Expected Value and Variance

Section 7.4

Section Summary

- Expected Value
- Linearity of Expectations
- ~~Average-Case Computational Complexity~~
- ~~Geometric Distribution~~
- ~~Independent Random Variables~~
- ~~Variance~~
- ~~Chebyshev’s Inequality~~

Expected Value

Definition: The *expected value* (or *expectation* or *mean*) of the random variable $X(s)$ on the sample space S is equal to

$$E(X) = \sum_{x \in S} p(s) X(s).$$

Expected Value

$$E(X) = \sum_{x \in S} p(s)X(s).$$

Example: **Expected Value of a Die**

Let X be the number that comes up when a fair die is rolled. What is the expected value of X?

Solution: The random variable X takes the values 1, 2, 3, 4, 5, or 6. Each has probability 1/6. It follows that

$$E(X) = \frac{1}{6} \cdot 1 + \frac{1}{6} \cdot 2 + \cdots \frac{1}{6} \cdot 6 = \frac{21}{6} = \frac{7}{2}.$$

Expected Value

Theorem 1: If X is a random variable and $p(X = r)$ is the probability that X = r, so that

$$p(X = r) = \sum_{s \in S, X(s) = r} p(s),$$ then

$$E(X) = \sum_{r \in X(S)} p(X = r) r.$$

Proof: Suppose that X is a random variable with range $X(S)$ and let $p(X = r)$ be the probability that X takes the value r. Consequently, $p(X = r)$ is the sum of the probabilities of the outcomes s such that $X(s) = r$. Hence,

$$E(X) = \sum_{r \in X(S)} p(X = r) r. \qquad \blacktriangleleft$$

Expected Value

Theorem 2: The expected number of successes when n mutually independent Bernoulli trials are performed, where, the probability of success on each trial, $p = np$.

Proof: Let X be the random variable equal to the number of success in n trials. By Theorem 2 of section 7.2, $p(X = k) = C(n,k)p^k q^{n-k}$. Hence,

$$E(X) = \sum_{k=1}^{n} kp(X = k) \quad \text{by Theorem 1}$$

continued →

Expected Value

$$\sum_{k=0}^{n} C(n,k)p^k q^{n-k} = (p+q)^n = 1$$

reminder: the binomial theorem

$$E(X) = \sum_{k=1}^{n} kp(X=k)$$ from previous page

$$= \sum_{k=1}^{n} kC(n,k)p^k q^{n-k}$$ by Theorem 2 in Section 7.2

$$= \sum_{k=1}^{n} nC(n-1,k-1)p^k q^{n-k}$$ by Exercise 21 in Section 6.4

$$= np\sum_{k=1}^{n} C(n-1,k-1)p^{k-1} q^{n-k}$$ factoring np from each term

$$= np\boxed{\sum_{j=0}^{n-1} C(n-1,j)p^j q^{n-1-j}}$$ shifting index of summation with $j = k - 1$

$$= np(p+q)^{n-1}$$ by the binomial theorem

$$= np.$$ because $p + q = 1$

We see that the expected number of successes in n mutually independent Bernoulli trials is np.

Linearity of Expectations

- The following theorem tells us that expected values are linear.
- For example, the expected value of the sum of random variables is the sum of their expected values.

Theorem 3: If X_i, $i = 1, 2, \ldots, n$ with n a positive integer, are random variables on S, and if a and b are real numbers, then

(i) $E(X_1 + X_2 + \ldots. + X_n) = E(X_1) + E(X_2) + \ldots. + E(X_n)$

(ii) $E(aX + b) = aE(X) + b$.

see the text for the proof

Linearity of Expectations

- **Example:** Find the expected value of the sum of the numbers that appear when a pair of fair dice is rolled (using linearity of expectations).

- **Solution:** Let X_1 and X_2 be the random variables, where X_1 is the number appearing on the first die and X_2 is the number appearing on the second die.
- It is easy to see that $E(X_1) = E(X_2) = 7/2$ because both equal $(1 + 2 + 3 + 4 + 5 + 6)/6 = 21/6 = 7/2$.
- The sum of the two numbers that appear when the two dice are rolled is the sum $X_1 + X_2$.
- By Theorem 3, the expected value of the sum is $E(X_1 + X_2) = E(X_1) + E(X_2) = 7/2 + 7/2 = 7$.

Linearity of Expectations

- In the proof of Theorem 2 we found the expected value of the number of successes when n independent Bernoulli trials are performed as np, where p is the probability of success on each trial by direct computation. S

Example: Show how Theorem 3 can be used to derive this result where the Bernoulli trials are not necessarily independent.

Solution:

- Let X_i be the random variable with $X_i((t_1, t_2, \ldots, t_n)) = 1$ if t_i is a success and $X_i((t_1, t_2, \ldots, t_n)) = 0$ if t_i is a failure.
- The expected value of X_i is $E(X_i) = 1 \cdot p + 0 \cdot (1 - p) = p$ for $i = 1, 2, \ldots, n$.
- Let $X = X_1 + X_2 + \cdots + X_n$, so that X counts the number of successes when these n Bernoulli trials are performed.
- Theorem 3, applied to the sum of n random variables, $E(X) = E(X_1) + E(X_2) + \cdots + E(X_n) = np$.

Linearity of Expectations

Expected Value in the Hatcheck Problem: A new employee checks the hats of n people at a restaurant, forgetting to put claim check numbers on the hats. When customers return for their hats, the checker gives them back hats chosen at random from the remaining hats. So, the n customers just receive a random hat from those remaining. What is the expected number of hats that are returned correctly?

Solution: Let X be the random variable that equals the number of people who receive the correct hat. Note that $X = X_1 + X_2 + \cdots + X_n$,
where $X_i = 1$ if the ith person receives the correct hat and $X_i = 0$ otherwise.

- Because it is equally likely that the checker returns any of the hats to the ith person, it follows that the probability that the ith person receives the correct hat is $1/n$. Consequently (by Theorem 1), for all i

$$E(X_i) = 1 \cdot p(X_i = 1) + 0 \cdot p(X_i = 0) = 1 \cdot 1/n + 0 = 1/n .$$

- By the linearity of expectations (Theorem 3), it follows that:

$$E(X) = E(X_1) + E(X_2) + \cdots + E(X_n) = n \cdot 1/n = 1.$$

Consequently, the average number of people who receive the correct hat is exactly 1. (Surprisingly, this answer remains the same no matter how many people have checked their hats!)