1 Banquet Arrangement

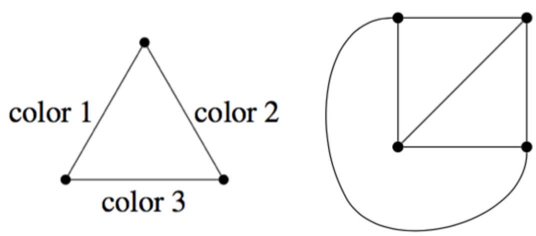
Suppose n people are attending a banquet, and each of them has at least m friends ($2 \leq m \leq n$), where friendship is mutual. Prove that we can put at least $m+1$ of the attendants on the same round table, so that each person sits next to his or her friends on both sides.

Solution: Let each person be a vertex and add an edge between two people if they are friends. Thus we have a graph with n vertices. Since each of them has at least m friends, we know that all the vertices in the graph have degree at least m. Suppose we find a cycle of length at least $m+1$ in this graph, say $C = \{v_0, v_1, \ldots, v_k\}$, where $k \geq m$. If we place these $k+1$ people at the round table in the order given by the cycle C, they observe that each person sits next to his or her friends since he/she has an edge with him/her in the corresponding graph. Thus we can rephrase the problem in graph theory terms as follows: given that all the vertices in an n-vertex graph have degree at least m, show that there exists a cycle containing at least $m+1$ vertices.

Let $P = v_0 v_1 \ldots v_l$ be a longest path in the graph. Such a path exists because the length of paths is bounded above by n. All neighbors of v_0 must be in P, since otherwise P can be extended to be even longer by appending this edge at the beginning of path P. Let k be the maximum index of neighbors of v_0 along P. Since v_0 has at least m neighbors, we must have $k \geq m$. Then $v_0 v_1 \ldots v_k v_0$ gives us the desired cycle.

2 Edge Colorings

An edge coloring of a graph is an assignment of colors to edges in a graph where any two edges incident to the same vertex have different colors. An example is shown on the left.



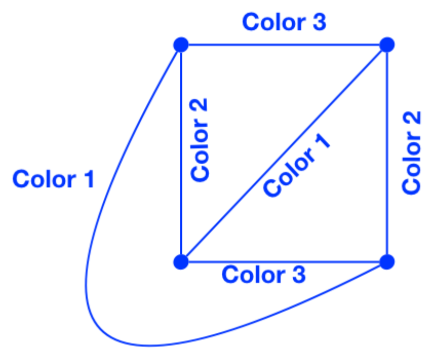
(a) Show that the 4 vertex complete graph above can be 3 edge colored. (Use the numbers 1, 2, 3 for colors. A figure is shown on the right.)

(b) Prove that any graph with maximum degree $d \geq 1$ can be edge colored with $2d-1$ colors.

(c) Show that a tree can be edge colored with d colors where d is the maximum degree of any vertex.

Solution:

(a) Three color a triangle. Now add the fourth vertex notice, call it vertex u. For any edge, say $\{u,v\}$ from this fourth vertex u, observe that the vertex v has two edges from before and hence there a third color available for the edge $\{u,v\}$.



(b) We will use induction on the number of edges n in the graph to prove the statement: If a graph G has $n \geq 0$ edges and the maximum degree of any vertex is d, then G can be colored with $2d-1$ colors.

Base case (n = 0). If there are no edges in the graph, then there is nothing to be colored and the statement holds trivially.

Inductive hypothesis. Suppose for $n = k \geq 0$, the statement holds.

Inductive step. Consider a graph G with $n = k+1$ edges. Remove an edge of your choice, say e from G. Note that in the resulting graph the maximum degree of any vertes is $d' \leq d$. By the inductive hypothesis, we can color this graph using $2d'-1$ colors and hence with $2d-1$ colors too. The removed edge is incident to two vertices each of which is incident to at most $d-1$ other edges, and thus at most $2(d-1) = 2d-2$ colors are unavailable for edge e. Thus, we can color edge e without any conflicts. This proves the statement for $n = k+1$ and hence by induction we get that the statement holds for all $n \geq 0$.

(c) We will use induction on the number of vertices n in the tree to prove the statement: For a tree with $n \geq 1$ vertices, if the maximum degree of any vertex is d, then the tree can be colored with d colors.

Base case (n=1). If there is only one vertex, then there are no edges to color, and thus can be colored with 0 colors.

Inductive hypothesis. Suppose the statement holds for $n = k \geq 1$.

Inductive Step. Remove any leaf v of your choice from the tree. We can then color the remaining tree with d colors by the inductive hypothesis. For any neighboring vertex u of vertex v, the degree of u is at most $d-1$ since we removed the edge $\{u,v\}$ along with the vertex v. Thus its incident edges use at most $d-1$ colors and there is a color available for coloring the edge {u,v}. This completes the inductive step and by induction we have that the statement holds for all $n \geq 1$.

3 Triangular Faces

Suppose we have a connected planar graph G with v vertices and e edges such that $e = 3v - 6$. Prove that in any planar drawing of G, every face must be a triangle; that is, prove that every face must be incident to exactly three edges of G.

Solution:

Suppose for the sake of contradiction that we have found a planar drawing of G such that one of the faces is incident on more than three edges. Choose an arbitrary vertex on that face to call v_0, and number the other vertices around the face $v_1, v_2, \ldots, v_k$ proceeding clockwise from v_0. Since this face has at least 4 sides, we know that v_0 and v_2 do not have an edge between them. Furthermore, we know that we can add this edge to the planar drawing of G without having it cross any existing edges by just letting it cross the face. Thus, adding an edge between v_0 and v_2 results in a planar graph with v vertices and $e + 1 = 3v - 5$ edges. But we know that a planar graph can have at most $3v - 6$ edges, so this is a contradiction. Thus, we must have that no such face exists; that is, we must have that every face in G is incident on exactly 3 edges.

4 True or False

(a) Any pair of vertices in a tree are connected by exactly one path.

(b) Adding an edge between two vertices of a tree creates a new cycle.

(c) Adding an edge in a connected graph creates exactly one new cycle.

(d) We can create a soccer ball by stitching together 10 pentagons and 20 hexagonal pieces, with three pieces meeting at each vertex.

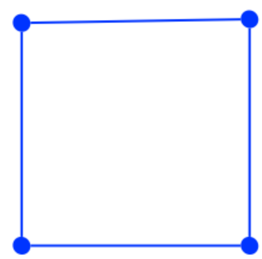
Solution:

(a) **True.**
Pick any pair of vertices x, y. We know there is a path between them since the graph is

connected. We will prove that this path is unique by contradiction: Suppose there are two distinct paths from x to y. At some point (say at vertex a) the paths must diverge, and at some point (say at vertex b) they must reconnect. So by following the first path from a to b and the second path in reverse from b to a we get a cycle. This gives the necessary contradiction.

(b) **True.**
Pick any pair of vertices x, y not connected by an edge. We prove that adding the edge $\{x,y\}$ will create a cycle. From part (a), we know that there is a unique path between x and y. Therefore, adding the edge $\{x,y\}$ creates a cycle obtained by following the path from x to y, then following the edge $\{x,y\}$ from y back to x.

(c) **False.**
In the following graph adding an edge creates two cycles.



(d) **False.**
If P pentagons and H hexagons are used, then there are $f = P+H$ faces, $v = (5P+6H)/3$ vertices, and $e = (5P+6H)/2$ edges. Since a soccer ball is a polyhedron without holes, by Euler's formula we have

$$2 = v + f - e = \frac{5P+6H}{3} + P + H - \frac{5P+6H}{2} = \frac{P}{6}.$$

Thus the number of pentagons must be 12 and not 10.