Last time

R: comm. ring

$f(x) \in R[x]$: polynomial

$$(f(x)) = \{ f(x)h(x) : h(x) \in R[x] \}$$

Fact ① $(f(x)) \leq R[x]$ subgroup under addition

2 There is a unique structure of a commutative ring on $R[x]/(f(x))$ s.t.

$$R[x] \longrightarrow R[x]/(f(x))$$
$$g(x) \longmapsto g(x) + (f(x))$$

is a ring homomorphism

Note: $\langle f(x) \rangle = \{ m \cdot f(x) : m \in \mathbb{Z} \}$

$\boxed{\text{subgroup}}$ generated by $f(x)$

$$\langle f(x) \rangle \neq (f(x))$$

$$\langle f(x) \rangle < (f(x))$$

Rmk: If $\varphi : R[x] \longrightarrow R'$ is a ring homomorphism & $\varphi(f(x)) = 0$

then

$$\begin{aligned}\varphi(f(x)h(x)) &= \varphi(f(x))\varphi(h(x)) \\ &= 0 \cdot \varphi(h(x)) \\ &= 0 \in R'\end{aligned}$$

So if φ **kills** $f(x)$

then it **kills** the whole ideal $(f(x))$ generated by $f(x)$

Fact ③

ring homomorphism

$\downarrow$

$\varphi : f(x) \in \ker \varphi$

$$\begin{tikzcd}
R[x] \arrow[d, "\pi"'] \arrow[dr, "\varphi = \bar{\varphi} \circ \pi"] & \\
R[x]/(f(x)) \arrow[r, "\exists! \, \bar{\varphi}"'] & R'
\end{tikzcd}$$

where $\bar{\varphi}$ is a ring homomorphism

Pf: Rmk $\Longrightarrow$ $[f(x) \in \ker \varphi]$

$$\Downarrow$$

$[(f(x)) \subseteq \ker \varphi]$

- $\bar{\varphi}\left(h(x)+(f(x))\right) = \varphi(h(x)) \in R'$

$\qquad\qquad ||$

$\bar{\varphi}\left(\pi(h(x))\right) = \varphi(h(x))$

$$\bar{\varphi}\left(\pi(h_1(x))\pi(h_2(x))\right) \overset{?}{=} \bar{\varphi}(\pi(h_1(x))) \cdot \bar{\varphi}(\pi(h_2(x)))$$

$||$ π is a ring hom.

$\bar{\varphi}\left(\pi(h_1(x)h_2(x))\right) \qquad\qquad ||$

$||$

$$\varphi(h_1(x)h_2(x)) \overset{\checkmark}{=} \varphi(h_1(x))\varphi(h_2(x))$$

- $\bar{\varphi}\left(1+(f(x))\right) = \varphi(1_{R[x]})$

$\qquad = 1_{R'} \in R'$

Example

$R = \mathbb{Z}$, $f(x) = x - 1$

$\mathbb{Z}[x]/(x-1)$

$$\begin{tikzcd}
\mathbb{Z}[x] \arrow[d] \arrow[dr, "\varphi :\ \varphi(x-1)=0"] & \\
\mathbb{Z}[x]/(x-1) \arrow[r, "\overline{\varphi}"'] & R'
\end{tikzcd}$$

$$\underset{\substack{\| \\ 0}}{\varphi(x-1)} = \varphi(x) - \varphi(1) = \varphi(x) - 1_{R'}$$

$$\Longrightarrow \varphi(x) = 1_{R'}$$

$$\varphi(x^2) = \varphi(x)^2 = 1_{R'}^2 = 1_{R'}$$

$$\varphi(x^n) = 1_{R'}$$

$$\varphi(a_n x^n + \dots + a_1 x + a_0)$$

$a_i \in \mathbb{Z}$

$= \varphi(a_n x^n) + \dots + \varphi(a_1 x) + \varphi(a_0)$

$= \varphi(a_n)\varphi(x)^n + \dots + \varphi(a_1)\varphi(x) + \varphi(a_0)$

$= \varphi(a_n)\cdot 1_{R'} + \dots + \varphi(a_1)\cdot 1_{R'} + \varphi(a_0)$

$= a_n \cdot 1_{R'} + a_{n-1}\cdot 1_{R'} + \dots + a_1 \cdot 1_{R'} + a_0 \cdot 1_{R'}$

$= (a_n + a_{n-1} + \dots + a_1 + a_0)\cdot 1_{R'}$

This is the unique such homomorphism.

Fact 4 $\varphi : \mathbb{Z}\cancel{[x]} \longrightarrow R'$

$\varphi(m) = m \cdot 1_{R'}$

$m \in \mathbb{Z}$

Notation

$m \in R'$ instead of

$m \cdot 1_{R'}$.

Structure of $R[x]/(f(x))$ when $f(x)$ is monic

$$f(x) = a_n x^n + \dots + a_1 x + a_0, \quad a_n \in R^\times$$

Fact 5 In this case, we have a bijection

$g(x) + (f(x))$

$$\{\underset{R[x]}{r(x)} : \deg < n\} \xrightarrow[\cong]{} R[x]/(f(x))$$

$$r(x) \longmapsto r(x) + (f(x))$$

Pf: $g(x) + (f(x))$

absorbed

$$= \boxed{q(x)f(x) + r(x)} + (f(x))$$

$\deg r(x) < n$

$$= r(x) + (f(x))$$

Surjectivity

injectivity: Uniqueness of remainder.

Example

$$\mathbb{Z} \xrightarrow{\cong} \frac{\mathbb{Z}[x]}{(x-1)}$$

$$m \longmapsto m + (x-1)$$

This is actually an isomorphism of rings.

$$\begin{tikzcd} \mathbb{Z}[x] \arrow[d] \arrow[dr, "\varphi"] & \\ \mathbb{Z}[x]/(x-1) \arrow[r, "\overline{\varphi}", "\cong ??"'] & \mathbb{Z} \end{tikzcd}$$

$\varphi : a_n x^n + \cdots + a_1 x + a_0 \longmapsto a_n + \cdots + a_0$

$$\ker \varphi = \{ f(x) : \underbrace{a_n + a_{n-1} + \cdots + a_0}_{f(1)} = 0 \}$$

$f(x) = a_n x^n + \cdots + a_1 x + a_0$

$$\ker\varphi \supseteq (x-1)$$

Need: $(x-1) \stackrel{?}{=} \ker \varphi$

$\ker \varphi \ni f(x) = (x-1)q(x) + r$

$r \in \mathbb{Z}$

$$\begin{array}{ccc} "f(1)" & = & r \\ \| & & \\ 0 & & \end{array}$$

$\Rightarrow r = 0 \Rightarrow f(x) \in (x-1).$