Problem Bank 3B: (from Rosen's book)
Predicates and Quantifiers

A new predicate *teaches*(p, s), representing that professor p teaches student s, can be defined using the Prolog rule

```
teaches(P,S) :- instructor(P,C), enrolled(S,C)
```

which means that *teaches*(p, s) is true if there exists a class c such that professor p is the instructor of class c and student s is enrolled in class c. (Note that a comma is used to represent a conjunction of predicates in Prolog. Similarly, a semicolon is used to represent a disjunction of predicates.)

Prolog answers queries using the facts and rules it is given. For example, using the facts and rules listed, the query

```
?enrolled(kevin,math273)
```

produces the response

```
yes
```

because the fact *enrolled* (kevin, math273) was provided as input. The query

```
?enrolled(X,math273)
```

produces the response

```
kevin
kiko
```

To produce this response, Prolog determines all possible values of X for which *enrolled*$(X,$ math273) has been included as a Prolog fact. Similarly, to find all the professors who are instructors in classes being taken by Juana, we use the query

```
?teaches(X,juana)
```

This query returns

```
patel
grossman
```

◀

Exercises

1. Let $P(x)$ denote the statement "$x \leq 4$." What are the truth values?
   a) $P(0)$ **b)** $P(4)$ **c)** $P(6)$
2. Let $P(x)$ be the statement "the word x contains the letter a." What are the truth values?
   a) $P(\text{orange})$ **b)** $P(\text{lemon})$
   c) $P(\text{true})$ **d)** $P(\text{false})$
3. Let $Q(x, y)$ denote the statement "x is the capital of y." What are these truth values?
   a) Q(Denver, Colorado)
   b) Q(Detroit, Michigan)
   c) Q(Massachusetts, Boston)
   d) Q(New York, New York)
4. State the value of x after the statement **if** $P(x)$ **then** $x := 1$ is executed, where $P(x)$ is the statement "$x > 1$," if the value of x when this statement is reached is
   a) $x = 0$. **b)** $x = 1$.
   c) $x = 2$.
5. Let $P(x)$ be the statement "x spends more than five hours every weekday in class," where the domain for x consists of all students. Express each of these quantifications in English.

a) $\exists x P(x)$ b) $\forall x P(x)$
c) $\exists x \neg P(x)$ d) $\forall x \neg P(x)$

6. Let $N(x)$ be the statement "x has visited North Dakota," where the domain consists of the students in your school. Express each of these quantifications in English.

a) $\exists x N(x)$ b) $\forall x N(x)$ c) $\neg\exists x N(x)$
d) $\exists x \neg N(x)$ e) $\neg\forall x N(x)$ f) $\forall x \neg N(x)$

7. Translate these statements into English, where $C(x)$ is "x is a comedian" and $F(x)$ is "x is funny" and the domain consists of all people.

a) $\forall x(C(x) \to F(x))$ b) $\forall x(C(x) \wedge F(x))$
c) $\exists x(C(x) \to F(x))$ d) $\exists x(C(x) \wedge F(x))$

8. Translate these statements into English, where $R(x)$ is "x is a rabbit" and $H(x)$ is "x hops" and the domain consists of all animals.

a) $\forall x(R(x) \to H(x))$ b) $\forall x(R(x) \wedge H(x))$
c) $\exists x(R(x) \to H(x))$ d) $\exists x(R(x) \wedge H(x))$

9. Let $P(x)$ be the statement "x can speak Russian" and let $Q(x)$ be the statement "x knows the computer language C++." Express each of these sentences in terms of $P(x)$, $Q(x)$, quantifiers, and logical connectives. The domain for quantifiers consists of all students at your school.

a) There is a student at your school who can speak Russian and who knows C++.
b) There is a student at your school who can speak Russian but who doesn't know C++.
c) Every student at your school either can speak Russian or knows C++.
d) No student at your school can speak Russian or knows C++.

10. Let $C(x)$ be the statement "x has a cat," let $D(x)$ be the statement "x has a dog," and let $F(x)$ be the statement "x has a ferret." Express each of these statements in terms of $C(x)$, $D(x)$, $F(x)$, quantifiers, and logical connectives. Let the domain consist of all students in your class.

a) A student in your class has a cat, a dog, and a ferret.
b) All students in your class have a cat, a dog, or a ferret.
c) Some student in your class has a cat and a ferret, but not a dog.
d) No student in your class has a cat, a dog, and a ferret.
e) For each of the three animals, cats, dogs, and ferrets, there is a student in your class who has one of these animals as a pet.

11. Let $P(x)$ be the statement "$x = x^2$." If the domain consists of the integers, what are the truth values?

a) $P(0)$ b) $P(1)$ c) $P(2)$
d) $P(-1)$ e) $\exists x P(x)$ f) $\forall x P(x)$

12. Let $Q(x)$ be the statement "$x + 1 > 2x$." If the domain consists of all integers, what are these truth values?

a) $Q(0)$ b) $Q(-1)$ c) $Q(1)$
d) $\exists x Q(x)$ e) $\forall x Q(x)$ f) $\exists x \neg Q(x)$
g) $\forall x \neg Q(x)$

13. Determine the truth value of each of these statements if the domain consists of all integers.

a) $\forall n(n + 1 > n)$ b) $\exists n(2n = 3n)$
c) $\exists n(n = -n)$ d) $\forall n(n^2 \geq n)$

14. Determine the truth value of each of these statements if the domain consists of all real numbers.

a) $\exists x(x^3 = -1)$ b) $\exists x(x^4 < x^2)$
c) $\forall x((-x)^2 = x^2)$ d) $\forall x(2x > x)$

15. Determine the truth value of each of these statements if the domain for all variables consists of all integers.

a) $\forall n(n^2 \geq 0)$ b) $\exists n(n^2 = 2)$
c) $\forall n(n^2 \geq n)$ d) $\exists n(n^2 < 0)$

16. Determine the truth value of each of these statements if the domain of each variable consists of all real numbers.

a) $\exists x(x^2 = 2)$ b) $\exists x(x^2 = -1)$
c) $\forall x(x^2 + 2 \geq 1)$ d) $\forall x(x^2 \neq x)$

17. Suppose that the domain of the propositional function $P(x)$ consists of the integers 0, 1, 2, 3, and 4. Write out each of these propositions using disjunctions, conjunctions, and negations.

a) $\exists x P(x)$ b) $\forall x P(x)$ c) $\exists x \neg P(x)$
d) $\forall x \neg P(x)$ e) $\neg\exists x P(x)$ f) $\neg\forall x P(x)$

18. Suppose that the domain of the propositional function $P(x)$ consists of the integers -2, -1, 0, 1, and 2. Write out each of these propositions using disjunctions, conjunctions, and negations.

a) $\exists x P(x)$ b) $\forall x P(x)$ c) $\exists x \neg P(x)$
d) $\forall x \neg P(x)$ e) $\neg\exists x P(x)$ f) $\neg\forall x P(x)$

19. Suppose that the domain of the propositional function $P(x)$ consists of the integers 1, 2, 3, 4, and 5. Express these statements without using quantifiers, instead using only negations, disjunctions, and conjunctions.

a) $\exists x P(x)$ b) $\forall x P(x)$
c) $\neg\exists x P(x)$ d) $\neg\forall x P(x)$
e) $\forall x((x \neq 3) \to P(x)) \vee \exists x \neg P(x)$

20. Suppose that the domain of the propositional function $P(x)$ consists of -5, -3, -1, 1, 3, and 5. Express these statements without using quantifiers, instead using only negations, disjunctions, and conjunctions.

a) $\exists x P(x)$ b) $\forall x P(x)$
c) $\forall x((x \neq 1) \to P(x))$
d) $\exists x((x \geq 0) \wedge P(x))$
e) $\exists x(\neg P(x)) \wedge \forall x((x < 0) \to P(x))$

21. For each of these statements find a domain for which the statement is true and a domain for which the statement is false.

a) Everyone is studying discrete mathematics.
b) Everyone is older than 21 years.
c) Every two people have the same mother.
d) No two different people have the same grandmother.

22. For each of these statements find a domain for which the statement is true and a domain for which the statement is false.

a) Everyone speaks Hindi.
b) There is someone older than 21 years.

c) Every two people have the same first name.
d) Someone knows more than two other people.

23. Translate in two ways each of these statements into logical expressions using predicates, quantifiers, and logical connectives. First, let the domain consist of the students in your class and second, let it consist of all people.
a) Someone in your class can speak Hindi.
b) Everyone in your class is friendly.
c) There is a person in your class who was not born in California.
d) A student in your class has been in a movie.
e) No student in your class has taken a course in logic programming.

24. Translate in two ways each of these statements into logical expressions using predicates, quantifiers, and logical connectives. First, let the domain consist of the students in your class and second, let it consist of all people.
a) Everyone in your class has a cellular phone.
b) Somebody in your class has seen a foreign movie.
c) There is a person in your class who cannot swim.
d) All students in your class can solve quadratic equations.
e) Some student in your class does not want to be rich.

25. Translate each of these statements into logical expressions using predicates, quantifiers, and logical connectives.
a) No one is perfect.
b) Not everyone is perfect.
c) All your friends are perfect.
d) At least one of your friends is perfect.
e) Everyone is your friend and is perfect.
f) Not everybody is your friend or someone is not perfect.

26. Translate each of these statements into logical expressions in three different ways by varying the domain and by using predicates with one and with two variables.
a) Someone in your school has visited Uzbekistan.
b) Everyone in your class has studied calculus and C++.
c) No one in your school owns both a bicycle and a motorcycle.
d) There is a person in your school who is not happy.
e) Everyone in your school was born in the twentieth century.

27. Translate each of these statements into logical expressions in three different ways by varying the domain and by using predicates with one and with two variables.
a) A student in your school has lived in Vietnam.
b) There is a student in your school who cannot speak Hindi.
c) A student in your school knows Java, Prolog, and C++.
d) Everyone in your class enjoys Thai food.
e) Someone in your class does not play hockey.

28. Translate each of these statements into logical expressions using predicates, quantifiers, and logical connectives.
a) Something is not in the correct place.
b) All tools are in the correct place and are in excellent condition.
c) Everything is in the correct place and in excellent condition.
d) Nothing is in the correct place and is in excellent condition.
e) One of your tools is not in the correct place, but it is in excellent condition.

29. Express each of these statements using logical operators, predicates, and quantifiers.
a) Some propositions are tautologies.
b) The negation of a contradiction is a tautology.
c) The disjunction of two contingencies can be a tautology.
d) The conjunction of two tautologies is a tautology.

30. Suppose the domain of the propositional function $P(x, y)$ consists of pairs x and y, where x is 1, 2, or 3 and y is 1, 2, or 3. Write out these propositions using disjunctions and conjunctions.
a) $\exists x\, P(x, 3)$
b) $\forall y\, P(1, y)$
c) $\exists y \neg P(2, y)$
d) $\forall x \neg P(x, 2)$

31. Suppose that the domain of $Q(x, y, z)$ consists of triples x, y, z, where $x = 0, 1$, or 2, $y = 0$ or 1, and $z = 0$ or 1. Write out these propositions using disjunctions and conjunctions.
a) $\forall y Q(0, y, 0)$
b) $\exists x Q(x, 1, 1)$
c) $\exists z \neg Q(0, 0, z)$
d) $\exists x \neg Q(x, 0, 1)$

32. Express each of these statements using quantifiers. Then form the negation of the statement so that no negation is to the left of a quantifier. Next, express the negation in simple English. (Do not simply use the words "It is not the case that.")
a) All dogs have fleas.
b) There is a horse that can add.
c) Every koala can climb.
d) No monkey can speak French.
e) There exists a pig that can swim and catch fish.

33. Express each of these statements using quantifiers. Then form the negation of the statement, so that no negation is to the left of a quantifier. Next, express the negation in simple English. (Do not simply use the words "It is not the case that.")
a) Some old dogs can learn new tricks.
b) No rabbit knows calculus.
c) Every bird can fly.
d) There is no dog that can talk.
e) There is no one in this class who knows French and Russian.

34. Express the negation of these propositions using quantifiers, and then express the negation in English.
a) Some drivers do not obey the speed limit.
b) All Swedish movies are serious.
c) No one can keep a secret.
d) There is someone in this class who does not have a good attitude.

35. Find a counterexample, if possible, to these universally quantified statements, where the domain for all variables consists of all integers.
   a) $\forall x(x^2 \geq x)$
   b) $\forall x(x > 0 \vee x < 0)$
   c) $\forall x(x = 1)$

36. Find a counterexample, if possible, to these universally quantified statements, where the domain for all variables consists of all real numbers.
   a) $\forall x(x^2 \neq x)$
   b) $\forall x(x^2 \neq 2)$
   c) $\forall x(|x| > 0)$

37. Express each of these statements using predicates and quantifiers.
   a) A passenger on an airline qualifies as an elite flyer if the passenger flies more than 25,000 miles in a year or takes more than 25 flights during that year.
   b) A man qualifies for the marathon if his best previous time is less than 3 hours and a woman qualifies for the marathon if her best previous time is less than 3.5 hours.
   c) A student must take at least 60 course hours, or at least 45 course hours and write a master's thesis, and receive a grade no lower than a B in all required courses, to receive a master's degree.
   d) There is a student who has taken more than 21 credit hours in a semester and received all A's.

Exercises 38–42 deal with the translation between system specification and logical expressions involving quantifiers.

38. Translate these system specifications into English where the predicate $S(x, y)$ is "x is in state y" and where the domain for x and y consists of all systems and all possible states, respectively.
   a) $\exists x S(x, \text{open})$
   b) $\forall x(S(x, \text{malfunctioning}) \vee S(x, \text{diagnostic}))$
   c) $\exists x S(x, \text{open}) \vee \exists x S(x, \text{diagnostic})$
   d) $\exists x \neg S(x, \text{available})$
   e) $\forall x \neg S(x, \text{working})$

39. Translate these specifications into English where $F(p)$ is "Printer p is out of service," $B(p)$ is "Printer p is busy," $L(j)$ is "Print job j is lost," and $Q(j)$ is "Print job j is queued."
   a) $\exists p(F(p) \wedge B(p)) \rightarrow \exists j L(j)$
   b) $\forall p B(p) \rightarrow \exists j Q(j)$
   c) $\exists j(Q(j) \wedge L(j)) \rightarrow \exists p F(p)$
   d) $(\forall p B(p) \wedge \forall j Q(j)) \rightarrow \exists j L(j)$

40. Express each of these system specifications using predicates, quantifiers, and logical connectives.
   a) When there is less than 30 megabytes free on the hard disk, a warning message is sent to all users.
   b) No directories in the file system can be opened and no files can be closed when system errors have been detected.
   c) The file system cannot be backed up if there is a user currently logged on.
   d) Video on demand can be delivered when there are at least 8 megabytes of memory available and the connection speed is at least 56 kilobits per second.

41. Express each of these system specifications using predicates, quantifiers, and logical connectives.
   a) At least one mail message, among the nonempty set of messages, can be saved if there is a disk with more than 10 kilobytes of free space.
   b) Whenever there is an active alert, all queued messages are transmitted.
   c) The diagnostic monitor tracks the status of all systems except the main console.
   d) Each participant on the conference call whom the host of the call did not put on a special list was billed.

42. Express each of these system specifications using predicates, quantifiers, and logical connectives.
   a) Every user has access to an electronic mailbox.
   b) The system mailbox can be accessed by everyone in the group if the file system is locked.
   c) The firewall is in a diagnostic state only if the proxy server is in a diagnostic state.
   d) At least one router is functioning normally if the throughput is between 100 kbps and 500 kbps and the proxy server is not in diagnostic mode.

43. Determine whether $\forall x(P(x) \rightarrow Q(x))$ and $\forall x P(x) \rightarrow \forall x Q(x)$ are logically equivalent. Justify your answer.

44. Determine whether $\forall x(P(x) \leftrightarrow Q(x))$ and $\forall x \; P(x) \leftrightarrow \forall x Q(x)$ are logically equivalent. Justify your answer.

45. Show that $\exists x(P(x) \vee Q(x))$ and $\exists x P(x) \vee \exists x Q(x)$ are logically equivalent.

Exercises 46–49 establish rules for **null quantification** that we can use when a quantified variable does not appear in part of a statement.

46. Establish these logical equivalences, where x does not occur as a free variable in A. Assume that the domain is nonempty.
   a) $(\forall x P(x)) \vee A \equiv \forall x(P(x) \vee A)$
   b) $(\exists x P(x)) \vee A \equiv \exists x(P(x) \vee A)$

47. Establish these logical equivalences, where x does not occur as a free variable in A. Assume that the domain is nonempty.
   a) $(\forall x P(x)) \wedge A \equiv \forall x(P(x) \wedge A)$
   b) $(\exists x P(x)) \wedge A \equiv \exists x(P(x) \wedge A)$

48. Establish these logical equivalences, where x does not occur as a free variable in A. Assume that the domain is nonempty.
   a) $\forall x(A \rightarrow P(x)) \equiv A \rightarrow \forall x P(x)$
   b) $\exists x(A \rightarrow P(x)) \equiv A \rightarrow \exists x P(x)$

49. Establish these logical equivalences, where x does not occur as a free variable in A. Assume that the domain is nonempty.
   a) $\forall x(P(x) \rightarrow A) \equiv \exists x P(x) \rightarrow A$
   b) $\exists x(P(x) \rightarrow A) \equiv \forall x P(x) \rightarrow A$

50. Show that $\forall x P(x) \vee \forall x Q(x)$ and $\forall x(P(x) \vee Q(x))$ are not logically equivalent.

51. Show that $\exists x P(x) \wedge \exists x Q(x)$ and $\exists x(P(x) \wedge Q(x))$ are not logically equivalent.

52. As mentioned in the text, the notation $\exists! x P(x)$ denotes

"There exists a unique x such that $P(x)$ is true."

If the domain consists of all integers, what are the truth values of these statements?

a) $\exists! x(x > 1)$ **b)** $\exists! x(x^2 = 1)$
c) $\exists! x(x + 3 = 2x)$ **d)** $\exists! x(x = x + 1)$

53. What are the truth values of these statements?

a) $\exists! x P(x) \rightarrow \exists x P(x)$
b) $\forall x P(x) \rightarrow \exists! x P(x)$
c) $\exists! x \neg P(x) \rightarrow \neg\forall x P(x)$

54. Write out $\exists! x P(x)$, where the domain consists of the integers 1, 2, and 3, in terms of negations, conjunctions, and disjunctions.

55. Given the Prolog facts in Example 28, what would Prolog return given these queries?

a) `?instructor(chan,math273)`
b) `?instructor(patel,cs301)`
c) `?enrolled(X,cs301)`
d) `?enrolled(kiko,Y)`
e) `?teaches(grossman,Y)`

56. Given the Prolog facts in Example 28, what would Prolog return when given these queries?

a) `?enrolled(kevin,ee222)`
b) `?enrolled(kiko,math273)`
c) `?instructor(grossman,X)`
d) `?instructor(X,cs301)`
e) `?teaches(X,kevin)`

57. Suppose that Prolog facts are used to define the predicates *mother*(M, Y) and *father*(F, X), which represent that M is the mother of Y and F is the father of X, respectively. Give a Prolog rule to define the predicate *sibling*(X, Y), which represents that X and Y are siblings (that is, have the same mother and the same father).

58. Suppose that Prolog facts are used to define the predicates *mother*(M, Y) and *father*(F, X), which represent that M is the mother of Y and F is the father of X, respectively. Give a Prolog rule to define the predicate *grandfather*(X, Y), which represents that X is the grandfather of Y. [*Hint:* You can write a disjunction in Prolog either by using a semicolon to separate predicates or by putting these predicates on separate lines.]

Exercises 59–62 are based on questions found in the book *Symbolic Logic* by Lewis Carroll.

59. Let $P(x)$, $Q(x)$, and $R(x)$ be the statements "x is a professor," "x is ignorant," and "x is vain," respectively. Express each of these statements using quantifiers; logical connectives; and $P(x)$, $Q(x)$, and $R(x)$, where the domain consists of all people.

a) No professors are ignorant.
b) All ignorant people are vain.
c) No professors are vain.
d) Does (c) follow from (a) and (b)?

60. Let $P(x)$, $Q(x)$, and $R(x)$ be the statements "x is a clear explanation," "x is satisfactory," and "x is an excuse," respectively. Suppose that the domain for x consists of all English text. Express each of these statements using quantifiers, logical connectives, and $P(x)$, $Q(x)$, and $R(x)$.

a) All clear explanations are satisfactory.
b) Some excuses are unsatisfactory.
c) Some excuses are not clear explanations.
***d)** Does (c) follow from (a) and (b)?

61. Let $P(x)$, $Q(x)$, $R(x)$, and $S(x)$ be the statements "x is a baby," "x is logical," "x is able to manage a crocodile," and "x is despised," respectively. Suppose that the domain consists of all people. Express each of these statements using quantifiers; logical connectives; and $P(x)$, $Q(x)$, $R(x)$, and $S(x)$.

a) Babies are illogical.
b) Nobody is despised who can manage a crocodile.
c) Illogical persons are despised.
d) Babies cannot manage crocodiles.
***e)** Does (d) follow from (a), (b), and (c)? If not, is there a correct conclusion?

62. Let $P(x)$, $Q(x)$, $R(x)$, and $S(x)$ be the statements "x is a duck," "x is one of my poultry," "x is an officer," and "x is willing to waltz," respectively. Express each of these statements using quantifiers; logical connectives; and $P(x)$, $Q(x)$, $R(x)$, and $S(x)$.

a) No ducks are willing to waltz.
b) No officers ever decline to waltz.
c) All my poultry are ducks.
d) My poultry are not officers.
***e)** Does (d) follow from (a), (b), and (c)? If not, is there a correct conclusion?

1.4 Nested Quantifiers

Introduction

In Section 1.3 we defined the existential and universal quantifiers and showed how they can be used to represent mathematical statements. We also explained how they can be used to translate

Successively applying the rules for negating quantified expressions, we construct this sequence of equivalent statements

$$\neg\forall\epsilon > 0\ \exists\delta > 0\ \forall x(0 < |x - a| < \delta \to |f(x) - L| < \epsilon)$$
$$\equiv \exists\epsilon > 0\ \neg\exists\delta > 0\ \forall x(0 < |x - a| < \delta \to |f(x) - L| < \epsilon)$$
$$\equiv \exists\epsilon > 0\ \forall\delta > 0\ \neg\forall x(0 < |x - a| < \delta \to |f(x) - L| < \epsilon)$$
$$\equiv \exists\epsilon > 0\ \forall\delta > 0\ \exists x\ \neg(0 < |x - a| < \delta \to |f(x) - L| < \epsilon)$$
$$\equiv \exists\epsilon > 0\ \forall\delta > 0\ \exists x(0 < |x - a| < \delta \wedge |f(x) - L| \geq \epsilon).$$

In the last step we used the equivalence $\neg(p \to q) \equiv p \wedge \neg q$, which follows from the fifth equivalence in Table 7 of Section 1.2.

Because the statement "$\lim_{x\to a} f(x)$ does not exist" means for all real numbers L, $\lim_{x\to a} f(x) \neq L$, this can be expressed as

$$\forall L \exists\epsilon > 0\ \forall\delta > 0\ \exists x(0 < |x - a| < \delta \wedge |f(x) - L| \geq \epsilon).$$

This last statement says that for every real number L there is a real number $\epsilon > 0$ such that for every real number $\delta > 0$, there exists a real number x such that $0 < |x - a| < \delta$ and $|f(x) - L| \geq \epsilon$. ◀

Exercises

1. Translate these statements into English, where the domain for each variable consists of all real numbers.
   a) $\forall x \exists y(x < y)$
   b) $\forall x \forall y(((x \geq 0) \wedge (y \geq 0)) \to (xy \geq 0))$
   c) $\forall x \forall y \exists z(xy = z)$

2. Translate these statements into English, where the domain for each variable consists of all real numbers.
   a) $\exists x \forall y(xy = y)$
   b) $\forall x \forall y(((x \geq 0) \wedge (y < 0)) \to (x - y > 0))$
   c) $\forall x \forall y \exists z(x = y + z)$

3. Let $Q(x, y)$ be the statement "x has sent an e-mail message to y," where the domain for both x and y consists of all students in your class. Express each of these quantifications in English.
   a) $\exists x \exists y Q(x, y)$
   b) $\exists x \forall y Q(x, y)$
   c) $\forall x \exists y Q(x, y)$
   d) $\exists y \forall x Q(x, y)$
   e) $\forall y \exists x Q(x, y)$
   f) $\forall x \forall y Q(x, y)$

4. Let $P(x, y)$ be the statement "student x has taken class y," where the domain for x consists of all students in your class and for y consists of all computer science courses at your school. Express each of these quantifications in English.
   a) $\exists x \exists y P(x, y)$
   b) $\exists x \forall y P(x, y)$
   c) $\forall x \exists y P(x, y)$
   d) $\exists y \forall x P(x, y)$
   e) $\forall y \exists x P(x, y)$
   f) $\forall x \forall y P(x, y)$

5. Let $W(x, y)$ mean that student x has visited website y, where the domain for x consists of all students in your school and the domain for y consists of all websites. Express each of these statements by a simple English sentence.
   a) W(Sarah Smith, www.att.com)
   b) $\exists x W(x$, www.imdb.org)
   c) $\exists y W$(Jose Orez, y)
   d) $\exists y(W$(Ashok Puri, $y) \wedge W$(Cindy Yoon, y))
   e) $\exists y \forall z(y \neq$ (David Belcher) $\wedge$ (W(David Belcher, z) $\to W(y, z)$))
   f) $\exists x \exists y \forall z((x \neq y) \wedge (W(x, z) \leftrightarrow W(y, z)))$

6. Let $C(x, y)$ mean that student x is enrolled in class y, where the domain for x consists of all students in your school and the domain for y consists of all classes being given at your school. Express each of these statements by a simple English sentence.
   a) C(Randy Goldberg, CS 252)
   b) $\exists x C(x$, Math 695)
   c) $\exists y C$(Carol Sitea, y)
   d) $\exists x(C(x$, Math 222) $\wedge C(x$, CS 252))
   e) $\exists x \exists y \forall z((x \neq y) \wedge (C(x, z) \to C(y, z)))$
   f) $\exists x \exists y \forall z((x \neq y) \wedge (C(x, z) \leftrightarrow C(y, z)))$

7. Let $T(x, y)$ mean that student x likes cuisine y, where the domain for x consists of all students at your school and the domain for y consists of all cuisines. Express each of these statements by a simple English sentence.
   a) $\neg T$(Abdallah Hussein, Japanese)
   b) $\exists x T(x$, Korean) $\wedge \forall x T(x$, Mexican)

c) $\exists y(T(\text{Monique Arsenault}, y) \vee T(\text{Jay Johnson}, y))$
d) $\forall x \forall z \exists y((x \neq z) \rightarrow \neg(T(x, y) \wedge T(z, y)))$
e) $\exists x \exists z \forall y(T(x, y) \leftrightarrow T(z, y))$
f) $\forall x \forall z \exists y(T(x, y) \leftrightarrow T(z, y))$

8. Let $Q(x, y)$ be the statement "student x has been a contestant on quiz show y." Express each of these sentences in terms of $Q(x, y)$, quantifiers, and logical connectives, where the domain for x consists of all students at your school and for y consists of all quiz shows on television.
   a) There is a student at your school who has been a contestant on a television quiz show.
   b) No student at your school has ever been a contestant on a television quiz show.
   c) There is a student at your school who has been a contestant on *Jeopardy* and on *Wheel of Fortune*.
   d) Every television quiz show has had a student from your school as a contestant.
   e) At least two students from your school have been contestants on *Jeopardy*.

9. Let $L(x, y)$ be the statement "x loves y," where the domain for both x and y consists of all people in the world. Use quantifiers to express each of these statements.
   a) Everybody loves Jerry.
   b) Everybody loves somebody.
   c) There is somebody whom everybody loves.
   d) Nobody loves everybody.
   e) There is somebody whom Lydia does not love.
   f) There is somebody whom no one loves.
   g) There is exactly one person whom everybody loves.
   h) There are exactly two people whom Lynn loves.
   i) Everyone loves himself or herself.
   j) There is someone who loves no one besides himself or herself.

10. Let $F(x, y)$ be the statement "x can fool y," where the domain consists of all people in the world. Use quantifiers to express each of these statements.
   a) Everybody can fool Fred.
   b) Evelyn can fool everybody.
   c) Everybody can fool somebody.
   d) There is no one who can fool everybody.
   e) Everyone can be fooled by somebody.
   f) No one can fool both Fred and Jerry.
   g) Nancy can fool exactly two people.
   h) There is exactly one person whom everybody can fool.
   i) No one can fool himself or herself.
   j) There is someone who can fool exactly one person besides himself or herself.

11. Let $S(x)$ be the predicate "x is a student," $F(x)$ the predicate "x is a faculty member," and $A(x, y)$ the predicate "x has asked y a question," where the domain consists of all people associated with your school. Use quantifiers to express each of these statements.
   a) Lois has asked Professor Michaels a question.
   b) Every student has asked Professor Gross a question.
   c) Every faculty member has either asked Professor Miller a question or been asked a question by Professor Miller.
   d) Some student has not asked any faculty member a question.
   e) There is a faculty member who has never been asked a question by a student.
   f) Some student has asked every faculty member a question.
   g) There is a faculty member who has asked every other faculty member a question.
   h) Some student has never been asked a question by a faculty member.

12. Let $I(x)$ be the statement "x has an Internet connection" and $C(x, y)$ be the statement "x and y have chatted over the Internet," where the domain for the variables x and y consists of all students in your class. Use quantifiers to express each of these statements.
   a) Jerry does not have an Internet connection.
   b) Rachel has not chatted over the Internet with Chelsea.
   c) Jan and Sharon have never chatted over the Internet.
   d) No one in the class has chatted with Bob.
   e) Sanjay has chatted with everyone except Joseph.
   f) Someone in your class does not have an Internet connection.
   g) Not everyone in your class has an Internet connection.
   h) Exactly one student in your class has an Internet connection.
   i) Everyone except one student in your class has an Internet connection.
   j) Everyone in your class with an Internet connection has chatted over the Internet with at least one other student in your class.
   k) Someone in your class has an Internet connection but has not chatted with anyone else in your class.
   l) There are two students in your class who have not chatted with each other over the Internet.
   m) There is a student in your class who has chatted with everyone in your class over the Internet.
   n) There are at least two students in your class who have not chatted with the same person in your class.
   o) There are two students in the class who between them have chatted with everyone else in the class.

13. Let $M(x, y)$ be "x has sent y an e-mail message" and $T(x, y)$ be "x has telephoned y," where the domain consists of all students in your class. Use quantifiers to express each of these statements. (Assume that all e-mail messages that were sent are received, which is not the way things often work.)
   a) Chou has never sent an e-mail message to Koko.
   b) Arlene has never sent an e-mail message to or telephoned Sarah.
   c) Jose has never received an e-mail message from Deborah.
   d) Every student in your class has sent an e-mail message to Ken.

e) No one in your class has telephoned Nina.
f) Everyone in your class has either telephoned Avi or sent him an e-mail message.
g) There is a student in your class who has sent everyone else in your class an e-mail message.
h) There is someone in your class who has either sent an e-mail message or telephoned everyone else in your class.
i) There are two different students in your class who have sent each other e-mail messages.
j) There is a student who has sent himself or herself an e-mail message.
k) There is a student in your class who has not received an e-mail message from anyone else in the class and who has not been called by any other student in the class.
l) Every student in the class has either received an e-mail message or received a telephone call from another student in the class.
m) There are at least two students in your class such that one student has sent the other e-mail and the second student has telephoned the first student.
n) There are two different students in your class who between them have sent an e-mail message to or telephoned everyone else in the class.

14. Use quantifiers and predicates with more than one variable to express these statements.
a) There is a student in this class who can speak Hindi.
b) Every student in this class plays some sport.
c) Some student in this class has visited Alaska but has not visited Hawaii.
d) All students in this class have learned at least one programming language.
e) There is a student in this class who has taken every course offered by one of the departments in this school.
f) Some student in this class grew up in the same town as exactly one other student in this class.
g) Every student in this class has chatted with at least one other student in at least one chat group.

15. Use quantifiers and predicates with more than one variable to express these statements.
a) Every computer science student needs a course in discrete mathematics.
b) There is a student in this class who owns a personal computer.
c) Every student in this class has taken at least one computer science course.
d) There is a student in this class who has taken at least one course in computer science.
e) Every student in this class has been in every building on campus.
f) There is a student in this class who has been in every room of at least one building on campus.
g) Every student in this class has been in at least one room of every building on campus.

16. A discrete mathematics class contains 1 mathematics major who is a freshman, 12 mathematics majors who are sophomores, 15 computer science majors who are sophomores, 2 mathematics majors who are juniors, 2 computer science majors who are juniors, and 1 computer science major who is a senior. Express each of these statements in terms of quantifiers and then determine its truth value.
a) There is a student in the class who is a junior.
b) Every student in the class is a computer science major.
c) There is a student in the class who is neither a mathematics major nor a junior.
d) Every student in the class is either a sophomore or a computer science major.
e) There is a major such that there is a student in the class in every year of study with that major.

17. Express each of these system specifications using predicates, quantifiers, and logical connectives, if necessary.
a) Every user has access to exactly one mailbox.
b) There is a process that continues to run during all error conditions only if the kernel is working correctly.
c) All users on the campus network can access all websites whose url has a .edu extension.
*d) There are exactly two systems that monitor every remote server.

18. Express each of these system specifications using predicates, quantifiers, and logical connectives, if necessary.
a) At least one console must be accessible during every fault condition.
b) The e-mail address of every user can be retrieved whenever the archive contains at least one message sent by every user on the system.
c) For every security breach there is at least one mechanism that can detect that breach if and only if there is a process that has not been compromised.
d) There are at least two paths connecting every two distinct endpoints on the network.
e) No one knows the password of every user on the system except for the system administrator, who knows all passwords.

19. Express each of these statements using mathematical and logical operators, predicates, and quantifiers, where the domain consists of all integers.
a) The sum of two negative integers is negative.
b) The difference of two positive integers is not necessarily positive.
c) The sum of the squares of two integers is greater than or equal to the square of their sum.
d) The absolute value of the product of two integers is the product of their absolute values.

20. Express each of these statements using predicates, quantifiers, logical connectives, and mathematical operators where the domain consists of all integers.
a) The product of two negative integers is positive.
b) The average of two positive integers is positive.

c) The difference of two negative integers is not necessarily negative.
d) The absolute value of the sum of two integers does not exceed the sum of the absolute values of these integers.

21. Use predicates, quantifiers, logical connectives, and mathematical operators to express the statement that every positive integer is the sum of the squares of four integers.

22. Use predicates, quantifiers, logical connectives, and mathematical operators to express the statement that there is a positive integer that is not the sum of three squares.

23. Express each of these mathematical statements using predicates, quantifiers, logical connectives, and mathematical operators.

a) The product of two negative real numbers is positive.
b) The difference of a real number and itself is zero.
c) Every positive real number has exactly two square roots.
d) A negative real number does not have a square root that is a real number.

24. Translate each of these nested quantifications into an English statement that expresses a mathematical fact. The domain in each case consists of all real numbers.

a) $\exists x \forall y(x + y = y)$
b) $\forall x \forall y(((x \geq 0) \wedge (y < 0)) \rightarrow (x - y > 0))$
c) $\exists x \exists y(((x \leq 0) \wedge (y \leq 0)) \wedge (x - y > 0))$
d) $\forall x \forall y((x \neq 0) \wedge (y \neq 0) \leftrightarrow (xy \neq 0))$

25. Translate each of these nested quantifications into an English statement that expresses a mathematical fact. The domain in each case consists of all real numbers.

a) $\exists x \forall y(xy = y)$
b) $\forall x \forall y(((x < 0) \wedge (y < 0)) \rightarrow (xy > 0))$
c) $\exists x \exists y((x^2 > y) \wedge (x < y))$
d) $\forall x \forall y \exists z(x + y = z)$

26. Let $Q(x, y)$ be the statement "$x + y = x - y$." If the domain for both variables consists of all integers, what are the truth values?

a) $Q(1, 1)$
b) $Q(2, 0)$
c) $\forall y Q(1, y)$
d) $\exists x Q(x, 2)$
e) $\exists x \exists y Q(x, y)$
f) $\forall x \exists y Q(x, y)$
g) $\exists y \forall x Q(x, y)$
h) $\forall y \exists x Q(x, y)$
i) $\forall x \forall y Q(x, y)$

27. Determine the truth value of each of these statements if the domain for all variables consists of all integers.

a) $\forall n \exists m(n^2 < m)$
b) $\exists n \forall m(n < m^2)$
c) $\forall n \exists m(n + m = 0)$
d) $\exists n \forall m(nm = m)$
e) $\exists n \exists m(n^2 + m^2 = 5)$
f) $\exists n \exists m(n^2 + m^2 = 6)$
g) $\exists n \exists m(n + m = 4 \wedge n - m = 1)$
h) $\exists n \exists m(n + m = 4 \wedge n - m = 2)$
i) $\forall n \forall m \exists p(p = (m + n)/2)$

28. Determine the truth value of each of these statements if the domain of each variable consists of all real numbers.

a) $\forall x \exists y(x^2 = y)$
b) $\forall x \exists y(x = y^2)$
c) $\exists x \forall y(xy = 0)$
d) $\exists x \exists y(x + y \neq y + x)$
e) $\forall x(x \neq 0 \rightarrow \exists y(xy = 1))$
f) $\exists x \forall y(y \neq 0 \rightarrow xy = 1)$
g) $\forall x \exists y(x + y = 1)$
h) $\exists x \exists y(x + 2y = 2 \wedge 2x + 4y = 5)$
i) $\forall x \exists y(x + y = 2 \wedge 2x - y = 1)$
j) $\forall x \forall y \exists z(z = (x + y)/2)$

29. Suppose the domain of the propositional function $P(x, y)$ consists of pairs x and y, where x is 1, 2, or 3 and y is 1, 2, or 3. Write out these propositions using disjunctions and conjunctions.

a) $\forall x \forall y P(x, y)$
b) $\exists x \exists y P(x, y)$
c) $\exists x \forall y P(x, y)$
d) $\forall y \exists x P(x, y)$

30. Rewrite each of these statements so that negations appear only within predicates (that is, so that no negation is outside a quantifier or an expression involving logical connectives).

a) $\neg \exists y \exists x P(x, y)$
b) $\neg \forall x \exists y P(x, y)$
c) $\neg \exists y(Q(y) \wedge \forall x \neg R(x, y))$
d) $\neg \exists y(\exists x R(x, y) \vee \forall x S(x, y))$
e) $\neg \exists y(\forall x \exists z T(x, y, z) \vee \exists x \forall z U(x, y, z))$

31. Express the negations of each of these statements so that all negation symbols immediately precede predicates.

a) $\forall x \exists y \forall z T(x, y, z)$
b) $\forall x \exists y P(x, y) \vee \forall x \exists y Q(x, y)$
c) $\forall x \exists y(P(x, y) \wedge \exists z R(x, y, z))$
d) $\forall x \exists y(P(x, y) \rightarrow Q(x, y))$

32. Express the negations of each of these statements so that all negation symbols immediately precede predicates.

a) $\exists z \forall y \forall x T(x, y, z)$
b) $\exists x \exists y P(x, y) \wedge \forall x \forall y Q(x, y)$
c) $\exists x \exists y(Q(x, y) \leftrightarrow Q(y, x))$
d) $\forall y \exists x \exists z(T(x, y, z) \vee Q(x, y))$

33. Rewrite each of these statements so that negations appear only within predicates (that is, so that no negation is outside a quantifier or an expression involving logical connectives).

a) $\neg \forall x \forall y P(x, y)$
b) $\neg \forall y \exists x P(x, y)$
c) $\neg \forall y \forall x(P(x, y) \vee Q(x, y))$
d) $\neg(\exists x \exists y \neg P(x, y) \wedge \forall x \forall y Q(x, y))$
e) $\neg \forall x(\exists y \forall z P(x, y, z) \wedge \exists z \forall y P(x, y, z))$

34. Find a common domain for the variables x, y, and z for which the statement $\forall x \forall y((x \neq y) \rightarrow \forall z((z = x) \vee (z = y)))$ is true and another domain for which it is false.

35. Find a common domain for the variables x, y, z, and w for which the statement $\forall x \forall y \forall z \exists w((w \neq x) \wedge (w \neq y) \wedge (w \neq z))$ is true and another common domain for these variables for which it is false.

36. Express each of these statements using quantifiers. Then form the negation of the statement so that no negation is to the left of a quantifier. Next, express the negation in simple English. (Do not simply use the words "It is not the case that.")

a) No one has lost more than one thousand dollars playing the lottery.
b) There is a student in this class who has chatted with exactly one other student.
c) No student in this class has sent e-mail to exactly two other students in this class.
d) Some student has solved every exercise in this book.
e) No student has solved at least one exercise in every section of this book.

37. Express each of these statements using quantifiers. Then form the negation of the statement so that no negation is to the left of a quantifier. Next, express the negation in simple English. (Do not simply use the words "It is not the case that.")

a) Every student in this class has taken exactly two mathematics classes at this school.
b) Someone has visited every country in the world except Libya.
c) No one has climbed every mountain in the Himalayas.
d) Every movie actor has either been in a movie with Kevin Bacon or has been in a movie with someone who has been in a movie with Kevin Bacon.

38. Express the negations of these propositions using quantifiers, and in English.

a) Every student in this class likes mathematics.
b) There is a student in this class who has never seen a computer.
c) There is a student in this class who has taken every mathematics course offered at this school.
d) There is a student in this class who has been in at least one room of every building on campus.

39. Find a counterexample, if possible, to these universally quantified statements, where the domain for all variables consists of all integers.

a) $\forall x \forall y(x^2 = y^2 \rightarrow x = y)$
b) $\forall x \exists y(y^2 = x)$
c) $\forall x \forall y(xy \geq x)$

40. Find a counterexample, if possible, to these universally quantified statements, where the domain for all variables consists of all integers.

a) $\forall x \exists y(x = 1/y)$
b) $\forall x \exists y(y^2 - x < 100)$
c) $\forall x \forall y(x^2 \neq y^3)$

41. Use quantifiers to express the associative law for multiplication of real numbers.

42. Use quantifiers to express the distributive laws of multiplication over addition for real numbers.

43. Use quantifiers and logical connectives to express the fact that every linear polynomial (that is, polynomial of degree 1) with real coefficients and where the coefficient of x is nonzero, has exactly one real root.

44. Use quantifiers and logical connectivesto express the fact that a quadratic polynomial with real number coefficients has at most two real roots.

45. Determine the truth value of the statement $\forall x \exists y(xy = 1)$ if the domain for the variables consists of

a) the nonzero real numbers.
b) the nonzero integers.
c) the positive real numbers.

46. Determine the truth value of the statement $\exists x \forall y(x \leq y^2)$ if the domain for the variables consists of

a) the positive real numbers.
b) the integers.
c) the nonzero real numbers.

47. Show that the two statements $\neg\exists x \forall y P(x, y)$ and $\forall x \exists y \neg P(x, y)$, where both quantifiers over the first variable in $P(x, y)$ have the same domain, and both quantifiers over the second variable in $P(x, y)$ have the same domain, are logically equivalent.

***48.** Show that $\forall x P(x) \vee \forall x Q(x)$ and $\forall x \forall y(P(x) \vee Q(y))$, where all quantifiers have the same nonempty domain, are logically equivalent. (The new variable y is used to combine the quantifications correctly.)

***49.** a) Show that $\forall x P(x) \wedge \exists x Q(x)$ is logically equivalent to $\forall x \exists y (P(x) \wedge Q(y))$, where all quantifiers have the same nonempty domain.
b) Show that $\forall x P(x) \vee \exists x Q(x)$ is equivalent to $\forall x \exists y(P(x) \vee Q(y))$, where all quantifiers have the same nonempty domain.

A statement is in **prenex normal form (PNF)** if and only if it is of the form

$$Q_1 x_1 Q_2 x_2 \cdots Q_k x_k P(x_1, x_2, \ldots, x_k),$$

where each Q_i, $i = 1, 2, \ldots, k$, is either the existential quantifier or the universal quantifier, and $P(x_1, \ldots, x_k)$ is a predicate involving no quantifiers. For example, $\exists x \forall y(P(x, y) \wedge Q(y))$ is in prenex normal form, whereas $\exists x P(x) \vee \forall x Q(x)$ is not (because the quantifiers do not all occur first).

Every statement formed from propositional variables, predicates, **T**, and **F** using logical connectives and quantifiers is equivalent to a statement in prenex normal form. Exercise 51 asks for a proof of this fact.

***50.** Put these statements in prenex normal form. [*Hint:* Use logical equivalence from Tables 6 and 7 in Section 1.2, Table 2 in Section 1.3, Example 19 in Section 1.3, Exercises 45 and 46 in Section 1.3, and Exercises 48 and 49 in this section.]

a) $\exists x P(x) \vee \exists x Q(x) \vee A$, where A is a proposition not involving any quantifiers.
b) $\neg(\forall x P(x) \vee \forall x Q(x))$
c) $\exists x P(x) \rightarrow \exists x Q(x)$

****51.** Show how to transform an arbitrary statement to a statement in prenex normal form that is equivalent to the given statement.

***52.** Express the quantification $\exists! x P(x)$, introduced on page 38, using universal quantifications, existential quantifications, and logical operators.

Solution: Let $P(n)$ denote "$n > 4$" and $Q(n)$ denote "$n^2 < 2^n$." The statement "For all positive integers n, if n is greater than 4, then n^2 is less than 2^n" can be represented by $\forall n(P(n) \rightarrow Q(n))$, where the domain consists of all positive integers. We are assuming that $\forall n(P(n) \rightarrow Q(n))$ is true. Note that $P(100)$ is true because $100 > 4$. It follows by universal modus ponens that $Q(n)$ is true, namely that $100^2 < 2^{100}$. ◀

Another useful combination of a rule of inference from propositional logic and a rule of inference for quantified statements is **universal modus tollens.** Universal modus tollens combines universal instantiation and modus tollens and can be expressed in the following way:

$$\begin{array}{l} \forall x(P(x) \rightarrow Q(x)) \\ \underline{\neg Q(a), \text{ where } a \text{ is a particular element in the domain}} \\ \therefore \neg P(a) \end{array}$$

We leave the verification of universal modus tollens to the reader (see Exercise 25). Exercise 26 develops additional combinations of rules of inference in propositional logic and quantified statements.

Exercises

1. Find the argument form for the following argument and determine whether it is valid. Can we conclude that the conclusion is true if the premises are true?

   If Socrates is human, then Socrates is mortal.
   Socrates is human.
   ∴ Socrates is mortal.

2. Find the argument form for the following argument and determine whether it is valid. Can we conclude that the conclusion is true if the premises are true?

   If George does not have eight legs, then he is not an insect.
   George is an insect.
   ∴ George has eight legs.

3. What rule of inference is used in each of these arguments?
   a) Alice is a mathematics major. Therefore, Alice is either a mathematics major or a computer science major.
   b) Jerry is a mathematics major and a computer science major. Therefore, Jerry is a mathematics major.
   c) If it is rainy, then the pool will be closed. It is rainy. Therefore, the pool is closed.
   d) If it snows today, the university will close. The university is not closed today. Therefore, it did not snow today.
   e) If I go swimming, then I will stay in the sun too long. If I stay in the sun too long, then I will sunburn. Therefore, if I go swimming, then I will sunburn.

4. What rule of inference is used in each of these arguments?
   a) Kangaroos live in Australia and are marsupials. Therefore, kangaroos are marsupials.
   b) It is either hotter than 100 degrees today or the pollution is dangerous. It is less than 100 degrees outside today. Therefore, the pollution is dangerous.
   c) Linda is an excellent swimmer. If Linda is an excellent swimmer, then she can work as a lifeguard. Therefore, Linda can work as a lifeguard.
   d) Steve will work at a computer company this summer. Therefore, this summer Steve will work at a computer company or he will be a beach bum.
   e) If I work all night on this homework, then I can answer all the exercises. If I answer all the exercises, I will understand the material. Therefore, if I work all night on this homework, then I will understand the material.

5. Use rules of inference to show that the hypotheses "Randy works hard," "If Randy works hard, then he is a dull boy," and "If Randy is a dull boy, then he will not get the job" imply the conclusion "Randy will not get the job."

6. Use rules of inference to show that the hypotheses "If it does not rain or if it is not foggy, then the sailing race will be held and the lifesaving demonstration will go on," "If the sailing race is held, then the trophy will be awarded," and "The trophy was not awarded" imply the conclusion "It rained."

7. What rules of inference are used in this famous argument? "All men are mortal. Socrates is a man. Therefore, Socrates is mortal."

8. What rules of inference are used in this argument? "No man is an island. Manhattan is an island. Therefore, Manhattan is not a man."

9. For each of these sets of premises, what relevant conclusion or conclusions can be drawn? Explain the rules of inference used to obtain each conclusion from the premises.
   a) "If I take the day off, it either rains or snows." "I took Tuesday off or I took Thursday off." "It was sunny on Tuesday." "It did not snow on Thursday."
   b) "If I eat spicy foods, then I have strange dreams." "I have strange dreams if there is thunder while I sleep." "I did not have strange dreams."
   c) "I am either clever or lucky." "I am not lucky." "If I am lucky, then I will win the lottery."
   d) "Every computer science major has a personal computer." "Ralph does not have a personal computer." "Ann has a personal computer."
   e) "What is good for corporations is good for the United States." "What is good for the United States is good for you." "What is good for corporations is for you to buy lots of stuff."
   f) "All rodents gnaw their food." "Mice are rodents." "Rabbits do not gnaw their food." "Bats are not rodents."

10. For each of these sets of premises, what relevant conclusion or conclusions can be drawn? Explain the rules of inference used to obtain each conclusion from the premises.
   a) "If I play hockey, then I am sore the next day." "I use the whirlpool if I am sore." "I did not use the whirlpool."
   b) "If I work, it is either sunny or partly sunny." "I worked last Monday or I worked last Friday." "It was not sunny on Tuesday." "It was not partly sunny on Friday."
   c) "All insects have six legs." "Dragonflies are insects." "Spiders do not have six legs." "Spiders eat dragonflies."
   d) "Every student has an Internet account." "Homer does not have an Internet account." "Maggie has an Internet account."
   e) "All foods that are healthy to eat do not taste good." "Tofu is healthy to eat." "You only eat what tastes good." "You do not eat tofu." "Cheeseburgers are not healthy to eat."
   f) "I am either dreaming or hallucinating." "I am not dreaming." "If I am hallucinating, I see elephants running down the road."

11. Show that the argument form with premises $p_1, p_2, \ldots, p_n$ and conclusion $q \rightarrow r$ is valid if the argument form with premises $p_1, p_2, \ldots, p_n, q$, and conclusion r is valid.

12. Show that the argument form with premises $(p \wedge t) \rightarrow (r \vee s)$, $q \rightarrow (u \wedge t)$, $u \rightarrow p$, and $\neg s$ and conclusion $q \rightarrow r$ is valid by first using Exercise 11 and then using rules of inference from Table 1.

13. For each of these arguments, explain which rules of inference are used for each step.
   a) "Doug, a student in this class, knows how to write programs in JAVA. Everyone who knows how to write programs in JAVA can get a high-paying job. Therefore, someone in this class can get a high-paying job."
   b) "Somebody in this class enjoys whale watching. Every person who enjoys whale watching cares about ocean pollution. Therefore, there is a person in this class who cares about ocean pollution."
   c) "Each of the 93 students in this class owns a personal computer. Everyone who owns a personal computer can use a word processing program. Therefore, Zeke, a student in this class, can use a word processing program."
   d) "Everyone in New Jersey lives within 50 miles of the ocean. Someone in New Jersey has never seen the ocean. Therefore, someone who lives within 50 miles of the ocean has never seen the ocean."

14. For each of these arguments, explain which rules of inference are used for each step.
   a) "Linda, a student in this class, owns a red convertible. Everyone who owns a red convertible has gotten at least one speeding ticket. Therefore, someone in this class has gotten a speeding ticket."
   b) "Each of five roommates, Melissa, Aaron, Ralph, Veneesha, and Keeshawn, has taken a course in discrete mathematics. Every student who has taken a course in discrete mathematics can take a course in algorithms. Therefore, all five roommates can take a course in algorithms next year."
   c) "All movies produced by John Sayles are wonderful. John Sayles produced a movie about coal miners. Therefore, there is a wonderful movie about coal miners."
   d) "There is someone in this class who has been to France. Everyone who goes to France visits the Louvre. Therefore, someone in this class has visited the Louvre."

15. For each of these arguments determine whether the argument is correct or incorrect and explain why.
   a) All students in this class understand logic. Xavier is a student in this class. Therefore, Xavier understands logic.
   b) Every computer science major takes discrete mathematics. Natasha is taking discrete mathematics. Therefore, Natasha is a computer science major.
   c) All parrots like fruit. My pet bird is not a parrot. Therefore, my pet bird does not like fruit.
   d) Everyone who eats granola every day is healthy. Linda is not healthy. Therefore, Linda does not eat granola every day.

16. For each of these arguments determine whether the argument is correct or incorrect and explain why.
   a) Everyone enrolled in the university has lived in a dormitory. Mia has never lived in a dormitory. Therefore, Mia is not enrolled in the university.
   b) A convertible car is fun to drive. Isaac's car is not a convertible. Therefore, Isaac's car is not fun to drive.
   c) Quincy likes all action movies. Quincy likes the movie *Eight Men Out*. Therefore, *Eight Men Out* is an action movie.

d) All lobstermen set at least a dozen traps. Hamilton is a lobsterman. Therefore, Hamilton sets at least a dozen traps.

17. What is wrong with this argument? Let $H(x)$ be "x is happy." Given the premise $\exists x H(x)$, we conclude that $H(\text{Lola})$. Therefore, Lola is happy.

18. What is wrong with this argument? Let $S(x, y)$ be "x is shorter than y." Given the premise $\exists s S(s, \text{Max})$, it follows that $S(\text{Max}, \text{Max})$. Then by existential generalization it follows that $\exists x S(x, x)$, so that someone is shorter than himself.

19. Determine whether each of these arguments is valid. If an argument is correct, what rule of inference is being used? If it is not, what logical error occurs?

a) If n is a real number such that $n > 1$, then $n^2 > 1$. Suppose that $n^2 > 1$. Then $n > 1$.

b) If n is a real number with $n > 3$, then $n^2 > 9$. Suppose that $n^2 \leq 9$. Then $n \leq 3$.

c) If n is a real number with $n > 2$, then $n^2 > 4$. Suppose that $n \leq 2$. Then $n^2 \leq 4$.

20. Determine whether these are valid arguments.

a) If x is a positive real number, then x^2 is a positive real number. Therefore, if a^2 is positive, where a is a real number, then a is a positive real number.

b) If $x^2 \neq 0$, where x is a real number, then $x \neq 0$. Let a be a real number with $a^2 \neq 0$; then $a \neq 0$.

21. Which rules of inference are used to establish the conclusion of Lewis Carroll's argument described in Example 26 of Section 1.3?

22. Which rules of inference are used to establish the conclusion of Lewis Carroll's argument described in Example 27 of Section 1.3?

23. Identify the error or errors in this argument that supposedly shows that if $\exists x P(x) \wedge \exists x Q(x)$ is true then $\exists x(P(x) \wedge Q(x))$ is true.

1. $\exists x P(x) \wedge \exists x Q(x)$	Premise
2. $\exists x P(x)$	Simplification from (1)
3. $P(c)$	Existential instantiation from (2)
4. $\exists x Q(x)$	Simplification from (1)
5. $Q(c)$	Existential instantiation from (4)
6. $P(c) \wedge Q(c)$	Conjunction from (3) and (5)
7. $\exists x(P(x) \wedge Q(x))$	Existential generalization

24. Identify the error or errors in this argument that supposedly shows that if $\forall x(P(x) \vee Q(x))$ is true then $\forall x P(x) \vee \forall x Q(x)$ is true.

1. $\forall x(P(x) \vee Q(x))$	Premise
2. $P(c) \vee Q(c)$	Universal instantiation from (1)
3. $P(c)$	Simplification from (2)
4. $\forall x P(x)$	Universal generalization from (3)
5. $Q(c)$	Simplification from (2)
6. $\forall x Q(x)$	Universal generalization from (5)
7. $\forall x(P(x) \vee \forall x Q(x))$	Conjunction from (4) and (6)

25. Justify the rule of universal modus tollens by showing that the premises $\forall x(P(x) \rightarrow Q(x))$ and $\neg Q(a)$ for a particular element a in the domain, imply $\neg P(a)$.

26. Justify the rule of **universal transitivity**, which states that if $\forall x(P(x) \rightarrow Q(x))$ and $\forall x(Q(x) \rightarrow R(x))$ are true, then $\forall x(P(x) \rightarrow R(x))$ is true, where the domains of all quantifiers are the same.

27. Use rules of inference to show that if $\forall x(P(x) \rightarrow (Q(x) \wedge S(x)))$ and $\forall x(P(x) \wedge R(x))$ are true, then $\forall x(R(x) \wedge S(x))$ is true.

28. Use rules of inference to show that if $\forall x(P(x) \vee Q(x))$ and $\forall x((\neg P(x) \wedge Q(x)) \rightarrow R(x))$ are true, then $\forall x(\neg R(x) \rightarrow P(x))$ is also true, where the domains of all quantifiers are the same.

29. Use rules of inference to show that if $\forall x(P(x) \vee Q(x))$, $\forall x(\neg Q(x) \vee S(x))$, $\forall x(R(x) \rightarrow \neg S(x))$, and $\exists x \neg P(x)$ are true, then $\exists x \neg R(x)$ is true.

30. Use resolution to show the hypotheses "Allen is a bad boy or Hillary is a good girl" and "Allen is a good boy or David is happy" imply the conclusion "Hillary is a good girl or David is happy."

31. Use resolution to show that the hypotheses "It is not raining or Yvette has her umbrella," "Yvette does not have her umbrella or she does not get wet," and "It is raining or Yvette does not get wet" imply that "Yvette does not get wet."

32. Show that the equivalence $p \wedge \neg p \equiv \mathbf{F}$ can be derived using resolution together with the fact that a conditional statement with a false hypothesis is true. [*Hint:* Let $q = r = \mathbf{F}$ in resolution.]

33. Use resolution to show that the compound proposition $(p \vee q) \wedge (\neg p \vee q) \wedge (p \vee \neg q) \wedge (\neg p \vee \neg q)$ is not satisfiable.

***34.** The Logic Problem, taken from *WFF'N PROOF, The Game of Logic*, has these two assumptions:

1. "Logic is difficult or not many students like logic."
2. "If mathematics is easy, then logic is not difficult."

By translating these assumptions into statements involving propositional variables and logical connectives, determine whether each of the following are valid conclusions of these assumptions:

a) That mathematics is not easy, if many students like logic.

b) That not many students like logic, if mathematics is not easy.

c) That mathematics is not easy or logic is difficult.

d) That logic is not difficult or mathematics is not easy.

e) That if not many students like logic, then either mathematics is not easy or logic is not difficult.

***35.** Determine whether this argument, taken from Kalish and Montague [KaMo64], is valid.

> If Superman were able and willing to prevent evil, he would do so. If Superman were unable to prevent evil, he would be impotent; if he were unwilling to prevent evil, he would be malevolent. Superman does not prevent evil. If Superman exists, he is neither impotent nor malevolent. Therefore, Superman does not exist.