1 True or False

(a) A graph with k edges and n vertices has a vertex of degree at least $2k/n$.

(b) If $e \leq 3v - 6$ holds for a graph G, then G is planar.

(c) An n-dimensional hypercube has an Eulerian cycle if and only if n is even.

(d) If all vertices of an undirected graph have degree 4, the graph must be the complete graph on five vertices, K_5.

Solution:

(a) **True.**
The sum of degrees is $2k$. Since there are n vertices, the average vertex degree is $2k/n$ and hence there is at least one vertex with degree at least $2k/n$.

(b) **False.**
The graph $K_{3,3}$ is not planar. It has $e = 9$ and $v = 6$ which satisfy the condition $e \leq 3v - 6$.

(c) **True.**
In the n-dimensional hypercube, every vertex has degree n. If n is odd, then by Euler's Theorem there can be no Eulerian tour. On the other hand, the hypercube is connected: we can get from any one bit-string x to any other y by flipping the bits they differ in one at a time. Therefore, when n is even, since every vertex has even degree and the graph is connected, there is an Eulerian tour.

(d) **False.**
Consider the 4-dimensional hypercube. Each vertex has exactly 4 neighbors, but it is not K_5.

2 Short Answers

(a) A connected planar simple graph has 5 more edges than it has vertices. How many faces does it have?

(b) How many edges need to be removed from a 3-dimensional hypercube to get a tree?

Solution:

(a) **7.**
Use Euler's formula $v+f=e+2$.

(b) **5.**
The 3-dimensional hypercube has $3(2^3)/2 = 12$ edges and $2^3 = 8$ vertices. A tree on 8 vertices has 7 edges, so one needs to remove 5 edges.

3 Coloring Trees

Prove that all trees with at least 2 vertices are *bipartite*: the vertices can be partitioned into two groups so that every edge goes between the two groups.

[*Hint:* Use induction on the number of vertices.]

Solution:

Proof using induction on the number of vertices n.

Base case $n=2$. A tree with two vertices has only one edge and is a bipartite graph by partitioning the two vertices into two separate parts.

Inductive hypothesis. Assume that all trees with k vertices for an arbitrary $k \geq 2$ is bipartite.

Inductive step. Consider a tree $T=(V,E)$ with $k+1$ vertices. We know that every tree must have at least two leaves, so remove one leaf u and the edge connected to u, say edge e. The resulting graph $T-u$ is a tree with k vertices and is bipartite by the inductive hypothesis. Thus there exists a partitioning of the vertices $V = R \cup L$ such that there does not exist an edge that connects two vertices in L or two vertices in R. Now when we add u back to the graph. If edge e connects u with a vertex in L then let $L' = L$ and $R' = R \cup \{u\}$. On the other hand if edge e connects u with a vertex in R then let $L' = L \cup \{u\}$ and $R' = R$. L' and R' gives us the required partition to show that T is bipartite. This completes the inductive step and hence by induction we get that all trees with at least 2 vertices are bipartite.