# BBM205 Midterm Exam II

Time: 9:00-11:00

14 November 2021

| Question: | 1 | 2 | 3 | 4 | 5 | 6 | 7 | Total |
|---|---|---|---|---|---|---|---|---|
| Points: | 12 | 18 | 10 | 15 | 15 | 10 | 20 | 100 |
| Score: | | | | | | | | |

1. (12 points) Use proof by induction to show that for any positive integer $n$, $13{\cdot}7^{n-1}{+}5^{2n-1}$ is divisible by 18.

> **Solution:** Let $P(n)$ to be the statement $13 \cdot 7^{n-1} + 5^{2n-1}$ is divisible by 18.
> **Base step:** (n=1) $13 \cdot 7^0 + 5^1 = 18$, hence divisible by 18.
> **Inductive Step:** Assuming that $P(n)$ is true (*Inductive Hypothesis*), we show that $P(n+1)$ is true.
> To show $13 \cdot 7^{(n+1)-1} + 5^{2(n+1)-1}$, we rewrite it as
>
> $$13 \cdot 7^{(n+1)-1} + 5^{2(n+1)-1} = 7(13 \cdot 7^{n-1} + 5^{2n-1}) + 18 \cdot 5^{2n-1}.$$
>
> Note that the expression inside the paranthesis is divisible by 18 because of the inductive hypothesis. So, also $P(n+1)$ is true.

2. (18 points) (a) Determine the following numbers by showing all your work:

$$2^{2021} \mod 11, \quad 3^{2021} \mod 11, \quad 4^{2021} \mod 11$$

(b) Determine the following numbers by showing all your work:

$$2^{2021} \mod 9, \quad 3^{2021} \mod 9, \quad 4^{2021} \mod 9$$

(c) An inverse of k modulo $n > 1$ is an integer $k^{-1}$ such that $k \cdot k^{-1} \equiv 1 \mod n$. Show that k has an inverse if and only if gcd(k, n) = 1.

(d) For any natural number a, show that 10x+3 and 7x+2 are coprime (relatively prime).

**Solution:**

(a)

$$
\begin{array}{lll}
2^0 \equiv 1 \mod 11, & 3^0 \equiv 1 \mod 11, & 4^0 \equiv 1 \mod 11, \\
2^1 \equiv 2 \mod 11, & 3^1 \equiv 3 \mod 11, & 4^1 \equiv 4 \mod 11, \\
2^2 \equiv 4 \mod 11, & 3^2 \equiv 9 \mod 11, & 4^2 \equiv 5 \mod 11, \\
2^3 \equiv 8 \mod 11, & 3^3 \equiv 5 \mod 11, & 4^3 \equiv 9 \mod 11, \\
2^4 \equiv 5 \mod 11 & 3^4 \equiv 4 \mod 11, & 4^4 \equiv 3 \mod 11 \\
2^5 \equiv 10 \mod 11 & 3^5 \equiv 1 \mod 11, ... & 4^5 \equiv 1 \mod 11, ... \\
2^6 \equiv 9 \mod 11 & & \\
2^7 \equiv 7 \mod 11 & & \\
2^8 \equiv 3 \mod 11 & & \\
2^9 \equiv 6 \mod 11, ... & & \\
2^{10} \equiv 1 \mod 11 & &
\end{array}
$$

Since the pattern $(1, 2, 4, ..., 7, 3, 6)$ repeats in $2^i \mod 11$ (every 10 steps) and $2021 \equiv 1 \mod 10$, we have $2^{2021} \mod 11 \equiv 2^1 \mod 11 \equiv 2$.

Since the pattern $(1, 3, 9, 5, 4)$ repeats in $3^i \mod 11$ (every 5 steps) and $2021 \equiv 1 \mod 5$, we have $3^{2021} \mod 11 \equiv 3^1 \mod 11 \equiv 3$.

Since the pattern $(1, 4, 5, 9, 3)$ repeats in $4^i \mod 11$ (every 5 steps) and $2021 \equiv 1 \mod 5$, we have $4^{2021} \mod 11 \equiv 4^1 \mod 11 \equiv 4$.

(b)

$$
\begin{array}{lll}
2^0 \equiv 1 \mod 9, & 3^0 \equiv 1 \mod 9, & 4^0 \equiv 1 \mod 9, \\
2^1 \equiv 2 \mod 9, & 3^1 \equiv 3 \mod 9, & 4^1 \equiv 4 \mod 9, \\
2^2 \equiv 4 \mod 9, & 3^2 \equiv 0 \mod 9, & 4^2 \equiv 7 \mod 9, \\
2^3 \equiv 8 \mod 9, & 3^3 \equiv 0 \mod 9, .. & 4^3 \equiv 1 \mod 9, ... \\
2^4 \equiv 7 \mod 9 & & \\
2^5 \equiv 5 \mod 9 & & \\
2^6 \equiv 1 \mod 9 & &
\end{array}
$$

As we observe above, $3^{2021} \equiv 0 \mod 9$.

Since the pattern $(1, 2, 4, 8, 7, 5)$ repeats in $2^i \mod 9$ (every 6 steps) and $2021 \equiv 5 \mod 6$, we have $2^{2021} \mod 9 \equiv 2^5 \mod 9 \equiv 5$.

Since the pattern $(1, 4, 7)$ repeats in $4^i \mod 9$ (every 3 steps) and $2021 \equiv 2 \mod 3$, we have $4^{2021} \mod 9 \equiv 4^2 \mod 9 \equiv 7$.

(c) In proving both parts, we use the fact that if $gcd(k, n) = 1$, then there exist integers $s$ and $t$ such that $sk + tn = 1$ by definition of the greatest common divisor.

If $k \cdot k^{-1} \equiv 1 \mod n$, then there is an integer $m$ such that $k \cdot k^{-1} + m \cdot n = 1$. Hence $gcd(k, n) = 1$.

Conversely, if $gcd(k,n) = 1$, then there exist integers $s$ and $t$ such that $sk + tn = 1$. Therefore, $sk \equiv 1 \mod n$ implying that $s$ is $k^{-1} \mod n$.

(d) By using Euclid's algorithm, we have

$$gcd(10x + 3, 7x + 2) = gcd(7x + 2, 3x + 1) = gcd(3x + 1, x) = gcd(x, 1) = 1.$$

3. (10 points) Determine whether the following are true or false.

**Solution:**

(False): Having the same number of vertices and edges are sufficient conditions for two graphs to be isomorphic.
(False): If more than half of the vertices of a graph $G$ have degree at most 1, then $G$ is not connected.
(True): If a bipartite graph $G$ has an Euler circuit, then this Euler circuit has even length.
(True): If a bipartite graph has a Hamilton cycle, then the two parts of this bipartite graph have equal number of vertices.
(False): If a graph has an Euler circuit, then it has a Hamilton cycle.

4. (15 points) Let P(n) be the statement that every positive integer that is greater than or equal to 18 can be written as a sum of 3's and 10's. As an example, P(19) is true since 19 can be written as 10+3+3+3=19.

Prove that P(n) is true using the principle of mathematical induction. Be sure to state your inductive hypothesis in the inductive step explicitly and include clear explanations on your reasoning for different cases.

**Solution:** Let P(n) be the statement that every positive integer n, where $n \geq 18$, can be written as a sum of 3's and 10's.
**Base step:** P(18) is true, since $18 = 3 + 3 + 3 + 3 + 3 + 3$.
**Inductive step:** Assume that $P(k)$ is true for $k \geq 18$ *(inductive hypothesis)*, i.e. we can write $k$ as a sum of 3s and 10s; we will show that $P(k+1)$ is also true.

There are two cases:
1- If the sum of $k$ includes two 10s (i.e. 10+10), then we replace them by seven 3s $7 \cdot 3 = 2 \cdot 10 + 1$.
2- If there is at least three 3s are included in the sum of $k$; $k$ is written either using

just 3s, or from one 10 and $(k-10)$ 3s. Because $k \geq 18$, there must be at least three 3s involved in either case. In this case, we replace three 3s (i.e. 3+3+3) by one 10, and we have formed $k+1$. (since $10 = 3 \cdot 3 + 1$).

5. (15 points) a) Find a recurrence relation $(a_n)$ for the number of strings of length n that contains two consecutive b's using only the letters from the set A={b, c, d, e}.
b) What are the initial conditions of the recurrence relation that is stated in (a) above?
c) Calculate the number of such strings of length 4, i.e. $a_4$, using the recurrence relation you obtained.

(a) Finding a recurrence relation $(a_n)$

**Solution:** Let an be the number of strings of length n that contains two consecutive b's using only the letters from the set $A = \{b, c, d, e\}$.
To construct such a string:
1) We could start with either $c$, $d$ or $e$ and follow with a string containing two consecutive $b$'s (this can be done in $3a_{n-1}$ ways), or
2) We could start with $bc$ or $bd$ or $be$, and follow with a string containing two consecutive b's (and this can be done in $3a_{n-2}$ ways),
3) We could start with $bb$ and follow with any string of length $n-2$ (of which there are $4^{n-2}$ ).

Therefore the recurrence relation, valid for all $n \geq 2$ , is:

$$a_n = 3a_{n-1} + 3a_{n-2} + 4^{n-2}$$

(b) The initial conditions of the recurrence relation:

**Solution:** $a_0 = a_1 = 0$.

(c) Calculation of $a_4$:

**Solution:**
$a_2 = 3a_1 + 3a_0 + 4^0 = 0 + 0 = 1$
$a_3 = 3a_2 + 3a_1 + 4^1 = 3 + 0 + 4 = 7$
$a_4 = 3a_3 + 3a_2 + 4^2 = 21 + 3 + 16 = 40$

6. (10 points) Consider the following inequality

$$(a+\theta)^t \geq (a + t\theta)$$

where $t \in \mathbb{N}, \theta \in \mathbb{R}^+$ and $a$ is arbitrary.
Using mathematical Induction, find a value for $a$ that the inequality given above must hold for every case.

**Solution:** Consider a=1, then inequality becomes

$$(1+\theta)^t \geq (1+t\theta)$$

Then, we can prove this by using induction on $t$.
**Base case:** When $t = 0$, the claim holds since

$$(1+\theta)^0 \geq (1+0\theta)$$

**Inductive Hypothesis:** Now, assume that $(1+\theta)^k \geq 1+k\theta$ holds for some value of $t = k$ where $k > 0$.
**Inductive Step:** For $t = k+1$, we have

$$\begin{aligned}
(1+\theta)^{k+1} &= (1+\theta)^k(1+\theta) \\
&\geq ((1+k\theta))(1+\theta) \quad \textit{(by inductive hypothesis)} \\
&\geq 1+k\theta+\theta+k\theta^2 \\
&\geq 1+(k+1)\theta+k\theta^2 \\
&\geq 1+(k+1)\theta \quad (k\theta^2 > 0 \textit{ since } k,\theta > 0)
\end{aligned}$$

q.e.d

7. (20 points) Consider the following sequence

$$\sqrt{x}, \sqrt{x\sqrt{y}}, \sqrt{x\sqrt{y\sqrt{x}}}, \ldots$$

where $x, y \in \mathbb{N} - \{0, 1\}$ and $x < y$ is arbitrary.
a) Find the recurrence relation for this sequence
b) Using mathematical Induction with recurrence relation in a) , show that it is an increasing sequence or a decreasing sequence.

**Solution:** **a)** The terms in the sequence given by the problem are denoted by $a_n$. Then, for every $n \in \mathbb{N}$, we can define followings.
$a_{2n+2=\sqrt{x\sqrt{ya_{2n}}}}$ (gives the even terms)
$a_{2n+1=\sqrt{x\sqrt{ya_{2n-1}}}}$ (gives the odd terms)

**b)** To prove that sequence is increasing or not, using induction, we will prove following inequality.

$$a_{2n-1} < a_{2n} < a_{2n+1} \quad \forall n \in \mathbb{N}$$

which implies $\{a_n\}$ is increasing.
**Base:** $n = 1$

$$a_1 = \sqrt{x} = \sqrt{x\sqrt{1}} < \sqrt{x} = \sqrt{x\sqrt{y}} = a_2 = \sqrt{x} = \sqrt{x\sqrt{y\sqrt{1}}} < \sqrt{x\sqrt{y\sqrt{x}}} = a_3$$
$$a_1 < a_2 < a_3$$

**Inductive Step:** Assume

$$a_{2k-1} < a_{2k}z < a_{2k+1}$$

Then we have

$$a_{2(k+1)-1} = a_{2k+1} = \sqrt{x\sqrt{ya_{2k-1}}} < \sqrt{x\sqrt{ya_{2k}}} = a_{2(k+1)}$$

Similarly we have

$$a_{2k+2} < a_{2k+3}$$

Thus, $\{a_n\}$ is increasing.