Thus P.I is.

$$y_p = \frac{1}{6}xe^{x} \cdot \frac{1}{2}x^2 + \frac{1}{2}x - \frac{3}{4}$$

Unit - IV

functions of Complex Variables.

* A Complex number $z = x + iy$ is defined as ordered pair (x, y) where x and y are real numbers.

let $P(x,y)$ represent the Complex no. $z = x + iy$.
let $\vec{OP}$ make an angle θ with ~~position~~ positive direction of OX and $OP = r$ and $\angle MOP = \theta$ and

~~$x = \overset{OM}{MP} = OP\sin\theta$~~

~~$x = r\cos\theta$~~

Sine law

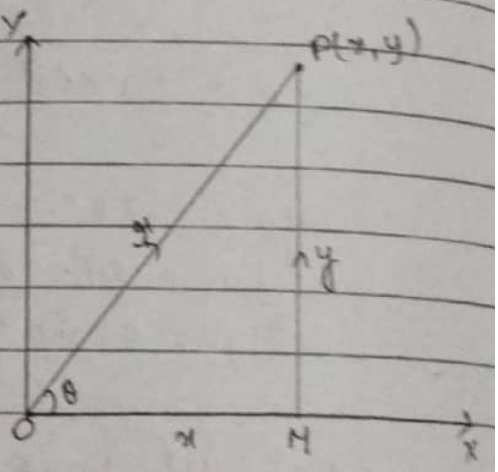
$y = MP$

$y = OP \sin\theta$

$y = r\sin\theta$ — (1)

$x = OM = OP\cos\theta$

$x = r\cos\theta$ — (2)



Squaring and adding 1 and 2.

$$x^2 + y^2 = r^2$$

and

$$\tan\theta = \frac{y}{x}$$

$$\theta = \tan^{-1}\frac{y}{x}$$ is Called the amplitude

argument

Unit

or argument of z.
The Value of θ staticfying $-\pi < \theta < \pi$
is called principal value. Hence,
$z = x+iy = r\cos\theta + ir\sin\theta = re^{i\theta}$.

* Some Trigonometric formulas:-

If $z = x+iy$ is trigonometric function of ~~of ordered pair (x,y) where x and y are~~ z are defined as.

1. $\cos z = \dfrac{e^{iz} + e^{-iz}}{2}$

2. $\sin z = \dfrac{e^{iz} - e^{iz}}{2i}$

3. $\tan z = \dfrac{\sin z}{\cos z} = \dfrac{e^{iz} - e^{-iz}}{i(e^{iz} + e^{-iz})}$

4. $\cot z = \dfrac{i(e^{iz} + e^{-iz})}{e^{iz} - e^{-iz}}$

5. $\sec z = \dfrac{2}{e^{iz} + e^{-iz}}$

6. $\operatorname{cosec} z = \dfrac{2i}{e^{iz} - e^{-iz}}$

Periodicity of Circular Function.

→ To prove that $\sin z$ is periodic function with 2π.

proof :- we know $\sin z = \dfrac{e^{iz} - e^{-iz}}{2i}$

Then,

$$\sin(z+2n\pi) = \frac{e^{i(z+2n\pi)} - e^{-i(z+2n\pi)}}{2i}$$

$$= \frac{e^{iz}\, e^{2n\pi i} - e^{-iz}\, e^{-2n\pi i}}{2i}$$

$$= \frac{e^{iz} - e^{-iz}}{2i}$$

$$= \sin z.$$

$e^{2n\pi i} = 1.$

$e^{i\theta}$

$\cos 2n\pi + \sin 2n\pi$

$(-1)^{2n} + 0$

$1.$

Q Seperate into real and imaginary function.

$$e^{(5+3i)^2}$$

$$(5+3i)^2 = 25 + 30i + (3i)^2$$
$$= 25 + 9i^2 + 30i$$
$$= 25 - 9 + 30i$$
$$= 16 + 30i$$

$$e^{(5+3i)^2} = e^{16+30i}$$
$$= e^{16} \cdot e^{30i}$$
$$= e^{16}(\cos 30 + i\sin 30)$$

Q. Prove that $[\sin(\alpha-\theta)+e^{-i\alpha}\sin\theta]^n = \sin^n\alpha\, e^{-in\theta}$

$\Rightarrow [(\sin\alpha\cos\theta-\cos\alpha\sin\theta)+((\cos\alpha-i\sin\alpha)\sin\theta]^n$

Soln.

$[\sin(\alpha-\theta)+e^{-i\alpha}\sin\theta]^n$

$[\sin\alpha\cos\theta-\cancel{\cos\alpha\sin\theta}+\cancel{\cos\alpha\sin\theta}-i\sin\alpha\sin\theta]^n$

$[\sin\alpha\cos\theta-i\sin\alpha\sin\theta]^n$

$\sin^n\alpha\,[\cos\theta-i\sin\theta]^n$

$\sin^n\alpha\, e^{-in\theta} \Rightarrow$ R.H.S.

Trigonometric Identities.

If z is a complex no. then.

(1) $\sin 2z = 2\sin z\cos z$

proof:

take R.H.S.

$2\sin z\cos z$.

$$2\left(\frac{e^{iz}-e^{-iz}}{2i}\right)\times\left(\frac{e^{iz}+e^{-iz}}{2}\right)$$

$$\frac{e^{2iz}-e^{iz-iz}+e^{iz-iz}-e^{-2iz}}{2i}$$

$$\frac{e^{2iz}-e^{-2iz}}{2i}$$

$\sin 2z =$ L.H.S.

2. $\cos 2z = \cos^2 z - \sin^2 z$
$= 1 - 2\sin^2 z$
$= 2\cos^2 z - 1.$

3. $\tan 2z = \dfrac{2\tan z}{1 - \tan^2 z}$

4. $\sin 3z = 3\sin z - 4\sin^3 z$

5. $\tan 3z = \dfrac{3\tan z - \tan^3 z}{1 - 3\tan^2 z}$

Hyperbolic functions.

1. $\sinh x = \dfrac{e^x - e^{-x}}{2.}$

2. $\cosh x = \dfrac{e^x + e^{-x}}{2.}$

3. $\tanh x = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

$\coth x = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$

$\operatorname{sech} x = \dfrac{2}{e^x + e^{-x}}$

$\operatorname{cosech} x = \dfrac{2}{e^x - e^{-x}}$

Trigonometric identities

1. $\cos 2z = \cos^2 z - \sin^2 z$.

R.H.S $= \cos^2 z - \sin^2 z$

$$\left[\frac{e^{iz}+e^{-iz}}{2}\right]^2 - \left[\frac{e^{iz}-e^{-iz}}{2i}\right]^2$$

$$= \frac{1}{4}\left[\left(e^{2iz}+e^{-2iz}+2\right)+\left(e^{2iz}+e^{-2iz}-2\right)\right]$$

$$= \frac{1}{4}\left[e^{2iz}+e^{2iz}+e^{-2iz}+e^{-2iz}\right]$$

$$= \frac{1}{\cancel{4}_2}\,\cancel{2}\left(e^{2iz}+e^{-2iz}\right)$$

$$= \frac{e^{2iz}-e^{-2iz}}{2}$$

2. $1-2\sin^2 z$.

$$1-2\left[\frac{e^{iz}-e^{-iz}}{2i}\right]^2$$

$$1+\frac{1}{2}\left(e^{2iz}+e^{-2iz}-2\right)$$

$$\frac{e^{2iz}+e^{-2iz}}{2}$$

$\cos 2z$.

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Page

2. $\cos^2 z - 1$

$$\cancel{2}\left[\frac{e^{iz} + e^{-iz}}{2\cancel{e}}\right]^2 - 1$$

$$\frac{1}{2}\left(e^{2iz} + e^{-2iz} + 2\right) - 1$$

$$\frac{e^{2iz} + e^{-2iz}}{2}$$

$\cos 2z.$

3. $\tan 2z = \dfrac{2\tan z}{1 - \tan^2 z}.$

$$R.H.S = \frac{2\tan z}{1 - \tan^2 z}$$

$$= \frac{2 \cdot \dfrac{e^{iz} - e^{-iz}}{i(e^{iz} + e^{-iz})}}{1 - \left[\dfrac{e^{iz} - e^{-iz}}{i(e^{iz} + e^{-iz})}\right]^2}$$

$$= \frac{2(e^{iz} - e^{-iz})(e^{iz} + e^{-iz})}{i[(e^{iz} + e^{-iz})^2 + (e^{iz} - e^{-iz})^2]}$$

$$= \frac{2(e^{2iz} - e^{-2iz})}{i \cdot 2(e^{2iz} + e^{-2iz})} = \frac{(e^{2iz} - e^{-2iz})}{i(e^{2iz} + e^{-2iz})}$$

$$= \tan 2z$$

4. $\sin 3z = 3\sin z - 4\sin^3 z.$

$$\sin 3z = \frac{e^{3iz} - e^{-3iz}}{2i} = \frac{x^3 - y^3}{2i}$$

$$\frac{(x-y)^3 + 3xy(x-y)}{2i}$$

$$= \frac{1}{2i}\left[(2i\sin z)^3 + 3(2i\sin z)\right]$$

$$= \frac{1}{2i}\left[-8i\sin^3 z + 6i\sin z\right]$$

$$= 3\sin z - 4\sin^2 z$$

5. $\tan 3z = \dfrac{3\tan z - \tan^3 z}{1 - 3\tan^2 z.}$

R.H.S. $$\frac{3\left[\dfrac{e^{iz} - e^{-iz}}{i(e^{iz} + e^{-iz})}\right] - \left[\dfrac{e^{iz} - e^{-iz}}{i(e^{iz} + e^{-iz})}\right]^3}{1 - 3\left[\dfrac{e^{iz} - e^{-iz}}{i(e^{iz} + e^{-iz})}\right]^2}$$

$$\Rightarrow \frac{3.\left[\dfrac{e^{iz} - e^{-iz}}{i(e^{iz} + e^{-iz})}\right] + \dfrac{1}{i}\left[\dfrac{e^{iz} - e^{-iz}}{e^{iz} + e^{-iz}}\right]^3}{1 + 3\left[\dfrac{e^{iz} - e^{-iz}}{(e^{iz} + e^{-iz})}\right]^2}$$

$$\Rightarrow \frac{3.\dfrac{x}{iy} + \dfrac{1}{i}\left(\dfrac{x}{y}\right)^3}{1 + 3\left(\dfrac{x}{y}\right)^3}$$ where, $x = e^{iz} - e^{-iz}$

$y = e^{iz} + e^{-iz}$

Trigonometric formula.

1. $\cos z = \dfrac{e^{iz} + e^{-iz}}{2}$

Sol^n

$$\cos(z + 2n\pi) = \frac{e^{i(z+2n\pi)} + e^{-i(z+2n\pi)}}{2}$$

$$= \frac{e^{iz} + e^{i2n\pi} + e^{-iz} + e^{-i2n\pi}}{2}$$

$$= \frac{e^{iz} + e^{-iz}}{2}$$

$$= \cos z.$$

3. $\tan z = \dfrac{e^{iz} - e^{-iz}}{i(e^{iz} + e^{-iz})}$

Sol^n

$$\tan(z + n\pi) = \frac{e^{i(z+n\pi)} + e^{-i(z+n\pi)}}{i[e^{i(z+n\pi)} + e^{-i(z+n\pi)}]}$$

$$= \frac{e^{iz} . e^{in\pi} - e^{-iz} . e^{-in\pi}}{i[e^{iz} . e^{in\pi} + e^{-iz} . e^{-in\pi}]}$$

$$= \frac{e^{iz} . e^{2n\pi i} - e^{-iz}}{i[e^{iz} . e^{2n\pi i} + e^{-iz}]}$$

$$= \frac{e^{iz} - e^{-iz}}{i(e^{iz} + e^{-iz})} = \tan z$$

4. $\cot z = \dfrac{i(e^{iz} + e^{-iz})}{e^{iz} - e^{-iz}}.$

$$\cot(z+n\pi) = \frac{i(e^{i(z+n\pi)} + e^{-i(z+n\pi)})}{e^{i(z+n\pi)} - e^{-i(z+n\pi)}}$$

$$= \frac{i[e^{iz}\cdot e^{in\pi} + e^{-iz}\, e^{-in\pi}]}{e^{iz}\cdot e^{in\pi} - e^{-iz}\cdot e^{-in\pi}}$$

$$= \frac{i[e^{iz}\cdot e^{2n\pi i} + e^{-iz}]}{e^{iz}\cdot e^{2n\pi i} - e^{-iz}}$$

$$= \frac{i(e^{iz} + e^{-iz})}{e^{iz} - e^{-iz}}$$

$$= \cot z.$$

5. $\sec z = \dfrac{2}{e^{iz} + e^{-iz}}.$

$$\sec(z+2n\pi) = \frac{2}{e^{i(z+2n\pi)} + e^{-i(z+2n\pi)}}$$

$$= \frac{2}{e^{iz} + e^{i2n\pi} + e^{-iz} + e^{-i2n\pi}}$$

$$= \frac{2}{e^{iz} + e^{-iz}}$$

$$= \sec z.$$

6. $\operatorname{Cosec} z = \dfrac{2i}{e^{iz} - e^{-iz}}$

~~Sec~~

$$\operatorname{Cosec}(z+2n\pi) = \frac{2i}{e^{i(z+2n\pi)} - e^{-i(z+2n\pi)}}$$

$$= \frac{2i}{e^{iz}.e^{2n\pi i} - e^{-iz}.e^{-2n\pi i}}$$

$$= \frac{2i}{e^{iz} - e^{-iz}}$$

$$= \operatorname{Cosec} z.$$

Hyperbolic function formula :-

1. $\sinh 2x = 2\sinh x \cosh x = \dfrac{2\tanh x}{1 - \tanh^2 x}$

2. $\cosh 2x = \cosh^2 x + \sinh^2 x = 2\cosh^2 x - 1$
$= 1 + 2\sinh^2 x, \quad = \dfrac{1 + \tanh^2 x}{1 - \tanh^2 x}$

3. $\tanh 2x = \dfrac{2\tanh x}{1 + \tanh^2 x}$

4. $\sinh 3x = 3\sinh x + 4\sinh^3 x.$

5. $\cosh 3x = 4\cosh^3 x - 3\cosh x.$

6. $\tanh 3x = \dfrac{3\tanh x - \tanh^3 x}{1 + 3\tanh^3 x}$

7). $\cosh x = \cos ix.$

8) $\sin ix = i \sinh x.$

Inverse hyperbolic function

Q. Prove that

$$\sinh^{-1} x = \log(x + \sqrt{x^2+1})$$

proof:-

let. $\sinh^{-1} x = y$

$x = \sinh y.$

$$x = \frac{e^y - e^{-y}}{2}$$

$$x = \frac{e^y - \frac{1}{e^y}}{2.}$$

$$x = \frac{e^{2y} - 1}{2e^y}$$

$$2xe^y = e^{2y} - 1$$

$$e^{2y} - 2xe^y - 1 = 0.$$

$$e^y = \frac{2x \pm \sqrt{4x^2+4}}{2.}$$

$$e^y = x \pm \sqrt{x^2+1}$$

Take +ve sign.

$$y = \log(x + \sqrt{x^2+1})$$

$$\sinh^{-1} x = \log(x + \sqrt{x^2+1})$$

Imp.

Q. Seperate into real and imaginary parts.

$\sin^{-1}(\cos\theta + i\sin\theta), 0 < \theta < \pi/2$

Sol^n,

let $\sin^{-1}(\cos\theta + i\sin\theta) = x + iy$

$\cos\theta + i\sin\theta = \sin(x+iy)$

$\cos\theta + i\sin\theta = \sin x \cos iy + \cos x \sin iy.$

$\cos\theta + i\sin\theta = \sin x \cosh y + \cos x \, i\sinh y.$

equate the real and imaginary parts

$\cos\theta = \sin x \cosh y \quad — (1)$

$i\sin\theta = i\cos x \sinh y$

$\sin\theta = \cos x \sinh y \quad — (2)$

Squaring and adding.

$\cos^2\theta + \sin^2\theta = (\sin^2 x \cosh^2 y) + (\cos^2 x \sinh^2 y)$

$1 = \sin^2 x(1+\sinh^2 y) + (1-\sin^2 x)\sinh^2 y$

$1 = \sin^2 x + \cancel{\sin^2 x \sinh^2 y} + \sinh^2 y - \cancel{\sin^2 x \sinh^2 y}$

$1 = \sin^2 x + \sinh^2 y$

$\sinh^2 y = 1 - \sin^2 x$

$\sinh^2 y = \cos^2 x.$

from (2) (squaring)

$\sin^2\theta = \cos^2 x \sinh^2 y$

$\sin^2\theta = \cos^2 x \cos^2 x$

$\sin^2\theta = \cos^4 x.$

$$\cos x = \sqrt{\sin\theta}$$

$$x = \left[\cos^{-1}\left(\sqrt{\sin\theta}\right)\right]$$

Again from (2)

$$\sinh y = \frac{\sin\theta}{\cos x} = \frac{\sin\theta}{\sqrt{\sin\theta}}$$

$$\sinh y = \sqrt{\sin\theta}$$

$$y = \sinh^{-1}\left(\sqrt{\sin\theta}\right)$$

Q. If $u = \log\tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right)$ then prove that

(1) $\tanh\frac{u}{2} = \tan\frac{\theta}{2}$.

(2) $\cosh u = \sec\theta$.

proof :-

$$u = \log\tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right)$$

$$e^u = \tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right)$$

$$\cancel{e^{u/2} \cdot e^{u/2} = \tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right)}$$

$$e^{u/2} \cdot e^{u/2} = \frac{1 + \tan\theta/2}{1 - \tan\theta/2}.$$

$$\frac{e^{u/2}}{e^{-u/2}} = \frac{1 + \tan\theta/2}{1 - \tan\theta/2}$$

Apply componendo and dividendo.

$$\frac{e^{u/2} - e^{-u/2}}{e^{u/2} + e^{-u/2}} = \frac{\cancel{1} + \tan\theta/2 - \cancel{1} + \tan\theta/2}{1 - \tan\theta/2 + 1 - \tan\theta/2}$$

$$\tanh\frac{u}{2} = \frac{2\tan\theta/2}{2}$$

$$\tanh\frac{u}{2} = \tan\frac{\theta}{2}.$$

2nd

$$\cosh u = \frac{1 + \tanh^2 u/2}{1 - \tanh^2 u/2}.$$

$$\cosh u = \frac{1 + \tan^2\theta/2}{1 - \tan^2\theta/2}.$$

$$\cosh u = \frac{1}{\cos\theta}$$

$$\cosh u = \sec\theta.$$

Q. If $\cosh(u+iv) = x+iy$ then prove that,

1. $$\frac{x^2}{\cosh^2 u} + \frac{y^2}{\sinh^2 u} = 1.$$

2. $$\frac{x^2}{\cos^2 v} - \frac{y^2}{\sin^2 v} = 1.$$

Solⁿ.

$$x+iy = \cosh(u+iv)$$
$$x+iy = \cos i(u+iv)$$
$$x+iy = \cos(iu-v)$$
$$x+iy = \cos iu \cos v + \sin iu \sin v.$$
$$x+iy = \cosh u \cos v + i \sinh u \sin v.$$

Equating.

$$x = \cosh u \cos v \quad \text{— 1}$$
$$iy = i \sinh u \sin v$$
$$y = \sinh u \sin v \quad \text{—— 2.}$$

from (1) and (2)

$$\cos v = \frac{x}{\cosh u} \; -(3) \qquad \sin v = \frac{y}{\sinh u} \; -(4)$$

Eq. Squaring and adding (3) and (4)

$$\frac{x^2}{\cosh^2 u} + \frac{y^2}{\sinh^2 u} = 1.$$

Now,

Again from (1) and (2)

$$\cosh u = \frac{x}{\cos v} \qquad \sinh u = \frac{x}{\sin v}$$

Squaring and substracting,

$$\frac{x^2}{\cos^2 v} - \frac{x}{\sin^2 v} = 1$$

proved

Logarithmic function of Complex Variables :-

If $z = x + iy$ and $w = u + iv$ be two complex variables such that $e^{w} = z$ then w is said to be logarithmic of z to the base 'e'

Properties :-

1. logarithm of a complex variable is multi valued function.

$$\text{let } e^{\omega} = z$$

$$\omega = \log z$$

Also,

$$z = e^{\omega} = e^{\omega} \cdot e^{2n\pi i}$$

$$z = e^{\omega + 2n\pi i}$$

$$\log z = \omega + 2n\pi i$$

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2. logarithm of a positive real no. is real.

3. $\operatorname{Log} z = 2n\pi i + \log_e z$.

Q. Seperate into real and imaginary parts of $\log(a+ib)$

Sol^n

$$\operatorname{Log}(a+ib) = 2n\pi i + \log(a+ib)$$

Put $a = r\cos\theta$ — (1)

$b = r\sin\theta$ — (2)

Square and add (1) and (2)

$$a^2+b^2 = r^2$$
$$r = \sqrt{a^2+b^2}$$

divide (2) by (1)

$$\theta = \tan^{-1}\frac{b}{a}.$$

$$\begin{aligned}\log(a+ib) &= 2n\pi i + \log(r\cos\theta + ir\sin\theta) \\ &= 2n\pi i + \log(re^{i\theta}) \\ &= 2n\pi i + \log r + i\theta \\ &= 2n\pi i + \log(a^2+b^2)^{1/2} + i\tan^{-1}\frac{b}{a}.\end{aligned}$$

Real part $= \frac{1}{2}\log(a^2+b^2)$

Imaginary part $= 2n\pi + \tan^{-1}\frac{b}{a}.$

Q. find the general value of $\log(-3)$.

Soln.

$$-3 = 3(-1) = 3(\cos\pi + i\sin\pi)$$
$$= 3e^{i\pi}$$

$$\log(-3) = 2n\pi i + \log(-3)$$
$$= 2n\pi i + \log(3e^{i\pi})$$
$$= 2n\pi i + \log 3 + i\pi$$
$$= \log 3 + (2n+1)i\pi$$

Q. If $i^{\alpha+i\beta} = \alpha + i\beta$ then prove that $\alpha^2 + \beta^2 = e^{-(4n+1)\pi\beta}$

Soln

$$\alpha + i\beta = i^{\alpha+i\beta}.$$
$$= e^{\log i^{\alpha+i\beta}}$$
$$= e^{(\alpha+i\beta)\log i}$$
$$= e^{(\alpha+i\beta)(2n\pi i + \log i)}$$
$$= e^{(\alpha+i\beta)(2n\pi i + i\pi/2)}$$

$i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = e^{i\pi/2}$

$\log i = i\frac{\pi}{2}$

$$= e^{\left(2n\pi i\alpha + \frac{i\pi}{2}\alpha - 2n\pi\beta - \frac{\beta\pi}{2}\right)}$$

$$= e^{\left(2n+\frac{1}{2}\right)i\pi\alpha} \cdot e^{-\left(2n+\frac{1}{2}\right)\pi\beta}$$

$$= \left[\cos\left(\frac{4n+1}{2}\right)\pi\alpha + i\sin\left(\frac{4n+1}{2}\right)\pi\alpha\right] e^{-\frac{(4n+1)\pi\beta}{2}}$$

Seperating real and Imaginary parts.

$$\alpha = e^{-\frac{(4n+1)\pi\beta}{2}} \cos\left(\frac{4n+1}{2}\right)\pi\alpha. \quad -(1)$$

$$\beta = e^{-\frac{(4n+1)\pi\beta}{2}} \sin\left(\frac{4n+1}{2}\right)\pi\alpha \quad -(2)$$

Square and add (1) and (2)

$$\alpha^2 + \beta^2 = e^{-(4n+1)\pi\beta}.$$

Analytic function :-

Limit :→

A Complex number 'l' is said to be the limit of $f(z)$ at z_0 and is denoted by

$$\lim_{z \to z_0} f(z) = l$$

If for every $\varepsilon > 0$ there exists $\delta > 0$ so that $|z - z_0| < \delta \Rightarrow |f(z) - l| < \varepsilon$

Continuity :→

A function $f(z)$ is said to be Continuous at point z_0 if.

1. $f(z_0)$ exists.
2. $\lim_{z \to z_0} f(z)$ exists.
3. $\lim_{z \to z_0} f(z) = f(z_0)$

Derivability :→

The function $f(z)$ is said to be derivable at point z_0 in its domain if the limit

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \text{ exists.}$$

Analytically → let $f(z)$ be singled valued function in D. The function $f(z)$ is said to be analytic at point z_0. If 'f' is derivable not only at z_0 but also every part of some neighbourhood of z_0.

An analytic function is also known as regular or ~~[illegible]~~ holomorphic function.

A function which is analytic everywhere is called entire function.

Most Important.

Cauchy Riemann Equation:- (C-R) equation.

Theorem:- The necessary and sufficient condition for a function $w = f(z) = u(x,y) + iv(x,y)$ to be analytic in region R

1. $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

2. $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are continuous in R.

Cauchy Riemann equation in polar form:-

let (r, θ) be the polar Co-ordinates of a point whose Cartesian co-ordinates are (x,y) then,

$$z = x + iy = re^{i\theta}$$

Now,

$$u + iv = f(z) = f(re^{i\theta})$$

$$u + iv = f(re^{i\theta}) \quad \text{—— (1)}$$

diff w.r.t 'r' and 'θ'

$$\frac{\partial u}{\partial r} + \frac{i\partial v}{\partial r} = f'(re^{i\theta}) . e^{i\theta} \quad \text{—— (2)}$$

and,

$$\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} = f'(re^{i\theta})\, ri\, e^{i\theta} \quad \text{—— (3)}$$

using (2) and (3)

$$\frac{\partial u}{\partial \theta} + \frac{i\partial v}{\partial \theta} = ri\left(\frac{\partial u}{\partial r} + \frac{i\partial v}{\partial r}\right)$$

$$\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} = ri\frac{\partial u}{\partial r} - \frac{r\partial v}{\partial r}$$

Seperate to real and Imaginary parts.

$$\frac{\partial u}{\partial \theta} = -\frac{r\partial v}{\partial r} \qquad \frac{\partial v}{\partial \theta} = r\frac{\partial u}{\partial r}$$

Congugate function :-

The real and Img parts of an analytic function are called Congugate function.

Harmonic function :-

If $f(z) = u(x,y) + iv(x,y)$ be an analytic function in domain D then $u + iv$ will satisfy the laplace equation

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$ and are called Harmonic function.

Application to flow problem :- If $w = f(z) = \phi(x,y) + i\psi(x,y)$ represent the flow pattern then $w(z)$ is known as complex potential function where $\phi(x,y)$ is velocity potential and $\psi(x,y)$ is stream function or flux function.

If $w = f(z) = \phi(x,y) + i\psi(x,y)$ represent heat flow pattern then $\phi(x,y)$ is called isothermal function and $\psi(x,y)$ is heat flow pattern.

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If $w = f(z) = \phi(x,y) + i\psi(x,y)$ represent the flow pattern then $w(z)$ is known as Complex potential function where $\phi(x,y)$ is velocity potential and $\psi(x,y)$ is stream function or flux function.

If $w = f(z) = \phi(x,y) + i\psi(x,y)$ represent heat flow pattern then $\phi(x,y)$ is called Isothermal function and $\psi(x,y)$ is heat flow pattern.

Q Determine the analytic function whose real part is $e^{2x}(x\cos 2y - y\sin 2y)$.

let $f(z) = u + iv$

where $u = e^{2x}(x\cos 2y - y\sin 2y)$

$$\frac{\partial u}{\partial x} = 2e^{2x}(x\cos 2y - y\sin 2y) + e^{2x}(\cos 2y)$$

$$= e^{2x}[2x\cos 2y - 2y\sin 2y + \cos 2y]$$

$$\cancel{= e^{2x}[-2x\cos 2y + 2y\sin 2y - \cos 2y]}$$

$$\frac{\partial u}{\partial y} = -e^{2x}(2x\sin 2y + \sin 2y + 2y\cos 2y)$$

Since $f(z)$ is analytic, it satisfy C-R eqn

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \Big| \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

since,

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = e^{2x}[2x\cos 2y - 2y\sin 2y + \cos 2y]$$

Integrate w.r. to y.

$$v = e^{2x}[x\sin 2y + y\cos 2y] + \phi(x)$$

$\rightarrow$ Constant

Now,

$$\frac{\partial v}{\partial x} = 2e^{2x}[x\sin 2y + y\cos 2y] + e^{2x}[\sin 2y] + \phi'(x)$$

But $\dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y}.$

$$\Rightarrow e^{2x}[2x\sin 2y + 2y\cos 2y + \sin 2y] + \phi'(x) = e^{2x}[2x\sin 2y + \sin 2y + 2y\cos 2y]$$

$$\phi'(x) = 0.$$

$$\phi(x) = 0.$$

$$v = e^{2x}[x\sin 2y + y\cos 2y] + c.$$

so,

$$f(z) = e^{2x}(x\cos 2y - y\sin 2y) + i(e^{2x}(x\sin 2y + y\cos 2y)) + c$$

Milne Thomson Method :-

1. If u is given find $f'(z) = \frac{du}{dx} - i\frac{\partial u}{\partial y}$

and if v is given find $f'(z) = \frac{\partial v}{\partial y} - i\frac{\partial v}{\partial x}$

2. Replace x by z and y by zero in $f'(z)$

3. Integrate $f'(z)$ w.r.t z.

Q. Find the analytic function whose imaginary part is $\frac{x-y}{x^2+y^2}$

Sol^n

$$v = \frac{x-y}{x^2+y^2}$$

$$\frac{\partial v}{\partial x} = \frac{(x^2+y^2) - (x-y)2x}{(x^2+y^2)^2}$$

$$= \frac{x^2+y^2-2x^2+2xy}{(x^2+y^2)^2}$$

$$= \frac{-x^2+y^2+2xy}{(x^2+y^2)^2}$$

$$\frac{\partial v}{\partial y} = \frac{-(x^2+y^2) - (x-y)\times 2y}{(x^2+y^2)^2}$$

$$= \frac{-x^2-y^2-2xy+2y^2}{(x^2+y^2)^2}$$

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$$= \frac{-x^2+y^2-2xy}{(x^2+y^2)^2}$$

$$f'(z) = \frac{-x^2+y^2-2xy}{(x^2+y^2)^2} - i \, \frac{-x^2+y^2+2xy}{(x^2+y^2)^2}$$

$$= \frac{-z^2}{z^4} - i \, \frac{-z^2}{z^4}$$

$$= \frac{-1}{z^2} + i \frac{1}{z^2}$$

$$= \frac{-1}{z^2} (1 - i)$$

$$\int f'(z)\, dz = -\int \frac{1}{z^2} (1 - i)$$

$$f(z) = \left[\frac{-1}{z} + \frac{i}{z} \right]$$

$$f(z) = -\frac{1}{z} (1 - i) + c.$$

Q. If the potential function is $\log(x^2+y^2)$ find the flux function and complex potential function.

Sol^n

since $\omega = f(z) = \phi + i\psi$

where, $\phi = \log(x^2+y^2)$

By Milne Thomson's method.

$$f'(z) = \frac{\partial \phi}{\partial x} - i\frac{\partial \phi}{\partial y}$$

~~Since~~,

$$\cancel{f(z)} \quad f'(z) = \frac{2x}{x^2+y^2} - i\frac{2y}{x^2+y^2}$$

put $x = z$, $y \Rightarrow 0$.

$$f'(z) = \frac{2z}{z^2} - i0$$

$$= \frac{2}{z}.$$

$$f'(z) = \frac{\partial \omega}{\partial z} = \frac{2}{z}$$

Integrate w. r. to z

$$\omega = 2\log z + c$$

$$= 2\log(x+iy) + c.$$

Put $x = r\cos\theta$; $y = r\sin\theta$

$$\omega = 2\log(x^2+y^2) + 2i\tan^{-1}\frac{y}{x} + c.$$

$$\varphi = 2\tan^{-1}\frac{y}{x}.$$

Q If $f(z)$ is a regular function of z then show that $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|$

Here,

$$f(z) = u + iv$$
$$|f(z)| = \sqrt{u^2 + v^2}$$
$$|f(z)|^2 = u^2 + v^2$$

let $f = u^2 + v^2$.

$$\frac{\partial f}{\partial x} = 2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x}$$

$$\frac{\partial^2 f}{\partial x^2} = 2\left[u\frac{\partial^2 u}{\partial x^2} + v\frac{\partial^2 v}{\partial x^2} + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right]$$

and.

$$\frac{\partial^2 f}{\partial y^2} = 2\left[u\frac{\partial^2 u}{\partial y^2} + \left(\frac{\partial u}{\partial y}\right)^2 + v\frac{\partial^2 v}{\partial y^2} + \left(\frac{\partial v}{\partial y}\right)^2\right]$$

Now,

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y} = 2\left[u\frac{\partial^2 y}{\partial x^2} + \left(\frac{\partial u}{\partial x}\right)^2 + v\frac{\partial^2 v}{\partial x^2} + \left(\frac{\partial v}{\partial x}\right)^2 + u\frac{\partial^2 v}{\partial y^2} + \left(\frac{\partial u}{\partial y}\right)^2 + v\frac{\partial^2 y}{\partial y^2} + \left(\frac{\partial v}{\partial y}\right)^2\right] \quad -(1)$$

since, $f(z)$ is regular function therefore its satisfies C-R eq^n as well as laplace equation.

i.e. $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$f(z) = u + iz$

$\frac{d}{dz}(u+iv) + \frac{du}{dz} + \frac{d}{dz}iv$

$\left[\frac{du}{dz} + i\frac{dv}{dz}\right]^2$

and

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}$$

putting all these values in (1)

$$\frac{\partial^2 E}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} = 2\left[u\left(\frac{\partial^2 u}{d x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + v\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right]$$

$$\frac{\partial^2 E}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 2\left[u(0) + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial x}\right)^2 + v(0) + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right.$$

$$= 2\left[2\left[\left(\frac{du}{dx}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right]\right]$$

$$= 4\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right] \quad \text{—— (2)}$$

Next,

$$f(z) = u + iv$$

$$|f'(z)|^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial x}\right)^2 \quad \text{—— (3)}$$

using (2) and (3).

$f(z) = u+iv$

and

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}$$

putting all these values in (1)

$$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} = 2\Big[u\Big(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\Big) + \Big(\frac{\partial u}{\partial x}\Big)^2 + \Big(\frac{\partial u}{\partial y}\Big)^2 + v\Big(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\Big) + \Big(\frac{\partial v}{\partial x}\Big)^2 + \Big(\frac{\partial v}{\partial y}\Big)^2\Big]$$

$$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} = 2\Big[u(0) + \Big(\frac{\partial u}{\partial x}\Big)^2 + \Big(\frac{\partial v}{\partial x}\Big)^2 + v(0) + \Big(\frac{\partial v}{\partial x}\Big)^2 + \Big(\frac{\partial v}{\partial x}\Big)^2$$

$$= 2\Big[2\Big[\Big(\frac{\partial u}{\partial x}\Big)^2 + \Big(\frac{\partial v}{\partial x}\Big)^2\Big]$$

$$= 4\Big[\Big(\frac{\partial u}{\partial x}\Big)^2 + \Big(\frac{\partial v}{\partial x}\Big)^2\Big] \quad \text{--- (2)}$$

Next,

$$f(z) = u + iv$$

$$|f'(z)|^2 = \Big(\frac{\partial u}{\partial x}\Big)^2 + \Big(\frac{\partial u}{\partial x}\Big)^2 \quad \text{--- (3)}$$

using (2) and (3).

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) f = 4|f'(z)|^2$$

Q. If $f(z) = u + iv$ is an analytic function of z and $u - v = \dfrac{\cos x + \sin x - e^{-y}}{2\cos x - 2\cosh y}$ than find $f(z)$ subject to the conduction $f\left(\frac{\pi}{2}\right) = 0$

Here,

$$f(z) = u + iv$$
$$if(z) = iu - v$$

Adding,

$$(1+i) f(z) = (u-v) + i(u+v)$$
$$(1+i) f(z) = u + iv$$

Where,

$$u = u - v$$
$$f(z) = u + iv$$

Now,

$$u = u - v = \frac{\cos x + \sin x - e^{-y}}{2\cos x - 2\cosh y}$$

$$\frac{\partial u}{\partial x} = \frac{1}{2}\left[\frac{(\cos x - \cosh y)(\sin x + \cos x) - (-\sin x)(\cos x + \sin x - e^{-y})}{(\cos x - \cosh y)^2}\right]$$

$$\frac{\partial u}{\partial y} = \frac{1}{2}\left[\frac{(\cos x - \cosh y)(e^{-y}) - (-\sinh y)(\cos x + \sin x - e^{-y})}{(\cos x - \cosh y)^2}\right]$$

$$f'(z) = \frac{\partial u}{\partial x} + \frac{i\,\partial v}{\partial x}$$

$$f'(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$$

Replace $x \to z$ and $y \to 0$

Then,

integrate $f'(z) \to f$