MATH 3311, FALL 2025: LECTURE 13, SEPTEMBER 26

Video: `https://youtu.be/lCNG-fWyARg`

We now return to the study of homomorphisms in general.

Definition 1. Let $f : G \to G'$ be a homomorphism of groups.

The **kernel** of f is given by

$$\ker f = \{g \in G : \ f(g) = e\} \subset G.$$

The **image** of f is given by

$$\{f(g) : g \in G\} = f(G) \subset G'$$

Proposition 1. *Suppose that $f : G \to G'$ is a homomorphism of groups. Then:*

(1) $\ker f \trianglelefteq G$ is a normal subgroup of G.
(2) $\mathrm{im}\ f \leq G'$ is a subgroup of G'.
(3) f is injective if and only if $\ker f = \{e\}$.
(4) f is surjective if and only if $\mathrm{im}\ f = G'$.

Proof. Let us check (1). We need to verify the following properties:

(1) $e \in \ker f$: This is because $f(e) = e$.
(2) If $h_1, h_2 \in \ker f$, then $g_1 g_2 \in \ker f$: This is because

$$f(h_1 h_2) = f(h_1) f(h_2) = e \cdot e = e.$$

(3) If $h \in \ker f$, then $h^{-1} \in \ker f$: This is because

$$f(h^{-1}) = f(h)^{-1} = e^{-1} = e.$$

(4) (Normality) If $h \in \ker f$ and $g \in G$, then $ghg^{-1} \in \ker f$: This is because

$$f(ghg^{-1}) = f(g)f(h)f(g)^{-1} = f(g)ef(g)^{-1} = f(g)f(g)^{-1} = f(gg^{-1}) = f(e) = e.$$

As for (2), we have to verify:

(1) $e \in \mathrm{im}\ f$: This is because $e = f(e)$.
(2) If $g_1', g_2' \in \mathrm{im}\ f$, then $g_1' g_2' \in \mathrm{im}\ f$: This is because if $g_1' = f(g_1)$ and $g_2' = f(g_2)$, then

$$g_1' g_2' = f(g_1) f(g_2) = f(g_1 g_2).$$

(3) If $g' \in \mathrm{im}\ f$, then $(g')^{-1} \in \mathrm{im}\ f$: This is because if $g' = f(g)$, then

$$(g')^{-1} = f(g)^{-1} = f(g^{-1}).$$

(4) is more or less immediate from the definitions, since surjectivity amounts to the statement that $\mathrm{im}\ f = f(G) = G'$.

For (3), we will see two proofs. We begin with the direct one. First, if f is injective, then it is clear that $e \in G$ must be the only element satisfying $f(e) = e$. Thus, $\ker f = \{e\}$. Conversely, if $\ker f = \{e\}$, then we must see that f is injective: that is, we must show that $f(g_1) = f(g_2)$ means that $g_1 = g_2$. For this we note:

$$\begin{aligned} f(g_1) = f(g_2) &\Leftrightarrow e = f(g_1)^{-1} f(g_2) \\ &\Leftrightarrow e = f(g_1^{-1} g_2) \\ &\Leftrightarrow g_1^{-1} g_2 \in \ker f \\ &\Leftrightarrow e = g_1^{-1} g_2 \\ &\Leftrightarrow g_1 = g_2. \end{aligned}$$

In the second to last equality, we have used our hypothesis that $\ker f = \{e\}$.

□

For the more indirect proof of (3), we will first obtain a more general statement. For this, we need a reinterpretation of the kernel and the image in terms of group actions.

Observation 1. If $f : G \to G'$ is a homomorphism of groups, then the assignment

$$G \times G' \xrightarrow{(g,g') \mapsto f(g)g'} G'$$

gives an action $G \curvearrowright G'$.

Proof. This is immediate from the fact that f is a homomorphism:

$$g_1 \cdot (g_2 \cdot g') = f(g_1)(f(g_2)g') = (f(g_1)f(g_2))g' = f(g_1g_2)g' = (g_1g_2) \cdot g'.$$

Moreover, $e \cdot g' = f(e)g' = eg' = g'$. □

Observation 2. If $f : G \to G'$, then under the action from Observation 1, we have:

(1) $\ker f = G_e$ is the stabilizer of $e \in G'$.
(2) $\operatorname{im} f = \mathcal{O}(e) \subset G'$ is the orbit of $e \in G'$.

Proof. This is essentially immediate from the definitions. □

Now, recall the general statement of orbit-stabilizer:

Proposition 2 (”Orbit-stabilizer theorem”)**.** *Suppose that $G \curvearrowright X$ and $x \in X$. Then there is a* bijection

$$G/G_x \to \mathcal{O}(x),$$

which carries each left coset gG_x to $g \cdot x$.

Applying this with the group action $G \curvearrowright G'$ associated with a homomorphism $f : G \to G'$ and with $x = e$, we find:

Proposition 3. *If $f : G \to G'$ is a homomorphism of groups, there is a natural bijection*

$$G/\ker f \xrightarrow{\simeq} \operatorname{im} f$$

which carries each left coset $g(\ker f)$ to $g \cdot e = f(g)e = f(g)$.

Remark 1. This gives another (though logically equivalent) way of seeing that $\ker f = \{e\}$ implies that f is injective: Each coset $g(\ker f)$ in this case consists of the single element $\{g\}$.

Remark 2. Note that $\operatorname{im} f$ is a subgroup of G' and so in particular is a group. This means that the above natural bijection also equips the set of cosets $G/\ker f$ with the structure of a *group*. In fact, since $f(g_1)f(g_2) = f(g_1g_2)$, the group structure is determined by the formula

$$g_1(\ker f) \cdot g_2(\ker f) = g_1g_2(\ker f).$$

One can now ask: Is the above remark true for *any* subgroup $H \leq G$? Is there a group operation on G/H given by

$$g_1H \cdot g_2H = g_1g_2H.$$

The problem here is that there is not a single way of writing down a coset of H in G in terms of a representative, and the above formula is not evidently independent of this choice. In fact, it turns out this is actually a real problem.

Example 1. Take $G = S_3 = \{e, \sigma, \sigma^2, \tau, \sigma\tau, \sigma^2\tau\}$ and $H = \langle\tau\rangle = \{e, \tau\}$. Then we have three cosets:

$$H = \{e, \tau\} = \tau H \ ; \ \sigma H = \{\sigma, \sigma\tau\} = \sigma\tau H \ ; \ \sigma^2 H = \{\sigma^2, \sigma^2\tau\} = \sigma^2\tau H.$$

Consider the ‘product’

$$\sigma H \cdot \sigma^2 H = \sigma^3 H = eH = H.$$

If we instead wrote the first term as $\sigma\tau H$, then we would get

$$\sigma\tau H \cdot \sigma^2 H = \sigma\tau\sigma^2 H = \sigma^2\tau H = \sigma^2 H \neq H.$$

Here, we have used the formula $\tau\sigma^2 = \sigma\tau$, which is valid in $S_3 = D_6$.

Therefore, depending on *how* we write down the coset $\sigma H = \sigma\tau H$, we get *different* answers for the possible product. This shows that the operation is not well-defined.

The difference between this and the case of a kernel is that the latter is always normal, while $H = \langle \tau \rangle \leq S_3$ is *not*:

$$\sigma\tau\sigma^{-1} = \sigma\tau\sigma^2 = \sigma\sigma\tau = \sigma^2\tau \notin H.$$