MATH 3311, FALL 2025: LECTURE 10, SEPTEMBER 17

Video: `https://youtu.be/Yx36GOmRDOM`

Stabilizers and orbits

Definition 1. Suppose that we have a group action $G \curvearrowright X$. For $x \in X$, the **stabilizer** of x is the subset

$$G_x = \{g \in G : \ g \cdot x = x\} \subset G.$$

If $g \cdot x = x$ so that $g \in G_x$, we will say that g **stabilizes** or **fixes** x.

Observation 1. $G_x \leq G$ is a *subgroup*.

Proof. We need to know that $e \in G_x$, which is clear since $e \cdot x = x$. We also need to know that, if $g_1, g_2 \in G_x$, then $g_1 g_2 \in G_x$. This is because

$$(g_1 g_2) \cdot x = g_1 \cdot (g_2 \cdot x) = g_1 \cdot x = x,$$

where the last two equalities are using the fact that $g_1, g_2 \in G_x$. Finally, we need to know that $g \in G_x$ implies that $g^{-1} \in G_x$. This follows from

$$g^{-1} \cdot x = g^{-1} \cdot (g \cdot x) = (g^{-1} g) \cdot x = e \cdot x = x.$$

□

Complementary to a stabilizer is:

Definition 2. The **orbit** of x is the subset

$$\mathcal{O}(x) = \{g \cdot x : \ g \in G\} \subset X.$$

This is the set of all elements of X that are 'reachable' from x via the paths provided by G.

Orbits are actually equivalence classes for an equivalence relation on X imposed by the group action by G.

Observation 2. Consider the relation $\sim_G$ on X given by: $x \sim_G y$ if there exists $h \in G$ such that $h \cdot x = y$. Equivalently, $x \sim_G y$ if $y \in \mathcal{O}(x)$. Then $\sim_G$ is an equivalence relation, and the equivalence classes are exactly the orbits of elements of X.

Proof. We have to check the three conditions for being an equivalence relation:

- (Reflexivity) $x \sim_G x$ because $e \cdot x = x$.
- (Symmetry) If $x \sim_G y$ with $h \cdot x = y$, then $h^{-1} \cdot y = x$. Therefore, $y \sim_G x$.
- (Transitivity) If $x \sim_G y$ and $y \sim_G z$, then we have $h \cdot x = y$ and $h' \cdot y = z$ for some $h, h' \in G$. Therefore, we see

  $$(h'h) \cdot x = h' \cdot (h \cdot x) = h' \cdot y = z$$

  which means that $x \sim_G z$.

□

Remark 1. This gives us an alternative definition of an orbit that doesn't privilege any particular element inside of it: It is simply an equivalence class for the relation $\sim_G$ on X.

Remark 2. Note in particular, the following consequence: $y \in \mathcal{O}(x)$ if and only if $x \in \mathcal{O}(y)$. In this case, we have $\mathcal{O}(x) = \mathcal{O}(y)$. This is a particular case of the general fact that equivalence classes for an equivalence relation are either equal or disjoint.

This gives us:

Observation 3. If we have a group action $G \curvearrowright X$ then X is a *disjoint* union of the orbits for this action.

Example 1. Let S_3 act on the set $\{1, 2, 3, 4\}$ by moving $1, 2, 3$ around in the usual way and by fixing 4. This action has two orbits $\{1, 2, 3\}$ and $\{4\}$.

Proposition 1 (Orbit-Stabilizer I). *Suppose that G is a finite group acting on a set X. Then for $x \in X$, $\mathcal{O}(x)$ is finite, and we have*

$$|G| = |G_x| \cdot |\mathcal{O}(x)|.$$

Corollary 1. *In the situation of Proposition 1, $|\mathcal{O}(x)|$ divides $|G|$.*