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{Mathematics}
unit - II.

Exact Differential Equation

$$Mdx + Ndy = 0$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

D.E
ODE $\frac{dy}{dx}$
PDE $\frac{\partial^2 z}{\partial y \partial x}$

$$\int_{y\ \text{constant}} M\,dx + \int \begin{pmatrix} N \text{ terms} \\ \text{not containing} \\ x \end{pmatrix} dy = c$$

Q Solve the D.E $\underbrace{(1+e^{x/y})}_{M}dx + \underbrace{e^{x/y}\left(1-\frac{x}{y}\right)}_{N}dy = 0$

$$\frac{\partial M}{\partial y} = 0 + e^{x/y}\left(\frac{-x}{y^2}\right)$$

$$\frac{\partial N}{\partial x} = e^{x/y}\left(0-\frac{1}{y}\right) + \left(1-\frac{x}{y}\right)e^{x/y}\left(\frac{1}{y}\right)$$

$$= \cancel{-\frac{e^{x/y}}{y}} + \cancel{\frac{e^{x/y}}{y}} - e^{x/y} \times \frac{x}{y^2}$$

$$= -e^{x/y} \times \frac{x}{y^2}$$

Hence $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$$\int_{y\ \text{constant}} M\,dx + \int N(\text{not } x)\,dy = c$$

$$\int_{y\ \text{const}} (1+e^{x/y})\,dx + \int 0\,dy = c.$$

$$x + \frac{e^{x/y}}{\frac{1}{y}} = c$$

$$\boxed{x + ye^{x/y} = c}$$

Q Solve the D.E

$$(x^4 - 2xy^2 + y^4)\,dx - (2x^2y - 4xy^3 + \sin y)\,dy = 0$$

$$\underbrace{(x^4 - 2xy^2 + y^4)}_{M}\,dx + \underbrace{(-2x^2y + 4xy^3 - \sin y)}_{N}\,dy = 0$$

$$\frac{\partial M}{\partial y} = -4xy + 4y^3$$

$$\frac{\partial N}{\partial x} = -4xy + 4y^3$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

The sol^n of D.E is

$$\int_{y\text{ const}} (x^4 - 2xy^2 + y^4)\,dx + \int -\sin y\,dy = c$$

$$\boxed{\frac{x^5}{5} - \frac{2x^2}{2}y^2 + xy^4 + \cos y = c}$$

* Equation Reducible to Exact D.E

* Integrating factor (I.F)

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Q using IF $\frac{1}{x^2}$ find the solution of D-E

$$\underbrace{(x^2+y^2)}_{M}dx - \underbrace{2xy}_{-N}\,dy = 0 \quad \rightarrow (1)$$

$$\frac{\partial M}{\partial y} = 2y$$

$$\frac{\partial N}{\partial x} = -2y$$

Not exact

Multiply IF $\frac{1}{x^2}$ to eqn (1)

$$\left(\frac{x^2+y^2}{x^2}\right)dx - \frac{2xy}{x^2}\,dy = 0$$

$$\left(1+\frac{y^2}{x^2}\right)dx - \frac{2y}{x}\,dy = 0$$

$$\frac{\partial M}{\partial y} = \frac{2y}{x^2} \qquad \frac{\partial N}{\partial x} = \frac{2y}{x^2}$$

exact

$$\int\left(1+\frac{y^2}{x^2}\right)dx + \int 0\,dy = c$$

$$x - \frac{y^2}{x} = c$$

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Rules for finding Integrating factor:-

Rule (1) If $M(x,y)$ and $N(x,y)$ are homogeneous function in $(x \& y)$ and the equation $Mdx + Ndy = 0$ is not exact then $\frac{1}{Mx+Ny}$ is an Integrating factor

Provided $Mx + Ny \neq 0$

Q Solve $\underbrace{x^2y}_{M}\,dx - \underbrace{(x^3+y^3)}_{N}\,dy = 0 \quad \rightarrow (1)$

$\frac{\partial M}{\partial y} = x^2 \qquad \frac{\partial N}{\partial x} = -3x^2$

Not exact

$Mx + Ny$

$x^3y - x^3y - y^4$

$= -y^4$

$\frac{1}{Mx+Ny} = \frac{-1}{y^4}$

Multiply I.F to eqn (1)

$-\frac{x^2y}{y^4}\,dx + \left(\frac{x^3}{y^4} + \frac{y^3}{y^4}\right)dy = 0$

$-\frac{x^2}{y^3}\,dx + \left(\frac{x^3}{y^4} + \frac{1}{y}\right)dy = 0$

$\frac{\partial M}{\partial y} = \frac{3x^2}{y^4} \;\Big|\; \frac{\partial N}{\partial x} = \frac{3x^2}{y^4}$

exact

Solu:-

$\int_{y\,const} -\frac{x^2}{y^3}\,dx + \int \frac{1}{y}\,dy = C$

$\boxed{\frac{-x^3}{3y^3} + \log y = c}$

Rule-2

Q

Rule-2 If the eq^n $Mdx + Ndy = 0$ is not exact and is of the form $f(xy)y\,dx + g(xy)x\,dy = 0$ then $\frac{1}{Mx - Ny}$ is and I.F provided

$Mx - Ny \neq 0$

Q Solve:- $\underbrace{(xy^2 + 2x^2y^3)}_{M} dx + \underbrace{(x^2y - x^3y^2)}_{N} dy = 0$ ↳ (1)

$$\frac{\partial M}{\partial y} = 2xy + 6x^2y^2$$

$$\frac{\partial N}{\partial x} = 2xy - 3x^2y^2$$

Not exact

$Mx - Ny = \cancel{x^2y^2} + 2x^3y^3 - \cancel{x^2y^2} + x^3y^3$

$Mx - Ny = 3x^3y^3$

$I.F = \frac{1}{3x^3y^3}$

Multiply I.F to eq^n (1)

$$\left(\frac{xy^2}{3x^3y^3} + \frac{2x^2y^3}{3x^3y^3}\right)dx + \left(\frac{x^2y}{3x^3y^3} - \frac{x^3y^2}{3x^3y^3}\right)dy = 0$$

$$\left(\frac{1}{3x^2y} + \frac{2}{3x}\right)dx + \left(\frac{1}{3xy^2} - \frac{1}{3y}\right)dy = 0$$

$$\frac{\partial M}{\partial y} = \frac{1}{-3x^2y^2} \qquad \frac{\partial N}{\partial x} = \frac{-1}{3x^2y^2}$$

Solve

$$\int_{y\ const} \frac{1}{3x^2y} + \frac{2}{3x} - \frac{1}{3}\int \frac{1}{y}\,dy = c$$

$$\int_{y\ cons} \frac{1}{3x^2y} + \int_{y\ const} \frac{2}{3x} + \frac{1}{3}\int \frac{1}{y}\,dy = c$$

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$$\frac{-1}{3xy} + \frac{2}{3}\log x - \frac{1}{3}\log y = c$$

$$\frac{1}{3}\left(\frac{-1}{xy} + 2\log x - \log y\right) = c$$

Q $(1+xy)y\,dx + (1-xy)x\,dy = 0$

$$\frac{\partial M}{\partial y} = y + xy^2 \qquad \frac{\partial N}{\partial x} = x - x^2 y$$

$$= 1 + 2xy \qquad \frac{\partial N}{\partial x} = 1 - 2xy$$

$$\frac{1}{Mx - Ny} = ? \qquad \text{Not exact}$$

$$Mx - Ny = \cancel{yx} + x^2y^2 - \cancel{xy} + x^2y^2$$

$$= 2x^2y^2$$

$$\frac{1}{Mx-Ny} = \frac{1}{2x^2y^2}$$

$$\left(\frac{y}{2x^2y^2} + \frac{xy^2}{2x^2y^2}\right)dx + \left(\frac{x}{2x^2y^2} - \frac{x^2y}{2x^2y^2}\right)dy = 0$$

$$\left(\frac{1}{2x^2y} + \frac{1}{2x}\right)dx + \left(\frac{1}{2xy^2} - \frac{1}{2y}\right)dy = 0$$

$$\frac{\partial M}{\partial y} = \frac{-1}{2y^2x^2} \qquad \frac{\partial N}{\partial x} = \frac{-1}{2x^2y^2}$$

exact

$$\int \frac{1}{2x^2y}$$

Rule-3) If $M\,dx + N\,dy = 0$ is not exact and

$\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}$ is function of x only $= f(x)$

Then $e^{\int f(x)\,dx}$ is an I.F

Rule-4 → If $M\,dx + N\,dy = 0$ is not exact and

$\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y} \Big/ M$ is function of y only $= f(y)$

Then $e^{\int f(y)\,dy}$ is an I.F

Q Solve

$$(x^2+y^2+2x)\,dx + 2y\,dy = 0$$

$\dfrac{\partial M}{\partial y} = 2y$ $\qquad$ $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}$

$\dfrac{\partial N}{\partial x} = 0$ $\qquad$ $\dfrac{2y-0}{2y} = \dfrac{2y}{2y} = 1 = x^0$

Not exact $\qquad$ $1 = f(x)$

If $e^{\int f(x)\,dx} = e^{\int 1\,dx} = e^x$

$$(x^2+y^2+2x)e^x\,dx + 2ye^x\,dy = 0$$

$\dfrac{\partial M}{\partial y} = 2ye^x$

$\dfrac{\partial N}{\partial x} = 2y\,e^x$

~~Not~~ exact

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$$\int_{y\ \text{const}} M\,dx + \int N \underline{\quad} x\ \text{only} = c$$

$$\int_{y\ \text{const}} (x^2e^x + y^2e^x + 2xe^x)\,dx + \int 0\,dy = c$$

$$x^2e^x - \cancel{\int 2xe^x\,dx} + y^2e^x + \cancel{\int 2xe^x\,dx} = c$$

$$e^x(x^2+y^2) = c$$

Q Solve

$$\underbrace{(3x^2y^4 + 2xy)}_{M}\,dx + \underbrace{(2x^3y^3 - x^2)}_{N}\,dy = 0$$

$$\frac{\partial M}{\partial y} = 4\times 3x^2y^3 + 2x = 12x^2y^3 + 2x$$

$$\frac{\partial N}{\partial x} = 6x^2y^3 - 2x$$

Not exact

$$\frac{\partial M}{\partial x} = 6xy^4 + 2y$$

$$\frac{\partial N}{\partial y} = 6x^3y^2$$

$$= \frac{6xy^4 + 2y - 6x^3y^2}{3x^2y^4 + 2xy} = \frac{6xy^4 + 2y - 6x^3y^2}{3x^2y^4 + 2xy}$$

$$= \frac{6xy^3 + 2 - 6x^3y}{3x^2y + 2x}$$

$$I.F = \frac{1}{y^2} \qquad 2xy^4$$

$$M = 3x^2y^2 + \frac{2x}{y} \qquad N = 2x^3y - \frac{x^2}{y^2}$$

$$\frac{\partial M}{\partial y} = 6x^2y - \frac{2x}{y^2} \qquad \frac{\partial N}{\partial x} = 6x^2y - \frac{2x}{y^2}$$

$$\int_{y\ \text{const}} M\,dx + \int N\,(\text{not } x)\,dy = c$$

$$\int 3x^2y^2\,dx + \int \frac{2x}{y}\,dx + \int 0\,dy = c$$

$$\cancel{3}y^2 \left| \frac{x^3}{\cancel{3}} \right| + \frac{\cancel{2}}{y} \left| \frac{x^2}{\cancel{2}} \right| \quad \cancel{0y} = c$$

$$x^3y^2 + \frac{x^2}{y} \cancel{+0y} = c$$

$$\frac{x^3y^3 + x^2}{y} \cancel{+0y} = c$$

$$\frac{x^3y^3 + x^2}{y} = c$$

$$\boxed{x^3y^3 + x^2 = cy}$$

Q Solve

$(x^2+y^2+x)\,dx + xy\,dy = 0$

Rule-3

$\frac{\partial M}{\partial y} = 2y$

$\frac{\partial N}{\partial x} = y$

Not exact

using Rule-3

$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2y-y}{xy} = \frac{\cancel{y}}{x\cancel{y}} = \frac{1}{x}$

$e^{\int \frac{1}{x} dx}$ $\quad e^{\log x} = x$

$I.F = x$

$M = x^3 + xy^2 + x^2$

$N = x^2 y$

$\frac{\partial M}{\partial y} = 2xy$

$\frac{\partial N}{\partial y} = 2xy$

$Sol^n = \int_{y\,const} M\,dx + \int N(\text{not } x)\,dy = c$

$\int x^3 + \int xy^2 + \int x^2 + \int 0\,dy = c$

$\cancel{3x^2 + y^2} + 2x$

$\frac{x^4}{4} + \frac{x^2y^2}{2} + \frac{x^3}{3} = c$

$\frac{3x^4 + 6x^2y^2 + 4x^3}{12} = c$

$3x^4 + 6x^2y^2 + 4x^3 = c$

$2 | 4 - 2 - 3$
$2 | 2 - 1 - 3$
$3 | 1 - 1 - 3$
$\;\;\; 1 - 1 - 1$

$= 12$

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Q. $2\cos(y^2)\,dx - xy\sin(y^2)\,dy = 0$

$M = 2\cos y^2 \qquad N = xy\sin y^2$

$\frac{\partial M}{\partial y} = -4y\sin y^2 \qquad \frac{\partial N}{\partial x} = -y\sin y^2$

Not exact

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{-4y\sin y^2 + y\sin y^2}{-xy\sin y^2} = \frac{3}{x} = f(x)$$

$$e^{\int f(x)dx}$$

$$= e^{3\log x} = x^3 = I.f$$

Solve $(x^2+y^2+2x)\,dx + 2y\,dy = 0$

$M = x^2+y^2+2x \qquad N = 2y$

$\frac{\partial M}{\partial y} = 2y \qquad \frac{\partial N}{\partial x} = 0$

Not exact

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Rule-5 If the E_q^n $Mdx + Ndy = 0$ can be expressed as $x^a y^b (my\,dx + nx\,dy) + x^c y^d (py\,dx + qx\,dy) = 0$ where a, b, c, d, m, n, p, q are constants and $\frac{m}{n} \neq \frac{p}{q}$, then $x^\alpha y^\beta$ is an I.f where $\alpha + \beta$ are so chosen that

$$= \frac{\alpha + a + 1}{m} = \frac{b + \beta + 1}{n}$$

and

$$\frac{c + \alpha + 1}{p} = \frac{d + \beta + 1}{q}$$

Q $(y^3 - 2yx^2)dx + (2xy^2 - x^3)dy = 0$

$(y^3 dx - 2xy^2 dy) + (-2yx^2 dy - x^3 dy) = 0$

$x^0 y^2 (y\,dx + 2x\,dy) + x^2 y^0 (-2y\,dx - x\,dy) = 0$

on comparing

$a = 0,\quad b = 2,\quad c = 2,\quad d = 0$

$m = 1,\quad n = 2,\quad p = -2,\quad q = -1$

$\frac{m}{n} \neq \frac{p}{q}$

$x^\alpha y^\beta$ is I.f

$\frac{a + \alpha + 1}{m} = \frac{b + \beta + 1}{n}$

$\frac{0 + \alpha + 1}{1} = \frac{2 + \beta + 1}{2}$

$2\alpha + 2 = 2 + \beta + 1$

$2\alpha - \beta = 1$ — (1)

$\frac{c + \alpha + 1}{p} = \frac{d + \beta + 1}{q}$

$\frac{2 + \alpha + 1}{-2} = \frac{0 + \beta + 1}{-1}$

$2 + \alpha + 1 = 2\beta + 2$

$2\beta - \alpha = 1$ — (2)

Solving (1) & (2) [(1) + (2) 2]

$2\alpha - \beta = 1$

$-2\alpha + 4\beta = 2$

$3\beta = 3$

$\boxed{\beta = 1}$

$\alpha = 1$

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$(y^3 - 2yx^2)\,dx + (2xy^2 - x^3)\,dy = 0$

xy is integrating factor.

$M = xy^4 - 2y^2x^3$ | $N = 2x^2y^3 - x^4y$

$\frac{\partial M}{\partial y} = 4xy^3 - 4yx^3$ | $\frac{\partial N}{\partial x} = 4xy^3 - 4yx^3$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \rightarrow \text{Exact}$$

$$\int_{y\ \text{const}} (xy^4 - 2y^2x^3)\,dx + \int 0\,dy = c$$

$$\frac{x^2y^4}{2} - \frac{y^2x^4}{2} = c$$

$$x^2y^4 - y^2x^4 = 2c$$

$$x^2y^4 - y^2x^4 = c'$$

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* Linear Differential Eq^n

A differential eq^n of the form $\frac{dy}{dx} + Py = Q$

is called linear differential equation, where P & Q are functions of x (but not of y) or constants

$e^{\int P dx}$ is called I.F

Solution of $y(I.F)$ is $\int Q \, I.F \, dx + c$

Q $(x+1)\frac{dy}{dx} - y = e^x(x+1)^2$

$$\frac{dy}{dx} - \frac{y}{x+1} = e^x(x+1)$$

$$P = \frac{-1}{x+1}, \quad Q = e^x(x+1)$$

$$I.F = e^{\int \frac{-1}{x+1} dx} = e^{-\log(x+1)} = e^{\log(x+1)^{-1}}$$

$$= \frac{1}{x+1}$$

$$I.F = \frac{1}{x+1}$$

$$y \cdot \frac{1}{x+1} = \int e^x (x+1) \frac{1}{(x+1)} dx + c$$

$$\frac{y}{(x+1)} = e^x + c$$

$$y = (e^x + c)(x+1)$$

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Solve:-

$$(x^3-x)\frac{dy}{dx}-(3x^2-1)y=x^5-2x^3+x$$

$$\frac{dy}{dx}-\frac{(3x^2-1)y}{(x^3-x)}=\frac{x^5-2x^3+x}{x^3-x}$$

$$\frac{dy}{dx}-\frac{3x^2-1}{x^3-x}y=\frac{x^4-2x^2+1}{x^2-1}$$

$$P=\frac{-(3x^2-1)}{x^3-x},\quad Q=\frac{x^4-2x^2+1}{x^2-1}=\frac{(x^2-1)^{\cancel{2}}}{\cancel{(x^2-1)}}$$

$$e^{\int-\frac{(3x^2-1)}{x^3-x}dx}$$

$$x^3-x=t$$

$$(3x^2-1=dt$$

$$e^{\int-\frac{1}{t}dt}$$

$$I.F=\frac{1}{t}=\frac{1}{x^3-x}$$

$$y\cdot\left(\frac{1}{x^3-x}\right)=\int(x^2-1)\left(\frac{1}{x^3-x}\right)dx+c$$

$$=\int\frac{(x^2-1)}{(x^3-x)}dx+c$$

$$=\int\frac{\cancel{x^2-1}}{x\cancel{(x^2-1)}}dx+c$$

$$y=(\log x+c)(x^3-x)$$

* Bernoulli's theorem

The equation of the form

$$\frac{dy}{dx} + Py = Qy^n$$

where P & Q are consts of function of x can be reduced to the linear form on dividing by y^n.

Q Solve $x^2 \frac{dy}{dx} + y(x+y) = 0$

$$x^2 \frac{dy}{dx} + xy + y^2 = 0$$

$$x^2 \frac{dy}{dx} + xy = -y^2$$

$$\frac{dy}{dx} + \frac{xy}{x^2} = \frac{-y^2}{x^2}$$

$$\frac{1}{y^2}\frac{dy}{dx} + \frac{1}{xy} = \frac{-1}{x^2}$$

$$\frac{1}{y} = z$$

$$\frac{-1}{y^2}\frac{dy}{dx} = \frac{dz}{dx}$$

$$\frac{1}{y^2}\frac{dy}{dx} = -\frac{dz}{dx}$$

$$-\frac{dz}{dx} + \frac{z}{x} = \frac{-1}{x^2}$$

$$\boxed{\frac{dy}{dx} + Py = Q}$$

$P = \frac{-1}{x}$; $Q = \frac{1}{x^2}$

$P = \frac{-1}{x}$. $Q = \frac{1}{x^2}$

$$I.F = e^{\int P\,dx} = e^{\int \frac{-1}{x} dx} = e^{\log x^{-1}} = x^{-1} = \frac{1}{x}$$

Sol^n

$$z \cdot I.F = \int Q \cdot I.F \, dx + c$$

$$z \cdot \frac{1}{x} = \int \frac{1}{x^2} \times \frac{1}{x} dx + c$$

$$\frac{z}{x} = \frac{-1}{2x^2} + c$$

$$\frac{1}{yx} = \frac{-1}{2x^2} + c$$

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Q Solve $x\frac{dy}{dx} + y\log y = xye^{x}$

$$\frac{dy}{dx} + \frac{y}{x}\log y = \frac{xye^{x}}{x}$$

$$\frac{1}{y}\frac{dy}{dx} + \frac{1}{x}\log y = e^{x}$$

$$\log y = z$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{dz}{dx}$$

$$\frac{dz}{dx} + \frac{1}{x}z = e^{x}$$

$$\frac{dy}{dx} + Py = Q$$

$$P = \frac{1}{x},\ Q = e^{x}$$

$$I.F = e^{\int \frac{1}{x}dx} = e^{\log x} = x$$

Sol^n

$$z \cdot I.F = \int Q \cdot I.F\, dx + c$$

$$z \cdot x = \int e^{x} \cdot x \cdot dx + c$$

$$zx = xe^{x} - e^{x} + c$$

$$\log y\, x = (x-1)e^{x} + c$$

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* Orthogonal trajectory.

A curve is called orthogonal trajectory if it cuts every member of given family of curves at right angles

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★ Orthogonal trajectories in Cartesian coordinates
working rule

(1) Let the equation of the family of given curve be
$f(x, y, c) = 0$ — (1)
Differentiate (1) w.r.t x and then eliminate the arbitrary constants between this derived eqn. & given eqn.

(2) Replace $\frac{dy}{dx}$ to $-\frac{dx}{dy}$ in eqn obtain in step (1)

(3) Integrate the eqn obtained in step II.

Q Find the orthogonal trajectories of family of parabolas

$$y = ax^2 \text{ — (1)}$$

Soln. Diff (1) w.r.t x

$$\frac{dy}{dx} = 2ax \text{ — (2)}$$

Eliminate 'a' from (1) & (2)

$$y = x^2 \times \frac{1}{2x} \frac{dy}{dx}$$

$$y = \frac{x}{2} \frac{dy}{dx} \qquad 2y\,dx = x\,dy$$

Step II Replace $\frac{dy}{dx}$ to $-\frac{dx}{dy}$

$$y = -\frac{x}{2} \frac{dx}{dy}$$

$$2y\,dy = -x\,dx \text{ — (3)}$$

Step III = Integrate (3)

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$$\int 2y\,dy = -\int x\,dx$$

$$\frac{2y^2}{2} = -\frac{x^2}{2} + c$$

$$y^2 = \frac{-x^2}{2} + c$$

$$y^2 + \frac{x^2}{2} = c$$

$$x^2 + 2y^2 = c$$

Q for Practice

(1) $y = ax^3$
(2) $y = ax^n$
(3) $x^2 + y^2 = a^2$
(4) $y^2 = 4ax$

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* Self orthogonal - If $f(x, y, c) = 0$ is a family of curves and its orthogonal trajectories are also the same family of curves $f(x, y, c) = 0$ then such a family of curves is called self orthogonal.

Q Show that the system of confocal conics $\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1$ is self orthogonal where λ is constant

Sol^n: Diff (1) w.r.t x

$$\frac{2x}{a^2+\lambda} + \frac{2y\frac{dy}{dx}}{b^2+\lambda} = 0$$

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$$xb^2 + x\lambda + a^2y\frac{dy}{dx} + \lambda y\frac{dy}{dx} = 0$$

$$\lambda\left(x + y\frac{dy}{dx}\right) = -xb^2 - a^2y\frac{dy}{dx}$$

$$\lambda = -\frac{\left(xb^2 + a^2y\frac{dy}{dx}\right)}{\left(x + y\frac{dy}{dx}\right)}$$

Now,

$$a^2 + \lambda = a^2 - \frac{\left(xb^2 + a^2y\frac{dy}{dx}\right)}{\left(x + y\frac{dy}{dx}\right)}$$

$$a^2 + \lambda = \frac{a^2x - b^2x}{x + y\frac{dy}{dx}} = \frac{(a^2 - b^2)x}{x + y\frac{dy}{dx}} \quad -(2)$$

$$b^2 + \lambda = b^2 - \frac{\left(xb^2 + a^2y\frac{dy}{dx}\right)}{x + y\frac{dy}{dx}}$$

$$b^2 + \lambda = \frac{-(a^2 - b^2)y\frac{dy}{dx}}{x + y\frac{dy}{dx}} \quad -(3)$$

Eliminating λ after Putting (2) & (3) in (1)

$$\frac{x^2}{\frac{(a^2 - b^2)x}{x + y\frac{dy}{dx}}} + \frac{y^2}{\frac{-(a^2 - b^2)y\frac{dy}{dx}}{x + y\frac{dy}{dx}}} = 1$$

$$\frac{\left(x+y\frac{dy}{dx}\right)}{a^2-b^2}\left[x-\frac{y}{\frac{dy}{dx}}\right]=1$$

$$\Rightarrow \left(x+y\frac{dy}{dx}\right)\left(x-y\frac{dx}{dy}\right)=a^2-b^2$$

$$\frac{dy}{dx} \Rightarrow -\frac{dx}{dy}$$

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Orthogonal trajectories for Polar co-ordinates

(1) Let the family of Polar curves be, $\phi(r, \theta, c) = 0$ —(1)

Diff (1) w.r.t θ

and eliminate Parameter

(2) Replace $\frac{dr}{d\theta} \rightarrow -r^2 \frac{d\theta}{dr}$.

(3) Then integrate

Q Find orthogonal trajectories of the cardiod $r = a(1-\cos\theta)$ —(1)

where (a) is parameter.

Solⁿ. $\frac{dr}{d\theta} = a\sin\theta$ —(2)

Eliminate (a) b/w (1) & (2)

$r = \frac{1}{\sin\theta} \frac{dr}{d\theta} (1-\cos\theta)$

$\frac{1}{r} \frac{dr}{d\theta} = \frac{\sin\theta}{1-\cos\theta} = \frac{2\sin\theta/2 \cos\theta/2}{2\sin^2\theta/2} = \cos\theta/2$

Step-2 Replace $\frac{dr}{d\theta} \rightarrow r^2 \frac{d\theta}{dr}$

$\frac{-r^2}{r} \frac{d\theta}{dr} = \cos\theta/2$

$-r \frac{d\theta}{dr} = \cos\theta/2$

$-\frac{d\theta}{\cos\theta/2} = \frac{dr}{r}$

$-\tan\theta/2 \, d\theta = \frac{dr}{r}$

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Step III Integrate

$$\int -\tan\theta/2 \, d\theta = \int \frac{dx}{x}$$

$$\log \frac{\cos\theta/2}{1/2} + c = \log x$$

$$2\log\cos\frac{\theta}{2} + c = \log x$$

$$\cancel{\log}\cos^2\frac{\theta}{2} + c = \cancel{\log} x$$

$$x = c\cos^2\frac{\theta}{2}$$

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☆ Equations not of first degree :-
Equations solvable for p. Here we put $\frac{dy}{dx} = p$

Q Solve for the D.E

$$\left(\frac{dy}{dx}\right)^2 - 5\frac{dy}{dx} + 6 = 0$$

$$p^2 - 5p + 6 = 0$$

$$(p-3)(p-2) = 0$$

$$p = 3, 2$$

$p = 3$

$$\frac{dy}{dx} = 3$$

$$\int dy = \int 3\,dx$$

$$y = 3x + c_1$$

$$(y - 3x - c_1) = 0$$

$p = 2$

$$\frac{dy}{dx} = 2$$

$$\int dy = \int 2\,dx$$

$$y = 2x + c_2$$

$$y - 2x - c_2 = 0$$

$$(y - 3x - c_1)\cdot(y - 2x - c_2) = 0$$

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Q Solve for D.E

$$p^3 + 2xp^2 - y^2p^2 - 2xy^2p = 0$$

$$p(p^2 + 2xp - y^2p - 2xy^2) = 0$$

$$p = 0 \quad \Big| \quad p(p+2x) - y^2(p-2x) = 0$$

$$(p - y^2)(p + 2x) = 0$$

$$p = 0 \;/\; p = y^2 \;/\; p = -2x$$

$$\int \frac{dy}{dx} = \int 0 \qquad \frac{dy}{dx} = y^2 \qquad \frac{dy}{dx} = -2x$$

$$y = c_1 \qquad \int \frac{dy}{y^2} = \int dx \qquad \int dy = \int -2x\,dx$$

$$y - c_1 = 0 \qquad -\frac{1}{y} = x + c_2 \qquad y = -\frac{2x^2}{2} + c_3$$

$\hookrightarrow$ (1)

$$-\frac{1}{y} - x - c_2 = 0 \qquad y + x^2 - c_3 = 0$$

$\hookrightarrow$ (3)

$$\frac{1}{y} + x + c_2 = 0$$

$\hookrightarrow$ (2)

$$(y - c_1)\left(\frac{1}{y} + x + c_2\right)(y + x^2 - c_3) = 0$$

☆ Equations solvable for y

Working Rule

(1) Express the given equation in the form of

$$y = f(x, p) \quad -(1)$$

(2) Diff (1) w.r.t x and replace $\frac{dy}{dx}$ by p. So that the equation has two variables p and x

(3) Solve the eqn obtained in step 2.

(4) Let the solution of eqn be $\phi(x,p,c) = 0$

(5) Eliminate p from step (1) & step (4)

Q Solve the D.E.

$$y = 2px + p^4x^2$$

$$\frac{dy}{dx} = 2p(1) + 2x\frac{dp}{dx} + p^4 \times 2x + x^2 \times 4p^3\frac{dp}{dx}$$

$$p = 2p + 2x\frac{dp}{dx} + 2p^4x + 4x^2p^3\frac{dp}{dx}$$

$$p + 2x\frac{dp}{dx} + 2p^4x + 4x^2p^3\frac{dp}{dx} = 0$$

$$\left(p + 2x\frac{dp}{dx}\right) + 2p^3x\left(p + 2x\frac{dp}{dx}\right) = 0$$

$$\left(p + 2x\frac{dp}{dx}\right)\cancel{\left(1 + 2p^3x\right)} = 0$$

↳ Eliminate because of absence of $\frac{dp}{dx}$

$$p + 2x\frac{dp}{dx} = 0$$

$$p = -2x\frac{dp}{dx}$$

$$\int -2\frac{dp}{p} = \int \frac{dx}{x}$$

$$-2\log p = \log x + \log c$$

$$\cancel{\log} p^{(-2)} = \cancel{\log} xc$$

$$\frac{1}{p^2} = xc$$

$$p^2 = \frac{1}{x\boxed{c}}$$

$$p^2 = \frac{1 \times c_1}{x} = \sqrt{\frac{c_1}{x}}$$

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Putting p value in eq ①

$$y = x\left(\sqrt{\frac{c_1}{x}}\right)^2 x + \left(\sqrt{\frac{c_1}{x}}\right)^4 x^2 \quad \underline{\text{Ans}}$$

Equation Solvable for x.

① Express the given eq^n in form of

$$x = f(y, p) \quad -①$$

② Diff ① w.r.t y and put $\frac{dx}{dy} = 1/p$

So that the eq^n has two variable p & y

③ Solve the eq^n obtained in step ②

④ Let the Sol^n be $\phi(y, p, c) = 0$

⑤ Eliminate p from ① & ④ If eliminate of p is not possible the values of x & y expressed in terms of parameter & together from the Sol^n of the eq^n

Q Solve $x = y + p^2$

$$\frac{dx}{dy} = 1 + 2p\frac{dp}{dy}$$

$$\frac{1}{p} = 1 + 2p\frac{dp}{dy}$$

$$\left(\frac{1-p}{p}\right) = 2p\frac{dp}{dy}$$

$$\frac{1-p}{2p^2} = \frac{dp}{dy}$$

$$\int dy = \int \frac{2p^2}{1-p}\,dp$$

$$y = -2\int \frac{p^2}{p-1}\,dp$$

$$y = -2\int \left(p+1+\frac{1}{p-1}\right)dp$$

$$y = -2\left[\frac{p^2}{2} + p + \log(p-1)\right] + c$$

$$y = c - p^2 - 2p - 2\log(p-1) \quad \text{(2)}$$

Put in (1)

Sol^n

$$x = c - p^2 - 2p - 2\log(p-1) + p^2$$

$$x = c - 2p - 2\log(p-1)$$

Q Solve.

$$y = 2px + p^3y^2$$

$$x = \frac{y}{2p} - \frac{y^2p^2}{2} \quad (1)$$

$$\frac{dx}{dy} = \frac{1}{2p} - \frac{y}{2p^2}\frac{dp}{dy} - yp^2 - py^2\frac{dp}{dy}$$

$$\frac{1}{p} = \frac{1}{2p} - yp^2 - \left(\frac{y}{2p^2} + py^2\right)\frac{dp}{dy}$$

$$\left(\frac{1}{2p} + yp^2\right) = -\frac{y}{p}\left(\frac{1}{2p} + yp^2\right)\frac{dp}{dy}$$

$$\left(1 + \frac{y}{p}\frac{dp}{dy}\right) = 0$$

$$\frac{dp}{p} + \frac{dy}{y} = 0$$

$$\log p + \log y = \log c$$

$$\cancel{\log} py = \cancel{\log} c$$

$$py = c$$

$$p = \frac{c}{y}$$

put value of p in (1)

$$x = \frac{y \times y}{2 \times c} + \frac{y^2 \times c^2}{2 \times y^2}$$

$$x = \frac{y^2}{2c} - \frac{c^2}{2} \quad \text{Ans}$$

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* Clairaut's E_q^n :-

The D.E of the type $y = px + f(p)$ is known as the clairaut's E_q^n

Q $(y-px)^2 = 1+p^2$ solve?

$$y - px = \sqrt{1+p^2}$$

$$y = px + \sqrt{1+p^2}$$

$$y = px + f(p) \quad \text{Ans}$$

Q $(y-px)(p+1) = p^2$

$$y - px = \frac{p^2}{p+1}$$

$$y = px + \frac{p^2}{p+1}$$

$$y = px + f(p) \quad \text{Ans}$$

Q Solve $p = \log(px-y)$

$$e^p = px - y$$

$$y = px - e^p$$

$$y = px + (-e)^p$$

$$y = px + f(p) \quad \text{Ans}$$

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★ Electric Circuit :-

Kirchhoff's Law :-

Let i be the current flowing in the circuit containing resistance R and inductance L in series with Voltage source E at any time t

By Voltage Law

$$Ri + L\frac{di}{dt} = E \qquad\qquad I.F = e^{\int \frac{R}{L} dt} = e^{\frac{R}{L}t}$$

$$\frac{di}{dt} + \frac{Ri}{L} = E$$

This is known as Linear D.E

Q. Solve the equation $L\frac{di}{dt} + Ri = E_0 \sin\omega t$ where L, R, E_0 are constants

Sol^n $\frac{di}{dt} + \frac{R}{L} i = \frac{E_0}{L} \sin\omega t$

$$I_F \quad e^{\int \frac{R}{L} dt} = e^{\frac{Rt}{L}}$$

Solution :-

$$i \cdot e^{\frac{R}{L}t} = \int \frac{E_0}{L} \sin\omega t \, e^{\frac{Rt}{L}} dt$$

$$i \cdot e^{\frac{R}{L}t} = \frac{E_0}{L} \frac{e^{\frac{R}{L}t}}{\sqrt{\frac{R^2}{L^2} + \omega^2}} \sin\left(\omega t - \tan^{-1}\frac{\omega L}{R}\right) + A$$

$$\left[\because \int e^{ax} \sin bx \, dx = \frac{e^{ax}}{\sqrt{a^2+b^2}} \sin\left(bx - \tan^{-1}\frac{b}{a}\right)\right]$$

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$$i = \frac{E_0 L}{L\sqrt{R^2+\omega^2L^2}} \sin\left(\omega t - \tan^{-1}\frac{\omega L}{R}\right) + A e^{-\frac{R}{L}t}$$

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Roots Ordinary D.E of higher order.

linear D.E of second and higher order, complete solⁿ, Complementary function and particular Integral

D.E of the form

$$P_0 \frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots P_n y = x$$

where $P_0, P_1 \cdots P_n$ and x are function of x or constant is liner D.E of n^{th} order

Differential equation

$$\frac{d}{dx^1} = D^1$$

$$\frac{d^2}{dx^2} = D^2$$

Roots	Complete Sol^n
case I :- If all roots are real and different i.e, $m_1, m_2, m_3 \cdots$	Then sol^n is $y = c_1 e^{m_1 x} + C_2 e^{m_2 x} + C_3 e^{m_3 x}$
case II If two roots are real and all other are different i.e $\rightarrow m_1, m_1, m_2, m_3$	Then sol^n is $y = (C_1 + C_2 x + C_3 x^2) e^{m_1 x} + C_4 e^{m_2 x} + C_5 e^{m_3 x}$
Case III If two roots are imaginary and all other are real i.e $= \alpha \pm i\beta ; m_3 \; m_4$	Then sol^n is $y = (C_1 \cos\beta x + C_2 \sin\beta x) e^{\alpha x} + C_3 e^{m_3 x} + C_4 e^{m_4 x}$

Q Solve the D.E

$$2\frac{d^3y}{dx^3} - 7\frac{d^2y}{dx^2} + 7\frac{dy}{dx} - 2y = 0$$

Symbolic form

$$2D^3y - 7D^2y + 7Dy - 2y = 0$$

$$y(2D^3 - 7D^2 + 7D - 2) = 0$$

Auxillary Eq^n

$$2D^3 - 7D^2 + 7D - 2 = 0$$

$$\begin{array}{c|cccc} 1 & 2 & -7 & 7 & -2 \\ & & 2 & -5 & 2 \\ \hline & 2 & -5 & 2 & \end{array}$$

$$2D^2 - 5D + 2 = 0$$

$$D = 1, 2, \tfrac{1}{2}$$

So

$$y = c_1e^{x} + c_2e^{2x} + c_3e^{\frac{1}{2}x} \quad \text{Ans}$$

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Type-I (only that $Q=0$)

Q. Solve the D.E

$(D^4+5D^2+6)y=0$

$D^4+3D^2+2D^2+6=0$

$D^2(D^2+3)+2(D^2+3)=0$

$(D^2+3)(D^2+2)=0$

$D^2=-3$ $\quad$ $D^2=-2$

$D=\pm\sqrt{3}\,i$ $\quad$ $D=\pm\sqrt{2}\,i$

$y=(C_1\cos\sqrt{3}x+C_2\sin\sqrt{3}x+(C_3\cos\sqrt{2}+C_4\sin\sqrt{2}x)$

Type-II

(Right side only exponential)

Q Solve the D.E

$\frac{d^2y}{dx^2}+\frac{6dy}{dx}+25y=104e^{3x}$

Sf (Symbolic form) $\quad D^2y+6Dy+25y=104e^{3x}$

A.E (Auxilliary equation) $\quad D^2+6D+25=0$

$\frac{-6\pm\sqrt{36-100}}{2}$

$\frac{-6\pm\sqrt{64}}{2}$

$\frac{-6\pm 8i}{2}$

$Cf=-3\pm 4i$

(Complementary function)

$C.f = (c_1 \cos 4x + c_2 \sin 4x)\, e^{-3x}$

Particular Integral

$P.I = \frac{1}{D^2+6D+25} \times 104 e^{+3x}$

$D \to 3$

$= \frac{1}{9+18+25} \times 104 e^{3x}$

$= \frac{1}{\cancel{52}} \times \cancel{104}^{2} e^{3x}$

$P.I = 2e^{3x}$

$y = C.f + I.f$

$= (c_1 \cos 4x + c_2 \sin 4x)\, e^{-3x} + 2e^{3x}$

Q Solve the D.E

$\frac{d^3y}{dx^3} + y = 3 + e^{-3x} + 5e^{2x}$

$S.F: \quad D^3y + y = 3 + e^{-3x} + 5e^{2x}.$

$A.E = D^3 + 1 = 0$

$(D+1)(D^2 - D + 1) = 0$

$D = -1 \quad \Big| \quad \frac{1 \pm \sqrt{1-4}}{2}$

$\frac{1 \pm \sqrt{3}\, i}{2}$

$D = -1, \; D = \frac{1}{2} \pm \frac{\sqrt{3}}{2} i$

$$C.F = c_1 e^{-x} + \left(c_2 \cos\frac{\sqrt{3}}{2}x + c_3 \sin\frac{\sqrt{3}}{2}x\right) e^{\frac{1}{2}x}$$

$$P.I = \frac{1}{D^3+1}(3 + e^{-x} + 5e^{2x})$$

$$= \frac{1}{D^3+1} 3e^{0x} + \frac{1}{D^3+1} e^{-1x} + \frac{1}{D^3+1} 5e^{2x}$$

$$\frac{1}{1} 3(1) + \frac{1}{-1+1} e^{-x} + \frac{1}{8+1} 5e^{2x}$$

$$3 + \frac{1}{0} e^{-x} + \frac{1}{9} 5e^{+2x}$$

$$3 + \frac{1}{3D^2} xe^{-x} + \frac{5}{9} e^{2x}$$

$$3 + \frac{1}{3(1)} xe^{-x} + \frac{5}{9} e^{2x}$$

$$3 + \frac{1}{3} xe^{-x} + \frac{5}{9} e^{2x}.$$

$$C.S = C.f + P.I$$

$$= c_1 e^{-x} + \left(c_2 \cos\frac{\sqrt{3}x}{2} + c_3 \sin\frac{\sqrt{3}x}{2}\right) e^{\frac{1}{2}x} + \left(3 + \frac{1}{3}xe^{-x} + \frac{5}{9}e^{2x}\right)$$

* Solve

$$\frac{d^2y}{dx^2} - y = \cosh x$$

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

Type - III

$\frac{1}{f(D^2)} \sin ax$ ~~or~~ $\frac{1}{f(D^2)} \cos ax$

then put $D^2 = -(a^2)$

Q. Solve the D.E

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = \sin 2x$$

$$D^2 + D + 1 = 0$$

$$P.I = \frac{1}{D^2+D+1} \sin 2x$$

$$\frac{1}{-4+D+1} \sin 2x$$

$$\frac{1}{D-3} \sin 2x$$

$$\frac{1}{D-3} \times \frac{D+3}{D+3} (\sin 2x)$$

$$\frac{D+3}{(D^2-9)} \sin 2x$$

$$D^2 = -4$$

$$\left(\frac{D+3}{(-4-9)}\right) \sin 2x$$

$$P.I = -\frac{1}{13} (2\cos 2x + 3\sin 2x)$$

$$y = C.f + P.I$$

Q. $\frac{d^2y}{dx^2} - 4y = e^x + \cos 2x$

$D^2 - 4 = e^x + \cos 2x$

$D^2 - 2^2 = 0.$

$\boxed{D = +2, -2.}$

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$$C.f = c_1 e^{2x} + c_1 e^{-2x}$$

$$P.I = \frac{1}{D^2-4} e^{2x} + \frac{1}{D^2-4} \cos 2x$$

$$= \frac{1}{1-4} e^{x} + \frac{1}{-4-4} \cos 2x$$

$$PI = -\frac{1}{3} e^{x} + \left(-\frac{1}{8} \cos 2x\right)$$

$$y = C.F + P.I$$

$$y = c_1 e^{2x} + c_1 e^{-2x} - \frac{1}{3} e^{x} - \frac{1}{8} \cos 2x$$

Type - 4

$\frac{1}{f(D)} x^m$, where m is +ve integer. The binomial expression used for these type of questions

(1) $(1-D)^{-1} = 1 + D + D^2 + D^3 + \text{—}$

(2) $(1+D)^{-1} = 1 - D + D^2 - D^3 + \text{—}$

Q) Solve the D.E

$$\frac{d^2y}{dx^2} - 4y = x^2$$

Solⁿ - $D^2 - 4 = 0$

$D = \pm 2$

$$C.F = c_1 e^{2x} + c_2 e^{-2x}$$

$$P.I = \frac{1}{D^2-4} x^2$$

$$= \frac{1}{-4\left[1 - \frac{D^2}{4}\right]} x^2$$

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$$= \frac{-1}{4}\left[1-\frac{D^2}{4}\right]^{-1} x^2$$

$$= \frac{-1}{4}\left[1+\frac{D^2}{4}+\left(\frac{D^2}{4}\right)^2+\cdots\right]x^2$$

$$= \frac{-1}{4}\left[x^2+\frac{2}{4}+0\right]$$

$$P.I = \frac{-1}{4}\left[x^2+\frac{1}{2}\right.$$

$$y = C.F + P.I$$

$$y = c_1e^{2x} + c_2e^{-2x} + \frac{-1}{4}\left[x^2+\frac{1}{2}\right] \text{ Ans}$$

Q Solve the D.E

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = x^2 + e^x + \cos 2x$$

$$C.F = D^2 - 4D + 4 = 0$$

$$D^2 - 2D - 2D + 4 = 0$$

$$D(D-2) - 2(D-2) = 0$$

$$(D-2)(D-2)$$

$$D = 2, 2.$$

$$\frac{1}{D^2-4D+4}(x^2+e^x+\cos 2x)$$

$$\frac{x^2}{D^2-4D+4} + \frac{1}{D^2-4D+4}e^x + \frac{1}{D^2-4D+4}\cos 2x$$

$$\frac{1}{4\left[1+\frac{D^2}{4}-\frac{4D}{4}\right]}x^2 + \frac{1}{1-4+4}e^x + \frac{1}{-4-4D+4}\cos 2x$$

$$= \frac{1}{4}\left[1+\left(\frac{D^2}{4}-D\right)\right]^{-1}x^2 + e^x - \frac{1}{4}\frac{\sin 2x}{2}.$$

$$= \frac{1}{4}\left[1-\left(\frac{D^2}{4}-D\right)+\left(\frac{D^2}{4}-D\right)^2+\cdots\right]x^2+e^x-\frac{1}{8}\sin 2x$$

$$P.I = \frac{1}{4}\left[x^2-\frac{1}{2}+2x+2\right]+e^x-\frac{\sin 2x}{8}$$

$$C.F = (c_1+c_2x)e^{2x}$$

$$y = (c_1+c_2x)e^{2x}+\frac{1}{4}\left[x^2-\frac{1}{2}+2x+2\right]+e^x-\frac{\sin 2x}{8}$$ Ans

Type-5 $\quad \frac{1}{f(D)}e^{ax}\cdot V = e^{ax}\frac{1}{f(D+a)}V$

where V is any func^n of x

Q $\quad \frac{d^2y}{dx^2}+y = x\cdot e^{2x}$

$$D^2+1=0$$
$$D=\pm 1i$$
$$C.F = c_1\cos x + c_2\sin x$$

$$P.I = \frac{1}{D^2+1}e^{2x}\cdot x$$

$$= e^{2x}\frac{1}{(D+2)^2+1}x = e^{2x}\frac{1}{D^2+4+4D+1}x$$

$$= e^{2x}\frac{1}{D^2+5+4D} = e^{2x}\frac{1}{\quad}$$

* Method of Variation of Parameter
working Rule.

(1) find two independent sol^n of the eq^n $\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=R$ and denote them y_1 & y_2

(2) Put $c.f = Ay_1 + By_2$ where A & B are arbitrary constant

(3) find $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

and $u = -\int \frac{y_2 R}{W}\, dx + c_1$

& $v = \int \frac{y_1 R}{W}\, dx + c_2$

(4) Replace arbitrary constant A & B in $cf = Ay_1 + By_2$ by functions u & v so, that the complete solution is $y = uy_1 + vy_2$

Q Apply the Method of Variation of Parameters to solve.

$$\frac{d^2y}{dx^2} + 4y = \tan 2x$$

$$D^2 + 4 = 0$$

$$D^2 = -4$$

$$D = \pm 2i$$

$$C.f = A\cos 2x + B\sin 2x$$

Let the complete sol^n

$$u\cos 2x + v\sin 2x$$

$$u\, y_1 + v\, y_2$$

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$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$= \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix}$$

$$= 2\cos^2 2x + 2\sin^2 2x$$

$$W = 2$$

$$u = -\int \frac{y_2 R}{W}$$

$$= -\int \frac{\sin 2x \ \mathrm{Tan}\, 2x}{2}$$

$$= -\int \frac{\sin 2x \sin 2x}{2 \cos 2x} \qquad = -\int \frac{1-\cos^2 2x}{2\cos 2x}$$

$$\therefore (\sin^2 x = 1-\cos^2 x)$$

$$u = -\int \frac{\sin 2x \cdot \tan 2x}{2}$$

$$= \int - \frac{\sin 2x \,(\sin 2x / \cos 2x)}{2}$$

$$= \frac{-1}{2}\int \sin 2x \cdot \frac{\sin 2x}{\cos 2x}$$

$$= \frac{-1}{2}\int (\sin^2 2x) \times \frac{1}{\cos 2x}$$

$$\therefore (\sin^2 x = 1-\cos^2 x)$$

$$= \frac{-1}{2}\int \frac{1-\cos^2 2x}{\cos 2x}$$

$$= \frac{-1}{2}\left[\int \frac{1}{\cos 2x} - \frac{\cos^2 2x}{\cos 2x}\right]$$

$$u = -\frac{1}{2}\int \sec 2x \, dx + \frac{1}{2}\int \cos 2x \, dx$$

$$u = \frac{-1}{2 \times 2} \log|\sec 2x + \tan 2x| + \frac{1}{2}\frac{\sin 2x}{2} + C_1$$

$$v = \int \frac{y_1 R}{W} = \int \frac{\cos 2x \ \mathrm{Tan}\, 2x}{2}$$

$$v = \int \frac{\cancel{\cos 2x} \times \sin 2x}{2 \cancel{\cos 2x}}$$

$$= \frac{1}{2}\frac{(-\cos 2x)}{2} \rightarrow v = \frac{-\cos 2x}{4} + C_2$$

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$y = uy_1 + vy_2$

$y = uy_1 + vy_2$ **Ans**

☆ Simultaneous linear equation with constant coefficient

Q Solve the simultaneous equation

$$\frac{dx}{dt} + 5x + y = e^t$$

$$\frac{dy}{dt} - x + 3y = e^{2t}$$

$$Dx + 5x + y = e^t$$
$$Dy - x + 3y = e^{2t}$$

$$(D+5)x + y = e^t \quad - \times (D+3)$$
$$-x + (D+3)y = e^{2t}$$

$$(D+5)(D+3)x + \cancel{(D+3)y} = e^t(D+3)$$
$$\underset{+}{-x} \quad \underset{-}{+} \cancel{(D+3)y} = \underset{-}{e^{2t}}$$

$$[D^2 + 8D + 15 + 1]x = e^t + 3e^t - e^{2t}$$
$$(D^2 + 8D + 16)x = 4e^t - e^{2t}$$
$$D^2 + 8D + 16 = 0$$
$$D^2 + 4D + 4D + 16 = 0$$
$$(D+4)^2 = 0$$
$$D = -4, -4$$
$$C.F = (C_1 + C_2 t)e^{-4t}$$

$$P.I = \frac{1}{D^2 + 8D + 16} \, 4e^t - e^{2t}$$

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$$\frac{1}{D^2+8D+16} 4e^t - \frac{1}{D^2+8D+16} e^{2t}$$

$D = 1$

$$4 \frac{1}{D^2+8D+16} e^t - \frac{1}{4+16+16} e^{2t}$$

$$\frac{4}{25} e^t - \frac{1}{36} e^{2t}$$

$C.S = x = C.F + P.I$

$$x = (c_1 + c_2 t)e^{-4t} + \frac{4}{25} e^t - \frac{1}{36} e^{2t}$$ Ans

Put x in any eq^n

$$\frac{dx}{dt} + 5x + y = e^t$$

$$y = e^t - \frac{dx}{dt} - 5x$$

$$y = e^t - \left[(c_1+c_2 t)(-4e^{-4t}) + e^{-4t} c_2 + \frac{4}{25} e^t - \frac{2}{36} e^{2t}\right] - 5(c_1 + c_2 t)e^{-4t} - \frac{20}{25} e^t + \frac{5}{36} e^{2t}$$