

Problem Bank 2A: (from Rosen's book)  
Permutation, Combination, Pigeonhole Principle

It is possible to prove some useful properties about Ramsey numbers, but for the most part it is difficult to find their exact values. Note that by symmetry it can be shown that  $R(m, n) = R(n, m)$  (see Exercise 28). We also have  $R(2, n) = n$  for every positive integer  $n \geq 2$  (see Exercise 27). The exact values of only nine Ramsey numbers  $R(m, n)$  with  $3 \leq m \leq n$  are known, including  $R(4, 4) = 18$ . Only bounds are known for many other Ramsey numbers, including  $R(5, 5)$ , which is known to satisfy  $43 \leq R(5, 5) \leq 49$ . The reader interested in learning more about Ramsey numbers should consult [MiRo91] or [GrRoSp90].

## Exercises

1. Show that in any set of six classes, each meeting regularly once a week on a particular day of the week, there must be two that meet on the same day, assuming that no classes are held on weekends.
2. Show that if there are 30 students in a class, then at least two have last names that begin with the same letter.
3. A drawer contains a dozen brown socks and a dozen black socks, all unmatched. A man takes socks out at random in the dark.
  - a) How many socks must he take out to be sure that he has at least two socks of the same color?
  - b) How many socks must he take out to be sure that he has at least two black socks?
4. A bowl contains 10 red balls and 10 blue balls. A woman selects balls at random without looking at them.
  - a) How many balls must she select to be sure of having at least three balls of the same color?
  - b) How many balls must she select to be sure of having at least three blue balls?
5. Show that among any group of five (not necessarily consecutive) integers, there are two with the same remainder when divided by 4.
6. Let  $d$  be a positive integer. Show that among any group of  $d + 1$  (not necessarily consecutive) integers there are two with exactly the same remainder when they are divided by  $d$ .
7. Let  $n$  be a positive integer. Show that in any set of  $n$  consecutive integers there is exactly one divisible by  $n$ .
8. Show that if  $f$  is a function from  $S$  to  $T$ , where  $S$  and  $T$  are finite sets with  $|S| > |T|$ , then there are elements  $s_1$  and  $s_2$  in  $S$  such that  $f(s_1) = f(s_2)$ , or in other words,  $f$  is not one-to-one.
9. What is the minimum number of students, each of whom comes from one of the 50 states, who must be enrolled in a university to guarantee that there are at least 100 who come from the same state?
- \*10. Let  $(x_i, y_i)$ ,  $i = 1, 2, 3, 4, 5$ , be a set of five distinct points with integer coordinates in the  $xy$  plane. Show that the midpoint of the line joining at least one pair of these points has integer coordinates.
- \*11. Let  $(x_i, y_i, z_i)$ ,  $i = 1, 2, 3, 4, 5, 6, 7, 8, 9$ , be a set of nine distinct points with integer coordinates in  $xyz$  space. Show that the midpoint of at least one pair of these points has integer coordinates.
12. How many ordered pairs of integers  $(a, b)$  are needed to guarantee that there are two ordered pairs  $(a_1, b_1)$  and  $(a_2, b_2)$  such that  $a_1 \bmod 5 = a_2 \bmod 5$  and  $b_1 \bmod 5 = b_2 \bmod 5$ ?
13. a) Show that if five integers are selected from the first eight positive integers, there must be a pair of these integers with a sum equal to 9.  
b) Is the conclusion in part (a) true if four integers are selected rather than five?
14. a) Show that if seven integers are selected from the first 10 positive integers, there must be at least two pairs of these integers with the sum 11.  
b) Is the conclusion in part (a) true if six integers are selected rather than seven?
15. How many numbers must be selected from the set  $\{1, 2, 3, 4, 5, 6\}$  to guarantee that at least one pair of these numbers add up to 7?
16. How many numbers must be selected from the set  $\{1, 3, 5, 7, 9, 11, 13, 15\}$  to guarantee that at least one pair of these numbers add up to 16?
17. A company stores products in a warehouse. Storage bins in this warehouse are specified by their aisle, location in the aisle, and shelf. There are 50 aisles, 85 horizontal locations in each aisle, and 5 shelves throughout the warehouse. What is the least number of products the company can have so that at least two products must be stored in the same bin?
18. Suppose that there are nine students in a discrete mathematics class at a small college.
  - a) Show that the class must have at least five male students or at least five female students.
  - b) Show that the class must have at least three male students or at least seven female students.
19. Suppose that every student in a discrete mathematics class of 25 students is a freshman, a sophomore, or a junior.
  - a) Show that there are at least nine freshmen, at least nine sophomores, or at least nine juniors in the class.



- b) Show that there are either at least three freshmen, at least 19 sophomores, or at least five juniors in the class.
20. Find an increasing subsequence of maximal length and a decreasing subsequence of maximal length in the sequence 22, 5, 7, 2, 23, 10, 15, 21, 3, 17.
21. Construct a sequence of 16 positive integers that has no increasing or decreasing subsequence of five terms.
22. Show that if there are 101 people of different heights standing in a line, it is possible to find 11 people in the order they are standing in the line with heights that are either increasing or decreasing.
- \*23. Describe an algorithm in pseudocode for producing the largest increasing or decreasing subsequence of a sequence of distinct integers.
24. Show that in a group of five people (where any two people are either friends or enemies), there are not necessarily three mutual friends or three mutual enemies.
25. Show that in a group of 10 people (where any two people are either friends or enemies), there are either three mutual friends or four mutual enemies, and there are either three mutual enemies or four mutual friends.
26. Use Exercise 25 to show that among any group of 20 people (where any two people are either friends or enemies), there are either four mutual friends or four mutual enemies.
27. Show that if  $n$  is a positive integer with  $n \geq 2$ , then the Ramsey number  $R(2, n)$  equals  $n$ .
28. Show that if  $m$  and  $n$  are positive integers with  $m \geq 2$  and  $n \geq 2$ , then the Ramsey numbers  $R(m, n)$  and  $R(n, m)$  are equal.
29. Show that there are at least six people in California (population: 36 million) with the same three initials who were born on the same day of the year (but not necessarily in the same year). Assume that everyone has three initials.
30. Show that if there are 100,000,000 wage earners in the United States who earn less than 1,000,000 dollars, then there are two who earned exactly the same amount of money, to the penny, last year.
31. There are 38 different time periods during which classes at a university can be scheduled. If there are 677 different classes, how many different rooms will be needed?
32. A computer network consists of six computers. Each computer is directly connected to at least one of the other computers. Show that there are at least two computers in the network that are directly connected to the same number of other computers.
33. A computer network consists of six computers. Each computer is directly connected to zero or more of the other computers. Show that there are at least two computers in the network that are directly connected to the same number of other computers. [Hint: It is impossible to have a computer linked to none of the others and a computer linked to all the others.]
34. Find the least number of cables required to connect eight computers to four printers to guarantee that four computers can directly access four different printers. Justify your answer.
35. Find the least number of cables required to connect 100 computers to 20 printers to guarantee that 20 computers can directly access 20 different printers. (Here, the assumptions about cables and computers are the same as in Example 9.) Justify your answer.
- \*36. Prove that at a party where there are at least two people, there are two people who know the same number of other people there.
37. An arm wrestler is the champion for a period of 75 hours. (Here, by an hour, we mean a period starting from an exact hour, such as 1 P.M., until the next hour.) The arm wrestler had at least one match an hour, but no more than 125 total matches. Show that there is a period of consecutive hours during which the arm wrestler had exactly 24 matches.
- \*38. Is the statement in Exercise 37 true if 24 is replaced by  
a) 2?    b) 23?    c) 25?    d) 30?
39. Show that if  $f$  is a function from  $S$  to  $T$ , where  $S$  and  $T$  are finite sets and  $m = \lceil |S|/|T| \rceil$ , then there are at least  $m$  elements of  $S$  mapped to the same value of  $T$ . That is, show that there are distinct elements  $s_1, s_2, \dots, s_m$  of  $S$  such that  $f(s_1) = f(s_2) = \dots = f(s_m)$ .
40. There are 51 houses on a street. Each house has an address between 1000 and 1099, inclusive. Show that at least two houses have addresses that are consecutive integers.
- \*41. Let  $x$  be an irrational number. Show that for some positive integer  $j$  not exceeding  $n$ , the absolute value of the difference between  $jx$  and the nearest integer to  $jx$  is less than  $1/n$ .
42. Let  $n_1, n_2, \dots, n_t$  be positive integers. Show that if  $n_1 + n_2 + \dots + n_t - t + 1$  objects are placed into  $t$  boxes, then for some  $i, i = 1, 2, \dots, t$ , the  $i$ th box contains at least  $n_i$  objects.
- \*43. A proof of Theorem 3 based on the generalized pigeonhole principle is outlined in this exercise. The notation used is the same as that used in the proof in the text.
- a) Assume that  $i_k \leq n$  for  $k = 1, 2, \dots, n^2 + 1$ . Use the generalized pigeonhole principle to show that there are  $n + 1$  terms  $a_{k_1}, a_{k_2}, \dots, a_{k_{n+1}}$  with  $i_{k_1} = i_{k_2} = \dots = i_{k_{n+1}}$ , where  $1 \leq k_1 < k_2 < \dots < k_{n+1}$ .
- b) Show that  $a_{k_j} > a_{k_{j+1}}$  for  $j = 1, 2, \dots, n$ . [Hint: Assume that  $a_{k_j} < a_{k_{j+1}}$ , and show that this implies that  $i_{k_j} > i_{k_{j+1}}$ , which is a contradiction.]
- c) Use parts (a) and (b) to show that if there is no increasing subsequence of length  $n + 1$ , then there must be a decreasing subsequence of this length.



Many identities involving binomial coefficients can be proved using combinatorial proofs. We now provide a combinatorial proof of Corollary 2.

**Proof:** Suppose that  $S$  is a set with  $n$  elements. Every subset  $A$  of  $S$  with  $r$  elements corresponds to a subset of  $S$  with  $n - r$  elements, namely  $\bar{A}$ . Consequently,  $C(n, r) = C(n, n - r)$ . ◀

**EXAMPLE 12** How many ways are there to select five players from a 10-member tennis team to make a trip to a match at another school?



**Solution:** The answer is given by the number of 5-combinations of a set with 10 elements. By Theorem 2, the number of such combinations is

$$C(10, 5) = \frac{10!}{5!5!} = 252. \quad \blacktriangleleft$$

**EXAMPLE 13** A group of 30 people have been trained as astronauts to go on the first mission to Mars. How many ways are there to select a crew of six people to go on this mission (assuming that all crew members have the same job)?

**Solution:** The number of ways to select a crew of six from the pool of 30 people is the number of 6-combinations of a set with 30 elements, because the order in which these people are chosen does not matter. By Theorem 2, the number of such combinations is

$$C(30, 6) = \frac{30!}{6!24!} = \frac{30 \cdot 29 \cdot 28 \cdot 27 \cdot 26 \cdot 25}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 593,775. \quad \blacktriangleleft$$

**EXAMPLE 14** How many bit strings of length  $n$  contain exactly  $r$  1s?

**Solution:** The positions of  $r$  1s in a bit string of length  $n$  form an  $r$ -combination of the set  $\{1, 2, 3, \dots, n\}$ . Hence, there are  $C(n, r)$  bit strings of length  $n$  that contain exactly  $r$  1s. ◀

**EXAMPLE 15** Suppose that there are 9 faculty members in the mathematics department and 11 in the computer science department. How many ways are there to select a committee to develop a discrete mathematics course at a school if the committee is to consist of three faculty members from the mathematics department and four from the computer science department?

**Solution:** By the product rule, the answer is the product of the number of 3-combinations of a set with nine elements and the number of 4-combinations of a set with 11 elements. By Theorem 2, the number of ways to select the committee is

$$C(9, 3) \cdot C(11, 4) = \frac{9!}{3!6!} \cdot \frac{11!}{4!7!} = 84 \cdot 330 = 27,720. \quad \blacktriangleleft$$

## Exercises

1. List all the permutations of  $\{a, b, c\}$ .
2. How many different permutations are there of the set  $\{a, b, c, d, e, f, g\}$ ?
3. How many permutations of  $\{a, b, c, d, e, f, g\}$  end with  $a$ ?

4. Let  $S = \{1, 2, 3, 4, 5\}$ .
  - a) List all the 3-permutations of  $S$ .
  - b) List all the 3-combinations of  $S$ .
5. Find the value of each of these quantities.
  - a)  $P(6, 3)$
  - b)  $P(6, 5)$

- c)  $P(8, 1)$                       d)  $P(8, 5)$   
 e)  $P(8, 8)$                       f)  $P(10, 9)$
6. Find the value of each of these quantities.  
 a)  $C(5, 1)$                       b)  $C(5, 3)$   
 c)  $C(8, 4)$                       d)  $C(8, 8)$   
 e)  $C(8, 0)$                       f)  $C(12, 6)$
7. Find the number of 5-permutations of a set with nine elements.
8. In how many different orders can five runners finish a race if no ties are allowed?
9. How many possibilities are there for the win, place, and show (first, second, and third) positions in a horse race with 12 horses if all orders of finish are possible?
10. There are six different candidates for governor of a state. In how many different orders can the names of the candidates be printed on a ballot?
11. How many bit strings of length 10 contain  
 a) exactly four 1s?  
 b) at most four 1s?  
 c) at least four 1s?  
 d) an equal number of 0s and 1s?
12. How many bit strings of length 12 contain  
 a) exactly three 1s?  
 b) at most three 1s?  
 c) at least three 1s?  
 d) an equal number of 0s and 1s?
13. A group contains  $n$  men and  $n$  women. How many ways are there to arrange these people in a row if the men and women alternate?
14. In how many ways can a set of two positive integers less than 100 be chosen?
15. In how many ways can a set of five letters be selected from the English alphabet?
16. How many subsets with an odd number of elements does a set with 10 elements have?
17. How many subsets with more than two elements does a set with 100 elements have?
18. A coin is flipped eight times where each flip comes up either heads or tails. How many possible outcomes  
 a) are there in total?  
 b) contain exactly three heads?  
 c) contain at least three heads?  
 d) contain the same number of heads and tails?
19. A coin is flipped 10 times where each flip comes up either heads or tails. How many possible outcomes  
 a) are there in total?  
 b) contain exactly two heads?  
 c) contain at most three tails?  
 d) contain the same number of heads and tails?
20. How many bit strings of length 10 have  
 a) exactly three 0s?  
 b) more 0s than 1s?  
 c) at least seven 1s?  
 d) at least three 1s?
21. How many permutations of the letters  $ABCDEFGH$  contain  
 a) the string  $BCD$ ?  
 b) the string  $CFGA$ ?  
 c) the strings  $BA$  and  $GF$ ?  
 d) the strings  $ABC$  and  $DE$ ?  
 e) the strings  $ABC$  and  $CDE$ ?  
 f) the strings  $CBA$  and  $BED$ ?
22. How many permutations of the letters  $ABCDEFGH$  contain  
 a) the string  $ED$ ?  
 b) the string  $CDE$ ?  
 c) the strings  $BA$  and  $FGH$ ?  
 d) the strings  $AB$ ,  $DE$ , and  $GH$ ?  
 e) the strings  $CAB$  and  $BED$ ?  
 f) the strings  $BCA$  and  $ABF$ ?
23. How many ways are there for eight men and five women to stand in a line so that no two women stand next to each other? [Hint: First position the men and then consider possible positions for the women.]
24. How many ways are there for 10 women and six men to stand in a line so that no two men stand next to each other? [Hint: First position the women and then consider possible positions for the men.]
25. One hundred tickets, numbered 1, 2, 3, ..., 100, are sold to 100 different people for a drawing. Four different prizes are awarded, including a grand prize (a trip to Tahiti). How many ways are there to award the prizes if  
 a) there are no restrictions?  
 b) the person holding ticket 47 wins the grand prize?  
 c) the person holding ticket 47 wins one of the prizes?  
 d) the person holding ticket 47 does not win a prize?  
 e) the people holding tickets 19 and 47 both win prizes?  
 f) the people holding tickets 19, 47, and 73 all win prizes?  
 g) the people holding tickets 19, 47, 73, and 97 all win prizes?  
 h) none of the people holding tickets 19, 47, 73, and 97 wins a prize?  
 i) the grand prize winner is a person holding ticket 19, 47, 73, or 97?  
 j) the people holding tickets 19 and 47 win prizes, but the people holding tickets 73 and 97 do not win prizes?
26. Thirteen people on a softball team show up for a game.  
 a) How many ways are there to choose 10 players to take the field?  
 b) How many ways are there to assign the 10 positions by selecting players from the 13 people who show up?



- c) Of the 13 people who show up, three are women. How many ways are there to choose 10 players to take the field if at least one of these players must be a woman?
27. A club has 25 members.
- How many ways are there to choose four members of the club to serve on an executive committee?
  - How many ways are there to choose a president, vice president, secretary, and treasurer of the club, where no person can hold more than one office?
28. A professor writes 40 discrete mathematics true/false questions. Of the statements in these questions, 17 are true. If the questions can be positioned in any order, how many different answer keys are possible?
- \*29. How many 4-permutations of the positive integers not exceeding 100 contain three consecutive integers  $k$ ,  $k + 1$ ,  $k + 2$ , in the correct order
- where these consecutive integers can perhaps be separated by other integers in the permutation?
  - where they are in consecutive positions in the permutation?
30. Seven women and nine men are on the faculty in the mathematics department at a school.
- How many ways are there to select a committee of five members of the department if at least one woman must be on the committee?
  - How many ways are there to select a committee of five members of the department if at least one woman and at least one man must be on the committee?
31. The English alphabet contains 21 consonants and five vowels. How many strings of six lowercase letters of the English alphabet contain
- exactly one vowel?
  - exactly two vowels?
  - at least one vowel?
  - at least two vowels?
32. How many strings of six lowercase letters from the English alphabet contain
- the letter  $a$ ?
  - the letters  $a$  and  $b$ ?
  - the letters  $a$  and  $b$  in consecutive positions with  $a$  preceding  $b$ , with all the letters distinct?
  - the letters  $a$  and  $b$ , where  $a$  is somewhere to the left of  $b$  in the string, with all the letters distinct?
33. Suppose that a department contains 10 men and 15 women. How many ways are there to form a committee with six members if it must have the same number of men and women?
34. Suppose that a department contains 10 men and 15 women. How many ways are there to form a committee with six members if it must have more women than men?
35. How many bit strings contain exactly eight 0s and 10 1s if every 0 must be immediately followed by a 1?
36. How many bit strings contain exactly five 0s and 14 1s if every 0 must be immediately followed by two 1s?
37. How many bit strings of length 10 contain at least three 1s and at least three 0s?
38. How many ways are there to select 12 countries in the United Nations to serve on a council if 3 are selected from a block of 45, 4 are selected from a block of 57, and the others are selected from the remaining 69 countries?
39. How many license plates consisting of three letters followed by three digits contain no letter or digit twice?
40. How many ways are there to seat six people around a circular table, where seatings are considered to be the same if they can be obtained from each other by rotating the table?
41. How many ways are there for a horse race with three horses to finish if ties are possible? (Note: Two or three horses may tie.)
- \*42. How many ways are there for a horse race with four horses to finish if ties are possible? (Note: Any number of the four horses may tie.)
- \*43. There are six runners in the 100-yard dash. How many ways are there for three medals to be awarded if ties are possible? (The runner or runners who finish with the fastest time receive gold medals, the runner or runners who finish with exactly one runner ahead receive silver medals, and the runner or runners who finish with exactly two runners ahead receive bronze medals.)
- \*44. This procedure is used to break ties in games in the championship round of the World Cup soccer tournament. Each team selects five players in a prescribed order. Each of these players takes a penalty kick, with a player from the first team followed by a player from the second team and so on, following the order of players specified. If the score is still tied at the end of the 10 penalty kicks, this procedure is repeated. If the score is still tied after 20 penalty kicks, a sudden-death shootout occurs, with the first team scoring an unanswered goal victorious.
- How many different scoring scenarios are possible if the game is settled in the first round of 10 penalty kicks, where the round ends once it is impossible for a team to equal the number of goals scored by the other team?
  - How many different scoring scenarios for the first and second groups of penalty kicks are possible if the game is settled in the second round of 10 penalty kicks?
  - How many scoring scenarios are possible for the full set of penalty kicks if the game is settled with no more than 10 total additional kicks after the two rounds of five kicks for each team?