Unit - II
Differential Equations

It attain dependent Variable and independent Variable and its derivative.

Two types:-

(i) Odinary diff eq^n $\left(\frac{dy}{dx}\right)$

(ii) Partial diff eq^n $\left(\frac{d^2 z}{dx dy}\right)$

Exact diff eq^n

$$m dx + N dy = 0$$

$$\frac{\partial m}{\partial y} = \frac{\partial N}{\partial x}$$

The Solution of exact differential equation is

$$\int_{y \text{ constant}} m dx + \int (\text{term in } N \text{ not Containing } x)\, dy = 0$$

Que Solve the differential equation.

$$(1 + e^{x/y})\, dx + e^{x/y}\left(1 - \frac{x}{y}\right) dy = 0$$

$$\underbrace{(1 + e^{x/y})}_{m} dx + \underbrace{e^{x/y}\left(1 - \frac{x}{y}\right)}_{N} dy = 0$$

$$\frac{\partial M}{\partial y} = 0 + e^{x/y}\left(\frac{-x}{y^2}\right)$$

$$\frac{\partial N}{\partial x} = e^{x/y}\left(\frac{-1}{y}\right) + \left(1 - \frac{x}{y}\right) e^{x/y}\left(\frac{1}{y}\right)$$

$$= \frac{-e^{x/y}}{\cancel{y}} + \frac{e^{x/y}}{\cancel{y}} - \frac{x e^{x/y}}{y^2}$$

$$= \frac{-x e^{x/y}}{y^2}$$

It is exact diff equation

The solution of D.E

$$\int M dx + \int 0 \, dy = c$$

$$\int_{y\,const} (1 + e^{x/y})\, dx = c$$

$$x + \frac{e^{x/y}}{1/y} = c.$$

Que Solve the diff eqⁿ.

$$\underbrace{(x^4 - 2xy^2 + y^4)}_{m} dx - \underbrace{(2x^2y - 4xy^3 + \sin y)}_{N} dy = 0$$

$$\frac{\partial M}{\partial y} = -4xy + 4y^3$$

Date
Page

$$\frac{\partial N}{\partial x} \Rightarrow -4xy + 4y^3$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

It is exact diff equation.

$$\int_{y\ \text{const}} M\,dx + \int N\ (\text{not containing } x)\ dy = C$$

$$\int_{y\ \text{const}} (x^4 - 2xy^2 + y^4)\,dx + \int -\sin y\,dy = C$$

$$\frac{x^5}{5} - \frac{2x^2y^2}{2} + xy^4 + \cos y = C.$$

Que $\underbrace{(e^y + 1)\cos x\,dx}_{M} + \underbrace{e^y \sin x\,dy}_{N} = 0$

$$\frac{\partial M}{\partial y} = \frac{\partial (e^y+1)}{\partial y} \cos x$$

$$\Rightarrow e^y \cos x + 0$$

$$\frac{\partial N}{\partial x} = \frac{\partial e^y \sin x}{\partial x} \qquad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\Rightarrow e^y \cos x$$

$\therefore$ it is exact solution.

$$\int_{y\ const} M\,dx + \int N(\text{not containing } x)\,dy = c.$$

$$\int (e^y \cos x + \cos x)\,dx + e^y \sin x + \sin x = c$$

EQUATION REDUCIBLE TO EXACT DIFFERENTIAL EQUATION.

INTEGRATING FACTOR (I.F)

Que using I.F $\frac{1}{x^2}$. find the solution of the DE $(x^2+y^2)dx - 2xy\,dy = 0$.

$$\frac{\partial M}{\partial y} = 2y \qquad \frac{\partial N}{\partial x} = -2y.$$

Not exact

multiply $\frac{1}{x^2}$ (I.F) to eq^n (1)

$$\left(\frac{x^2+y^2}{x^2}\right)dx - \frac{2xy}{x^2}\,dy = 0$$

$$\underbrace{\left(1 + \frac{y^2}{x^2}\right)}_{m} dx - \underbrace{\frac{2y}{x}}_{n}\,dy = 0$$

$$\frac{\partial m}{\partial y} = \frac{2y}{x^2} \qquad \frac{\partial N}{\partial x} = \frac{2y}{x^2}$$

exact sol^n.

$$\int_{y\,const} \left(1 + \frac{y^2}{x^2}\right) dx + \int 0\, dy = c.$$

$$x - \frac{y^2}{x} = c$$

Rules for finding integrating factor:-

<u>Rule 1</u> :- If $m(x,y)$ & $N(x,y)$ are homogenous function in $x+y$ the eq^n $mdx + ndy = 0$ is not exact then $\frac{1}{mx + Ny}$ is an I.F provided $Mx + Ny \neq 0$

<u>Que</u> <u>Solve</u> $x^2y\,dx - (x^3 + y^3)\,dy = 0$

$$\frac{\partial M}{\partial y} = x^2 \qquad \frac{\partial N}{\partial x} = -3x^2$$

Not exact solution.

$$Mx + Ny = \cancel{x^3y} - \cancel{x^3y} - y^4$$
$$= -y^4$$

$$I.F = \frac{-1}{y^4}$$

$$-\frac{x^2y}{y^4}\,dx + \left(\frac{x^3}{y^4} + \frac{y^3}{y}\right) dy = 0$$

$$-\frac{x^2}{y^3}\,dx + \left(\frac{x^3}{y^4} + \frac{1}{y}\right) dy = 0$$

$$\frac{\partial M}{\partial y} = \frac{3x^2}{y^4} \qquad \frac{\partial N}{\partial x} = \frac{3x^2}{y^4}$$

exact Solution.

$$\int_{y\ const} M dx + \int (\text{terms of } N \text{ not Containing } x)\, dy.$$

$$\int_{y\ const} \frac{-x^2 y}{y^4}\, dx + \int \frac{y^3}{y^4}\, dy = c$$

$$\frac{-x^3}{3y^3} + \log y = c.$$

Rule 2 :- If the eq^n $Mdx + Ndy = 0$ is the exact and is of the form $f(x,y)\,y\,dx + g(xy)\,x\,dy = 0$ then $\frac{1}{Mx - Ny}$ is an integrating factor, provided $Mx - Ny \neq 0$.

Que $\underbrace{xy^2 + 2x^2y^3}_{m}\, dx + \underbrace{x^2y - x^3y^2}_{n}\, dy = 0$

y common   x common

$$\frac{\partial M}{\partial y} = \frac{\partial xy^2}{\partial y} + \frac{\partial}{\partial y} 2x^2y^3$$

$$\Rightarrow 2yx + 2x^2 3y^2$$
$$\Rightarrow 2xy + 6x^2y^2.$$

$$\frac{\partial N}{\partial x} = \frac{\partial x^2 y}{\partial x} - \frac{\partial}{\partial x} x^3y^2$$

Date
Page

$\Rightarrow \quad 2xy - 3x^2y^2$

Not exact.

$$Mx - Ny = x^2y^2 + 2x^3y^3 - x^2y^2 + x^3y^3 \neq 0$$

$$I.F = \frac{1}{3x^3y^3}$$

Multiplying (1) by I.F.

$$\left(\frac{xy^2}{3x^3y^3} + \frac{2x^2y^3}{3x^3y^3}\right)dx + \left(\frac{x^2y}{3x^3y^3} - \frac{x^3y^3}{3x^3y^3}\right)dy = 0$$

$$\underbrace{\left(\frac{1}{3x^2y} + \frac{2}{3x}\right)}_{m} dx + \underbrace{\left(\frac{1}{3xy^2} - \frac{1}{3y}\right)}_{n} dy = 0$$

$$\frac{\partial m}{\partial y} = \left(\frac{\partial}{\partial y}\frac{1}{3x^2y} + \frac{\partial}{\partial y}\frac{2}{3x}\right)$$

$$= \frac{1}{3x^2}\frac{\partial}{\partial y}\frac{1}{y} + 0.$$

$$= -\frac{1}{3x^2y}$$

$$\frac{\partial N}{\partial x} \Rightarrow \frac{\partial}{\partial x}\frac{1}{3xy^2} - \frac{\partial}{\partial x}\frac{1}{3y}.$$

$$= \frac{1}{3y^2}\left(-\frac{1}{x^2}\right) + 0.$$

exact.

$$\int_{y\,const} m\,dx + \int (\text{Terms not contain } x)\,dy \Rightarrow c.$$

$$\int_{y\,const} \left(\frac{1}{3x^2y} + \frac{2}{3x} \right) dx + \int \frac{-1}{3y}\,\partial y \Rightarrow c.$$

$$-\frac{1}{3xy} + \frac{2}{3}\log x - \frac{1}{3}\int \frac{1}{y}\,dy \Rightarrow c.$$

$$-\frac{1}{3xy} + \frac{2}{3}\log x - \frac{\log y}{3} \Rightarrow c.$$

$$-\frac{1}{xy} + 2\log x - \log y \Rightarrow 3c.$$

Que $\underbrace{(1+xy)y}_{m}\,dx + \underbrace{(1-xy)x}_{n}\,dy \Rightarrow 0$ —— (1)

$$\frac{\partial M}{\partial y},\ \frac{\partial}{\partial y}(1+xy)y$$

$$1 + \frac{\partial}{\partial y} xy^2$$

$$\frac{\partial M}{\partial y} \Rightarrow 1 + x2y$$

$$\frac{\partial N}{\partial x} \Rightarrow \frac{\partial(1-xy)}{\partial x}$$

$$= 1 - \frac{\partial x^2y}{\partial x}$$

Date
Page

$$\frac{\partial N}{\partial x} \Rightarrow 1 - 2xy.$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

Not exact

$$Mx - Ny \Rightarrow xy + x^2y^2 - (xy - x^2y^2)$$
$$\Rightarrow xy + x^2y^2 - xy + x^2y^2$$
$$\Rightarrow 2x^2y^2.$$

$$I.F = \frac{1}{2x^2y^2}$$

Multiplying (1) by I.F

$$\frac{(y + xy^2)dx}{2x^2y^2} + \frac{(x - x^2y)}{2x^2y}dy = 0$$

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(\frac{1}{2x^2y} + \frac{1}{2x}\right)$$
$$= \frac{1}{2x^2}\left(\frac{-1}{y^2}\right)$$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{1}{2xy^2} - \frac{1}{2y}\right)$$

$$= \frac{1}{2y^2}\left(\frac{-1}{x^2}\right)$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Exact.

The solution is.

$$\int_{y\,const} M\,dx + \int (\text{Terms of } N \text{ not containing } x)\,dy.$$

$$\int_{y\,const} \left(\frac{1}{2x^2y} + \frac{1}{2x}\right)dx + \int\left(\frac{-1}{2y}\right)dy = C.$$

$$\frac{-1}{2y} \times \frac{1}{x} + \frac{\log x}{2} - \frac{1}{2}\log y = C.$$

$$\frac{-1}{2xy} + \frac{\log x}{2} - \frac{\log y}{2} \Rightarrow C$$

$$\frac{-1}{xy} + \log x - \log y \Rightarrow 2C.$$

Rule 3 :- If the equation $M dx + N dy = 0$ is not exact and

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} \text{ is function of } x \text{ only}$$

$f(x)$ say then $e^{\int f(x)dx}$ is an I.F.

Rule 4:- If the eq^n $Mdx + Ndy = 0$ is not exact and $\dfrac{\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}}{M}$ is function of y only $f(y)$

then $e^{\int f(y)\,dy}$ is an integrating factor.

Q Solve $(x^2 + y^2 + 2x)\,dx + 2y\,dy = 0$.

$$\frac{\partial M}{\partial y} = 2y$$

$$\frac{\partial N}{\partial x} = 0$$

Not exact.

$$\frac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N} = \frac{2y - 0}{2y} = 1 = x^0 = f(x).$$

$$e^{\int F(x)\,dx}$$

$$e^{\int 1\,dx}$$

$$e^x$$

$$(x^2 + y^2 + 2x)e^x\,dx + 2ye^x\,dy = 0$$

$$\frac{\partial M}{\partial y} = 2ye^x$$

$$\frac{\partial N}{\partial x} = 2ye^x$$

exact.

Now,

product rule

$$\int_{y\,\text{const}} (x^2e^x + y^2e^x + 2xe^x)\,dx + \int 0\,dy = c$$

$$= x^2e^x - \int \cancel{2xe^x}\,dx + y^2e^x + \int \cancel{2xe^x}\,dx = c.$$

$$= x^2e^x + y^2e^x = c$$

$$= e^x(x^2+y^2) = c.$$

Q. Solve $\underbrace{(3x^2y^4 + 2xy)}_{m}dx + \underbrace{(2x^3y^3 - x^2)}_{n}dy$

$$\frac{\partial M}{\partial y} = 12x^2y^3 + 2x$$

$$\frac{\partial N}{\partial x} = 6x^2y^3 - 2x$$

Not exact.

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = \frac{-12x^2y^3 \mp 2x + 6x^2y^3 \mp 2x}{3x^2y^4 + 2xy}$$

$$= \frac{6x^2y^3 + 4x}{3x^2y^4 + 2xy}.$$

$$= \cancel{6x}$$

$$= \frac{2x\,\cancel{(3xy^3+2)}}{xy\,\cancel{(3xy^3+2)}}$$

$$= -\frac{2}{y}$$

$$\text{I.F.} = e^{\int -\frac{2}{y}dy} =$$

$$= e^{-2\int y^{-1}dy}$$

$$= e^{-2\log y}$$

$= e^{\log y^{-2}}$

$= e^{\log\left(\frac{1}{y}\right)^2}$

$= \left(\frac{1}{y}\right)^2$

$\frac{1}{y^2}(3x^2y^4 + 2xy)\,dx + \frac{1}{y^2}(2x^3y^3 - x^2) = 0$

$\frac{\partial M}{\partial y} = 6yx^2 + \cancel{\frac{2x \log y}{y^2}}$

$\frac{\partial N}{\partial x} = 6x^2y - \frac{2x}{y^2}$

exact

$\int_{y\,const}\left(3x^2y^2 + \frac{2x}{y}\right)dx + \int 0\,dy = c.$

$\cancel{6xy^2 + \frac{2}{y} = c}$

$\int_{y\,const} 3x^2y^2 + \int_{y\,const} \frac{2x}{y}\,dx = c$

$3y^2 \int x^2\,dx + \frac{2x^2}{y^2} = c$

$3y^2 \frac{x^3}{3} + \frac{x^2}{y} = c$

$\frac{x^{2+1}}{2+1}$

$x^3y^2 + \frac{x^2}{y} = c$

Rule 5 :- If the equation $Mdx + Ndy = 0$ can be expressed as $x^a y^b (my\,dx + nx\,dy) + x^c y^d (py\,dx + qx\,dy) = 0$ where a, b, c, d, m, n, p, q are constants and $\frac{m}{n} \neq \frac{p}{q}$ then $x^\alpha y^\beta$ is an I.F where α and β are so chosen that

$$\frac{a+\alpha+1}{m} = \frac{b+\beta+1}{n} \quad \text{and.}$$

$$\frac{c+\alpha+1}{p} = \frac{d+\beta+1}{q}$$

Q Solve $(2x^2y - 3y^4)dx + (3x^3 + 2xy^3)dy = 0$

$2x^2y\,dx - 3y^4dx + 3x^3dy + 2xy^3dy = 0$

$yx^2(2y\,dx + 3x\,dy) + y^3(-3y\,dx + 2x\,dy) = 0$

comparing,

$a = 2, b = 0, m = 2, n = 3, c = 0, d = 3, p = 3, q = 2.$

$$\frac{2+\alpha+1}{2} = \frac{0+\beta+1}{3}$$

$$\frac{2+\alpha}{2} = \frac{\beta+1}{3}$$

$6 + 3\alpha = 2\beta + 2$

$3\alpha - 2\beta = -4 \quad \text{—— (1)}$

$$\frac{0+\alpha+1}{-3} = \frac{3+\beta+1}{2}$$

$2\alpha + 2 = -12 - 3\beta$

$2\alpha + 3\beta = -14 \quad \text{—— (2)}$

Solving (1) and (2)

$$\alpha = \frac{-49}{13}, \quad \beta = \frac{-28}{13}$$

$$I.F = x^{\alpha} y^{\beta}$$

$$= x^{\frac{-49}{13}} y^{\frac{-28}{13}}$$

$$\left(2x^2y\, x^{\frac{-49}{13}} y^{\frac{-28}{13}} - 3y^4 x^{\frac{-49}{13}} y^{\frac{-28}{13}}\right)dx +$$

$$\left(3x^2 x^{\frac{-49}{13}} y^{\frac{-28}{13}} + 2xy^3 x^{-49/13} y^{-28/13}\right)dy = 0$$

$$= \left(2x\, x^{-23/13} y^{-15/13} - 3x^{\frac{-49}{13}} y^{24/13}\right)dx +$$

$$\left(3x^{-10/13} y^{-28/13} + 2x^{-36/13} y^{11/13}\right)dy = 0$$

$$\frac{\partial M}{\partial y} = -\frac{30}{13} x^{-23/13} y^{-28/13} - \frac{72}{13} x^{-49/13} y^{11/13}$$

$$\frac{\partial N}{\partial x} = \frac{-30}{13} x^{-23/13} y^{-28/13} - \frac{72}{13} x^{-49/13} y^{11/13}.$$

exact.

$$\int_{y\,const.} \left(3x^{-23/13} y^{-15/13} - 3x^{-49/13} y^{24/13}\right)dx = C.$$

$$\frac{-13\times 3}{10} y^{-15/13} x^{-10/13} + \frac{3\times 13 x}{36}^{-36/13} y^{24/13} = C$$

Date
Page

Q. Solve $(y^3 - 2yx^2)\,dx + (2xy^2 - x^3)\,dy = 0$

$$x^a y^b (my\,dx + nx\,dy) + x^c y^d (py\,dx + qx\,dy) = 0.$$

$$y^3dx - 2yx^2dx + 2xy^2\,dy - x^3dy$$
$$y^2(y\,dx + 2x\,dy) + x^2(-2y\,dx - x\,dy)$$

$a=0,\ b=2,\ m=1,\ n=2,\ c=2,\ d=0,\ p=-2,\ q=-1.$

$$\frac{m}{n} \neq \frac{p}{q}$$

$$\frac{1}{2} \neq \frac{-2}{-1}$$

$$I.F = x^{\alpha} y^{\beta}$$

$$\frac{a+\alpha+1}{m} = \frac{b+\beta+1}{n}$$

$$\frac{\alpha+1}{1} = \frac{2+\beta+1}{2}$$

$$2(\alpha+1) = (3+\beta)$$
$$2\alpha+2 = 3+\beta$$
$$2\alpha - \beta = 1. \quad \text{——} (1)$$

$$\frac{c+\alpha+1}{p} = \frac{d+\beta+1}{q}$$

$$\frac{2+\alpha+1}{-2} = \frac{\beta+1}{-1}$$

$$\frac{3+\alpha}{2} = \beta+1$$

$$3+\alpha = 2\beta+2$$
$$\alpha - 2\beta = -1$$

$$x^2y^2(y^2-x^2)=C$$

$2\alpha - \beta = 1$

$\alpha - 2\beta = -1 \qquad \times 2.$

$2\alpha - \beta = 1$

$-2\alpha - 4\beta = -2$

$3\beta = 3$

$\beta = 1.$

$2\alpha - 1 = 1$

$2\alpha = 2$

$\alpha = 1.$

$\alpha = 1, \quad \beta = 1.$

$I.F = x^{\alpha} y^{\beta}$

$= x^1 y^1.$

$= xy.$

$$(xy^4 - 2x^3y^2)dx + (2x^2y^3 - x^4y)dy = 0.$$

~~$\frac{\partial M}{\partial y} = \frac{xy^5}{5} - 2x^3$~~

$$\frac{\partial M}{\partial y} = 4xy^3 - \cancel{6x^3y}\ 4x^3y.$$

$$\frac{\partial N}{\partial x} = 4xy^3 - 4x^3y$$

NOW,

$$\int_{y\ \text{const.}} (xy^4 - 2x^3y^2)\,dx = C.$$

$$\frac{x^2y^4}{2} - \frac{\cancel{2}y^2x^4}{\cancel{4}2} = C$$

$$x^2y^4 - y^2x^4 = C$$

$$x^2y^2(y^2-x^2) = C.$$

Linear differential equations :-

A linear differential equation of the form

$$\frac{dy}{dx} + Py = Q$$

is called linear differential equation where P and Q are functions of x (but not of y) or constants.

→ $e^{\int P dx}$ is an integrating factor of diff eq^n.

→ solution is $y \times I.F = \int Q \times I.F \, dx + c$.

Q. Solve $(x+1)\frac{dy}{dx} - y = e^x (x+1)^2$

$$\frac{dy}{dx} - \frac{y}{x+1} = e^x (x+1)$$

$$P = \frac{-1}{x+1} \quad , \quad Q = e^x (x+1)$$

$$I.F. = e^{\int \frac{-1}{x+1} dx}$$

$$= e^{-\log(x+1)}$$

$$= e^{\log\left(\frac{1}{x+1}\right)}$$

$$= \frac{1}{x+1}.$$

$$y \times \frac{1}{x+1} = \int e^x (x+1) \times \frac{1}{(x+1)} \, dx + c$$

$$\frac{y}{x+1} = \int e^x \, dx + c.$$

$$\frac{y}{x+1} = e^x + c.$$

$$y = e^x (x+1) + c(x+1).$$

Q $x^3 - x \frac{dy}{dx} - (3x^2 - 1) y = x^5 - 2x^3 + x.$

$$\frac{dy}{dx} - \frac{(3x^2-1)}{(x^3-x)} y = \frac{x^5 - 2x^3 + x}{x^3 - x}$$

$$\frac{dy}{dx} - \frac{3x^2-1}{x(x^2-1)} y = \frac{\cancel{x}(x^4 - 2x^2 + 1)}{\cancel{x}(x^2-1)}$$

$$P = -\frac{3x^2-1}{x(x^2-1)}, \quad Q = \frac{x^4 - 2x^2 + 1}{(x^2-1)} = \frac{(x^2-1)\cancel{(x^2-1)}}{\cancel{(x^2-1)}}$$

$$\text{I.F.} = e^{\int \frac{-3x^2-1}{\cancel{x(x^3-x)}} dx}$$

$x^3 - x = t$

$dt = (3x^2 - 1) dx$

$$e^{-\int \frac{dt}{t}} = e^{-\log t} = \frac{1}{t} = \frac{1}{x^3 - x}$$

solution

$$y \times \frac{1}{x^3 - x} = \int (x^2 - 1) \times \frac{1}{x(x^2-1)} \, dx$$

$$\frac{y}{x^3-x} = \log x + c.$$

$$y = (\log x + c)(x^3 - x)$$

Q. Solve.

1. $$\frac{dy}{dx} + \frac{1}{x}y = x^3 - 3.$$

$$P = \frac{1}{x}, \quad Q = x^3 - 3.$$

$$\text{I.F} = e^{\int \frac{1}{x}dx}$$
$$= e^{\log x}$$
$$= x.$$

$$y \times x = \int (x^3 - 3) \times x \, dx + c$$

$$xy = \int x^4 dx - \int 3x \, dx + c$$

$$xy = \frac{x^5}{5} - \frac{3x^2}{2} + c.$$

2. $$(2y - 3x)\,dx + x\,dy = 0.$$

$$\frac{(2y-3x)\cancel{dx}}{x \cdot \cancel{dx}} + \frac{\cancel{x}\,dy}{\cancel{x}\,dx} = 0$$

$$\frac{2}{x}y - 3 + \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} + \frac{2}{x}y = 3.$$

$$P = \frac{2}{x} \quad , Q = 3.$$

$$I.f = e^{\int P dx}$$
$$= e^{2\int \frac{1}{x} dx}$$
$$= e^{2\log x}$$
$$= e^{\log x^2}$$
$$= x^2$$

$$y \times x^2 = \int \cancel{x^2} \, 3x^2 + c.$$

$$y x^2 = \cancel{3} \cdot \frac{x^3}{\cancel{3}} + c$$

$$y x^2 = x^3 + c.$$

8. $$\frac{dy}{dx} + y \cot x = \cos x$$

$$P = \cot x \qquad Q = \cos x.$$

$$I.f. = e^{\int P dx}$$
$$= e^{\int \cot dx}$$
$$= e^{\log_e \sin x}$$
$$= \sin x.$$

$$y \times \sin x = \int \cos x \times \sin x \, dx + c.$$

let $t = \sin x$
$dt = \cos x \, dx$

$$y \times \sin x = \int t\,dt + c.$$

$$y\sin x = \frac{t^2}{2} + c.$$

$$y\sin x = \frac{\sin^2 x}{2} + c.$$

4. $$\frac{dy}{dx} + y\sec x = \tan x.$$

$P = \sec x \qquad Q = \tan x.$

$$I.F = e^{\int P\,dx}$$
$$= e^{\int \sec dx}$$
$$= e^{\log(\sec x + \tan x)}$$
$$= \sec x + \tan x.$$

$$y \times (\sec x + \tan x) = \int \tan x(\sec x + \tan x)\,dx + c$$
$$y \times (\sec x + \tan x) = \int \tan x \sec x\,dx + \int \cancel{(1 + \sec^2 x)}(\sec^2 x - 1)\,dx + c$$

$$y \times (\sec x + \tan x) = \sec x - x + \tan x + c.$$
$$y = \frac{\sec x + \tan x}{\sec x + \tan x} - \frac{x + c}{\sec x + \tan x}$$

$$y = 1 + \frac{-x + c}{\sec x + \tan x}$$

$$y = \frac{c - x}{\sec x + \tan x} + 1.$$

5. $\cos^2 x \frac{dy}{dx} + y = \tan x.$

$$\frac{dy}{dx} + \frac{1}{\cos^2 x} y = \frac{\tan x}{\cos^2 x}$$

$$P = \frac{1}{\cos^2 x} \qquad Q = \frac{\tan x}{\cos^2 x}$$

$$e^{\int P dx}.$$

$$e^{\int \frac{1}{\cos^2 x} dx}. = e^{\int \sec^2 x \, dx}.$$

$$\cancel{e^{\int \cos^{-2} x \, dx}}. = e^{\tan x} + c.$$

$$\cancel{e^{\int \frac{-2\cos^{-3} x}{-3} x \, dx}}.$$

$$y \times e^{\tan x} = \int \frac{\tan x}{\cos^2 x} \times e^{\tan x} + c.$$

$$\cancel{y \times e^{\tan x} = \int \frac{\sin x}{\cos^3 x} \times e^{\tan x} + c}.$$

$$y \times e^{\tan x} = \int \tan x \sec^2 x \, e^{\tan x} dx + c.$$

ILATE

$t_1 = \tan x$
$dt_1 = \sec^2 x \, dx$

$$y \times e^{\tan x} \int t_1 e^{t_1} dt_1 + c.$$

$$y \times e^{\tan x} = t_1 \int e^{t_1} - \int \left[\frac{dt_1}{dt_1} \int e^{t_1} \right] dt_1 + c.$$

$$y \times e^{\tan x} = t e^{t} - \int e^{t} dt + c.$$

$$y \times e^{\tan x} = t e^{t} - e^{t} + c.$$

$$y \times e^{\tan x} = \tan x \, e^{\tan x} - e^{\tan x} + c.$$

$$y \times e^{\tan x} = (\tan x - 1) e^{\tan x} + c.$$

$$y = \frac{(\tan x - 1) \cancel{e^{\tan x}}}{\cancel{e^{\tan x}}} + \frac{c}{e^{\tan x}}$$

$$y = (\tan x - 1) + c e^{-\tan x}.$$

6. $(x+a)\frac{dy}{dx} - 3y = (x+a)^5$

$$\frac{dy}{dx} - \frac{3}{(x+a)} y = (x+a)^4$$

$$P = \frac{-3}{x+a} \qquad Q = (x+a)^4$$

$$e^{\int P dx}.$$

$$e^{\int \frac{-3}{x+a} dx}.$$

$$e^{-3\log(x+a) \cancel{dx}}$$

$$\frac{1}{(x+a)^3}$$

$$y \times \frac{1}{(x+a)^3} = \int (x+a)^4 \times \frac{1}{(x+a)^3} + c.$$

$$y \times \frac{1}{(x+a)^3} = \int (x+a) + c.$$

$$y \times \frac{1}{(x+a)^3} = \int x\,dx + \int a\,dx + c.$$

$$y \times \frac{1}{(x+a)^3} = \frac{x^2}{2} + ax + c.$$

$$y \times \frac{1}{(x+a)^3} = \frac{(x+a)^2}{2} + c.$$

$$2y = (x+a)^5 + c(x+a)^3.$$

7. $x\cos x \frac{dy}{dx} + y(x\sin x + \cos x) = 1.$

$$\frac{dy}{dx} + \frac{x\sin x + \cos x}{x\cos x} y = \frac{1}{x\cos x}.$$

$$P = \frac{x\sin x + \cos x}{x\cos x} \qquad Q = \frac{1}{x\cos x}.$$

$$= \tan x + \frac{1}{x}.$$

$$I.F = e^{\int P\,dx}.$$

$$= e^{\int \left(\tan x + \frac{1}{x}\right) dx}.$$

$$= e^{\log \sec x + \log x}$$

$$= e^{\log(\sec x * x)}$$

$$= \sec x * x = x \sec x.$$

$$y \times x \sec x = \int \frac{1}{x\cos x} \times x\sec x \, dx + c$$

$$y \times x \sec x = \int \sec^2 x \, dx + c.$$

$$xy \sec x = \tan x + c.$$

$$xy = \frac{\sin x}{\cos x} \times \cos x + C\cos x.$$

$$xy = \sin x + C\cos x.$$

8. $x\log x \frac{dy}{dx} + y = 2\log x$

$$\frac{dy}{dx} + \frac{1}{x\log x} y = \frac{2\log x}{x\log x}$$

$$\frac{dy}{dx} + \frac{1}{x\log x} y = \frac{2}{x}$$

$$P = \frac{1}{x\log x} \qquad Q = \frac{2}{x}$$

$$I.F = e^{\int P dx}$$

$$e^{\int \frac{1}{x\log x} dx}.$$

$u = \log x$

$du = \frac{1}{x} dx.$

$$e^{\int \frac{1}{u} du}$$

$$e^{\log u}.$$

$$\log x$$

$$y \times \log x = \int \frac{2}{x} \log x \, dx + c$$

$$y\log x = 2\int u \, du + c$$

$$y\log x = \frac{2u^2}{2} + c.$$

$$y\log x = (\log x)^2 + c.$$

9. $x\frac{dy}{dx} + 2y = x^2\log x$.

$$\frac{dy}{dx} + \frac{2}{x}y = x\log x$$

$P = \frac{2}{x}$ $\quad\quad Q = x\log x$.

$$I.F = e^{2\int \frac{1}{x}dx}.$$

$$e^{\log x^2}$$

$$x^2$$

$$y \times x^2 = \int x\log x \times x^2$$

$$yx^2 = \log x\int x^3 - \int\left[\frac{d\log x}{dx} \cdot \int x^3\right]dx + c.$$

$$yx^2 = \frac{\log x\, x^4}{4} - \int \frac{1}{x} \times \frac{x^4}{4}\, dx. + c$$

$$yx^2 = \frac{x^4\log x}{4} - \frac{1}{4}\frac{x^4}{4} + c$$

$$yx^2 = \frac{x^4\log x}{4} - \frac{x^4}{16} + c$$

10. $\frac{dy}{dx} + y\cos x = \frac{1}{2}\sin 2x$

$P = \cos x \quad\quad Q = \frac{1}{2}\sin 2x$.

$$e^{\int Pdx}. = e^{\int \cos x\, dx} = e^{\sin x}$$

$$y \times e^{\sin x} = \int \frac{1}{2} \sin 2x \times e^{\sin x} dx + c.$$

$$\cancel{y \times e^{\sin x} = \frac{1}{2}\left[\sin 2x \int e^{\sin x} - \int\left[\frac{d \sin 2x}{dx} \int e^{\sin x}\right] dx + c.\right.}$$

$$\cancel{y \times e^{\sin x} = \frac{1}{2}\left[\sin 2x \; e^{\sin x} - \int\left[\frac{\cos 2x}{2} \times e^{\sin x} dx + c\right.\right.}$$

$u = \sin x$

$du = \cos x \, dx$

$$y \times e^{\sin x} = \frac{1}{2} \int 2 \sin x \cos x \; e^{\sin x} dx + c$$

$$y \times e^{\sin x} = \frac{2}{2} \int e^{u} u \, du + c.$$

$$y \times e^{\sin x} = u \int e^{u} - \int\left[\frac{du}{du} \int e^{u}\right] dx + c.$$

$$y \times e^{\sin x} = u e^{u} - e^{u} + c.$$

$$y \times e^{\sin x} = \sin x e^{\sin x} - e^{\sin x} + c.$$

$$y = \frac{(\sin x - 1) e^{\sin x}}{e^{\sin x}} + \frac{c}{e^{\sin x}}.$$

$$y = (\sin x - 1) + c e^{-\sin x}$$

12. $$(1 - x^2)\frac{dy}{dx} + 2xy = x(1 - x^2)^{1/2}$$

$$\frac{dy}{dx} + \frac{2x}{1 - x^2} y = \frac{x}{\sqrt{1 - x^2}}$$

$P = \dfrac{2x}{1-x^2}$ $\qquad Q = \dfrac{x}{\sqrt{1-x^2}}$

$$I.F = e^{\int P\,dx}$$
$$= e^{\int \frac{2x}{1-x^2}\,dx}$$
$$= e^{\log(1-x^2)}$$
$$= 1-x^2$$

$$y \times (1-x^2) = \cancel{\int \frac{x}{(1-x^2)} \times (1-x^2)\,dx + c} \int \frac{x}{\sqrt{1-x^2}} \times (1-x^2)\,dx$$

$$y \times (1-x^2) = \cancel{\frac{x^2}{2} + c} \int x(1-x^2)^{1/2}\,dx$$

$$\cancel{y \times (1-x^2) =}$$

let $t = 1-x^2$

$dt = -2x$

$y \times (1-x^2) =$

13. ~~$\sec x \frac{dy}{dx} - \frac{1}{\sec x} y$~~

13. $\sec x \frac{dy}{dx} = y + \sin x.$

$$\frac{dy}{dx} - \frac{1}{\sec x} y = \frac{\sin x}{\sec x}.$$

$$P = -\frac{1}{\sec x} \qquad Q = \frac{\sin x}{\sec m}$$

I.F. $e^{\int P dx}$

$e^{\int \cos x \, dx}$

$e^{-\sin x}$

$$y \times e^{-\sin x} = \int \sin x \, e^{-\sin x} \cos x \, dx$$

put $\sin x = u$ ~~put $e^{\sin x} = u$~~

$\frac{du}{dx} = \cos x$ ~~$du = e^{\sin x} \times \cos x$~~

$$y \times e^{-\sin x} = \int u e^{-u} du.$$

$$y \times e^{-\sin x} = -ue^{-u} - e^{-u} + c.$$

$$y \times e^{-\sin x} = \sin x e^{-\sin x} - e^{-\sin x} + c.$$

$$y \cancel{\times e^{-\sin x}} = -(\sin x + 1) + ce^{\sin x}$$

$$y = -\sin x - 1 + ce^{\sin x}$$

14. $\frac{dy}{dx} + y\tan x = \cos x.$

$P = \tan x \qquad Q = \cos x.$

$I.F = e^{\int P dx}$

$= e^{\int \tan x dx}$

$= e^{\log(\sec x)}$

$= \sec x.$

$y \times \sec x = \int \cos x \times \sec x \, dx + c.$

$y \times \sec x = x + c.$

$y = x\cos x + c.$

Bernoulli's Equation

The equation of the form $\frac{dy}{dx} + Py = Qy^n$

where P and Q are Constant or functions of x can be reduced to the linear by dividing y^n.

Q. Solve.

$$x^2 \frac{dy}{dx} + y(x+y) = 0.$$

$$x^2 \frac{dy}{dx} + xy = -y^2$$

$$\frac{dy}{dx} + \frac{y}{x} = -\frac{y^2}{x^2}$$

$$\frac{1}{y^2}\frac{dy}{dx} + \frac{1}{xy} = -\frac{1}{x^2}$$

put $\frac{1}{y} = z$

$$\frac{1}{y^2}\frac{dy}{dx} = \frac{-dz}{dx}$$

$$\frac{-dz}{dx} + \frac{z}{x} = -\frac{1}{x^2}$$

$$\frac{dz}{dx} - \frac{z}{x} = \frac{1}{x^2}$$

$P = -\frac{1}{x}$ $\qquad$ $Q = \frac{1}{x^2}$

$$I.F = e^{\int P\,dx}$$
$$= e^{-\int \frac{1}{x}\,dx}$$
$$= e^{\log(\frac{1}{x})}$$
$$= \frac{1}{x}$$

~~$y \times \frac{1}{}$~~

$$z \times \frac{1}{x} = \int \frac{1}{x^2} \times \frac{1}{x}\,dx + c.$$

$$\frac{z}{x} = \frac{x^{-2}}{-2} + c$$

$$\frac{z}{x} = \frac{-1}{2x^2} + c.$$

Q. $x\frac{dy}{dx} + y\log y = xye^{x}$

$$\frac{dy}{dx} + \frac{y}{x}\log y = ye^{x}$$

$$\frac{1}{y}\frac{dy}{dx} + \frac{1}{x}\log y = e^{x}$$

$$z = \log y$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{dz}{dx}$$

$$\frac{dz}{dx} + \frac{1}{x}z = e^{x}$$

$P = \frac{1}{x}$ $Q = e^x$

$I.F. = e^{\int \frac{1}{x} dx}$

$= x.$

$z \times x = \int e^x \times x \, dx + c$

$xz = xe^x - \int e^x dx + c.$

$xz = xe^x - e^x + c$

$xz = e^x(x-1) + c.$

$x \log y = e^x(x-1) + c.$

Q $\frac{dy}{dx} - \frac{\tan y}{1+x} = (1+x) e^x \sec y$

$\frac{1}{\sec y} \frac{dy}{dx} - \frac{1}{(1+x)} \frac{\tan y}{\sec y} = (1+x) e^x.$

$z = \frac{\tan y}{\sec y}$

$z = \sin y$

$\frac{dz}{dx} = \cos y \frac{dy}{dx}$

$\frac{dz}{dx} = \frac{1}{\sec y} \frac{dy}{dx}.$

$\frac{dz}{dx} - \frac{z}{(1+x)} = (1+x) e^x$

$P = \frac{-1}{1+x}$ $Q = (1+x)e^x$

$I.F = e^{\int P dx}$

$= e^{\log\left(\frac{1}{1+x}\right)}$

$= \frac{1}{1+x}.$

$y \times \frac{1}{1+x} = \int (\cancel{1+x}) e^x \times \frac{1}{(\cancel{1+x})} dx + c.$

$y \times \frac{1}{1+x} = e^x + c$

$y = e^x (1+x) + c$

Q. Solve $\tan y \frac{dy}{dx} + \tan x = \cos y \cos^2 x$

Sol^n

$\tan y \frac{dy}{dx} + \tan x = \cos y \cos^2 x$

$\frac{1}{\cos y} \tan y \frac{dy}{dx} + \frac{1}{\cos y} \tan x = \cos^2 x$

$\sec y \tan y \frac{dy}{dx} + \sec y \tan x = \cos^2 x$

put $z = \sec y$

$\frac{dz}{dx} = \sec y \tan y \frac{dy}{dx}$

$$\frac{dz}{dx} + z\tan x = \cos^2 x$$

$P = \tan x$ $\qquad$ $Q = \cos^2 x$

$$I.F = e^{\int P dx}$$
$$= e^{\int \tan x\, dx}$$
$$= e^{\log(\sec x)}$$
$$= \sec x$$

$$z \times \sec x = \int \cos^2 x \cdot \frac{1}{\cos x}\, dx + c.$$

$$z \times \sec x = \int \cos x\, dx + c$$

$$z \times \sec x = \sin x + c.$$
$$\sec y = \sin x \cos x + c\cos x$$
$$\sec y = \cos x(\sin x + c)$$

Example-17.

Q. $$x\left[\frac{dx}{dy} + y\right] = 1 - y.$$

$$\frac{x\,dx}{dy} + yx\frac{dx}{dy} = 1 - y$$

$$x\left(\frac{dy}{dx} + y\right) = (1 - y)$$

$$\frac{dy}{dx} + y = \frac{1}{x} - \frac{y}{x}$$

$$\frac{dy}{dx} + \left(1 + \frac{1}{x}\right)y = \frac{1}{x}$$

$P = \left(1 + \frac{1}{x}\right)$, $Q = \frac{1}{x}$.

$I.F = e^{\int P dx}$

$= e^{\int \left(1 + \frac{1}{x}\right) dx}$.

$= e^{x + \log x}$

$= e^{x} \cdot e^{\log x}$

$= x e^{x}$

$y \times x e^{x} = \int \frac{1}{\cancel{x}} \times \cancel{x} e^{x} \, dx$

$y \times x e^{x} = e^{x} + c$.

$y = \frac{1}{x} + \frac{c}{x} e^{-x}$.

Ex - 3.6

1. $\frac{1}{y^2}\frac{dy}{dx} - \frac{1}{y} = 2xe^{-x}$

let $-\frac{1}{y} = z$

$\frac{dz}{dx} = +\frac{1}{y^2}\frac{dy}{dx}$

$\frac{dz}{dx} + z = 2xe^{-x}$

$P = 1$, $Q = 2xe^{-x}$

$e^{\int P dx}$

$e^{\int 1 dx}$

e^{x}

$z \times e^{x} = \int 2xe^{-x} \times e^{x} dx + c$

$z \times e^{x} = \int 2x \, dx + c$

$z \times e^{x} = x^2 + c$

$-\frac{1}{y} \cancel{2xe^{-x}} e^{x} = x^2 + c$

$-e^{x} = x^2 y + c$

$x^2 y + e^{x} + cy = 0$

2. $3\frac{dy}{dx} + 3\frac{y}{x} = 2x^4y^4$

$\frac{3}{y^4}\frac{dy}{dx} + \frac{3y}{y^4}\frac{1}{x} = 2x^4$

let $\frac{3}{y^3} = z$

$\frac{3}{y^4}\frac{dy}{dx} = \frac{dz}{dx}$

Now,

$\frac{dz}{dx} + \frac{z}{x} = 2x^4$

$P = \frac{1}{x}$ $\quad Q = 2x^4$

$I.F = e^{\int \frac{1}{x} dx}$

$= e^{\log x}$

$= x$

$\therefore$ $z \times x = \int 2x^4 \times x\, dx + c$

$\frac{3x}{y^3} = \frac{\cancel{2}x^6}{\cancel{6}3} + c$ $\qquad \frac{1}{y^3} = \frac{x^5}{3} + \frac{cx^{-1}}{3}$

$3x = \frac{y^3x^6}{3} + cy^3$

$9x = y^3x^6 + 3cy^3$

$x^6y^3 - 9x + 3cy^3 = 0$

Orthogonal Trajectory.

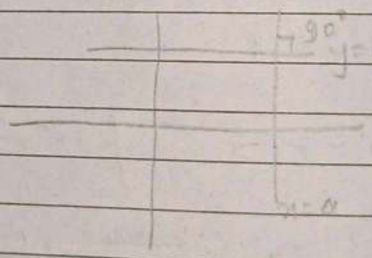
A curve is called Orthogonal trajectory if it cuts every member of a given family of curves at right angle.

→ Orthogonal trajectory in Cartesian Co-ordinates.

Step 1:- let the equation of the family of given curves be $f(x, y, c) = 0$ —— (i)
Differentiate (1) w.r.t x and then eliminate the arbitary Constants between the derived eq^n and given eq^n (1)

Step 2:- Replace $\frac{dy}{dx}$ to $-\frac{dx}{dy}$ in eq^n obtained in step 1.

Step 3:- Integrate the eq^n obtained in step 2.



$m_1 \times m_2 = -1$

$\frac{dy}{dx} \times -\frac{dx}{dy}$

-1

$\frac{dy}{dx} \times m_2 = -1$

$m_2 = -\frac{dx}{dy}$

Q find the Orthogonal trajectories of the family of parabola. $y = ax^2$

$$y = ax^2 \quad \text{——(1)}$$

diff w. r. t x.

$$\frac{dy}{dx} = 2ax.$$

$$a = \frac{1}{2x}\frac{dy}{dx}$$

putting value of a in eq^n (1)

$$y = \frac{1}{2x}\frac{dy}{dx} \times x^2$$

$$y = \frac{x}{2}\frac{dy}{dx}$$

Replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$.

$$y = -\frac{x}{2}\frac{dx}{dy}.$$

$$2y\,dy = -x\,dx \quad \text{—(2)}$$

Integrating eq^n (2).

$$\int 2y\,dy = -\int x\,dx.$$

$$\frac{\cancel{2}y^2}{\cancel{2}} = \frac{-x^2}{2} + c$$

$$2y^2 = -x^2 + c.$$

$$2y^2 + x^2 = c.$$

$m_1 \times m_2 = -1$

$\frac{dy}{dx} \times -\frac{dx}{dy}$

-1

$\frac{dy}{dx} \times m_2 = -1$

$m_2 = -\frac{dx}{dy}$

Q Find the Orthogonal trajectories of the family of parabola. $y = ax^2$

$$y = ax^2 \quad \text{—— (1)}$$

diff w. r. t x.

$$\frac{dy}{dx} = 2ax.$$

$$a = \frac{1}{2x}\frac{dy}{dx}$$

putting value of a in eq^n (1)

$$y = \frac{1}{2x}\frac{dy}{dx} \times x^2$$

$$y = \frac{x}{2}\frac{dy}{dx}$$

Replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$.

$$y = -\frac{x}{2}\frac{dx}{dy}.$$

$$2y\,dy = -x\,dx \quad \text{— (2)}$$

Integrating eq^n (2).

$$\int 2y\,dy = -\int x\,dx.$$

$$\frac{\cancel{2}y^2}{\cancel{2}} = -\frac{x^2}{2} + c$$

$$2y^2 = -x^2 + c.$$

$$2y^2 + x^2 = c.$$

Self Orthogonal:

If $f(x,y,c)=0$ is a family of curves and its orthogonal trajectories are also the same family of curves $f(x,y,c)=0$ then such a family of curves is called self orthogonal.

Imp Q. Show that the system of confocal conics $\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1$ is self orthogonal.

Soln.

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1. \quad \text{--- (1)}$$

diff w.r.t x.

$$\frac{2x}{a^2+\lambda}+\frac{2y\frac{dy}{dx}}{b^2+\lambda}=0$$

$$b^2x+x\lambda+a^2y\frac{dy}{dx}+y\lambda\frac{dy}{dx}=0$$

$$\lambda\left(x+y\frac{dy}{dx}\right)=-b^2x-a^2y\frac{dy}{dx}$$

$$\lambda=\frac{-\left(b^2x+a^2y\frac{dy}{dx}\right)}{\left(x+y\frac{dy}{dx}\right)}$$

Now,

$$a^2+\lambda=a^2-\frac{\left(b^2x+a^2y\frac{dy}{dx}\right)}{x+y\frac{dy}{dx}}$$

$$a^2+\lambda = \frac{(a^2-b^2)x}{x+y\frac{dy}{dx}} \quad -(2)$$

and

$$b^2+\lambda = b^2 - \frac{\left(b^2x + a^2y\frac{dy}{dx}\right)}{x+y\frac{dy}{dx}}$$

$$b^2+\lambda = \frac{-(a^2-b^2)y\frac{dy}{dx}}{x+y\frac{dy}{dx}} \quad -(3)$$

Eliminate λ from eq^n (1)

$$\frac{x^2}{\frac{(a^2-b^2)x}{x+y\frac{dy}{dx}}} + \frac{y^2}{\frac{-(a^2-b^2)y\frac{dy}{dx}}{x+y\frac{dy}{dx}}} = 1.$$

$$\left(x+y\frac{dy}{dx}\right)\left[\frac{x}{(a^2-b^2)} - \frac{y}{(a^2-b^2)\frac{dy}{dx}}\right] = 1.$$

$$\frac{\left(x+y\frac{dy}{dx}\right)}{a^2-b^2}\left[\frac{x}{1} - \frac{y}{\frac{dy}{dx}}\right] = 1.$$

$$\cancel{\left(x+y\frac{dy}{dx}\right)\left(\frac{x\frac{dy}{dx} - y}{\frac{dy}{dx}}\right)} = a^2-b^2$$

$$\left(x+y\frac{dy}{dx}\right)\left[x - y\frac{dx}{dy}\right] = a^2-b^2 \quad -(4)$$

Replacing $\frac{dy}{dx} = -\frac{dx}{dy}$.

$$\left(x - y\frac{dx}{dy}\right)\left(x + y\frac{dy}{dx}\right) = a^2 - b^2$$

$$\left(x + y\frac{dy}{dx}\right)\left(x - y\frac{dx}{dy}\right) = a^2 - b^2 \quad - (5)$$

Since (4) and (5) are same. They are self orthogonal.

Example 2.

Q. find the orthogonal trajectories of the family of co-axal circles $x^2+y^2+2gx+c=0$. where g is the parameter and c is the constant.

Sol^n

The given equation is.

$$x^2+y^2+2gx+c=0. \quad \rightarrow 1$$

diff w. r. to x.

$$2x + 2y\frac{dy}{dx} + 2g = 0$$

$$x + y\frac{dy}{dx} + g = 0$$

$$g = -\left(x + y\frac{dy}{dx}\right)$$

putting value of g in eq^n − 1.

$$x^2+y^2-2x\left(x+y\frac{dy}{dx}\right)+c=0$$

$$x^2+y^2-2x^2+2xy\frac{dy}{dx}+c=0$$

$$-x^2+y^2+2xy\frac{dy}{dx}=-c$$

$$-x^2+2xy\frac{dy}{dx}=-c-y^2$$

put $x^2=t$

$$2x\frac{dx}{dy}=\frac{dt}{dy}$$

$$y\frac{dt}{dy}-t=-c-y^2$$

$$\frac{dt}{dy}-\frac{1}{y}t=-\frac{c}{y}-y$$

which is linear diff eq^n in t.

$$P=\frac{-1}{y} \qquad Q=-\frac{c}{y}-y.$$

$$\cancel{t\times\frac{1}{y}=\int\left(\frac{-c}{y}-y\right)\times\frac{1}{y}}$$

$$I.f=e^{\int -1/y}$$

$$=\frac{1}{y}.$$

$$t\times\frac{1}{y}=\int\left(-\frac{c}{y}-y\right)\times\frac{1}{y}\,dy+c.$$

$$\frac{x^2}{y}=\frac{+c}{y}-y+c_1$$

$$x^2=c-y^2+c_1y$$

$$x^2+y^2-c_1y-c=0$$

Date
Page

Q. Find the orthogonal trajectories of the following family of curves.

(i) $y = ax^3$

diff w. r. to x.

$$\frac{dy}{dx} = 3ax^2$$

$$a = \frac{1}{3x^2}\frac{dy}{dx}.$$

$$y = \frac{1}{3\cancel{x^2}}\frac{dy}{dx} \times \cancel{x^3}\, x$$

$$y = \frac{x}{3}\frac{dy}{dx}.$$

$$\cancel{\frac{1}{x}\,dx = \frac{1}{3y}\,dy}$$

~~tog~~ Replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$.

$$y = -\frac{x}{3}\frac{dx}{dy}$$

$$y\,dy = -\frac{x}{3}\,dx.$$

$$\int y\,dy = -\frac{1}{3}\int x\,dx.$$

$$\frac{y^2}{\cancel{2}} + c = -\frac{1}{3}\frac{x^2}{\cancel{2}} + c$$

$$3y^2 + x^2 = c$$

$$x^2 + 3y^2 = c.$$

ii) $y = ax^n$

diff w. r. t x.

$$\frac{dy}{dx} = a n x^{n-1}$$

$$a = \frac{1}{nx^{n-1}} \frac{dy}{dx}.$$

$$y = a \frac{1}{nx^{n-1}} \frac{dy}{dx} \times x^n$$

$$y = \frac{1}{n \cancel{x^n} . x^{-1}} \frac{dy}{dx} \times \cancel{x^n}$$

$$y = \frac{x}{n} \frac{dy}{dx}.$$

Replacing $\frac{dy}{dx} = -\frac{dx}{dy}.$

$$y = -\frac{x}{n} \frac{dx}{dy}$$

$$\int y \, dy = -\frac{1}{n} \int x \, dx.$$

$$\frac{y^2}{\cancel{2}} = -\frac{1}{n} \frac{x^2}{\cancel{2}} + c.$$

$$ny^2 + x^2 = c.$$

$$x^2 + ny^2 = c.$$

iii) $ax^2 + y^2 = 1$

diff w.r. to x.

$$2ax + 2y\frac{dy}{dx} = 1$$

$$2ax = 1 - 2y\frac{dy}{dx}$$

$$a = \frac{1}{2x}\left(1 - 2y\frac{dy}{dx}\right)$$

$$\frac{x^{\cancel{2}}}{2\cancel{x}}\left(1 - 2y\frac{dy}{dx}\right)$$

$$\frac{x}{2} - xy\frac{dy}{dx}.$$

$$x\left(\frac{1}{2} - y\frac{dy}{dx}\right) + y^2 = 1.$$

$$\frac{x}{2} - xy\frac{dy}{dx} + y^2 = 1.$$

$$-xy\frac{dy}{dx} + \frac{x}{2} = -y^2$$

$$xy\frac{dy}{dx} - \frac{x}{2} = y^2$$

$$\frac{dy}{dx} - \frac{1}{2y} = \frac{y^2}{x}$$

$$\frac{1}{y}\frac{dy}{dx} - \frac{1}{2y^2} = \frac{1}{x}.$$

let $y^p = \frac{-1}{2y^2}$

$$\frac{du}{dx} = \frac{1}{y}\frac{dy}{dx}$$

$$\frac{du}{dx} + \frac{1}{2}du = \frac{1}{x}.$$

$P = \frac{1}{2}$ $\quad\quad$ $Q = \frac{1}{x}.$

$$e^{\int \frac{1}{2}dx}$$

$$e^{\frac{1}{2}x}$$

$$u \times e^{1/2x} = \int \frac{1}{x} \times e^{1/2x}\, dx + c$$

Orthogonal Trajectory for polar Co-ordinates.

Step 1.

Let the family of polar curves be $\phi(r, \theta, c) = 0$ — (1)

diff (1) w.r.t θ and eliminate the parameter c.

Step 2.

Replace $\frac{dr}{d\theta} \to -r^2 \frac{d\theta}{dr}$

Step 3.

Integrate.

Q. Find the orthogonal trajectories of the cardioid $r = a(1 - \cos\theta)$ where 'a' is parameter.

Soln

$$r = a(1 - \cos\theta)$$

$$\frac{dr}{d\theta} = a\sin\theta$$

$$a = \frac{1}{\sin\theta}\frac{dr}{d\theta}$$

$$r = \frac{1}{\sin\theta}\frac{dr}{d\theta}(1 - \cos\theta)$$

$$\cancel{\frac{\sin\theta \; d\theta}{1-\cos\theta}} =$$

$$r = \frac{1}{\sin\theta} - r^2 \frac{d\theta}{dr}(1 - \cos\theta)$$

$$\frac{r}{r^2}dr = -\frac{(1 - \cos\theta)}{\sin\theta}d\theta$$

$\int \tan\theta = \log(\sec\theta)$

but if $\int -\tan\theta = \log(\cos\theta)$

$$\frac{dr}{r} = \frac{2\sin^2\theta/2}{2\sin\theta/2\cos\theta/2}\, d\theta$$

$$\frac{dr}{r} = -\frac{\sin\theta/2}{\cos\theta/2}\, d\theta.$$

$$\int \frac{dr}{r} = -\int \tan\theta/2 \; d\theta.$$

$$\log r = \frac{\log\cos\theta/2}{1/2} + c.$$

$$\log r = 2\log\cos\theta/2 + \log c.$$

$$\log r = \log(\cos^2\theta/2 + c)$$

$$r = \cos^2\frac{\theta}{2} + c.$$

Equations not of first degree equations solveable for P.

Here,

$$\frac{dy}{dx} = P.$$

Q Solve the differential eq^n.

$$\left(\frac{dy}{dx}\right)^2 - 5\frac{dy}{dx} + 6 = 0$$

$\cancel{p = 3, 2}$

$\cancel{\frac{dy}{dx} = 3}$

$$p^2 - 5p + 6 = 0$$

$$p^2 - 3p - 2p + 6 = 0$$

$$p(p-3) - 2(p-3) = 0$$

$$(p-3)(p-2) = 0$$

$p = 3, 2$

for $p = 3$.

$$\frac{dy}{dx} = 3$$

$$dy = 3\,dx$$

$$\int dy = 3\int dx$$

$$y = 3x + c$$

for $p = 2$.

$$\frac{dy}{dx} = 2$$

$$dy = 2dx$$

$$\int dy = 2\int dx$$

$$y = 2x + c$$

$$(y - 3x + c)(y - 2x + c) = 0$$

Q.

$$p^3 + 2xp^2 - y^2p^2 - 2xy^2p = 0$$

$$p(p^2 + 2xp - y^2p - 2xy^2) = 0$$

$$p(p(p+2x) - y^2(p + 2x)) = 0$$

$$p(p+2x)(p-y^2) = 0$$

$$p = 0, -2x, y^2$$

$p = 0$ | $p = -2x$ | $p = y^2$

$\frac{dy}{dx} = 0$ | $\int dy = -2\int x\,dx$ | $\frac{dy}{dx} = y^2$

$\int dy = \int 0$ | $y = \frac{-2x^2}{2}$ | $\int \frac{1}{y^2}\,dy = \int dx.$

$y = c.$ | $y = -x^2 + c$ | $\frac{-1}{y} = x + c.$

$$(y-c)(y+x^2-c)(x+1/y+c) = 0$$

Equation Solvable for y.

- Express the given equation in $y = f(x, p)$
- Diff w.r.t x and replace $\frac{dy}{dx}$ to p so that eq^n has two variables p and x.
- ~~Soln~~ Solve the equation obtained in step 2.
- let the solution of eq^n be $\phi(x, p, c) = 0$
- Eliminate p from eq^n of step I and step IV

Q. Solve the D.E $y = 2px + p^4x^2$

Sol^n

$$\frac{dy}{dx} = 2p + 2x\frac{dp}{dx} + 2xp^4 + 4p^3x^2\frac{dp}{dx}$$

$$p = 2p + 2x\frac{dp}{dx} + 2xp^4 + 4p^3x^2\frac{dp}{dx}$$

$$p + 2x\frac{dp}{dx} + 2xp^4 + 4p^3x^2\frac{dp}{dx} = 0$$

$$\left(p + 2x\frac{dp}{dx}\right)(1 + 2p^3x) = 0$$

Is term mei $\frac{dp}{dx}$ nhi h neglect that

$$p + 2x\frac{dp}{dx} = 0.$$

$$2x\frac{dp}{dx} = -p$$

$$\int \frac{1}{p}\,dp = \int \frac{1}{2x}\,dx$$

$$\log p = -\frac{1}{2}\log x$$

$$2\log p = -\log x + \log c.$$

$$2\log p = \log\left(\frac{c}{x}\right)$$

$$p^2 = \frac{c}{x}$$

$$p = \sqrt{\frac{c}{x}}$$

$$y = 2\sqrt{\frac{c}{x}}\,x + \left(\sqrt{\frac{c}{x}}\right)^4 x^2$$

$$= 2\sqrt{\frac{c}{x}}\,x + \frac{c^2}{\cancel{x^2}}\cancel{x^2}$$

$$= 2\sqrt{\frac{c}{x}}\,x + c^2$$

Equation solvable for x.

Working Rule:-

1. Express the given equation in form of $x = f(y, p)$

2. Diff (1) w.r.t y and put $\frac{dx}{dy} = \frac{1}{p}$ so that eqⁿ has two variables y and p.

3. Solve the equation obtained in step 2.

4. let the solution of eqⁿ be $\phi(y, p, c) = 0$.

5. Eliminate p from step 4 and step 1. we get the solution of (1). If elimination of p is not possible, then values of x and y expressed in terms of parameter 'p' together form the solution of equations.

Q. Solve the D.E $x = y + p^2$.

$$x = y + p^2$$

$$\frac{dx}{dy} = 1 + 2p\frac{dp}{dy}$$

$$\frac{1}{p} = 1 + 2p\frac{dp}{dy}$$

$$\frac{1}{p} - 1 = 2p\frac{dp}{dy}$$

$$\frac{1-p}{2p^2} = \frac{dp}{dy}$$

$$\int dy = -\int \frac{2p^2}{p-1}\, dp$$

$$y = -2\left(p+1+\frac{1}{p-1}\right) dp.$$

$$y = -2\left[p^2+p+\log(p-1)\right]+c.$$

$$y = c - p^2 - 2p - 2\log(p-1)$$

Now,

$$x = c - \cancel{p^2} - 2p - 2\log(p-1) + \cancel{p^2}$$

$$x = c - 2p - 2\log(p-1)$$

$\left(p+1+\frac{1}{p-1}\right)$ $\quad p-1 \,)\, 2p^2 \,(\, p+1$

$-p^2 \mp p$

$\cancel{p}$

$\cancel{p} \mp 1$

1

Q. Solve the D.E $y = 2px + y^2p^3$

$$2px = y - y^2p^3$$

$$x = \frac{y}{2p} - \frac{y^2p^3}{2p}$$

$$x = \frac{y}{2p} - \frac{y^2p^2}{2} \quad \text{— (1)}$$

diff (1) w.r.to y.

$$\frac{dx}{dy} = \frac{1}{2p} - \frac{y}{2p^2} \cdot \frac{dp}{dy} - yp^2 - py^2 \frac{dp}{dy}.$$

$$\frac{1}{p} = \frac{1}{2p} - \frac{y}{2p^2} \frac{dp}{dy} - yp^2 - py^2 \frac{dp}{dy}$$

$$\frac{1}{p} - \frac{1}{2p} + yp^2 = -\frac{dp}{dy}\left(\frac{y}{2p^2} + py^2\right)$$

$$\frac{1}{2p} + yp^2 = -\frac{y}{p}\left(\frac{1}{2p} + p^2 y\right)\frac{dp}{dy}$$

$$\frac{1}{2p} + yp^2 + \frac{y}{p}\left(\frac{1}{2p} + yp^2\right)\frac{dp}{dy} = 0$$

$$\left(\frac{1}{2p} - yp^2\right)\left(1 + \frac{y}{p}\frac{dp}{dy}\right) = 0.$$

neglect the term because it doesn't contain $\frac{dp}{dy}$

$$1 + \frac{y}{p}\frac{dp}{dy} = 0$$

$$\frac{dy}{y} + \frac{dp}{p} = 0$$

$$\int \frac{dy}{y} + \int \frac{dp}{p} = 0$$

$$\log y + \log p = \log c.$$

$$\log yp = \log c$$

$$yp = c$$
$$p = \frac{c}{y}$$

putting in eqn 1.

$$x = \frac{y}{2 \times c} \times y - \frac{y^2 \times c^3}{2y^2}$$

$$x = \frac{y^2}{2c} - \frac{c^2}{2}.$$

Clairaut's Equation.

Defination - The D.E of the type $y = px + f(p)$ is known as clairaut's eq^n.

Q Solve the D.E.

$$(y - px)^2 = 1 + p^2$$

$$y - px = \sqrt{1 + p^2}.$$

$$y = px + \sqrt{1+p^2}$$ Which is in the form $y = px + f(p)$.

Q. $(y - px)(p+1) = p^2$

$$y - px = \frac{p^2}{p+1}$$

$$y = px + \frac{p^2}{p+1}$$

$$y = px + f(p)$$

Q. $p = \log(px - y)$

$\cancel{p = \frac{\log px}{\log y}}$ $\cancel{p = \log px - \log y}$.

~~log y~~

$$e^p = px - y$$

$$y = px - e^p$$

$$y = px + f(p).$$

Q $\sin px \cos y = \cos px \sin y + p$.

$$\sin px \cos y - \cos px \sin y = p$$
$$\sin(px - y) = p.$$
$$(px - y) = \sin^{-1} p$$
$$y = px - \sin^{-1} p.$$

Application to linear equation electric circuit.

let i be the current flowing in the circuit containing resistance R and inductance L, with voltage E at any time t.

Then by voltage law,

$$Ri + L\frac{di}{dt} = E.$$

$$\frac{di}{dt} + \frac{R}{L} i = \frac{E}{L}$$

Then,
$$I.F = e^{\int \frac{R}{L} dt} = e^{\frac{R}{L} t}.$$

Q. Solve the equation.

$$L\frac{di}{dt} + Ri = E_0 \sin \omega t.$$

Where L, R and E_0 are constants.

$$\frac{di}{dt} + \frac{R}{L} i = \frac{E_0}{L} \sin \omega t$$

$$I.F = e^{\int \frac{R}{L} dt}$$

$$= e^{\frac{R}{L}t}$$

$$I \times e^{\frac{R}{L}t} = \int \frac{E_0}{L} \sin \omega t \times e^{\frac{R}{L}t} \, dt$$

$a = \frac{R}{L}$ $\quad b = \omega$

$$I \times e^{\frac{R}{L}t} = \frac{E_0}{L} \int \sin \omega t \, e^{\frac{R}{L}t} \, dt$$

$$I \times e^{\frac{R}{L}t} = \frac{E_0}{L} \left[\frac{e^{R/L\, t}}{\sqrt{\frac{R^2}{L} + \omega^2}} \sin\left(\omega t - \tan^{-1} \frac{L\omega}{R}\right) + A \right]$$

$$I = \frac{E_0}{L \cancel{e^{R/L t}}} \left[\frac{\cancel{e^{\frac{R}{L}t}}}{\sqrt{R^2/_L + \omega^2}} \times L \sin\left[\omega t - \tan^{-1} \frac{b}{a}\right] + A e^{-\frac{R}{L}t} \right]$$

$$\left[\because \int e^{ax} \sin bx \, dx = \frac{e^{ax}}{\sqrt{a^2+b^2}} \cdot \sin\left(bx - \tan^{-1} \frac{b}{a}\right) \right]$$

$$I = \frac{E_0}{\sqrt{\frac{R^2}{L} + \omega^2}} \sin\left[\omega t - \tan^{-1} \frac{b}{a}\right] + A e^{-R/L\, t}$$