Unit - I

Complex Variables.

Double Integral.

A double integral is the Counterpart, in two dimensions of the definite integral of a function of single Variable and is denoted by

$$\iint_R f(x,y)\,dx\,dy.$$

Q Verify that $\int_1^2 \int_3^4 (xy+e^y)\,dy\,dx = \int_3^4 \int_1^2 (xy+e^y)\,dx\,dy$

Soln,

L.H.S $= \int_1^2 \int_3^4 (xy+e^y)\,dy\,dx.$

$$= \int_1^2 \left(\frac{xy^2}{2}+e^y\right)_3^4 dx.$$

$$= \int_1^2 \left(\frac{16x}{2}+e^4-\frac{9x}{2}-e^3\right)dx.$$

$$= \int_1^2 \left(8x+e^4-e^3-\frac{9x}{2}\right)dx$$

$$= \int_1^2 \left[\frac{7}{2}x+(e^4-e^3)\right]dx$$

$$= \left[\frac{7}{2}\frac{x^2}{2}+(e^4-e^3)\right]_1^2.$$

$$\frac{21}{4} + (e^4 - e^3)$$

R.H.S $= \int_3^4 \int_1^2 (xy + e^y)\, dx\, dy.$

$$\cancel{= \int_3^4 \int_1^2 \left(\frac{x^2 y}{2} + e^y\right)}$$

$$= \int_3^4 \left(\frac{x^2 y}{2} + e^y x\right)_1^2 dy$$

$$= \int_3^4 \left(2y + 2e^y - \frac{y}{2} - e^y\right) dy$$

$$= \int_3^4 \left(\frac{3y}{2} + e^y\right) dy$$

$$\left[\frac{3y^2}{4} + e^y\right]_3^4$$

$$\frac{12}{6} + e^4 - \frac{27}{4} - e^3$$

$$12 + e^4 - \frac{27}{4} - e^3$$

$$\frac{21}{4} + e^4 - e^3$$

Q

$$\iint_{x^2+y^2\le 1} x^2y^2\,dx$$

put $y=0$

$x^2 \le 1$ $\qquad$ $y^2 \le 1-x^2$

$x \le \pm 1$ $\qquad$ $y \le \pm\sqrt{1-x^2}$

$-1 \le x \le 1$ $\qquad$ $-\sqrt{1-x^2} \le y \le \sqrt{1-x^2}$

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} x^2y^2\,dy\,dx$$

$$\int_{1}^{1}\left[\frac{x^2y^3}{3}\right]_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dx$$

$$\int_{1}^{1} x^2\left[2(\sqrt{1-x^2})^3\right)dx$$

$$\frac{2}{3}\int_{1}^{1} x^2(1-x^2)^{3/2}\,dx.$$

put $x = \sin\theta$

$dx = \cos\theta\,d\theta$

$$\frac{2}{3}\int_{-\pi/2}^{\pi/2} \sin^2\theta\,(\cos^2\theta)^{3/2}\cos\theta\,d\theta.$$

$$\frac{2}{3}\int_{\frac{\pi}{0}}^{\frac{\pi}{2}} \sin^2\theta\,\cos^4\theta\,d\theta.$$

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formula,

$$\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx & \rightarrow \text{even} \\ 0 & \rightarrow \text{odd.} \end{cases}$$

and,

$$\int_0^{\pi/2} \sin^m x \cos^n x \, dx$$

$$= \frac{(m-1)(m-3)\cdots(n-1)(n-3)}{(m+n)(m+n-2)\cdots} \times \frac{\pi}{2} \text{ (even)}.$$

$$\frac{(m-1)(m-3)\cdots(n-1)(n-3)}{(m+n)(m+n-2)\cdots} \text{ (odd)}$$

$$\Rightarrow 2 \times \frac{2}{3} \int_0^{\pi/2} \sin^2\theta \cos^4\theta \, d\theta.$$

$$\Rightarrow \frac{(2-1)(4-1)(4-3)}{(2+4)(2+4-2)(2+4-4)}$$

$$\frac{4}{3} \times \frac{1 \times 3 \times 1}{6 \times 4 \times 2} \times \frac{\pi}{2}$$

$$\frac{\pi}{24}$$

Imp.

Q. Evaluate $\iint_R y\,dx\,dy$ where R is the Region bounded by the parabola. $y^2 = 4x$ and $x^2 = 4y$.

Soln,

The given Curves are :-

$$y^2 = 4x \quad — 1$$
$$x^2 = 4y. \quad — 2$$

from 2

$$y = \frac{x^2}{4} \quad —(3)$$

from 1.

$$y = \cancel{\sqrt{4x}}\ 2\sqrt{x}. \quad —(4)$$

put the Value of y from (3) in (1)

$$\left(\frac{x^2}{4}\right)^2 = 4x$$

$$x^4 - 64x = 0$$
$$x(x^3 - 64) = 0$$
$$x = 0, 4,$$
$$y = \frac{x^2}{4}, 2\sqrt{x}.$$

$$\iint_R y\,dx\,dy = \int_0^4 \int_{\frac{x^2}{4}}^{2\sqrt{x}} y\,dy\,dx.$$

$$\int_0^4 \left(\frac{y^2}{2}\right)_{x^2/4}^{2\sqrt{x}}$$

$$\frac{1}{2}\int_0^4 (2\sqrt{x})^2 - \left(\frac{x^2}{4}\right)^2 dx.$$

$$\frac{1}{2}\int_0^4 \left(4x - \frac{x^4}{16}\right) dx$$

$$\frac{1}{2}\left[\frac{4x^2}{2} - \frac{x^5}{80}\right]_0^4$$

$$\frac{1}{2}\left[32 - \frac{4\times4\times4\times4\times4}{80}\right]$$

$$\frac{1}{2}\left[32 - \frac{64}{80}\right]$$

$$\frac{48}{5}$$ //

Substitution method for double integral:-

In problems whose region is $x^2+y^2 \le a^2$ and ~~integral~~ integrand is a function of x^2+y^2, we use the transformation $x = r\cos\theta$, $y = r\sin\theta$

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r.$$

If R' is transformed region of R then

$$\iint_R f(x,y)\,dx\,dy = \iint_R f(r\cos\theta, r\sin\theta)\,|J|\,dr\,d\theta$$

$$= \iint_{R'} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

Q Prove that $\iint_R (x^2+y^2)^{7/2}\,dx\,dy = \frac{2\pi}{9}$ where R is the ~~region~~ interior of circle $x^2+y^2=1$.

Sol^n

put $x = r\sin\theta$
$y = r\cos\theta$

$$J = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$= r$

let the region R be mapped into R'

$R = \{(x,y),\ x^2+y^2 \le 1\}$
$R' = \{(r,\theta),\ r^2 \le 1;\ 0 \le \theta \le 2\pi\}$
$R' = \{(r,\theta),\ 0 \le r \le 1,\ 0 < \theta \le 2\pi\}$

$$I = \iint_{x^2+y^2 \le 1} (x^2+y^2)^{7/2}\,dx\,dy.$$

$$\int_0^1 \int_0^{2\pi} (r^2)^{7/2} \, |J| \, d\theta \, dr$$

$$\int_0^1 \int_0^{2\pi} r^7 r^1 \, d\theta \, dr.$$

$$\int_0^1 \left[r^8 \theta \right]_0^{2\pi} dr$$

$$\int_0^1 r^8 \, 2\pi \, dr$$

$$\left[\frac{2\pi r^9}{9} \right]_0^1$$

$$\frac{2\pi}{9}$$

Q Evaluate $\iint \sqrt{a^2-x^2-y^2}\, dx\, dy$ over the semi circle $x^2+y^2=ax$ in the positive quadrant.

Soln.

put $x = r\cos\theta$

$y = r\sin\theta$

$x^2+y^2 \le ax$

$r^2 \le ar\cos\theta$

$r \le a\cos\theta$

let the region R be mapped into R'.

$R' = \{(r,\theta) ; 0 \le r \le a\cos\theta, \ 0 \le \theta \le \frac{\pi}{2}\}$

$$I = \iint_R \sqrt{a^2-(x^2+y^2)}\; dx.dy.$$

$$= \iint \int_0^{\pi/2} \int_0^{a\cos\theta} \sqrt{a^2-r^2}\; |J|\, dr\, d\theta.$$

$$= \int_0^{\pi/2} \int_0^{a\cos\theta} \sqrt{a^2-r^2}\; (r)\, dr\, d\theta$$

$$= -\frac{1}{2} \int_0^{\pi/2} \int_0^{a\cos\theta} \sqrt{a^2-r^2}\; (-2r)\, dr\, d\theta.$$

$$= -\frac{1}{2} \int_0^{\pi/2} \left[\frac{(a^2-r^2)^{3/2}}{3/2} \right]_0^{a\cos\theta}$$

$$= -\frac{1}{3} \int_0^{\pi/2} \left((a^2 - a^2\cos^2\theta)^{3/2} - (a^2)^{3/2} \right) d\theta.$$

$$= -\frac{1}{3} \int_0^{\pi/2} (a^3 \sin^3\theta - a^3)\, d\theta.$$

$$= -\frac{a^3}{3} \left[\int_0^{\pi/2} \sin^3\theta - \int_0^{\pi/2} 1\, d\theta \right]$$

$$= -\frac{a^3}{3} \left[\frac{2}{3} - \theta \right]_0^{\pi/2} \qquad \left[\frac{(m-1)(m-3)\cdots}{m(m-2)(m-4)\cdots} \right]$$

$$= -\frac{a^3}{3} \left[\frac{2}{3} - \frac{\pi}{2} \right]$$

Imp.

Change of order of Integration:-

Q. Change of order of integration in,

$\int_0^a\int_y^a \frac{x\,dx\,dy}{x^2+y^2}$ and hence evaluate the same.

Soln,

limits of integration are $0 \le y \le a$, $y \le x \le a$

Divide the region OAB into vertical strips.

⇒ After change of order of integration. limits are.

$$0 \le y \le x$$

and $0 \le x \le a$

$$I = \int_0^a\int_0^x \frac{x}{x^2+y^2}\,dy\,dx.$$

$$= \int_0^a\int_0^x x\left(\left(\frac{1}{x^2+y^2}\right)dy\right)dx$$

$$\int\frac{1}{a^2+b^2} = \frac{1}{a}\tan^{-1}\frac{b}{a}$$

$$= \int_0^a x\left[\frac{1}{x}\tan^{-1}\frac{y}{x}\right]_0^x dx.$$

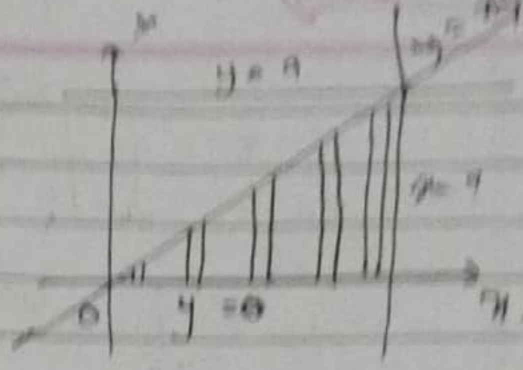
$$= \int_0^a \left(\tan^{-1}\frac{x}{x} - \tan^{-1}0\right)dx.$$

$$= \int_0^a \left(\tan^{-1}1 - 0\right)dx.$$

$$= \int_0^a \frac{\pi}{4}\, dx.$$

$$= \left[\frac{\pi}{4} x\right]_0^a$$

$$= \frac{\pi}{4} a$$



Q change of order of integration in

$$\int_0^1 \int_{4y}^{4} e^{x^2}\, dx\, dy$$ and hence evaluate,

Sol^n

Here the limits are

$4y \le x \le 4$

$0 \le y \le 1$

for $x = 4y$.

y	0	1	2
x	0	4	8.

divide the region into vertical strips.

∴ After changing the order of integration, limits are.

$$0 \le x \le 4, \quad 0 \le y \le \frac{x}{4}$$

$$I = \int_0^4 \int_0^{x/4} e^{x^2}\, dy\, dx$$

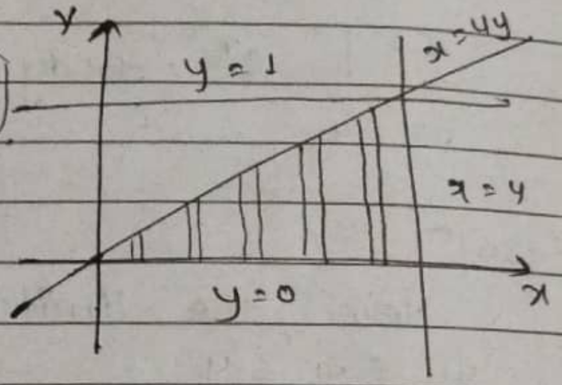
$$= \int_0^4 e^{x^2} \left[y \right]_0^{x/4}$$

$$= \int_0^4 e^{x^2} \frac{x}{4}\, dx$$

put $x^2 = t$

$2x\,dx = dt$ $x=0,\ t=0$

$x\,dx = \frac{dt}{2}$ $x=4,\ t=16$



$$= \int_0^{16} e^t\, dt$$

$$= \left[\frac{e^t}{8} \right]_0^{16}$$

$$= \frac{e^{16} - e^0}{8}$$

$$= \frac{e^{16} - 1}{8}.$$

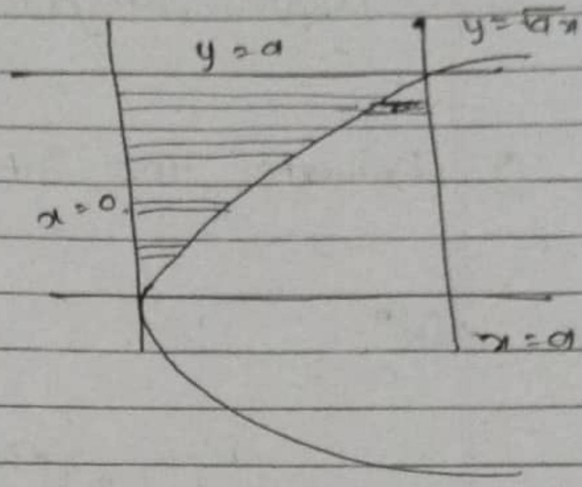
Q $\displaystyle \int_0^a \int_{\sqrt{ax}}^{a} \frac{y^2}{\sqrt{y^4 - a^2x^2}}\, dy\, dx.$

Sol^n

limits are $\sqrt{ax} \le y \le a$

$0 \le x \le a$

divide the region into horizontal strips.



limits are

$0 \le x \le \frac{y^2}{a}$.

$0 \le y \le a$

$$I = \int_0^a \int_0^{y^2/a} \frac{y^2\,dx\,dy}{\sqrt{y^4 - a^2x^2}}.$$

$$= \frac{1}{a}\int_0^a \int_0^{y^2/a} \frac{dx}{\sqrt{\left(\frac{y^2}{a}\right)^2 - x^2}}$$

$$= \frac{1}{a}\int_0^a \sin^{-1}\left[\frac{xa}{y}\right]_0^{y^2/a} dy$$

$$= \frac{1}{a}\int_0^a (\sin^{-1} 1 - \sin^{-1} 0)\,dy.$$

$$= \frac{1}{a}\int_0^a \frac{\pi}{2}\, y^2\,dy$$

$$= \frac{\pi}{2a}\left[\frac{y^3}{3}\right]_0^a$$

$$= \frac{a^3\pi}{6a}$$

$$= \frac{\pi a^2}{6}.$$

. Triple Integral

Q Evaluate the integral $\int_0^1 \int_0^x \int_0^{1+x+y} 1 \, dz\, dy\, dx$.

Soln.

$$\int_0^1 \int_0^x \Big[z \Big]_0^{1+x+y} dy\, dz.$$

$$\int_0^1 \int_0^x (1+x+y)\, dy\, dz.$$

$$\int_0^1 \left[\frac{y^2}{2}\right]_0^x \qquad \int_0^1 \left(y + xy + \frac{y^2}{2}\right)_0^x.$$

$$\int_0^1 x + \frac{x^2}{2}\, dz. \qquad \int_0^1 \left(x + x^2 + \frac{x^2}{2}\right) dz.$$

$$\int_0^1 \frac{2x+x^2}{2}\, dz. \qquad \int_0^1 \left[\frac{2x + 3x^2}{2}\right]_0^1$$

$$\frac{1}{2}\left[\int_0^1 2x + x^2\, dz.\right] \qquad \frac{1}{2}\left(\int_0^1 \frac{\cancel{2}x^2}{\cancel{2}} + \int_0^1 \frac{\cancel{3}x^3}{\cancel{3}}\right]_0^1$$

$$\frac{1}{2}\left[\frac{2x^2}{2} + \frac{x^3}{3}\right]_0^1 \qquad \frac{1}{2}(1+1) = 1.$$

$$\frac{1}{2}\left[1 + \frac{1}{3}\right] = \frac{1}{2}\left[\frac{4}{3}\right] = \frac{1}{6}.$$

Q. Evaluate $\iiint (x+y+z)\, dx\, dy\, dz$ over the tetrahedron bounded by the planes $x=0, y=0, z=0$ and $x+y+z=1$

Sol

$x=0,\ y=0,\ z=0$

$x+y+z=1.$

for x

$x+0+0=1$

$x=1$

for y

$x+y+0=1$

$y=1-x$

for z

$x+y+z=1$

$z=1-x-y$

limits,

$0 \le x \le 1$

$0 \le y \le 1-x$

$0 \le z \le 1-x-y.$

$$\int_0^1 \int_0^{1-x} \int_0^{1-x-y} (x+y+z)\, dz\, dy\, dx$$

$$\int_0^1 \int_0^{1-x} \left[\frac{(x+y+z)^2}{2} \right]_0^{1-x-y} dy\, dx$$

$$\int_0^1 \int_0^{1-x} \frac{(\cancel{x}+\cancel{y}+1-\cancel{x}-\cancel{y})^2}{2} - \frac{(x+y)^2}{2}\, dy\, dx.$$

$$\int_0^1 \int_0^{1-x} \frac{1}{2}\left(1-(x+y)^2\right) dy\, dx$$

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$$= \frac{1}{2}\int_0^1 \left[y - \frac{(x+y)^3}{3} \right]_0^{1-x}.$$

$$= \frac{1}{2}\int_0^1 \left(1 - x - \frac{(\cancel{x} + 1 - \cancel{x})^3}{3} + \frac{x^3}{3} \right) dx.$$

$$= \frac{1}{2}\int_0^1 \left(1 - x - \frac{1}{3} + \frac{x^3}{3} \right) dx.$$

$$= \frac{1}{2}\left[x - \frac{x^2}{2} - \frac{1}{3}x + \frac{x^4}{12} \right]_0^1$$

$$= \frac{1}{2}\left[1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{12} \right]$$

$$= \frac{1}{8}$$

Change of Variable (Cartesian to polar):

1. When cartesian Co-ordinates (x,y,z) are changed into cylindrical Co-ordinates (r,θ,z) then put $x = r\cos\theta$, $y = r\sin\theta$, $z = z$

Here,

$$J = \begin{vmatrix} \partial x/\partial r & \partial x/\partial \theta & \partial x/\partial z \\ \partial y/\partial r & \partial y/\partial \theta & \partial y/\partial z \\ \partial z/\partial r & \partial z/\partial \theta & \partial z/\partial z \end{vmatrix} = r$$

The $dx\, dy\, dz$ is replaced by $r\,dr\,d\theta\,dz$.

$$\iiint_V f(x,y,z)\,dx\,dy\,dz = \iiint f(r\cos\theta, r\sin\theta, z)\,dr\,d\theta\,dz$$

If the given region is cylindrical then the limits of integration are $0 \le r \le a$;
$0 \le \theta \le 2\pi$
$0 \le z \le h$.

2. When Cartesian Co-ordinates changed into Spherical Co-ordinates (r, θ, ϕ) then

put $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$; $z = r\cos\theta$

Thus $dx\,dy\,dz$ is replaced by $r^2\sin\theta\,dr\,d\theta\,d\phi$

$$J = r^2\sin\theta$$

Note:- ① If region is whole Sphere the limits are $0 \le r \le a$, $0 \le \theta \le \pi$, $0 \le \phi \le 2\pi$.

② If region is positive octant then, limits are $0 \le r \le a$, $0 \le \theta \le \frac{\pi}{2}$, $0 \le \phi \le \frac{\pi}{2}$.

$x^2, y^2, z^2 \rightarrow$ spherical.
$x^2, y^2, z \rightarrow$ cylindrical.

Q Evaluate $\iiint_R z(x^2+y^2)\, dx\, dy\, dz$

where,

$R = \{(x,y,z) ; -1 \le z \le 1, x^2+y^2 \le 4\}$

Soln

Change into cylindrical Co-ordinates

put $x = r\cos\theta$

$y = r\sin\theta$

$z = z$

$x^2+y^2 \le 4$

$r^2 \le 4$

$|r| \le 2.$

The space R is changed into R'

$R' = \{(r,\theta,z) ; 0 \le r \le 2, 0 \le \theta \le 2\pi, -1 \le z \le 1\}.$

$$\int_{-1}^{1}\int_{0}^{2\pi}\int_{0}^{2} z(r^2)\, J\, dr\, d\theta\, dz.$$

$$\int_{-1}^{1}\int_{0}^{2\pi}\int_{0}^{2} z\, r^2 r\, dr\, d\theta\, dz.$$

$$\int_{-1}^{1}\int_{0}^{2\pi} z\, r^3\, dr\, d\theta\, dz.$$

$$\int_{-1}^{1}\int_{0}^{2\pi}\left[z\ \frac{r^4}{4}\right]_{0}^{2}$$

$$\int_{-1}^{1}\int_{0}^{2\pi} \frac{16z}{4}\, d\theta\, dz$$

$$\int_{-1}^{1} (4z\theta)_{0}^{2\pi}\, dz.$$

$$\int_{-1}^{1} 4 \times z \times 2\pi\, dz.$$

$$\left[\frac{8\pi z^2}{2}\right]_{-1}^{1}$$

$$0$$

Q $$\iiint_{x^2+y^2+z^2 \le 1} (x^2+y^2+z^2)\, dx\, dy\, dz.$$

Sol^n

change into Spherical Coordinates.

put $x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

$z = r\cos\theta$

$x^2+y^2+z^2 \pm r^2[\sin^2\theta\cos^2\phi + \sin^2\phi\sin^2\phi + \cos^2\theta]$

$x^2+y^2+z^2 = r^2$

$x^2+y^2+z^2 \le 1$

$x^2+y^2+z^2 \le 1$

$r^2 \le 1.$

$|r| \le 1.$

The Space. R is changed into R'.

$0 \leq r \leq 1$

$0 \leq \theta \leq \pi$

$0 \leq \phi \leq 2\pi$

Application for finding Area and Volume of surfaces.

for area

$$\iint_{T} 1 \, dx\,dy.$$

for volume.

$$\iiint_{V} 1 \, dx\,dy\,dz.$$

Q. find the area of ellipse.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

The given ellispe is $\frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1$

Here,

we put $y = 0$

$$\frac{x^2}{a^2} \leq 1$$

$$x^2 \leq a$$

$$-a \leq x \leq a$$

and

$$\frac{y^2}{b^2} \leq 1 - \frac{x^2}{a^2}$$

$$y^2 \leq \frac{a^2-x^2}{a^2} \times b^2$$

$$|y| \leq \frac{b}{a}\sqrt{a^2-x^2}$$

$$-\frac{b}{a}\sqrt{a^2-x^2} \leq y \leq \frac{b}{a}\sqrt{a^2-x^2}$$

Area of ellipse is $\displaystyle\iint_{\frac{x^2}{a^2}+\frac{y^2}{b^2} \leq 1} 1 \, dx\, dy.$

$$= \int_{-a}^{a}\int_{-\frac{b}{a}\sqrt{a^2-x^2}}^{\frac{b}{a}\sqrt{a^2-x^2}} 1 \, dy\, dx.$$

$$= 2\int_{-a}^{a}\int_{0}^{\frac{b}{a}\sqrt{a^2-x^2}} 1 \, dy\, dx$$

$$= 2\int_{-a}^{a} [y]_{0}^{\frac{b}{a}\sqrt{a^2-x^2}} \, dx.$$

$$= 2\int_{-a}^{a} \frac{b}{a}\sqrt{a^2-x^2} \, dx.$$

$$= \frac{2b}{a}\left[\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_{-a}^{a}.$$

$= \pi ab.$

Find the area of the region bounded by

$y^2 = 25x$

$x^2 = 16y$ using double Integral.

- $x^2 = 16y$

$y = \frac{x^2}{16}.$

$\cancel{y = \frac{(18.56)^2}{16}}$

$\cancel{y = 21.52}$

$\left(\frac{x^2}{16}\right)^2 = 25x$

$\left(\frac{x}{2}\right)^4 = 25x$

$y^2 = 25x$

$y = \sqrt{25x}$

$\frac{x^4}{256} = 25\cancel{x}$

also,

$y = \frac{x^2}{16}.$

$x^3 = 6400$

$x = 18.56$

$\frac{x^2}{16} \le y \le 5\sqrt{x}.$

$0 \le x \le 4(100)^{1/3}$

$0 \le x \le 18.56.$

Now,

$$\int_0^{18.56} \int_{\frac{x^2}{16}}^{5\sqrt{x}} 1 \, dy \, dx.$$

$$\int_0^{18} [y]_{x^2/16}^{5\sqrt{x}}.$$

$$\int_0^{18.56} \left[5\sqrt{x} - \frac{x^2}{16}\right]$$

$$\int_0^{18.56}\left[5x^{1/2} - \frac{x^2}{16}\right]$$

$$\left[\frac{5x^{3/2}}{3/2} - \frac{x^3}{16\times 3}\right]_0^{18.56}$$

$$\left[\frac{5\times(18.56)^{3/2}}{3/2} - \frac{(18.56)^3}{16\times 3}\right]$$

$$\left[266.52 - 133.19\right]$$

$$\left[133.33\right]$$

Q. find the Volume of the Sphere of radius 'a' i.e. $x^2+y^2+z^2 \le a^2$, $a>0$.

Soln.

$\cancel{x^2+y^2+z^2 \le a^2}$

$\cancel{x^2+0+0 \le a^2}$

put

$x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

$z = r\cos\phi$

$x^2+y^2+z^2 \le a^2$

$r^2 \le a^2$

$r \le a$

$0 \le r \le a$, $0 \le \theta \le \pi$, $0 \le \phi \le 2\pi$

$\frac{4\pi a^3}{3}$

Now, $\int_0^{2\pi}\int_0^{\pi}\int_0^{a} 1\, r^2 \sin\theta \, dr\, d\theta\, d\phi$

$\int_0^{2\pi}\int_0^{\pi}\int_0^{a} r^2 \sin\theta \, dr\, d\theta\, d\phi$

$\int_0^{2\pi}\int_0^{\pi}\left[\sin\theta \frac{r^3}{3}\right]_0^a d\theta\, d\phi$

$\int_0^{2\pi}\int_0^{\pi}\left[\sin\theta \frac{a^3}{3}\right] d\theta\, d\phi$

$\int_0^{2\pi} \frac{a^3}{3}\left[-\cos\theta\right]_0^{\pi} d\phi$

$\int_0^{2\pi} \frac{a^3}{3}\left[-\cos\pi + \cos 0\right] d\phi$

$\frac{a^3}{3}\int_0^{2\pi} 2\, d\phi$

$\frac{2a^3}{3}\int_0^{2\pi} d\phi$

$\frac{2a^3}{3} \times 2\pi$

$\frac{4\pi a^3}{3}$

Line Integral :

An integral which is to be evaluated along a curve is called a line integral

$$\int_c \vec{F}\cdot d\vec{R} \quad \text{or} \quad \int_c \vec{F}\,\frac{d\vec{R}}{dt}\,dt.$$

where $\vec{R} = x\hat{i} + y\hat{j} + z\hat{k}$

→ If the parameteric equation of the curve c are $x = x(t)$, $y = y(t)$ $z = z(t)$ at $t = t_1$, at A and $t = t_2$ at B then

$$\int_c \vec{F}\,d\vec{R} = \int_{t_1}^{t_2}\left(f_1\frac{dx}{dt} + F_2\frac{dy}{dt} + f_3\frac{dz}{dt}\right)dt.$$

→ If c is closed curve then the integral sign, $\int_c$ is replaced by $\oint_c$.

Q If $\vec{F} = 3xy\hat{i} - y^2\hat{j}$ evaluate $\int_c \vec{F}\,d\vec{r}$ where C is the arc of the parabola $y = 2x^2$ from $(0,0)$ to $(1,2)$

Solⁿ

Since the integration is performed in xy-plan $z = 0$ we take

$$\vec{r} = x\hat{i} + y\hat{j}$$
$$d\vec{r} = dx\hat{i} + dy\hat{j}$$

On curve C : $y = 2x^2$ from $(0,0)$ to ~~(x,2)~~ $(1,2)$

$$\vec{F} \cdot d\vec{r} = (3xy\hat{i} - y^2\hat{j}) \cdot (dx\hat{i} + dy\hat{j})$$
$$= 3xy\,dx - y^2 dy$$
$$= 3x(2x^2)\,dx - 4x^4 \times 4x\,dx$$
$$= (6x^3 - 16x^5)\,dx.$$

Also x varies from 0 to 1.

$$\int_C \vec{F} \cdot d\vec{r}$$

$$\int_0^1 (6x^3 - 16x^5)\,dx.$$

$$\left[\frac{6x^4}{4} - \frac{16x^6}{6}\right]_0^1$$

$$\frac{6}{4} - \frac{16}{6} = \frac{-7}{6}.$$

πa^2

Q A vector field is given by $\vec{F} = (\sin y)\hat{i} + x(1$ Evaluate the integral over the circular path given by $x^2+y^2=a^2, z=0$

Sol^n

$$\vec{r} = x\hat{i} + y\hat{j}$$
$$dr = dx\hat{i} + dy\hat{j}$$

$$x^2+y^2=a^2$$

put $x = a\sin t \quad dx = a\cos t$

$y = a\cos t. \quad dy = -a\sin t$

$F.dr = (\sin y)\hat{i} + x(1-\cos y)\hat{j} \cdot (a\cos t\,\hat{i} - a\sin t\,\hat{j})$

$= a\sin y\cos t\,\hat{i} + x a\sin t\,\hat{j} - a x\cos y\,\cancel{\cos t}\,\sin t\,\hat{j}$

$= a$

Q. If $\vec{F} = 2y\hat{i} - z\hat{j} + x\hat{k}$ evaluate $\oint_c \vec{F} \times d\vec{r}$ along the curve $x = \cos t$, $y = \sin t$, $z = 2\cos t$ from $t = 0$ to $\frac{\pi}{2}$.

Sol^n.

$$\vec{F} \times d\vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2y & -z & x \\ dx & dy & dz \end{vmatrix}$$

$$= \hat{i}(-zdz - xdy) + \hat{j}(xdx - 2ydz) + k(2ydy + zdx)$$

from $t = 0$ to $\frac{\pi}{2}$.

put $x = \cos t$, $y = \sin t$, $z = 2\cos t$.

$$\vec{F} \times d\vec{r} = \hat{i}[-2\cos t \times -2\sin t - \cos t\cos t)dt +$$
$$j(-\cos t\sin t - 2\sin t \times -2\sin t)dt +$$
$$k(2\sin t\cos t + 2\cos t(-\sin t))dt$$

$$\int_0^{\pi/2} \vec{F} \times d\vec{r} = \int_0^{\pi/2} \left[(4\cos t\sin t - \cos^2 t)\hat{i} + (-\cos t\sin t + 4\sin^2 t)\hat{j} + k(\cancel{\sin^2 t} - \cancel{\sin^2 t})\hat{k}\right]dt$$

$$= \int_0^{\pi/2} (4\cos t\sin t - \cos^2 t)\hat{i}dt + \int_0^{\pi/4} \sin t\cos t\,dt$$

$$+ \int_0^{\pi/4} -\cos t\, 4\sin^2 t\, dt.$$

Date 10/04/25

Surface Integral.

Any integral which is to be evaluated over a surface is called surface integral.

$$\iint_S F \cdot ds = \iint_S (\vec{F} \cdot \hat{n})\, ds$$

where. $\hat{n} = \dfrac{\nabla \phi}{|\nabla \phi|}$

Note :- ① If R is the projection of S on xy plane then $ds = \dfrac{dx\,dy}{|\hat{n}\,\hat{k}|}$

② If R is the projection of S on yz plane then $ds = \dfrac{dy\,dz}{|\hat{n}\,\hat{i}|}$

③ If R is the projection of S on xz plane then $ds = \dfrac{dx\,dz}{|\hat{n}\,\hat{j}|}$

Q Evaluate $\iint_S (yz\hat{i} + zx\hat{j} + xy\hat{k})\, ds$ where S is the surface of sphere $x^2+y^2+z^2=1$ in the 1st Octant.

Soln

$$\phi = x^2+y^2+z^2-1$$

$$\nabla\phi = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z} \right)(x^2+y^2+z^2-1)$$

$$= 2x\hat{i} + 2y\hat{j} + 2z k.$$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$$

$$= \frac{2x\hat{i} + 2y\hat{j} + 2z\hat{k}}{\sqrt{4x^2+4y^2+4z^2}}$$

$$= \frac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2+y^2+z^2}}$$

$$= x\hat{i} + y\hat{j} + z\hat{k}$$

$$\hat{n}\cdot\hat{k} = (x\hat{i} + y\hat{j} + z\hat{k})\,\hat{k}$$

$$\hat{n}\cdot\hat{k} = z$$

$$\vec{F}\cdot\hat{n} = (yz\hat{i} + zx\hat{j} + xy\hat{k})\cdot(x\hat{i} + y\hat{j} + z\hat{k})$$

$$= xyz + xyz + xyz$$

$$= 3xyz.$$

Since it lies in 1st Octant, $z = 0$.

$$x^2+y^2 = 1$$

$$x^2 = 1$$

$$x = \pm 1$$

limits of $x = 0$ to 1.

$$y = \sqrt{1-x^2}$$

limits of y : 0 to $\sqrt{1-x^2}$

$$\iint_S 3xy\cancel{z}\,\frac{dx\,dy}{\cancel{z}}$$

$$\int_0^1 \int_0^{\sqrt{1-x^2}} 3xy \, dy \, dx.$$

$$\int_0^1 3x \left[\frac{y^2}{2}\right]_0^{\sqrt{1-x^2}} dx.$$

$$\int_0^1 3x \left[\frac{1-x^2}{2}\right] \cdot dx.$$

$$\frac{3}{2} \int_0^1 x \, dx - \int_0^1 x^3 \, dx$$

$$\frac{3}{2} \left[\frac{x^2}{2}\right]_0^1 - \left[\frac{x^4}{4}\right]_0^1$$

$$\frac{3}{2} \times \left[\frac{1}{2} - \frac{1}{4}\right]$$

$$\frac{3}{2} \times \frac{1}{4}$$

$$\cancel{\frac{3}{12} = \frac{1}{4}} \qquad \frac{3}{8}$$

Q. Evaluate $\iint_S \vec{F} \cdot \hat{n} \, ds$ where $\vec{F} = (x+y^2)\hat{i} - 2x\hat{j} + 2yz\hat{k}$ and S is the surface of the plane $2x+y+2z=6$ in the first Octant.

Sol^n.

$$\phi = 2x + y - 2z - 6.$$

$$\nabla\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(2x + y - 2z - 6)$$

$$= (2\hat{i} + \hat{j} + 2\hat{k})$$

$$\hat{n} = \frac{(2\hat{i} + \hat{j} + 2k)}{\sqrt{4+1+4}}$$

$$= \frac{2\hat{i} + \hat{j} + 2k}{3.}$$

$$n \cdot \hat{k} = \frac{1}{3}(2\hat{i} + \hat{j} + 2k) \cdot \hat{k}$$

$$= \frac{2}{3}$$

$$\vec{F} \cdot \hat{n} = \frac{1}{3}\left[(x+y^2)\hat{i} - 2x\hat{j} + 2yz\hat{k}\right]\left[2i + j + 2k\right]$$

$$= \frac{1}{3}\left[\cancel{2x} + 2y^2 - \cancel{2x} + 4yz\right]$$

$$= \frac{1}{3}(2y^2 + 4yz)$$

$$= \frac{2y}{3}(y + 4z)$$

<u>formula</u>

$$\iint_S \frac{2y}{3}(y + 4z) \cdot \frac{dx\,dy}{\frac{2}{3}}$$

limits.

$$2x + y + 2z = 6$$
$$2x + y = 6$$

for x

$$2x = 6$$
$$x = 3.$$

$x = 0$ to 3.

for y,

$$y = 6 - 2x$$

$y = 0$ to $(6-2x)$

$$\int_0^3 \int_0^{6-2x} \frac{2y}{3}(y+4z) \frac{dy\,dx}{\cancel{2} \cdot \frac{2}{3}}$$

$$\frac{2}{3} \times \frac{3}{\cancel{2}} \int_0^3 \int_0^{6-2x} y^2 + 4yz \; dy\,dx$$

$$\int_0^3 \int_0^{6-2x} y^2 + 4yz \; dy\,dx.$$

$$\int_0^3 \left[\frac{y^3}{3}\right]_0^{6-2x} + 4z \left[\frac{y^2}{2}\right]_0^{6-2x} dx$$

$$\int_0^3 \frac{(6-2x)^3}{3} + \cancel{4}^{2} z \frac{(6-2x)^2}{\cancel{2}} \, dx.$$

$$\int_0^3 \frac{(6-2x)^3}{3} + \frac{\cancel{2}(6-2x-y)(6-2x)^2}{\cancel{2}}\, dx$$

$$\int_0^3 (6)^3 \mp 8x^3 \mp 3\times(6)^2 2x + 3\times 6\times(2x)^2 dx +$$

$$\int_0^3 (6-2x-y)(36+4x^2-24x)\, dx$$

Volume Integral :-

Any integral which is to be evaluated over a volume is called Volume integral.

$$\iiint_V \vec{F}\, dv.$$

Q. If $\vec{F} = (2x^2 - 3z)\hat{i} - 2xy\hat{j} - 4x\hat{k}$ then evaluate $\iiint \nabla \cdot \vec{F}\, dv$ where v is bounded by planes $x=0$, $y=0$, $z=0$ and $2x + 2y + z = 4$.

Sol^n

$$\nabla \cdot \vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} \right) (6x^2 - 3z)\hat{i} - 2xy\hat{j} - 4x\hat{k}$$

$$= 4x + (-2x) - 0$$

$$= 2x.$$

$$2x + 2y + z = 4$$

put $y, z = 0$.

$$x = 2$$

$$2x + 2y + 0 = 4$$

$$y = 2 - x$$

$$z = 4 - 2x - 2y.$$

$$\int_0^2 \int_0^{2-x} \int_0^{4-2x-2y} 2x \, dz\,dy\,dx.$$

$$\int_0^2 \int_0^{2-x} 2x \left[z\right]_0^{4-2x-2y} \cdot dy\,dx$$

$$\int_0^2 \int_0^{2-x} 2x\,(4-2x-2y)\,dy\,dx.$$

$$\int_0^2 \int_0^{2-x} 8x - 4x^2 - 4xy \; dy\,dx.$$

$$\int_0^2 8x\left[y\right]_0^{2-x} - 4x^2\left[y\right]_0^{2-x} - \cancel{4}^{2}x\left[\frac{y^2}{\cancel{2}}\right]_0^{2-x} dx$$

$$\int_0^2 8x(2-x) - 4x^2(2-x) - 2x(2-x)^2\,dx$$

$$\int_0^2 16x - 8x^2 - \cancel{8x^2} + 4x^3 - 8x - 2x^3 + \cancel{8x^2}\,dx.$$

$$\int_0^2 16x - \cancel{16x^2} + 2x^3 - 8x^2 \; \cancel{dx} - 8x\,dx$$

$$16\left[\frac{x^2}{2}\right]_0^2 + 2\left[\frac{x^4}{4}\right]_0^2 - 8\left[\frac{x^3}{3}\right]_0^2 - 8\left[\frac{x^2}{2}\right]_0^2 dx.$$

$$\cancel{16}^{8}\times\frac{4}{\cancel{2}_1} + \cancel{2}\cancel{6}^{3}\times\frac{\cancel{16}^{8}}{\cancel{4}\cancel{2}_1} - \frac{\cancel{48}^{16}\times 8}{\cancel{3}_1} \quad 8\times 4 - \frac{8\times 8}{3} + \frac{16}{2} - 16$$

$$\cancel{32\;24 + 24 - 16\times 8} \quad 32 - \frac{64}{3} + 8 - 16$$

$$\frac{8}{3}.$$

Q. If $\vec{A} = 2xz\hat{i} - x\hat{j} + y^2\hat{k}$. evaluate

$\iiint \vec{A}\, dv$ where v is the region bounded by the surface $x=0, y=0, x=2, y=6, z=x^2, z=4$

Soln.

$$\int_0^2\int_0^6\int_{x^2}^4 2xz\hat{i} - x\hat{j} + y^2\hat{k}\; dz\,dy\,dx$$

$$\int_0^2\int_0^6 2x\left[\frac{z^2}{2}\right]_{x^2}^4\hat{i} - x[z]_{x^2}^4\hat{j} + y^2[z]_{x^2}^4\hat{k}\; dy\,dx$$

$$\int_0^2\int_0^6 2x^2\left[\frac{x^4}{24} - \frac{16}{2}\right]\hat{i} - x[x^2-4]\hat{j} + y^2[x^2-4]\hat{k}\,dy\,dx$$

$$\int_0^2\int_0^6 (x^4-16)\hat{i} - (x^3-4x)\hat{j} + (x^2y^2-4y^2)\hat{k}\,dy\,dx$$

$$\int_0^2 x^4[y]_0^6\hat{i} - 16[y]_0^6\hat{i} - x^3[y]_0^6\hat{j} - 4x[y]_0^6\hat{j} + x^2\left[\frac{y^3}{3}\right]_0^6\hat{k} - 4\left[\frac{y^3}{3}\right]_0^6\hat{k}\; dx.$$

$$\int_0^2 6x^4\hat{i} - 96\hat{i} - 6x^3\hat{j} - 24x\hat{j} + 72x^2\hat{k} - 288\hat{k}\; dx.$$

$$6\frac{x^5}{5}\hat{i} - 96\hat{i} - 6\frac{x^4}{4}\hat{j} - 24\frac{x^2}{2}\hat{j} + 72\frac{x^3}{3}\hat{k} - 288\hat{k}$$

$$\left(\frac{192}{5} - 96\right)\hat{i} - (24-48)\hat{j} + (192-288)\hat{k}$$

Most Imp.

Green's Theorem :-

statement :- If $m(x,y)$ and $N(x,y)$ be continuous function of x and y having continuous partial derivatives $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$ in a region R of the xy plane bounded by a closed curve c then.

$$\oint_c (M\,dx + N\,dy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy.$$

Q Verify Green's Theorem in the plane for

$\oint_c (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy$ where c is the boundary of the region defined by

(a) $y = \sqrt{x}$, $y = x^2$

(b) $x = 0$, $y = 0$, $x + y = 1$.

Here,

$$m = 3x^2 - 8y^2, \quad N = 4y - 6xy.$$

$$\frac{\partial M}{\partial y} = -16y, \quad \frac{\partial N}{\partial x} = -6y$$

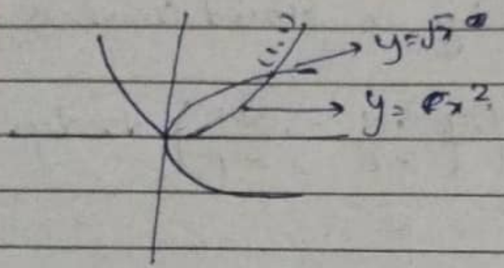
$$\cancel{\frac{\partial N}{\partial x} = 8x}, \quad \cancel{\partial M}$$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = -6y + 16y = 10y.$$

If R is the region bounded by c.

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \int_0^1 \int_{x^2}^{\sqrt{x}} 10y \, dy\,dx.$$

$$= \int_0^1 10\left[\frac{y^2}{2}\right]_{x^2}^{\sqrt{x}}$$



$$= \int_0^1 \frac{10}{2}(x - x^4)\,dx.$$

$$= 5\left[\frac{x^2}{2} - \frac{x^5}{5}\right]_0^1$$

$$= \frac{3}{2}$$

Now,

$$\oint_C M dx + N dy = \int_{C_1} M dx + N dy + \int_{C_2} M dx + N dy$$

Along C_1, $y = x^2$,
$dy = 2x dx$,
the limits of x are from 0 to 1.

Along C_1 = $\int_0^1 (3x^2 - 8x^4)\,dx + (4x^2 - 6x^3) 2x\,dx.$

$$= \int_0^1 (3x^2 + 8x^3 - 20x^4)\,dx$$

$$= -1.$$

Along C_2: $y^2 = x$, $2y\,dy = dx$

the limits of y are from 1 to 0.

Along C_2 $= \int_{C_2} M\,dx + N\,dy.$

$$= \int_1^0 (3y^4 - 8y^2)\,2y\,dy + (4y - 6y^3)\,dy$$

$$= \int_1^0 (4y - 22y^3 + 6y^5)\,dy = \frac{5}{2}.$$

Now,

$$\oint_C M\,dx + N\,dy = \int_{C_1} M\,dx + N\,dy + \int_{C_2} M\,dx + N\,dy.$$

$$= -1 + \frac{5}{2} = \frac{3}{2}.$$

Theorem Verified

(b) Here,

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy.$$

Region $\Rightarrow$ $x = 0,\ y = 0\ ;\ x + y = 1.$

$$\int_0^1 \int_0^{1-x} 10y\,dy\,dx.$$

$$\int_0^1 10\left(\frac{y^2}{2}\right)_0^{1-x} dx.$$

$$5 \int_0^1 (1-x)^2\,dx$$

$$= 5\left[\frac{(1-x)^3}{-3}\right]_0^1$$

$$= \frac{5}{3}.$$

$$\oint_C Mdx + Ndy = \int_{OA} Mdx + Ndy + \int_{AB} Mdx + Ndy + \int_{BO} Mdx + Ndy.$$

Along OA:

$y = 0$, $dy = 0 \Rightarrow x$ from 0 to 1.

$$\int_{OA} Mdx + Ndy = \int_0^1 3x^2 dx = 3\left[\frac{x^3}{3}\right]_0^1 = 1.$$

Along AB,

$y = 1 - x$, $dy = -dx$ limits of x from 1 to 0

$$\int_{AB} Mdx + Ndy = \int_1^0 [3x^2 - 8(1-x)^2] + 4(1-x) - 6x(1-x)dx$$

$$= \frac{8}{3}.$$

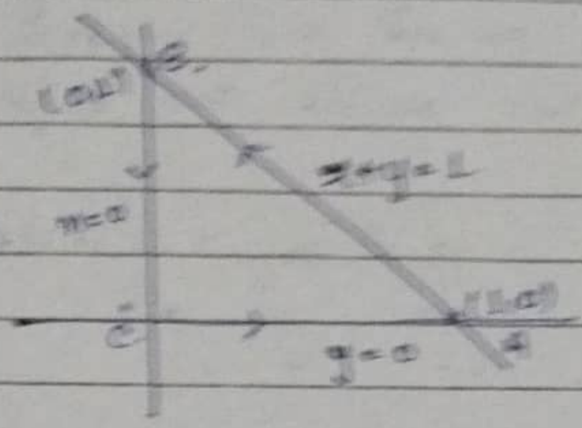
Along BO $\Rightarrow$

$x = 0$, $dx = 0$, limits $y = 1$ to 0.

$$\int_{BO} Mdx + Ndy = \int_1^0 4y\,dy = 2[y^2]_0^1 = -2.$$

$$\oint_C M\,dx + N\,dy = 1 + \frac{5}{3} - 2$$

$$= \frac{2}{3}.$$



Stoke's Theorem:-

The surface integral of the curl of a function over a surface bounded by a closed surface is equal to the line integral of the particular vector function around that surface

$$\therefore \oint_c \vec{F}\cdot d\vec{r} = \iint_S \text{Curl}\,\vec{F}\cdot\hat{n}\,ds.$$

Q Verify stokes theorem for

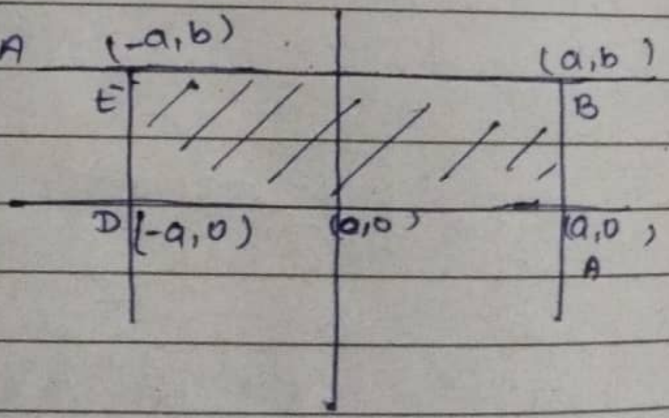
$\vec{F} = (x^2+y^2)\hat{i} - 2xy\hat{j}$ taken around the rectangle bounded by the lines $x=\pm a$, $y=0$, $y=b$.

Sol^n

$$\oint_c \vec{F}\cdot d\vec{r} = \oint [[(x^2+y^2)\hat{i} - 2xy\hat{j}]\cdot(dx\,\hat{i} + dy\,\hat{j})]$$

$$= \oint_c (x^2+y^2)\,dx - 2xy\,dy$$

The curve consist of four lines AB, BE, ED, DA



Along AB, $x=a$, $dx=0$,

$$\oint_{AB} (x^2+y^2)\,dx - 2xy\,dy$$

$$\int_0^b -2ay\,dy$$

nction

s

ector

$$= -a\left[y^2\right]_0^b$$

$$= -ab^2$$

Along BE, $y = b$, x varies from a to $-a$

$$\oint_{BE} (x^2+y^2)dx - 2xy\,dy = \int_a^{-a} \cancel{x^2+y^2}\ (x^2+b^2)dx$$

$$= \frac{x^3}{3} + b^2\left[x\right]_a^{-a}$$

$$= -\frac{2a^2}{3} - 2ab^2.$$

Along ED, $x = -a$, $dx = 0$.

$$\oint_{ED} (x^2+y^2)dx - 2xy\,dy = \int_b^0 2ay\,dy.$$

$$= -ab^2.$$

Along DA, $y = 0$, $dy = 0$.

$$\oint_{DA} (x^2+y^2)dx - 2xy\,dy = \int_{-a}^{a} x^2 dx.$$

$$= \frac{2a^3}{3}.$$

Adding above all lines.

$$\oint_C \vec{F}\cdot d\vec{r} = -ab^2 - \cancel{\frac{2a^3}{3}} - 2ab^2 - ab^2 + \cancel{\frac{2a^3}{3}}$$

$$= -4ab^2$$

Now,

$$\iint_S \text{curl } \vec{F} \cdot \hat{n} \, ds.$$

$$\text{curl } f = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2+y^2 & -2xy & 0 \end{vmatrix}.$$

$$= -4y\hat{k}$$

Here, $\hat{n} = \hat{k}$

$$\text{curl } \hat{F} \cdot \hat{n} = -4y \, \hat{k} \cdot \hat{k}$$

$$= -4y.$$

$$\iint \text{curl } \vec{F} \cdot \hat{n} \, ds = \int_0^b \int_{-a}^{a} -4y \, dy \, dx$$

$$= \int_0^b -4y \left[x\right]_{-a}^{a} dx.$$

$$= -8a \int_0^b y \, dy.$$

$$= -4ab^2$$

Verified.