BBM 205 - Discrete Structures: Midterm 1 Solutions Date: 10.12.2019, Time: 16:00 - 17:30

Name:

Student ID:

Question:	1	2	3	4	5	6	7	8	9	Total
Points:	6	12	12	18	8	8	7	14	15	100
Score:										

1. (6 points) According to the RSA encryption system below, decryption works if for any message m, the following is true:

$$m^{de} \equiv m \mod pq$$

Use Euler's Theorem to prove that $m^{de} \equiv m \mod pq$ for all messages m relatively prime to pq. (Euler's Theorem says that if k is relatively prime to n then $k^{\Phi(n)} \equiv 1 \mod n$.)

RSA Public Key Encryption

Beforehand The receiver creates a public key and a secret key as follows.

- 1. Generate two distinct primes, p and q.
- 2. Let n = pq.
- 3. Select an integer e such that $\gcd(e,(p-1)(q-1))=1$. The *public key* is the pair (e,n). This should be distributed widely.
- 4. Compute d such that $de \equiv 1 \pmod{(p-1)(q-1)}$. The secret key is the pair (d,n). This should be kept hidden!

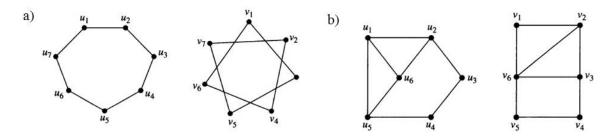
Encoding The sender encrypts message m to produce m^\prime using the public key:

$$m' = m^e \operatorname{rem} n$$
.

 $\textbf{Decoding}\;$ The receiver decrypts message m' back to message m using the secret key:

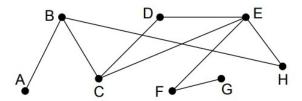
$$m = (m')^d \text{ rem } n.$$

2. (12 points) For each pair of graphs below, determine whether the pair is isomorphic or not. If yes, provide an isomorphism function. Otherwise, explain why they are not isomorphic.



3. (12 points) Use extended Euclidean algorithm to find $\gcd(9888,6060)$ and to write $\gcd(9888,6060)$ as a linear combination of 9888 and 6060.

- 4. The distance between two vertices in a graph is the length of the shortest path between them. The diameter of a graph is the distance between two vertices that are farthest apart.
 - (a) (6 points) What is the diameter of the following graph? Briefly explain your answer.



(b) (6 points) What is the chromatic number of this graph? Prove it.

(c) (6 points) Suppose that every vertex in a graph is within distance t of a vertex v. Prove that the diameter of the graph is at most 2t.

5. (8 points) Show that every planar graph has a vertex of degree at most 5.

6. (8 points) Prove that a graph is bipartite if and only if it is 2-colorable.

7. (7 points) Let $S^k = 1^k + 2^k + \cdots + (p-1)^k$, where p is an odd prime and k is a positive multiple of p-1. Use Fermat's Theorem to prove that $S^k \equiv -1 \mod p$. (Fermat's Theorem says that $a^{p-1} \equiv 1 \mod p$ if p is a prime that does not divide a.)

8.	(a) (7 points) Prove that for any number n , if the number formed by the last two digits
	of n is divisible by 4 then n is divisible by 4.

(b) (7 points) Prove that for any number n, if the sum of the digits of n are divisible by 3, then n is divisible by 3.

- 9. A tree is a connected graph with no cycles. A leaf in a tree is a vertex with degree 1.
 - (a) (8 points) Consider a tree with $n \geq 3$ vertices. What is the largest possible number of leaves the tree could have? Prove that this maximum m is possible to achieve, and further that there cannot exist a tree with more than m leaves.

(b) (7 points) Prove that every tree on $n \ge 2$ vertices must have at least two leaves.