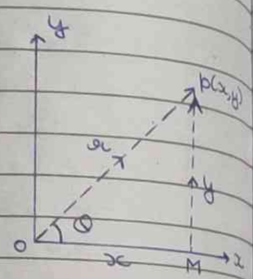
UNIT - 04 (Function of complex variable)

→ Let $P(x,y)$ represent the complex No $Z = x + iy$. Let $\vec{OP}$ makes an angle θ with +ve direction of OX and $OP = r$ & $\angle MOP = \theta$ then $X = OM = OP\cos\theta$ where $OP = r$ is known as radius vector and $y = MP = OP\sin\theta$



$\therefore x = r\cos\theta$, $y = r\sin\theta$

$\Rightarrow x^2 + y^2 = r^2$ & $\tan\theta = y/x$

$\Rightarrow r = \sqrt{x^2+y^2}$ is called modulus of z &

$\theta = \tan^{-1}(y/x)$ is called amplitude or argument of z.

The value of θ satisfying $-\pi < \theta < \pi$ is called the principle value of amplitude. Hence,

$Z = x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}$

* Exponential f^{x^n} of a complex variable

→ the exponential fx^n of a complex variable z is defined as

$$e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots\cdots + \frac{z^n}{n!} + \cdots\cdots$$

Property of Exponential fx^n

a) $e^z = e^{x+iy} = e^x(\cos y + i\sin y)$

$$e^{x+iy} = 1 + (x+iy) + \frac{(x+iy)^2}{2!} + \frac{(x+iy)^3}{3!} + \cdots\cdots$$

Put $x = 0$ or $e^{x+iy} = e^x(e^{iy})$

$$e^{iy} = 1 + iy + \frac{(iy)^2}{2!} + \frac{(iy)^3}{3!} + \cdots\cdots$$

$$= 1 + iy - \frac{y^2}{2!} - \frac{iy^3}{3!} + \frac{y^4}{4!} + \cdots$$

$$= \left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} + \cdots\right) + i\left(y - \frac{y^3}{3!} + \cdots\cdots\right)$$

$e^{iy} = \cos y + i\sin y$

$e^{x+iy} = e^x + e^{iy}$

$= e^x(\cos x + i\sin y)$

b) e^z is a periodic fx^n of $2\pi i$

→ since, $e^z = e^{x+iy} = e^x \cdot e^{iy}$

$= e^x(\cos y + i\sin y)$

also, $\cos(y + 2n\pi) = \cos y$

$\sin(y + 2n\pi) = \sin y$

$e^z = e^x \cdot e^{iy} = e^x(\cos y + i\sin y) = e^x(\cos(y+2n\pi) + i\sin(y+2n\pi))$

$= e^x e^{i(y+2n\pi)}$

$= e^{x+iy+2n\pi i} = e^{z+2n\pi i}$

e^z remain unchanged where z is Ted by multiple of $2\pi i$

c) Trigonometric fx^n of complex variable

→ WKT ⇒ $e^{ix} = \cos x + i\sin x$ -①

& $e^{ix} = \cos x - i\sin x$ -②

add and substract ① & ②

$\cos x = \dfrac{e^{ix} + e^{-ix}}{2}$ | $\sin x = \dfrac{e^{ix} - e^{-ix}}{2i}$

these are called the euler exponential value of $\sin x$ & $\cos x$. If $z = x + iy$ then trigonometric fx^n of z are defined as follow →

$\cos z = \dfrac{e^{iz} + e^{-iz}}{2}$, $\sin z = \dfrac{e^{iz} - e^{-iz}}{2i}$

$\tan z = \dfrac{\sin z}{\cos z} = \dfrac{e^{iz} - e^{-iz}}{i(e^{iz} + e^{-iz})}$, $\cot z = \dfrac{i(e^{iz} + e^{-iz})}{(e^{iz} - e^{-iz})}$

$\operatorname{cosec} z = \dfrac{1}{\sin z} = \dfrac{2i}{e^{iz} - e^{-iz}}$, $\sec z = \dfrac{1}{\cos z} = \dfrac{2}{e^{iz} + e^{-iz}}$

Q1 $e^{(5+3i)^2}$ → solve for real & imaginary value

→ $(5+3i) = 16 + 30i$

$e^{(16+3i)} = e^{16} \times e^{30i}$

$= e^{16}(\cos 30 + i \sin 30)$

Q2 $(\sin(\alpha-\theta) + e^{i\alpha}\sin\theta)^n = \sin^n\alpha \cdot e^{-in\theta}$ (Prove it)

LHS →

→ $[\sin\alpha\cos\theta - \sin\theta\cos\alpha + \sin\theta((\cos\alpha - i\sin\alpha))]^n$

$[\sin\alpha\cos\theta - \cancel{\sin\theta\cos\alpha} + \cancel{\sin\theta\cos\alpha} - i\sin\alpha\sin\theta]^n$

$[\sin\alpha\,[\cos\theta - i\sin\theta]\,]^n$

$[\sin\alpha \cdot e^{-i\theta}]^n$

$\sin^n\alpha \cdot e^{-in\theta} = RHS$

Trigonometry Identity

→ If z is a complex variable then

① $\sin 2z = 2\sin z\cos z$

Take RHS

$2\sin z\cos z = \cancel{2} \times \frac{e^{iz} - e^{-iz}}{\cancel{2}i} \times \frac{e^{iz} + e^{-iz}}{2}$

$= \frac{(e^{iz})^2 - (e^{-iz})^2}{2i} = \frac{e^{2iz} - e^{-2iz}}{2i}$

$= \sin 2z = RHS$

② $\cos 2z = 2\cos^2 z - 1$

Take RHS, $= 2 \cdot \left(\frac{e^{iz} + e^{-iz}}{2}\right)^2 - 1$

$= \frac{e^{2iz} + e^{-2iz} + 2e^{iz}\cdot e^{-iz}}{2} - 1$

$= \frac{e^{2iz} + e^{-2iz} + \cancel{2}e^{\cancel{iz-iz}}}{2} - \cancel{2}$

$= \frac{e^{2iz} + e^{-2iz}}{2} = \cos 2z$

$= LHS$

③ $\cos 2z = \cos^2 z - \sin^2 z$

Take RHS, $= \frac{e^{2iz} + e^{-2iz} + 2}{4} + \frac{(e^{2iz} + e^{-2iz} - 2)}{4i^2 = 4(-1)}$

$= \frac{e^{2iz} + e^{-2iz} + \cancel{2} + e^{2iz} + e^{-2iz} - \cancel{2}}{4}$

$= \frac{2e^{2iz} + 2e^{-2iz}}{4} = \frac{e^{2iz} + e^{-2iz}}{2} = \cos 2z$

$= LHS$

④ $\cos 2z = 1 - 2\sin^2 z$

Take RHS, $= 1 - \cancel{2}\cdot\frac{(e^{iz} - e^{-iz})^2}{(2i)^{\cancel{2}}} = 1 + \frac{e^{2iz} + e^{-2iz} - 2}{2}$

$= \frac{e^{2iz} + e^{-2iz}}{2} = \cos 2z = LHS$

⑤ $\tan 2z = \frac{2\tan z}{1 - \tan^2 z}$

Take RHS $= \frac{2\cdot(e^{iz} - e^{-iz})}{i(e^{iz} + e^{-iz})\cdot\left(1 - \left(\frac{e^{iz} - e^{-iz}}{e^{iz} + e^{-iz}}\right)^2 \frac{1}{i^2}\right)}$

$= \frac{1}{i}\left[\frac{2(e^{iz} - e^{-iz})(e^{iz} + e^{-iz})^{\cancel{2}}}{(\cancel{e^{iz} + e^{-iz}})(e^{2iz} + e^{-2iz} + \cancel{2} + e^{2iz} + e^{-2iz} - \cancel{2})}\right]$

$= \frac{1}{i}\left[\frac{\cancel{2}(e^{2iz} - e^{-2iz})}{\cancel{2}(e^{2iz} + e^{-2iz})}\right] = \tan 2z$

$= LHS$

⑥ $\sin 3z = 3\sin z - 4\sin^3 z$

take RHS $= 3\cdot\frac{e^{iz} - e^{-iz}}{2i} - 4\left(\frac{e^{iz} - e^{-iz}}{2i}\right)^3$

$= \frac{e^{iz} - e^{-iz}}{2i}\left[3 + (e^{2iz} + e^{-2iz} - 2)\right]$

$= \frac{e^{3iz} + \cancel{e^{-iz}} + \cancel{e^{iz}} - \cancel{e^{iz}} - e^{-3iz} - \cancel{e^{-iz}}}{2i}$

$= \frac{e^{3iz} - e^{-3iz}}{2i} = \sin 3z$

⑦ $\tan 3z = \dfrac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}$

$= $ Take RHS $= \dfrac{e^{iz}-e^{-iz}}{i(e^{iz}+e^{-iz})}\left[\dfrac{3(e^{2iz}+e^{-2iz}+2) + e^{2iz}+e^{-2iz}-2}{(e^{iz}+e^{-iz})^2}\right]$

$\dfrac{e^{2iz}+e^{-2iz}+2+3(e^{2iz}+e^{-2iz}-2)}{(e^{iz}+e^{-iz})^2}$

$= \dfrac{(e^{iz}-e^{-iz})}{i(e^{iz}+e^{-iz})}\left[\dfrac{4e^{2iz}+4e^{-2iz}+4}{4e^{2iz}+4e^{-2iz}-4}\right]$

$= \dfrac{(e^{iz}-e^{-iz})}{i(e^{iz}+e^{-iz})}\left[\dfrac{e^{2iz}+e^{-2iz}+1}{e^{2iz}+e^{-2iz}-1}\right]$

$= \dfrac{e^{3iz}+\cancel{e^{-iz}}+\cancel{e^{iz}}-\cancel{e^{iz}}-e^{-3iz}-\cancel{e^{-iz}}}{i(e^{3iz}+\cancel{e^{-iz}}-\cancel{e^{iz}}+\cancel{e^{iz}}+e^{-3iz}-\cancel{e^{-iz}})}$

$= \dfrac{e^{3iz}-e^{-3iz}}{i(e^{3iz}+e^{-3iz})}$

$= \tan 3z$

Hyperbolic $f x^n$

① $\sinh x = \dfrac{e^x - e^{-x}}{2}$

② $\cosh x = \dfrac{e^x + e^{-x}}{2}$

③ $\tanh x = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

④ $\operatorname{sech} x = \dfrac{2}{e^x + e^{-x}}$

⑤ $\operatorname{cosech} x = \dfrac{2}{e^x - e^{-x}}$

⑥ $\coth x = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$

⑦ $\cosh^2 x - \sinh^2 x = 1$

⑧ $\operatorname{sech}^2 x + \tanh^2 x = 1$

⑨ $\coth^2 x - \operatorname{cosech}^2 x = 1$

⑩ $\sinh 2x = 2\sinh x \cosh x$

⑪ $\cosh 2x = \cosh^2 x + \sinh^2 x$

$= 2\cosh^2 x - 1$

$= 1 + 2\sinh^2 x$

$= \dfrac{1+\tanh^2 x}{1-\tanh^2 x}$

⑫ $\tanh 2x = \dfrac{2\tanh x}{1+\tanh^2 x}$

⑬ $\sinh 3x = 3\sinh x + 4\sinh^3 x$

⑭ $\cosh 3x = 4\cosh^3 x - 3\cosh x$

⑮ $\tanh 3x = \dfrac{3\tanh x + \tanh^3 x}{1 + 3\tanh^2 x}$

⑯ $\cosh x = \cos ix$

⑰ $\sin ix = i\sinh x$

$[\sinh x = -\sin ix]$

Inverse hyperbolic function

→ prove that $\sinh^{-1} x = \log(x + \sqrt{x^2+1})$

proof = let $\sinh^{-1} x = y$

then $x = \sinh y$

$x = \dfrac{e^y - e^{-y}}{2} = \dfrac{e^y - 1/e^y}{2} = \dfrac{e^{2y}-1}{2e^y}$

$2xe^y = e^{2y} - 1$

$e^{2y} - 2xe^y - 1 = 0$

$e^y = \dfrac{2x \pm \sqrt{4x^2+4}}{2} = x \pm \sqrt{x^2+1}$

take +ve sign

i.e. $e^y = x + \sqrt{x^2+1}$

take log on BS

$\therefore y = \log(x + \sqrt{x^2+1})$

$\sinh^{-1} x = \log(x + \sqrt{x^2+1})$

→ prove that $\cosh^{-1} x = \log(x + \sqrt{x^2-1})$

proof = let $\cosh^{-1} x = y$

$\cosh y = x$

$x = \dfrac{e^y + e^{-y}}{2} = \dfrac{e^{2y}+1}{2e^y}$

$2xe^y = e^{2y} + 1$

$e^{2y} - 2xe^y + 1 = 0$

$e^y = \dfrac{+2x \pm \sqrt{4x^2-4}}{2}$

$e^y = x \pm \sqrt{x^2-1}$

take log on Both side (& take +ve one)

$y = \log(x + \sqrt{x^2-1})$

$\cosh^{-1} x = \log(x + \sqrt{x^2-1})$ ✓

→ prove $\tanh^{-1} x = \frac{1}{2} \log\left(\frac{1+x}{1-x}\right)$

proof → let $y = \tanh^{-1} x$

$x = \tanh y$

$\frac{x}{1} = \frac{e^y - e^{-y}}{e^y + e^{-y}}$

apply componendo & dividendo ie $\frac{a}{b} = \frac{b+a}{b-a}$

$\frac{1+x}{1-x} = \frac{e^y + e^{-y} + e^y - e^{-y}}{e^y + e^{-y} - e^y + e^{-y}}$

$\frac{1+x}{1-x} = \frac{2e^y}{2e^{-y}} = e^{2y}$

take log on BS

$2y = \log\left(\frac{1+x}{1-x}\right)$

$\tanh^{-1} x = y = \log\sqrt{\frac{1+x}{1-x}}$ or $\frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$

Q separate into real and imaginary parts

$\sin^{-1}(\cos\theta + i\sin\theta)$; $0 < \theta < \pi/2$

→ let $\sin^{-1}(\cos\theta + i\sin\theta) = x + iy$

$\cos\theta + i\sin\theta = \sin(x + iy)$

$= \sin x \cos iy + \sin iy \cos x$

$= \sin x \cosh y + i \sinh y \cos x$

equate real and imaginary parts

$\cos\theta = \sin x \cosh y$ — ①

$\sin\theta = \sinh y \cos x$ — ②

squaring and adding ①, ②

$1 = \sin^2 x \cosh^2 y + \sinh^2 y \cos^2 x$

$1 = \sin^2 x(1 + \sinh^2 y) + (1 - \sin^2 x)\sinh^2 y$

$= \sin^2 x + \cancel{\sin^2 x \sinh^2 y} + \sinh^2 y - \cancel{\sinh^2 y \sin^2 x}$

$1 - \sin^2 x = \sinh^2 y$

$\sinh^2 y = \cos^2 x$ — ③

from ②

$\sin^2\theta = \cos^2 x \sinh^2 y$

$\sin^2\theta = \cos^2 x \times \cos^2 x$ — using ③

$\sin^2\theta = \cos^4 x$

$\cos^2 x = \sin\theta$

$x = \cos^{-1}(\sin\theta)^{1/2}$

again from ②

$\sinh y = \frac{\sin\theta}{\cos x}$

$= \frac{\sin\theta}{\sqrt{\sin\theta}} = \sqrt{\sin\theta}$

$y = \sinh^{-1}(\sin\theta)^{1/2}$

since,

$\sinh^{-1} x = \log(x + \sqrt{x^2+1})$

$\therefore y = \log(\sqrt{\sin\theta} + \sqrt{1 + \sin\theta})$ ✓

Q If $u = \log(\tan(\pi/4 + \theta/2))$ then prove that

(i) $\tanh\frac{u}{2} = \tan\frac{\theta}{2}$

(ii) $\cosh u = \sec\theta$

→ (i) $u = \log\tan(\pi/4 + \theta/2)$

$e^u = \tan(\pi/4 + \theta/2)$

$e^{u/2} \cdot e^{u/2} = \frac{1 + \tan\theta/2}{1 - \tan\theta/2}$

$\frac{e^{u/2}}{e^{-u/2}} = \frac{1 + \tan\theta/2}{1 - \tan\theta/2}$

apply componendo & dividendo

ie $\frac{e^{u/2} - e^{-u/2}}{e^{u/2} + e^{-u/2}} = \frac{1 + \tan\theta/2 - 1 + \tan\theta/2}{1 + \tan\theta/2 + 1 - \tan\theta/2}$

$\tanh\frac{u}{2} = \frac{2\tan\theta/2}{2} = \tan\frac{\theta}{2}$

(ii) Take LHS

$\cosh u = \cancel{\frac{1+\tan^2 h^2 \theta/2}{1-\tan h^2 \theta/2}} \quad \frac{1+\tanh^2 u/2}{1-\tanh^2 u/2}$

also, $\frac{1}{\cos\theta} = \sec\theta$

from previous part, $\tanh u/2 = \tan\theta/2$

$\therefore \cosh u = \frac{1+\tan^2\theta/2}{1-\tan^2\theta/2}$

$= \frac{1}{\cos\theta} = \sec\theta$

Hence, $\left[\cosh u = \sec\theta\right]$ ✓

Q If $\cosh(u+iv) = x+iy$ then prove

a) $\frac{x^2}{\cosh^2 u} + \frac{y^2}{\sinh^2 u} = 1$

b) $\frac{x^2}{\cos^2 v} - \frac{y^2}{\sin^2 v} = 1$

→ $x + iy = \cosh(u+iv)$

$= \cos i(u+iv)$

$= \cos(iu - v)$

$= \cos iu \cos v + \sin iu \sin v$

$= \cosh u \cos v + i \sinh u \sin v$

equate Real & Imaginary part

$x = \cosh u \cos v$ -(1)

$y = \sinh u \sin v$ -(2)

also, $\frac{x}{\cosh u} = \cos v$; $\frac{y}{\sinh u} = \sin v$

squaring & adding

ie $\left[\frac{x^2}{\cosh^2 u} + \frac{y^2}{\sinh^2 u} = 1\right]$ ✓

from (1) and (2)

$\frac{x}{\cos v} = \cosh u$ & $\frac{y}{\sin v} = \sinh u$

squaring and substracting.

$\left[\frac{x^2}{\cos^2 v} - \frac{y^2}{\sin^2 v} = \cosh^2 u - \sinh^2 u = 1\right]$ ✓

Q If $\cos(\theta + i\phi) = \cos\alpha + i\sin\alpha$

prove that $\sin^2\theta = \pm\sin\alpha$

→ $\cos\alpha + i\sin\alpha = \cos(\theta+i\phi)$

$= \cos\theta\cos i\phi - \sin\theta\sin i\phi$

$= \cos\theta\cosh\phi - i\sin\theta\sinh\phi$

equal real & Imaginary parts

$\cos\alpha = \cos\theta\cosh\phi$ -(3) $\quad \sin\alpha = -\sin\theta\sinh\phi$ -(2)

squaring & adding

$\cos^2\theta\cosh^2\phi + \sin^2\theta\sinh^2\phi = 1$

$\cos^2\theta\cosh^2\phi + (1-\cos^2\theta)\sinh^2\phi = 1$

$\cos^2\theta\cosh^2\phi + \sinh^2\phi - \cos^2\theta\sinh^2\phi = 1$

$\cos^2\theta(\cosh^2\phi - \sinh^2\phi) + \sinh^2\phi = 1$

$\cos^2\theta + \sinh^2\phi = 1$

$\sinh^2\phi = 1 - \cos^2\theta = \sin^2\theta$ -(1)

also, from (2)

$\sin^2\alpha = \sin^2\theta\sinh^2\phi$

$\sin^2\alpha = \sin^2\theta \cdot \sin^2\theta$ (from (1))

$\sin^2\alpha = \sin^4\theta$

$\left.\sin^2\theta = \pm\sin\alpha\right]$ ✓

* $e^{i(2n\pi)} = \cos(2n\pi) + i\sin(2n\pi) = 1 + 0 = 1$

Logarithmic $f x^n$ of complex variable 1.No.

→ ⊙ If $z = x + iy$ & $w = u + iv$ be two complex variable such that $e^w = z$ then w is said to be logarithmic of z to base e.

properties →

① logarithmic of a complex variable is multivalued function ;

$$\text{Let } e^w = z \qquad \left[\because e^{2n\pi i} = 1\right]$$

$$w = \log z$$

also $z = e^w = e^w \cdot e^{2n\pi i} = e^{w + 2n\pi i}$

② Logarithmic of a positive real no. is real

③ $\log i = \frac{\pi}{2} i$

④ $\underset{\text{big}}{\text{Log}}\, z = 2n\pi i + \underset{\text{small}}{\log_e z}$

Ques1 Seperate into real and imaginary parts of $\text{Log}(a+ib)$

→ $\because \text{Log}(a+ib) = 2n\pi i + \log_e(a+ib)$

Put $a = r\cos\theta$ —① , $b = r\sin\theta$ —②

also $r = \sqrt{a^2+b^2}$

divide ② by ①

$\frac{b}{a} = \tan\theta$, $\theta = \tan^{-1}\left(\frac{b}{a}\right)$

$$\begin{aligned}\text{Log}(a+ib) &= 2n\pi i + \log(r\cos\theta + ir\sin\theta) \\ &= 2n\pi i + \log(r e^{i\theta}) \\ &= 2n\pi i + \log_e r + i\theta \\ &= 2n\pi i + \log(a^2+b^2)^{1/2} + i\tan^{-1}\left(\frac{b}{a}\right)\end{aligned}$$

* $i = e^{\pi/2\, i} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = 0 + i(1) = i$

$$= \frac{1}{2}\log(a^2+b^2) + i\left(2n\pi + \tan^{-1}\frac{b}{a}\right)$$

$\therefore$ Real part of $\text{Log}(a+ib) = \frac{1}{2}\log(a^2+b^2)$

Img part of $\text{Log}(a+ib) = 2n\pi + \tan^{-1}\left(\frac{b}{a}\right)$

Ques2 find the general value of $\text{Log}(-3)$

→ $(-3) = 3(-1) = 3(\cos\pi + i\sin\pi) = 3e^{i\pi}$

i.e. $\text{Log}(-3) = \text{Log}(3e^{i\pi})$

$$\begin{aligned}&= 2n\pi i + \log 3e^{i\pi} \\ &= 2n\pi i + \log 3 + \log_e e^{i\pi} \\ &= \log 3 + i\pi(2n+1)\end{aligned}$$

Ques3 If $i^{\alpha+i\beta} = \alpha + i\beta$ prove that $\alpha^2 + \beta^2 = e^{-(4n+1)\beta\pi}$

→ $\alpha + i\beta = i^{\alpha+i\beta}$

$= e^{\text{Log}\, i^{(\alpha+i\beta)}}$ → general , or $e^{\log i^{\alpha+i\beta}}$ → not used for general result as it can't expand

$$\begin{aligned}&= e^{(\alpha+i\beta)\text{Log}\, i} \\ &= e^{(\alpha+i\beta)\left[2n\pi i + \log_e\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)\right]} \\ &= e^{(\alpha+i\beta)\left[2n\pi i + \log_e e^{\pi/2\, i}\right]} \\ &= e^{(\alpha+i\beta)\left[2n\pi i + i\pi/2\right]} \\ &= e^{\left[2n\pi i\alpha + \alpha i\pi/2 - 2n\pi\beta - \beta\pi/2\right]} \\ &= e^{-\beta\pi\left[2n+\frac{1}{2}\right] + i\alpha\pi\left[2n+\frac{1}{2}\right]} \\ &= e^{-\beta\pi\left(\frac{4n+1}{2}\right) + i\alpha\left(\frac{4n+1}{2}\right)\pi} \\ &= e^{-\beta\pi(4n+1)/2} \cdot e^{i\alpha(4n+1)\pi/2} \\ &= e^{-\beta\pi(4n+1)/2}\left[\cos\left(\frac{4n+1}{2}\right)\pi\alpha + i\sin\left(\frac{4n+1}{2}\right)\pi\alpha\right] \\ &= e^{-\beta\pi(4n+1)/2}\end{aligned}$$

seprate real & imaginary parts

i.e. $\alpha = e^{-\beta\pi(4n+1)/2} \cdot \cos\frac{(4n+1)}{2}\pi\alpha$ — ①

$\beta = e^{-\beta\pi(4n+1)/2} \cdot \sin\frac{(4n+1)}{2}\pi\alpha$ — ②

squaring & adding ① & ②

ie $\alpha^2 + \beta^2 = e^{-\beta\pi(4n+1)}\left[\cos^2\frac{(4n+1)\pi}{2} + \sin^2\frac{(4n+1)\pi}{2}\right]$

$= e^{-(4n+1)\beta\pi}\ (1)$

$= e^{-(4n+1)\beta\pi}$

hence proved.

Ques) Prove $\text{Log}(4+3i) = 2n\pi i + \log(4+3i)$

→ put $4 = r\cos\theta$

$3 = r\sin\theta$

$16 + 9 = r^2$; $[r = 5]$

$\therefore \tan\theta = \frac{3}{4}$; $\theta = \tan^{-1}\left(\frac{3}{4}\right)$

$\text{Log}(4+3i) = 2n\pi i + \log(\cancel{\ } r\cos\theta + i r\sin\theta)$

$= 2n\pi i + \log(re^{i\theta})$

$= 2n\pi i + \log r + \log e^{i\theta}$

$= 2n\pi i + \log 5 + i\theta$

$= 2n\pi i + \log 5 + i\tan^{-1}\left(\frac{3}{4}\right)$

$\therefore$ real part $= \log 5$

Imaginary part $= 2n\pi + \tan^{-1}\left(\frac{3}{4}\right)$

Analytic fx^ns

1) Limit - a complex No. 'l' is said to be the limit of $f(z)$ at z_0 and is denoted by $\lim_{z\to z_0} f(z) = l$

(OR)

if for every $\epsilon > 0$ there exist $\delta > 0$ (delta) S.T.

$|z - z_0| < \delta \Rightarrow |f(z) - l| < \epsilon$

2) Continuity - A fx^n $f(z)$ is said to be continuous at a point z_0 if

(i) $f(z_0)$ exist

(ii) $\lim_{z\to z_0} f(z)$ exist

(iii) $\lim_{z\to z_0} f(z) = f(z_0)$

3) Derivability - the fx^n $f(z)$ is said to be derivable at point z_0 in it's domain if the limit.

$\lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0}$ exists

4) Analyticity → let $f(z)$ be the single valued fx^n in domain. the fx^n $f(z)$ is said to be analytic at point z_0 if 'f' is derivable not only at z_0 but also on every point of some neighbourhood of z_0.

→ An analytic fx^n is also known as regular or holomorphic fx^n

→ A function $f(z)$ which is analytic everywhere is called entire fx^n.

Cauchy - Riemann Equation → CR eq^n

→ <u>Theorem</u> - The necessary and sufficient condition for a function

$$w = f(z) = u(x,y) + iv(x,y)$$

to be analytic in a region R are

(i) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$; $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

(ii) $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are continous in R.

Polar form of C - R eq^n - let (r, θ) be the polar coordinate of a point whose cartesian coordinates are (x,y) then

$$z = x + iy = re^{i\theta}$$

Now, $u + iv = f(z) = f(re^{i\theta})$

$$u + iv = f(re^{i\theta}) \quad -①$$

diff ① w.r.t r & θ

ie $\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} = f'(re^{i\theta}).e^{i\theta} \quad -②$

$$\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} = f'(re^{i\theta}).re^{i\theta}i$$

$$\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} = ri\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)$$

$$= ri\frac{\partial u}{\partial r} - r\frac{\partial v}{\partial r}$$

seprate real & imaginary parts

$\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$ | $\frac{\partial v}{\partial \theta} = r\frac{\partial u}{\partial r}$

$\boxed{\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}}$ | $\boxed{\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}}$

Conjugate fx^n → The real and Imaginary parts of an analytic fx^n are called conjugate fx^n.

Harmonic fx^n → If $f(z) = u(x,y) + iv(x,y)$ be an analytic fx^n in domain D then u & v will satisfy the laplace eq^n

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

and are called harmonic fx^n

Application to flow problem

① If $w = f(z) = \phi(x,y) + i\psi(x,y)$ represent the flow pattern then $w(z)$ is known as complex potential fx^n where $\phi(x,y)$ is velocity potential fx^n and $\psi(x,y)$ is stream fx^n or flux fx^n

② If $w = f(z) = \phi(x,y) + i\psi(x,y)$ represents the heat flow pattern then $\phi(x,y)$ is called isothermal fx^n & $\psi(x,y)$ is heat flow fx^n.

<u>Ques1</u> determine the analytic fx^n whose real part is $e^{2x}(x\cos 2y - y\sin 2y)$

→ let $f(z) = u + iv$ be the analytic fx^n where

$$u = e^{2x}(x\cos 2y - y\sin 2y)$$

$$\frac{\partial u}{\partial x} = e^{2x}(2x\cos 2y - 2y\sin 2y + \cos 2y)$$

& $\frac{\partial u}{\partial y} = -e^{2x}(2x\sin 2y + \sin 2y + 2y\cos 2y)$

since $f(z)$ is analytic ∴ u & v satisfy C-R eq^n.

ie $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ | $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

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$\therefore \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = e^{2x}(2x\cos 2y - 2y\sin 2y + \cos 2y)$

Integrate WRT y

$v = e^{2x}\left[\frac{\cancel{2}x\sin 2y}{\cancel{2}} + y\cos 2y\right] + \phi(x)$

also,

$\frac{\partial v}{\partial x} = 2e^{2x}\left[x\sin 2y + y\cos 2y\right] + e^{2x}\sin 2y + \phi'(x)$

but $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$

$\Rightarrow e^{2x}(2x\sin 2y + 2y\cos 2y + \cancel{\sin 2y}) + \phi'(x)$
$= e^{2x}(2x\sin 2y + \cancel{\sin 2y} + 2y\cos 2y)$

$\phi'(x) = 0$

$\phi(x) = 0$

$\left[v = e^{2x}(x\sin 2y + y\cos 2y) + c\right]$ ✓

→ required imaginary part

Milne thomson's Method

→ ① If u is given, find $f'(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$

and if v is given $f'(z) = \frac{\partial v}{\partial y} - i\frac{\partial v}{\partial x}$

② Replace x by z and y be 0 in $f'(z)$

③ Integrate $f'(z)$ wrt z.

Ques Find the analytic fx^n whose imaginary part is $\frac{x-y}{x^2+y^2}$

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→ let $f(z) = u + iv$

$v = \frac{x-y}{x^2+y^2}$

$\frac{\partial v}{\partial x} = \frac{(x^2+y^2)(1) - (x-y)(2x)}{(x^2+y^2)^2} = \frac{x^2+y^2-2x^2+2xy}{(x^2+y^2)^2}$

$= \frac{-x^2+y^2+2xy}{(x^2+y^2)^2}$

$\frac{\partial v}{\partial y} = \frac{(x^2+y^2)(-1) - (x-y)(2y)}{(x^2+y^2)^2} = \frac{-x^2-y^2-2xy+2y^2}{(x^2+y^2)^2}$

$= \frac{-x^2+y^2-2xy}{(x^2+y^2)^2}$

since $f(z) = u + iv$

$f'(z) = \frac{\partial v}{\partial y} - i\frac{\partial v}{\partial x}$

$= \frac{-x^2+y^2-2xy}{(x^2+y^2)^2} - i\frac{(-x^2+y^2+2xy)}{(x^2+y^2)^2}$

replace x by z & y be 0

ie $f'(z) = \frac{-z^2}{z^4} - i\frac{(-z^2)}{z^4}$

$= -\frac{1}{z^2} + i\left(\frac{1}{z^2}\right) = \frac{-(1-i)}{z^2}$

$f'(z) = \frac{-(1-i)}{z^2}$

integrate WRT z

$\int f'(z)\,dz = \int \frac{-(1-i)}{z^2}\,dz + c$

$f(z) = \frac{-(1-i)\cdot z^{-2+1}}{(-2+1)} + c$

$\left[f(z) = \frac{(1-i)}{z} + c\right]$ ✓

regular means analytic
Laplace means Harmonic fxn

Ques2 If $f(z)$ is a regular fxn of z, then prove that $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4(f'(z))^2$

$\Rightarrow$ $f(z) = u + iv$, $|f(z)| = \sqrt{u^2+v^2}$

$|f(z)|^2 = u^2+v^2$

let $F(x,y) = u^2+v^2$

$$\frac{\partial F}{\partial x} = 2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x}$$

$$\frac{\partial^2 F}{\partial x^2} = 2\left[u\frac{\partial^2 u}{\partial x^2}+\left(\frac{\partial u}{\partial x}\right)^2 + v\frac{\partial^2 v}{\partial x^2}+\left(\frac{\partial v}{\partial x}\right)^2\right]$$

llly $$\frac{\partial^2 F}{\partial y^2} = 2\left[u\frac{\partial^2 u}{\partial y^2}+\left(\frac{\partial u}{\partial y}\right)^2 + v\frac{\partial^2 v}{\partial y^2}+\left(\frac{\partial v}{\partial y}\right)^2\right]$$

add both eqns

$$\frac{\partial^2 F}{\partial x^2}+\frac{\partial^2 F}{\partial y^2} = 2\left[u\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)+\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2\right]+ v\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)+\left[\left(\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2\right]\right] \quad -(1)$$

since $f(z)$ is a regular fxn therefore it satisfies C-R eqn as well as Laplace eqn

$\Rightarrow$ $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ | $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

and $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = 0 = \frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}$

put all these in eq $-(1)$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)F = 2\left[u.0 + \left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2 + v(0) + \left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2\right]$$

$$\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right]F = 2\left[2\left(\frac{\partial u}{\partial x}\right)^2+2\left(\frac{\partial v}{\partial x}\right)^2\right]$$

$$= 4\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2\right] \quad -(2)$$

Also $f(z) = u+iv$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$|f'(z)|^2 = \left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2 \quad -(3)$$

from 2 & 3

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)F = 4.|f'(z)|^2$$

Ques3 If $f(z) = u+iv$ is an analytic fxn of z and $u - v = \frac{\cos x + \sin x - e^{-y}}{2\cos x - 2\cosh y}$ then find $f(z)$ subject to condition $f(\pi/2)=0$

$\Rightarrow$ $f(z) = u+iv$

$if(z) = iu - v$

Adding both

$f(z)+if(z) = u+iv+iu+(-v)$

$f(z)(1+i) = u-v + i(u+v)$

$F(z) = f(z)(1+i) = U + iV$

where $U = u-v$

$V = u+v$

and $F(z) = U+iV$

Now, $U = u-v = \frac{\cos x + \sin x - e^{-y}}{2\cos x - 2\cosh y}$

using Milne theorem, (as we have U)

ie $$\frac{\partial U}{\partial x} = \frac{1}{2}\left[\frac{(-\sin x+\cos x)(\cos x-\cosh y)-(\cos x+\sin x-e^{-y})(-\sin x)}{(\cos x-\cosh y)^2}\right]$$

$$\frac{d(\cos hy)}{dy} = \sinh y \quad \text{not } -\sinh y$$

$$\frac{\partial u}{\partial y} = \frac{1}{2}\left[\frac{(e^{-y})((\cos x - \cosh y) - (+\sinh y)(\cos x + \sin x - e^{-y})}{(\cos x - \cosh y)^2}\right]$$

Now

$$F'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$$

$$= \frac{1}{2(\cos x - \cosh y)^2}\Big[(\cos x - \cosh y)(-\sin x + \cos x) + \sin x(\cos x + \sin x) - i\left[(\cos x - \cosh y)(e^{-y}) - \sinh y(\cos x + \sin x - e^{-y})\right]$$

replace $x \to z$

$y \to 0$

we get

$$F'(z) = \frac{1}{2(\cos z - 1)^2}\Big[(\cos z - 1)(-\sin z + \cos z) + \sin z(\sin z + \cos z) - i\left[(\cos z - 1)(1) + 0\right]\Big]$$

$$= \cancel{\frac{(\cos z - 1)}{2(\cos z - 1)^2}\left[(1 - \sin z) + \sin\right]}$$

$$F'(z) = \frac{-\sin z + 1}{2(\cos z - 1)} + \frac{\sin z(\sin z + 2)}{2(\cos z - 1)^2} - \frac{1}{2(\cos z - 1)}\, i$$

int it w.r.t z

$$\int F'(z) = \frac{1}{2(\cos z - 1)^2}\Big((\cos z - 1)(-\sin z + \cos z) + \sin z \cos z + \sin^2 z - \sin z - i(\cos z - 1)\Big)$$

$$\text{"} \int \cancel{-\cos z \sin z + \sin z + \cos^2 z - \cos z + \sin z \cos z + \sin^2 z - \sin z} - i(\cos z - 1)$$

$$\text{"} \int \frac{1 - \cos z - i(\cos z - 1)}{2(\cos z - 1)^2}$$

$$= \frac{(1 - \cos z) - i(\cos z - 1)}{2(\cos z - 1)^2}$$

$$f'(z) = \frac{\cancel{(1 - \cos z)}\,[1 + i]}{2(1 - \cos z)^{\cancel{2}}}$$

$$\int f'(z) = \frac{1}{2}\int \frac{1}{(1 - \cos z)}(1 + i) = \frac{1}{2}\int \frac{1}{2\sin^2 z/2}(1 + i)\,dz$$

$$= \frac{(1+i)}{2}\int \csc^2\frac{z}{2}\,dz$$

$$= \left(\frac{(1+i)}{2}\cdot -\cot\frac{z}{2}\cdot\frac{2}{1}\right)\frac{1}{2}$$

$$= \left(-\cot\frac{z}{2}\right)\frac{(1+i)}{2} + C$$

$$F(z) = (1+i)[\;\;]$$

$$F(z) = f(z)\,(1+i)$$

$$(1+i)[\;\;] = f(z)\,(1+i)$$

$$z = \frac{\pi}{2}$$

$$\therefore F(z) = (1+i)\left(-\cot\frac{z}{2}\right)$$

also, $F(z) = (1+i)\,\phi(z)$

$$\therefore \cancel{(1+i)}\,\phi(z) = \cancel{(1+i)}\left(-\cot\frac{z}{2}\right)\frac{1}{2}$$

$$\boxed{\phi(z) = -\frac{1}{2}\left(\cot\frac{z}{2}\right)}$$

$$\phi(\pi/2) = -\frac{1}{2}\cot\left(\frac{\pi/2}{2}\right) + C$$

$$0 = -\frac{1}{2}(1) + C \quad ; \quad C = \frac{1}{2}$$

$$\therefore \phi(z) = -\frac{1}{2}\cot\left(\frac{z}{2}\right) + \frac{1}{2}$$

$$\Rightarrow \frac{1}{2}\left(1 - \cot\left(\frac{z}{2}\right)\right)$$

Practise Questions

Que1 $f(z) = u + iv$ be an analytic $f x^n$ of z

If $U = e^{2x}(x\cos 2y - y\sin 2y)$

find the value of v

→ according to C R eq^n

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

also, according to milne theorem

$$f'(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \quad -①$$

ie $\frac{\partial u}{\partial x} = e^{2x}(\cos 2y) + (x\cos 2y - y\sin 2y).e^{2x}.2$

$\frac{\partial u}{\partial y} = e^{2x}x(-2\sin 2y) - e^{2x}\sin 2y - 2e^{2x}y\cos 2y$

put value in ①

ie $f'(z) = e^{2x}(\cos 2y) + 2e^{2x}x\cos 2y - y\sin 2y.e^{2x}.2$

$-i(e^{2x}.x(-2\sin 2y) - e^{2x}\sin 2y - 2e^{2x}y\cos 2y)$

Replace $x \to z$

$y \to 0$

$f'(z) = e^{2z}(1) + 2e^{2z}.z(1) - 0 - i(0+0+0)$

integrate $f'(z)$ WRT z

ie

$$\int f'(z)\,dz = \int(e^{2z} + 2e^{2z}(z))\,dz$$

$$f(z) = \int e^{2z}\,dz + 2\int e^{2z}(z)\,dz$$

$$= \frac{e^{2z}}{2} + 2[$$