm{an}}$ over $U \overset{\mathrm{defn}}{=} \mathrm{Spec}\,\mathfrak{S}_T\backslash V(p,E(u))$ obtained from the analytic prismatic F-crystal realization of Q extends (necessarily uniquely) to a $\mathcal{G}$-torsor over $\mathfrak{S}_T$—or, equivalently, that it is trivial.

Consider the map $\mathcal{G} \to \overline{\mathcal{G}} \overset{\mathrm{defn}}{=} \mathcal{G}^{\mathrm{ad}} \times \mathcal{G}^{\mathrm{ab}}$: This is a finite flat cover of reductive group schemes, and Lemmas 5.1.9 and 5.1.12 together tell us that the theorem is valid with $\mathcal{G}$ replaced with $\overline{\mathcal{G}}$ and

Q replaced with the associated $\overline{\mathcal{G}}(\mathbb{Z}_p)$-torsor over $R[1/p]$. Therefore, $\mathcal{Q}_T^{\mathrm{an}}$ is trivial after change of structure group to $\overline{\mathcal{G}}$, and so admits a reduction of structure group to a $\mathcal{Z}$-torsor over U, where $\mathcal{Z} = \ker(\mathcal{G} \to \overline{\mathcal{G}})$. However, by purity of branch locus for torsors under finite flat group schemes [66, Theorem 3.1], all $\mathcal{Z}$-torsors over U extend uniquely to $\mathcal{Z}$-torsors over $\mathfrak{S}_T$. This implies that $\mathcal{Q}_T^{\mathrm{an}}$ also extends to a $\mathcal{G}$-torsor over $\mathfrak{S}_T$, as desired. □

mprovement

Remark 5.1.14 (The Hodge type case)**.** If $(\mathcal{G}, \mu)$ is of *Hodge type*—that is, if there exists a map $(\mathcal{G}, \mu) \to (\mathrm{GL}_n, \mu_d)$ such that the underlying map $\mathcal{G} \to \mathrm{GL}_n$ is a closed immersion—then we can improve the bound to $p < e - 1$: Indeed, we can replace the use of the adjoint representation with the faithful representation given by such a map, and this will have μ-weights in $\{0, 1\}$. The case where $\mathcal{G} = \mathrm{GL}_n$ amounts to the assertion that any crystalline local system over $R[1/p]$ with Hodge-Tate weights in $\{0, 1\}$ arises from a p-divisible group. This is a result of Liu-Moon [55, Theorem 1.2].

:versality

5.2. **Versality.** When we obtain an aperture using Theorem 5.1.4, versality is not automatic. In applications, it can be understood via a Kodaira-Spencer map, which we now discuss.

al_de_rham

Remark 5.2.1 (The formal de Rham realization and a versality condition)**.** Suppose that we have a map $f : \mathcal{X} \to \mathrm{BT}_\infty^{\mathcal{G},\mu,\mathrm{alg}}$ of stacks over $\mathscr{O}_{E,(v)}$. On the level of formal stacks, we obtain a filtered de Rham realization

$$\widehat{\mathcal{X}} \to \mathrm{BT}_\infty^{\mathcal{G},\mu} \xrightarrow{T_{\mathrm{dR}}} \mathrm{Gr}_\mu/\mathcal{G}_{\mathscr{O}_{E,(v)}}.$$

By the discussion in Remark 3.4.3, this gives a map of p-complete complexes

$$\mathbb{L}^{\wedge}_{(\mathrm{Gr}_\mu/\mathcal{G}_{\mathscr{O}_{E,(v)}})/B\mathcal{G}_{\mathcal{O}}}|_{\widehat{\mathcal{X}}} \to \mathbb{L}^{\wedge}_{\mathcal{X}/\mathscr{O}_{E,(v)}},$$

which is an isomorphism if and only if the map $\widehat{\mathcal{X}} \to \mathrm{BT}_\infty^{\mathcal{G},\mu}$ is formally étale.

ra-spencer

Remark 5.2.2 (The Kodaira-Spencer map)**.** Suppose that $\mathcal{X}$ is a smooth scheme over $\mathscr{O}_{E,(v)}$. We can then dualize the map of p-complete cotangent complexes from the previous remark to obtain a map

_space_map

$$\mathbb{T}^{\wedge}_{\mathcal{X}/\mathscr{O}_{E,(v)}} \to \mathbb{T}^{\wedge}_{(\mathrm{Gr}_\mu/\mathcal{G}_{\mathscr{O}_{E,(v)}})/B\mathcal{G}_{\mathcal{O}}}|_{\widehat{\mathcal{X}}} \tag{5.2.2.1}$$

of vector bundles over $\widehat{\mathcal{X}}$. The left hand side here is the p-completed tangent bundle of $\mathcal{X}$ while the right hand side can be described explicitly: The aperture $\mathfrak{Q}$ classified by f admits a Hodge-filtered de Rham realization $\mathrm{Fil}^\bullet_{\mathrm{Hdg}} T_{\mathrm{dR}}(\mathfrak{Q})$ equipped with a topologically nilpotent integrable connection. Let $\mathfrak{g}$ be the Lie algebra of $\mathcal{G}$, equipped with the adjoint representation. Twisting this by $\mathrm{Fil}^\bullet_{\mathrm{Hdg}} T_{\mathrm{dR}}(\mathfrak{Q})$, we obtain a filtered vector bundle $\mathrm{Fil}^\bullet \mathfrak{g}_{\mathrm{twist}}$ over $\widehat{\mathcal{X}}$. There is now a canonical isomorphism

$$\mathbb{T}^{\wedge}_{(\mathrm{Gr}_\mu/\mathcal{G}_{\mathscr{O}_{E,(v)}})/B\mathcal{G}_{\mathcal{O}}}|_{\widehat{\mathcal{X}}} \simeq \mathfrak{g}_{\mathrm{twist}}/\mathrm{Fil}^0 \mathfrak{g}_{\mathrm{twist}}.$$

Now, (5.2.2.1) now gives a map $\mathbb{T}^{\wedge}_{\mathcal{X}/\mathscr{O}_{E,(v)}} \to \mathfrak{g}_{\mathrm{twist}}/\mathrm{Fil}^0 \mathfrak{g}_{\mathrm{twist}}$. This is just the Kodaira-Spencer map associated with the integrable connection on $\mathrm{Fil}^\bullet_{\mathrm{Hdg}} T_{\mathrm{dR}}(\mathfrak{Q})$, and so versality comes down to knowing that this is an isomorphism.

ture_group

5.3. Apertures via reduction of structure group.

Setup 5.3.1. Let κ be a perfect characteristic p field over $\mathcal{O}$, and let $R_1 \to R_2$ be a map of complete local normal rings that are flat and essentially of finite type over $W(\kappa)$. We will asume that both R_1 and R_2 have residue field κ. Suppose that we have a closed immersion $(\mathcal{G}_2, \mu_2) \to (\mathcal{G}_1, \mu_1)$ of two pairs as in Setup 3.1.1, with μ_1 and μ_2 both defined over $\mathcal{O}$, as well as $\mathsf{Q}_i \in \mathrm{Loc}_{\mathcal{G}_i(\mathbb{Z}_p)}(R_i[1/p])$ such that the restriction of Q_1 over $\mathrm{Spec}\, R_2[1/p]$ is isomorphic to the change of structure group of Q_2 along the map $\mathcal{G}_1(\mathbb{Z}_p) \to \mathcal{G}_2(\mathbb{Z}_p)$.

_reduction

Remark 5.3.2 (Type of reduction at every point)**.** Suppose that Q_1 lifts to $\mathfrak{Q}_1 \in \mathrm{BT}_\infty^{\mathcal{G}_1,\mu_1}(R_1)$, and that we have a finite extension $K/W(\kappa)[1/p]$, and a map $x : R_2 \to \mathscr{O}_K$ of complete local rings. The $\mathcal{G}_1(\mathbb{Z}_p)$-local system $\mathsf{Q}_{1,x}$ has a reduction of structure to a $\mathcal{G}_2(\mathbb{Z}_p)$-local system $\mathsf{Q}_{2,x}$, which is in

$\mathrm{Loc}^{\mathrm{crys},\mu_2'}_{\mathcal{G}_2(\mathbb{Z}_p)}(\mathscr{O}_K)$ for some cocharacter μ_2' mapping to the $\mathcal{G}_1$-conjugacy class of μ_1. By Proposition 5.1.3, it corresponds to a lift $\mathfrak{Q}_{2,x} \in \mathrm{BT}^{\mathcal{G}_2,\mu_2'}_\infty(\mathscr{O}_K)$ of $\mathsf{Q}_{2,x}$. We will refer to the conjugacy class of μ_2' as the **type of** x.

cial_fiber **Remark 5.3.3** (Detecting the type over the special fiber)**.** With the setup as in the previous remark, let κ_x be the residue field of $\mathscr{O}_K$: this is a finite extension of κ. Let $\mathfrak{Q}_{2,x,0}$ be the aperture over κ_x obtained from $\mathfrak{Q}_{2,x}$. Then by definition (and Remark 3.1.4), $\mathfrak{Q}_{2,x,0}$ is of type μ_2' if and only if $\mathfrak{Q}_{2,x}$ is so.

alizations **Remark 5.3.4** (Compatibility between specializations of points)**.** Suppose that we have two points $x : R_2 \to \mathscr{O}_K$ and $y : R_2 \to \mathscr{O}_L$ as in Remark 5.3.2 for finite extensions K and L of $W(\kappa)$. We can assume without loss of generality that $\kappa_x = \kappa_y$. Then $\mathfrak{Q}_{2,x}$ and $\mathfrak{Q}_{2,y}$ yield $\mathcal{G}_2$-bundles $\mathfrak{Q}_{2,x,0}$ and $\mathfrak{Q}_{2,y,0}$ over κ_x^{syn}, which are both canonically reductions of structure group for the $\mathcal{G}_1$-torsor $\mathfrak{Q}_1|_{\kappa_x^{\mathrm{syn}}}$. In particular, it makes sense to ask for them to be *equal*. If this is the case, then Remark 5.3.3 tells us that both $\mathfrak{Q}_{2,x}$ and $\mathfrak{Q}_{2,y}$ have the same type.

def:limpid **Definition 5.3.5.** Suppose that Q_1 lifts to $\mathfrak{Q}_1 \in \mathrm{BT}^{\mathcal{G}_1,\mu_1}_\infty(R_1)$, and let $\mathfrak{Q}_{1,0}$ be the resulting aperture over κ. We will say that the reduction of structure Q_2 is **limpid of type** μ_2 if there exists a reduction of structure $\mathfrak{Q}_{2,0} \in \mathrm{BT}^{\mathcal{G}_2,\mu_2}_\infty(\kappa)$ for $\mathfrak{Q}_{1,0}$ such that, for all finite extensions $K/W(\kappa)[1/p]$ and all maps $x : R_2 \to \mathscr{O}_K$, we have

$$\mathfrak{Q}_{2,x,0} = \mathfrak{Q}|_{\kappa_x^{\mathrm{syn}}} \subset \mathfrak{Q}_{1,0}|_{\kappa_x^{\mathrm{syn}}}.$$

ture_group **Proposition 5.3.6.** *Suppose that* Q_1 *lifts to* $\mathfrak{Q}_1 \in \mathrm{BT}^{\mathcal{G}_1,\mu_1}_\infty(R_1)$ *and that* Q_2 *is a limpid reduction of structure of type* μ_2*. Then:*

(1) Q_2 *lifts to* $\mathfrak{Q}_2 \in \mathrm{BT}^{\mathcal{G}_2,\mu_2}_\infty(R_2)$.
(2) Suppose that the following conditions hold:
   (a) The classifying map $\mathrm{Spf}\, R_1 \to \mathrm{BT}^{\mathcal{G}_1,\mu_1}_\infty$ *for* $\mathfrak{Q}_1$ *is unramified;*
   (b) The map $R_1 \to R_2$ *is the normalization of a surjection;*
   (c) $\dim R_2 = 1 + \dim \mathcal{G}_2 - \dim \mathcal{P}^-_{\mu_2}$.

   Then $R_1 \to R_2$ *is in fact surjective and the classifying map* $\mathrm{Spf}\, R_2 \to \mathrm{BT}^{\mathcal{G}_2,\mu_2}_\infty$ *for* $\mathfrak{Q}_2$ *induces an isomorphism* $\widehat{\mathscr{O}}_{\mathrm{BT}^{\mathcal{G}_2,\mu_2}_\infty,\mathfrak{Q}_{2,0}} \xrightarrow{\simeq} R_2$.

Proof. The limpidity ensures that there is a canonical object $\mathfrak{Q}_{2,0} \in \mathrm{BT}^{\mathcal{G}_2,\mu_2}_\infty(\kappa)$ giving a reduction of structure group for $\mathfrak{Q}_{1,0}$. Let $\tilde{R}_i$ be the universal deformation ring for $\mathfrak{Q}_{i,0}$ for $i = 1, 2$ (see Remark 3.1.6). Then we have a map $\tilde{R}_1 \to R_1$ classifying $\mathfrak{Q}_1$ as a deformation of $\mathfrak{Q}_{1,0}$.

Now, the limpidity condition also tells us that, for every $x : R_2 \to \mathscr{O}_K$, the composition $\tilde{R}_1 \to R_1 \to R_2 \to \mathscr{O}_K$ factors through $\tilde{R}_2$. This implies that in fact the composition $\tilde{R}_1 \to R_1 \to R_2$ factors through $\tilde{R}_2$: In other words, we have a deformation $\mathfrak{Q}_2$ over R_2 of $\mathfrak{Q}_{2,0}$ specializing to the various $\mathfrak{Q}_{2,x}$.

If the classifying map for $\mathfrak{Q}_1$ is formally unramified, then the map $\tilde{R}_1 \to R_1$ is surjective, and so the map $\tilde{R}_2 \to R_2$ is the normalization of a surjection. Since $\dim \tilde{R}_2 = 1 + \dim \mathcal{G}_2 - \dim \mathcal{P}^-_{\mu_2} = \dim R_2$ by hypothesis, we find that $\tilde{R}_2 \xrightarrow{\simeq} R_2$. This of course shows that $R_1 \to R_2$ is surjective, and identifies R_2 with the deformation ring of $\mathrm{BT}^{\mathcal{G}_2,\mu_2}_\infty$ at $\mathfrak{Q}_{2,0}$. □

via_lambda **Remark 5.3.7** (Explication of type via a choice of representation)**.** The existence of $\mathfrak{Q}_1$ tells us that we have an F-gauge over R_1 lifting $(\Lambda)_{\mathsf{Q}_1}$. By Remark 3.2.15, for every $x : R_2 \to \mathscr{O}_K$, we now obtain a tuple $(\mathcal{L}_x, \{s_{\alpha,\mathcal{L}_x}\}, \mathrm{Fil}^\bullet L_x)$ over $\mathfrak{S}_x = W(\kappa_x)[|u|]$, where $\mathrm{Fil}^\bullet L_x$ is a filtration split by a cocharacter of an inner form of $\mathcal{G}_{\mathscr{O}_K}$. The type of $\mathfrak{Q}_{2,x}$ is given by the geometric conjugacy class of this cocharacter.

osed_point **Remark 5.3.8** (Explication over the closed point)**.** In the context of the previous remark, the base-change of the F-gauge lifting $(\Lambda)_{\mathsf{Q}_1}$ over κ gives rise to a pair $(L_0, \mathrm{Fil}^\bullet(\kappa \otimes_{W(\kappa)} L_0))$ where L_0 is a finite locally free module over $W(\kappa)$ étale locally isomorphic to $W(\kappa)\otimes_{W(\kappa)}\Lambda$, with the isomorphism well-defined up to conjugation by a section of $\mathcal{G}_1$, and $\mathrm{Fil}^\bullet(\kappa\otimes_{W(\kappa)}L_0)$ is a filtration split by a cocharacter of $\mathcal{G}_{1,\kappa}$ that is geometrically conjugate to μ_1. Furthermore, for every $x: R_1\to\mathscr{O}_K$, Remark **??** tells us that there are canonical isomorphisms

$$W(\kappa_x)\otimes_{\mathfrak{S}_x}\mathcal{L}_x\xrightarrow{\simeq}W(\kappa_x)\otimes_{W(\kappa)}\mathcal{L}\ ;\ \kappa_x\otimes_{\mathscr{O}_K}\mathrm{Fil}^\bullet L_x\xrightarrow{\simeq}\kappa_x\otimes_\kappa\mathrm{Fil}^\bullet(\kappa\otimes_{W(\kappa)}L_0).$$

In particular, if x factors through R_2, then the tensors $\{s_{\alpha,\mathcal{L}_x}\}$ from Remark 5.3.7 map to φ-invariant tensors $\{s_{\alpha,x,0}\}$ in $W(\kappa_x)\otimes_{W(\kappa)}L_0^{\otimes}$.

We now obtain:

_criterion **Lemma 5.3.9** (Criterion for limpidity)**.** *The reduction of structure Q_2 is limpid of type μ_2 if and only if the following conditions hold:*

(1) The collection of tensors $\{s_{\alpha,x,0}\}\subset W(\kappa_x)\otimes_{W(\kappa)}L_0^{\otimes}$ is independent of the choice of $x: R_2\to\mathscr{O}_K$. In particular, there is a canonical collection of φ-invariant tensors $\{s_{\alpha,0}\}\subset L_0^{\otimes}$ whose stabilizer $\mathcal{G}_{2,0}$ is a pure inner form of $\mathcal{G}_{2,W(\kappa)}$;

(2) For some (hence every) $x: R_2\to\mathscr{O}_K$, Q_x is in $\mathrm{Loc}^{\mathrm{crys},\mu_2}_{\mathcal{G}_2(\mathbb{Z}_p)}(K)$.

_limpidity **Remark 5.3.10** (Families of F-isocrystals)**.** Suppose that R_1 is formally smooth over $W(\kappa)$, equipped with a Frobenius lift $\varphi: R_1\to R_1$. Then the rigid analytic space $\widehat{U}_1^{\mathrm{an}}$ associated with $\widehat{U}_1=\mathrm{Spf}\,R_1$ is an open ball. Moreover, the F-gauge $\mathfrak{L}$ lifting $(\Lambda)_{\mathsf{Q}_1}$ yields a filtered F-isocrystal over R_1/pR_1, which realizes to a filtered vector bundle $\mathrm{Fil}^\bullet\mathcal{L}_{\mathbb{Q}}$ over $\widehat{U}_1^{\mathrm{an}}$, equipped with an integrable connection, and specializing to $\mathrm{Fil}^\bullet L_x[1/p]$ at every classical point $x\in\widehat{U}_1^{\mathrm{an}}(K)$. Set

$$\mathcal{L}_{\mathbb{Q}}^{\nabla}\overset{\mathrm{defn}}{=}H^0(\widehat{U}_1^{\mathrm{an}},\mathcal{L}_{\mathbb{Q}})^{\nabla=0}.$$

This is a finite dimensional vector space over $W(\kappa)[1/p]$, and inherits the structure of an F-isocrystal from that on $\mathcal{L}_{\mathbb{Q}}$. Moreover, we have a canonical isomorphism of F-isocrystals

$$\delta:\mathscr{O}_{\widehat{U}_1^{\mathrm{an}}}\otimes_{W(\kappa)[1/p]}\mathcal{L}_{\mathbb{Q}}^{\nabla}\xrightarrow{\simeq}\mathcal{L}_{\mathbb{Q}}$$

over $\widehat{U}_1^{\mathrm{an}}$. This is a consequence of Dwork's trick; see for instance the discussion in [90, §3.7] in a somewhat more general logarithmic context.

_limpidity **Remark 5.3.11** (Application of the de Rham functor of Liu-Zhu)**.** By Remark 3.7.2, $\mathrm{Fil}^\bullet\mathcal{L}_{\mathbb{Q}}$ is also obtained from the filtered F-isocrystal associated with the $\mathbb{Z}_p$-local system $(\Lambda)_{\mathsf{Q}_1}$. If $\widehat{U}_2^{\mathrm{an}}$ is a *smooth* rigid analytic space over $W(\kappa)[1/p]$, then the restriction of $\mathrm{Fil}^\bullet\mathcal{L}_{\mathbb{Q}}$ over $\widehat{U}_2^{\mathrm{an}}$ is once again a filtered vector bundle with integrable connection, and is *associated* with $(\Lambda)_{\mathsf{Q}_2}$ via the D_{dR} functor of Liu-Zhu [54, Theorem 3.9]. In particular, this functor carries the global sections

$$\{s_{\alpha,\mathsf{Q}_2}\}\subset H^0(\mathrm{Spec}\,R_2[1/p],(\Lambda)^{\otimes}_{\mathsf{Q}_2})$$

to a collection

$$\{s_{\alpha,\mathcal{L}_{\mathbb{Q}}}\}\subset H^0(\widehat{U}_2^{\mathrm{an}},\mathrm{Fil}^0\mathcal{L}_{\mathbb{Q}}^{\otimes})^{\nabla=0}.$$

riterion_2 **Lemma 5.3.12** (Second criterion for limpidity)**.** *In the situation of Remark 5.3.10, suppose further that the rigid analytic subspace $\widehat{U}_2^{\mathrm{an}}\subset\widehat{U}_1^{\mathrm{an}}$ is also smooth over $W(\kappa)[1/p]$, and that, for* some *x as above, the filtration $\mathrm{Fil}^\bullet L_x$ is split by a cocharacter of $\mathcal{G}_{\mathscr{O}_K}$ in the geometric conjugacy class of μ_2. Then Q_2 is limpid of type μ_2.*

Proof. To begin, note that there is a canonical isomorphism of F-isocrystals $L_0[1/p]\xrightarrow{\simeq}\mathcal{L}_{\mathbb{Q}}^{\nabla}$. Suppose that we have $x: R_2\to\mathscr{O}_K$. The tensors $\{s_{\alpha,\mathcal{L}_x}\}$ map to φ-invariant tensors $\{s_{\alpha,x,0}\}$ in $W(\kappa_x)\otimes_{W(\kappa)}L_0^{\otimes}$, and these in turn yield a collection of parallel φ-invariant tensors

$$\{\tilde{s}_{\alpha,\mathcal{L}_{\mathbb{Q}}}\overset{\mathrm{defn}}{=}\delta(1\otimes s_{\alpha,x,0})\}\subset W(\kappa_x)\otimes_{W(\kappa)}H^0(\widehat{U}_1^{\mathrm{an}},\mathcal{L}_{\mathbb{Q}}^{\otimes})^{\nabla}.$$

The restriction of this collection over the base-change of $\widehat{U}_2^{\mathrm{an}}$ over $W(\kappa_x)[1/p]$ agrees with $\{s_{\alpha,\mathcal{L}_{\mathbb{Q}}}\}$ at x and so must agree everywhere. This shows that the tensors $\{s_{\alpha,x,0}\}$ are independent of the choice of x, and so we conclude using Lemma 5.3.9. □

Setup 5.3.13. Suppose that $\mathcal{X}$ is a normal Deligne-Mumford stack flat of finite type over $\mathscr{O}_{E,(v)}$ with generic fiber X, and that we have an object $\mathsf{Q} \in \mathrm{Loc}_{\mathcal{G}(\mathbb{Z}_p)}(X)$. We will give a pointwise criterion for the existence of an object $\mathfrak{Q} \in \mathrm{BT}_\infty^{\mathcal{G},\mu,\mathrm{alg}}(\mathcal{X})$ lifting Q under the étale realization. Such an object is uniquely determined by Theorem 4.1.5.

ture_group **Assumption 5.3.14.** There exists a commuting diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & \mathrm{Loc}_{\mathcal{G}(\mathbb{Z}_p)} = \mathrm{BT}_\infty^{\mathcal{G},\mu,\mathrm{alg}}[1/p] \arrow[d] \\
\mathcal{X} \arrow[d] & \mathrm{BT}_\infty^{\mathcal{G},\mu,\mathrm{alg}} \arrow[d] \\
\mathcal{X}^\sharp \arrow[r] & \mathrm{BT}_\infty^{\mathcal{G}^\sharp,\mu^\sharp,\mathrm{alg}}
\end{tikzcd}$$

where the top horizontal arrow classifies Q, the bottom left vertical arrow is the normalization of a closed immersion and the bottom horizontal arrow is formally unramified after v-adic completion. Moreover, we require that the following condition hold: Let $\mathsf{Q}^\sharp$ be the $\mathcal{G}^\sharp(\mathbb{Z}_p)$-torsor over $X^\sharp = \mathcal{X}^\sharp[1/p]$ classified by the generic fiber of the bottom arrow. Then, for all closed points x of $\mathcal{X}$, the reduction of structure for $\mathsf{Q}^\sharp$ over $\widehat{\mathscr{O}}_{\mathcal{X},x}$ given by Q is limpid of type μ.

ion_normal **Proposition 5.3.15.** *Suppose that, for every finite extension F/E_v and every $x \in \mathcal{X}(\mathscr{O}_F)$, Q_x lies in $\mathrm{Loc}^{\mathrm{crys},\mu}_{\mathcal{G}(\mathbb{Z}_p)}(F)$. Suppose also that Assumption 5.3.14 holds and $\mathcal{X}$ is equidimensional of relative dimension $\dim \mathcal{G} - \dim \mathcal{P}_\mu^-$ over $\mathscr{O}_{E,(v)}$. Then Q lifts to $\mathfrak{Q} \in \mathrm{BT}_\infty^{\mathcal{G},\mu,\mathrm{alg}}(\mathcal{X})$, and the classifying map $\widehat{\mathcal{X}} \to \mathrm{BT}_\infty^{\mathcal{G},\mu}$ is formally étale.*

Proof. We can reduce to the case where $\mathcal{X}$ is affine. Proposition 5.3.6 tells us that the complete local ring $\hat{R}_x$ of $\mathcal{X}$ at any closed points x is formally smooth over $\mathscr{O}_{E_v}$ and that in fact Q lifts to a reduction of structure $\mathfrak{Q}_x \in \mathrm{BT}_\infty^{\mathcal{G},\mu}(\hat{R}_x)$ such that the classifying map $\mathrm{Spf}\,\hat{R}_x \to \mathrm{BT}_\infty^{\mathcal{G},\mu}$ is formally étale. In particular, $\widehat{\mathcal{X}}$ is p-completely smooth over $\mathscr{O}_{E_v}$, and the conditions of Lemma 5.1.10 have been verified. This shows the existence of $\mathfrak{Q}$.

That the resulting classifying map on $\widehat{\mathcal{X}}$ is formally étale now follows from Proposition 5.3.6, and Lemma 5.3.16 below. □

_etaleness **Lemma 5.3.16.** *Suppose that we have a p-completely flat ring R and:*

- *A p-completely flat p-adic formal scheme $\mathfrak{X}$ over $\mathrm{Spf}\,R$ such that the cotangent complex $\mathbb{L}_{\mathfrak{X}\otimes\mathbb{F}_p/(R/pR)}$ is a vector bundle over $\mathfrak{X}\otimes\mathbb{F}_p$;*
- *A formally cohesive[20] p-adic formal prestack $\mathfrak{Y}$ over $\mathrm{Spf}\,R$ admitting a p-completed cotangent complex $\mathbb{L}^\wedge_{\mathfrak{Y}/R}$ that is in fact a vector bundle over $\mathfrak{Y}$; and*
- *A map $\varpi : \mathfrak{X} \to \mathfrak{Y}$*

with the following property: For every closed point $x : \mathrm{Spec}\,\kappa \to \mathfrak{X}$ with κ perfect, the map of complete local rings $\hat{\mathscr{O}}_{\mathfrak{Y},\varpi(x)} \to \hat{\mathscr{O}}_{\mathfrak{X},x}$ is an isomorphism. Then ϖ is formally étale: that is, the formal cotangent complex $\mathbb{L}^\wedge_{\mathfrak{X}/\mathfrak{Y}}$ is nullhomotopic.

[20] See [58, Definition 6.2.1].

Proof. Note that our hypotheses imply that $\mathfrak{Y}$ admits universal deformation rings, so the given property is sensible; see [58, Corollary 6.2.14], which is just a classical criterion of Schlessinger.

We can of course assume that $\mathfrak{X} = \operatorname{Spf} S$ for some p-complete R-algebra S with $\mathbb{L}_{(S/pS)/(R/pR)}$ finite locally free over S/pS. Via the fundamental sequence

$$\varpi^*\mathbb{L}^{\wedge}_{\mathfrak{Y}/R} \to \mathbb{L}^{\wedge}_{\mathfrak{X}/R} \to \mathbb{L}^{\wedge}_{\mathfrak{X}/\mathfrak{Y}}$$

of p-completed cotangent complexes and our hypotheses, we find that $\mathbb{L}^{\wedge}_{\mathfrak{X}/\mathfrak{Y}}$ is a perfect complex over S with Tor amplitude in $[-1, 0]$. Therefore, to prove the lemma, it is enough to know that, for any map $x : S \to \kappa$ with κ perfect and with kernel a maximal ideal, we have $x^*\mathbb{L}^{\wedge}_{\mathfrak{X}/\mathfrak{Y}} \simeq 0$ as perfect complexes over κ; but this is guaranteed by our hypothesis on complete local rings. □

:canonical

6. Integral canonical models

We will be following the conventions for Shimura varieties as laid out in [19]. Given a Shimura datum (G, X), for $x \in X$, the conjugacy class $\{\mu_x\}$ of the associated Shimura cocharacter μ_x [21]is defined over $E \subset \mathbb{C}$ and is independent of the choice of x. We will denote this E-rational class by $\{\mu\}$. For a compact open subgroup $K \subset G(\mathbb{A}_f)$, let Sh_K be the canonical model over E for the complex Shimura variety with complex points

$$\mathrm{Sh}_K(\mathbb{C}) = G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K.$$

ion_global

6.1. Étale realizations on Shimura varieties.

Definition 6.1.1 ([48, Definition 1.5.4, Lemma 1.5.5])**.** A torus T over $\mathbb{Q}$ is **cuspidal** if it is isogenous to a product of a $\mathbb{Q}$-split torus and a $\mathbb{Q}$-torus that is anisotropic over $\mathbb{R}$.

For a general $\mathbb{Q}$-torus T, the **anti-cuspidal part** $T_{ac} \subset T$ is the minimal $\mathbb{Q}$-subtorus such that T/T_{ac} is cuspidal.

i-cuspidal **Remark 6.1.2.** The anti-cuspidal part is also characterized as the smallest subtorus of the maximal anisotropic subtorus $T_a \subset T$ that contains the maximal $\mathbb{R}$-split subtorus of $T_{a,\mathbb{R}}$: in other words, it is the $\mathbb{Q}$-Zariski closure in T of the maximal $\mathbb{R}$-split subtorus of $T_{a,\mathbb{R}}$.

Definition 6.1.3 ([48, §1.5.6])**.** We will define $G^c = G/Z(G)_{ac}$, where $Z(G) \subset G$ is the center, and where $Z(G)_{ac}$ is the anti-cuspidal part of the connected component $Z(G)^\circ \subset Z(G)$: we will refer to it as the **cuspidal quotient** of G.

Remark 6.1.4. As noted in the footnote on p. 34 of *loc. cit.*, this definition differs from the one used for instance by Lovering in [57], since we are not assuming Milne's axiom requiring that $Z(G)^\circ$ splits over a CM field. *If* we made this assumption, then we could identify $Z(G)_{ac}$ with the maximal anisotropic subtorus of $Z(G)$ that is $\mathbb{R}$-split.

ctoriality **Lemma 6.1.5.** *The assignment $(G, X) \mapsto G^c$ is functorial for maps between Shimura data.*

Proof. See [38, Lemma 3.8]. □

ied_tuples **Definition 6.1.6** (Unramified tuples)**.** Fix a prime p. **p-unramified Shimura tuple** or simply **unramified tuple**, (since p will remain fixed) is a tuple $(G, \mathcal{G}, X, K)$ where (G, X) is a Shimura datum, $\mathcal{G}$ is a reductive model for G over $\mathbb{Z}_p$, and $K \subset G(\mathbb{A}_f)$ is a compact open subgroup of the form K^pK_p where $K^p \subset G(\mathbb{A}_f^p)$ and $K_p = \mathcal{G}(\mathbb{Z}_p)$. Such a tuple is **neat** if K is neat.

Setup 6.1.7. As in the introduction, we will fix a prime p, and an unramified tuple $(G, \mathcal{G}, X, K)$. This yields a reductive model $\mathcal{G}_{(p)}$ for G over $\mathbb{Z}_{(p)}$. We will write $\mathcal{G}^c$ for the induced reductive model for G^c.

lem:GQK **Lemma 6.1.8.** *Set $Z(G)_K \overset{\text{defn}}{=} Z(G)(\mathbb{Q}) \cap K$. Then:*

[21]For any representation W of $G_{\mathbb{R}}$, x yields a Hodge structure W_x on W and μ_x acts on $W_x^{p,q}$ by $z \mapsto z^{-p}$.

(1) *$X\times G(\mathbb{A}_f)$ is a complex analytic covering space of* $\mathrm{Sh}_K(\mathbb{C})$ *with Galois group* $(G(\mathbb{Q})\times K)/Z(G)_K$.
(2) *For any normal subgroup $K'\subset K$ with $K'=K^pK'_p\subset K^pK_p=K$, $\mathrm{Sh}_{K'}\to\mathrm{Sh}_K$ is a finite Galois cover with Galois group $K/K'Z(G)_K\simeq K_p/K'_pZ(G)_{K_p}$, where $Z(G)_{K_p}$ is the image of $Z(G)_K$ in K_p.*
(3) *For any normal subgroup $K'\subset K$ with $K'=K^{',p}K_p\subset K^pK_p$, $\mathrm{Sh}_{K'}\to\mathrm{Sh}_K$ is a finite Galois cover with Galois group $K^p/K^pZ(G)_{K^p}$, where $Z(G)_{K^p}$ is the image of $Z(G)_K$ in K^p.*
(4) *For K* neat*, we have $Z(G)_K\subset Z(G)_{ac}(\mathbb{Q})$; in particular, we have canonical maps*

$$Z(G)_K\backslash G(\mathbb{Q})\to G^c(\mathbb{Q})\ ;\ K_p/Z(G)^-_{K_p}\to G^c(\mathbb{Q}_p)\ ;\ K^p/Z(G)^-_{K^p}\to G^c(\mathbb{A}_f^p).$$

Proof. Only the last assertion—where $Z(G)^-_{K_p}\subset K_p$ is the closure of $Z(G)_{K_p}$—needs proof. This follows from [48, Lemma 1.5.7]. □

ealization **Remark 6.1.9** (p-adic étale realization)**.** Fix a prime p, By assertion (2) of Lemma 6.1.8, we have a cofiltered system of Galois covers

$$\{\mathrm{Sh}_{K^pK'_p}\to\mathrm{Sh}_K\}_{K'_p\triangleleft K_p}$$

where each cover has Galois group $K_p/K'_pZ(G)_{K_p}$. The inverse limit of this system is a pro-Galois cover of Sh_K with Galois group $K_p/Z(G)^-_{K_p}$, which by assertion (3) of the same lemma admits a map to $G^c(\mathbb{Q}_p)$. In fact, its image lands inside $\mathcal{G}^c(\mathbb{Z}_p)$, and so we obtain a canonical pro-étale $\mathcal{G}^c(\mathbb{Z}_p)$-torsor over Sh_K, which we denote by $\mathbf{Et}_{K,p}$.

ealization **Remark 6.1.10** (Prime-to-p étale realization)**.** Similarly, we can consider the cofiltered system of Galois covers

$$\{\mathrm{Sh}_{K^{',p}K_p}\to\mathrm{Sh}_K\}_{K^{',p}\subset G(\mathbb{A}_f)}$$

where each cover has Galois group $K^p/K^{',p}Z(G)_{K^p}$. The inverse limit is represented by a scheme $\mathrm{Sh}_{K_p}\to\mathrm{Sh}_K$ that is a pro-Galois cover with Galois group $K^p/Z(G)^-_{K^p}$. Just like with the p-primary version, this admits a map to $G^c(\mathbb{A}_f^p)$, and so, for every prime $\ell\neq p$, we obtain a K^c_ℓ-torsor $\mathbf{Et}_{K,\ell}$ over Sh_K, where K^c_ℓ is the projection of K^p into $G^c(\mathbb{Q}_\ell)$.

to-p_hecke **Remark 6.1.11** (Prime-to-p Hecke correspondences and connected components)**.** There is an action of $G(\mathbb{A}_f^p)$ on the underlying cofiltered system of E-schemes $\{\mathrm{Sh}_{K^{',p}K_p}\}$: Given $g\in G(\mathbb{A}_f^p)$, we obtain a canonical isomorphism $\mathrm{Sh}_{K^{',p}K_p}\xrightarrow{\simeq}\mathrm{Sh}_{gK^{',p}g^{-1}K_p}$ induced over $\mathbb{C}$ by the endomorphism $(x,h)\mapsto(x,hg^{-1})$ of $X\times G(\mathbb{A}_f)$. This gives an action of $G(\mathbb{A}_f^p)$ on Sh_{K_p}. By [19, p. 2.1.3] and [44, Lemma (2.2.6)], the induced action on the connected components of $\mathrm{Sh}_{K_p,\overline{\mathbb{Q}}}$ factors through a simply transitive action of the quotient $G(\mathbb{A}_f^p)/(\bigcap_{K^{',p}}G(\mathbb{Z}_{(p)})_+K^{',p})$, which is in fact an abelian group.

_varieties

6.2. **The de Rham realization on Shimura varieties.** We recall here some properties of the de Rham realization on Shimura varieties. We will maintain the notation and hypotheses stemming from Definition 6.1.6.

Construction 6.2.1. By definition of the reflex field, the geometric conjugacy class $[\mu]$ in G of the Shimura cocharacters

$$\mu_x:\mathbb{G}_{m,\mathbb{C}}\xrightarrow{z\mapsto(z,1)}\mathbb{G}_{m,\mathbb{C}}\times\mathbb{G}_{m,\mathbb{C}}\xleftarrow[\simeq]{(az,a\overline{z})\leftarrow a\otimes z}\mathbb{S}_\mathbb{C}\xrightarrow{h_x}G_\mathbb{C}$$

is defined over $E\subset\mathbb{C}$. This implies the following: There exists a canonical projective homogeneous variety $\mathrm{Gr}_{[-\mu]}$ over E parameterizing filtrations on representations of G[22] that are étale locally split by a cocharacter of G in the geometric conjugacy class $[-\mu]$.

[22] More precisely, filtrations on the canonical fiber functor $\mathrm{Rep}_E(G)\to\mathrm{Vect}_E$.

Remark 6.2.2 (The de Rham realization)**.** As explained in [20, §5.2], there is a canonical functor from $\mathrm{Rep}_{\mathbb{Q}}(G^c)$ to the category of variations of $\mathbb{Q}$-Hodge structures over the complex analytic space $\mathrm{Sh}^{\mathrm{an}}_{K,\mathbb{C}}$: Its fiber at any lift $(x,g) \in X \times G(\mathbb{A}_f)$ is canonically isomorphic to the Hodge structure on V arising from the Deligne cocharacter $h_x : \mathbb{S} \to G_{\mathbb{R}}$ at x.[23] Moreover, the underlying filtered vector bundle with integrable connection for each object in the image of the functor is algebraic. That is, with every $V \in \mathrm{Rep}_{\mathbb{Q}}(G^c)$, we can associate a filtered vector bundle $\mathrm{Fil}^\bullet_{\mathrm{Hdg}}\,\mathbf{dR}_{K,\mathbb{C}}(V)$ over $\mathrm{Sh}_{K,\mathbb{C}}$, equipped with an integrable connection satisfying Griffiths transversality. This functorial association yields in particular a map $\mathrm{Sh}_{K,\mathbb{C}} \to \mathrm{Gr}_{-\mu,\mathbb{C}}/G_{\mathbb{C}}$ classifying a filtered G-torsor $\mathrm{Fil}^\bullet_{\mathrm{Hdg}}\,\mathbf{dR}_{K,\mathbb{C}}$ over $\mathrm{Sh}_{K,\mathbb{C}}$, and this filtered G-torsor is equipped with an integrable connection satisfying Griffiths transversality; compare with Remark 3.4.3. By [67, Theorems 4.3, 5.1], this descends canonically to a filtered G-torsor over $\mathrm{Fil}^\bullet_{\mathrm{Hdg}}\,\mathbf{dR}_{K,\mathbb{Q}}$ over Sh_K, and the integrable connection also descends. Therefore, for any $V \in \mathrm{Rep}_{\mathbb{Q}}(G)$, we obtain a canonical filtered vector bundle $\mathrm{Fil}^\bullet_{\mathrm{Hdg}}\,\mathbf{dR}_K(V)$ over Sh_K with an integrable connection satisfying Griffiths transversality.

p_rational **Remark 6.2.3** (The Kodaira-Spencer map)**.** Let $\mathbb{T}_{\mathrm{Sh}_K/E}$ be the tangent bundle for Sh_K. By Griffiths transversality, for any $V \in \mathrm{Rep}_{\mathbb{Q}}(G^c)$, we obtain a map of vector bundles

$$\mathbb{T}_{\mathrm{Sh}_K/E} \to \oplus_i \underline{\mathrm{Hom}}\left(\mathrm{gr}^i_{\mathrm{Hdg}}\,\mathbf{dR}_K(V), \mathrm{gr}^{i-1}_{\mathrm{Hdg}}\,\mathbf{dR}_K(V)\right).$$

This arises from a single map $\mathbb{T}_{\mathrm{Sh}_K/E} \to \mathbf{dR}_K(\mathfrak{g}_{\mathbb{Q}})/\mathrm{Fil}^0\,\mathbf{dR}_K(\mathfrak{g}_{\mathbb{Q}})$, where $\mathfrak{g}$ is the Lie algebra of $\mathcal{G}_{(p)}$ equipped with the adjoint action, and we refer to this latter map as the **Kodaira-Spencer map** for Sh_K. By the construction of the Shimura variety, this map is an isomorphism. See [19, Proposition 1.1.14] for a discussion of this over $\mathbb{C}$.

rem:lz **Remark 6.2.4** (de Rham-ness of the p-adic étale realization)**.** Fix a place $v|p$ of E, and let $\mathrm{Sh}^{\mathrm{an}}_{K,E_v}$ be the rigid analytification of Sh_K over E_v. Any $V \in \mathrm{Rep}_{\mathbb{Q}}(G)$, viewed as a $\mathbb{Q}_p$-representation of G, can be twisted by $\mathbf{Et}_{K,p}$ to obtain a $\mathbb{Q}_p$-local system $\mathbf{Et}_{K,p}(V)$ over Sh_K. In [54], Liu and Zhu showed that the restriction over $\mathrm{Sh}^{\mathrm{an}}_{K,E_v}$ of $\mathbf{Et}_{K,p}(V)$ is *de Rham*. More precisely, they showed that there is a tensor-functorially associated filtered vector bundle $\mathrm{Fil}^\bullet_{\mathrm{Hdg}}\,\mathbf{dR}^{\mathrm{LZ,an}}_{K,v}(V)$ over $\mathrm{Sh}^{\mathrm{an}}_{K,E_v}$ equipped with an integrable connection satisfying Griffiths transversality, such that $\mathbf{Et}_{K,p}(V)$ is *associated* with it in the sense of [78, Definition 8.3]. In particular, for every classical point x of $\mathrm{Sh}^{\mathrm{an}}_{K,E_v}$, $\mathbf{Et}_{K,p,x}(V)$ corresponds to a de Rham Galois representation, and the fiber of $\mathrm{Fil}^\bullet_{\mathrm{Hdg}}\,\mathbf{dR}^{\mathrm{LZ,an}}_{K,v}(V)$ at x is canonically isomorphic to the image of $\mathbf{Et}_{K,p,x}(V)$ under Fontaine's D_{dR} functor.

rem:dllz **Remark 6.2.5** (Work of Diao-Lan-Liu-Zhu)**.** In [20], we find the following improvements to the results of [54]:

- First, using toroidal compactifications, it is shown that $\mathrm{Fil}^\bullet_{\mathrm{Hdg}}\,\mathbf{dR}^{\mathrm{LZ,an}}_{K,v}(V)$ admits a canonical algebraization $\mathrm{Fil}^\bullet_{\mathrm{Hdg}}\,\mathbf{dR}^{\mathrm{LZ}}_{K,v}(V)$ over Sh_{K,E_v} compatible with tensor structures; see the discussion on pp. 536–537 of *loc. cit.*
- Next, the authors show in Theorem 5.3.1 of *loc. cit.* that the tensor functors

$$V \mapsto \mathrm{Fil}^\bullet_{\mathrm{Hdg}}\,\mathbf{dR}^{\mathrm{LZ}}_{K,v}(V)$$
$$V \mapsto \mathrm{Fil}^\bullet_{\mathrm{Hdg}}\,\mathbf{dR}_K(V)|_{\mathrm{Sh}_{K,E_v}}$$

are canonically isomorphic. Here, we view both as functors from $\mathrm{Rep}_{\mathbb{Q}}(G)$ to filtered vector bundles over Sh_{K,E_v} with integrable connection.

[23] Here, we follow Deligne's convention from [19] that $h_x(z)$ acts on the (p,q)-part of the Hodge decomposition via $z^{-p}\overline{z}^{-q}$.

l_aperture

6.3. **Consequences of having a universal aperture.**

Setup 6.3.1. Fix an unramified tuple $(G, \mathcal{G}, X, K)$. Fix a place $v|p$ of the reflex field E. Set $\hat{\mathcal{O}} = \mathscr{O}_{E_v}$, and let k be the residue field of $\hat{\mathcal{O}}$. If we view $\{\mu\}$ as an E_v-rational conjugacy class, then we in fact have a representative $\mu_v : \mathbb{G}_{m,\mathcal{O}} \to \mathcal{G}_{\hat{\mathcal{O}}}$ for $\{\mu\}$: The existence of a representative over E_v follows from the quasi-splitness of G_{E_v} [51, Lemma 1.1.3], while that over $\hat{\mathcal{O}}$ now follows from [44, Proposition 1.1.4]. We will view μ_v as a cocharacter of $\mathcal{G}^c_{\hat{\mathcal{O}}}$. Recall from § 4.1 the pro-algebraic stack $\mathrm{BT}^{\mathcal{G}^c,-\mu_v,\mathrm{alg}}_\infty$ over $\mathscr{O}_{E,(v)}$ with generic fiber $\mathrm{Loc}_{\mathcal{G}^c(\mathbb{Z}_p)}$ and v-adic completion $\mathrm{BT}^{\mathcal{G}^c,-\mu_v}_\infty$.

rmal_model **Definition 6.3.2.** Suppose that we have a map of schemes $U \to \mathrm{Sh}_K$, and suppose that $\mathcal{U}$ is a flat integral model for U over $\mathscr{O}_{E,(v)}$: By this, we mean that $\mathcal{U}$ is a flat separated algebraic space over $\mathscr{O}_{E,(v)}$ equipped with an identification $\mathcal{U} \otimes_{\mathscr{O}_{E,(v)}} E \xrightarrow{\simeq} U$. We will say that $\mathcal{U}$ is **apertile** if there is an extension $\mathcal{U} \to \mathrm{BT}^{\mathcal{G}^c,-\mu_v,\mathrm{alg}}_\infty$ of the classifying map $U \to \mathrm{Loc}_{\mathcal{G}^c(\mathbb{Z}_p)}$ for $\mathbf{Et}_{K,p}|_U$.

m:apertile **Remark 6.3.3.** By Theorem 4.1.5, such an extension is unique up to unique isomorphism if $\mathcal{U}$ is η-normal. In particular, in this case, being apertile is a *property* of the integral model $\mathcal{U}$.

wise_lifts **Remark 6.3.4** (Type of $\mathbf{Et}_{K,p}$)**.** The existence of the aperture implies in particular that the restriction of $\mathbf{Et}_{K,p}$ over $\hat{\mathcal{U}}_\eta$ is a crystalline $\mathcal{G}(\mathbb{Z}_p)$-local system of type $-\mu_v$ at every classical point. Conversely, if we knew *a priori* that $\mathbf{Et}_{K,p}$ is crystalline at every classical point of $\hat{\mathcal{U}}_\eta$, then, by Remarks 6.2.4 and 6.2.5, we see that its type at any such point has to be $-\mu_v$.

al_de_rham **Construction 6.3.5** (The integral de Rham realization)**.** Consider the composition

e_rham_map
$$\widehat{\mathcal{U}} \to \mathrm{BT}^{\mathcal{G},-\mu_v}_\infty \to \mathrm{Gr}_{-\mu_v}/\mathcal{G}_{\mathscr{O}_{E_v}} \tag{6.3.5.1}$$

where the second map is as in Remark 3.4.3. This map classifies a filtered $\mathcal{G}$-bundle over $\widehat{\mathcal{U}}$, whose restriction over $\widehat{\mathcal{U}}_\eta$ is canonically isomorphic to the restriction of $\mathrm{Fil}^\bullet_{\mathrm{Hdg}}\,\mathbf{dR}_{K,\mathbb{Q}}$: One checks this using Remarks 6.2.5 and 3.7.3. This tells us that there exists a map

$$\mathcal{U} \to \mathrm{Gr}_{-\mu_v}/\mathcal{G}_{\mathscr{O}_{E_v}}$$

whose formal completion is (6.3.5.1) and whose generic fiber classifies $\mathrm{Fil}^\bullet_{\mathrm{Hdg}}\,\mathbf{dR}_{K,\mathbb{Q}}$. Denote the associated filtered $\mathcal{G}$-bundle over $\mathscr{S}_K$ by $\mathrm{Fil}^\bullet_{\mathrm{Hdg}}\,\mathbf{dR}_K$.

braization **Remark 6.3.6** (The integral Kodaira-Spencer map)**.** Suppose that $\mathcal{U}$ is *smooth* over $\mathscr{O}_{E,(v)}$. Then the filtered $\mathcal{G}$-bundle from the previous remark gives us a Kodaira-Spencer map

pencer_map
$$\mathbb{T}_{\mathcal{U}/\mathscr{O}_{E,(v)}} \to \mathbf{dR}_K(\mathfrak{g})/\mathrm{Fil}^0\,\mathbf{dR}_K(\mathfrak{g}) \tag{6.3.6.1}$$

whose restriction over the generic fiber is described in Remark 6.2.3 and whose v-adic completion is described in Remark 5.2.2.

_condition **Lemma 6.3.7** (A versality condition)**.** *The following are equivalent:*

(1) The map $\widehat{\mathcal{U}} \to \mathrm{BT}^{\mathcal{G}^c,-\mu_v}_\infty$ is formally étale
(2) $\mathcal{U}$ is smooth over $\mathscr{O}_{E,(v)}$ and the map (6.3.6.1) *is an isomorphism.*

Proof. Immediate from Remark 5.2.2. ☐

cal_models

6.4. **Integral canonical models.**

We can now give the definition of integral canonical models and formulate their mapping properties.

defn:ICM **Definition 6.4.1** (Integral canonical models)**.** An apertile integral model $\mathscr{S}_K$ for Sh_K over $\mathscr{O}_{E,(v)}$ is a **crystalline integral canonical model**—or Cr-ICM for short—if it satisfies the equivalent conditions of Lemma 6.3.7, and if, for any mixed characteristic $(0,p)$ complete discrete valuation field F over $\mathscr{O}_{E_v}$ with perfect residue field, the following are equivalent for $x \in \mathrm{Sh}_K(F)$:

(1) $x \in \mathscr{S}_K(\mathscr{O}_F)$;
(2) $\mathbf{Et}_{K,p,x}$ is crystalline;
(3) $\mathbf{Et}_{K,p,x}$ is potentially crystalline.

It is an **étale integral canonical model**—or Ét-ICM for short—if, instead, we replace the last condition with: For any mixed characteristic $(0,p)$ discrete valuation field F over $\mathscr{O}_{E_v}$, the following are equivalent for $x \in \mathrm{Sh}_K(F)$:

(1) $x \in \mathscr{S}_K(\mathscr{O}_F)$;
(2) $\mathbf{Et}_{K,\ell,x}$ is unramified for all $\ell \neq p$;
(3) $\mathbf{Et}_{K,\ell,x}$ is potentially unramified for all $\ell \neq p$.

If the context is agnostic to the precise type of integral canonical model involved, we will simply call $\mathscr{S}_K$ an ICM for Sh_K.

n_ell-adic

Remark 6.4.2 (Extension of ℓ-adic local systems)**.** The pointwise condition in the definition of an Ét-ICM implies that, for all $\ell \neq p$, $\mathbf{Et}_{K,\ell}$ extends to a local system over $\mathscr{S}_K$: One applies the unramifiedness to the local ring at any codimension 1 point and then uses purity.

Let us restate Theorem G in the following form, which is now immediate from Proposition 4.3.1.

g_property

Theorem 6.4.3 (Mapping property of (weak) integral canonical models)**.** *Let $\mathcal{Y}$ be an excellent algebraic space flat over $\mathscr{O}_{E,(v)}$ with generic fiber Y. Suppose that $\widehat{\mathcal{Y}}$ is η-normal in the sense of [1, Appendix A].*

(1) If $\mathscr{S}_K$ is a Cr*-ICM for* Sh_K *over $\mathscr{O}_{E,(v)}$, then giving a map $\tilde{f} : \mathcal{Y} \to \mathscr{S}_K$ is equivalent to giving maps $\eta : \mathcal{Y} \to \mathrm{BT}_\infty^{\mathcal{G}^c,-\mu_v,\mathrm{alg}}$ and $f : Y \to \mathrm{Sh}_K$ such that the generic fiber $Y \to \mathrm{Loc}_{\mathcal{G}(\mathbb{Z}_p)}$ of η lifts f along the map $\mathrm{Sh}_K \to \mathrm{Loc}_{\mathcal{G}(\mathbb{Z}_p)}$ classifying $\mathbf{Et}_{K,p}$.*

(2) If $\mathscr{S}_K$ is an Ét*-ICM, then giving a map $\tilde{f} : \mathcal{Y} \to \mathscr{S}_K$ is equivalent to giving maps $\eta : \mathcal{Y} \to \mathrm{BT}_\infty^{\mathcal{G}^c,-\mu_v,\mathrm{alg}}$ and $f : Y \to \mathrm{Sh}_K$ such that:*

(a) The generic fiber $Y \to \mathrm{Loc}_{\mathcal{G}(\mathbb{Z}_p)}$ of η lifts f along the map $\mathrm{Sh}_K \to \mathrm{Loc}_{\mathcal{G}(\mathbb{Z}_p)}$ classifying $\mathbf{Et}_{K,p}$;

(b) For all $\ell \neq p$, $\mathbf{Et}_{K,\ell}|_Y$ extends to a K_ℓ^c-local system over $\mathcal{Y}$.

In both cases, we have canonical isomorphisms of classifying maps $\varpi \circ \tilde{f} \simeq \eta$.

uniqueness

Corollary 6.4.4 (Uniqueness)**.** *An ICM $\mathscr{S}_K$ for Sh_K is unique up to unique isomorphism.*

Definition 6.4.5 (Maps between unramified tuples)**.** A map $(G_1, \mathcal{G}_1, X_1, K_1) \to (G_2, \mathcal{G}_2, X_2, K_2)$ between unramified tuples is the data of a map of $\mathbb{Q}$-groups $f : G_1 \to G_2$ and an element $g \in G_2(\mathbb{A}_f^p)$ with the following properties:

- $f_{\mathbb{Q}_p}$ extends to a map of $\mathbb{Z}_p$-group schemes $\mathcal{G}_1 \to \mathcal{G}_2$;
- f induces a map of Shimura data $(G_1, X_1) \to (G_2, X_2)$;
- We have $gf(K_1^p)g^{-1} \subset K_2^p$.

rom_tuples

Remark 6.4.6. Let $E_2 \subset E_1 \subset \mathbb{C}$ be the reflex fields of (G_2, X_2) and (G_1, X_1), respectively. there is a unique map of Shimura varieties

$$\mathrm{Sh}_{K_1} \to E_1 \otimes_{E_2)} \mathrm{Sh}_{K_2}$$

whose evaluation on $\mathbb{C}$-points is given by

$$G_1(\mathbb{Q})\backslash X_1 \times G_1(\mathbb{A}_f)/K_1 \xrightarrow{[(x,h)]\mapsto[(f(x),f(h)g^{-1})]} G_2(\mathbb{Q})\backslash X_2 \times G_2(\mathbb{A}_f)/K_2.$$

ctoriality

Corollary 6.4.7 (Functoriality)**.** *Suppose that we have a map of tuples as above. Then the map $\mathrm{Sh}_{K_1} \to E_1 \otimes_{E_2} \mathrm{Sh}_{K_2}$ extends to a map $\mathscr{S}_{K_1} \to \mathscr{O}_{E_1,(v_1)} \otimes_{\mathscr{O}_{E_2,(v_2)}} \mathscr{S}_{K_2}$.*

:cm_points 6.5. **CM points and canonical models for tori.**

_varieties **Construction 6.5.1** (CM Shimura varieties)**.** Let T be a torus over $\mathbb{Q}$ equipped with a map $h_T : \mathbb{S} \to T_{\mathbb{R}}$ and let $\mu_T \in X_*(T)$ be the Shimura cocharacter associated with h. Let $E_T \subset \overline{\mathbb{Q}}$ be the field of definition of μ_T. Then, for any neat compact open subgroup $K_T \subset T(\mathbb{A}_f)$, the tuple $(T, \{h\}, K_T)$ yields a zero-dimensional Shimura variety Sh_{K_T} over E with

$$\mathrm{Sh}_{K_T}(\mathbb{C}) \simeq T(\mathbb{Q})\backslash T(\mathbb{A}_f)/K.$$

Fix a place $w|p$ of E_T, and let $\mathscr{S}_{K_T}$ be the normalization of $\operatorname{Spec} \mathscr{O}_{E_T,(w)}$ in Sh_{K_T}.

etale_tori **Remark 6.5.2** (Étale realizations for tori)**.** Suppose that we have a finite extension $F/E_{T,w}$ and a point $x \in \mathrm{Sh}_{K_T}(F)$. As discussed in [48, §4.3.13, 5.7], for each prime ℓ, we can describe the Galois representation associated with the local system $\mathbf{Et}_{K_T,\ell,x}$ as follows: Consider the reflex norm

$$r_{\mu_T} : \operatorname{Res}_{E_T/\mathbb{Q}} \mathbb{G}_m \xrightarrow{\operatorname{Res}_{E_T/\mathbb{Q}} \mu_T} \operatorname{Res}_{E_T/\mathbb{Q}} T \xrightarrow{\mathrm{Nm}_{E_T/\mathbb{Q}}} T.$$

This induces a map

$$E_T^\times \backslash \mathbb{A}_{E_T}^\times \xrightarrow{r_\mu(\mathbb{A}_f)} T(\mathbb{Q})\backslash T(\mathbb{A}) \to \pi_0(T^c(\mathbb{Q})\backslash T^c(\mathbb{A})) \to T^c(\mathbb{Q})\backslash T^c(\mathbb{A}_f)$$

that factors via the global reciprocity map (with the geometric normalization) through a homomorphism

$$\tau(\mu_T) : \mathrm{Gal}(E_T^{\mathrm{ab}}/E_T) \to T^c(\mathbb{Q})\backslash T^c(\mathbb{A}_f)$$

The composition

$$\mathrm{Gal}(\overline{\mathbb{Q}}_p/F) \to \mathrm{Gal}(E_T^{\mathrm{ab}}/E_T) \to T^c(\mathbb{Q})\backslash T^c(\mathbb{A}_f) \to T^c(\mathbb{Q}_\ell)$$

factors through $K^c_{T,\ell}$, and the local system associated with it is canonically isomorphic to $\mathbf{Et}_{K_T,\ell,x}$.

stallinity **Remark 6.5.3** (Crystallinity at p)**.** The description in Remark 6.5.2, combined with [48, Proposition 4.3.14], shows that the $T(\mathbb{Q}_p)$-local system underlying $\mathbf{Et}_{K_T,p,x}$ is crystalline of type $-\mu_T$.

unramified **Remark 6.5.4** (Unramifiedness at $\ell \neq p$)**.** For $\ell \neq p$, the description from Remark 6.5.2 shows that the local system $\mathbf{Et}_{K_T,\ell,x}$ is unramified. This follows from the usual argument, noting that $\mathscr{O}_F^\times$ admits a finite index pro-p subgroup, while $K^c_{T,\ell}$ is a pro-ℓ subgroup.

Integral canonical models always exist for CM Shimura varieties:

s_for_tori **Proposition 6.5.5** (ICMs for tori)**.** *Suppose that $T_{\mathbb{Q}_p}$ extends to a torus $\mathcal{T}$ over $\mathbb{Z}_p$ such that $K_{T,p} = \mathcal{T}(\mathbb{Z}_p)$. Then $\mathscr{S}_{K_T}$ is both a* Cr-*ICM and an* Ét-*ICM for* Sh_{K_T}.

Proof. In this case, $\mathscr{S}_{K_T}$ is a finite étale scheme over $\mathscr{O}_{E_T,(w)}$. Moreover, the stacks $\mathrm{BT}_n^{\mathcal{T},-\mu_T}$ are étale over $\mathscr{O}_{E_{T,w}}$ and are in fact non-canonically isomorphic to the $\mathscr{O}_{E_{T,w}}$-stack $\mathrm{Loc}_{\underline{\mathcal{T}(\mathbb{Z}/p^n\mathbb{Z})}}$ of $\mathcal{T}(\mathbb{Z}/p^n\mathbb{Z})$-local systems; see [24, Proposition 10.4.1]. Therefore, *any* map $\widehat{\mathscr{S}}_{K_T,w} \to \mathrm{BT}_\infty^{\mathcal{T},-\mu_T}$ will be formally étale. Proposition 5.1.3 now reduces us to checking that, for every finite extension $L/E_{T,w}$ and every $x \in \mathscr{S}_K(\mathscr{O}_L)$, the following statements hold:

(1) $\mathbf{Et}_{K_T,p,x}$ belongs to $\mathrm{Loc}^{\mathrm{crys},-\mu_T}_{\mathcal{T}(\mathbb{Z}_p)}(L)$;
(2) For $\ell \neq p$, $\mathbf{Et}_{K_T,\ell,x}$ is unramified.

These follow from Remarks 6.5.3 and 6.5.4. ∎

imura_data **Proposition 6.5.6.** *Suppose that in the situation of Corollary 6.4.7, the map $G_1 \to G_2$ is a central cover. Then an integral model $\mathcal{U}_1$ for*

:cm_points **Construction 6.5.7.** Suppose that we have an unramified tuple $(G, \mathcal{G}, X, K)$ and an ICM $\mathscr{S}_K$ for Sh_K over $\mathscr{O}_{E,(v)}$ for a place $v|p$. As a special case of Construction 6.5.1, suppose that we have a torus T over $\mathbb{Q}$ and a closed immersion $i : T \to G$ over $\mathbb{Q}$ such that, for some $x \in X$, the map $h_x : \mathbb{S} \to G_{\mathbb{R}}$ factors through $i_{\mathbb{R}}$ via a map $h_T : \mathbb{S} \to T_{\mathbb{R}}$. Given $g \in G(\mathbb{A}_f^p)$, set $K_{T,g} = i^{-1}(gKg^{-1}) \subset T(\mathbb{A}_f)$. Then, for any place $w|p$ of E_T above v, we obtain an integral model $\mathscr{S}_{K_{T,g},(w)}$ for $\mathrm{Sh}_{K_{T,g}}$ over $\mathscr{O}_{E_T,(w)}$ as the normalization of $\mathrm{Spec}\,\mathscr{O}_{E,(v)}$ in $\mathrm{Sh}_{K_{T,g}}$.

_cm_points **Lemma 6.5.8** (Extension of CM points)**.** *The map $\mathrm{Sh}_{K_{T,g}} \to E_T \otimes_E \mathrm{Sh}_K$ of canonical models given on $\mathbb{C}$-points by*

$$T(\mathbb{Q})\backslash T(\mathbb{A}_f)/K_{T,g} \xrightarrow{[t]\mapsto[(x_T, i(t)g]} G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K$$

extends canonically to a map of integral models $\mathscr{S}_{K_{T,g},(w)} \to \mathscr{O}_{E_T,(w)} \otimes_{\mathscr{O}_{E,(v)}} \mathscr{S}_K$.

Proof. To alleviate notation, set $K_T = K_{T,g}$. Given Theorem 6.4.3, it is enough to know the following:

(1) The restriction of $\mathbf{Et}_{K,p}$ over Sh_{K_T} lifts to a map

$$\mathscr{S}_{K_T,(w)} \to \mathrm{BT}_\infty^{\mathcal{G},-\mu_v,\mathrm{alg}}.$$

(2) For all $\ell \neq p$, the restriction of $\mathbf{Et}_{K,\ell}$ over Sh_{K_T} extends to a local system over $\mathscr{S}_{K_T}$.

Since $\mathscr{S}_{K,T,(w)}$ is a product of spectra of localizations of rings of integers of number fields, we reduce to knowing (by Lemma 5.1.10 in the p-adic case) that, for every complete local ring $\hat{\mathscr{O}}$ of $\mathscr{S}_{K,T,(w)}$, the restriction of $\mathbf{Et}_{K,p}$ (resp. $\mathbf{Et}_{K,\ell}$) over $\hat{\mathscr{O}}[1/p]$ lifts to $\mathrm{BT}_\infty^{\mathcal{G},-\mu_v}(\hat{\mathscr{O}})$ (resp. extends to a local system over $\hat{\mathscr{O}}$).

By Proposition 5.1.3, the p-adic condition is equivalent to checking that the restriction of $\mathbf{Et}_{K,p}$ over $\hat{\mathscr{O}}[1/p]$ is crystalline of type $-\mu_v$, which is immediate from Remark 6.5.3. The unramifiedness of the ℓ-adic realizations is similarly immediate from Remark 6.5.4. ∎

Not sure if this definition is unnecessarily baroque.

Definition 6.5.9 (CM points)**.** For any $\mathscr{O}_{E,(v)}$-algebra R, a point $x \in \mathscr{S}_K(R)$ is a **CM point** if there exist a finite extension F/E, a place $w'|p$ of F above v, a tuple (T, i, h_T, g) as in Construction 6.5.7, a place $w|p$ of E_T, a faithfully flat map $R \to R'$, and a commuting diagram of schemes

$$\begin{tikzcd}
\mathrm{Spec}\, R' \arrow[r] \arrow[dr] & \mathrm{Spec}\, \mathscr{O}_{F,(w')} \arrow[r] \arrow[d] & \mathscr{S}_{K_{T,g},(w)} \arrow[d] \\
& \mathrm{Spec}\, R \arrow[r, "x"] & \mathscr{S}_K
\end{tikzcd}$$

where the right vertical arrow is the obtained from Lemma 6.5.8.

bal_mu-ord

6.6. The global μ-ordinary locus.

Here, we will state some results about the μ-ordinary loci in integral canonical models. Proofs will appear in [62] .

Definition 6.6.1 (The global μ-ordinary locus)**.** Let $\mathscr{S}_K$ be an ICM for Sh_K over $\mathscr{O}_{E,(v)}$.

_ord_locus **Theorem 6.6.2** (Properties of the μ-ordinary locus)**.**

(1) *The μ-ordinary locus is open and fiber-wise dense in $\mathscr{S}_K$.*

(2) *For any perfect field κ in characteristic p and $x \in \mathscr{S}_K^{\mathrm{ord}}(\kappa)$, the completion $\widehat{\mathscr{S}}_{K,x}$ of $\mathscr{S}_K$ at x has the structure of a cascade over $W(\kappa)$ in the sense of [69].*

(3) *If κ is algebraic over $k(v)$ and $\mathscr{S}_K$ is a Cr-ICM, then every torsion point of the cascade $\widehat{\mathscr{S}}_{K,x}$ is a CM point of $\mathscr{S}_K$.*

ral_covers

6.7. **Central covers and reductions of structure group.** This subsection—the heart of our approach to the construction of integral canonical models—establishes some very useful group theoretic functoriality properties for the existence of such models. The main goal is to prove the following theorems that imply Theorems D and Theorem C.

ture_group

Theorem 6.7.1 (Reduction of structure group)**.** *Suppose that* $(G,\mathcal{G},X,K)\to(G^\sharp,\mathcal{G}^\sharp,X^\sharp,K^\sharp)$ *is a map of neat unramified tuples such that* $\mathcal{G}\to\mathcal{G}^\sharp$ *is a closed immersion. If* $\mathrm{Sh}_{K^\sharp}$ *admits an ICM* $\mathscr{S}_{K^\sharp}$*, then the normalization of* $\mathscr{S}_{K^\sharp}$ *in* Sh_K *gives an ICM for* Sh_K*.*

ral_covers

Theorem 6.7.2 (Insensitivity to central covers)**.** *Suppose that* $(G,\mathcal{G},X,K)\to(\overline{G},\overline{\mathcal{G}},\overline{X},\overline{K})$ *is a map of unramified tuples such that* $\mathcal{G}\to\overline{\mathcal{G}}$ *is a surjective map of reductive group schemes with central kernel. Then:*

1. *If* Sh_K *admits an ICM over* $\mathscr{O}_{E,(v)}$*, then* $\mathrm{Sh}_{\overline{K}}$ *admits one over* $\mathscr{O}_{\overline{E},\overline{v}}$*.*
2. *If* $\mathrm{Sh}_{\overline{K}}$ *admits an* Ét*-ICM over* $\mathscr{O}_{\overline{E},\overline{v}}$ *and the map* $\mathcal{G}^{\mathrm{der}}\to\overline{\mathcal{G}}^{\mathrm{der}}$ *on derived subgroups is an isomorphism, then* Sh_K *admits an* Ét*-ICM over* $\mathscr{O}_{E,(v)}$*.*
3. *If* $\mathrm{Sh}_{\overline{K}}$ *admits a* Cr*-ICM over* $\mathscr{O}_{\overline{E},\overline{v}}$*, then* Sh_K *admits a* Cr*-ICM over* $\mathscr{O}_{E,(v)}$*.*

In all these cases, the map $\mathscr{S}_K\to\mathscr{S}_{\overline{K}}$ *of ICMs is finite étale.*

We have the following immediate corollaries:

-to-p_part

Corollary 6.7.3 (Insensitivity to prime-to-p level and Hecke correspondences)**.** *Suppose that* $K^{',p}\subset K^p$ *is a compact open subgroup and set* $K'=K_pK^{',p}$*. Then:*

1. $\mathrm{Sh}_{K'}$ *admits an ICM over* $\mathscr{O}_{E,(v)}$ *if and only if* Sh_K *does.*
2. *If (1) holds, and* $g\in G(\mathbb{A}_f^p)$ *is such that* $gK^{',p}g^{-1}\subset K^p$*, then the map* $i_g:\mathrm{Sh}_{K'}\to\mathrm{Sh}_K$ *determined on complex points by*

$$G(\mathbb{Q})\backslash X\times G(\mathbb{A}_f)/K'\xrightarrow{[(x,h)]\mapsto[(x,hg^{-1})]}G(\mathbb{Q})\backslash X\times G(\mathbb{A}_f)/K$$

extends to a finite étale map of ICMs $\mathscr{S}_{K'}\to\mathscr{S}_K$*.*

_canonical

Corollary 6.7.4. *Suppose that* $\mathscr{S}_K$ *is a* Cr*-ICM for* Sh_K*. Then:*

1. *For all* $\ell\neq p$, $\mathbf{Et}_{K,\ell}$ *extends to a local system over* $\mathscr{S}_K$*;*
2. *Any* Ét*-ICM for* Sh_K *is canonically isomorphic to* $\mathscr{S}_K$*.*

Proof. Assertion (1) is just a reinterpretation of part of assertion (2) from Corollary 6.7.3. If $\mathscr{S}'$ is an Ét-ICM for Sh_K, then (1), along with Theorem 6.4.3, tells us that we have a map $\mathscr{S}_K\to\mathscr{S}'$ extending the identity for Sh_K. On the other hand, it is clear from the definitions and *loc. cit.* that we have a map $\mathscr{S}'\to\mathscr{S}_K$ extending the identity for Sh_K. These maps are now inverses to each other. □

The proofs of the theorems will be interwoven with each other. We begin by proving some important special cases. Here is the first of these:

_immersion

Proposition 6.7.5. *Theorem 6.7.1 holds when the map* $\mathrm{Sh}_K\to\mathrm{Sh}_{K^\sharp}$ *is a closed immersion.*

Proof. We have the following diagram

$$\begin{tikzcd} \mathrm{Sh}_K \arrow[r] \arrow[d] & \mathrm{Loc}_{\mathcal{G}(\mathbb{Z}_p)} = \mathrm{BT}_\infty^{\mathcal{G},-\mu_v,\mathrm{alg}}[1/p] \arrow[d] \\ \mathscr{S}_K \arrow[r, dotted, "?\exists"] \arrow[d] & \mathrm{BT}_\infty^{\mathcal{G},-\mu_v,\mathrm{alg}} \arrow[d] \\ \mathscr{S}_{K^\sharp} \arrow[r] & \mathrm{BT}_\infty^{\mathcal{G}^\sharp,-\mu_v,\mathrm{alg}} \end{tikzcd}$$

The bottom left vertical arrow is the normalization of a closed immersion by construction, and the bottom horizontal arrow is formally étale. Therefore, to fill in the middle dotted arrow and also know that it is formally étale, we can appeal to Proposition 5.3.15. The only thing that needs to be checked is the limipidity criterion from Lemma 5.3.12, which is immediate from Remark 6.3.4.

To finish, we need to know that $\mathscr{S}_K$ satisfies the pointwise conditions for ICMs: This just amounts to the observation that by construction for any discrete valuation field F over E, we have

$$\mathscr{S}_K(\mathscr{O}_F) = \mathrm{Sh}_K(F) \cap \mathscr{S}_{K^\sharp}(\mathscr{O}_F) \subset \mathrm{Sh}_{K^\sharp}(F).$$

□

For the remaining proofs, it will be useful to have a generalization of the notion of an ICM to étale schemes over Sh_K. This is purely for technical reasons, and is not necessarily the 'correct' such definition for for all such schemes, so we would suggest that the reader not take it too seriously.

CM_general **Definition 6.7.6** ((Generalized) Integral canonical models)**.** Suppose that we have a finite extension E'/E in which p is unramified, a place $v'|v$ of E, and an étale map $U \to E' \otimes_E \mathrm{Sh}_K$. An apertile integral model $\mathcal{U}$ for U over $\mathscr{O}_{E',(v')}$ is a **crystalline integral canonical model**—or Cr-ICM for short—if it satisfies the equivalent conditions of Lemma 6.3.7, and if, for any mixed characteristic $(0,p)$ complete discrete valuation field F over $\mathscr{O}_{E'_{v'}}$ with perfect residue field, the following are equivalent for $x \in U(F)$:

(1) $x \in \mathcal{U}(\mathscr{O}_F)$;
(2) $\mathbf{Et}_{K,p,x}$ is crystalline;
(3) $\mathbf{Et}_{K,p,x}$ is potentially crystalline.

It is an **étale integral canonical model**—or Ét-ICM for short—if, instead, we replace the last condition with: For any mixed characteristic $(0,p)$ discrete valuation field F over $\mathscr{O}_{E'_{v'}}$, the following are equivalent for $x \in U(F)$:

(1) $x \in \mathcal{U}(\mathscr{O}_F)$;
(2) $\mathbf{Et}_{K,\ell,x}$ is unramified for all $\ell \neq p$;
(3) $\mathbf{Et}_{K,\ell,x}$ is potentially unramified for all $\ell \neq p$.

If the context is agnostic to the precise type of integral canonical model involved, we will simply call $\mathcal{U}$ an ICM for U.

lized_icms **Remark 6.7.7** (Properties of generalized ICMs)**.** The proofs from § 6.4 work *mutatis mutandis* to show:

(1) An ICM for U is unique up to unique isomorphism.
(2) Suppose that we have a map of unramified tuples $(G_1, \mathcal{G}_1, X_1, K_1) \to (G_2, \mathcal{G}_2, X_2, K_2)$, p-unramified extensions E'_2/E_2 and E'_1/E_1 with $E'_2 \subset E'_1$, and that we have a commuting

diagram

$$\begin{tikzcd} U_1 \arrow[r] \arrow[d] & U_2 \arrow[d] \\ E_1' \otimes_{E_1} \mathrm{Sh}_{K_1} \arrow[r] & E_2' \otimes_{E_2} \mathrm{Sh}_{K_2} \end{tikzcd}$$

where the vertical arrows are étale, and where the bottom arrow is obtained as a base-change of the map from Remark 6.4.6. Suppose that $v_1'|p$ is a place of E_1' lying above a place v_2' of E_2', and that $\mathcal{U}_1$ (resp. $\mathcal{U}_2$) is an ICM for U_1 over $\mathscr{O}_{E_1',(v_1')}$ (resp. for U_2 over $\mathscr{O}_{E_2',(v_2')}$). Then the map $U_1 \to U_2$ extends to a map $\mathcal{U}_1 \to \mathcal{U}_2$.

s_gen_ICMs **Lemma 6.7.8** (Descent for coefficients)**.** *Suppose that E''/E' is a further p-unramified extension, and that $v''|v'$ is a place of E'' such that $E'' \otimes_{E'} U$ admits an ICM $\mathcal{U}''$ over $\mathscr{O}_{E'',(v'')}$. Then U admits an ICM already over $\mathscr{O}_{E',(v')}$.*

Proof. We can assume without loss of generality that E''/E' is finite Galois.

It suffices to show that $\widehat{\mathcal{U}}''$, along with the classifying map $\widehat{\mathcal{U}}'' \to \mathrm{BT}_\infty^{\mathcal{G},-\mu_v}$, descends to a formal scheme $\widehat{\mathcal{U}}$ over $\mathscr{O}_{E'_{v'}}$. Indeed, this would show—by the Beauville-Laszlo gluing results of [2] for instance—the existence of an apertile integral model $\mathcal{U}$ over $\mathscr{O}_{E',(v')}$ satisfying the equivalent conditions of Lemma 6.3.7, and the pointwise conditions would follow immediately from those for $\mathcal{U}''$.

To show that $\widehat{\mathcal{U}}''$ descends, it suffices to establish that the action of $\mathrm{Gal}(E''_{v''}/E'_{v'}) \leq \mathrm{Gal}(E''/E')$ on $E'' \otimes_{E'} U$ extends to an action on $\mathcal{U}''$, which follows from (2) of Remark 6.7.7. $\square$

scent_icms **Lemma 6.7.9** (Galois descent for ICMs)**.** *In the situation of (2) of Remark 6.7.7, suppose that $\mathcal{G}_1 \to \mathcal{G}_2$ is a central cover, and that the map $U_1 \to E_1' \otimes_{E_2'} U_2$ is finite Galois. If U_1 admits an ICM over $\mathscr{O}_{E_1',v_1'}$, then U_2 admits one over $\mathscr{O}_{E_2',v_2'}$.*

Proof. By Lemma 6.7.9, we can assume that $E_2' = E_1'$. We can also assume that U_2 and U_1 are connected. Let Δ be the Galois group for the cover $U_1 \to U_2$. If $\mathcal{U}_1$ is the ICM for U_1 over $\mathscr{O}_{E_1',v_1'}$, then by the functoriality explained in (2) of Remark 6.7.7, the action of Δ on $\mathcal{U}_1$ extends to one on $\mathcal{U}_1$. Consider the Deligne-Mumford stack $[\mathcal{U}_1/\Delta]$: Its generic fiber is U_2, and, by Proposition 4.4.4, we obtain an extension $[\mathcal{U}_1/\Delta] \to \mathrm{BT}_\infty^{\mathcal{G}_2,-\mu_{v_2},\mathrm{alg}}$ whose p-adic completion is formally étale. By Lemma 4.6.1, we see that $\mathcal{U}_2 \overset{\mathrm{defn}}{=} \mathcal{U}_1/\Delta$ is in fact a separated algebraic space.

We now claim that it is an ICM for U_2. Let us consider the situation of a Cr-ICM; the argument for an Ét-ICM is similar. We have to verify that if $F/E'_{1,v_1'}$ is a complete discrete valuation field with perfect residue field, and $x \in U_2(F)$ is such that the $\mathcal{G}_2^c(\mathbb{Z}_p)$-local system $\mathbf{Et}_{K_2,p,x}$ is potentially crystalline, then $x \in \mathcal{U}_2(\mathscr{O}_F)$. For this, after replacing F with a finite extension if necessary we can assume that x lifts to $y \in U_1(F)$. We claim that the $\mathcal{G}_1^c(\mathbb{Z}_p)$-local system $\mathbf{Et}_{K_1,p,y}$ is also potentially crystalline: This would imply that we have $y \in \mathcal{U}_1(\mathscr{O}_F)$ and hence that its image x lies in $\mathcal{U}_2(\mathscr{O}_F)$.

To prove the claim, consider the map $\mathcal{G}_1 \to \mathcal{G}_2 \times \mathcal{G}^{\mathrm{ab}}$: This has finite kernel, and so to know that a $\mathcal{G}_1^c(\mathbb{Z}_p)$-local system over F is potentially crystalline, it is enough to know that the associated local systems for $\mathcal{G}_2^c(\mathbb{Z}_p)$ and $\mathcal{G}^{\mathrm{ab},c}(\mathbb{Z}_p)$-local systems are potentially crystalline. For $\mathbf{Et}_{K_1,p,y}$, we are guaranteed the potential crystalline of the first by hypothesis, and that of the second follows from Remark 6.5.3. $\square$

ral_covers **Remark 6.7.10** (CM points and central covers)**.** Suppose that we have (T, i, h_T) as in Construction 6.5.7 with $\overline{T} \subset \overline{G}$ the image of T, and $\overline{g} \in \overline{G}(\mathbb{A}_f^p)$. Then we can consider the fiber product

$$\mathrm{Sh}_K \times_{\mathrm{Sh}_{\overline{K}}} \mathrm{Sh}_{\overline{K}_{\overline{T},\overline{g}}} \to \mathrm{Sh}_{\overline{K}_{\overline{T},\overline{g}}}.$$

By looking at $\mathbb{C}$-points, one finds that this is a disjoint union of CM Shimura varieties of the form $\mathrm{Sh}_{K_{T',g'}}$, where $i' : (T', h_{T'}) \hookrightarrow (G, X)$ is a $\overline{G}(\mathbb{Q})$-conjugate of i.

The next result proves the most non-trivial part of Theorem 6.7.2, whose notation we will be using here.

ral_covers **Proposition 6.7.11.** *Suppose that $\mathcal{G} \to \overline{\mathcal{G}}$ has finite kernel and that we have a finite p-unramified extension E'/E and an étale map $V \to E' \otimes_{\overline{E}} \mathrm{Sh}_{\overline{K}}$. If V admits a* Cr*-ICM $\mathcal{V}$ over $\mathscr{O}_{\overline{E},(\overline{v})}$, then $V \times_{\mathrm{Sh}_{\overline{K}}} \mathrm{Sh}_K$ also admits a* Cr*-ICM over $\mathscr{O}_{E,(v)}$.*

Proof. The map $\tilde{V} \overset{\text{defn}}{=} V \times_{\mathrm{Sh}_{\overline{K}}} \mathrm{Sh}_K \to V$ is finite étale. Let $\tilde{\mathcal{V}}$ be the normalization of $\mathcal{V}$ in $\tilde{V}$. We claim that $\tilde{\mathcal{V}}$ is the desired canonical model for $\tilde{V}$.

We begin by checking that $\tilde{\mathcal{V}} \to \mathcal{V}$ is finite étale and that the classifying map for $\mathbf{Et}_{K,p}|_{\tilde{V}}$ admits an extension $\tilde{\mathcal{V}} \to \mathrm{BT}_\infty^{\mathcal{G},-\mu_v,\mathrm{alg}}$ lifting the composition

$$\tilde{\mathcal{V}} \to \mathcal{V} \to \mathrm{BT}_\infty^{\overline{\mathcal{G}},\overline{\mu}_v^{-1},\mathrm{alg}}.$$

To do this, we apply the criteria from Proposition 4.5.8. We will see that the μ-ordinary locus is the dense open subspace $U \subset \mathcal{V} \otimes k(v)$ with the required properties. Talk about canonical lifts

At this point, we know that $\tilde{\mathcal{V}} \to \mathcal{V}$ is finite étale and that $\tilde{\mathcal{V}}$ is an apertile integral model for V and 2-torsion points.
satisfying the conditions of Lemma 6.3.7. It only remains to check that, for F as in the definition of integral canonical models, $\tilde{\mathcal{V}}(\mathscr{O}_F) \subset \tilde{V}(F)$ contains all potentially crystalline points, but this is immediate from the corresponding assertion for $\mathcal{V}$. □

gral_model **Construction 6.7.12.** Suppose that we are given a finite p-unramified extension $E^\heartsuit/E$, a place $v^\heartsuit|v$, and an open and closed subscheme

$$\mathrm{Sh}_K^\heartsuit \subset E^\heartsuit \otimes_E \mathrm{Sh}_K\,.$$

Given a normal integral model $\mathscr{S}_K$ for Sh_K over $\mathscr{O}_{E,(v)}$, we obtain a normal integral model $\mathscr{S}_K^\heartsuit$ for $\mathrm{Sh}_K^\heartsuit$ over $\mathscr{O}_{E^\heartsuit,(v^\heartsuit)}$ by taking the Zariski closure of $\mathrm{Sh}_K^\heartsuit$ in $\mathscr{O}_{E^\heartsuit,(v^\heartsuit)} \otimes_{\mathscr{O}_{E,(v)}} \mathscr{S}_K$. It is easy to see that $\mathscr{S}_K^\heartsuit$ is an ICM for $\mathrm{Sh}_K^\heartsuit$ if $\mathscr{S}_K$ is one for Sh_K.

_canonical **Lemma 6.7.13** (Canonical models via Hecke translates)**.** *If $\mathrm{Sh}_K^\heartsuit$ admits an ICM $\mathscr{S}_K^\heartsuit$ over $\mathscr{O}_{E^\heartsuit,v^\heartsuit}$, then Sh_K admits an ICM $\mathscr{S}_K$ over $\mathscr{O}_{E,(v)}$ such that $\mathscr{S}_K$ is carried to $\mathscr{S}_K^\heartsuit$ by Construction 6.7.12.*

Proof. We will argue as in [57, Lemma 2.5.16]. Let us first note that, for all $K' \subset K$ of the form $K_pK'^{,p}$, $\mathrm{Sh}_{K'}^\heartsuit \overset{\text{defn}}{=} \mathrm{Sh}_K^\heartsuit \times_{\mathrm{Sh}_K} \mathrm{Sh}_{K'}$ admits an ICM $\mathscr{S}_{K'}^\heartsuit$: For Ét-ICMs, this is immediate from the fact that the local systems $\mathbf{Et}_{K,\ell}|_{\mathrm{Sh}_K^\heartsuit}$ extend over $\mathscr{S}_K^\heartsuit$. For Cr-ICMs, this is a consequence of Proposition 6.7.11 applied to the maps $(G, \mathcal{G}, X, K') \to (G, \mathcal{G}, X, K)$ and $V = \mathrm{Sh}_K^\heartsuit \subset E_K^\heartsuit \otimes_E \mathrm{Sh}_K$.

For any $g \in G(\mathbb{A}_f^p)$, we have an isomorphism of Shimura varieties $\iota_{K',g} : \mathrm{Sh}_{K'} \xrightarrow{\simeq} \mathrm{Sh}_{gK'g^{-1}}$ given on $\mathbb{C}$-points by

$$[(x,h)] \mapsto [(x, hg^{-1}].$$

This isomorphism carries the restriction of $\mathbf{Et}_{K,p}$ to the source to that over the target.

Then we see that $\iota_{K',g}(\mathrm{Sh}_{K'}^\heartsuit) \subset E^\heartsuit \otimes_E \mathrm{Sh}_{gK'g^{-1}}$ also admits an ICM over $\mathscr{O}_{E^\heartsuit,v^\heartsuit}$, which is in fact isomorphic to $\mathscr{S}_{K'}^\heartsuit$. If $K'_g = K' \cap gK'g^{-1}$, this in turn implies that

$$\mathrm{Sh}_{K'_g}^\heartsuit \overset{\text{defn}}{=} \iota_{K',g}(\mathrm{Sh}_{K'}^\heartsuit) \times_{\mathrm{Sh}_{gK'g^{-1}}} \mathrm{Sh}_{K'_g}$$

admits an ICM model $\mathscr{S}_{K'_g}^\heartsuit$; see the argument from the second paragraph of the proof.

By Lemma 6.7.9, we find that the image of $\mathrm{Sh}_{K'_g}^\heartsuit$ in $E^\heartsuit \otimes_E \mathrm{Sh}_K$ admits an ICM over $\mathscr{O}_{E^\heartsuit,v^\heartsuit}$.

Now, using Remark 6.1.11, we find that there is a finite set of pairs $\Xi = \{(K',g)\}$ such that the map

$$\bigsqcup_{(K',g)\in\Xi} \mathrm{Sh}_{K'_g}^\heartsuit \to E^\heartsuit \otimes_E \mathrm{Sh}_K$$

is surjective.

Therefore, $E^{\heartsuit}\otimes_E \mathrm{Sh}_K$ is covered by open and closed subschemes, each of which admits an ICM over $\mathscr{O}_{E^{\heartsuit},v^\sharp}$. Using (1) of Remark 6.7.7 and Zariski descent, we see that $E^{\heartsuit}\otimes_E \mathrm{Sh}_K$ admits an ICM over $\mathscr{O}_{E^{\heartsuit},v^\sharp}$. We now conclude using Lemma 6.7.8. $\square$

ral_covers **Proposition 6.7.14.** *Assertion (1) of Theorem 6.7.2 holds.*

Proof. The map $\mathrm{Sh}_K\to E\otimes_{\overline{E}}\mathrm{Sh}_{\overline{K}}$ is a finite Galois cover over its image $\mathrm{Sh}^{\heartsuit}_{\overline{K}}\subset E\otimes_{\overline{E}}\mathrm{Sh}_{\overline{K}}$. Therefore, Lemma 6.7.9 tells us that $\mathrm{Sh}^{\heartsuit}_{\overline{K}}$ admits an ICM over $\mathscr{O}_{E,(v)}$, and Lemma 6.7.13 tells us that Sh_K admits an ICM over $\mathscr{O}_{\overline{E},(\overline{v})}$. $\square$

Proof of Theorem 6.7.1. By [44, Lemma (2.1.2)], we can find compact open subgroups $K_1^p\subset K^p$ and $K_1^{\sharp,p}\subset K^{\sharp,p}$ such that, with $K_1=K_1^pK_p$ and $K_1^\sharp=K_1^{\sharp,p}K_p^\sharp$, we have $K_1=K_1^\sharp\cap G(\mathbb{A}_f^p)$ and the map $\mathrm{Sh}_{K_1}\to\mathrm{Sh}_{K_1^\sharp}$ is a closed immersion. By Propositions 6.7.14 and 6.7.5, it is enough to know that $\mathrm{Sh}_{K_1^\sharp}$ admits a Cr-ICM (resp. weak integral canonical model) whenever $\mathrm{Sh}_{K^\sharp}$ does so.

If $\mathrm{Sh}_{K^\sharp}$ admits an Ét-ICM over $\mathscr{O}_{E^\sharp,(v^\sharp)}$, then one can see this using Remark 6.4.2.

If $\mathrm{Sh}_{K^\sharp}$ admits a Cr-ICM, then one appeals to Proposition 6.7.11 instead. $\square$

Proof of Theorem 6.7.2. Assertion (1) has already been shown in Proposition 6.7.14.

Let us move on to (2): As we already observed in the proof of *loc. cit.*, the natural map $\mathcal{G}\to\overline{\mathcal{G}}\times\mathcal{G}^{\mathrm{ab}}$ has finite kernel, and our hypothesis on derived subgroups tells us that it is in fact a closed immersion. Therefore, we conclude using Proposition 6.5.5, our hypothesis, and Theorem 6.7.1.

Assertion (3) lies a bit deeper. Here, by replacing $\mathcal{G}$ with the image of $\mathcal{G}$ in $\overline{\mathcal{G}}\times\mathcal{G}^{\mathrm{ab}}$, and using the argument from the previous paragraph, we can reduce to the case where $\mathcal{G}\to\overline{\mathcal{G}}$ has finite kernel, where everything follows from Proposition 6.7.11. $\square$

elian_type

6.8. Shimura varieties of pre-abelian type.

Definition 6.8.1. A Shimura datum (G,X) is of **Hodge type** if G admits a symplectic representation H on which some (hence any) element of X induces a Hodge structure with weights $(-1,0),(0,-1)$. It is of **abelian type** if there exists a Shimura datum (G_1,X_1) of Hodge type and a map $G_1^{\mathrm{der}}\to G_2^{\mathrm{der}}$ of derived groups inducing an isomorphism on adjoint quotients. It is of **pre-abelian type** if the adjoint Shimura datum $(G^{\mathrm{ad}},X^{\mathrm{ad}})$ is of abelian type.

vs_abelian **Remark 6.8.2** (Pre-abelian vs. abelian vs. Hodge)**.** .

Add some remarks here about the differences

hm:abelian **Theorem 6.8.3** (Canonical models of pre-abelian type)**.** *Suppose that (G,X) is of pre-abelian type. Then $(G,\mathcal{G},X)$ admits a* Cr*-ICM over $\mathscr{O}_{E,(v)}$, and this is also an* Ét*-ICM.*

Proof. Given Theorem 6.7.2, we can assume without loss of generality that $(G,\mathcal{G},X)$ is of Hodge type, meaning that (G,X) admits a closed immersion into a Siegel Shimura datum $(G^\sharp,X^\sharp)$. Using Zarhin's trick, we can assume that the latter can be extended to an unramified tuple $(G^\sharp,\mathcal{G}^\sharp,X^\sharp)$: Concretely, this means that we can choose a symplectic representation H of G such that $H_{\mathbb{Q}_p}$ admits a self-dual lattice $H_{\mathbb{Z}_p}$ stabilized by $\mathcal{G}$, and such that $-\mu_v$ acts on H with weights $0,1$.

Now, for a suitable choice of level $K^{\sharp,p}\subset G^\sharp(\mathbb{A}_f^p)$, $\mathrm{Sh}_{K^\sharp}$ admits an integral model $\mathscr{S}_{K^\sharp}$ over $\mathbb{Z}_{(p)}$ given by an open and closed subspace of the moduli of abelian varieties equipped with a prime-to-p polarization and some prime-to-p level structure. By Theorem 6.7.1, it is enough to know that $\mathscr{S}_{K^\sharp}$ is both a Cr-ICM and an Ét-ICM. Therefore, we can assume that $(G,\mathcal{G},X)$ is itself of Siegel type.

Here, by Theorem 2.4.2, we have an F-gauge $\mathfrak{H}$ over $\mathscr{S}_K$ associated with the p-divisible group $\mathcal{A}[p^\infty]$ of the universal abelian scheme $\mathcal{A}\to\mathscr{S}_K$. We can actually view this as a (GL_{2g},μ_g)-aperture over $\mathscr{S}_K$, where g is the relative dimension of $\mathcal{A}$, and μ_g is the cocharacter of GL_{2g} splitting a subspace in middle dimension. Now, Proposition 5.3.15 applies with

$$\mathcal{X}=\mathcal{X}^\sharp=\mathscr{S}_K;\ (\mathcal{G}^\sharp,\mu^\sharp)=(\mathrm{GL}_{2g},\mu_g).$$

The pointwise condition is immediate from Remark 6.3.4 and the fact that the map $\widehat{\mathscr{S}}_{K,v}\to\mathrm{BT}_\infty^{\mathrm{GL}_{2g},\mu_g}$ is formally unramified follows from classical Serre-Tate theory and Theorem 2.4.2 once again. $\square$

properties

6.9. **Mapping properties for smooth inputs.** In this subsection, we will assume that $p > 3$. This will allow us to give more concrete mapping properties for (weak) canonical models in terms of pointwise criteria.

_canonical

Proposition 6.9.1 (A pointwise mapping property for smooth inputs)**.** *Suppose that $\mathcal{X}$ is a smooth separated $\mathscr{O}_{E,(v)}$-scheme with generic fiber X, and suppose that we have a map $f : X \to \mathrm{Sh}_K$.*

(1) If $\mathscr{S}_K$ is a Cr*-ICM for* Sh_K *over $\mathscr{O}_{E,(v)}$, f extends to a map $\mathcal{X} \to \mathscr{S}_K$ if and only if, for all finite extensions F/E_v and $x \in \mathcal{X}(F)$, $\mathbf{Et}_{K,p,f(x)}$ is crystalline.*

(2) If $\mathscr{S}_K$ is an Ét*-ICM, then f extends to a map $\mathcal{X} \to \mathscr{S}_K$ if and only if, for all finite extensions F/E_v and $x \in \mathcal{X}(F)$, $\mathbf{Et}_{K,p,f(x)}$ is crystalline, and moreover $\mathbf{Et}_{K,\ell,f(x)}$ is unramified for all $\ell \neq p$.*

Proof. Given Theorem 6.4.3, it is enough to know that $\mathbf{Et}_{K,p}|_X$ lifts to a $(\mathcal{G}, -\mu_v)$-aperture over $\mathcal{X}$. But this is immediate from Theorem 5.1.4 and Remark 6.3.4. ☐

r_neronian

Corollary 6.9.2 (Néronian property for proper models)**.** *The following statements are equivalent for an integral model $\mathscr{S}_K$ for* Sh_K*:*

(1) $\mathscr{S}_K$ is proper over $\mathscr{O}_{E,(v)}$ and is a Cr*-ICM for* Sh_K*.*

(2) $\mathscr{S}_K$ is proper over $\mathscr{O}_{E,(v)}$ and is an Ét*-ICM for* Sh_K*.*

If these equivalent conditiosn hold, then $\mathscr{S}_K$ is in fact a Nèron model for Sh_K*: In the context of Proposition 6.9.1, every map $X \to \mathrm{Sh}_K$ extends to a map $\mathcal{X} \to \mathscr{S}_K$.*

Proof. Suppose that Sh_K admits a proper integral model $\mathscr{S}_K$. Then, for any finite extension F/E_v, we have $\mathscr{S}_K(\mathscr{O}_F) = \mathrm{Sh}_K(F)$. Therefore, if $\mathscr{S}_K$ is apertile, then for all $x \in \mathrm{Sh}_K(F)$, $\mathbf{Et}_{K,p,x}$ is crystalline. If $\mathscr{S}_K$ is now an Ét-ICM, then it follows immediately that it is also a Cr-ICM. Conversely, if $\mathscr{S}_K$ is a Cr-ICM, then Corollary 6.7.4 tells us that, for all $x \in \mathrm{Sh}_K(F)$ and $\ell \neq p$, $\mathbf{Et}_{K,\ell,x}$ is also unramified. Therefore $\mathscr{S}_K$ is also an Ét-ICM.

That $\mathscr{S}_K$ is a Nèron model for Sh_K is now immediate from Proposition 6.9.1. ☐

Remark 6.9.3 (Proper models of pre-abelian type)**.** A necessary condition for properness is that $G/Z(G)$ be anisotropic. With this condition in hand, it follows from [64, Theorem 1] that, when (G, X) is of pre-abelian type, then the integral canonical models from Theorem 6.8.3 are proper and so are Nèron models of their generic fibers.

Diao-Yao stuff says enough to check at one point. Alex wanted to say something here.

Remark 6.9.4. Suppose one knew that $\mathbf{Et}_{K,p}$ is a *semistable* local system on Sh_K.

6.10. **Canonicity of the models of Bakker-Shankar-Tsimerman.** Here, we will find a proof of Theorem F. We can assume that the rank of G is at least 2, since otherwise the Shimura datum is of pre-abelian type and so is already addresed by Theorem 6.8.3. .

Proof of Theorem F. We will take a prime $p > 3$ large enough such that the following holds:

(1) (Log smooth compactification) First, we can assume that p is sufficiently large so that Sh_K admits a smooth model $\mathscr{S}_K$ over $\mathscr{O}_{E,(v)}$ with a log smooth compactification;

(2) (Versal Kodaira-Spencer map) Furthermore, by increasing the lower bound for p, we can also assume that the filtered G-bundle with integrable connection $\mathrm{Fil}^\bullet_{\mathrm{Hdg}}\,\mathbf{dR}_K$ extends to a filtered $\mathcal{G}_{(p)}$-bundle with integrable connection on $\mathscr{S}_K$, which we denote by the same symbol. We can also assume that this satisfies Griffiths' transversality and that the associated Kodaira-Spencer map
$$\mathbb{T}_{\mathscr{S}_K/\mathscr{O}_{E,(v)}} \to \mathbf{dR}_K(\mathfrak{g})/\,\mathrm{Fil}^0_{\mathrm{Hdg}}\,\mathbf{dR}_K(\mathfrak{g})$$
is an isomorphism.

(3) (Existence of ℓ-adic local systems) By work of Klevdal-Patrikis [49, Theorem 1.3] and Patrikis [72, Corollary 3.4], we can assume that, for all $\ell \neq p$, $\mathbf{Et}_{K,\ell}$ extends to an ℓ-adic local system on $\mathscr{S}_K$, which again we denote by the same symbol.
(4) (Pointwise crystallinity) By [73, Theorem 7.1], we can also assume that, for some choice of faithful representation Λ of $\mathcal{G}$, the filtered de Rham realization $\mathrm{Fil}^\bullet_{\mathrm{Hdg}}\,\mathbf{dR}_K(\Lambda)$ lifts to a Fontaine-Laffaille module (up to suitable twist) over $\mathscr{S}_K$. This implies in particular that $\mathbf{Et}_{K,p}$ is crystalline at every classical point of $\widehat{\mathscr{S}}_{K,\eta}$

At this point, we are almost there. Theorem 5.1.4, combined with Remark 6.3.4 and Lemma 6.3.7, tells us that we have an extension $\mathscr{S}_K \to \mathrm{BT}_\infty^{\mathcal{G}^c,-\mu_v,\mathrm{alg}}$ whose v-adic completion is formally étale.

Let us now check that $\mathscr{S}_K$ satisfies condition (1) for being an Ét-ICM. That is, we must know that, if $x \in \mathrm{Sh}_K(F)$ is such that $\mathbf{Et}_{K,\ell,x}$ is potentially unramified for all $\ell \neq p$, then $x \in \mathscr{S}_K(\mathscr{O}_F)$. This follows from the argument in [73, Lemma 8.4].

Finally, suppose that $G/Z(G)$ is anisotropic. In this case, Sh_K is a proper variety over E, and, since $\mathscr{S}_K$ admits a log smooth compactification, it follows that $\mathscr{S}_K$ is also proper over $\mathscr{O}_{E,(v)}$. We now conclude using Corollary 6.9.2. □

rjectivity

6.11. **Surjectivity.** The following result of Vasiu [86, Main Theorem A], which he terms the **crystalline boundedness principle**, will be used in the proof of Theorem I.

oundedness

Theorem 6.11.1 (Vasiu). *Let the notation be as in Setup 3.1.1. Then there exists $m(\mathcal{G},\mu) \geq 1$ such that, for any algebraically closed field κ over $\mathcal{O}$, and any $n \geq m(\mathcal{G},\mu)$, the map of groupoids*

$$\mathrm{BT}_\infty^{\mathcal{G},\mu}(\kappa) \to \mathrm{BT}_n^{\mathcal{G},\mu}(\kappa)$$

induces a bijection on isomorphism classes.

Proof. This is just a reformulation of the cited result of Vasiu, using the quotient descriptions of $\mathrm{BT}_n^{\mathcal{G},\mu}(\kappa)$ and $\mathrm{BT}_\infty^{\mathcal{G},\mu}(\kappa)$ given to us by Remark 3.2.19.

See also [89, Theorem 6.31] for the equal characteristic version: That argument can be combined with work of Zhu on the Witt vector affine Grassmannian [93] to give an alternate proof of the theorem. □

Proof of Theorem I. Note that, for every n, the classifying map $\widehat{\mathscr{S}}_{K,v} \to \mathrm{BT}_n^{\mathcal{G}^c,-\mu_v}$ is a formally smooth map of p-completely smooth stacks over $\mathcal{O}$, and is therefore p-completely smooth.

By smoothness, the map $|\widehat{\mathscr{S}}_{K,v}| \to |\mathrm{BT}_n^{\mathcal{G}^c,-\mu_v}|$ of the underlying topological spaces is open, and its surjectivity amounts to saying that it is surjective on connected components and that its image is closed under specialization. The first is easy, since $\mathrm{BT}_n^{\mathcal{G}^c,-\mu_v}$ is connected.[24]

For the surjectivity, observe the following consequence of Theorem 6.11.1: $|\mathrm{BT}_\infty^{\mathcal{G}^c,-\mu_v}| \to |\mathrm{BT}_n^{\mathcal{G}^c,-\mu_v}|$ is a homeomorphism for n sufficiently large. The proof is now completed by the following relative completeness result:

Lemma 6.11.2. *Suppose that C is a smooth curve over a finite extension $\kappa/k(v)$ equipped with a κ-valued point $c \in C(\kappa)$, and suppose that we have a map $f : C\backslash\{c\} \to \mathscr{S}_K \otimes k(v)$ such that the composition with $\mathscr{S}_K \otimes k(v) \to \mathrm{BT}_\infty^{\mathcal{G},-\mu_v} \otimes k(v)$ extends to a map $h : C \to \mathrm{BT}_\infty^{\mathcal{G},-\mu_v} \otimes k(v)$. Then f in fact extends to a map $C \to \mathscr{S}_K \otimes k(v)$.*

Proof. We can assume that $C = \operatorname{Spec} R$ is affine and geometrically connected.

Suppose first that we have a smooth compactification $\overline{\mathscr{S}}_K$ such that f extends to a map $\overline{f} : C \to \overline{\mathscr{S}}_K$. Choose a p-completely smooth $W(\kappa)$-algebra $\tilde{R}$ lifting R. Then, by the smoothness of $\overline{\mathscr{S}}_K$, we can lift $\overline{f}$ to a map of formal schemes $\tilde{\overline{f}} : \operatorname{Spf} \tilde{R} \to \overline{\mathscr{S}}^\wedge_{K,v}$. By shrinking C if necessary we can assume that it factors through an affine open, and hence we can even algebraize $\tilde{\overline{f}}$ to a map $\operatorname{Spec} \tilde{R} \to \overline{\mathscr{S}}_K$, which we will refer to by the same symbol.

[24]We expect that the syntomic realization exists and the surjectivity holds even if one works with possibly disconnected reductive groups, though technically speaking such groups cannot appear as part of a Shimura datum as defined in [19].

It is of course enough to show that $\tilde{\overline{f}}$ factors through $\mathscr{S}_K$. In fact, it is enough to show that every $W(\kappa)$-valued point of $\operatorname{Spf}\tilde{R}$ factors through $\mathscr{S}_K$, and in turn this amounts to knowing that every $W(\kappa)[1/p]$-valued point of the adic fiber of $\operatorname{Spf}\tilde{R}$ maps to the crystalline locus of $\mathrm{Sh}_K^{\mathrm{an}}$.

For this, note that we can also lift h to a map of formal stacks $\tilde{h} : \operatorname{Spf}\tilde{R} \to \mathrm{BT}_\infty^{\mathcal{G},-\mu_v}$. Clearly, the associated $\mathcal{G}(\mathbb{Z}_p)$-local system over $\operatorname{Spec}\tilde{R}[1/p]$ is now crystalline at every classical point. However, this is the *same* as the one associated with the map $\tilde{\overline{f}}$ above: Indeed, it suffices to check that they agree after completion at any maximal ideal, and we can choose one corresponding to a point specializing to a point in $C\backslash\{c\}$.

Now, let us consider the general case: If the map f extends to a finite étale cover of $\mathscr{S}_K \otimes k(v)$, then it in fact descends to a map to $\mathscr{S}_K \otimes k(v)$. Therefore, we can assume that we have a map of tuples $(G, \mathcal{G}, X, K) \to (G^\sharp, \mathcal{G}^\sharp, X^\sharp, K^\sharp)$ with $\mathcal{G} \to \mathcal{G}^\sharp$ a closed immersion, and with $\mathrm{Sh}_{K^\sharp}$ admitting a canonical model $\mathscr{S}_{K^\sharp}$ equipped with a log smooth compactification. In this case, the argument above shows that we have an extension $C \to \mathscr{S}_{K^\sharp} \otimes k(v)$, and this must necessarily lift to the desired extension to $\mathscr{S}_K \otimes k(v)$. ☐

☐

-emptiness

6.12. Non-emptiness of strata.

Construction 6.12.1 (Newton strata)**.** By Construction 3.3.1, for any perfect $k(v)$-algebra R, we obtain canonical maps

$$\mathscr{S}_K(R) \to \mathrm{BT}_\infty^{\mathcal{G},\mu}(R) \to \mathrm{Isoc}_G(R).$$

For any class $[b] \in B(G)$, this gives a canonical topological subspace $\mathscr{S}_{K,k(v),[b]} \subset \mathscr{S}_{K,k(v)}$ defined by the property that, for any perfect field κ, the κ-points of $\mathscr{S}_K$ contained in this subset are the ones

The next result proves Theorem J

_emptiness

Theorem 6.12.2. *The Newton stratum $\mathscr{S}_{K,k(v),[b]}$ is non-empty if and only if $[b] \in B(G^c_{\mathbb{Q}_p}, -\mu_v)$.*

Proof. If the model satisfies Assumption 1.3.3, then, by Theorem I, it suffices to verify that there exists $\mathcal{Q} \in \mathrm{BT}_\infty^{\mathcal{G}^c_p,-\mu_v}(\overline{\mathbb{F}}_p)$ with $[b_{\mathcal{Q}}] = [b]$. This follows from Lemma 3.3.3.

Otherwise, let's observe that, by Lemma 6.5.8, every special point $x : \operatorname{Spec} F \to \mathrm{Sh}_K$ extends to a map $\operatorname{Spec}\mathscr{O}_{F,(w)} \to \mathscr{S}_K$ for some place $w|v$ of F. Now, the argument in [46, Proposition 1.3.10] applies and gives the desired conclusion. ☐

onst:fzips

Construction 6.12.3. By

-emptiness

Proposition 6.12.4. *Suppose that $\mathscr{S}_K$ satisfies Assumption 1.3.3. Then every Ekedahl-Oort stratum of $\mathscr{S}_{K,k(v)}$ is non-empty.*

Proof. Given the surjectivity of the syntomic realization, this comes down to the observation that the map

$$\mathrm{BT}_1^{\mathcal{G}^c,-\mu_v} \otimes k(v) \to \mathcal{G}^c\text{-zip}_{-\mu_v}$$

is surjective, which follows from the fact that it is in fact a gerbe banded by a finite flat group scheme; see the proof of [24, Theorem 9.3.2]. ☐

cohomology

6.13. Applications to cohomology.

ura_stacks

6.14. Integral canonical models for Shimura stacks.

f_integers

6.15. Integral canonical models over global rings of integers.

eriod_maps

6.16. Period maps.

ric_levels

6.17. **Integral canonical models at parahoric level.** Here we explain that any unramified Shimura datum $(G, X, \mathcal{G})$ which admits an integral canonical in the sense of Definition 6.4.1 automatically admits a Pappas–Rapoport integral canonical model as studied in [71]. We then apply results of Takaya eriod_maps from [83] to obtain Pappas–Rapoport integral canonical models for subhyperspecial (e.g., Iwahori) level. Finally, we give an analogue of the mapping property from Theorem 6.4.3 in this parahoric situation.

Notation 6.17.1. In the following we denote by $\mathbf{Perf}_A$, for a p-adically complete ring A, the category of characteristic p perfectoid spaces S equipped with a map $S \to \mathrm{Spd}(A)$. For any other undefined piece of notation or terminology, we direct the reader to the comprehensive discussion in [71, §2-3].

Definition 6.17.2 (Tame local Shimura datum)**.** A **tame local Shimura datum** is a triple $(\mathcal{G}, b, \{\mu\})$ where $\mathcal{G}$ is a parahoric group $\mathbb{Z}_p$-scheme with generic fiber G, $\{\mu\}$ is a conjugacy class of minuscule cocharacters of $G_{\overline{\mathbb{Q}}_p}$, and b is an element of $G(\breve{\mathbb{Q}}_p)$ inducing an element of $B(G, \mu^{-1})$.

Remark 6.17.3 (Parahoricity assumption)**.** We have our restricted our attention to the tame situation, i.e., where $\mathcal{G}$ is a parahoric group scheme, because this is all that we will presently need. That said, most of the following setup readily extends to arbitrary smooth affine group $\mathbb{Z}_p$-schemes.

ra-variety **Definition 6.17.4** (Local Shimura variety)**.** Given a parahoric local Shimura datum $(\mathcal{G}, b, \{\mu\})$ with reflex field F, we obtain a presheaf

$$\mathcal{M}^{\mathrm{int}}_{\mathcal{G},b,\{\mu\}} \colon \mathbf{Perf}_{\mathscr{O}_{\breve{F}}} \to \mathbf{Set},$$

assigning to $S \to \mathrm{Spd}(\mathscr{O}_{\breve{F}})$ the set of isomorphism classes of tuples $(S^\sharp, \mathscr{P}, \phi_{\mathscr{P}}, i_r)$, where:

- $S^\sharp$ is the untilt of S over $\mathscr{O}_{\breve{E}}$ associated with $S \to \mathrm{Spd}(\mathscr{O}_{\breve{F}})$,
- $(\mathscr{P}, \phi_{\mathscr{P}})$ is a $\mathcal{G}$-shtuka on S with one leg along $S^\sharp$ bounded by $\{\mu\}$ (see [71, Definition 2.4.3]),

By [80, §25.1], the presheaf $\mathcal{M}^{\mathrm{int}}_{\mathcal{G},b,\{\mu\}}$ is a small v-sheaf (in the sense of [77, Definition 12.1]), which is called the **integral local Shimura variety** associated to the tame local Shimura datum $(\mathcal{G}, b, \{\mu\})$.

Definition 6.17.5 (Tame Shimura datum/tuple)**.** A **tame Shimura datum** is a triple $(G, X, \mathcal{G})$ where (G, X) is a Shimura datum and $\mathcal{G}$ is a parahoric model of $G_{\mathbb{Q}_p}$. A **tame Shimura tuple** is a quadruple $(G, X, \mathcal{G}, K)$ where $(G, X, \mathcal{G})$ is a tame Shimura datum and $K \subseteq G(\mathbb{A}_f)$ is a compact open subgroup with $K = K_p K^p$ and where $K_p = \mathcal{G}(\mathbb{Z}_p)$. Se say $(G, X, \mathcal{G}, K)$ is **neat** if K^p is.

Remark 6.17.6. Fix a tame Shimura tuple $(G, X, \mathcal{G}, K)$ with reflex field E and a place v of E of lying over p. In the following we shall abusively write Sh_K for $\mathrm{Sh}_K(G, X)_{E_v}$. We will write $\{\mu_v\}$ for the conjugacy class of $G_{\overline{E}_v}$ determined by the Hodge cocharacter of (G, X).

As in Remark 6.1.9 for one may build proétale $\mathcal{G}^c(\mathbb{Z}_p)$-torsor $\mathbf{Et}_{K,p}$ on Sh_K (see [15, §4.2]). Here $\mathcal{G}^c$ is the parahoric model of G^c induced by the parahoric model $\mathcal{G}$ of G (e.g., see [18, §4.1]). As in the reductive case, we treat $\{\mu_v\}$ also as a conjugacy class of cocharacters of $G^c_{\overline{E}_v}$.

ric-shtuka **Definition 6.17.7** (Generic shtuka associated to a tame Shimura datum)**.** We denote by $\mathscr{P}_{K,E_v}$ the $\mathcal{G}^c$-shtuka bounded by $\{-\mu_v\}$ on $\mathrm{Sh}_K^\lozenge \to \mathrm{Spd}(E)$ defined by $U_{\mathrm{sht}}(\mathbf{Et}_{K,p})$ with notation as in [36, §3.2.2].

Construction 6.17.8 (Formal neighborhoods of local Shimura varieties associated to $\overline{k}(v)$-points of integral model of Sh_K)**.** Let $\mathscr{S}_K$ be flat normal $\mathscr{O}_{E_v}$-model of Sh_K equipped with an extension $\mathscr{P}_K$ to $\mathscr{S}_K^{\lozenge/}$ (see [71, Defintion 2.1.9]) of the $\mathcal{G}^c$-shtuka $\mathscr{P}_{K,E_v}$.[25] For any point x of $\mathscr{S}_K(\overline{k}(v))$, the pullback $x^*\mathscr{P}_K$ defines a $\mathcal{G}^c$-shtuka over $\mathrm{Spd}(\overline{k}(v))$ which by [71, Example 2.4.9] naturally determines an object $(\mathcal{P}_x, \phi_x)$ of $\mathrm{BT}_\infty^{\mathcal{G}^c,\mu_v^{-1}}(\overline{k}(v))$ and by Construction 3.3.1 an element b_x in $G(\breve{\mathbb{Q}}_p)$ inducing an element in the Kottwitz set $B(G^c, \mu_v^{-1})$.

Then the triple $(\mathcal{G}^c, b_x, \{\mu_v^{-1}\})$ defines a parahoric local Shimura datum, and we can consider the integral local Shimura variety $\mathcal{M}^{\mathrm{int}}_{\mathcal{G}^c,b_x,\{\mu_v\}}$. Moreover, the pair $(\mathcal{P}_x, \mathrm{id})$ determines a point x_0 of the

[25]Note that $\mathscr{P}_K$ is necessarily bounded by $\{\mu\}$ by [15, Lemma 2.1].

affine Deligne–Lusztig set $X_{\mathcal{G}^c}(b_x, \mu_v^{-1})(\bar{k}(v))$ (e.g., as in [71, Definition 3.3.1]), and so a point of $\mathcal{M}^{\mathrm{int}}_{\mathcal{G}^c, b_x, \{\mu_v\}}(\bar{k}(v))$ via the identification given in [26, Proposition 2.61.(1)]. One may then consider the formal neighborhood $(\mathcal{M}^{\mathrm{int}}_{\mathcal{G}^c, b_x, \{\mu_v\}})^{\wedge}_{/x_0}$ as in [27, Definition 4.18]

Definition 6.17.9 (Pappas–Rapoport integral models)**.** A system $\{\mathscr{S}_K\}$ of normal flat $\mathscr{O}_{E_v}$-models of Sh_K, as one ranges over all neat tame Shimura tuples $(G, X, \mathcal{G}, K)$, is a **Pappas–Rapoport integral canonical model** for $(G, X, \mathcal{G})$ if it satisfies conditions (i), (ii), (iii) and (iv) of [18, Definition 4.3]. We say a Pappas–Rapoport integral canonical model is **strong** if, for any K, and for any complete discrete valuation field F with mixed characteristic $(0, p)$ with perfect residue field, the following are equivalent for $x \in \mathrm{Sh}_K(F)$:

1. $x \in \mathscr{S}_K(\mathscr{O}_F)$;
2. $\mathbf{Et}_{K,p,x}$ is crystalline;
3. $\mathbf{Et}_{K,p,x}$ is potentially crystalline.

re-PR ICMS **Proposition 6.17.10.** *Suppose that $(G, X, \mathcal{G})$ is an unramified Shimura datum admitting an* Ét*-ICM $\mathscr{S}_K$ for each (equiv. any) neat unramified Shimura tuple $(G, X, \mathcal{G}, K)$. Then:*

(1) The collection $\{\mathscr{S}_K\}$ is a Pappas–Rapoport integral canonical model for $(G, X, \mathcal{G})$.
(2) If $\mathscr{S}_K$ is a Cr*-ICM, then the Pappas–Rapoport integral model $\{\mathscr{S}_K\}$ is strong.*

Proof. For (1), we check the conditions from [18, Definition 4.3]

For (i), we have to show that, given a mixed characteristic discrete valuation field F, and an $x\colon \mathrm{Spec}(F) \to \varprojlim_K \mathrm{Sh}_K$, the projection of x_K to Sh_K lifts to an $\mathscr{O}_F$-point of $\mathscr{S}_K$. But the existence of the lift x shows that, for all $\ell \neq p$, the local system $\mathbf{Et}_{K,\ell,x_K}$ is unramified (in fact, trivial). Therefore, the desired assertion follows from the definition of an Ét-ICM.

Condition (ii) follows from Theorem 6.7.2. For condition (iii), we observe that by definition of being an Ét-ICM, each $\mathscr{S}_K$ carries a prismatic F-crystal in the sense of [36, Definition 3.18] with de Rham local system $\mathbf{Et}_{K,p}$. Thus, applying the shtuka realization functor from [36, Construction 3.19] to this prismatic F-crystal yields a shtuka $\mathscr{P}_K$ on $\mathscr{S}_K^{\Diamond/}$. That $\mathscr{P}_K$ models $\mathscr{P}_{K,E}$ is clear as $(\mathscr{P}_K)_E$ is (by definition) $U_{\mathrm{sht}}(\mathbf{Et}_{K,p}) = \mathscr{P}_{K,E}$. Finally, to check (iv), we may use the formal étaleness of ϖ (as in the definition of an Ét-ICM) and [38, Proposition 3.32] (see also [24, §10.2]) to reduce ourselves to constructing an isomorphism

$$\Theta_x\colon \left(\mathcal{M}^{\mathrm{int}}_{\mathcal{G}^c, b_x, \{\mu_v\}}\right)^{\wedge}_{/x_0} \xrightarrow{\sim} \mathrm{Spf}(R_{\mathcal{G},\{\mu_v\}}),$$

where the target using notation as in loc. cit., which pulls back the shtuka realization of the universal aperture on the target to the universal $\mathcal{G}^c$-shtuka on the source. But, this is precisely the content of [40, Theorem 5.3.5].

(2) is now immediate from the definition of a Cr-ICM; see also Corollary 6.7.4. □

Definition 6.17.11 (Subhyperspecial parahoric)**.** Let $F/\mathbb{Q}_p$ be a finite extension and let H be a reductive group over F. A subgroup of $H(F)$ is called **subhyperspecial parahoric** if it is a parahoric subgroup in the sense of [14] (see also [41, Definition 4.1.4]) which is contained in $\mathcal{H}(\mathscr{O}_F)$ for some reductive $\mathscr{O}_F$-model $\mathcal{H}$ of H.

Remark 6.17.12 (Explicit description of subhyperspecial parahorics)**.** For a reductive model $\mathcal{H}$ of H over $\mathscr{O}_F$, the subhyperspecial parahorics contained in $\mathcal{H}(\mathscr{O}_F)$ are precisely those of the form $\mathrm{red}^{-1}(P(k))$ where k is the residue field of F, $P \subseteq \mathcal{H}_k$ is a parabolic subgroup, and

$$\mathrm{red}\colon \mathcal{H}(\mathscr{O}_F) \to \mathcal{H}(k)$$

is the reduction map. Equivalently, the subhyperspecial parahoric subgroups of $H(K)$ are $\mathcal{H}'(\mathscr{O}_K)$ for a parahoric group $\mathscr{O}_K$-scheme $\mathcal{H}'$ (see [18, Definition 2.6]) obtained as the dilatation (see [41, §A.5]) of a reductive $\mathscr{O}_F$-model $\mathcal{H}$ along a parabolic subgroup $P \subseteq \mathcal{H}_k$. We call these $\mathscr{O}_K$-models $\mathcal{H}'$ of H **subhyperspecial parahoric group schemes minorizing** $\mathcal{H}$.

Example 6.17.13 (Iwahoris are subhyperspecial)**.** Considering $\mathrm{red}^{-1}(B(k))$ for $B \subseteq \mathcal{H}_k$ a Borel subgroup we obtain the (or more precisely *a*) Iwahori subgroup of $H(F)$ as in [41, Defintion 4.1.3].

minorizing **Proposition 6.17.14.** *Suppose that $(G, X, \mathcal{G})$ is an unramified Shimura datum satisfying **(SV5)** and admitting* **Ét***-ICMs $\{\mathscr{S}_K\}_K$. Then, for any subhyperspecial parahoric group $\mathscr{O}_K$-scheme $\mathcal{G}'$ minorizing $\mathcal{G}$ the tame Shimura datum $(G, X, \mathcal{G}')$ admits a Pappas–Rapoport integral canonical model at every level K.*

Proof. Immediate from Proposition 6.17.10 and [83, Theorem 6.20]. the only thing to prove is that each Pappas–Rapoport integral canonical model $\mathscr{S}_K$ is strong. But, □

aya_strong **Remark 6.17.15.** If $\mathscr{S}_K$ is in fact a Cr-ICM, then the Pappas–Rapoport model for $(G, X, \mathcal{G}')$ is actually strong. Indeed, if K^p is a neat level away from p, $K = K_pK^p$ and $K' = K'_pK^p$, then the natural map $\mathscr{S}_{K'} \to \mathscr{S}_K$ is proper by [83, Proposition 6.23]. Using this, we can lift the crystalline criterion for good reduction at hyperspecial level to one at subhyperspecial level.

nd-large-p **Corollary 6.17.16.** *Suppose that $(G, X, \mathcal{G}')$ is a tame parahoric Shimura datum with $\mathcal{G}'$ a subhyperspecial parahoric group scheme such that either:*

(1) (G, X) is of pre-abelian type,
(2) p is sufficiently large and G is anisotropic mod center.

Then, $(G, X, \mathcal{G}')$ admits a strong Pappas–Rapoport integral canonical model at every level.

Remark 6.17.17 (Agreement with Kisin–Pappas–Zhou)**.** When $(G, X, \mathcal{G}')$ is of abelian type and $p > 3$ the models constructed by Corollary 6.17.16 agree with the Kisin–Pappas–Zhou models constructed in [43] and [47]. Indeed, this follows from the unicity of Pappas–Rapoport integral canonical models (see [71, Theorem 4.3.1]) and the results of [18], building on previous work from [71] and [17].

Definition 6.17.18 (Moduli stack of $(\mathcal{G}, \mu)$-shtukas)**.** Let notation be as in Definition 6.17.4. We denote by $\mathrm{Sht}_{\mathcal{G},\{\mu\}}$ the v-stack on $\mathbf{Perf}_{\mathscr{O}_F}$ associating to $S \to \mathrm{Spd}(\mathscr{O}_F)$ the groupoid of $\mathcal{G}$-shtukas over S with leg along $S^\sharp$ bounded by $\{\mu\}$ (cf. loc. cit.).

The proof of the following result is along the same lines as that of Theorem 6.4.3. One has to replace the use of Theorem 4.1.5 with [71, Theorem 2.7.7].

g-property **Theorem 6.17.19.** *Let $(G, X, \mathcal{G}, K)$ be a neat tame Shimura tuple admitting a strong Pappas–Rapoport integral canonical model $\mathscr{S}_K$. Suppose that $\mathcal{X}$ is a flat, η-normal algebraic space of finite type over $\mathscr{O}_{E,(v)}$. Then a map $\mathcal{X}_\eta \to \mathrm{Sh}_K$ extends to a map $\mathcal{X} \to \mathscr{S}_K$ if and only if the composition $\mathcal{X}_\eta^\diamondsuit \to \mathrm{Sh}_K^\diamondsuit \to \mathrm{Sht}_{\mathcal{G},\{-\mu_v\}}$ lifts to a map[26] $\mathcal{X}^{\diamondsuit/} \to \mathrm{Sht}_{\mathcal{G},\{-\mu_v\}}$.*

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Keerthi Madapusi, Department of Mathematics, Maloney Hall, Boston College, Chestnut Hill, MA 02467, USA

Email address: `madapusi@bc.edu`

Alex Youcis, Department of Mathematics, Bahen Centre, University of Toronto, Toronto, ON, M5S 2E4, Canada

Email address: `alex.youcis@gmail.com`