MATH 3311, FALL 2025: LECTURE 33, NOVEMBER 17

Video: `https://youtu.be/w7sdNVacKL0?si=SU-pDA11C5cCWG7-`
Recall the two following propositions from last time:

Proposition 1. *Suppose that G is an abelian group. Then the following are equivalent:*

(1) G is finitely generated.
(2) There exists $m \geq 1$ and a surjective *homomorphism $f : \mathbb{Z}^m \to G$.*
(3) There exists $m \geq 1$ and a subgroup $H \trianglelefteq \mathbb{Z}^m$[1] *such that we have an isomorphism*

$$\mathbb{Z}^m/H \xrightarrow{\simeq} G.$$

Proposition 2. *Let $H \leq \mathbb{Z}^m$ be a subgroup. Then $H \simeq \mathbb{Z}^n$ for some $n \leq m$.*

The composition $\mathbb{Z}^n \simeq H \leq \mathbb{Z}^m$ can be viewed as a(n injective) homomorphism from $\mathbb{Z}^n$ to $\mathbb{Z}^m$. So what these two propositions together tell us is that, to understand finitely generated abelian groups, we just have to understand homomorphisms $f : \mathbb{Z}^n \to \mathbb{Z}^m$, and more particularly the quotient $\mathbb{Z}^m/\mathrm{im}\, f$ for such homomorphisms.

Observation 1. Suppose that $n \leq m$ and that $d_1, \ldots, d_n \in \mathbb{Z}$ are integers. Consider the homomorphism $f : \mathbb{Z}^n \to \mathbb{Z}^m$ satisfying $f(\vec{\mathbf{e}}_i) = d_i$. Then we have

$$\mathbb{Z}/\mathrm{im}\, f \simeq \mathbb{Z}/d_1\mathbb{Z} \times \cdots \times \mathbb{Z}/d_n\mathbb{Z} \times \mathbb{Z}^{m-n}.$$

Proof. The point is that the image of f is exactly the subgroup

$$d_1\mathbb{Z} \times \cdots \times d_n\mathbb{Z} \times \{0\} \leq \underbrace{\mathbb{Z} \times \cdots \mathbb{Z}}_{n} \times \mathbb{Z}^{m-n},$$

and the quotient can now be computed factor by factor. □

Remark 1. We used the following fact above: If $H_1 \trianglelefteq G_1$ and $H_2 \trianglelefteq G_2$ are normal subgroups, then $H_1 \times H_2 \leq G_1 \times G_2$ is also normal and we have

$$(G_1 \times G_2)/(H_1 \times H_2) \xrightarrow{\simeq} G_1/H_1 \times G_2/H_2.$$

This pops out of applying the factoring triangle to the homomorphism $G_1 \times G_2 \xrightarrow{(g_1,g_2) \mapsto (g_1H_1, g_2H_2)} G_1/H_1 \times G_2/H_2$.

Observation 2. Suppose that we have a diagram

$$\begin{tikzcd} \mathbb{Z}^n \arrow[r, "f"] & \mathbb{Z}^m \arrow[d, "\psi", "\simeq"'] \\ \mathbb{Z}^n \arrow[u, "\varphi"', "\simeq"] \arrow[r, "\psi \circ f \circ \varphi"'] & \mathbb{Z}^m \end{tikzcd}$$

Then there is an isomorphism

$$\mathbb{Z}^m/\mathrm{im}\, f \xrightarrow{\simeq} \mathbb{Z}^m/\mathrm{im}\,(\psi \circ f \circ \varphi).$$

[1] Note that any subgroup of the abelian group $\mathbb{Z}^m$ is automatically normal.

Proof. The point is that we have an isomorphism $\varphi^{-1} : \mathbb{Z}^n \xrightarrow{\simeq} \mathbb{Z}^n$ sitting in a diagram

$$\begin{tikzcd}
\mathbb{Z}^n \arrow[r, "f"] \arrow[d, "\varphi^{-1}", "\simeq"'] & \operatorname{im} f & \leq \mathbb{Z}^m \arrow[d, "\psi", "\simeq"'] \\
\mathbb{Z}^n \arrow[r, "\psi \circ f \circ \varphi"'] & \operatorname{im}\,(\psi \circ f \circ \varphi) & \leq \mathbb{Z}^m
\end{tikzcd}$$

Contemplating this for a bit shows that ψ restricts to an isomorphism

$$\operatorname{im} f \xrightarrow{\simeq} \operatorname{im}\,(\psi \circ f \circ \varphi)$$

and so gives rise to an isomorphism

$$\mathbb{Z}^m/\operatorname{im} f \xrightarrow{\simeq} \mathbb{Z}^m/\operatorname{im}\,(\psi \circ f \circ \varphi).$$

of quotients as desired. $\square$

Observation 3. Giving a homomorphism $f : \mathbb{Z}^n \to \mathbb{Z}^m$ is equivalent to specifying the $m \times n$-matrix

$$A = \begin{pmatrix} f(\overrightarrow{\mathbf{e}}_1) & \cdots & f(\overrightarrow{\mathbf{e}}_n) \end{pmatrix} \in M_{m\times n}(\mathbb{Z}).$$

Here, we are viewing elements of $\mathbb{Z}^m$ as being $m \times 1$ column vectors.

Example 1. The homomorphism from Observation 1 corresponds to the matrix

$$A_f = \begin{pmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & d_n \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

Definition 1. For $m \geq 1$, we set

$$\mathrm{GL}_m(\mathbb{Z}) = \{A \in M_{m\times m}(\mathbb{Z}) :\ \det(A) = \pm 1\}.$$

$$A = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots \\ 0 & 0 & \cdots & d_s \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \begin{matrix} 0 \\ 0 \\ \vdots \\ d_s s \text{ rows} \\ 0 \\ \vdots \\ 0 r - s \text{ rows,} \end{matrix}$$

Remark 2. The condition $\det(A) = \pm 1$ is equivalent to saying that A is invertible *and* A^{-1} is also a matrix in $M_{m\times m}(\mathbb{Z})$ with integer entries. Indeed, we have

$$1 = \det(I_m) = \det(AA^{-1}) = \det(A)\det(A^{-1}).$$

If A and A^{-1} are integer matrices, then the right hand side is a product of integers, and the only way for such product to be 1 is if the integers are either both 1 or both -1.

Unwinding the relationship between compositions of homomorphisms with multiplication of matrices gives us:

Observation 4. In Observation 2, ψ corresponds to an invertible matrix $A_\psi \in \mathrm{GL}_m(\mathbb{Z})$ and φ to an invertible matrix $A_\varphi \in \mathrm{GL}_n(\mathbb{Z})$, and we have

$$A_{\psi\circ f\circ\varphi} = A_\psi A_f A_\varphi \in M_{m\times n}(\mathbb{Z}).$$

So we are now faced with the following question: How simple can I make the computation of $\mathrm{im}\ f$ by replacing A_f with a product by invertible matrices on both the right and the left? The answer to this is given by the *Smith Normal Form*, which we will consider in the next lecture.