

MATH 3311, FALL 2025: LECTURE 3, AUGUST 29

Video: <https://youtu.be/fkJs-0GeNR4>

In the last lecture, we considered the symmetries of the set $\{1, \zeta_5, \zeta_5^2, \zeta_5^3, \zeta_5^4\}$ equipped with its *multiplicative* structure and noticed that there are exactly four such symmetries, $\sigma_1, \sigma_2, \sigma_3, \sigma_4$, given by raising to the power in the subscript.

Now, two bijections from a set to itself can be composed to get another such bijection. In particular, we can compose $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ with themselves in various ways to get other bijections. But in fact, everything we obtain in this way is once again one of the σ_i . In other words, the set of bijections $\{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}$ is *closed* under compositions. This is because we have

$$\sigma_i \circ \sigma_j = \sigma_{ij}.$$

(Why?)

This gives us the following composition table.

\circ	σ_1	σ_2	σ_3	σ_4
σ_1	σ_1	σ_2	σ_3	σ_4
σ_2	σ_2	σ_4	σ_1	σ_3
σ_3	σ_3	σ_1	σ_4	σ_2
σ_4	σ_4	σ_3	σ_2	σ_1

We can also remember just the subscripts, and note that all we are doing is multiplying them and taking the remainder under division by 5. In other words, we are now doing mod-5 arithmetic again, except that we are using multiplication instead of addition. This gives us the following table:

.	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

What makes these symmetries special? After all, there are $4! = 24$ ways of shuffling around the four non-trivial zeros. Why did we pick out these four in particular? For this, we need to make the observation that there is additional *structure* here that we have not explicated yet. The fifth roots of unity are not a *random* collection of five things. They possess a *multiplicative structure*: If we multiply two roots, we get another one: $\zeta_5^i \cdot \zeta_5^j = \zeta_5^{i+j}$.

Note that $\zeta_5^5 = 1$, so that when we take the i^{th} -power, only the remainder that i leaves when divided by 5 matters. We can write down a ‘multiplication table’ for this as follows

.	1	ζ_5	ζ_5^2	ζ_5^3	ζ_5^4
1	1	ζ_5	ζ_5^2	ζ_5^3	ζ_5^4
ζ_5	ζ_5	ζ_5^2	ζ_5^3	ζ_5^4	1
ζ_5^2	ζ_5^2	ζ_5^3	ζ_5^4	1	ζ_5
ζ_5^3	ζ_5^3	ζ_5^4	1	ζ_5	ζ_5^2
ζ_5^4	ζ_5^4	1	ζ_5	ζ_5^2	ζ_5^3

If we extract out just the exponents (thinking of 1 as ζ_5^0), we see that this can also be represented as a table for *addition*, where we only keep track of the remainder when divided by 5 (this is called **mod-5 arithmetic** or **arithmetic modulo 5**).

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

These are the ‘same’ table up to some relabeling of the rows and columns. We will formalize what this means in future lectures.

Observation 1. The four symmetries considered above (trivial, squaring, cubing, fourth power) are precisely those *bijections*

$$\sigma : \{1, \zeta_5, \zeta_5^2, \zeta_5^3, \zeta_5^4\} \rightarrow \{1, \zeta_5, \zeta_5^2, \zeta_5^3, \zeta_5^4\}$$

that *preserve the multiplicative structure*. More precisely, they satisfy

$$\sigma(\zeta_5^i)\sigma(\zeta_5^j) = \sigma(\zeta_5^{i+j})$$

for all $i, j \in \{0, 1, 2, \dots, 4\}$.

Proof. First, note that a symmetry σ that preserves multiplicative structure (a *multiplicative symmetry* for short) must satisfy $\sigma(1) = 1$. This is because we have

$$\sigma(1)\sigma(\zeta_5^j) = \sigma(\zeta_5^0)\sigma(\zeta_5^j) = \sigma(\zeta_5^{0+j}) = \sigma(\zeta_5^j).$$

Since $\sigma(\zeta_5^j) \neq 0$, we can cancel it from both sides, leaving us with

$$\sigma(1) = 1.$$

Now, we have four options for $\sigma(\zeta_5)$: $\zeta_5, \zeta_5^2, \zeta_5^3, \zeta_5^4$ (1 is not an option, since it is the value of $\sigma(1)$, and a bijection cannot have repeated outputs). Picking any of these options completely determines what σ does to the rest of the powers of ζ_5 : If $\sigma(\zeta_5) = \zeta_5^k$, then for instance we have

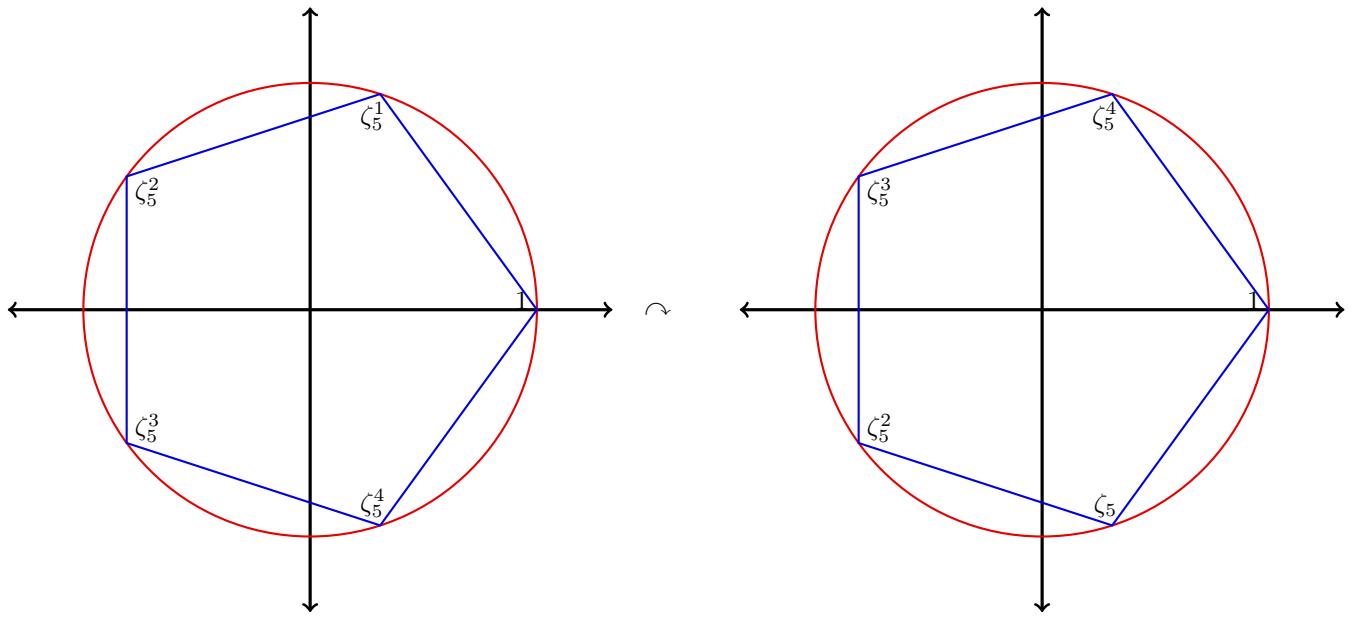
$$\sigma(\zeta_5^2) = \sigma(\zeta_5)\sigma(\zeta_5) = \zeta_5^k \cdot \zeta_5^k = \zeta_5^{2k}.$$

Similarly, you can check that we have $\sigma(\zeta_5^3) = \zeta_5^{3k}$ and $\sigma(\zeta_5^4) = \zeta_5^{4k}$.

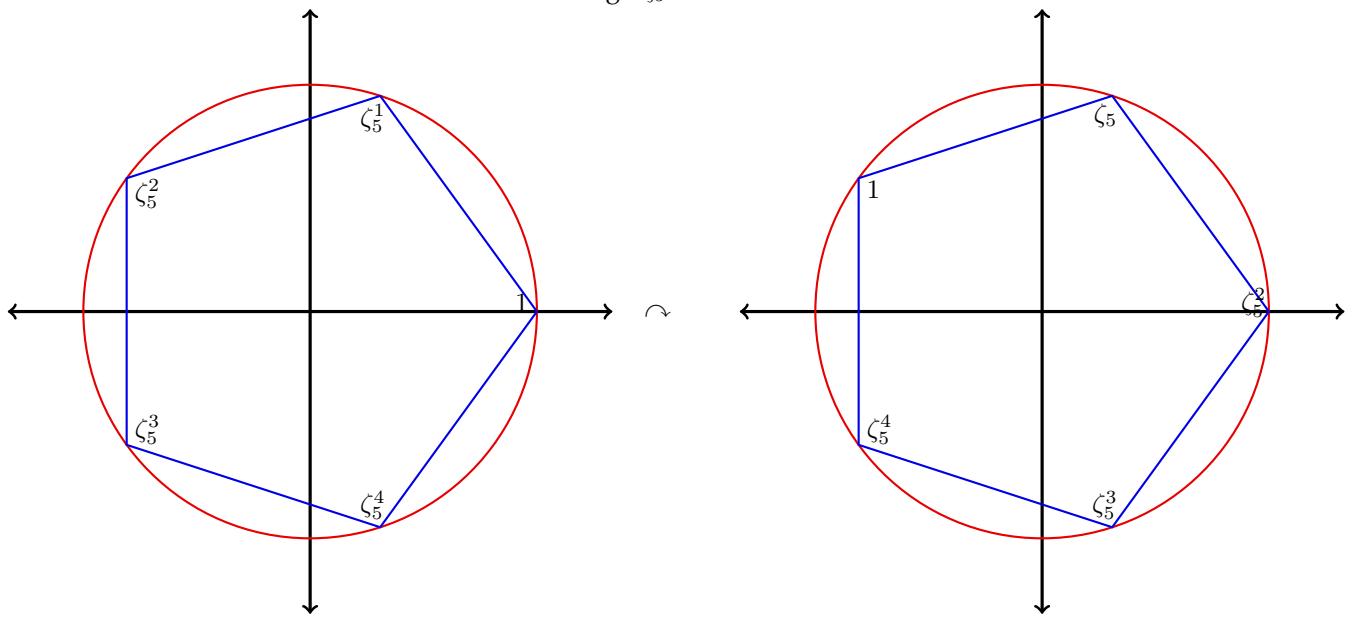
In other words, the four options for $\sigma(\zeta_5)$ are yielding, in order, the trivial, squaring, cubing and fourth power symmetries, respectively. \square

There is also a different, more geometric, structure on this set: They are the vertices of a regular pentagon. We can therefore consider the *rigid* symmetries of this pentagon that keep the perimeter unchanged, so that if we have an ant traveling along it, it will still be doing so after applying the symmetry, but just in a different location on the pentagon, relative to the plane on which it lies.

For instance, we can *reflect* it across the median through the vertex 1. This would simply give us complex conjugation, which we have already seen in the form of the symmetry σ_4 .

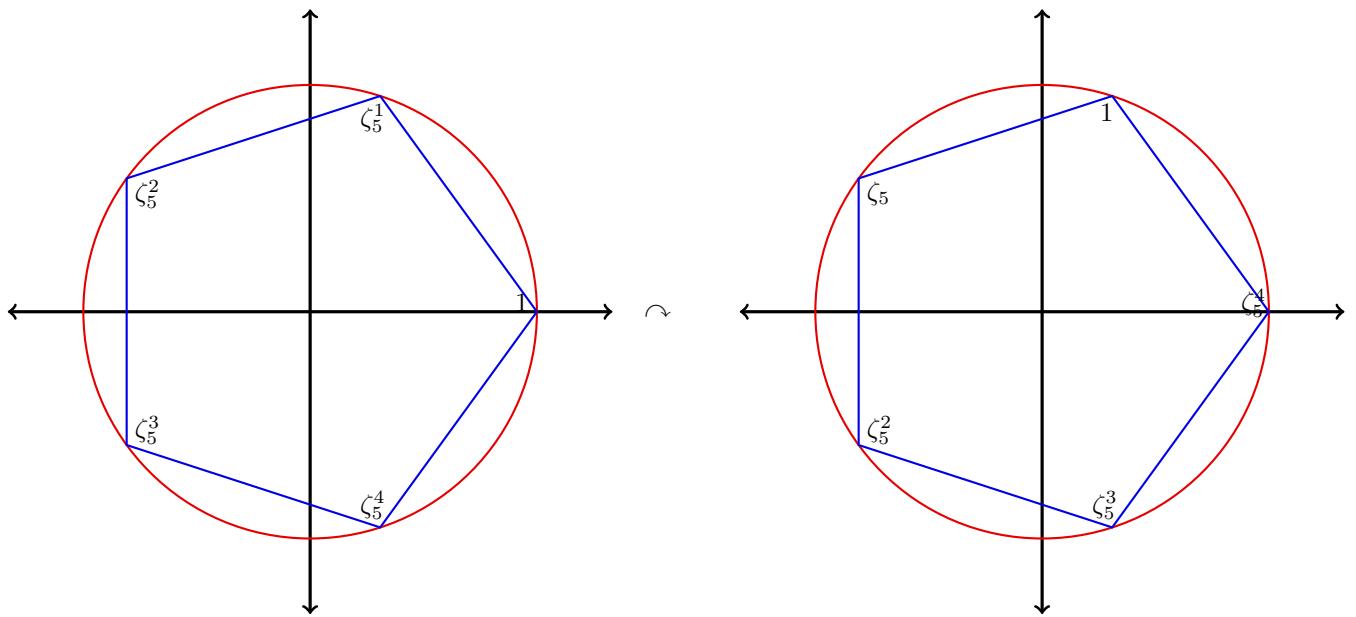


This fixes 1 and flips the other four in pairs.
But we could also reflect it across the median through ζ_5 .



This fixes ζ_5 and flips 1 with ζ_5^2 , and ζ_5^3 with ζ_5^4 .

Another thing we can do is rotate everything by some integer multiple of $2\pi/5$. For instance, if we rotate counterclockwise by $2\pi/5$, we get:



Let us call this rotation σ , and let us write τ for the complex conjugation symmetry (or reflection across the median through 1) from the beginning.