Unit - III

Odinary differential equation of higher order.

↳ linear D.E of second and higher order, Complete solution, Complementary functions and particular Integral.

→ A. D.E of the form

$$P_0 \frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + P_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots + P_n y = X$$

Where $P_0, P_1, P_2 \ldots P_n$ and X are functions of x or Constant is called linear differential equation of n^{th} order.

Values of Operator.

$$\frac{d^1}{dx} = D^1$$

$$\frac{d^2}{dx^2} = D^2$$

$$\frac{d^3}{dx^3} = D^3 \ldots$$

Type - I

Q. Solve the D.E

$$\frac{2d^3y}{dx^3} - 7\frac{d^2y}{dx^2} + \frac{7dy}{dx} - 2y = 0$$

Symbolic form.

$$2D^3y - 7D^2y + 7Dy - 2y = 0$$

Auxillary eq^n.

$$2D^3 - 7D^2 + 7D - 2 = 0.$$

put $D = 1$.

$$2 - 7 + 7 - 2 = 0$$
$$0 = 0.$$

1	2	-7	7	-2
		2	-5	2
	2	-5	2	0

$D = 1$ or, $2D^2 - 5D + 2 = 0$

$$(2D-1)(D-2) = 0.$$

$$\left(D = 1, \frac{1}{2}, 2\right)$$

Complementary function.

$$y = c_1e^{x} + c_2e^{\frac{1}{2}x} + c_3e^{2x}$$

	Condition.	Solution.
1.	When all roots are different and real $m_1, m_2, m_3 \ldots$	The solution can be written as. $y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x} + \ldots$
2.	When two roots are equal and all other are different i.e m_1, m_1, m_1, m_2, m_3.	The Solution is. $y = (c_1 + c_2 x + c_3 x^2) e^{m_1 x} + c_4 e^{m_2 x} + c_5 e^{m_3 x}$
3.	When two roots are imaginary and other are real and different i.e. $\alpha \pm i\beta$, m_3, m_4.	The Solution is. $y = (c_1 \cos\beta x + c_2 \sin\beta x) e^{\alpha x} + c_3 e^{m_3 x} + c_4 e^{m_4 x}$

Q Solve the D.E

$$\frac{d^4y}{dx^4} - \frac{d^3y}{dx^3} - 9\frac{d^2y}{dx^2} - 11\frac{dy}{dx} - 4y = 0.$$

Symbolic form.

$$D^4y - D^3y - 9D^2y - 11Dy - 4y = 0.$$

Auxillary eq^n.

$$D^4 - D^3 - 9D^2 - 11D - 4 = 0$$

put $D = -1$.

$1 + 1 - 9 - 11 - 4 = 0$

$0 = 0$.

Synthetic division.

-1	1	-1	-9	-11	-4
		-1	2	7	4
	1	-2	-7	-4	0

$D = -1 \quad (D^3 - 2D^2 - 7D - 4) = 0$.

put $D = -1$.

$-1 - 2 + 7 - 4 = 0$

$0 = 0$.

$D = -1, -1$

-1	1	-2	-7	-4
		-1	3	4
	1	-3	-4	0.

$D = -1, -1 \quad , (D^2 - 3D - 4 = 0)$

$D^2 - (4-1)D - 4 = 0$

$D^2 - 4D + D - 4 = 0$

$D(D-4) + 1(D-4) = 0$

$(D+1)(D-4) = 0$.

$\therefore D = -1, -1, -1, 4$

Complementary function.

$$y = (c_1 + c_2 x + c_3 x^2) e^{-x} + c_4 e^{4x}$$

Q $\frac{d^3y}{dx^3} - 7\frac{dy}{dx} - 6y = 0.$

Symbolic form.

$$D^3y - 7Dy - 6y = 0.$$

Auxillary eq^n

$$D^3 - 7D - 6 = 0.$$

$-1 + 7 - 6$

put $D = -1$

$$-1 + 7 - 6 = 0$$
$$0 = 0.$$

Synthetic division.

-1	1	0	-7	-6
		-1	1	6
	1	-1	-6	0

$$D^2 - D - 6 = 0.$$
$$D^2 - (3-2)D - 6 = 0$$
$$D^2 - 3D + 2D - 6 = 0$$
$$D(D-3) + 2(D-3) = 0$$
$$(D+2)(D-3) = 0.$$

$\therefore D = -1, -2, 3.$

Complementary Solution.

$$y = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{3x}$$

Q. $(D^3 - 2D^2 - 4D + 8)y = 0.$

Auxillary eq^n.

$$D^3 - 2D^2 - 4D + 8 = 0$$

put $D = 2.$

$$8 - 8 - 8 + 8 = 0$$
$$0 = 0.$$

$$\begin{array}{c|cccc|}
2 & 1 & -2 & -4 & 8 \\
 & & 2 & 0 & -8 \\
\hline
 & 1 & 0 & -4 & 0.
\end{array}$$

$D = 2$, $\cancel{D^2 - 4D = 0}$ $\quad D^2 - (2)^2 = 0$

$\cancel{(D-4)(D+4) = 0}$ $\quad (D-2)(D+2) = 0$

$D = 2, -2.$

$\cancel{D = 2, -4, 4}$

~~Complementary Solution~~,

$\cancel{y = (c_1 + c_2 x)e}$

$$y = c_1 e^{4x} + c_2 e^{-4x} + c_3 e^{2x}$$

Complementary solⁿ.

$$y = (c_1 + c_2 x)e^{2x} + c_3 e^{-2x}$$

Q. Solve the D.E.

$$(D^4 + 5D^2 + 6)y = 0.$$

Auxillary eqⁿ

$$D^4 + 5D^2 + 6 = 0$$
$$D^4 + 3D^2 + 2D^2 + 6 = 0$$
$$D^2(D^2+3) + 2(D^2+3) = 0$$
$$(D^2+3)(D^2+2) = 0$$

$$D^2 + 3 = 0 \qquad\qquad D^2 + 2 = 0$$
$$D^2 = -3 \qquad\qquad D^2 = -2$$
$$D = \pm\sqrt{3}\, i \qquad\qquad D = \pm\sqrt{2}\, i$$

$$y = (c_1 \cos\sqrt{3}x + c_2 \sin\sqrt{3}x) + (c_3 \cos\sqrt{2}x + c_4 \sin\sqrt{2}x)$$

Type II :-

Q. Solve the D.E.

$$\frac{d^2y}{dx^2} + \frac{6\,dy}{dx} + 25y = 104e^{3x}.$$

Symbolic form.

$$D^2y + 6Dy + 25y = \cancel{104e^{3x}}. \; 104e^{3x}$$

Auxillary equation.

$$D^2 + 6D + 25 = 0$$

$$\frac{-6 \pm \sqrt{36-100}}{2}$$

$$\frac{-6 \pm \sqrt{-64}}{2}.$$

$$\frac{-6 \pm 8i}{2}$$

$$-3 \pm 4i$$

Complementary function $= (c_1 \cos 4x + c_2 \sin 4x)e^{-3x}.$

particular integral $= \dfrac{1}{D^2+6D+25} \times 104e^{3x}$

$$= \frac{1}{9+18+25} \times 104e^{3x}.$$

$$= \frac{1}{\cancel{52}_1} \times \cancel{104}^{2} e^{3x}$$

$= 2e^{3x}.$

Complete Solution,

$$y = (c_1 \cos 4x + c_2 \sin 4x) e^{-3x} + 2e^{3x}.$$

Q $\dfrac{d^3y}{dx^3} + y = 3 + e^{-x} + 5e^{2x}$

Symbolic form,

$$D^3 y + y = 3 + e^{-x} + 5e^{2x}$$

A.E.

$$D^3 + 1 = 0.$$

$$(D+1)(D^2 - D + 1) = 0$$

$$D = -1, \quad \frac{1 \pm \sqrt{1-4}}{2}$$

$$\frac{1 \pm \sqrt{3}i}{2}$$

$$\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$$

$$\therefore \text{C.F} = c_1 e^{-x} + \left(c_2 \cos \frac{\sqrt{3}}{2}x + c_3 \sin \frac{\sqrt{3}}{2}x\right) e^{x/2}$$

$$\text{P.I} = \frac{1}{D^3+1} \times (3 + e^{-x} + 5e^{2x})$$

$$= \frac{3}{D^3+1} + \frac{e^{-x}}{D^3+1} + \frac{5e^{2x}}{D^3+1}.$$

$$= \frac{3e^{0x}}{D^3+1} + \frac{e^{-x}}{D^3+1} + \frac{5e^{2x}}{D^3+1}$$

$$\frac{1}{0+1}\times 3+\frac{1}{-1+1}e^{-x}+\frac{1}{8+1}5e^{2x}$$

$$3+\frac{1}{0}e^{-x}+\frac{1}{9}5e^{2x}.$$

Since $\frac{1}{0}$ is not a finite value. so, derivate the term

$$3+\frac{1}{3D^2+0}xe^{-x}+\frac{5}{9}e^{2x}$$

$$3+\frac{1}{3(-1)^2}xe^{-x}+\frac{5}{9}e^{2x}$$

$$3+\frac{1}{3}xe^{-x}+\frac{5}{9}e^{2x}$$

Complete Solⁿ.

$$y=c_1e^{-x}+\left(c_2\cos\frac{\sqrt{3}}{2}x+c_3\sin\frac{\sqrt{3}}{2}x\right)e^{x/2}$$
$$+3+\frac{1}{3}xe^{-x}+\frac{5}{9}e^{2x}.$$

Q1. $\frac{2d^3y}{dx^3}-\frac{3d^2y}{dx^2}+y=e^x+1.$

Q2. $\frac{d^2y}{dx^2}-y=\cosh x.$

Q3. $\frac{d^2y}{dx^2}+4\frac{dy}{dx}+4y=2\sinh 2x.$

Date
Page

Q1 $\frac{2d^3y}{dx^3} - \frac{3d^2y}{dx^2} + y = e^x + 1$

Symbolic form

$$2D^3y - 3D^2y + y = e^x + 1.$$

Auxillary form,

$$2D^3 - 3D^2 + 1 = 0.$$

Synthetic division,

$D = 1.$

1	2	-3	0	1
	↓	2	-1	-1
	2	-1	-1	0.

$D = 1 \quad , \quad D = 2D^2 - D - 1 = 0$

$$2D^2 - 2D + D - 1 = 0$$

$$(D-1)(2D+1) = 0$$

$$D = 1, 1, -\frac{1}{2}.$$

$$y = (c_1 + c_2 x)e^x + c_3 e^{-1/2 x}.$$

Now,

$$P.I = \frac{1}{2D^3 - 3D^2 + 1} \times e^x + 1$$

$$\frac{e^x}{2D^3 - 3D^2 + 1} + \frac{1\, e^{0x}}{2D^3 - 3D^2 + 1}$$

$$\frac{xe^x}{2-3+1} + \frac{1}{1}.$$

$$\frac{e^x}{0} + 1.$$

$$\frac{xe^x}{6D^2-6D} + 1.$$

$$\frac{xe^x}{0} + 1.$$

$$\frac{x^2e^x}{12D-6} + 1$$

$$\frac{x^2e^x}{6} + 1.$$

Complete Solution $= (c_1+c_2x)e^x + c_3e^{-1/2x} + \frac{x^2e^x}{6}+1$

Q2 $\frac{d^2y}{dx^2} - y = \cosh x.$

Sol^n

S.F $D^2y - y = \cosh x$

A.E $D^2 - 1 = 0$

$D = 1, -1.$

Complementary Sol^n =

$$y = c_1e^x + c_2e^{-x}$$

$$P.I = \cosh x$$
$$= \frac{e^x + e^{-x}}{2}.$$

$$\frac{1}{D^2 - 1} \times \left(\frac{e^x + e^{-x}}{2} \right)$$

$$\frac{e^x}{2D^2 - 2} + \frac{e^{-x}}{2D^2 - 2}$$

$$\frac{e^x}{2 - 2} + \frac{e^{-x}}{2 - 2}$$

$$\frac{e^x}{4D} + \frac{e^{-x}}{4D}.$$

$$\frac{e^x}{4} \mp \frac{e^{-x}}{4}$$

complete Solution,

$$y = C_1 e^x + C_2 e^{-x} + \left(\frac{e^x}{4} + \frac{e^{-x}}{4} \right)$$

Q3 $\frac{d^2y}{dx^2} + \frac{4dy}{dx} + \frac{4y}{y} = 2\sinh 2x.$

Symbolic form

$$D^2y + 4Dy + 4y = 2\sinh 2x.$$

Auxillary eq^n.

$$D^2 + 4D + 4 = 0$$
$$D(D+2) + 2(D+2) = 0$$
$$(D+2)(D+2) = 0$$
$$D = -2, -2$$

$$y = (C_1 + C_2 x) e^{-2x}$$

Date
Page

$$P.I = 2\sinh 2x$$

$$= \cancel{2}\times \frac{e^{2x}-e^{-2x}}{\cancel{2}}$$

$$= e^{2x}-e^{-2x}$$

$$= \frac{1}{D^2+4D+4}\left(e^{2x}-e^{-2x}\right)$$

$$= \frac{e^{2x}}{4+8+4} - \frac{e^{-2x}}{4-8+4}$$

$$= \frac{e^{2x}}{16} - \frac{e^{-2x}}{0}$$

$$= \frac{e^{2x}}{16} - \frac{e^{-2x}}{2D+4}$$

$$= \frac{e^{2x}}{16} - \frac{e^{-2x}}{-4+4}$$

$$= \frac{e^{2x}}{16} - \frac{e^{-2x}}{2.}$$

Complete solution $= (c_1+c_2x)e^{-2x} + \left(\frac{e^{2x}}{16} - \frac{e^{-2x}}{2}\right)$

$D^3 + D^2 + D + 1$

Type - III

$\frac{1}{f(D^2)} \sin ax$ or $\frac{1}{f(D^2)} \cos ax$

put $D^2 = -a^2$.

Q Solve the D.E.

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = \sin 2x.$$

Symbolic form.

$$D^2y + Dy + y = \sin 2x.$$

Auxillary eq^n

$$D^2 + D + 1 = 0.$$

$$\frac{-1 \pm \sqrt{1-4}}{2.1}.$$

$$(-1 \pm \sqrt{-3})/2$$

$$\frac{-1 \pm \sqrt{3}i}{2} \qquad -\frac{1}{2} + \frac{\sqrt{3}i}{2}, \frac{-1}{2} - \frac{\sqrt{3}}{2}i$$

$$y = (c_1 \cos^{\sqrt{3}/2} x + c_2 \sin \sqrt{3}/2^{i}) + e^{-1/2 x} c_3.$$

$$P.I = \frac{1}{D^2 + D + 1} \sin 2x.$$

$$D^2 = -4.$$

$$\frac{1}{-4 + D + 1} \sin 2x$$

$$\frac{1}{D-3}\sin 2x.$$

$$\frac{1}{D-3}\times\frac{D+3}{D+3}\sin 2x.$$

$$\frac{D+3}{D^2-9}\sin 2x$$

$$\frac{(D+3)\sin 2x}{-13}$$

$$\frac{D\sin 2x}{-13}+\frac{3\sin 2x}{\cancel{-13}}.$$

$$\frac{2\cos 2x+3\sin 2x}{-13}.$$

Complete solution:- $\left(c_1\cos\frac{\sqrt{3}}{2}x+c_2\sin\frac{\sqrt{3}}{2}x\right)+c_3e^{-1/2x}+$

$$\frac{2\cos 2x+3\sin 2x}{-13}.$$

Q. $\frac{d^2y}{dx^2}-4y=e^x+\cos 2x.$

Symbolic form.

$$D^2y-4y=e^x+\cos 2x.$$

Auxillary eq^n

$$D^2-4=0$$
$$(D+2)(D-2)=0$$
$$D=2,-2.$$

C.F. $y = c_1 e^{2x} + c_2 e^{-2x}$

$$P.I = \frac{1}{D^2-4} \times (e^x + \cos 2x)$$

$$= \frac{e^x}{D^2-4} + \frac{\cos 2x}{D^2-4} \qquad D^2 = -4$$

$$= \frac{e^x}{1-4} + \frac{\cos 2x}{-4-4}$$

$$= \frac{e^x}{-3} + \frac{\cos 2x}{-8}$$

$$= \frac{e^x}{-3} \mp \frac{\cos 2x}{8}$$

~~$y = (c_1 \cos \frac{\sqrt{3}}{2} x + c_2$~~

$$y = c_1 e^{2x} + c_2 e^{-2x} - \frac{e^x}{3} - \frac{\cos 2x}{8}$$

Type - IV

$$\frac{1}{f(D)} x^m$$

Solve this type by binomial expansion.

1. $(1-D)^{-1} = 1 + D + D^2 + D^3 + \dots$
2. $(1+D)^{-1} = 1 - D + D^2 - D^3 + \dots$

Q. Solve the D.E.

$$\frac{d^2y}{dx^2} - 4y = x^2.$$

Sol^n

Symbolic form.

$$D^2y - 4y = x^2$$

Auxillary eq^n

$$D^2 - 4 = 0$$

$$D = 2, -2.$$

$$y = C_1 e^{2x} + C_2 e^{-2x}.$$

$$P.I = \cancel{x^2} \frac{1}{D^2 - 4} \times x^2$$

$$= \frac{1}{-4\left(1 - \frac{D^2}{4}\right)} \times x^2$$

$$= -\frac{1}{4}\left(1 - \frac{D^2}{4}\right)^{-1} \times x^2.$$

$$= -\frac{1}{4}\left[1 + \frac{D^2}{4} + \left(\frac{D^2}{4}\right)^2 + \cdots\right] x^2.$$

$$\cancel{= \frac{1}{4}\left[1 + D^2\right.}$$

$$= -\frac{1}{4}\left[x^2 + \frac{2}{4} + 0\right]$$

$$= \frac{-1}{4}\left[x^2 + \frac{1}{2}\right]$$

Complete solution :-

$$y = c_1 e^{2x} + c_2 e^{-2x} - \frac{1}{4}\left[x^2 + \frac{1}{2}\right]$$

Imp

Q. $(D^2 - 4D + 4)y = x^2 + e^x + \cos 2x$

Solⁿ.

Symbolic form,

$$(D^2 - 4D + 4)y = x^2 + e^x + \cos 2x$$

Auxillary eqⁿ.

$$D^2 - 4D + 4 = 0$$

$$D^2 - (2+2)D + 4 = 0$$

$$D^2 - 2D - 2D + 4 = 0$$

$$D(D-2) - 2(D-2) = 0$$

$$(D-2)(D-2) = 0$$

$$D = 2, 2.$$

C.f $y = (C_1 + C_2 x)e^{2x}$

$$P.I. := \frac{1}{D^2 - 4D + 4} \times x^2 + \frac{e^x}{D^2 - 4D + 4} + \frac{\cos 2x}{D^2 - 4D + 4}.$$

$$= \frac{x^2}{D^2 - 4D + 4} + \frac{e^x}{1 - \cancel{4} + \cancel{4}} + \frac{\cos 2x}{-\cancel{4} - 4D + \cancel{4}}$$

$$= \frac{x^2}{4\left(1 - \frac{D^2}{4} - \frac{\cancel{4}D}{\cancel{4}}\right)} + \frac{e^x}{1} + \frac{-1}{4D} \times \cos 2x$$

$$\frac{x^2}{\frac{1}{4}\left(1+\left(\frac{D^2}{4}-D\right)\right)} + e^x - \frac{1}{4}\frac{\sin 2x}{2}.$$

$$\frac{1}{4}x^2\left(1+\left(\frac{D^2}{4}-D\right)\right)^{-1} + e^x - \frac{1}{8}\sin 2x$$

$$\frac{1}{4}x^2\left[1-\left(\frac{D^2}{4}-D\right)+\left(\frac{D^2}{4}+D\right)^2-\cdots\right] + e^x - \frac{1}{8}\sin 2x.$$

$$\frac{1}{4}\left[x^2-\left(\frac{2}{4_2}-2x\right)+\left(\frac{0}{4}+2\right)-\cdots\right] + e^x - \frac{1}{8}\sin 2x.$$

$$\frac{1}{4}\left[x^2-(-x)+2\cdots\right] + e^x - \frac{1}{8}\sin 2x.$$

Complete Solution:-

$$y=(C_1+C_2x)e^{2x} + \frac{1}{4}\left[x^2+x+2\right] + e^x - \frac{1}{8}\sin 2x.$$

Type V

$$\frac{1}{f(D)} e^{ax} \cdot V = e^{ax} \frac{1}{f(D+a)} V \quad \text{where}$$

V is an function of x.

Q. Solve the D.E.

$$\frac{d^2y}{dx^2} + y = xe^{2x}$$

$$P.I = \frac{1}{D^2+1} xe^{2x}$$

$$e^{2x} \times \frac{1}{(D+2)^2+1} x$$

$$e^{2x} \frac{1}{D^2+4+4D+1} x$$

$$e^{2x} \frac{1}{D^2+4D+5} x$$

$$\frac{e^{2x}}{5} \frac{1}{\left[1+\left(\frac{D^2}{5}+\frac{4D}{5}\right)\right]} x$$

$$\frac{e^{2x}}{5} \left[1+\left(\frac{D^2}{5}+\frac{4D}{5}\right)\right]^{-1} x$$

Date
Page

$$\frac{e^{2x}}{5}\left[1+\left(\frac{D^2}{5}+\frac{4D}{5}\right)\right]^{-1} x.$$

$$\frac{e^{2x}}{5}\left[1-\frac{D^2}{5}-\frac{4D}{5}+\left(\frac{D^2}{5}+\frac{4D}{5}\right)^2+\cdots\right]x.$$

$$\frac{e^{2x}}{5}\left[x-0-4/5\right]$$

$$P.I = \frac{e^{2x}}{5}\left[x-\frac{4}{5}\right]$$

$$D^2y + y = xe^{2x}$$

$$D^2+1=0$$

$$\cancel{D=-1,+1}. \quad D=\sqrt{-1} = \pm i$$

$$y = (c_1\cos x + c_2\sin x)e^{0x}$$

$$= (c_1\cos x + c_2\sin x)$$

Complete sol^n $= (c_1\cos x + c_2\sin x) + \frac{e^{2x}}{5}\left[x-\frac{4}{5}\right]$

Q. $\dfrac{d^2y}{dx^2} + \dfrac{2dy}{dx} + y = e^x \sin 2x.$

Sol^n

$$D^2y + 2Dy + y = e^x \sin 2x.$$

$$D^2 + 2D + 1 = 0.$$
$$D^2 + \cancel{2}(\overset{1}{\cancel{2}} + 1)D + 1 = 0$$
$$D^2 + \cancel{2}D - \cancel{2}D + 1 = 0$$
$$D(D - \cancel{2}1) - 1(D - 1) = 0$$
$$(D-1)(D-1) = 0$$
$$D = 1, 1.$$

C.F. $y = (c_1 + c_2 x)\, e^x.$

$$P.I = \frac{1}{D^2 + 2D + 1} \times e^x \sin 2x$$
$$= e^x \frac{1}{(D+1)^2 + 2(D+1) + 1} \times \sin 2x$$
$$= e^x \frac{1}{D^2 + 2D + 1 + 2D + 2 + 1} \times \sin 2x$$
$$= \frac{e^x}{D^2 + 4D + 4} \times \sin 2x.$$

$D^2 = -4.$

$$= \frac{e^x}{4}\left[\frac{-\cos 2x}{2}\right]$$

$$e^x \times \frac{1}{-4 + 4D + 4} \times \sin 2x.$$

$$e^x \times \frac{1}{4D} \times \sin 2x$$

Imp

Type VI :-

$$\frac{1}{F(D)} x V = x \frac{1}{F(D)} V + \frac{d}{dD}\left[\frac{1}{F(D)}\right] V.$$

Q. $\frac{d^2y}{dx^2} + 4y = x\sin x.$

P.I : $\frac{1}{D^2+4} x \sin x.$

$$= \frac{x \times 1}{D^2+4} \sin x + \frac{d}{dD}\left(\frac{1}{D^2+4}\right) \sin x.$$

$D^2 = -1.$

$$x \cdot \frac{1}{3} \sin x \mp \frac{2D}{(D^2+4)^2} \sin x.$$

$$\frac{x}{3} \sin x \mp \frac{2D}{9} \sin x$$

$$\frac{x}{3} \sin x - \frac{2}{9} \cos x.$$

C.f $= D^2y + 4y = x\sin x$

$D^2 + 4 = 0.$

$D^2 = -4$

$D = \pm\sqrt{-4}$

$D = \pm 2i$

$y = (c_1 \cos 2x + c_2 \sin 2x) e^x$

Complete solution :-

$$y = (c_1 \cos 2x + c_2 \sin 2x) e^{x} + \frac{x}{3} \sin x - \frac{2}{9} \cos x$$

$$P.I = \frac{1}{D^2 - 5D + 6} e^{4x}$$

$$= \frac{1}{(D-3)(D-2)} e^{4x}$$

$$= \left(\frac{A}{D-3} + \frac{B}{D-2} \right) e^{4x}$$

$$\frac{1}{(D-3)} e^{4x} - \left(\frac{1}{D-2} \right) e^{4x}$$

$$= e^{3x} \int e^{-3x} e^{4x} dx - e^{2x} \int e^{-2x} e^{4x} dx.$$

$$= e^{3x} \int e^{x} dx - e^{2x} \int e^{2x} dx$$

$$= e^{3x} e^{x} - \frac{e^{2x} e^{2x}}{2}$$

$$= e^{4x} - \frac{e^{4x}}{2}$$

$$= \frac{1}{2} e^{4x}.$$

$$\frac{1}{D+a} x = e^{-ax} \int e^{ax} \times dx$$

Date
Page

$$\frac{1}{D-a} x = e^{ax} \int e^{-ax} \times dx$$

Q. $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = e^{4x}$

Sol^n.

$$D^2y - 5Dy + 6 = e^{4x}.$$

$$D^2 - 5D + 6 = \cancel{e^{4x}}.0$$

$$D^2 - D(6-1) + 6 = 0$$
$$D^2 - 6D + D + 6 = 0$$
$$D(D-6) - 1(D-6) = 0$$
$$(D-6)(D-1) = 0$$
$$D = 6, 1$$

$$D^2 - D(2+3) + 6 = 0$$
$$D^2 - 2D - 3D + 6 = 0$$
$$D(D-2) - 3(D-2) = 0$$
$$D = +2, +3.$$

$$y = c_1e^{2x} + c_2e^{3x}.$$

$$P.I = \frac{1}{D^2-5D+6} \times e^{4x}$$

$$= \frac{1}{4^2 - 5\times4 + 6} e^{4x}$$

$$= \frac{1}{16-20+6} e^{4x}$$

$$= \frac{1}{2} e^{4x}.$$

Complete Solution :-

$$y = c_1 e^{2x} + c_2 e^{3x} + \frac{1}{2} e^{4x}.$$

Q $\dfrac{d^2y}{dx^2} + a^2 y = \sec ax.$

Sol^n

$$D^2 y + a^2 y = \sec ax.$$
$$D^2 + a^2 = 0$$
$$D^2 = \pm ai$$

$$y = (c_1 + c_2 x) e^{ax}$$

$$P.I = \frac{1}{D^2 + a^2} \times \sec ax.$$

Method of Variation of parameters :-

Working Rule :-

let the eqn be $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$ — (1)

Where, P, Q and R are the function of x only.

(1) find two independent solutions of the equation (1) and denote them by y_1 and y_2.

(2) find C.F $= Ay_1 + By_2$ where A and B are arbitary Constant.

(3) find $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$, $u = -\int \frac{y_2 R}{W} dx + c_1$.

$v = \int \frac{y_1 R}{W} dx + c_2$.

(4) Replace the arbitary constants = A and B in C.F $= Ay_1 + By_2$ by functions u and v so that the complete solution is $y = uy_1 + vy_2$

Date
Page

Q. Apply the method of variation of parameters to solve $\frac{d^2y}{dx^2} + 4y = \tan 2x$

Sol^n

$$D^2 + 4 = 0$$
$$D^2 = -4$$
$$D = \pm 2i$$

$$y = (A\cos 2x + B\sin 2x)$$
$$= Ay_1 + By_2.$$

let the complete solution be.
$$y = uy_1 + vy_2.$$

~~Thus, the complete~~

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$= \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix}$$

$$= 2(1)$$

$$u = -\int \frac{y_2 R}{W}\, dx + C_1$$

$$= -\int \frac{\sin 2x \times \tan 2x}{2}\, dx + C_1.$$

$$= -\int \frac{\sin^2 2x}{2\cos 2x}\, dx + C_1$$

$$= -\int \frac{1 - \cos^2 x}{2\cos 2x}\, dx + C_1$$

$$= -\frac{1}{2}\left[\int (\sec 2x - \cos 2x)\, dx\right] + c_1$$

$$= -\frac{1}{2}\left[\frac{\log|\sec 2x + \tan 2x|}{2} - \frac{\sin 2x}{2}\right] + c_1.$$

$$v = \int \frac{y_1 R}{W}\, dx + c_1$$

$$= \int \frac{\cos 2x \times \tan 2x}{2.}\, dx + c_2$$

$$= \int \frac{\sin 2x}{2}\, dx + c_2.$$

$$= \frac{1}{2} \cdot \frac{-\cos 2x}{2} + c_2.$$

$$= -\frac{\cos 2x}{4} + c_2.$$

Thus, the complete solution is.

$$y = \left[-\frac{1}{2}\left(\frac{\log|\sec 2x + \tan 2x|}{2} - \frac{\sin 2x}{2}\right) + c_1\right]\cos 2x + \left(-\frac{\cos 2x}{4} + c_2\right)\sin 2x.$$

Simultaneous linear Equation with Constant Coefficient :-

Q. $\frac{dx}{dt} + 5x + y = e^t$

$\frac{dy}{dt} - x + 3y = e^{2t}$

Sol^n

In Operator form,

$$Dx + 5x + y = e^t$$
$$Dy - x + 3y = e^{2t}$$

$$(D+5)x + y = e^t$$
$$-x + (D+3)y = e^{2t} \quad — \times (D+5)$$

$$\cancel{(D+5)}x + y = e^t$$
$$-\cancel{(D+5)}x + (D+3)(D+5) = 2e^{2t} + 5e^{2t}$$

$$[1 + D^2 + 8D + 15)y = 2e^{2t} + 5e^{2t} + e^t$$
$$(D^2 + 8D + 16)y = 7e^{2t} + e^t$$

$$D^2 + 8D + 16 = 0$$
$$D = -4, -4$$

$$C.F = (C_1 + C_2 t)e^{-4t}$$

$$P.I = \frac{1}{D^2 + 8D + 16}(7e^{2t}) + \frac{1}{D^2 + 8D + 16}e^t$$

$$= \frac{1}{4+16+16} 7e^{2t} + \frac{1}{1+8+16} e^{t}.$$

$$\frac{7e^{2t}}{36} + \frac{1}{25} e^{t}.$$

$$y = (c_1 + c_2 t)e^{2t} + \frac{7e^{2t}}{36} + \frac{1}{25} e^{t}$$

putting value of y in eq^n (2).

$$\frac{dy}{dt} + 3y - e^{2t} = x.$$

$$x = \frac{d(c_1+c_2t)e^{2t}}{dt} + 3\left((c_1+c_2t)e^{2t} + \frac{7e^{2t}}{36} + \frac{1}{25}e^{t}\right) - e^{2t}.$$

$$= \left[(c_1+c_2t)\frac{de^{2t}}{dt} + e^{2t}\frac{d(c_1+c_2t)}{dt}\right] + 3(c_1+c_2t)e^{2t} +$$

$$\frac{1}{12} \times 7e^{2t} + \frac{3}{25}e^{t} - e^{2t}$$

$$= 2(c_1+c_2t)e^{2t} + e^{2t}(0 + c_2) + 3(c_1+c_2t)e^{2t} +$$

$$\frac{1}{12} \times 7e^{2t} + \frac{3}{25} e^{t} - e^{2t}.$$

$$= 5(c_1+c_2t)e^{2t} + c_2e^{2t} + \frac{7}{12}e^{2t} + \frac{3}{25}e^{t} - e^{2t}$$

$$= 5(c_1+c_2t)e^{2t} + e^{2t}\left(c_2 + \frac{7}{12} - 1\right) + \frac{3e^{t}}{25}$$