Fourier Series:

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C$$

$$\int \sin x \, dx = -\cos x + C.$$

$$\int \cos x \, dx = \sin x + C.$$

Integeration By parts:/Euler Formula

$$\int u \cdot v \, dx = u \int v \, dx - \int \left(u' \int v \, dx \right)$$

OR

$$\int u \cdot v \, dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \cdots$$

Some Important Formulas:

1) $\int_c^{c+2\pi} \cos mx \cdot \cos nx \, dx$

$$= \begin{cases} 0, & m \neq n \\ \pi, & m = n \end{cases}$$

eg :- $\int_0^{2\pi} \cos 6x \cdot \cos 7x \, dx = 0$

Ques) Show that the series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots\infty$ is condn convergent.

Soln Here

$\cancel{a_n} \sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}$ $\cancel{a_{n+1}=(-1)}$

$a_n=\frac{1}{n}$ $a_{n+1}=\frac{1}{n+1}$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{1}{n} = \frac{1}{\infty} = 0.$

Now

$n < n+1$ for all n

$\frac{1}{n^2} > \frac{1}{n+1}$

$\Rightarrow \frac{1}{n+1} < \frac{1}{n}$ $a_{n+1} < a_n.$

∴ By Leibnitz test, the $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$ is cgt.

Now, $\sum_{n=1}^{\infty}|a_n| = \sum_{n=1}^{\infty}|(-1)^{n-1}a_n|$

$= \sum_{n=1}^{\infty}\frac{1}{n}$

Here $p=1$ so, By p test it is not convergent. / divergent.

∴ Hence, the series is conditional convergent.

TEACHER SIGNATURE

Even Funcn & Odd Funcn

Odd Funcn:

A Funcn is said to be odd function if $f(-x) = -f(x)$ for all x

eg → $f(x) = x^3$
$f(-x) = (-x)^3$
$= -f(x)$

$f(x) = \sin x$
$f(-x) = \sin(-x)$
$= -\sin x$
$= -f(x)$

Even Funcn:

A funcn is said to be even funcn if $f(-x) = f(x)$ for all x

eg → $f(x) = x^2$
$f(-x) = (-x)^2$
$= x^2$
$= f(x)$.

$f(x) = \cos x$
$f(-x) = \cos(-x)$
$= \cos x$
$= f(x)$

$$= \frac{1}{4\pi} \left[\frac{(\pi - x)^3}{3(-1)} \right]_0^{3\pi}$$

$$= \frac{-1}{12\pi} \left[(\pi - 2\pi)^3 - (\pi - 0)^3 \right]$$

$$= \frac{-1}{12\pi} \left[-\pi^3 - \pi^3 \right].$$

$$= \frac{\cancel{2}\pi^{\cancel{3}2}}{6\cancel{12\pi}} = \frac{\pi^2}{6}$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} \frac{1}{4} (\pi - x)^2 \cdot \cos nx \, dx.$$

$$= \frac{1}{4\pi} \left[(\pi - x^2)^2 \frac{\sin nx}{n} - 2(\pi - x)(-1) \left(\frac{-\cos nx}{n^2} \right) + 2(-1)(-1) \left(-\frac{\sin nx}{n^3} \right) \right]_0^{2\pi}$$

$$= \frac{1}{4\pi} \left[(\pi - x^2) \frac{\sin nx}{n} - 2(\pi - x) \frac{\cos nx}{n^2} - \frac{2 \sin nx}{n^3} \right]_0^{2\pi}$$

$$= \frac{1}{4\pi} \left[\left(0 - 2(\pi - 2\pi) \frac{\cos 2n\pi}{2} - 0 \right) - \left(0 - 2\pi \frac{\cos 0}{n^2} - 0 \right) \right]$$

$$= \frac{1}{4\pi} \left[+ \frac{2\pi}{n^2} + \frac{2\pi}{n^2} \right] = \frac{1}{n^2}$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} \frac{1}{4} (\pi - x)^2 \sin x \, dx$$

$$=$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

where,

$$a_0 = \frac{1}{\pi}\int_{c}^{c+2\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_{c}^{c+2\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_{c}^{c+2\pi} f(x)\sin nx\,dx$$

Ques 1) Find the fourier series of the funcn $f(x) = \frac{1}{4}(\pi - x^2)$ $\quad 0 < x < 2\pi$

Soln $f(x) = \frac{1}{4}(\pi - x)^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$ — (1)

Now,

$$a_0 = \frac{1}{\pi}\int_{c}^{c+2\pi} f(x)\,dx.$$

$$= \frac{1}{\pi}\int_{0}^{2\pi} \frac{1}{4}(\pi - x^2)\,dx = \frac{1}{4\pi}\int_{0}^{2\pi} (\pi - x)^2\,dx$$

$$b_n = -\frac{2(-1)^n}{n} = \frac{2(-1)^{n+1}}{n}$$

Put these values in eqn ①

So,
From Eqn ① we get

$$f(x) = x = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin nx$$

Soln. The given function

$$f(x) = x$$
$$\Rightarrow f(-x) = -x$$
$$= -f(x)$$

∴ The function is odd function.

So therefore,

$$a_0 = 0 \quad \&$$
$$a_n = 0$$

& fourier series will be

$$f(x) = x = \sum_{n=1}^{\infty} b_n \sin nx \quad \text{— (1)}$$

Now,

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx \, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin nx \, dx$$

$$= \frac{1}{\pi}\left[\frac{-x\cos nx}{n} + \frac{\sin nx}{n^2}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{-\pi\cos n\pi}{n} \cancel{+ \sin n\pi} - \left(-\pi\left(\frac{-\cos n(-\pi)}{n}\right)\right)\right]$$

$$= \frac{1}{\pi}\left[\frac{-\pi(-1)^n}{n} - \frac{\pi(-1)^n}{n}\right]$$

Fourier Series for even and odd funcⁿ

Case I → If $f(x)$ is a even funcⁿ on a close interval $[-\pi, \pi]$ then $b_n = 0$ and fourier series can be represented as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

→ Cosine Series.

Case II → If $f(x)$ is odd function on a close interval $[-\pi, \pi]$ than $a_0 = 0$, $a_n = 0$ and fourier series can be represented

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

→ Sine Series

Q) Find the fourier series of the function

$$f(x) = x \text{ in } [-\pi, \pi]$$

Put $x = 0$ in eqn ②

$$f(0) = -\frac{\pi}{4} - \frac{2}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots \infty\right]$$

$$0 = -\frac{\pi}{4} - \frac{2}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots \infty\right].$$

$$\frac{\pi}{4} = -\frac{2}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots \infty\right].$$

$$\frac{-\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots \infty$$

$$= \frac{1}{\pi}\left[\left[-\pi \frac{\sin nx}{n}\right]_{-\pi}^{0} + \left[\frac{x \sin nx}{n} - 1\left(\frac{-\cos nx}{n^2}\right)\right]_{0}^{\pi}\right]$$

$$= \frac{1}{n^2\pi}(\cos n\pi - 1)$$

$$= \begin{cases} 0, & \text{when } n \text{ is even.} \\ -2/n^2\pi, & \text{when } n \text{ is odd.} \end{cases}$$

$$b_n = \frac{1}{n}\left[1 - 2(-1)^n\right]$$

$$= \begin{cases} -1/n, & n \text{ is even.} \\ 3/n, & n \text{ is odd.} \end{cases}$$

From Eqn (1), Fourier series will be

$$f(x) = \frac{-\pi}{4} + \sum_{n=1}^{\infty} \begin{cases} 0, & n \text{ is even} \\ -2/n^2\pi, & n \text{ is odd} \end{cases} \cos nx + \sum_{n=1}^{\infty} \begin{cases} -1/n, & n \text{ is even} \\ 3/n, & n \text{ is odd} \end{cases} \sin nx$$

$$f(x) = \frac{-\pi}{4} + \left(\frac{-2}{1^2\pi}\cos x\right) - \frac{2}{3^2\pi}\cos 3x - \frac{2}{5\pi}\cos 5x \ldots \infty) + \left(3\sin x - \frac{1}{2}\sin 2x + \sin 3x + \ldots \infty\right)$$

$$= \frac{-\pi}{4} - \frac{2}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\right] + \left(3\sin x - \frac{1}{2}\sin 2x + \sin 3x \ldots \infty\right)$$

— (2)

Fourier Series ~~no~~ of $f(x)$ having pt of dis continuity:

Q1 Obtain the fourier expansion for the $f(x) = \begin{cases} -\pi & , -\pi < x < 0 \\ x & , 0 < x < \pi \end{cases}$

hence deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}$

Soln $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \sin nx + \sum_{n=1}^{\infty} b_n \sin nx$ — (1)

Now,

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi\,dx + \int_{0}^{\pi} x\,dx\right]$$

$$= \frac{1}{\pi}\left[-\pi x\Big|_{-\pi}^{0} + \left|\frac{x^2}{2}\right|_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[-\pi^2 + \frac{\pi^2}{2}\right] = \frac{-\pi}{2}$$

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi \cos nx\,dx + \int_{0}^{\pi} x \cos nx\,dx\right]$$

~~$a_n = \frac{1}{\pi}\left[\int\right.$~~

Change of Interval:

$[c, c+2\pi]$

$[0, 2\pi]$ $\quad$ $[-\pi, \pi]$

$[\lambda, \lambda+2c]$

$[0, 2c]$ $\quad$ $[-c, c]$

$[0, 2]$ $\quad$ $[-1, 1]$

The fourier series of a $f(x)$ in the interval $(\lambda, \lambda+2c)$ is given by (c is constant)

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{c} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{c}}$$

where $a_0 = \frac{1}{c}\int_{\lambda}^{\lambda+2c} f(x)\,dx$

$$a_n = \frac{1}{\cancel{2}c}\int_{\lambda}^{2+2c} f(x)\cos\frac{n\pi x}{c}\,dx$$

$$b_n = \frac{1}{c}\int_{\lambda}^{2+2c} f(x)\sin\frac{n\pi x}{c}\,dx$$

Note → ① If $\lambda = 0$ in interval $[\lambda, \lambda+2c]$ then new interval will be $[0, 2c]$

② If $\overset{\lambda=-c}{\cancel{\lambda=c}}$ in $[\lambda+\lambda+2c]$ then new interval will lie $[-1, 1]$

③ If $d = -c$ in $[d, d+2c]$ than new interval will lie $[-c, c]$.

Ques1) Find the fourier series for the funcn $f(x) = \begin{cases} \pi x, & 0 \le x \le 1 \\ \pi(2-x), & 1 \le x \le 2 \end{cases}$

$[0, 2]$

$[2, cc]$

$c = 1$

Soln Here the value of $c = 1$
so now,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{1} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{1} \quad \text{—(1)}$$

so now,

$$a_0 = \frac{1}{1}\int_0^2 f(x)\,dx = \int_0^1 \pi x\,dx + \int_1^2 \pi(2-x)\,dx$$

$$a_0 = \pi$$

$$a_n = \frac{1}{1}\int_0^2 f(x)\cos\frac{n\pi x}{1}\,dx$$

$$= \int_0^1 \pi x \cos n\pi x\,dx + \int_1^2 \pi(2-x)\cos n\pi x$$

$$a_n = \frac{2}{n^2\pi}\left[(-1)^n - 1\right].$$

$$b_n = \frac{1}{c}\int_0^2 f(x)\sin\frac{n\pi x}{c}\,dx$$

$$= \int_0^1 \pi x \sin n\pi x\,dx + \int_1^2 \pi(2-x)\sin n\pi x\,dx$$

so from here

$$b_n = 0$$

Put all these value in eq^n (1).

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi}\left[(-1)^n - 1\right]\cos n\pi x$$

$$\cancel{= \frac{\pi}{2} + \Big[}$$

Half Range Series:

$\rightarrow [0, c]$

$[0, \pi]$ $[0, l]$

If $f(x)$ is a funcn on a interval and $(0, c)$ then

Sine Series

$a_0 = 0$

$a_n = 0$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{c}$$

Cosine Series

$b_n = 0$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{c}$$

where

$$b_n = \frac{2}{c} \int_0^c f(x) \sin \frac{n\pi x}{c} \, dx$$

$$a_0 = \frac{2}{c} \int_0^c f(x) \, dx$$

$$a_n = \frac{2}{c} \int_0^c f(x) \cos \frac{n\pi x}{c} \, dx$$

Ques 1:- Find the half range sine series of the $f(x) = x$ in $0 < x < 2$.

Sol^n Here $c=2$

we have to find sine half range sine series

$\therefore a_n = 0, \& a_0 = 0$

so,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{2} \quad -(1)$$

where,

$$b_n = \frac{2}{2}\int_0^2 x \sin\frac{n\pi x}{2}\,dx$$

$$= \left| x\left(\frac{-\cos\frac{n\pi x}{2}}{n\pi/2}\right) - 1\left(\frac{-\sin\frac{\pi n x}{2}}{(n\pi/2)^2}\right)\right|_0^2$$

$$b_n = \frac{4}{n\pi}(-1)^{n+1}$$

Put all these values in Eq^n (1)

$$f(x) = \sum_{n=1}^{\infty} \frac{4}{n\pi}(-1)^{n+1}\sin\frac{n\pi x}{2}$$

x———x