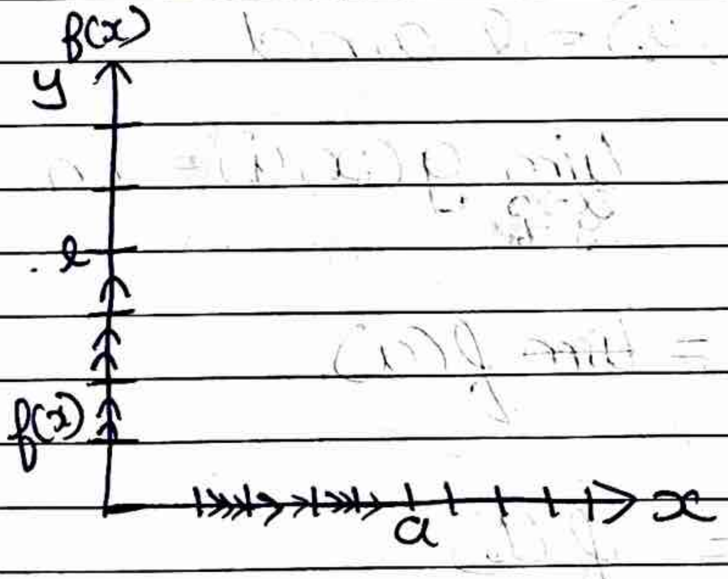
Unit - 3

Calculas - II

Limit and Continuity

Limit → function
Limit → point

$\lim_{x \to a} f(x) = l$ → finite + unique



$x \to a$

Def^n of Limits:

The function $f(x, y)$ is said to tend to the limit 'l' as $x \to a$ and $y \to b$ if and only if the limit 'l'

is independent of the path followed by the point (x, y) as $x \to a$ and $y \to b$ and it is written by

$$\boxed{\lim_{\substack{x \to a \\ y \to b}} f(x, y) = l}$$

or

$$\boxed{\lim_{(x,y) \to (a,b)} f(x, y) = l}$$

Ques) <u>Some properties of Limits</u>:

→ Let $\lim_{\substack{x \to a \\ y \to b}} f(x, y) = l$ and

$\lim_{\substack{x \to a \\ y \to b}} g(x, y) = m$

→ $\text{Limit}_{\substack{x \to a \\ y \to b}} f(x) = \cancel{\lim} f(a)$

→ $\text{Limit}_{\substack{x \to a \\ y \to b}} f(y) = f(b)$

→ $\lim_{\substack{x \to a \\ y \to b}} [f(x, y) \pm g(x, y)] = l \pm m$

→ $\lim_{\substack{x \to a \\ y \to b}} [K f(x, y)] = K l.$

where K is constant

$\rightarrow \lim\limits_{\substack{x \to a \\ y \to b}} [f(x,y) \cdot g(x,y)] = lm.$

$\rightarrow \lim\limits_{\substack{x \to a \\ y \to b}} \left[\dfrac{f(x,y)}{g(x,y)}\right] = \dfrac{l}{m}$

provided $g(x,y) \neq 0$

$\rightarrow \lim\limits_{\substack{x \to a \\ y \to b}} [f(x,y)]^{r/s} = (l)^{r/s}$

Case 1:

$(x,y) \nrightarrow (0,0)$

Put direct limits in given function.

Put Limit of x first, than limit of y 2nd.

Case 2:

$(x,y) \rightarrow (0,0)$

Then solve the function by seperation method ~~or~~ or by along the different curves.

Ques 1.) Find $\lim\limits_{\substack{x \to 1 \\ y \to 2}} x^6 y + 2xy$

Sol^n $\lim_{\substack{x \to 1 \\ y \to 2}} x^6 y + \lim_{\substack{x \to 1 \\ y \to 2}} 2xy$

$\lim_{y \to 2} (1)^6 y + \lim_{y \to 2} 2(1) y.$

~~lim~~ $= (1)(2) + 2(2)$
$= 4 + 2$
$= 6$ //.

Ques2) Find $\lim_{\substack{x \to 0 \\ y \to 0}} \frac{x^3 - y^3}{x - y}$

Sol^n Apply Seperation Method

$\lim_{\substack{x \to 0 \\ y \to 0}} \frac{\cancel{(x-y)}(x^2 + xy + y^2)}{\cancel{(x-y)}}$

$\lim_{\substack{x \to 0 \\ y \to 0}} x^2 + xy + y^2$

$\lim_{y \to 0} 0 + 0 + y^2$ ~~=t~~

$\lim_{y \to 0} y^2$

~~lt~~ $= 0$ //.

Case-3

By Using Polar Coordinate:

we will find limit of $f(x,y)$ by using the value of $x = r\cos\theta$ and $y = r\sin\theta$

Ques3) By Using Polar Coordinate Find $\lim_{\substack{x\to 0\\ y\to 0}} \frac{x^3+y^3}{x^2+y^2}$

Soln Put $x = r\cos\theta$
$y = r\sin\theta$
So, the given limit will be

$$\lim_{\substack{x\to 0\\ y\to 0}} \frac{x^3+y^2}{x^2+y^2} = \lim_{r\to 0} \frac{(r\cos\theta)^3+(r\sin\theta)^3}{(r\cos\theta)^2+(r\sin\theta)^2}$$

$$\lim_{r\to 0} \frac{r^3\cos^3\theta + r^3\sin^3\theta}{r^2\cos^2 + r^2\sin^2\theta}$$

$$\lim_{r\to 0} \frac{r^{\cancel{3}}(\cos^3\theta + \sin^3\theta)}{\cancel{r^2}}$$

$$\lim_{r\to 0} r(\cos^3\theta + \sin^3\theta)$$

$$= 0$$

TEACHER SIGNATURE

Ques4.) Find $\lim\limits_{\substack{x\to 0\\ y\to 0}} \dfrac{xy^2}{x^2+y^2}$ by polar form.

Ques5.) Find $\lim\limits_{\substack{x\to \sqrt{3}\\ y\to -4}} x+y^3$

Ques6.) Find $\lim\limits_{\substack{x\to 9\\ y\to 1}} \dfrac{x^{3/2}}{\sqrt{x^2+y^2+9}}$

Ques7.) Find $\lim\limits_{\substack{x\to 0\\ y\to 0}} \dfrac{x^2-4xy}{\sqrt{x}-2\sqrt{y}}$

Ques8.) Find $\lim\limits_{\substack{x\to \infty\\ y\to 2}} \dfrac{xy+3}{x^2+4}$

Along the Path or Curve:

$$(x, y) \to (0, 0)$$

$\lim_{x \to a} f(x) \to$ exist

RHL $\lim_{x \to a^+} f(x) = L_1$

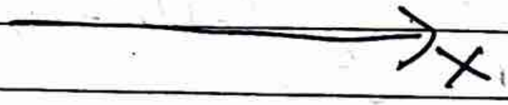
LHL $\lim_{x \to a^-} f(x) = L_2$

$L_1 = L_2$.
Limit Exist

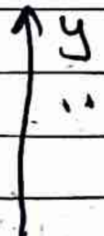
$L_1 \neq L_2$
Limit doesn't exist

$\lim_{(x,y) \to (0,0)} f(x, y) = L_1$ (say)

along the x-axis
i.e. $y = 0$



$\lim_{(x,y) \to (0,0)} f(x, y) = L_2$ (say)



Along the y-axis i.e. $x = 0$

If $L_1 = L_2$ than limit may be exist or may be not.

If $L_1 \neq L_2$ than limit of func^n does not exist.

There are mainly 6 path to check the limit of f(x, y)

(i) Along the x-axis i.e. $y=0$

(ii) Along the y-axis i.e. $x=0$

(iii) Along the $y=x$

(iv) Along the $y=-x$

(v) Along the $y=x^2$

(vi) Along the $x=y^2$

Ques 1.) Show wether or not the limit ~~$x \to 0$ $y \to 0$~~

$$\lim_{\substack{x \to 0 \\ y \to 0}} \frac{x^2-y^2}{x^2+y^2} \text{ exists?}$$

Sol^n. Approches (0, 0) to along the path x-axis : $y=0$.

$$\lim_{\substack{(x,y) \to (0,0) \\ y=0}} \frac{x^2-y^2}{x^2+y^2} = \lim_{\substack{x \to 0 \\ y \to 0}} \frac{x^2-0}{x^2+0}$$

$$= \lim_{(x,y)\to(0,0)} \frac{\cancel{x^2}}{\cancel{x^2}}$$

$$= 1$$

Approaches (0,0) along the path y-axis: $x=0$.

$$\lim_{\substack{x\to 0\\ y\to 0\\ x=0}} \frac{x^2-y^2}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{0-y^2}{0+y^2}$$

$$= \lim_{(x,y)\to(0,0)} \frac{-\cancel{y^2}}{\cancel{y^2}}$$

$$= -1$$

i.e. $L_1 \neq L_2$, so limit does not exist. because
since the limit along the x axis and y-axis are different
∴ limit of $f(x,y)$ doesn't exist at (0,0).

Ques2) Check the limit

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$$

exist or not?

Soln Approches (0,0) along the x-axis: $y=0$

$$\lim_{\substack{(x,y)\to(0,0)\\ y=0}} \frac{xy}{x^2+y^2} = \frac{x(0)}{x^2+0}$$

$$= 0$$

Approches (0,0) along the y-axis $x=0$

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = \frac{(0)y}{0+y^2}$$

$$= 0$$

Approches (0,0) along the $y=x$

$$\lim_{\substack{(x,y)\to(0,0)\\ \cancel{x}y=x}} \frac{\cancel{x}\,\cancel{x^2}}{2\cancel{x^2}} \quad \frac{x^2}{2x^2}$$

$$= \frac{1}{2}$$

The limit along the path are not same
So Limit does not exist.

Continuous

→ function
→ point
→ limit

Defn:

The function $f(x,y)$ is said to be continuous at (a,b) if it is satisfies the following condition

(i) ~~f(x)~~ $f(a,b)$ is defined

(ii) if $\lim_{(x,y)\to(a,b)} f(x,y)$ exists.

(iii) $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$

Q1) Show that $f(x,y) = 4x^2y - 3xy^2$ is continuous at pt $(0,0)$

Soln $f(0,0) = 4(0)^2(0) - 3(0)(0)^2$
$= 0.$

$\lim_{(x,y)\to(0,0)} 4x^2y - 3xy^2$

$\lim_{y\to0} 4(0)^2y - 3(0)y^2 = 0$

So, $\lim_{(x,y)\to(0,0)} 4x^2y\, f(x,y) = f(0,0)$.

$\therefore f(x,y)$ is continoues.

Ques2.) Show that $f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 2 & \text{if } (x,y)=(0,0) \end{cases}$ is continoues.

Soln ~~f(0,0)~~ given point is $(0,0)$

$\therefore f(0,0) = 2$

Limit along the path x-axis: $y=0$.

$\displaystyle \lim_{\substack{(x,y)\to(0,0)\\ y=0}} \frac{xy}{x^2+y^2} = 0$

Along the y-axis: $x=0$

$\displaystyle \lim_{\substack{(x,y)\to(0,0)\\ x=0}} \frac{xy}{x^2+y^2} = 0$

Along the path $y=x$

$\displaystyle \lim_{\substack{(x,y)\to(0,0)\\ x=0}} \frac{xy}{x^2+y^2} = \frac{1}{2}$

$\lim_{x\to0,\,y\to0} f(x,y)$ does not exist

$f(x)$ is not continoues at a pt $(0,0)$.

Ques 3) Check the continuity of

$$f(x,y) = \begin{cases} \dfrac{\sin(x^2+y^2)}{x^2+y^2}, & (x,y) \neq (0,0) \\ 1, & (x,y) = (0,0) \end{cases}$$

Sol^n Put $x = R\cos\theta$
$y = R\sin\theta$

$\rightarrow f(0,0) = \underline{1}$

$\rightarrow \lim\limits_{(x,y)\to(0,0)} \dfrac{\sin(x^2+y^2)}{x^2+y^2}$

$= \lim\limits_{r\to 0} \dfrac{\sin(r^2)}{r^2}$

$= \cancel{\lim\limits_{r\to 0}}$ Apply L'Hôpital Rule

$= \lim\limits_{r\to 0} \dfrac{\cancel{2r}\cos(r^2)}{\cancel{2r}}$

$= 1$ //.

$\rightarrow f(0,0) = \lim\limits_{(x,y)\to(0,0)} f(x,y)$

So, $f(x,y)$ is continuous at a point $(0,0)$.

Ques4) Check the continuity of

$$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Soln $\rightarrow f(0,0) = 0$

$\rightarrow \lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{\sqrt{x^2+y^2}}$

$x = r\cos\theta \qquad y = r\sin\theta.$

$\Rightarrow \lim\limits_{r\to 0} \dfrac{r^{\cancel{2}}\cos\theta\sin\theta}{\cancel{r}}$

$= 0$//.

$\rightarrow f(0,0) = \lim\limits_{(x,y)\to(0,0)} f(x,y) = 0$

therefore,

the function is continous on the point $(0,0)$

Hence proved.

Partial Derivatives:

$$y = f(x)$$

Dependent Variable ← y ; x → Independent Variable

$$\frac{d}{dx}$$

$$z = f(x, y)$$

D.V. ← z ; (x, y) → I.V.

$\frac{\partial}{\partial x} f(x, y)$ = diff. of x variable & variable y will be considered as constant.

$\frac{\partial}{\partial y} f(x, y)$ = diff of y variable & variable x will be considered as constant.

Partial Derivatives will be used when function has a two or more than two ~~in d~~ independent variable and It is denoted by the symbol '∂'

eg:- $z = f(x, y) \Rightarrow \frac{\partial z}{\partial x}$ or $\frac{\partial z}{\partial y}$

Ques1.) Find $\frac{\partial z}{\partial x}$ & $\frac{\partial z}{\partial y}$ when the function is given by $z = x^3y + y^2x + y\sin x$

Sol^n ~~∂z~~ Partial Derivation of given ~~fC~~ function w.r.t x, we get

$$\boxed{\frac{\partial z}{\partial x} = 3x^2y + y^2 + y\cos x}$$

Partial Derivative of given ~~ge~~ function w.r.t y, we get

$$\boxed{\frac{\partial z}{\partial y} = x^3 + 2yx + \sin x}$$

Partial Derivatives Notation:

$z = f(x, y)$ $\qquad\qquad$ $f(x,y)$

$\frac{\partial z}{\partial x}$ $\quad$ $\frac{\partial z}{\partial y}$ $\qquad\qquad$ $f_x = \frac{\partial f}{\partial x}$ $\quad$ $f_y = \frac{\partial f}{\partial y}$

$u = f(x,y)$

$\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$ $\qquad$ u_x, u_y

Q) If $u = (x-y)(y-z)(z-x)$
$u = f(x, y, z)$

Prove that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$.

Sol^n $\frac{\partial u}{\partial x} = (y-z)\left[(x-y)\frac{\partial}{\partial x}(z-x) + (z-x)\frac{\partial}{\partial x}(x-y)\right]$

$$= (y-z)\left[(x-y)(-1) + (z-x)(1)\right].$$
$$= (y-z)\left[(y-x) + z - x\right]$$
$$= (y-z)(y-x+z-x).$$
$$= (y-z)(-2x+y+z).$$

$$\frac{\partial u}{\partial y} = (z-x)\left[(x-y)(1) + (y-z)(-1)\right]$$
$$= (z-x)\left[x-y-y+z\right].$$
$$= (z-x)(-2y+z+x)$$

$$\frac{\partial u}{\partial z} = (x-y)\left[(z-x)(-1) + (y-z)(1)\right]$$
$$= (x-y)\left[-z+x+y-z\right]$$
$$= (x-y)(-2z+x+y)$$

L.H.S

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}$$

$$= (y-z)(-2x+y+z) + (z-x)(x-2y+z) + (x-y)$$

$-2\cancel{xy} + \cancel{y^2} + \cancel{yz} + \cancel{2xz} - \cancel{zy} - \cancel{z^2}$
$+\cancel{zx} - \cancel{2yz} + \cancel{z^2} - \cancel{x^2} + \cancel{2xy} - \cancel{zx}$
$+\cancel{xy} - 2\cancel{xz} + \cancel{x^2} - \cancel{y^2} + \cancel{2zy}$
$-\cancel{xy}$

$= 0 = R.H.S$

Hence proved.

Partial Derivative of 2nd order:

Let $z = f(x,y)$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)$$

Curly two z over curly x square.

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)$$

$f(x,y) = f_{xx}, f_{yy}, f_{xy}, f_{yx}$

$u(x,y) = u_{xx}, u_{yy}, u_{xy}, u_{yx}$

Q3 If u is given as a $u = f(x,y)$ $u = \log(x^2+y^2)$.

Prove that $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$

Solⁿ

LHS:

$$\frac{\partial u}{\partial y} = \frac{1 \cdot 2y}{x^2+y^2}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial^2 y}{\partial x \partial y} = 2x(-1)(x^2+y^2)^{-2}(2x)$$

$$= \frac{-4xy}{(x^2+y^2)^2}$$

R.HS:

$$\frac{\partial u}{\partial x} = \frac{1 \cdot 2x}{x^2+y^2}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 y}{\partial y \partial x}$$

$$\frac{\partial^2 y}{\partial y \partial x} = \frac{-4xy}{(x^2+y^2)^2}$$

$$\frac{\partial^2 y}{\partial x \partial y} = \frac{\partial^2 y}{\partial y \partial x}$$

$$LHS = RHS$$

Hence proved.

SR

Ques4) If $u = \log\sqrt{x^2+y^2}$
than prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

Ques5) If $u = x^y$ and show that
$U_{xy} = U_{yx}$

Soln $\cancel{\frac{\partial u}{\partial y} =}$ ~~LHS~~

LHS.
$u = x^y$
Partial Derivative w.r.t. y
$$\frac{\partial u}{\partial y} = x^y \log x$$

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = U_{xy} = \frac{x^y}{x} + \log x \, y x^{(y-1)}$$

$$\frac{\partial u}{\partial x} = y x^{(y-1)}$$

$$\frac{\partial}{dy}\left(\frac{\partial u}{dx}\right) = x y x^{(y-1)} + \log \ldots y x^{y-1}$$

$$x^{(y-1)} + \log x \cdot y x^{(y-1)} =$$

$$U_{xy} = U_{yx}$$

Hence proved.

Composite Function of Partial Derivati

$$Z = f(x, y)$$

$$x = \phi(r, s)$$
$$y = \psi(r, s)$$

$\frac{\partial z}{\partial s}$

$$\begin{tikzcd}[column sep=small] & Z \arrow[dl] \arrow[dr] & \\ x \arrow[dr, no head] & & y \arrow[dl, no head] \\ & r, s. & \end{tikzcd}$$

$\frac{\partial z}{\partial r}$

$$\Rightarrow \frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

$$\Rightarrow \frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r}$$

Q1) If $u = x^2 - y^2$ and $x = 2r - 3s + 4$, $y = -r + 8s - 5$
Find $\frac{\partial u}{\partial s}$ and $\frac{\partial u}{\partial r}$?

Solⁿ $\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y} + \frac{\partial y}{\partial s}$ — (1)

$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} + \frac{\partial y}{\partial r}$.

$\frac{\partial u}{\partial x} = 2x - 0 \qquad \frac{\partial u}{\partial y} = -2y$

$\frac{\partial x}{\partial r} = \cancel{6-3s}\, 2 \qquad \frac{\partial x}{\partial s} = -3$

$\frac{\partial y}{\partial s} = 8 \qquad \frac{\partial y}{\partial r} \; -1$

Put these values in eqn ① & ②.

$\frac{\partial u}{\partial s} = 2x(-3) + (-2y)(8)$.

$$\boxed{\frac{\partial u}{\partial s} = -6x - 16y.}$$

$\frac{\partial u}{\partial r} = 2x(2) + (-2y)(-1)$

$$\boxed{\frac{\partial u}{\partial r} = 4x + 2y}$$

Ques2) Find $\frac{\partial u}{\partial r}$ & $\frac{\partial u}{\partial s}$ of $u = x + 2y + z$

① Find $\frac{\partial u}{\partial s}, \frac{\partial u}{\partial r}$? $\quad x = r/s$

$y = r^2 + \log z$

$z = 2r.$

Solⁿ $\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial r} + \frac{\partial u}{\partial z} \cdot \frac{\partial z}{\partial x}$.

$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x} * \frac{\partial x}{\partial s} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial s} + \frac{\partial u}{\partial z} \cdot \frac{\partial z}{\partial s}$

Now we will find all the values and put in eqⁿ ① & ②

and we will get the values of $\frac{\partial u}{\partial r}$ and $\frac{\partial u}{\partial s}$.

Homogeneous Function

Def^n → If the function $f(x, y)$ can be represent in the form of $x^n \phi\left(\frac{y}{x}\right)$ then the function is known as homogeneous function of degree n.

eg:- $f(x, y) = x^2 + y^2$

$$= x^2\left(1 + \frac{y^2}{x^2}\right)$$

$$= x^2 \phi\left(\frac{y}{x}\right)$$

∴ $f(x, y)$ is a homogenous function of degree 2.

Def^n 2: The function $f(x, y)$ is said to be homogeneous function of degree n if $f(tx, ty)$

$$f(tx, ty) = t^n f(x, y).$$

Where t is non-zero, real no.

eg:- $f(x,y) = x^2+y^2$

$$f(tx, ty) = t^2x^2 + t^2y^2$$
$$= t^2(x^2+y^2)$$

$$f(xt, yt) = t^2 f(x,y)$$

$f(x,y)$ is a homogeneous function of degree 2.

Q1 Check the function $f(x,y) = 2x^3 - x^2y + 5y^3$ is homogeneous or not

Solⁿ $f(x,y) = 2x^3 - x^2y + 5y^3$

$$f(tx, ty) = 2t^3x^3 - t^3x^2y + 5t^3y^3$$
$$= t^3(2x^3 - x^2y + 5y^3)$$
$$= t^3 f(x,y)$$ yes, it is hom. funcⁿ deg = 3.

Q2, $f(x,y) = \log\left(\frac{x^2+y^2}{x+y}\right)$ check hom. or not.

Q3, If $u = \sin^{-1}\left(\frac{x+2y}{x^7+y^7}\right)$ then check u is homogeneous or not?

Euler's Theoram:

$z = f(x, y)$

If Z is a homogeneous function ~~than~~ function of degree 'n' in x & y then

$$\boxed{x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz}$$

Proof:

Since z is a homogneous function of degree n in x, y.

$$Z = x^n \phi\left(\frac{y}{x}\right) \quad -① $$

Partial Derivative of eqn ① w.r.t x we get

$$\frac{\partial z}{\partial x} = x^n \phi'\left(\frac{y}{x}\right) \cdot y\left(\frac{-1}{x^2}\right) + \phi\left(\frac{y}{x}\right) n x^{n-1}$$

$$= -x^{n-2} \cdot y \phi'\left(\frac{y}{x}\right) + n x^{n-1} \phi\left(\frac{y}{x}\right) \quad -②$$

Partial Derivative of eqn ① w.r.t y.

$$\frac{\partial z}{\partial y} = x^n \phi'\left(\frac{y}{x}\right)\left(\frac{1}{x}\right)$$

$$= x^{n-1} \phi'\left(\frac{y}{x}\right) \quad - ③$$

Put ② & ③ in L.H.S.

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}$$

$$= x\left(-x^{n-2}\, y\, \phi'\left(\frac{y}{x}\right) + n x^{n-1} \phi\left(\frac{y}{x}\right)\right) + y\left(x^{n-1} \phi'\left(\frac{y}{x}\right)\right)$$

$$= -x^{n-1}\cancel{y\phi'\left(\frac{y}{x}\right)} + n x^n \phi\left(\frac{y}{x}\right) + x^{n-1}\cancel{y\phi'\left(\frac{y}{x}\right)}$$

$$= n x^n \phi\left(\frac{y}{x}\right)$$ from Eqn ①.

$$= nz = R.H.S$$

Hence, proved. $\boxed{x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz}$

Ques 1) If $u = \frac{x+y}{x^2+y^2}$ then find the value of $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$

Soln We need to prove u is homogeneous function

Put $x = xt$ $y = yt$ in func^n u.

$$u(xt, yt) = \frac{xt+yt}{x^2t^2+y^2t^2}$$

$$= \frac{t(x+y)}{t^2(x^2+y^2)}$$

$$= t^{-1}\frac{(x+y)}{(x^2+y^2)}$$

$$= t^{-1}u(x,y)$$

∴ The given function $u(x,y)$ is homogeneous of degree -1.

∴ By Euler's Theorem.

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu.$$

$$\therefore \boxed{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -1\left(\frac{x+y}{x^2+y^2}\right)}$$

Relations Between 2nd order derivative of homogeneous function

1) If u is homogeneous function of degree n in x & y.

$\rightarrow$ $$x\frac{\partial^2 u}{\partial^2 x} + y\frac{\partial^2 u}{\partial x\partial y} = (n-1)\frac{\partial u}{\partial x}$$

$\rightarrow$ $$x\frac{\partial^2 u}{\partial y\partial x} + y\frac{\partial^2 u}{\partial y^2} = (n-1)\frac{\partial u}{\partial y}$$

$\rightarrow$ Euler of Second order

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u.$$

Q) If $u = \frac{xy}{x+y}$, Show that
$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 y}{\partial y^2} = 0,$$

Soln We need to prove u is a homogeneous function.

Put $x = xt$. $y = yt$

TEACHER SIGNATURE

$$u(xt, yt) = \frac{xt\,yt}{xt+yt}$$

$$= \frac{t^2 xy}{t(x+y)}$$

$$= t\left(\frac{xy}{x+y}\right)$$

$$= t^1 u(x,y)$$

$\therefore$ The given function $u(x,y)$ is homogeneous of degree 1.

$\therefore$ By Euler 2nd order theorem

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)z$$

$\therefore$ ~~$x^2\frac{\partial^2 u}{\partial}$~~ $n = 1$

~~$= 1(1$~~

$$= 1(1-1)\frac{xy}{(x+y)}$$

$$= 0$$

Hence

$$\boxed{x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 0}$$

Hence proved.

Euler Modification Method:

$$u = \sin^{-1} \phi(x, y)$$
$$u = \cos^{-1} \phi(x, y)$$
$$u = \tan^{-1} \phi(x, y)$$
$$u = \operatorname{cosec}^{-1} \phi(x, y)$$
$$u = \sec^{-1} \phi(x, y)$$
$$u = \cot^{-1} \phi(x, y).$$
$$u = \log . \phi(x, y)$$

$\Rightarrow$ u is not homogeneous.

$$z = \sin u = \phi(x, y)$$
$$z = \cos u = \phi(x, y)$$
$$z = \tan u = \phi(x, y)$$
$$z = \operatorname{cosec} u = \phi(x, y)$$
$$z = \sec = \phi(x, y)$$
$$z = \cot = \phi(x, y)$$
$$z = e^u = \phi(x, y)$$

z is homogeneous
Sin u is homogenous.
Cos u ——

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu.$$

$$x\frac{\partial(\sin u)}{\partial x} + y\frac{\partial(\sin u)}{\partial y} = nu.$$

Ques3) If ~~Sin~~ $u = \sin^{-1}\left(\frac{x^2+y^2}{x+y}\right)$ than Prove $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \tan u$.

Soln Put $x = xt$ $\quad y = yt$ in u,

$$u = \sin^{-1}\left(\frac{x^2t^2+y^2t^2}{xt+yt}\right)$$

$$= \sin^{-1}\left(t\left(\frac{x^2+y^2}{x+y}\right)\right)$$

u is not homogeneous function.

$$\sin u = \frac{x^2+y^2}{x+y} = z$$

Put $x = xt$, $\quad y = yt$ in $\sin u$ or z.

$$z(xt, yt) = \frac{x^2t^2+y^2t^2}{xt+yt}$$

$$= \frac{t^{\cancel{2}}}{\cancel{t}}\left(\frac{x^2+y^2}{x+y}\right)$$

$$= t\left(\frac{x^2+y^2}{x+y}\right)$$

$$= t\, z(x, y)$$

∴ Z or Sin u is homogeneous function of degree 1

By Euler Theoram

$$x\frac{\partial(\sin u)}{\partial y}+y\frac{\partial(\sin u)}{\partial y}=n\sin u.$$

$$\boxed{x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\tan u}$$

<u>Implicit</u> <u>Function:</u>

If the function $f(x,y)=0$ or constant

$$f(x,y)=x^2+y^2+2ab=0.$$

$$u=f(x,y)$$
$$z=f(x,y)$$

then $f(x)$ is known as the implicit Function.

To find $\frac{dy}{dx}$ & $\frac{d^2y}{dx^2}$

$$\boxed{\frac{dy}{dx}=-\frac{\partial f/\partial x}{\partial f/\partial y}}$$ or $-\frac{f_x(x)}{f_y}$ provided $f_y\neq0$.

$$\boxed{\frac{d^2y}{dx^2} = -\frac{f_{xx} f_y^2 - f_{xy} f_y f_x + f_{yy} f_x^2}{f_y^3}}$$

Ques 4) If $x^3 + y^3 - 3axy = 0$ is a function than find $\frac{dy}{dx}$ & $\frac{d^2y}{dx^2}$.

Sol^n

$$\frac{\partial f}{\partial x} = 3x^2 - 3ay$$

$$\frac{\partial f}{\partial y} = 3y^2 - 3ax$$

$$\frac{dy}{dx} = \frac{-\partial f/\partial x}{\partial f/\partial y} = -\frac{\cancel{3}(x^2 - ay)}{\cancel{3}(y^2 - ax)}$$

$$\boxed{\frac{dy}{dx} = \frac{-(x^2 - ay)}{y^2 - ax}}$$

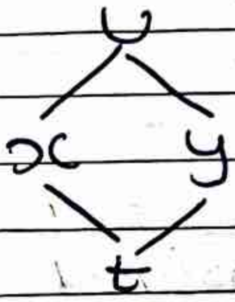
$$\frac{d^2y}{dx^2} = \frac{-6x(3y^2 - 3ax)^2 - 2(-3a)(3x^2 - 3ay)(3y^2 - 3ax) + 6y(3x^2 - 3ay)^2}{(3y - 3ax)^3}$$

Ques 5) If $y^2 - 3ax^2 + x^3 = 0$. P.T. $\frac{d^2y}{dx^2} + \frac{2a^2x^2}{y^5} = 0$

TEACHER SIGNATURE .....

Total Derivative:

→ If $u = \varphi(x, y)$



where $x = \phi_1(t)$

$y = \phi_2 = (t)$

$$\cancel{\frac{du}{dt}}$$

$$\boxed{\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}}$$

$\therefore \frac{du}{dt}$ = total derivative of u.

→ Implicit Function:

If $u = \varphi(x, y)$ and also given $f(x, y) = 0$ or constant

$$\frac{du}{dx} = \frac{\partial u}{\partial x} \frac{dx}{dx} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx}$$

$$\boxed{\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx}}$$

where $\frac{dy}{dx} = \frac{-\partial f/\partial x}{\partial f/\partial y}$

Ques1.) Find the total derivative of $u = e^x \sin y$ where $x = \log t$ and $y = t^2$ and also verify the result?

Sol^n $\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}$ — (1)

$\frac{\partial u}{\partial x} = e^x \sin y$

$\frac{\partial u}{\partial y} = e^x \cos y$

$\frac{dx}{dt} = \frac{1}{t}$

$\frac{dy}{dt} = 2t$

Put all these values in (1)

$$\frac{du}{dt} = \frac{e^x \sin y}{t} + \frac{e^x \cos y \cdot 2t}{\cancel{2t}} \quad -(2)$$

Put the values of x and y in (2).

$$\frac{du}{dt} = e^{\log t} \sin(t^2)\left(\frac{1}{t}\right) + e^{\log t} \cos(t)^2 \cdot 2t$$

$$\boxed{\frac{du}{dt} = \sin t^2 + 2t^2 \cos t^2} \quad -(2)$$

To verify:

$u = e^x \sin y$

& $x = \log t \quad y = t^2$

$u = e^{\log t} \sin t^2 \Rightarrow t \sin t^2$

$$\boxed{\frac{du}{dt} = \sin t^2 + 2t^2 \cos t^2} \quad (3)$$

Hence Eqn (2) & (3) are equal so our derivation is correct and verified.

Hence proved.

Ques2) If $u = x\log(xy)$ where $x^3+y^3+3xy=1$ then find du/dx

$$x^3+y^3+3xy-1=0.$$

Soln $\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{-\partial f/\partial x}{\partial f/\partial y}$$

$$\frac{\partial f}{\partial x} = 3x^2+3y \qquad \frac{\partial f}{\partial y} = 3y^2+3x$$

$$\frac{dy}{dx} = \frac{-(x^2+y)}{y^2+x} = \frac{-x^2-y}{y^2+x}$$

$$\frac{\partial u}{\partial x} = \log(xy) + x\cdot\frac{1}{\cancel{x}y}\cdot y$$
$$= \log(xy)+1$$

$$\frac{\partial u}{\partial y} = x\cdot\frac{1}{\cancel{x}y}\,x = \frac{x}{y}$$

$$\frac{du}{dx} = \log(xy)+1+\frac{x}{y}\left(\frac{-(x^2+y)}{y^2+x}\right)$$

$$\boxed{\frac{du}{dx} = \cancel{\log}\ 1+\log(xy) - \frac{x^3+xy}{y^3+xy}}$$

Maxima, Minima:

Working Rule:

If $f(x,y)$ is given

Step 1 - Find f_x, f_y, f_{xx}, f_{yy} and f_{xy}

Step 2 - Find the stationary / critical point

i.e. $f_x = 0$ & $f_y = 0$

and find the values of x and y.

Step 3 - $A = f_{xx}$ $B = f_{xy}$ $C = f_{yy}$

Case 1 (i) If $AC - B^2 > 0$ & $A > 0$

we get Minima. At

~~Case 2(ii)~~ minima pt $f(x,y)$ gives minimum value

Case 2 (ii) If $AC - B^2 > 0$ & $A < 0$ we get maxima point and at maxima pt the $f(x,y)$ gives the maximum value

Case 3 (iii) If $AC - B^2 < 0$, then we get saddle points i.e. neither maxima nor minima.

Case 4 (iv) If $AC - B^2 = 0$ this case is doubtful so further investigation is required

Ques 1) Examine the function
$f(x,y) = x^3 + y^3 - 3x - 12y + 20$ ~~for~~
For the maxima & minima value

Soln $\frac{\partial f}{\partial x} = 3x^2 - 3 = f_x$

$f_y = \frac{\partial f}{\partial y} = 3y^2 - 12$

$f_{xx} = \frac{\partial^2 f}{\partial x^2} = 6x \qquad f_{yy} = \frac{\partial^2 f}{\partial y^2} = 6y$

$f_{xy} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\right) = 0$

$A = 6x \qquad \cancel{B = 6y}\ C = 6y \qquad B = 0$

$f_x = 0$ & $f_y = 0$

$3x^2 - 3 = 0 \qquad 3y^2 - 12 = 0$

$\boxed{x = \pm 1} \qquad \boxed{y = \pm 2}$

$\therefore (1,2)\ (1,-2),\ (-1,2),\ (-1,-2)$
are the critical points / stationary

At $(1,2)$

$AC - B^2 = 36xy - 0 = 36(1)(2)$
$= 72 > 0$

$A = 6x = 6(1) = 6 > 0$

$\therefore (1,2)$ is a minima ~~functi~~ point

SR

TEACHER SIGNATURE

Minima value

$$f(1,2) = (1)^3 + (2)^3 + 3(1) + 12(2) + 20$$
$$= 1 + 8 - 3 - 24 + 20$$
$$= 2$$

At $(1,-2)$

$AC - B^2 = 36xy = 36(1)(-2) = -72 < 0$

$(1,-2)$ is a saddle point i.e. we neither get maxima nor minima.

At $(-1,2)$

$AC - B^2 = 36xy = 36(-1)(2) = -72 < 0$

$(-1,2)$ is a saddle point i.e. we neither get maxima nor minima.

At $(-1,-2)$

$AC - B^2 = 36xy = 36(-1)(-2) = 72 > 0$

$A = 6x = 6(-1) = -6 < 0$

$\therefore (-1,-2)$ is a maxima point

Maximum value.

$$f(-1,-2) = (-1)^3 + (-2)^3 - 3(-1) - 12(-2) + 20$$
$$= -1 - 8 + 3 + 24 + 20$$
$$= 38$$

2. Examine the function

$$f(x,y) = y^2 + x^2y + x^4$$

for the maxima & minima value?

Soln $\frac{\partial f}{\partial x} = 2xy + 4x^3$

$\frac{\partial f}{\partial y} = 2y + x^2$

$f_{xx} = A = 2y + 12x^2$

$f_{yy} = C = 2$

$f_{xy} = 2x = B$

$f_x = 0$
$2xy + 4x^3 = 0$
$2x(y + 2x^2) = 0$
$\boxed{x = 0}$
$y + 2x^2 = 0$

$f_y = 0$
$2y + x^2 = 0 \times 2$
$\cancel{xy} + 2x^2 = 0$
$2y + 4x^2 = 0$
$4y + 2x^2 = 0$

$-2y = 0 \Rightarrow \boxed{y = 0}$

$\therefore (0,0)$ is a stationary/Critical point.

At $(0,0)$
$AC - B^2 = 4y + 12x^2 - 4x^2 = 4(0) + 12(0) + 4(0)$
$= 0$

$\therefore$ This case is doubtful i.e. further investigation is required.

Assume For Small h & k

$f(x_1 + h, y_1 + k) - f(x_1, y_1)$

$$f(0+h, 0+k) - f(0,0)$$
$$= f(h,k) - f(0,0)$$
$$= f(h,k) - 0$$

$$f(h,k) = k^2 + h^2 k + (h)^4$$

$$= k^2$$ (Neglect the higher power of h & k)

The value is $f(x_1+h, y_1+h) - f(x_1,y_1) = k^2 > 0$

$\therefore$ The point $(0,0)$ is a minimum point

Minimum value

$$f(0,0) = 0$$

k

+ve: Minima

−ve: Maxima

Langrange's Multiplier:

Step I → Let $f(x,y) =$
$\phi(x,y) =$

Step II → $F(x,y) = f(x,y) + \lambda\phi(x,y)$

Step III → $dF = \left[\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy\right]$

$$+\lambda\left[\frac{\partial \theta}{\partial x}dx + \frac{\partial \theta}{\partial y}dy\right]$$

Step IV → Put $dF = 0$ and find x and y in term of λ.

Step V → Put the values of x and y in $\phi(x,y) = 0$ & find value of λ.

Step VI → Put value of λ in x and ~~x~~ y

Ques1.) Find the value of $x^2 + y^2$ subjected to the condition and the condition is $x + y = 2$ by Langrange's Multipliers?

Sol^n Let $f(x,y) = x^2+y^2$
$\phi(x,y) = x+y-2$

Now,

$$F(x,y) = f(x,y) + \lambda\phi(x,y)$$
$$= x^2+y^2+\lambda(x+y-2)$$

$$dF = \left[\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy\right] + \lambda\left[\frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy\right]$$

$$dF = [2x\,dx + 2y\,dy] + \lambda[dx+dy]$$
$$= [2x+\lambda]dx + [2y+\lambda]dy$$

$$dF = 0 = (2x+\lambda)dx + (2y+\lambda)dy$$

By comparing

$$(2x+\lambda) = 0 \qquad (2y+\lambda) = 0$$
$$x = -\lambda/2 \qquad y = -\lambda/2$$

Put the values of λ & y in

$$\phi(x,y) = 0.$$
$$\frac{-\lambda}{2} - \frac{\lambda}{2} - 2 = 0 \Rightarrow \frac{-\lambda-\lambda-4}{2} = 0$$
$$-2\lambda = 4 \Rightarrow \boxed{\lambda = -2}$$

$$\boxed{x = y = 1}$$

$\therefore (1,1)$ is the extreme point. For

$f(1,1) = 1^2 + 1^2 = 2$ //

Scalar & Vector Function:

Scalar Function:

It is a function that assigns a real number to a set of a real ~~no.~~ variables. It is denoted by.

$$\boxed{u = u(x_1, x_2, x_3, \ldots, x_n)}$$

eg: $f = 3xy$

Vector Function:

A vector Function is a function that assigns a vectors to a set of real variables. It is denoted by

$$\boxed{F = f(x_1, x_2, \ldots)\hat{i} + f(x_1, x_2, \ldots)\hat{j} + f(x_1, x_2, \ldots)\hat{k}}$$

Del Operator (∇):

The operator 'Del' (∇) is defined as $\boxed{\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}}$

Gradient:

If $f(x,y)$ is a scalar function then gradient of $f(x,y)$ is defined by

$$\boxed{\nabla f = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) f}$$

$$\boxed{\nabla f = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}} = (\text{grad } f)$$

Q) If $\phi = xz^2 - 5yz + xz$ Then find $\nabla\phi$ at a point $(1,-1,2)$

Solⁿ $\nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$

$$= \hat{i}\frac{\partial}{\partial x}(xz^2 - 5yz + xz) + \hat{j}\frac{\partial}{\partial y}(xz^2 - 5yz + xz) + \hat{k}\frac{\partial}{\partial z}(xz^2 - 5yz + xz)$$

$$\nabla\phi = \hat{i}(z^2 + z) + \hat{j}(-5z) + \hat{k}(2xz - 5y + x)$$

Now,

$$\nabla\phi_{(1,-1,2)} = \hat{i}((2)^2 + 2) + \hat{j}(-5(2)) + \hat{k}(2(1)(2) - 5(-1) + (1))$$

$$\boxed{\nabla\phi = 6\hat{i} - 10\hat{j} + 10\hat{k}}$$

Directional Derivatives:

Direction Derivative of ϕ in the direction $\vec{a}$

$$\underset{\vec{a}}{\;} = \nabla\phi \cdot \frac{\vec{a}}{|\vec{a}|} = \nabla\phi \cdot \hat{a}$$

$\hat{a}$ ↳ unit vector.

$$\boxed{\text{Maximum Direc. D} = |\nabla\phi|}$$

or

$|\text{grad}\,\phi|$

Ques 1.) Find the directional derivative of $\phi = 4xz^3 - 3x^2y^2z$ at a point $(2,-1,2)$ in the dir^n of $2\hat{i} + 3\hat{j} + 6\hat{k}$

Sol^n $\nabla\phi = \hat{i}\frac{d\phi}{dx} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{dz}$

$= \hat{i}(4z^3 - 6xy^2z) + \hat{j}(-6x^2yz) + \hat{k}(12xz^2 - 3x^2y^2)$

$\nabla\phi_{(2,-1,2)} = \hat{i}(4(2)^3 - 6(2)(-1)^2(2)) + \hat{j}(-6(2)^2(-1)(2)) + \hat{k}(12(2)(2)^2 - 3(2)^2(-1)^2)$

$\nabla\phi_{(2,-1,2)} = 8\hat{i} + 48\hat{j} + 84\hat{k}$

$\cancel{DD = \nabla\phi \cdot \frac{\vec{a}}{|\vec{a}|}}$

SR

TEACHER SIGNATURE

Now,

D.D of ϕ in the dirn $(2\hat{i}+3\hat{j}+6\hat{k}) = \vec{a}$

$$= \nabla\phi \cdot \frac{\vec{a}}{|\vec{a}|}$$

$$= \frac{(8\hat{i}+48\hat{j}+84\hat{k})\cdot(2\hat{i}+3\hat{j}+6\hat{k})}{\sqrt{(2)^2+(3)^2+(6)^2}}$$

$$= \frac{16+144+504}{7}$$

Q2 Find the D.D of $\phi = x^2+y^2+z^2$ in the dirn of the line $\frac{x}{3} = \frac{y}{4} = \frac{z}{5}$ at a point $(1,2,3)$.

Soln From the given line ~~the ve~~ we can form a vector $\vec{a} = 3\hat{i}+4\hat{j}+5\hat{k}$

$$|\vec{a}| = \sqrt{9+16+25} = 5\sqrt{2}$$

$$\nabla\phi = 2x\hat{i}+2y\hat{j}+2z\hat{k}$$
$$= 2\hat{i}+4\hat{j}+6\hat{k}$$

$$D.D = \frac{6+16+30}{5\sqrt{2}} = \frac{52}{5\sqrt{2}}$$

Q3 Find the D.D of $\phi = x^4+y^4+z^4$ at the point $A(1,-2,1)$ in the direction of AB where the B point is $(2,6,-1)$

Soln: Let $\vec{a} = \hat{i} - 2\hat{j} + \hat{k}$

$\vec{b} = 2\hat{i} + 6\hat{j} - \hat{k}$

$\vec{AB} = (2-1)\hat{i} + (6+2)\hat{j} + (-1-1)\hat{k}$

$= \hat{i} + 8\hat{j} - 2\hat{k}$

$|\vec{AB}| = \sqrt{1+64+4} = \sqrt{69} = 8.3.$

$\nabla\phi = 4x^3\hat{i} + 4y^3\hat{j} + 4z^3\hat{k}$

$\Rightarrow$ point $= (1, -2, 1)$

$\nabla\phi = 4\hat{i} - 32\hat{j} + 4\hat{k}.$

$D.D = \nabla\phi \cdot \frac{\vec{AB}}{|\vec{AB}|}$

Angle b/w two Surfaces:

$\boxed{\cancel{\cos\theta = (\nabla\phi_1) \cdot (\nabla\phi_2)}}$

$$\cos\theta = \frac{(\nabla\phi_1) \cdot (\nabla\phi_2)}{|\nabla\phi_1| |\nabla\phi_2|}$$

Ques 4) Find the angle of intersection at $(4, -3, 2)$ of spheres $x^2+y^2+z^2=x$ and $x^2+y^2+z^2+4x-6y-8z=0$.

SR

TEACHER SIGNATURE

Solⁿ Let $\phi_1 = x^2+y^2+z^2-29$

$\phi_2 = x^2+y^2+z^2+4x-6y-8z$

$\nabla\phi_1 = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$

$\nabla\phi_{1(4,-3,2)} = 8\hat{i} - 6\hat{j} + 4\hat{k}$

$\nabla\phi_2 = (2x+4)\hat{i} + (2y-6)\hat{j} + (2z-8)\hat{k}$

$\nabla\phi_{2(4,-3,2)} = 12\hat{i} - 12\hat{j} - 4\hat{k}$

$|\nabla\phi_1| = \sqrt{64+36+16} = \sqrt{116}$

$|\nabla\phi_2| = \sqrt{144+144+16} = \sqrt{304}.$

$$\cos\theta = \frac{96+72+16}{\sqrt{116}\times\sqrt{304}}$$

Divergence of Vector Function:

Let $\vec{f} = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$ be the differentiable vector function where f_1, f_2 and f_3 are functions of x, y and z than the divergence of $\vec{f}$ can be written as $\nabla \cdot \vec{f}$ or div $\vec{f}$ and defined as

$$\nabla \cdot \vec{f} = \frac{\partial f_1}{}\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)$$

$$\cdot\left(f_1\hat{i} + f_2\hat{j} + f_3\hat{k}\right)$$

$$\boxed{\nabla \cdot \vec{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}}$$

Note: (i) Divergence ~~is~~ of vector function is always be a scalar quantity.

(ii) ~~If ∇~~ The vector $\vec{f}$ is said to be solenoidal if div $\vec{f} = 0$ or $\nabla \cdot \vec{f} = 0$.

1.) Evaluate divergence of $\vec{f}$ vector

SR

TEACHER SIGNATURE

where
$\vec{f} = 2x^2\hat{i} \mp xy^2 z\hat{j} + 3y^2 x\hat{k}$ at the point (1, 1, 1).

Solⁿ div $\vec{f} = \nabla\cdot\vec{f} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot$

$\left(2x^2 z\hat{i} \mp xy^2 z\hat{j} + 3y^2 x\hat{k}\right)$

$= \frac{\partial}{\partial x}(2x^2 z) + \frac{\partial}{\partial y}(xy^2 z) + \frac{\partial}{\partial z}(3y^2 x)$

$= 4xz - 2xyz + 0.$

$= 4(1)(1) - 2(1)(1)(1) = 4 - 2$

$= 2$

Q2> Determine the constant a so that the vector $\vec{f} = (x+3y)\hat{i} + (y-2z)\hat{j} + (x+az)\hat{k}$ is a solenoidal.

Solⁿ For solenoidal $\nabla\cdot\vec{f} = 0$.

$\nabla\cdot\vec{f} = \frac{\partial}{\partial x}(x+3y) + \frac{\partial}{\partial y}(y-2z) + \frac{\partial}{\partial z}(x+az) = 0$

$1 + 1 + a = 0$

$2 + a = 0$

$\boxed{\cancel{a = 2}}$ $\boxed{a = -2}$

Property of Divergence:

→ $\text{div}(\vec{f}+\vec{g}) = \text{div}\,\vec{f} + \text{div}\,\vec{g}$

or $\nabla\cdot(\vec{f}+\vec{g}) = \nabla\cdot\vec{f} + \nabla\cdot\vec{g}$

→ divergence of $\vec{f}$ is zero if $\vec{f}$ is constant and known as solenoidal.

→ $\text{div}(\phi\vec{f}) = (\text{grad}\,\phi)\cdot\vec{f} + \phi(\text{div}\,\vec{f})$

or

$\nabla\cdot(\phi\vec{f}) = (\nabla\phi)\cdot\vec{f} + \phi(\nabla\cdot\vec{f})$

where

ϕ – is a scaler funcn.

$\vec{f}$ – is a vector funcn.

Curl of the Vector Function

Let $\vec{f}$ be ~~a vector~~ any given differentiable vector function then the curl of $\vec{f}$ vector is denoted by.

$\nabla \times \vec{f}$ and defined by

$$\nabla \times \vec{f} = \text{curl}\vec{f} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times \vec{f}$$

$$= \hat{i} \times \frac{\partial \vec{f}}{\partial x} + \hat{j} \times \frac{\partial \vec{f}}{\partial y} + \hat{k} \times \frac{\partial \vec{f}}{\partial z}$$

Note –

1. Curl of vector function is always be a vector quantity.

2. If $\nabla \times \vec{f} = 0$ then $\vec{f}$ is known as a irrotational.

Q1) If $\vec{f} = xy^2\hat{i} + 2x^2yz\hat{j} - 3yz^2\hat{k}$ then find curl$\vec{f}$ at a point $(1,-1,1)$.

Sol^n $\nabla \times \vec{f} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xy^2 & 2x^2yz & -3yz^2 \end{vmatrix}$ — $\hat{i}, \hat{j}, \hat{k}$ — ∇ — $\vec{f}$

$$= \hat{i}\left[\frac{\partial}{\partial y}(-3yz^2) - \frac{\partial}{\partial z}(2x^2yz)\right] - \hat{j}\left[\frac{\partial}{\partial x}(-3yz^2) - \frac{\partial}{\partial z}(xy^2)\right] + \hat{k}\left[\frac{\partial}{\partial x}(2x^2yz) - \frac{\partial}{\partial y}(xy^2)\right]$$

$$= (-3z^2 - 2x^2y)\hat{i} - 0\hat{j} + (4xyz - 2xy)\hat{k}$$

$$= [-3(1)^2 - 2(1)^2(-1)]\hat{i} + [4(1)(-1)^2 - 2(1)(-1)]\hat{k}$$

$$= (-3+2)\hat{i} + (-4+2)\hat{k}$$

$$= \cancel{-\hat{i} + 6\hat{k}} = -\hat{i} - 2\hat{k}$$

$$= -(\hat{i} + 2\hat{k})$$

Ques2.) Show that curl(curl $\vec{f}$) = 0 where $\vec{f} = z\hat{i} + x\hat{j} + y\hat{k}$

Ques3.) Find the value of a so that $\vec{f}$ is $(axy - z^3)\hat{i} + (a-2)x^2\hat{j} + (1-a)xz^2\hat{k}$ is irrotational.

Soln $\nabla \times \vec{f} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ axy - z^3 & (a-2)x^2 & (1-a)xz^2 \end{vmatrix} = 0.$

~~so~~ ~~that~~

that means

$\nabla \times \vec{f} = 0.$

$$\left[\frac{\partial}{\partial y}(1-a)xz^2 - \frac{\partial}{\partial z}(a-2)x^2\right]\hat{i} \mp \left[\frac{\partial}{\partial x}(1-a)xz^2 - \frac{\partial}{\partial z}(axy-z^3)\right]\hat{j} + \left[\frac{\partial}{\partial x}(a-2)x^2 - \frac{\partial}{\partial y}(axy-z^3)\right]\hat{k}$$

$\cancel{= -2xz\hat{i} -}$

we will consider $\vec{O}$ vector

$= 0\hat{i} + 0\hat{j} + 0\hat{k}$

than compare & we get

$\boxed{a=4}$

<u>Properties of Curl:</u>

$\rightarrow \nabla \times (\vec{f} + \vec{g}) = \nabla \times \vec{f} + \nabla \times \vec{g}$

$\rightarrow \nabla \times (\phi \vec{f}) = (\nabla \phi) \times \vec{f} + \phi(\nabla \times \vec{f})$

$\cancel{\rightarrow \nabla \times (\vec{f} \times \vec{g}) = \vec{f}(\nabla \cdot \vec{g}) + \vec{g} - \vec{f}(\nabla \times \vec{g}(\nabla \cdot \vec{f})}$

$\rightarrow \nabla \times (\vec{f} \times \vec{g}) = \vec{f}(\nabla \cdot \vec{g}) - (\vec{f} \cdot \nabla)\vec{g} + (\vec{g} \cdot \nabla)\vec{f} - \vec{g}(\nabla \cdot \vec{f})$

$\rightarrow \nabla \cdot (\vec{f} \times \vec{g}) = \vec{g} \cdot (\nabla \times \vec{f}) - \vec{f} \cdot (\nabla \times \vec{g})$

$\rightarrow$ If $\nabla \times \vec{f} = 0 \Rightarrow \vec{f}$ is ϑ constant.