Linear algebra

1-dim'l linear equation

$$5x = 3$$

$$\big\downarrow \cdot \tfrac{1}{5}$$

$$\Rightarrow \quad x = \frac{3}{5}$$

$$\left(5 \mid 3 \right)$$

$$\big\downarrow \cdot \tfrac{1}{5}$$

$$\left(1 \;\middle|\; \frac{3}{5} \right)$$

2-dim'l linear equations

$$5x + 2y = 3 \qquad (R1)$$

$$2x + y = 1 \qquad (R2)$$

$$R1 - 2 \times R2$$

$$x = 1$$

$$\left(\begin{array}{cc|c} 5 & 2 & 3 \\ 2 & 1 & 1 \end{array} \right)$$

Plug this into $R2$

$$y = -1$$

$$\big\downarrow R1 - 2 \times R2$$

$$\left(\begin{array}{cc|c} 1 & 0 & 1 \\ 2 & 1 & 1 \end{array} \right)$$

2 equations in 3 variables

$$2x + 3y + 5z = 5$$
$$x + y + 2z = 2$$

In this system, z can have <u>any</u> value.

<u>e.g.</u> if $z = 1$, then the system is

$$2x + 3y = 0$$
$$x + y = 0$$

$$\Rightarrow \boxed{x = 0,\ y = 0,\ z = 1}$$

<u>e.g.</u> If $z = 0$, then the system is

$$2x + 3y = 5$$
$$x + y = 2$$

Row reduction

Suppose we have a system of m equations in n variables $\underbrace{}_{x_1, x_2, \ldots, x_n}$

$$
\begin{aligned}
a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n &= b_1 \\
a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n &= b_2 \\
&\vdots \\
a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n &= b_m
\end{aligned}
$$

Here, $\boxed{a_{i,j} \in \mathbb{R}}$ $1 \le i \le m$, $1 \le j \le n$

Isolate coefficients

$$
\left(\begin{array}{cccc|c}
a_{1,1} & a_{1,2} & \cdots & a_{1,n} & b_1 \\
a_{2,1} & a_{2,2} & \cdots & a_{2,n} & b_2 \\
\vdots & & & \vdots & \vdots \\
a_{m,1} & a_{m,2} & \cdots & a_{m,n} & b_m
\end{array}\right)
$$

Row operations

(1) Multiply a row by a real number

(2) Add a multiple of a row to a different row

(3) Switch one row with another

Example

$$2x + 3y + 5z = 5$$
$$x + y + 2z = 2$$

$$\left(\begin{array}{ccc|c} 2 & 3 & 5 & 5 \\ 1 & 1 & 2 & 2 \end{array}\right)$$

$$\Big\downarrow R1 - 2 * R2$$

$$\left(\begin{array}{ccc|c} 0 & 1 & 1 & 1 \\ 1 & 1 & 2 & 2 \end{array}\right)$$

$$\Big\downarrow \; R1 \leftrightarrow R2$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 0 & 1 & 1 & 1 \end{array}\right)$$

$$\Big\downarrow \; R1 - R2$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{array}\right)$$

$$x + z = 1$$

$$y + z = 1$$

$$\iff \quad x = 1 - z, \quad y = 1 - z$$

z is arbitrary

Example

$$\mathbb{Z}/2\mathbb{Z} = \{0, 1\}$$

"$\mathbb{Z}$-mod-$2\mathbb{Z}$"

"zee-mod-two-zee"

$+$	0	1
0	0	1
1	1	0

$\cdot$	0	1
0	0	0
1	0	1

$$x + y + z = 1$$
$$y + z = 0$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right)$$

$\downarrow R1 - R2$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right)$$

$$\boxed{\begin{array}{l} x = 1 \\ y + z = 0 \end{array}}$$

z is <u>arbitrary</u>

$$\boxed{\begin{array}{l} x = 1,\ y = 0,\ z = 0 \\ \\ x = 1,\ y = 1,\ z = 1 \end{array}}$$

Example

$$\mathbb{Z}/3\mathbb{Z} = \{0, 1, 2\}$$

"zee-mod-three-zee"

$+$	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

$\cdot$	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

$2 \cdot 2 = 1$ ($\Leftrightarrow$ "$\frac{1}{2}$" $= 2$)

$\Rightarrow$ Dividing by 2 is possible in this arithmetic and it's just multiplying by 2!!