BBM205 Midterm Exam I

Time: 9:00-11:00

2 November 2021

Question:	1	2	3	4	5	6	7	8	9	10	Total
Points:	20	10	10	10	10	10	10	12	8	10	110
Score:											

1. (20 points) Determine whether the following statements are true or false.

Solution:

(a) True:
A function f is bijective only if it is surjective.

(b) False:
A function f is bijective if it is injective.

(c) True:
If a function has an inverse, then it is injective.

(d) False:
Suppose that f(g(x)) is a bijection, where $f : X \to Y$, $g : Y \to Z$. Then f is bijective.

(e) True:
Let $A = \mathbb{N}$ and $B = \mathbb{Q}$. Then $A \cap B = A$. ($\mathbb{N}$ is the set of natural numbers and $\mathbb{Q}$ is the set of rational numbers.)

(f) False:
If $f = x^2 + 3x + 1$ with domain and codomain being the set of real numbers, $\mathbb{R}$, then f is injective.

(g) True:
If $f = \lfloor x^2 + 3x + 0.1 \rfloor$ with domain and codomain being the set of natural numbers, $\mathbb{N}$, then f is surjective.

(h) True:

$$\exists x \forall y P(x,y) \implies \forall y \exists x P(x,y)$$

(i) False:

$$\forall x(P(x) \vee Q(x)) \equiv \forall x P(x) \vee \forall x Q(x)$$

(j) True:

$$\exists x(P(x) \vee Q(x)) \equiv \exists x P(x) \vee \exists x Q(x)$$

2. (10 points) Prove or disprove the following:

   (a) For any integer x and y, if xy is a multiple of 10, then x is a multiple of 10 or y is a multiple of 10.

   Solution: False: Let x=2 and y=5.

   (b) The contrapositive of part (a).

   Solution: Contrapositve of part (a) is equivalent to (a). Hence, it is false by part (a).

   (c) The converse of part (a).

   Solution: The converse of part (a) states: "For any integer x and y, if x is a multiple of 10 or y is a multiple of 10, then xy is a multiple of 10." This is true. Without loss of generality, say x=10k for some integer k. Then xy = 10ky.

3. (10 points) Express the following sets as cartesian products of the sets of bits, digits and letters:

   (a) bitstrings of length n.

   Solution: Let $A_i = \{0,1\}$ for all $i = 1,2,...,n$. Then, $A_1 \times A_2 \times ... \times A_n$ represents the set of bitstrings of length n.

   (b) license plates consisting two digits followed by three letters and four digits.

   Solution: Let $A_1 = A_2 = A_3 = A_4 = A_5 = A_6 = \{0,1,...,9\}$ and $B_1 = B_2 = B_3 = \{a,b,c,...,z\}$. Then the set of the license plates is given by $A_1 \times A_2 \times B_1 \times B_2 \times B_3 \times A_3 \times A_4 \times A_5 \times A_6$.

4. (10 points) Show that if $3r^3 + 4r^2 + 5$ is irrational, then r is irrational.

Solution: Proof by contrapositive:
Let $r = a/b$ for some integers a and b. Then,

$$3r^3 + 4r^2 + 5 = 3\left(\frac{a}{b}\right)^3 + 4\left(\frac{a}{b}\right)^2 + 5 = \frac{3a^3 + 4a^2b + 5b^3}{b^3}$$

Since $3a^3 + 4a^2b + 5b^3$ and b^3 are integers, we are done.

5. (10 points) Are these system specifications consistent? Justify your answer.

"Incoming messages will not be stored in a file. Emails will be sent if and only if the socket is not disabled. If the socket is not disabled, then incoming messages will be redirected. If incoming messages are not redirected, then they will be stored in a file. If the socket is not disabled, then incoming messages will be stored in a file."

Solution: Let the propositions be:
A: "The socket is disabled"
B: "Messages will be stored in a file"
C: "Incoming messages will be redirected"
D: "The emails will be sent"

Then the given specifications are: $\neg B, D \leftrightarrow \neg A, \neg A \to C, \neg C \to B, \neg A \to B$.

The given specifications are consistent. There is only one assignment to each variable that make it consistent:

For consistency, we need to set B false in order that $\neg B$ be true. Then, we need to set both A and C be true, by the two conditional statements that have B as their consequence. (Setting the cond. statements in the form F $\to$ T, which is true.)

Finally, the bi-conditional $D \leftrightarrow \neg A$ can be satisfied by taking D to be false. Thus this set of specifications is consistent.

6. (10 points) Using the rules of inference, show that each of the following compound propositions is a tautology.

Solution:

(a) (5 points)

$$[p \wedge (p \to q)] \to q$$

$$
\begin{aligned}
&\equiv [p \wedge (\neg p \vee q)] \to q \\
&\equiv [(p \wedge \neg p) \vee (p \wedge q)] \to q \\
&\equiv [F \vee (p \wedge q)] \to q \\
&\equiv (p \wedge q) \to q \\
&\equiv \neg(p \wedge q) \vee q \\
&\equiv \neg p \vee \neg q \vee q \\
&\equiv \neg p \vee (\neg q \vee q) \\
&\equiv \neg p \vee T \\
&\equiv T
\end{aligned}
$$

(b) (5 points)

$$
\begin{aligned}
&[(p \vee q) \wedge (p \to r) \wedge (q \to r)] \to r \\
&\equiv [(p \vee q) \wedge (\neg p \vee r) \wedge (\neg q \vee r)] \to r \\
&\equiv [(p \vee q) \wedge (r \vee (\neg p \wedge \neg q))] \to r \\
&\equiv [(p \vee q) \wedge (r \vee \neg(p \vee q))] \to r
\end{aligned}
$$

(call $p \vee q$ as k)

$$
\begin{aligned}
&\equiv [k \wedge (r \vee \neg k)] \to r \\
&\equiv [(k \wedge r) \vee (k \wedge \neg k)] \to r \\
&\equiv [(k \wedge r) \vee F] \to r \\
&\equiv (k \wedge r) \to r \\
&\equiv \neg(k \wedge r) \vee r \\
&\equiv \neg k \vee \neg r \vee r \\
&\equiv \neg k \vee T \\
&\equiv T
\end{aligned}
$$

7. (10 points) Let O(x) be the statement "x is in your office", W(x) be "x is a woman", let M(x) be the statement "x is a man", and let G (x) be the statement "x wears glasses".

   Express each of these statements in terms of O(x), W(x), M(x), G(x), quantifiers, and logical connectives. Let the domain consist of all people.

   a) Someone in your office wears glasses.
   b) Not everyone in your office wears glasses.
   c) No woman in your office wears glasses.
   d) All men in your office wears glasses.

Solution: a)

$$\exists x[O(x) \wedge G(x)]$$

b)

$$\neg \forall x[O(x) \rightarrow G(x)]$$

c)

$$\forall x[(O(x) \wedge W(x)) \rightarrow \neg G(x)]$$

d)

$$\forall x[(O(x) \wedge M(x)) \rightarrow G(x)]$$

8. (12 points) Let an operation f is defined by
$f : A \rightarrow B$ s.t. $f(x) = x^2$. $A, B \in \mathbb{R}^+$ where $\mathbb{R}^+$ denotes positive real numbers.

Solution:

(a) (3 points) Is $f(x)$ a function? why?
Yes, $f(x)$ is a function, since every element in the co-domain $\mathbb{R}^+$ is covered by this function (consider $y = x^2 \in \mathbb{R}^+$ and then since we are working with positive real numbers $\forall y, \exists \sqrt{y} \in \mathbb{R}^+$)

(b) (3 points) Is $f^{-1}(x)$ a function? why?
$f(x)^{-1}$ is not a function if the codomain of the $f(x)$ would be $\mathbb{R}$ i.e., $x = 1$ and $x = -1$ yields the same element that violated the definition of functions. However, codomain of the $f(x)$ and thus the domain of the $f(x)^{-1}$ is $\mathbb{R}^+$. Therefore
$f(x)^{-1}$ is a function.

(c) (6 points) Prove that, $\forall y \in B$ there exists a unique $x \in A$ such that $f(x) = y$
proof. first, we need to consider the existence property to show the uniqueness. Thus, for each $y \in B$, let $x = \sqrt{y}$. Then, with the help of working in positive real numbers $f(x) = f(\sqrt{y}) = (\sqrt{y})^2 = y.$,

uniqueness: assume, $\exists a, b \in \mathbb{R}^+$ such that $a \neq b$. Then, $f(a) = f(b)$
$a^2 = b^2$
$|a| = |b|$ and since $a, b > 0$, we have $a = b$ and which is a contradiction. q.e.d

9. (8 points) Let $A = \{x \in \mathbb{R} \quad | \quad 0 \leq x \leq 1\}$ and $B \subseteq A$, $A \subseteq B$ and define a function f as $f : A \rightarrow B$ and define

$$p = \forall a, b, \in A \qquad \text{s.t.} \qquad f(a) = f(b) \Rightarrow a = b$$

$$q = \forall y \in B,\ \exists x \in A \qquad \text{s.t.} \qquad f(x) = y$$

Solution:

(a) (4 points) Find a function $f(x)$ that satisfies p = true , $\neg q$ = true We are asked to find a function which is injective but not bijective by definition. This condition may be satisfied by several functions. For instance, $f(x) = x/n$ where $n \in \mathbb{R}^+ + 1$. Plot the function.

(b) (4 points) Find a function $f(x)$ that satisfies $\neg p$ = true , q = true

We are asked to find a function which is surjective but not bijective by definition. This condition may be satisfied by several functions again . For instance, $f(x) = \sin(5x)$

10. (10 points) Consider the following graph of operation g.

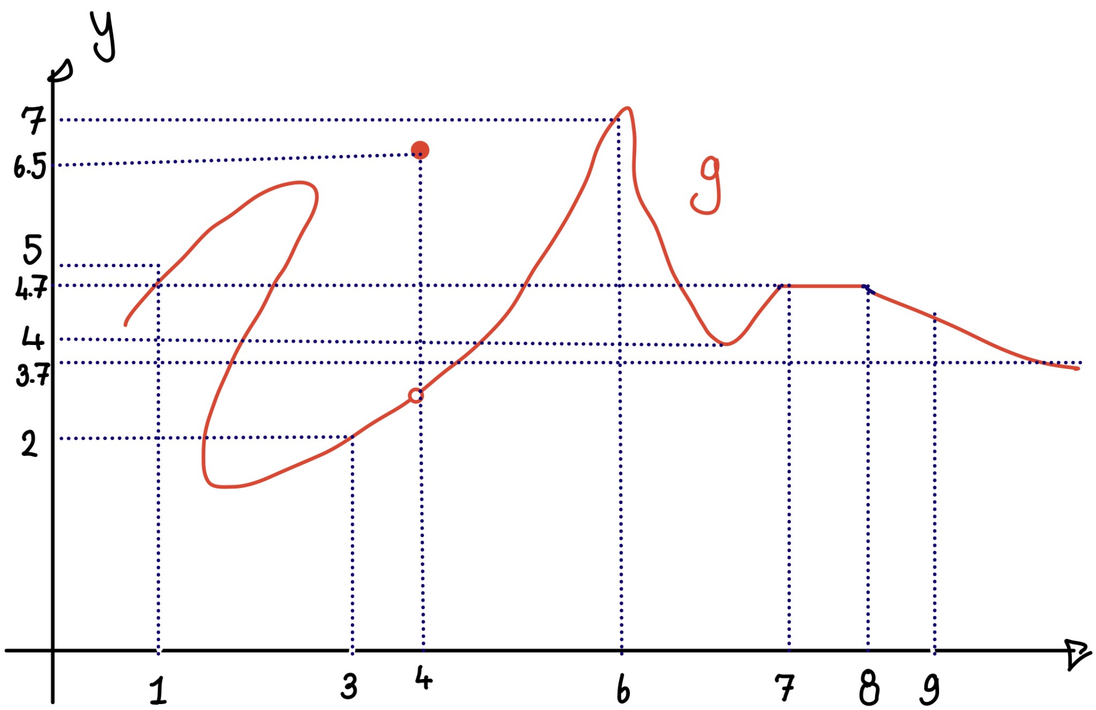

Figure 1: Caption

Solution:

(a) (5 points) Define a domain and a codomain where operation g is a function.

the domain can be taken as [3,9]
the the co-domain is [2,7]

(b) (5 points) What is the range of the function g.

for the domain that is defined by (a), the range is [2,7]