MATH 3311, FALL 2025: LECTURE 20, OCTOBER 15

Video: `https://youtu.be/dxugDqzoNXI`

As in the last lecture, we will fix a finite group G and a prime p. Let $m \geq 0$ be the integer such that p^m is the largest power of p dividing the order $|G|$.

Last time, we basically gave a full proof of

Theorem 1 (Sylow Theorem A)**.** *There exists a subgroup $Q \leq G$ of order $|Q| = p^m$.*

This leads to the following definition.

Definition 1. In this situation, we will say that Q is a **Sylow p-subgroup** of G. Equivalently, $Q \leq G$ is a subgroup satisfying two conditions:

(1) Q is a p-group;
(2) $|G|/|Q| = [G : Q]$ is prime to p.

Definition 2. We will write $\mathrm{Syl}_p(G)$ for the set of Sylow p-subgroups of G.

Remark 1. We can reformulate Theorem A as saying that $\mathrm{Syl}_p(G)$ is *non-empty*.

See the notes from the last lecture for the proof of Theorem A, which uses essentially every key notion we have learned this semester.

Before we state the second Sylow theorem, we will need the following observation.

Observation 1. If $P \in \mathrm{Syl}_p(G)$ and $g \in G$, then gPg^{-1} is also in $\mathrm{Syl}_p(G)$. That is, G *acts* on $\mathrm{Syl}_p(G)$ via conjugation.

Proof. This is just because conjugation is an *automorphism* of the group and so takes subgroups to subgroups and also preserves orders of subgroups. In fact, as we will use later, the function $x \mapsto gxg^{-1}$ from P to gPg^{-1} is an *isomorphism* of groups. $\square$

The second important theorem is:

Theorem 2 (Sylow Theorem B)**.** *The conjugation action of G on $\mathrm{Syl}_p(G)$ is* transitive. *That is, if $P, Q \in \mathrm{Syl}_p(G)$ are two Sylow p-subgroups of G, then there exists $g \in G$ such that $gPg^{-1} = Q$.*

Before we see the proof of the theorem, let us record some consequences.

Corollary 1. *If $P, Q \in \mathrm{Syl}_p(G)$ are two Sylow p-subgroups that P is* isomorphic *to Q.*

Proof. Since $Q = gPg^{-1}$ for some $g \in G$, the function

$$P \xrightarrow{x \mapsto gxg^{-1}} Q$$

is a bijective *homomorphism*[1], with inverse given by conjugation by g^{-1}. $\square$

Corollary 2. *If $P \in \mathrm{Syl}_p(G)$, then the following are equivalence:*

(1) $P \trianglelefteq G$ is normal *in G;*
(2) P is the unique *Sylow p-subgroup.*

Proof. This is because a subgroup is normal precisely when it is equal to all its conjugates, and we know that all Sylow p-subgroups are conjugate by Theorem B. $\square$

Corollary 3. *If G is* abelian, *then G has a unique Sylow p-subgroup.*

Proof. This follows from Corollary 2, because every subgroup of G is automatically normal. $\square$

[1]The homomorphism property just amounts to the identity $g(xy)g^{-1} = (gxg^{-1})(gyg^{-1})$ which you should have already used in Homework 6.

Proof of Theorem B. We want to show that, for $P, Q \in \mathrm{Syl}_p(G)$, there exists $g \in G$ such that $Q = gPg^{-1}$.

This in turn is equivalent to knowing

$$Q \leq gPg^{-1}.$$

Indeed, Q and gPg^{-1} both have order p^m by hypothesis.

We now observe that $gPg^{-1} = G_{gP}$ where $G_{gP} \leq G$ is the stabilizer of the coset $gP \in G/P$ for the left multiplication action[2] Therefore, we have to show:

There exists $g \in G$ such that $Q \leq G_{gP}$, which is equivalent to knowing that the action of Q on G/P via left multiplication has a *fixed point*.

That is, we need to know that $(G/P)^Q$ is non-empty, or equivalently that

$$|(G/P)^Q| \neq 0.$$

For this, we will prove something that is a bit stronger. Indeed, since Q is a p-group, we can apply our fundamental congruence for group actions by p-groups on finite sets to deduce that we have

$$|(G/P)^Q| \equiv |G/P| \pmod{p}.$$

Now, we finally use our hypothesis that P is a *Sylow* p-subgroup. This implies that $|G/P| = [G : P]$ is *not* divisible by p. Therefore, we have

$$|(G/P)^Q| \not\equiv 0 \pmod{p}.$$

In particular, $(G/P)^Q$ is *non-empty*, and hence there is $gP \in G/P$ that is fixed by Q. As we established above, this means that $Q = gPg^{-1}$. $\square$

[2] Quick proof: $hgP = h(gP) = gP \Leftrightarrow g^{-1}hg \in P \Leftrightarrow h \in gPg^{-1}$.