Lecture 2: Proofs
BBM205

Exercises (from Rosen's book)

Chapter 4, including mathematical induction, which can be used to prove results that hold for all positive integers. In Chapter 5 we will introduce the notion of combinatorial proofs.

In this section we introduced several methods for proving theorems of the form $\forall x(P(x) \rightarrow Q(x))$, including direct proofs and proofs by contraposition. There are many theorems of this type whose proofs are easy to construct by directly working through the hypotheses and definitions of the terms of the theorem. However, it is often difficult to prove a theorem without resorting to a clever use of a proof by contraposition or a proof by contradiction, or some other proof technique. In Section 1.7 we will address proof strategy. We will describe various approaches that can be used to find proofs when straightforward approaches do not work. Constructing proofs is an art that can be learned only through experience, including writing proofs, having your proofs critiqued, and reading and analyzing other proofs.

Exercises

1. Use a direct proof to show that the sum of two odd integers is even.

2. Use a direct proof to show that the sum of two even integers is even.

3. Show that the square of an even number is an even number using a direct proof.

4. Show that the additive inverse, or negative, of an even number is an even number using a direct proof.

5. Prove that if $m + n$ and $n + p$ are even integers, where m, n, and p are integers, then $m + p$ is even. What kind of proof did you use?

6. Use a direct proof to show that the product of two odd numbers is odd.

7. Use a direct proof to show that every odd integer is the difference of two squares.

8. Prove that if n is a perfect square, then $n + 2$ is not a perfect square.

9. Use a proof by contradiction to prove that the sum of an irrational number and a rational number is irrational.

10. Use a direct proof to show that the product of two rational numbers is rational.

11. Prove or disprove that the product of two irrational numbers is irrational.

12. Prove or disprove that the product of a nonzero rational number and an irrational number is irrational.

13. Prove that if x is irrational, then $1/x$ is irrational.

14. Prove that if x is rational and $x \neq 0$, then $1/x$ is rational.

15. Use a proof by contraposition to show that if $x + y \geq 2$, where x and y are real numbers, then $x \geq 1$ or $y \geq 1$.

16. Prove that if m and n are integers and mn is even, then m is even or n is even.

17. Show that if n is an integer and $n^3 + 5$ is odd, then n is even using

a) a proof by contraposition.

b) a proof by contradiction.

18. Prove that if n is an integer and $3n + 2$ is even, then n is even using

a) a proof by contraposition.

b) a proof by contradiction.

19. Prove the proposition $P(0)$, where $P(n)$ is the proposition "If n is a positive integer greater than 1, then $n^2 > n$." What kind of proof did you use?

20. Prove the proposition $P(1)$, where $P(n)$ is the proposition "If n is a positive integer, then $n^2 \geq n$." What kind of proof did you use?

21. Let $P(n)$ be the proposition "If a and b are positive real numbers, then $(a + b)^n \geq a^n + b^n$." Prove that $P(1)$ is true. What kind of proof did you use?

22. Show that if you pick three socks from a drawer containing just blue socks and black socks, you must get either a pair of blue socks or a pair of black socks.

23. Show that at least 10 of any 64 days chosen must fall on the same day of the week.

24. Show that at least 3 of any 25 days chosen must fall in the same month of the year.

25. Use a proof by contradiction to show that there is no rational number r for which $r^3 + r + 1 = 0$. [*Hint:* Assume that $r = a/b$ is a root, where a and b are integers and a/b is in lowest terms. Obtain an equation involving integers by multiplying by b^3. Then look at whether a and b are each odd or even.]

26. Prove that if n is a positive integer, then n is even if and only if $7n + 4$ is even.

27. Prove that if n is a positive integer, then n is odd if and only if $5n + 6$ is odd.

28. Prove that $m^2 = n^2$ if and only if $m = n$ or $m = -n$.

29. Prove or disprove that if m and n are integers such that $mn = 1$, then either $m = 1$ and $n = 1$, or else $m = -1$ and $n = -1$.

30. Show that these three statements are equivalent, where a and b are real numbers: (*i*) a is less than b, (*ii*) the average of a and b is greater than a, and (*iii*) the average of a and b is less than b.

31. Show that these statements about the integer x are equivalent: (*i*) $3x + 2$ is even, (*ii*) $x + 5$ is odd, (*iii*) x^2 is even.

32. Show that these statements about the real number x are equivalent: (*i*) x is rational, (*ii*) $x/2$ is rational, and (*iii*) $3x - 1$ is rational.

33. Show that these statements about the real number x are equivalent: (*i*) x is irrational, (*ii*) $3x + 2$ is irrational, (*iii*) $x/2$ is irrational.

34. Is this reasoning for finding the solutions of the equation $\sqrt{2x^2 - 1} = x$ correct? (*1*) $\sqrt{2x^2 - 1} = x$ is given; (*2*) $2x^2 - 1 = x^2$, obtained by squaring both sides of (1); (*3*) $x^2 - 1 = 0$, obtained by subtracting x^2 from both sides of (2); (*4*) $(x - 1)(x + 1) = 0$, obtained by factoring the left-hand side of $x^2 - 1$; (*5*) $x = 1$ or $x = -1$, which follows because $ab = 0$ implies that $a = 0$ or $b = 0$.

35. Are these steps for finding the solutions of $\sqrt{x + 3} = 3 - x$ correct? (*1*) $\sqrt{x + 3} = 3 - x$ is given; (*2*) $x + 3 = x^2 - 6x + 9$, obtained by squaring both sides of (1); (*3*) $0 = x^2 - 7x + 6$, obtained by subtracting $x + 3$ from both sides of (2); (*4*) $0 = (x - 1)(x - 6)$, obtained by factoring the right-hand side of (3); (*5*) $x = 1$ or $x = 6$, which follows from (4) because $ab = 0$ implies that $a = 0$ or $b = 0$.

36. Show that the propositions p_1, p_2, p_3, and p_4 can be shown to be equivalent by showing that $p_1 \leftrightarrow p_4$, $p_2 \leftrightarrow p_3$, and $p_1 \leftrightarrow p_3$.

37. Show that the propositions p_1, p_2, p_3, p_4, and p_5 can be shown to be equivalent by proving that the conditional statements $p_1 \to p_4$, $p_3 \to p_1$, $p_4 \to p_2$, $p_2 \to p_5$, and $p_5 \to p_3$ are true.

38. Find a counterexample to the statement that every positive integer can be written as the sum of the squares of three integers.

39. Prove that at least one of the real numbers $a_1, a_2, \ldots, a_n$ is greater than or equal to the average of these numbers. What kind of proof did you use?

40. Use Exercise 39 to show that if the first 10 positive integers are placed around a circle, in any order, there exist three integers in consecutive locations around the circle that have a sum greater than or equal to 17.

41. Prove that if n is an integer, these four statements are equivalent: (*i*) n is even, (*ii*) $n + 1$ is odd, (*iii*) $3n + 1$ is odd, (*iv*) $3n$ is even.

42. Prove that these four statements about the integer n are equivalent: (*i*) n^2 is odd, (*ii*) $1 - n$ is even, (*iii*) n^3 is odd, (*iv*) $n^2 + 1$ is even.

1.7 Proof Methods and Strategy

Introduction

Assessment

In Section 1.6 we introduced a variety of different methods of proof and illustrated how each method is used. In this section we continue this effort. We will introduce several other important proof methods, including proofs where we consider different cases separately and proofs where we prove the existence of objects with desired properties.

In Section 1.6 we only briefly discussed the strategy behind constructing proofs. This strategy includes selecting a proof method and then successfully constructing an argument step by step, based on this method. In this section, after we have developed a wider arsenal of proof methods, we will study some additional aspects of the art and science of proofs. We will provide advice on how to find a proof of a theorem. We will describe some tricks of the trade, including how proofs can be found by working backward and by adapting existing proofs.

When mathematicians work, they formulate conjectures and attempt to prove or disprove them. We will briefly describe this process here by proving results about tiling checkerboards with dominoes and other types of pieces. Looking at tilings of this kind, we will be able to quickly formulate conjectures and prove theorems without first developing a theory.

We will conclude the section by discussing the role of open questions. In particular, we will discuss some interesting problems either that have been solved after remaining open for hundreds of years or that still remain open.

Exhaustive Proof and Proof by Cases

Sometimes we cannot prove a theorem using a single argument that holds for all possible cases. We now introduce a method that can be used to prove a theorem, by considering different cases

The $3x + 1$ conjecture has an interesting history and has attracted the attention of mathematicians since the 1950s. The conjecture has been raised many times and goes by many other names, including the Collatz problem, Hasse's algorithm, Ulam's problem, the Syracuse problem, and Kakutani's problem. Many mathematicians have been diverted from their work to spend time attacking this conjecture. This led to the joke that this problem was part of a conspiracy to slow down American mathematical research. See the article by Jeffrey Lagarias [La85] for a fascinating discussion of this problem and the results that have been found by mathematicians attacking it. ◀

In Chapter 3 we will describe additional open questions about prime numbers. Students already familiar with the basic notions about primes might want to explore Section 3.4, where these open questions are discussed. We will mention other important open questions throughout the book.

Additional Proof Methods

In this chapter we introduced the basic methods used in proofs. We also described how to leverage these methods to prove a variety of results. We will use these proof methods in Chapters 2 and 3 to prove results about sets, functions, algorithms, and number theory. Among the theorems we will prove is the famous halting theorem which states that there is a problem that cannot be solved using any procedure. However, there are many important proof methods besides those we have covered. We will introduce some of these methods later in this book. In particular, in Section 4.1 we will discuss mathematical induction, which is an extremely useful method for proving statements of the form $\forall n P(n)$, where the domain consists of all positive integers. In Section 4.3 we will introduce structural induction, which can be used to prove results about recursively defined sets. We will use the Cantor diagonalization method, which can be used to prove results about the size of infinite sets, in Section 2.4. In Chapter 5 we will introduce the notion of combinatorial proofs, which can be used to prove results by counting arguments. The reader should note that entire books have been devoted to the activities discussed in this section, including many excellent works by George Pólya ([Po61], [Po71], [Po90]).

Finally, note that we have not given a procedure that can be used for proving theorems in mathematics. It is a deep theorem of mathematical logic that there is no such procedure.

Exercises

1. Prove that $n^2 + 1 \geq 2^n$ when n is a positive integer with $1 \leq n \leq 4$.
2. Prove that there are no positive perfect cubes less 1000 that are the sum of the cubes of two positive integers.
3. Prove that if x and y are real numbers, then $\max(x, y) + \min(x, y) = x + y$. [*Hint:* Use a proof by cases, with the two cases corresponding to $x \geq y$ and $x < y$, respectively.]
4. Use a proof by cases to show that $\min(a, \min(b, c)) = \min(\min(a, b), c)$ whenever a, b, and c are real numbers.
5. Prove the **triangle inequality**, which states that if x and y are real numbers, then $|x| + |y| \geq |x + y|$ (where $|x|$ represents the absolute value of x, which equals x if $x \geq 0$ and equals $-x$ if $x < 0$).
6. Prove that there is a positive integer that equals the sum of the positive integers not exceeding it. Is your proof constructive or nonconstructive?
7. Prove that there are 100 consecutive positive integers that are not perfect squares. Is your proof constructive or nonconstructive?
8. Prove that either $2 \cdot 10^{500} + 15$ or $2 \cdot 10^{500} + 16$ is not a perfect square. Is your proof constructive or nonconstructive?
9. Prove that there exists a pair of consecutive integers such that one of these integers is a perfect square and the other is a perfect cube.
10. Show that the product of two of the numbers $65^{1000} - 8^{2001} + 3^{177}$, $79^{1212} - 9^{2399} + 2^{2001}$, and $24^{4493} - 5^{8192} +$

7^{1777} is nonnegative. Is your proof constructive or nonconstructive? [*Hint:* Do not try to evaluate these numbers!]

11. Prove or disprove that there is a rational number x and an irrational number y such that x^y is irrational.

12. Prove or disprove that if a and b are rational numbers, then a^b is also rational.

13. Show that each of these statements can be used to express the fact that there is a unique element x such that $P(x)$ is true. [Note that we can also write this statement as $\exists! x P(x)$.]

a) $\exists x \forall y (P(y) \leftrightarrow x = y)$
b) $\exists x P(x) \wedge \forall x \forall y (P(x) \wedge P(y) \rightarrow x = y)$
c) $\exists x (P(x) \wedge \forall y (P(y) \rightarrow x = y))$

14. Show that if a, b, and c are real numbers and $a \neq 0$, then there is a unique solution of the equation $ax + b = c$.

15. Suppose that a and b are odd integers with $a \neq b$. Show there is a unique integer c such that $|a - c| = |b - c|$.

16. Show that if r is an irrational number, there is a unique integer n such that the distance between r and n is less than $1/2$.

17. Show that if n is an odd integer, then there is a unique integer k such that n is the sum of $k - 2$ and $k + 3$.

18. Prove that given a real number x there exist unique numbers n and ϵ such that $x = n + \epsilon$, n is an integer, and $0 \leq \epsilon < 1$.

19. Prove that given a real number x there exist unique numbers n and ϵ such that $x = n - \epsilon$, n is an integer, and $0 \leq \epsilon < 1$.

20. Use forward reasoning to show that if x is a nonzero real number, then $x^2 + 1/x^2 \geq 2$. [*Hint:* Start with the inequality $(x - 1/x)^2 \geq 0$ which holds for all nonzero real numbers x.]

21. The **harmonic mean** of two real numbers x and y equals $2xy/(x + y)$. By computing the harmonic and geometric means of different pairs of positive real numbers, formulate a conjecture about their relative sizes and prove your conjecture.

22. The **quadratic mean** of two real numbers x and y equals $\sqrt{(x^2 + y^2)/2}$. By computing the arithmetic and quadratic means of different pairs of positive real numbers, formulate a conjecture about their relative sizes and prove your conjecture.

***23.** Write the numbers $1, 2, \ldots, 2n$ on a blackboard, where n is an odd integer. Pick any two of the numbers, j and k, write $|j - k|$ on the board and erase j and k. Continue this process until only one integer is written on the board. Prove that this integer must be odd.

***24.** Suppose that five ones and four zeros are arranged around a circle. Between any two equal bits you insert a 0 and between any two unequal bits you insert a 1 to produce nine new bits. Then you erase the nine original bits. Show that when you iterate this procedure, you can never get nine zeros. [*Hint:* Work backward, assuming that you did end up with nine zeros.]

25. Formulate a conjecture about the decimal digits that appear as the final decimal digit of the fourth power of an integer. Prove your conjecture using a proof by cases.

26. Formulate a conjecture about the final two decimal digits of the square of an integer. Prove your conjecture using a proof by cases.

27. Prove that there is no positive integer n such that $n^2 + n^3 = 100$.

28. Prove that there are no solutions in integers x and y to the equation $2x^2 + 5y^2 = 14$.

29. Prove that there are no solutions in positive integers x and y to the equation $x^4 + y^4 = 625$.

30. Prove that there are infinitely many solutions in positive integers x, y, and z to the equation $x^2 + y^2 = z^2$. [*Hint:* Let $x = m^2 - n^2$, $y = 2mn$, and $z = m^2 + n^2$, where m and n are integers.]

31. Adapt the proof in Example 4 in Section 1.6 to prove that if $n = abc$, where a, b, and c are positive integers, then $a \leq \sqrt[3]{n}$, $b \leq \sqrt[3]{n}$, or $c \leq \sqrt[3]{n}$.

32. Prove that $\sqrt[3]{2}$ is irrational.

33. Prove that between every two rational numbers there is an irrational number.

34. Prove that between every rational number and every irrational number there is an irrational number.

***35.** Let $S = x_1y_1 + x_2y_2 + \cdots + x_ny_n$, where $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ are orderings of two different sequences of positive real numbers, each containing n elements.

a) Show that S takes its maximum value over all orderings of the two sequences when both sequences are sorted (so that the elements in each sequence are in nondecreasing order).

b) Show that S takes its minimum value over all orderings of the two sequences when one sequence is sorted into nondecreasing order and the other is sorted into nonincreasing order.

36. Prove or disprove that if you have an 8-gallon jug of water and two empty jugs with capacities of 5 gallons and 3 gallons, respectively, then you can measure 4 gallons by successively pouring some of or all of the water in a jug into another jug.

37. Verify the $3x + 1$ conjecture for these integers.

a) 6 **b)** 7 **c)** 17 **d)** 21

38. Verify the $3x + 1$ conjecture for these integers.

a) 16 **b)** 11 **c)** 35 **d)** 113

39. Prove or disprove that you can use dominoes to tile the standard checkerboard with two adjacent corners removed (that is, corners that are not opposite).

40. Prove or disprove that you can use dominoes to tile a standard checkerboard with all four corners removed.

41. Prove that you can use dominoes to tile a rectangular checkerboard with an even number of squares.

42. Prove or disprove that you can use dominoes to tile a 5×5 checkerboard with three corners removed.

43. Use a proof by exhaustion to show that a tiling using dominoes of a 4×4 checkerboard with opposite corners removed does not exist. [*Hint:* First show that you can assume that the squares in the upper left and lower right corners are removed. Number the squares of the original checkerboard from 1 to 16, starting in the first row, moving right in this row, then starting in the leftmost square in the second row and moving right, and so on. Remove squares 1 and 16. To begin the proof, note that square 2 is covered either by a domino laid horizontally, which covers squares 2 and 3, or vertically, which covers squares 2 and 6. Consider each of these cases separately, and work through all the subcases that arise.]

***44.** Prove that when a white square and a black square are removed from an 8×8 checkerboard (colored as in the text) you can tile the remaining squares of the checkerboard using dominoes. [*Hint:* Show that when one black and one white square are removed, each part of the partition of the remaining cells formed by inserting the barriers shown in the figure can be covered by dominoes.]

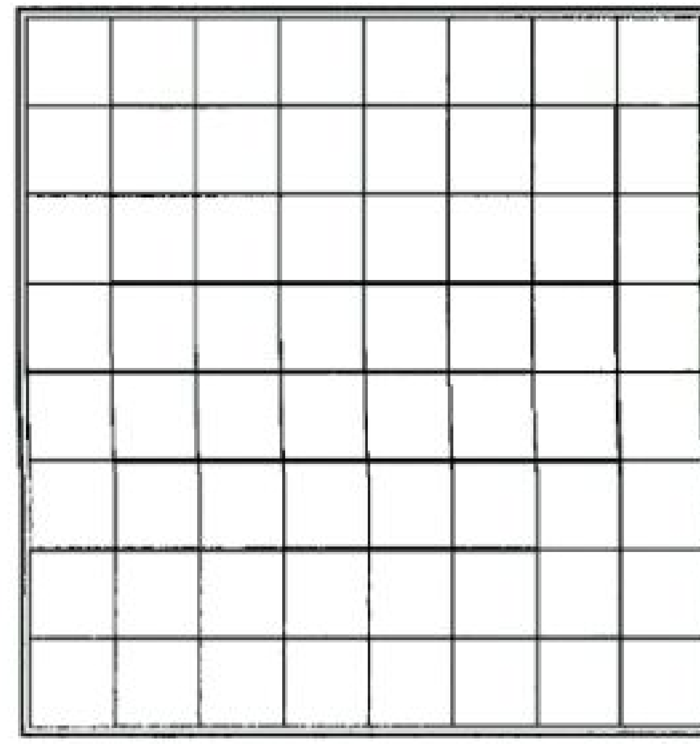

Figure for Exercise 44.

45. Show that by removing two white squares and two black squares from an 8×8 checkerboard (colored as in the text) you can make it impossible to tile the remaining squares using dominoes.

***46.** Find all squares, if they exist, on an 8×8 checkerboard so that the board obtained by removing one of these square can be tiled using straight triominoes. [*Hint:* First use arguments based on coloring and rotations to eliminate as many squares as possible from consideration.]

***47. a)** Draw each of the five different tetrominoes, where a tetromino is a polyomino consisting of four squares.

b) For each of the five different tetrominoes, prove or disprove that you can tile a standard checkerboard using these tetrominoes.

***48.** Prove or disprove that you can tile a 10×10 checkerboard using straight tetrominoes.

Key Terms and Results

TERMS

proposition: a statement that is true or false
propositional variable: a variable that represents a proposition
truth value: true or false
$\neg p$ (negation of p): the proposition with truth value opposite to the truth value of p
logical operators: operators used to combine propositions
compound proposition: a proposition constructed by combining propositions using logical operators
truth table: a table displaying the truth values of propositions
$p \vee q$ (disjunction of p and q): the proposition "p or q," which is true if and only if at least one of p and q is true
$p \wedge q$ (conjunction of p and q): the proposition "p and q" which is true if and only if both p and q are true
$p \oplus q$ (exclusive or of p and q): the proposition "p XOR q" which is true when exactly one of p and q is true
$p \rightarrow q$ (p implies q): the proposition "if p, then q," which is false if and only if p is true and q is false
converse of $p \rightarrow q$: the conditional statement $q \rightarrow p$
contrapositive of $p \rightarrow q$: the conditional statement $\neg q \rightarrow \neg p$
inverse of $p \rightarrow q$: the conditional statement $\neg p \rightarrow \neg q$
$p \leftrightarrow q$ (biconditional): the proposition "p if and only if q," which is true if and only if p and q have the same truth value
bit: either a 0 or a 1
Boolean variable: a variable that has a value of 0 or 1
bit operation: an operation on a bit or bits
bit string: a list of bits
bitwise operations: operations on bit strings that operate on each bit in one string and the corresponding bit in the other string
tautology: a compound proposition that is always true
contradiction: a compound proposition that is always false
contingency: a compound proposition that is sometimes true and sometimes false
consistent compound propositions: compound propositions for which there is an assignment of truth values to the variables that makes all these propositions true
logically equivalent compound propositions: compound propositions that always have the same truth values