Unit - 4

Sequence & Series

Test for Divergence of Sequence & Series.

Sequence:

The order set of real no. $a_1, a_2, a_3, ---, a_n$ is called a sequence. and it is denoted by (a_n) or $\langle a_n \rangle$. If the terms of the sequence are unlimited, then this sequence is known as the infinite sequence. i.e. $a_1, a_2, a_3 ---, a_n, --- \infty$

eg → ① $(n) = 1, 2, 3, 4, 5, --- n, --- \infty$
$n \in N$
② $(2n) = 2, 4, 6, 8, --- n, --- \infty$
$n \in N$

Limit:

A sequence is said to tend to the limit 'l' if for $\epsilon > 0$, a value capital N of n can be found

such that
$|a_n - l| < \epsilon$ for $n \geq N$
or
$\lim_{n\to\infty} a_n = l$

Convergence, Divergence and oscillation of infinite sequence:

1) Convergence: If $\lim_{n\to\infty} a_n = l$ is finite and unique than the sequence a_n is convergent.

2) Divergence: If $\lim_{n\to\infty} a_n = \pm\infty$ than the sequence a_n is divergence

3) oscillation: If $\lim_{n\to\infty} a_n =$ not unique than the sequence a_n is ossillatory.

eg:- (1) Check the convergency of the sequence $a_n = \frac{n^2-2n}{3n^2+n}$.

Soln. $a_n = \frac{n^2(1-2/n)}{n^2(3+1/n)} = \frac{(1-2/n)}{(3+1/n)}$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{1-2/n}{3+1/n}\right) = \frac{1}{3}$ = finite & unique ∴ The sequence is con

Series:

If $a_1, a_2, a_3, ---, a_n ---$ be the infinite sequence of real no. than the sum of these
$$a_1 + a_2 + a_3 + a_4 --- + a_n + --\infty$$
is known as the series. It is denoted by Σa_n.

Partial Sum:

$S_n =$
$S_1 = a_1$
$S_2 = a_1 + a_2$
$\vdots$
$S_n = a_1 + a_2 + --- + a_n$

Then This is known as partial sum.

Convergence, divergence & oscillation of infinite series:

Let the infinite series is $a_1 + a_2 + --- --- + a_n + --- \infty$ and let the sum of first n terms of series be
$$S_n = a_1 + a_2 + --- + a_n.$$
Then

(1) If $\lim_{n\to\infty} S_n =$ finite & unique $= l$ then the series Σa_n is said to be convergent.

(2) If $\lim_{n\to\infty} S_n = \pm\infty$ the series $\sum a_n$ is divergence.

(3) If $\lim_{n\to\infty} S_n =$ not unique then the series $\sum a_n$ is oscillatory.

eg:- (2) Examine the convergence of the series $1+2+3+\cdots+n+\cdots\infty$

Soln From the given series.

$$S_1 = 1$$
$$S_2 = 1+2$$
$$S_3 = 1+2+3$$
$$S_n = 1+2+3+\cdots+n.$$

the n^{th} partial sum is

$$S_n = 1+2+3+\cdots+n.$$
$$= \frac{n(n+1)}{2}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{n(n+1)}{2}$$
$$= \lim_{n\to\infty} \frac{n^2(1+1/n)}{2}$$
$$= \infty = \text{infinite}$$

$\therefore$ The series is divergence.

Geometric Series:

If the series $1+r+r^2+r^3 --- +r^n + --- \infty$.

than

(i) the series is convergent when $|r|<1$

(ii) the ~~divergence~~ series is divergence when $r \geq 1$

(iii) the series is oscillatory when $r \leq -1$

Ques1) Examine the convergence of $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+ --- +\infty$

Soln $1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+ --- +\infty$

The common ratio, $r = \frac{1}{2} \div 1$

$= \frac{1}{2}$

$= \frac{1}{\cancel{2}} \times \frac{\cancel{2}}{1} = \frac{1}{2}$

$\therefore r = |r| = \left|\frac{1}{2}\right| < 1$

Acc. to geometric series

$\therefore$ The given series is convergent.

Comparision Test

~~1. If two positive terms series ΣU_n & ΣV_n~~

Positive terms Series → An infinite series in which all the terms after some particular terms are positive is a known as a positive term series.

eg → $-2, -1, 0, 1, 2, 3, 4, - - - \infty$

1) If two positive terms series ΣU_n & ΣV_n be such that

→ ΣV_n Converges

→ $U_n \leq V_n$

then ΣU_n is called a convergence.

2) If two positive terms series ΣU_n & ΣV_n be such that

→ ΣV_n diverges

→ $U_n \geq V_n$

then ΣU_n is divergence.

3) If two positive terms series ΣU_n & ΣV_n be such that

$\lim_{n \to \infty} \frac{U_n}{V_n}$ = finite ($\neq 0$) then the ΣU_n & ΣV_n are convergent or divergent together.

Ques 1) Test the convergency of the series $\frac{1}{1\cdot 2 3}+\frac{3}{2\cdot 3\cdot 4}+\frac{5}{3\cdot 4\cdot 5}+ \dots \infty$

Soln Let $\sum_{n=1}^{\infty} U_n = \sum_{n=1}^{\infty} \frac{(2n-1)}{n\cdot(n+1)\cdot(n+2)}$

So, $U_n = \frac{(2n-1)}{n(n+1)(n+2)}$

Let $V_n = \frac{\cancel{n}}{n^{\cancel{3}2}}$ — highest power of n in numerator
— highest power of n in deno

$$V_n = \frac{1}{n^2}$$

So

$$\lim_{n\to\infty} \frac{U_n}{V_n} \; \frac{\cancel{\lim}(2n-1)}{\cancel{n\to\infty}\, n(n+1)(n+2)} \times n^2$$

$$\lim_{n\to\infty} \frac{U_n}{V_n} = \lim_{n\to\infty} \frac{\cancel{n}(2-1/n)}{\cancel{n^3}(1+1/n)(1+2/n)} \times \cancel{n^2}$$

$$\lim_{n\to\infty} \frac{U_n}{V_n} = \lim_{n\to\infty} \frac{(2-1/n)}{(1+1/n)(1+2/n)}$$

$$= 2 = \text{finite } (\neq 0)$$

∴ By comparision test, ΣU_n & ΣV_n are convergent & divergent together.

Now,

$$\sum_{n=1}^{\infty} V_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

~~W~~here $p=2>1$

then the series $\sum V_n$ is convergent.
$\therefore$ the series $\sum_{n=1}^{\infty} U_n$ is also convergent.

P-Series Test:

The series ~~$\frac{1}{n^p}$~~ $\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots$

then convergent if $p>1$
Divergent if $p \leq 1$

eg:- ① $\sum_{n=1}^{\infty} \frac{1}{n}$ Here $p=1$

$\therefore$ The series is divergent.

Ratio Test / D'Alembert's Ratio Test:

If $\sum_{n=1}^{\infty} U_n$ is a series of positive term such that

$$\boxed{\lim_{n\to\infty} \frac{U_n}{U_{n+1}} = l}$$

than,

(i) Convergent if $l > 1$
(ii) Divergent if $l < 1$
(iii) Test failed if $l = 1$
(iv) Divergent if $l = \infty$

Ques:) Test the Convergency of the series $\frac{1!}{5} + \frac{2!}{5^2} + \frac{3!}{5^3} + \ldots \infty$

Soln let the series $\sum_{n=1}^{\infty} U_n = \sum_{n=1}^{\infty} \frac{n!}{5^n}$

so,

$$U_n = \frac{n!}{5^n} \qquad U_{n+1} = \frac{(n+1)!}{5^{(n+1)}}$$

~~so,~~

$$\cancel{\lim_{n\to\infty} \frac{n!}{5^n} \times 5^{(n+1)}}$$

$$U_n = \frac{n!}{5^n}$$

$$U_{n+1} = \frac{(n+1)!}{5^{(n+1)}}$$

$$\lim_{n\to\infty} \frac{U_n}{U_{n+1}}$$

$$= \lim_{n\to\infty} \frac{n!}{5^n} \times \frac{5^{n+1}}{(n+1)!}$$

$$= \lim_{n\to\infty} \frac{\cancel{n!}}{\cancel{5^n}} \times \frac{\cancel{5^n}\cdot 5}{(n+1)\cancel{n!}}$$

$$= \lim_{n\to\infty} \frac{5}{(n+1)}$$

$$= 0 < 1$$

∴ The series is divergent.

Cauchy's Root Test:

(a) If $\sum_{n=1} U_n$ is a positive term series and

$$\boxed{\lim_{n\to\infty} (U_n)^{1/n} = l}$$

(i) then, the series is convergent when $l < 1$

(ii) the series is divergent whe $l > 1$.

(b) If the $\lim_{n\to\infty} (U_n)^{1/n} = \infty$ - Divergent

Ques1) Test the Convergency of the series $\sum_{n=2} \frac{1}{(\log(\log n))^n}$

Soln Here, $U_n = \frac{1}{(\log(\log n))^n}$

$\lim_{n\to\infty} (U_n)^{1/n} = \lim_{n\to\infty} \left[\frac{1}{[\log(\log n)]^n}\right]^{1/n}$

$\lim_{n\to\infty} \frac{1}{\log(\log n)} = \frac{1}{\log(\log\infty)} = \frac{1}{\log\infty}$

$= 1/\infty = 0 < 1$

$\therefore$ Acc to root test the series is convergent.

Raabe's Test:

If $\sum_{n=1}^{\infty} U_n$ is a series of positive term such that

$$\boxed{\lim_{n\to\infty} n\left[\frac{U_n}{U_{n+1}}-1\right]=l}$$

then the series is

1) Convergent if $l>1$
2) Divergent if $l<1$
3)

Logarithms Test:

If $\sum_{n=1}^{\infty} U_n$ is a series of positive term such that

$$\boxed{\lim_{n\to\infty}\left(\log\frac{U_n}{U_{n+1}}\right)n=l}$$

the the series is

1) Convergent if $l>1$
2) Divergent if $l<1$

Ques2) Test the convergency of the series $x + \frac{2^2x^2}{2!} + \frac{3^3x^3}{3!} + \frac{4^4x^4}{4!} + \cdots \infty$

Sol^n Let $u_n = \frac{n^n x^n}{n!}$

$$u_{n+1} = \frac{(n+1)^{n+1} x^{n+1}}{(n+1)!}$$

$$\frac{u_n}{u_{n+1}} = \frac{n^n x^n}{n!} \times \frac{(n+1)!}{(n+1)^{n+1} x^{n+1}}$$

$$= \frac{n^n \cancel{x^n}}{\cancel{n!}} \times \frac{\cancel{(n+1)}\, \cancel{n!}}{(n+1)^n \cancel{(n+1)}\, \cancel{x^n}\, x}$$

$$\frac{u_n}{u_{n+1}} = \frac{n^n}{(n+1)^n x}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{n^n}{(n+1)^n x}$$

$$= \lim_{n\to\infty} \frac{\cancel{n^n}}{\cancel{n^n}(1+1/n)^n} \cdot \frac{1}{x}$$

$$= \lim_{n\to\infty} \frac{1}{(1+1/n)^n} \frac{1}{x}$$

$$= \frac{1}{\lim_{n\to\infty}(1+1/n)^n} \frac{1}{x} = \frac{1}{ex}$$

By ratio test the series is convergent

if $1/ex > 1$ i.e. $ex < 1$.

Divergent - if $1/ex < 1$ i.e. $ex > 1$.

If $1/ex = 1$ i.e. $ex = 1$ or $x = 1/e$ then test fails

TEACHER SIGNATURE

At $x=1$

$$\frac{U_n}{U_{n+1}} = \frac{3n+7}{3n+3}$$

$$\frac{U_n}{U_{n+1}} - 1 = \frac{3n+7}{3n+3} - 1$$

$$= \frac{\cancel{3n}+7-\cancel{3n}-3}{3n+3}$$

$$\frac{U_n}{U_{n+1}} - 1 = \frac{4}{3n+3}$$

$$\lim_{n\to\infty} n\left[\frac{U_n}{U_{n+1}} - 1\right] = \frac{4n}{3n+3}$$

$$\lim_{n\to\infty} n\left[\frac{U_n}{U_{n+1}} - 1\right] = \lim_{n\to\infty} \frac{4\cancel{n}}{\cancel{n}(3+3/n)}$$

$$= \frac{4}{3} > 1$$

$\therefore$ ~~The is~~ By Raabe's Test the series is convergent.

Hence the series is convergent if $x \leq 1$ and divergent if $x > 1$

Ques 1) Test the convergency of the series $1+\frac{3}{7}x+\frac{3\cdot 6}{7\cdot 10}x^2+\frac{3\cdot 6\cdot 9}{7\cdot 10\cdot 13}x^3+ \cdots \infty$

Sol^n Let $U_n = \frac{3\cdot 6\cdot 9 \cdots 3n}{7\cdot 10\cdot 13 \cdots (3n+4)} x^n$

$$U_{n+1} = \frac{3\cdot 6\cdot 9 \cdots 3n(3n+3)}{7\cdot 10\cdot 13 \cdots (3n+7)} x^{n+1}$$

$$\frac{U_n}{U_{n+1}} = \cancel{\frac{3\cdot 6\cdot 9 \cdots 3n}{7\cdot 10\cdot 13 \cdots (3n+4)}} x^n \times \frac{\cancel{7\cdot 10\cdot 10 \cdots} (3n+7)}{\cancel{3\cdot 6\cdot 9 \cdots 3n}(3n+3)\, x^{(n+1)}}$$

$$\frac{U_n}{U_{n+1}} = \frac{x^n(3n+7)}{(3n+3)\cancel{x^n}\cdot x}$$

$$\frac{U_n}{U_{n+1}} = \frac{(3n+7)}{(3n+3)} \cdot \frac{1}{x}$$

$$\lim_{n\to\infty} \frac{U_n}{U_{n+1}} = \lim_{n\to\infty} \frac{\cancel{n}(3+7/n)}{\cancel{n}(3+3/n)} \frac{1}{x}$$

$$\lim_{n\to} = \frac{\cancel{3}}{\cancel{3}} \times \frac{1}{x} = \frac{1}{x}$$

By ratio test if $1/x > 1$ i.e. $x < 1$ then series is convergent.
if $1/x < 1$ i.e. $x > 1$ then series is divergent
~~1/x~~ if $1/x = 1$ i.e. $x = 1$ then the test fails.

Alternating Series:

A series of the form $a_1 - a_2 + a_3 - a_4 + a_5 - a_6 - \cdots \infty$, where $a_n > 0$ for all $n \in N$ is called an alternating series and it is denoted by $\sum_{n=1}^{\infty} (-1)^{n-1} a_n$

eg:- $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} - \cdots \infty$

$$\cancel{a_n} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$$

Leibnitz's Test:

The alternating series $\sum_{n=1}^{\infty} (-1)^{n-1} a_n$

$= a_1 - a_2 + a_3 - \cdots \infty$

($a_n > 0$ for all n) is convergent if

~~(i) $a_{n+1} \not> a_n$~~

(i) $a_{n+1} \leq a_n$

(ii) $\lim_{n \to \infty} a_n = 0$

Q1) Test the convergency of series

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots \infty$$

$$= \frac{1}{2} - \frac{1}{3n} + \frac{1}{4n^3} + \cdots \infty$$

$$\lim_{n\to\infty} n\left[\log \frac{u_n}{u_{n+1}}\right]$$

$$= \lim_{n\to\infty}\left[\frac{1}{2} - \frac{1}{3n} + \cdots \infty\right]$$

$$= \frac{1}{2} < 1$$

$\therefore$ By Logarithm test the series is divergent

Hence the series is convergent $xe < 1$ i.e. $x < \frac{1}{e}$

& divergent if $xe \geq 1$ i.e. $x \geq \frac{1}{e}$

At $x = \frac{1}{e}$

$$\frac{U_n}{U_{n+1}} = \frac{1}{(1+1/n)^n} \cdot e$$

$$\log \frac{U_n}{U_{n+1}} = \log\left(\frac{e}{(1+1/n)^n}\right)$$

$$= \log e - \log\left(1+\frac{1}{n}\right)^n$$

$$= 1 - n\log\left(1+\frac{1}{n}\right)$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \infty$$

$$= 1 - n\left[\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \frac{1}{4n^4} + \cdots \infty\right]$$

$$= \cancel{1} - \cancel{1} + \frac{1}{2n} - \frac{1}{3n^2} \cdots \infty.$$

$$\log \frac{U_n}{U_{n+1}} = \frac{1}{2n} - \frac{1}{3n^2} + \frac{1}{4n^3} - \cdots \infty$$

$n \cancel{\log}$

$$n\left[\log \frac{U_n}{U_{n+1}}\right]$$

$$= n\left[\frac{1}{2n} - \frac{1}{3n^2} + \frac{1}{4n^3} - \cdots \infty\right]$$

Soln From the given series
$$\sum_{n=1}^{\infty}(-1)^{n-1}a_n=\sum_{n=1}^{\infty}\cancel{(n+1)}(-1)^{n-1}\frac{1}{n^2}$$

Here,
$$a_n=\frac{1}{n^2},\quad a_{n+1}=\frac{1}{(n+1)^2}$$

$$\lim_{n\to\infty}a_n=\lim_{n\to\infty}\frac{1}{n^2}=\frac{1}{\infty}=0$$

Now,
$$n<n+1 \quad \text{for all } n.$$
$$n^2<(n+1)^2$$
$$\frac{1}{n^2}>\frac{1}{(n+1)^2}$$

$$\Rightarrow \frac{1}{(n+1)^2}<\frac{1}{n^2}$$
$$a_{n+1}<a_n$$
$\therefore$ By Lebinitz test, the $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$ is convergent.

Now, $$\sum_{n=1}^{\infty}|a_n|=\sum_{n=1}^{\infty}|(-1)^{n-1}a_n|$$
$$=\sum_{n=1}^{\infty}\left|(-1)^{n-1}\frac{1}{n^2}\right|$$
$$=\sum_{n=1}^{\infty}\frac{1}{n^2}$$

Here $p=2>1$, so By p test the series is cgt & absolute cgt.

 TEACHER SIGNATURE

Absolute Convergent

A series $\sum_{n=1}^{\infty} a_n$ is said to be absolute convergent if the series $\sum_{n=1} |a_n|$ is convergent.

Conditional Convergent:

A series $\sum_{n=1}^{\infty} a_n$ is said to be conditional convergent

(i) if $\sum_{n=1}^{\infty} a_n$ is convergent.

(ii) if $\sum_{n=1}^{\infty} a_n$ is not absolute convergent

i.e. $\sum_{n=1}^{\infty} |a_n|$ is not convergent.

Ques) Show that the series

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^2} = \frac{1}{1} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots \infty$$

is absolute convergent or not?

Soln $= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{(2n-1)}$

$= \sum_{n=1}^{\infty} (-1)^{n-1} a_n$

Here, $a_n = \frac{1}{2n-1}$

$a_{n+1} = \frac{1}{2(n+1)-1} = \frac{1}{2n+1}$

Now, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{2n-1} = \frac{1}{\infty} = 0$

Since,

$2n-1 < 2n+1$ for all n.

$\frac{1}{2n-1} > \frac{1}{2n+1}$

$\Rightarrow \frac{1}{2n+1} < \frac{1}{2n-1}$

$a_{n+1} < a_n$

By Leibniz test, the series is convergent.

2) $\int_{c}^{c+2\pi} \sin mx \cos nx \, dx = 0$ for $m = n$, $m \neq n$

3) $\int_{c}^{c+2\pi} \sin mx \sin nx \, dx = \begin{cases} 0, & m \neq n \\ \pi, & m = n \end{cases}$

4) $\int_{-a}^{a} f(x) \, dx = \begin{cases} 0, & f(x) \text{ is odd fun}^n \\ 2\int_0^a f(x)\,dx, & f(x) \text{ is even func}^n \end{cases}$

5) $\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2}\left[a \sin bx - b \cos bx\right]$

6) $\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2+b^2}\left[a \cos bx + b \sin bx\right]$

7) $\sin n\pi = 0$

8) $\sin 2n\pi = 0$

9) $\cos n\pi = (-1)^n$

1) $\cos 2n\pi = 1$

<u>Def^n</u>: Let $f(x)$ be a real value bounded and integrable on $[c, c+2\pi]$ such that $f(x$