MATH 3311, FALL 2025: LECTURE 36, NOVEMBER 24

Video: `https://youtu.be/Bv_Vjjxm_rg`

The existence part of the Fundamental Theorem for finitely generated abelian groups

As mentioned last time, we now have all the ingredients needed to establish the existence part of the fundamental theorem.

(1) First, given such a group G, we can choose a set of generators $\{x_1, \ldots, x_m\} \subset G$ and use this to write down a *surjective* homomorphism $\mathbb{Z}^m \to G$ carrying $\vec{\mathbf{e}}_i$ to x_i.
(2) Such a homomorphism gives an isomorphism $\mathbb{Z}^m/H \xrightarrow{\simeq} G$ for some subgroup $H \leq \mathbb{Z}^m$.
(3) The subgroup H is isomorphic to $\mathbb{Z}^n$ for some $n \geq m$, and so we can find n vectors $\vec{\mathbf{v}}_1, \ldots, \vec{\mathbf{v}}_n \in H$ that generate H. More conceptually, we have a homomorphism $\alpha : \mathbb{Z}^n \to \mathbb{Z}^m$ whose image is H.
(4) Write down the $m \times n$-matrix
$$A_\alpha = \begin{pmatrix} \vec{\mathbf{v}}_1 & \cdots & \vec{\mathbf{v}}_n \end{pmatrix}$$
(5) Find the Smith Normal Form for A_α: this is of the form $A_\psi A_\alpha A_\varphi$, where A_ψ and A_φ are invertible matrices associated with *isomorphisms* $\psi : \mathbb{Z}^m \xrightarrow{\simeq} \mathbb{Z}^m$ and $\varphi : \mathbb{Z}^n \to \mathbb{Z}^n$.
(6) If $d_1 \mid d_2 \mid \cdots \mid d_n$ are the diagonal entries of $A_\psi A_\alpha A_\varphi$, then see that
$$\mathbb{Z}^m/\mathrm{im}\,(\psi \circ \alpha \circ \varphi) \simeq \mathbb{Z}/d_1\mathbb{Z} \times \cdots \times \mathbb{Z}/d_n\mathbb{Z} \times \mathbb{Z}^{m-n}.$$
(7) Observe that this has to be isomorphic to $\mathbb{Z}^m/\mathrm{im}\,\alpha = \mathbb{Z}^m/H \xrightarrow{\simeq} G$.

Example 1. Consider $G = \mathbb{Z}^2/\langle(8,16)\rangle$. Here there is just one generator, and the subgroup it generates is the image of the homomorphism corresponding to the matrix
$$\begin{pmatrix} 8 \\ 16 \end{pmatrix}$$
The SNF for this matrix is easily seen to be
$$\begin{pmatrix} 8 \\ 0 \end{pmatrix}$$
Therefore, $\mathbb{Z}^2/\langle(8,16)\rangle$ is isomorphic to $\mathbb{Z}^2/\langle(8,0)\rangle$ which is isomorphic to $\mathbb{Z}/8\mathbb{Z} \times \mathbb{Z}$.

Example 2. Now, consider $G = \mathbb{Z}^2/\langle(8,16),(2,3)\rangle$. The subgroup now is the image of the homomorphism corresponding to the matrix
$$\begin{pmatrix} 8 & 2 \\ 16 & 3 \end{pmatrix}.$$
The gcd of the entries is 1 and the determinant is -8. Using this one can check that the SNF is
$$\begin{pmatrix} 1 & 0 \\ 0 & 8 \end{pmatrix}$$
The quotient is therefore isomorphic to $\mathbb{Z}/1\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z} \simeq \mathbb{Z}/8\mathbb{Z}$.

The uniqueness part

What we have to show: If we have
$$G \simeq \mathbb{Z}/d_1\mathbb{Z} \times \cdots \times \mathbb{Z}/d_m\mathbb{Z} \times \mathbb{Z}^r$$
and
$$G \simeq \mathbb{Z}/d_1'\mathbb{Z} \times \cdots \times \mathbb{Z}/d_{m'}'\mathbb{Z} \times \mathbb{Z}^{r'}$$
with $2 \leq d_1 \mid \cdots \mid d_m$ and $2 \leq d_1' \mid \cdots \mid d_m'$ then in fact $m' = m$ and $r' = r$.

Let's take care of the rank part first.

Proposition 1. *In the above situation, we have $r' = r$.*

Proof. In Homework 11, we saw: If $f : G \xrightarrow{\simeq} H$ is an isomorphism, then we obtain isomorphisms

$$G^{\mathrm{tors}} \xrightarrow{\simeq} H^{\mathrm{tors}}\ ;\ G^{\mathrm{tf}} \xrightarrow{\simeq} H^{\mathrm{tf}}.$$

Moreover, it is not difficult to see that, if

$$G \simeq \mathbb{Z}/d_1\mathbb{Z} \times \cdots \times \mathbb{Z}/d_m\mathbb{Z} \times \mathbb{Z}^r$$

then $G^{\mathrm{tf}} \simeq \mathbb{Z}^r$. Therefore, applying this to the two different presentations of G, we find that $\mathbb{Z}^r \simeq \mathbb{Z}^{r'}$. This can only happen if $r = r'$: see problem 3 on Homework 12. $\square$