Induction

Example

$$* \left(1+2+3+4+\dots+n = \frac{n(n+1)}{2} \right)$$

- $\underline{n=1}$

$$1 = \frac{1\cdot 2}{2} \checkmark$$

- $\underline{n=2}$

$$1+2 = \frac{2\cdot 3}{2} \checkmark$$

Principle of induction

Suppose that we have a sequence of assertions

$$P(1), P(2), P(3), \dots, P(n), \dots$$

[Example: $P(n)$ could be the statement]

["$*$ is true"]

Then: Suppose that the following holds:

(base case) (i) $P(1)$ is true

(inductive step) $\boxed{\text{(ii) If } P(n) \text{ is true then } P(n+1) \text{ is true}}$

Then: $P(n)$ is true for all $n \geq 1$

Pf: Uses the

$\boxed{\text{Well-ordering principle}}$

Every non-empty subset of $\mathbb{N}$ has a smallest element.

Take

$$X = \{ n \in \mathbb{N} : P(n) \text{ is false} \}$$

Assume $X \neq \emptyset$ (non-empty)

Then X contains a smallest element $s \in X$.

(i) $\Rightarrow s \neq 1$

$\Rightarrow s-1 \in \mathbb{N}$

$\Rightarrow s-1 \notin X$

$\Rightarrow P(s-1)$ is true

(ii) $\Rightarrow P(s)$ is true !!

$\Rightarrow s \notin X$!!

Contradiction!!

$\Rightarrow X = \emptyset$

Example:

$P(n)$

$P(n)$: $1+2+3+\dots+n = \frac{n(n+1)}{2}$

(i) $P(1)$ is true ✓

(ii) $P(n) \Longrightarrow P(n+1)$

"implies"

$$(1+2+\dots+n)+(n+1) = \frac{n(n+1)}{2}+n+1$$

$$= \frac{n(n+1)+2(n+1)}{2}$$

$$= \frac{n^2+n+2n+2}{2}$$

$$= \frac{n^2+3n+2}{2} = \frac{(n+1)(n+2)}{2}$$

$\Rightarrow$ $P(n+1)$ is true ✓

$$\boxed{1 + \cdots + n + (n+1) = \frac{(n+1)(n+2)}{2}}$$

$P(n+1)$

Example

$$P(n) : \quad 1 + x + \cdots + x^n = \frac{x^{n+1} - 1}{x - 1}$$

(i) $P(1)$

$$1 + x \overset{?}{=} \frac{x^2 - 1}{x - 1}$$

$\downarrow (x-1)\cdot$

$$(x-1)(1+x) \overset{?}{=} x^2 - 1$$

$\parallel \qquad \parallel$ ✓

$$(x-1)(x+1) = x^2 - x + x - 1$$

(ii) $P(n) \Longrightarrow P(n+1)$

$$(1+x+\cdots+x^n) + x^{n+1} = \frac{x^{n+1}-1}{x-1} + x^{n+1}$$

$$= \frac{x^{n+1}-1+x^{n+1}(x-1)}{x-1}$$

$$= \frac{\cancel{x^{n+1}}-1+x^{n+2}-\cancel{x^{n+1}}}{x-1}$$

$$= \frac{x^{n+2}-1}{x-1}$$

$P(n+1)$: $1+x+\cdots+x^n+x^{n+1} = \dfrac{x^{n+2}-1}{x-1}$ ✓

Example:

$P(2)$: 2 is a product of primes

$P(n)$: n is a product of prime numbers

n is <u>not</u> a prime

$\Rightarrow \quad n = a\,b, \quad a, b < n$

<u>Rmk:</u> The base case for induction can be <u>any</u> natural number $a \in \mathbb{N}$. The implication would be that $P(n)$ is true for all $n \geq a$.

<u>Base case:</u> $n=2$, $P(2)$ is true.

<u>Inductive step:</u>

If n is prime then we are done; otherwise

$n = ab$, $a < n$, $a < b$

Strong induction

(see next page)

$\Rightarrow P(a), P(b)$ are true (inductive hypothesis)

$\Rightarrow a$ & b are products of prime numbers

$\Rightarrow n$ is a product of prime numbers

$P(n)$ is true, for $n \geq 2$

We need the principle of

Strong induction

(i) $P(1)$ is true

(ii) If $P(1), P(2), \ldots, P(n)$ are all true, then $P(n+1)$ is true.

Reformulation of (ii)

(ii') For any $n > 1$

if $P(a)$ is true $\forall a < n$

then $P(n)$ is true.