Simple Harmonic Motion and Oscillator.

Oscillation:- The back and forth variation of the physical quantities (Including displacement) is called oscillator.

SHM:- It is defined as the motion of an oscillating particle moving back and forth about an equilibrium position through a restoring force which is directly proportional to the displacement but opposite to it in direction.

Ex: back and forward swing of pendulum is S.H.M.

Differential Equation of SHM or SHO can be derived as

$$x = a\sin\omega t + b\cos\omega t \quad \text{--- (1)}$$

Put $a = A\cos\phi$ and $b = A\sin\phi$ in # (1)

$$x = A\sin\omega t\cos\phi + A\cos\omega t\cdot\sin\phi$$

$$x = A\sin(\omega t + \phi) \quad \text{--- (2)}$$

If we put $a = -A\sin\phi$ & $b = A\cos\phi$ in # (1)

$$x = A\cos(\omega t + \phi) \quad \text{--- (3)}$$

On differentiating # (3) twice w.r.t t.

$$\frac{dx}{dt} = -A\omega\sin(\omega t + \phi)$$

& $$\frac{d^2x}{dt^2} = -A\omega^2\cos(\omega t + \phi)$$

$$= -\omega^2[A\cos(\omega t + \phi)]$$

or $$\boxed{\frac{d^2x}{dt^2} = -\omega^2 x} \quad \text{--- (A)}$$

In SHM x represents displacement and $\frac{d^2x}{dt^2}$ shows the acceleration of the particle. Hence, S·H·M can be defined as that motion in which acceleration is directly proportional to the displacement and is directed opposite to the displacement.

Time period of S·H·M is given by

$$\boxed{T = \frac{2\pi}{\omega}}$$

Relationship between displacement, Velocity + Acceleration in SHM:

The equation of motion of particle executing S·H·M is

$$x = A \sin \omega t \quad \text{—} (1)$$

On differentiating # (1) w·r·t t

$$v = \frac{dx}{dt} = A \omega \cos \omega t \quad \text{—} (11)$$

$$= A\omega \sqrt{\cos^2 \omega t} = A\omega \sqrt{1 - \sin^2 \omega t} \quad [\because \cos^2\theta + \sin^2\theta = 1]$$

$$= A\omega \sqrt{1 - x^2/A^2}$$

$$\boxed{v = \frac{dx}{dt} = \omega \sqrt{A^2 - x^2}} \quad \text{—} (111)$$

Again on differentiating # (3) wrt t.

$$a = \frac{d^2x}{dt^2} = -A\omega^2 \sin \omega t$$

$$= -\omega^2 (A \sin \omega t)$$

$$\boxed{a = \frac{d^2x}{dt^2} = -\omega^2 x} \quad \text{—} (IV)$$

Phase:- For S.H.M (oscillator)

$$x = A\cos(\omega t+\phi)$$

Term $(\omega t+\phi)$ is called phase of SHM. The ϕ is called initial phase or phase at $t=0$.

Find expression for the time period of Simple Harmonic oscillations:

We know

$$x = A\cos\omega t$$

$$\therefore \frac{dx}{dt} = A\omega\sin\omega t$$

or $a = \frac{d^2x}{dt^2} = -\omega^2 x$. [−ve sign only relates direction so can be ignored]

$$\therefore \omega^2 = \frac{a}{x}.$$

$$\omega = \sqrt{a/x}$$

But $\omega = 2\pi/T$

$$\therefore \frac{2\pi}{T} = \sqrt{\frac{a}{x}}$$

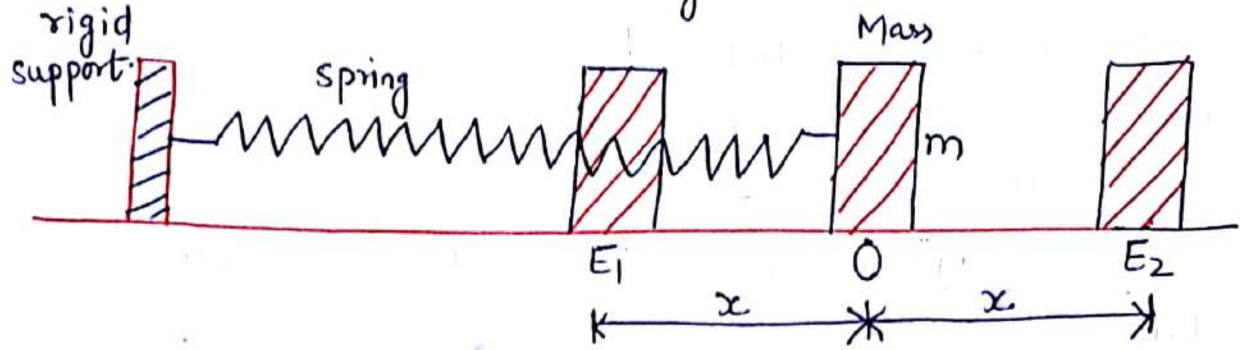
$$T = 2\pi\sqrt{\frac{x}{a}}$$

$\therefore$ Time period of SHM

$$\boxed{T = 2\pi\sqrt{\frac{\text{displacement}}{\text{acceleration}}}}$$

Imp

Mechanical Oscillator:- A mass attached with spring is a common example of mechanical oscillator. Let us consider a mass m attached to a spring. Suppose that the mass slides on a frictionless surface. O is the equilibrium position of the mass. The mass is at rest at O, when it is not oscillating.



Let E_1 and E_2 are the two extremes position of the mass. If the mass is pulled to the E_2 position, the spring is stretched. In this position a restoring force will be set up in the spring due to elasticity. This force will tend to restore the mass to equilibrium position. The work done in displacing the mass from O to E_2 will store in the spring as its potential Energy. If the mass is released at position E_2, the force will accelerate it towards to position O. After reaching O, the mass would have acquired K.E and due to inertia of motion will continue its journey towards E_1. In this way compression of spring will commence and it will again set up a restoring force in the spring. The motion of the mass will continue till whole of its K.E at O is converted into P.E of the compression of spring at E_1. At E_1, the mass will be momentarily at rest and restoring force in the spring will set it in motion toward O. On reaching O, it will again possess K.E & over shoot the mean position & moves towards E_2. In this way, the mass will oscillate between the extremes position E_1 and E_2.

Differential Equation of Mechanical Oscillator:

According to Hook's law, the restoring force (F) is directly proportional to the extension of spring (x).

$$\therefore F \propto (-x) \quad -(1)$$

$$\therefore F = -Sx$$ [where S is proportionality constant also called spring constant].

Spring constant is defined as restoring force per unit displacement.

Let m be the mass attached to the spring. Then according to Newton's second law of motion

$$F = ma = m \cdot \frac{d^2x}{dt^2} \quad -(2)$$

From (1) and (2)

$$m \cdot \frac{d^2x}{dt^2} = -Sx$$

$$\boxed{\frac{d^2x}{dt^2} = -\frac{S}{m}x} \quad -(3)$$

Put $S/m = \omega_0^2$ where ω_0 is another constant for oscillator.

$$\frac{d^2x}{dt^2} = -\omega_0^2 x$$

$$\boxed{\frac{d^2x}{dt^2} + \omega_0^2 x = 0}$$

This is called the differential equation of S.H.M.

The equation of motion of mass is given by

$$x = A \cos \omega_0 t$$

$A \rightarrow$ Amplitude, $\omega_0 = \sqrt{S/m}$.

Energy of the mechanical Oscillator:- The particle executing SHM posseses K.E as well as potential Energies.

Kinetic Energy: The kinetic energy of the mass m, when possessing a velocity $v = \frac{dx}{dt}$.

$$U_k = \frac{1}{2}mv^2 = \frac{1}{2}m\left(\frac{dx}{dt}\right)^2$$

Since $x = A\cos\omega_0 t$ and $v = dx/dt = -A\omega\sin\omega t$

$$\therefore U_k = \frac{1}{2}m(-A.\omega_0\sin\omega_0 t)^2$$

$$\boxed{U_k = \frac{1}{2}mA^2\omega_0^2\sin^2\omega_0 t}$$

Potential Energy: Suppose the particle is displaced by dx. then the workdone stored in the form of Potential Energy.

$$dU_p = -F.dx.$$

For the mass attached to a spring, $F = -Sx$.

$$dU_x = -(-Sx\,dx) = Sx\cdot dx.$$

The potential Energy of the oscillator, when the displacement is x,

$$U_p = \int dU_p = \int Sx\cdot dx.$$

$$U_p = \frac{1}{2}Sx^2$$

Put $x = A\cos\omega_0 t$

$$\therefore U_p = \frac{1}{2}SA^2\cos^2\omega_0 t$$

Total Energy of the Oscillator:- The total mechanical Energy of the oscillator is

$$U_m = U_k + U_p$$

$$= \frac{1}{2}mA^2\omega_0^2\sin^2\omega_0 t + \frac{1}{2}SA^2\cos^2\omega_0 t$$

But $S/m = \omega_0^2 \quad \therefore S = \omega_0^2 m$

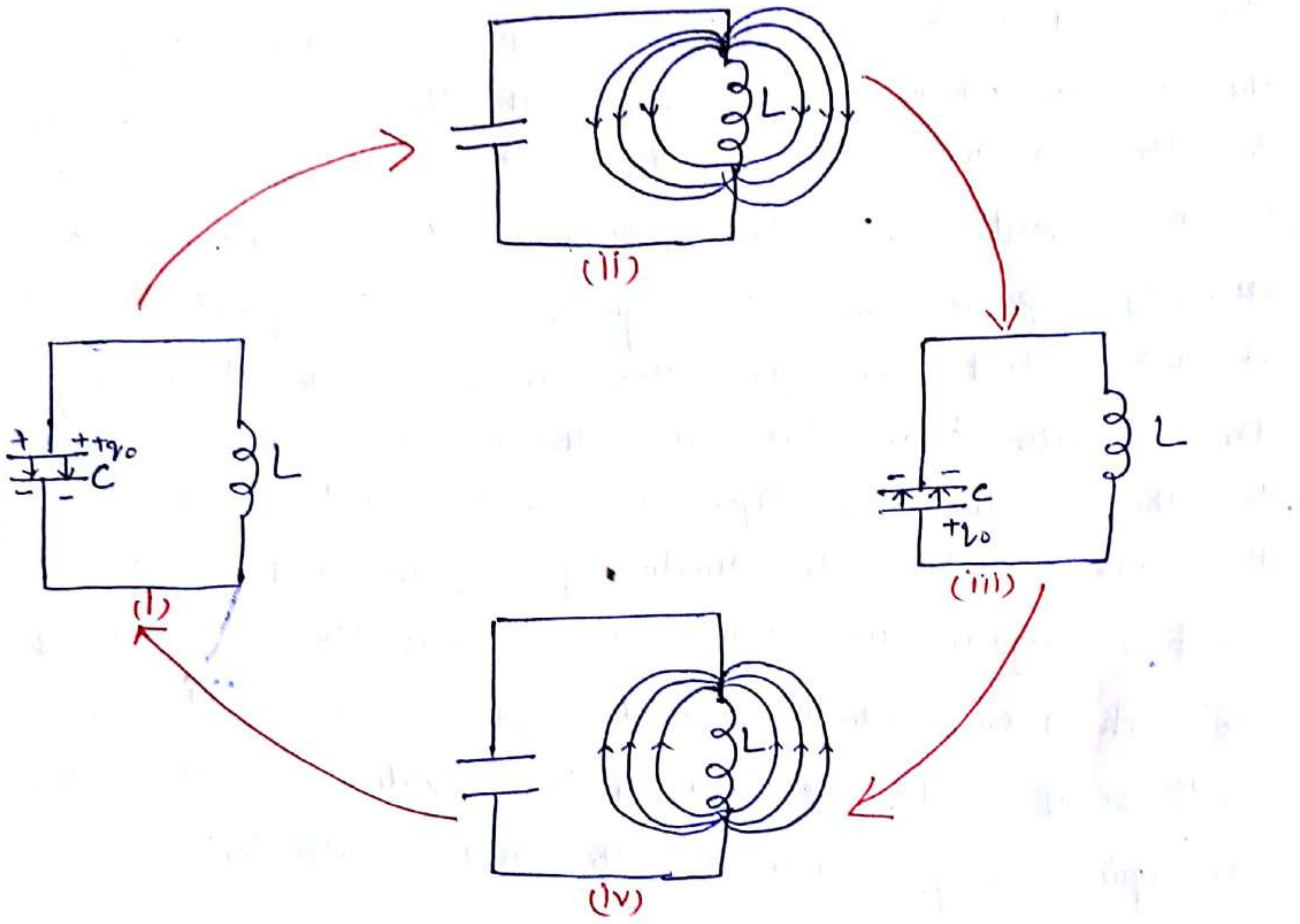
$$U_m = \frac{1}{2}mA^2\omega_0^2\sin^2\omega_0 t + \frac{1}{2}m\omega_0^2A^2\cos^2\omega_0 t$$

$$U_m = \frac{1}{2}mA^2\omega_0^2(\sin^2\omega_0 t + \cos^2\omega_0 t)$$

$$\left[U_m = \frac{1}{2}mA^2\omega_0^2\right]$$

This shows that the total energy of the mechanical oscillator is constant or is conserved and is independent of the location of the particle. It depends upon m, A, ω_0 or s.

Electrical Oscillator :-



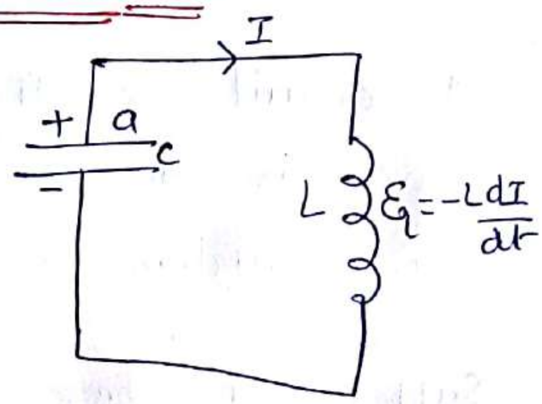
Electrical Oscillator.

A circuit consisting of Inductance (L) and Capacitance (C) serve as an electric oscillator. The various stages of the oscillations of charge q on the capacitor.

Suppose a charge q_0 is placed on the plates of the

Capacitor is shown in fig.(i) As the circuit is completed (Inductance L is connected across the capacitor), the capacitor begins to discharge through the Inductor. The flow of charge through the Indicator constitutes a current and magnetic field is set up in the coil L. The fig (ii) shows the state of oscillator, when the capacitor is completely discharged and maximum magnetic field is built up in the Inductor. The variation of the magnetic flux linked with the coil sets up an Induced emf across the Inductor which in accordance with the Lenz's law is opposite to the P.D across the capacitor. This emf charges the capacitor in the opposite sense. Again charge q_0 is collected on the plates of capacitor, but the sign of charge is opposite to that it was start with (i). Now the process of discharging of the capacitor begins but the direction of current is opposite to the earlier case. Again a magnetic field is set up across the Inductor, but the direction of magnetic field is opposite to setup in fig (ii). The variation of magnetic flux again sets up an Induced emf which charges the capacitor, obtaining the original state setup in (i). One oscillation is completed and the system is again ready to proceed with next oscillation.

Differential Equation for the electrical Oscillator:-



As per the Faraday's law, the induced emf in the Inductor is $\mathcal{E}_L = -L\frac{dI}{dt}$ where $I \rightarrow$ current in circuit

$\therefore$ $I = dq/dt$

$\therefore$ $\mathcal{E}_L = -L \dfrac{d^2q}{dt^2}$

The potential drop across the capacitor is $\mathcal{E}_C = q/C$

Applying Kirchoff's law (IInd)

$$\mathcal{E}_L = \mathcal{E}_C$$

$$\therefore \quad -L\frac{d^2q}{dt^2} = \frac{q}{C}$$

$$\text{or} \quad L\frac{d^2q}{dt^2} + \frac{q}{C} = 0$$

$$\text{or} \quad \frac{d^2q}{dt^2} + \frac{1}{LC}\cdot q = 0 \qquad -(1)$$

This is the differential Equation for the electrical oscillator, which is is

The equation of the simple harmonic oscillator, the solution of # (1) is

$$q = q_0 \cos \omega_0 t \qquad -(2)$$

$q_0 \rightarrow$ amplitude,

Energy of ~~Mechani~~ electrical oscillator

In the electrical oscillator, the energy exist as the electrical energy of capacitor + magnetic energy of Inductor.

$\therefore$ The total Energy stored in the Capacitor, when charge on the plate is q is given by $\quad \left[dU_C = \frac{q}{C}\cdot dq \right]$

$$U_C = \int dU_C = \int \frac{q}{C}\cdot dq$$

$$\therefore \boxed{U_C = \frac{1}{2}\frac{q^2}{C}}$$

The energy on the Inductor is $\quad \left[\because dU_L = \mathcal{E}_L \cdot I \cdot dt = L\cdot\frac{dI}{dt}\cdot I\cdot dt = LI\cdot dI \right]$

$$U_L = \int dU_L = \int LI\cdot dI = \frac{1}{2}LI^2$$

Since $I = dq/dt$ $\quad \therefore \boxed{U_L = \frac{1}{2}L\left(\frac{dq}{dt}\right)^2}$

Total Energy:

The total energy of the electrical oscillator is

$$U_{em} = U_L + U_C$$

$$U_{em} = \frac{1}{2} L \left(\frac{dq}{dt}\right)^2 + \frac{1}{2}\frac{q^2}{C}$$

Put $q = q_0 \cos \omega_0 t$

$$U_{em} = \frac{1}{2} L \left(-q_0 \omega_0 \sin \omega_0 t\right)^2 + \frac{1}{2}\frac{q_0^2}{C}\cos^2\omega_0 t$$

$$= \frac{1}{2} L q_0^2 \omega^2 \sin^2 \omega_0 t + \frac{1}{2}\frac{q_0^2}{C}\cos^2 \omega_0 t$$

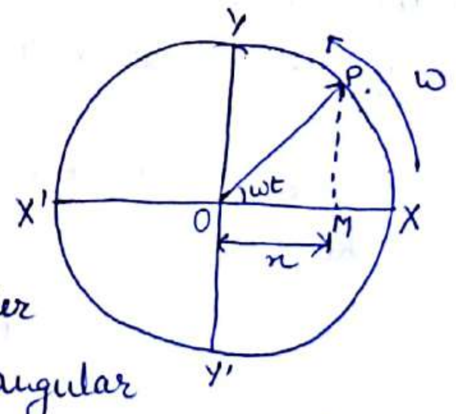
But $\omega_0^2 = 1/LC \quad \therefore \quad L\omega_0^2 = \frac{1}{C}$

$$\therefore \quad U_{em} = \frac{1}{2} L q_0^2 \omega_0^2 \cdot \sin^2 \omega_0 t + \frac{1}{2} q_0^2 L \omega_0^2 \cos^2 \omega_0 t$$

$$= \frac{1}{2} L\omega_0^2 q_0^2 (\sin^2 \omega_0 t + \cos^2 \omega_0 t)$$

$$\boxed{U_{em} = \frac{1}{2} L \omega_0^2 q_0^2}$$

PHASOR REPRESENTATION OF SHM :-



SHM can be represented as the projection of uniform circular motion on any one of the diameter. Consider a vector OP rotating with constant angular speed ω about point O in anticlkwise direction. Suppose OP coincides with OX. Then $\angle MOP = \omega t$

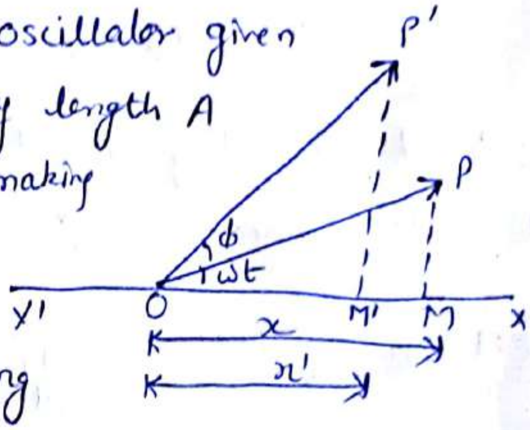
The projection of OP on X-axis is given by

$$OM = x = A \cos \omega t \quad —(1)$$

where $OP = A$ (length of rotating vector)

Such a vector of constant magnitude rotating with a constant angular speed in a plane about its own tail is called phasor. It is called so because its phase (the angle made by it with the axis) changes continuously with time.

To represent the simple harmonic oscillator given by $x = A\cos\omega t$, draw a vector of length A with its tail at origin O and making angle ωt with X-axis. Consider



OP' is another phasor, also rotating at angular speed ω.

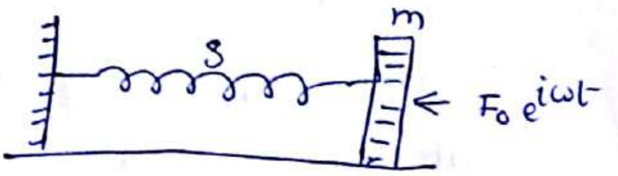
$$\therefore \; x' = OM' = A\cos(\omega t + \phi)$$

$\therefore$ If the two phasors differ in phase by angle ϕ, then they can be drawn from common origin and inclined to each other at an angle ϕ. This is the way of representing SHM phasor.

Forced Mechanical and Electrical Oscillator :- An oscillator to which a continuous excitation is provided by some external energy is called forced oscillator. The driving system (external agency) and the driven system are coupled to each other.

Equation of Forced Oscillator:

Mechanical Oscillator:



Consider a mass m attached to the End of a spring of spring Constant S. Other End of the spring is attached to rigid support.

Let the driving force acting on the system is $\vec{F} = F_0 e^{i\omega t}$

where $F_0 \rightarrow$ max value of driving force

$\omega \rightarrow$ frequency of oscillation.

The forces acting on the system is

① Restoring force :- $-Sx^*$ (x^* - displacement of the mass from equilibrium position).

② Damping force :- $-r \cdot \frac{d\vec{x}}{dt}$ (r is damping Constant)

③ Driving force :- $\vec{F} = F_0 e^{i\omega t}$

The negative sign in restoring force and damping force acts opposite to the displacement.

According to Newton's Law $F = m.a$

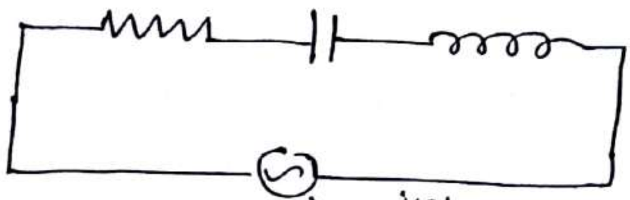
$$m \cdot \frac{d^2\vec{x}}{dt^2} = -S\vec{x} - b\frac{dx}{dt} + F_0 e^{i\omega t}$$

where $\frac{d^2\vec{x}}{dt^2}$ is the acceleration of the mass. On rearranging the above equation:

$$\boxed{m \cdot \frac{d^2x}{dt^2} + r \cdot \frac{dx}{dt} + Sx = F_0 e^{i\omega t}}$$

This is the equation of motion for the forced mechanical oscillator.

Electrical Oscillator:



$\vec{\mathcal{E}} = \mathcal{E}_0 e^{i\omega t}$

The electrical oscillator consist of capacitor (C) connected in series with an Inductor (L), resistor (R) and a source of emf $\vec{\mathcal{E}} = \mathcal{E}_0 e^{i\omega t}$

The instantaneous voltages in the circuit are

① Across the resistor $\vec{\mathcal{E}} = R\vec{I} = R\frac{d\vec{q}}{dt}$ [$\vec{I} \to$ instantaneous Current, $\vec{q} \to$ charge].

② Across the capacitor $\mathcal{E}_C = q/C$

③ Across the Inductor $\mathcal{E}_L = L \cdot \frac{dI}{dt} = L\frac{d^2q}{dt^2}$

Applying Kirchoff's second law to the ckt:

$$\boxed{L\frac{d^2\vec{q}}{dt^2} + R \cdot \frac{d\vec{q}}{dt} + \frac{\vec{q}}{C} = \mathcal{E}_0 e^{i\omega t}}$$

This is the equation of the electrical oscillations.

Solution of Equation of forced Mechanical Oscillator:

The equation of the forced mechanical oscillator is

$$m \cdot \frac{d^2x}{dt^2} + r \cdot \frac{dx}{dt} + Sx = F_0 e^{i\omega t} \quad \text{—①}$$

The response of the ~~f~~ damped oscillator to the driving force $\vec{F} = F_0 e^{i\omega t}$ consist of two parts

① Transient Behaviour

② Steady state Behaviour

Transient Behaviour: This ~~&~~ behaviour will persist only for a short interval of time in the beginning. During this tym, the natural oscillations of the system will dominate and the system will behave as if no external force is acting on the system.

$$\therefore \quad m \cdot \frac{d^2x}{dt^2} + r \cdot \frac{dx}{dt} + Sx = 0 \quad -(2)$$

Steady State behaviour :- Depending Upon the magnitude of damping, the transient oscillations die out and external driving force dominates the oscillator. Then the displacement in this state is the solution of # #(1).

Let the solution is $\vec{x} = \vec{A} e^{i\omega t}$

On differentiating it twice, we get

$$\frac{d\vec{x}}{dt} = \vec{A} i\omega e^{i\omega t}$$

$$\frac{d^2x}{dt^2} = -\vec{A}\omega^2 e^{i\omega t}$$

By putting these values in # (1)

$$(-m\omega^2\vec{A} + i\omega r\vec{A} + S\vec{A})e^{i\omega t} = F_0 e^{i\omega t}$$

Which gives $\quad A\omega\left[ir + \left(\frac{S}{\omega} - m\omega\right)\right] = F_0$

or $\quad A = \dfrac{\vec{F_0}}{\omega[ir + (S/\omega - m\omega)]}$

On Multiplying + dividing the R.H.S by $-i$

$$\vec{A} = \frac{-i\vec{F_0}}{\omega[r + i(m\omega - S/\omega)]} \quad -(A)$$

Writing $r + i(m\omega - S/\omega) = \vec{Z_m}$

$$\therefore \boxed{\vec{A} = \frac{-iF_0}{\omega\vec{Z_m}}} \quad -(B)$$

Here $\boxed{\vec{Z_m} = r + i\left(\omega m - \frac{s}{\omega}\right)}$ is called Mechanical Impedance of the oscillator.

It consist of two parts:

(i) $r \rightarrow$ resistance to the oscillatory motion due to friction

(ii) $X_m = (\omega m - s/\omega) \rightarrow$ reactance due to the combined effect of inertia and elasticity.

Here $\omega m \rightarrow$ inertial reactance and $s/\omega \rightarrow$ elastic reactance.

<u>Electrical Impedance</u>: The equation of forced electrical oscillator is given by:

$$L\frac{d^2q}{dt^2} + R\cdot\frac{dq}{dt} + \frac{q}{C} = \mathcal{E}_0 e^{i\omega t} \quad \text{—} (\mathrm{I})$$

Just like mechanical oscillator, the response of electrical oscillator to the driving emf $\vec{\mathcal{E}} = \mathcal{E}_0 e^{i\omega t}$ consist of two parts:

(1) Transient Behaviour: $L\frac{d^2q}{dt^2} + R\cdot\frac{dq}{dt} + \frac{q}{C} = 0$ [$\because$ external force is not acting]

(2) Steady state Behaviour: Let the solution of #(I) is

$$\vec{q} = \vec{A} e^{i\omega t} \quad \text{—} (\mathrm{II})$$

On differentiating #(I) twice

$$\frac{d\vec{q}}{dt} = \vec{A} i\omega e^{i\omega t}$$

$$\frac{d^2q}{dt^2} = -\vec{A} L\omega^2 e^{i\omega t}$$

Put these values in #(I)

$$\left(-L\omega^2\vec{A} + i\omega R\vec{A} + \vec{A}/C\right) e^{i\omega t} = \mathcal{E}_0 e^{i\omega t}$$

$$\vec{A} = \frac{\mathcal{E}_0}{\omega\left[iR + (1/\omega C - \omega L)\right]}$$

Put $R+i(\omega L - 1/\omega C) = \vec{Z_e}$, # ① becomes

$$\boxed{\vec{A} = \frac{-i\,\varepsilon_0}{\omega Z_e}}$$ where $\vec{Z_e} \rightarrow$ complex Impedance of electrical oscillator.

It consist of two parts:

① $R \rightarrow$ electrical resistance of the circuit

② $X_e = \omega L - 1/\omega C$ — reactance due to combined effect of Inductance & capacitance.

<u>Power Absorbed By Forced Oscillator</u>: The power is defined as the rate of doing work. Suppose an oscillator is driven by a force $F = F_0 \cos\omega t$, then the displacement is

$$x = \frac{F_0}{\omega Z_m} \sin(\omega t - \phi) \quad \text{—①}$$

where $Z_m \rightarrow$ mechanical Impedance

$$Z_m = \left[r^2 + (\omega m - s/\omega)^2 \right]^{1/2}$$

The workdone in displacing the oscillator through a distance dx is

$$dW = F \cdot dx$$

Power is given as $P = \frac{dW}{dt} = F \cdot \frac{dx}{dt}$ — ②

From # ①

$$\frac{dx}{dt} = \frac{F_0}{\omega Z_m} \omega \cos(\omega t - \phi) = \frac{F_0}{Z_m} \cos(\omega t - \phi)$$

Putting these values in # ②

$$P = (F_0 \cos\omega t) \cdot \frac{F_0}{Z_m} \cos(\omega t - \phi) = \frac{F_0^2}{Z_m} \cos\omega t \cdot \cos(\omega - \phi).$$

Average Power over one time period is

$$P_{av} = \frac{1}{T} \int_0^T P \cdot dt = \frac{1}{T} \int^T \frac{F_0^2}{Z_m} \cos\omega t \cdot \cos(\omega t - \phi) \cdot dt \quad \text{—③}$$

$$= \frac{F_0^2}{T Z_m} \int_0^T \cos\omega t (\cos\omega t \cdot \cos\phi + \sin\omega t \cdot \sin\phi) \cdot dt$$

$$= \frac{F_0^2}{T Z_m} \int_0^T (\cos^2\omega t \cdot \cos\phi + \cos\omega t \cdot \sin\omega t \cdot \sin\phi) dt$$

$$= \frac{F_0^2}{T Z_m} \int_0^T \left(\frac{1+\cos 2\omega t}{2} \cos\phi + \frac{\sin 2\omega t}{2} \sin\phi \right) dt$$

$$P_{av} = \frac{F_0^2}{2 T Z_m} \left(\cos\phi \int_0^T dt + \cos\phi \int_0^T \cos 2\omega t \cdot dt + \sin\phi \int_0^T \sin 2\omega t \cdot dt \right)$$

But $\int_0^T \cos 2\omega t \cdot dt = \int_0^T \sin 2\omega t \cdot dt = 0$

$$\therefore \quad P_{av} = \frac{F_0^2}{2 T Z_m} \left[(\cos\phi) t \right]_0^T$$

$$P_{av} = \frac{F_0^2}{2 T Z_m} (\cos\phi) T$$

$$\boxed{P_{av} = \frac{F^2}{2 Z_m} \cos\phi}$$

$\cos\phi \rightarrow$ power factor

$\phi \rightarrow$ phase difference between force and displacement.