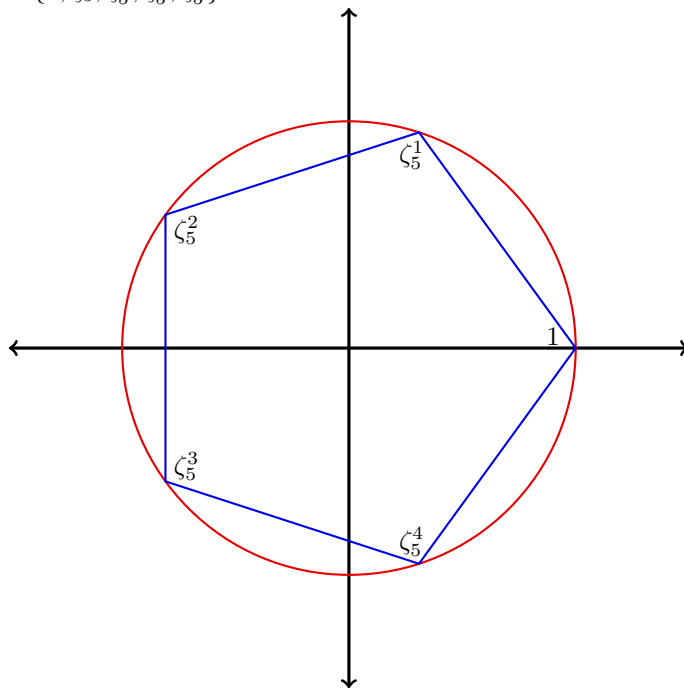


MATH 3311, FALL 2025: LECTURE 2, AUGUST 27

Lecture video:¹ <https://youtu.be/qMeZ9SV4tbc>

Let us return to the example from the end of the last lecture, where we considered the set of fifth roots of one: $\{1, \zeta_5, \zeta_5^2, \zeta_5^3, \zeta_5^4\}$.



We found the following ‘symmetries’ of the subset $\{\zeta_5, \zeta_5^2, \zeta_5^3, \zeta_5^4\}$ of the non-trivial fifth roots:

- (Complex conjugation) $\zeta_5 \leftrightarrow \zeta_5^4; \zeta_5^2 \leftrightarrow \zeta_5^3$.
- (Cubing) $\zeta_5 \mapsto \zeta_5^3 \mapsto \zeta_5^4 \mapsto \zeta_5^2 \mapsto \zeta_5$.

But we can add more along similar lines:

- (Squaring) $\zeta_5 \mapsto \zeta_5^2 \mapsto \zeta_5^4 \mapsto \zeta_5^3 \mapsto \zeta_5$.
- (Fourth power) $\zeta_5 \mapsto \zeta_5^4 \mapsto \zeta_5; \zeta_5^2 \mapsto \zeta_5^3 \mapsto \zeta_5^2$.
- (Zeroth power = fifth power = *trivial* symmetry) Every power of ζ_5 is kept where it is.

Note that taking the fourth power yields the same symmetry as complex conjugation. The property that makes these functions ‘symmetries’ is that they are *bijections*. Let’s recall the relevant definitions, just so we can all be on the same page.

Definition 1. A **function** $f : X \rightarrow Y$ is a rule assigning to every element x in a set X a unique element y in the set Y .

Example 1. The rule assigning to every person their name is a function, while the rule assigning to each person their grandfather’s name is *not*, since the output of such a rule is usually not uniquely determined (or sometimes determined at all!)

Definition 2. A function $f : X \rightarrow Y$ is **one-to-one** or **injective** if, for all $x_1, x_2 \in X$, $f(x_1) = f(x_2)$ implies that $x_1 = x_2$. Equivalently, for all $y \in Y$, there is *at most one* $x \in X$ such that $f(x) = y$.

Definition 3. A function $f : X \rightarrow Y$ is **onto** or **surjective** if, for all $y \in Y$ there exists some $x \in X$ such that $f(x) = y$.

¹Unfortunately, because of a recording snafu, only about a somewhat disjointed half of the lecture was captured.

Definition 4. A function $f : X \rightarrow Y$ is a **bijection** if it is both injective and surjective. Equivalently, for all $y \in Y$ there exists a *unique* $x \in X$ such that $f(x) = y$.

All the powering functions defined above are *bijections* from the set $\{\zeta_5, \zeta_5^2, \zeta_5^3, \zeta_5^4\}$ to itself.

Example 2 (Non-example). If we look at the 5th power function, then we have $(\zeta_5^k)^5 = \zeta_5^{5k} = 1$ for all k . That is, this function is not injective, but also that its outputs are no longer in the original set of the non-trivial fifth roots of 1!

Now that we have the powering symmetries, we can look at *composing* them. Recall:

Definition 5. If $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are two functions with the domain of g matching up with the codomain of f , then we can define their **composition** $g \circ f : X \rightarrow Z$ given by $(g \circ f)(x) = g(f(x))$.

Next time, we will look at what happens when we compose the powering symmetries, and we will also look at why we isolated them in particular.