# MATH 2211, SPRING 2026: HOMEWORK 1

**Notation 1.** We will use *summation notation*: The symbol

$$\sum_{k=1}^{n} a_k$$

has to be read like a for loop: The entries $a_1, a_2, \ldots, a_n$ are numbers that are indexed by their subscripts, and the symbol above is short for the iterated sum

$$a_1 + a_2 + \cdots + a_n.$$

(1) Suppose that we have $a, b \in \mathbb{Z}$.
   (a) If $a, b$ each leave a remainder of $1$ after dividing by $4$, what can you say about the remainder that $ab$ leaves when divided by $4$?
   (b) If $a, b$ each leave a remainder of $3$ after dividing by $4$, what can you say about the remainder that $ab$ leaves when divided by $4$?

(2) Use the previous problem to show that there are infinitely many primes (such as $3, 7, 11, 19, \ldots$) that leave a remainder of $3$ after dividing by $4$
   *Hint: This uses a variation of the construction used in Euclid's proof of the infinitude of all primes: If $p_1, \ldots, p_r$ is a finite set of primes, consider the prime factorization of $N = 4p_1 \cdots p_r - 1$.*

(3) Prove the following formulas using induction:
   (a)
$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6};$$
   (b)
$$\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}.$$

(4) Prove that $2^{2n+1} + 1$ is a multiple of $3$ for every $n \in \mathbb{N}$.

(5) Prove that for every $n \in \mathbb{N}$, we have
$$\sum_{k=1}^{n} \frac{1}{k^2} \leq 2 - \frac{1}{n}.$$

(6) Use l'Hôpital's rule and induction to prove that
$$\lim_{n\to\infty} \frac{x^n}{e^x} = 0$$
for all $n \in \mathbb{N}$. You may assume that the equality is true when $n = 0$.