Advanced Counting Techniques

Chapter 8

With Question/Answer Animations

Chapter Summary

- Applications of Recurrence Relations
- Solving Linear Recurrence Relations
  - Homogeneous Recurrence Relations
  - Nonhomogeneous Recurrence Relations
- Divide-and-Conquer Algorithms and Recurrence Relations
- Generating Functions
- Inclusion-Exclusion
- Applications of Inclusion-Exclusion

Applications of Recurrence Relations

Section 8.1

Section Summary

- Applications of Recurrence Relations
  - Fibonacci Numbers
  - The Tower of Hanoi
  - Counting Problems
- Algorithms and Recurrence Relations (*not currently included in overheads*)

Recurrence Relations
(recalling definitions from Chapter 2)

Definition: A *recurrence relation* for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms of the sequence, namely, $a_0, a_1, \ldots, a_{n-1}$, for all integers n with $n \geq n_0$, where n_0 is a nonnegative integer.

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- A sequence is called a *solution* of a recurrence relation if its terms satisfy the recurrence relation.
- The *initial conditions* for a sequence specify *the terms that precede the first term where the recurrence relation takes effect*.

Rabbits and the Fiobonacci Numbers

Example: A young pair of rabbits (one of each gender) is placed on an island:

- *A pair of rabbits does not breed until they are 2 months old*.
- After they are 2 months old, *each pair of rabbits produces another pair each month*.

Find a recurrence relation for *the number of pairs of rabbits on the island after n months*, assuming that rabbits never die.

This is the original problem considered by Leonardo Pisano (Fibonacci) in the thirteenth century.

Rabbits and the Fiobonacci Numbers (*cont.*)

Reproducing pairs (at least two months old)	Young pairs (less than two months old)	Month	Reproducing pairs	Young pairs	Total pairs
		1	0	1	1
		2	0	1	1
		3	1	1	2
		4	1	2	3
		5	2	3	5
		6	3	5	8

Modeling the Population Growth of Rabbits on an Island

Rabbits and the Fibonacci Numbers (*cont.*)

Solution: Let f_n be the *the number of pairs of rabbits after n months*.

- There are $f_1 = 1$ pairs of rabbits on the island at the end of the first month.
- We also have $f_2 = 1$ because the pair does not also breed during the second month.
- To find the number of pairs on the island after n months, *add the number on the island after the previous month*, f_{n-1}, and *the number of newborn pairs*, which equals f_{n-2}, *because each newborn pair comes from a pair at least two months old*.

Rabbits and the Fibonacci Numbers (*cont.*)

Solution: Let f_n be the the number of pairs of rabbits after n months.

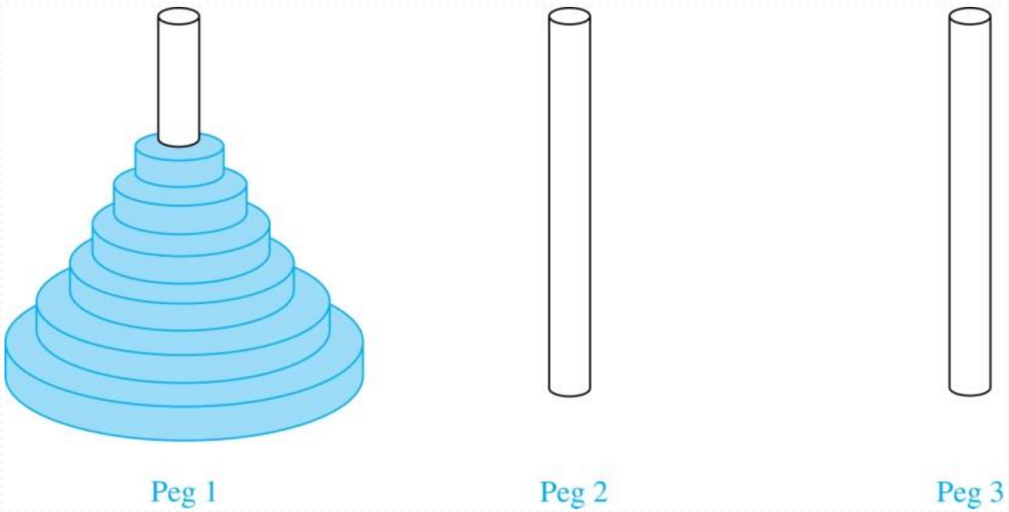
- There are is $f_1 = 1$ pairs of rabbits on the island at the end of the first month.
- We also have $f_2 = 1$ because the pair does not breed during the first month.
- To find the number of pairs on the island after n months, add the number on the island after the previous month, f_{n-1}, and the number of newborn pairs, which equals f_{n-2}, because each newborn pair comes from a pair at least two months old.

Consequently the sequence $\{f_n\}$ satisfies the recurrence relation $f_n = f_{n-1} + f_{n-2}$ for $n \geq 3$ with the initial conditions $f_1 = 1$ and $f_2 = 1$.

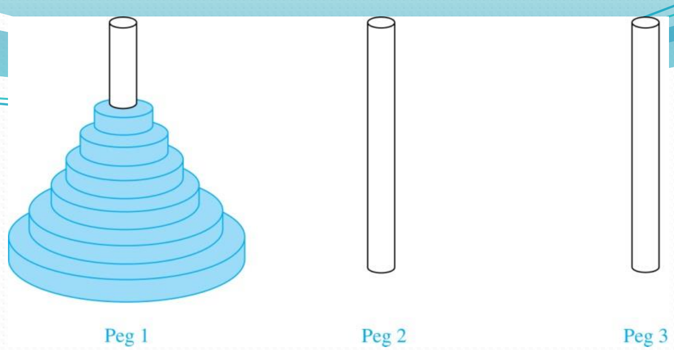
The number of pairs of rabbits on the island after n months is given by the *nth Fibonacci number*.

The Tower of Hanoi

- Three pegs on a board with disks of different sizes.
- Initially all of the disks are on the first peg *in order of size*, with the largest on the bottom.



The Tower of Hanoi

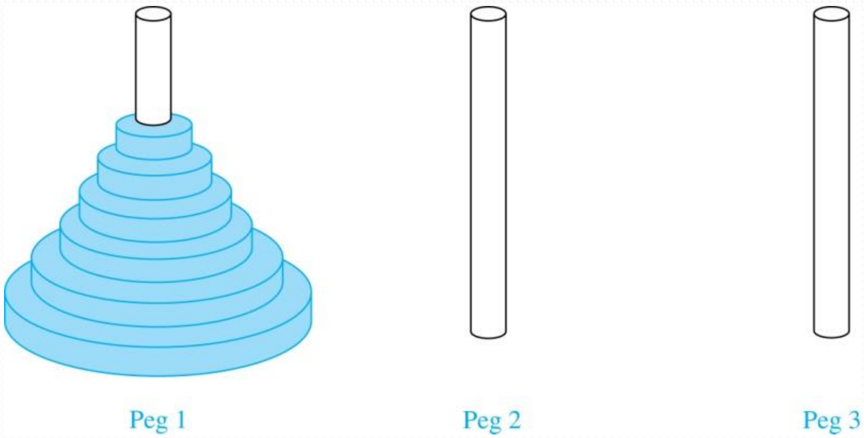


Rules:
You are allowed to move the disks one at a time from one peg to another as long as a larger disk is never placed on a smaller.

Goal:
Using allowable moves, end up with all the disks on the second peg in order of size with largest on the bottom.

The Tower of Hanoi (*continued*)

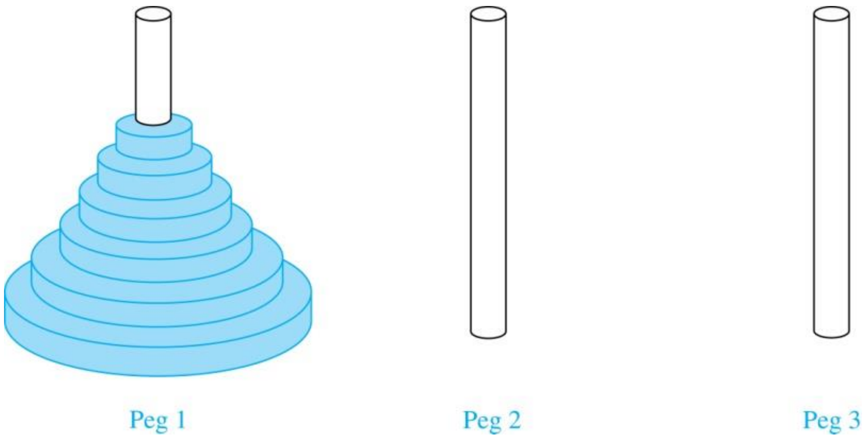
$\{H_n\}$: the number of moves needed to solve the Tower of Hanoi Puzzle with n disks.
Solution: Set up a recurrence relation for the sequence $\{H_n\}$.



The Initial Position in the Tower of Hanoi Puzzle

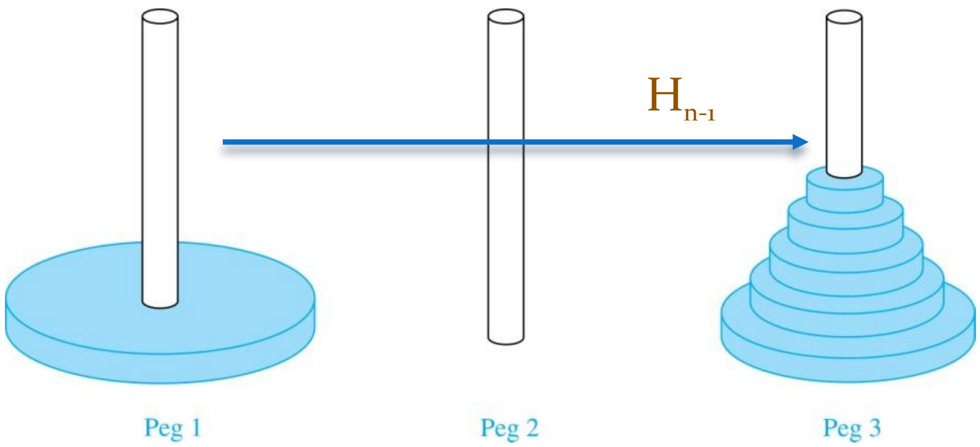
The Tower of Hanoi (*continued*)

- Begin with n disks on peg 1.



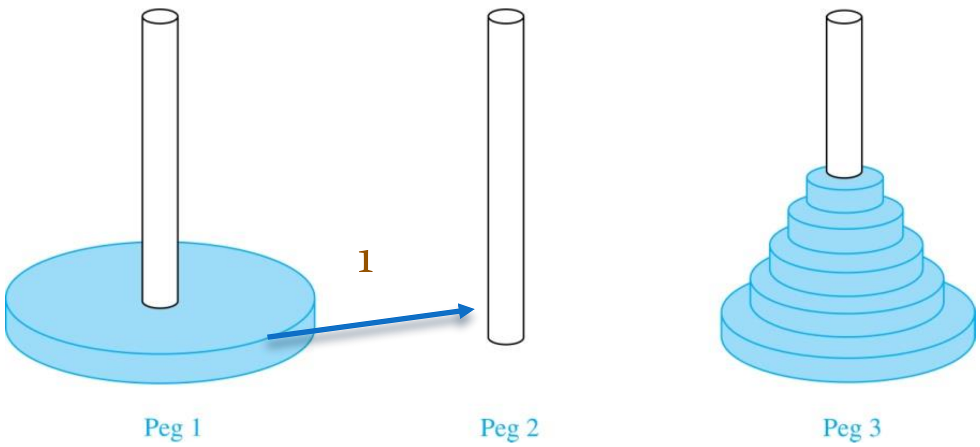
The Tower of Hanoi (*continued*)

- Begin with n disks on peg 1.
- We can transfer the top $n - 1$ disks, following the rules of the puzzle, to peg 3 using H_{n-1} moves.



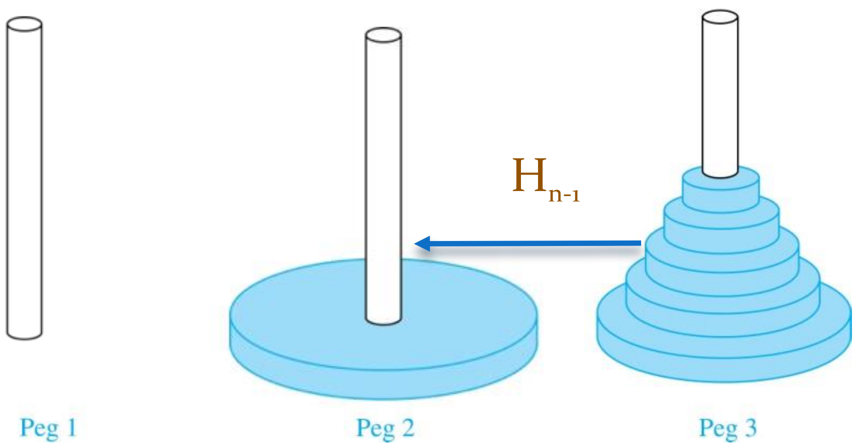
The Tower of Hanoi (*continued*)

- First, we use 1 move to transfer the largest disk to the second peg.



The Tower of Hanoi (*continued*)

- Then we transfer the $n-1$ disks from peg 3 to peg 2 using H_{n-1} additional moves.

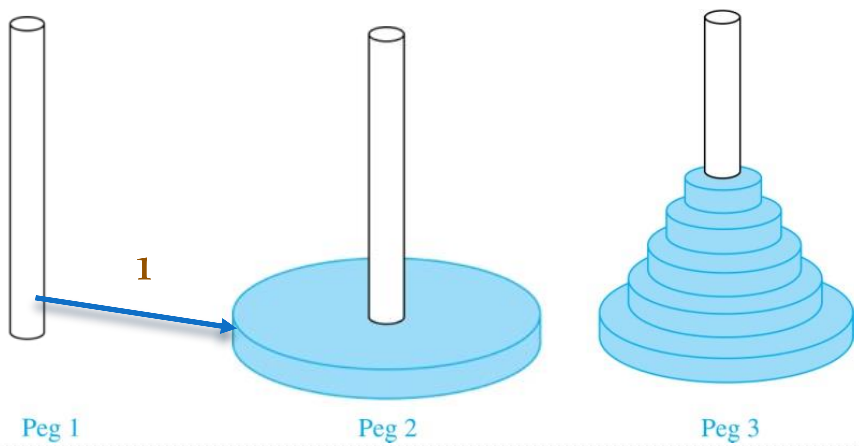


The Tower of Hanoi (*continued*)

- Then we transfer the $n-1$ disks from peg 3 to peg 2 using H_{n-1} additional moves.

$$\boxed{H_n = 2H_{n-1} + 1.}$$

- The initial condition is $H_1 = 1$
  - a single disk can be transferred from peg 1 to peg 2 in one move.



The Tower of Hanoi (*continued*)

- Iterative approach to solve this recurrence relation
  - by repeatedly expressing H_n in terms of the previous terms of the sequence.

$H_n = 2H_{n-1} + 1$

$= 2(2H_{n-2} + 1) + 1 = 2^2 H_{n-2} + 2 + 1$

$= 2^2(2H_{n-3} + 1) + 2 + 1 = 2^3 H_{n-3} + 2^2 + 2 + 1$

$\vdots$

$= 2^{n-1}H_1 + 2^{n-2} + 2^{n-3} + \dots + 2 + 1$

$= 2^{n-1} + 2^{n-2} + 2^{n-3} + \dots + 2 + 1$ because $H_1 = 1$

$= 2^n - 1$ using the formula for the sum of the terms of a geometric series

Counting Bit Strings

- Find the number of bit strings of length *n* without two consecutive 0s.
  - Find a recurrence relation and give initial conditions
  - How many such bit strings are there of length five?

Counting Bit Strings

Number of bit strings of length n without two consecutive 0s.

- a_n denote the number of bit strings of length n without two consecutive 0s.

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Number of bit strings of length n without two consecutive 0s.

- a_n denote the number of bit strings of length n without two consecutive 0s.

- The number of bit strings of length n that do not have two consecutive 0s:
  - the number of bit strings ending with a 0
  - the number of such bit strings ending with a 1

Counting Bit Strings

Number of bit strings of length n without two consecutive 0s.

- a_n denote the number of bit strings of length n without two consecutive 0s.

- Now assume that $n \geq 3$.
- The bit strings of length n ending with 1 without two consecutive 0s are the bit strings of length n -1 with no two consecutive 0s with a 1 at the end.
  - Hence, there are a_{n-1} such bit strings.

- The bit strings of length n ending with 0 without two consecutive 0s are the bit strings of length n -2 with no two consecutive 0s with 10 at the end.
  - Hence, there are a_{n-2} such bit strings.

Counting Bit Strings

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- Now assume that $n \geq 3$.
- The bit strings of length n ending with 1 without two consecutive 0s are the bit strings of length n −1 with no two consecutive 0s with a 1 at the end.
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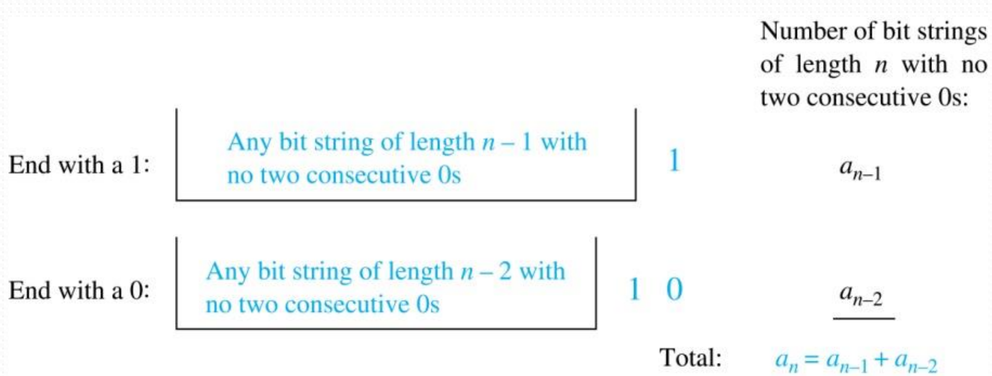
- The bit strings of length n ending with 0 without two consecutive 0s are the bit strings of length n −2 with no two consecutive 0s with 10 at the end.
  - Hence, there are a_{n-2} such bit strings.

$$a_n = a_{n-1} + a_{n-2} \text{ for } n \geq 3.$$



Bit Strings (*continued*)

$a_n = a_{n-1} + a_{n-2}$ for $n \geq 3$.

The initial conditions are:

- $a_1 = 2$, since both the bit strings 0 and 1 do not have consecutive 0s.
- $a_2 = 3$, since the bit strings 01, 10, and 11 do not have consecutive 0s, while 00 does.

To obtain a_5, we use the recurrence relation three times to find that:

- $a_3 = a_2 + a_1 = 3 + 2 = 5$
- $a_4 = a_3 + a_2 = 5 + 3 = 8$
- $a_5 = a_4 + a_3 = 8 + 5 = 13$

Note that $\{a_n\}$ satisfies the same recurrence relation as the Fibonacci sequence. Since $a_1 = 2 = f_3$ and $a_2 = 3 = f_4$, we conclude that $a_n = f_{n+2}$

.

Counting the Ways to Parenthesize a Product

- Find a recurrence relation for C_n:
  - The number of ways to parenthesize the product of $n + 1$ numbers, $x_0 \cdot x_1 \cdot x_2 \cdot \cdots \cdot x_n$, to specify the order of multiplication.
    - For example, $C_3 = 5$, since all the possible ways to parenthesize 4 numbers are

$$((x_0 \cdot x_1) \cdot x_2) \cdot x_3,$$
$$(x_0 \cdot (x_1 \cdot x_2)) \cdot x_3,$$
$$(x_0 \cdot x_1) \cdot (x_2 \cdot x_3),$$
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Counting the Ways to Parenthesize a Product

$((x_0 \cdot x_1) \cdot x_2) \cdot x_3$,
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Solution: Note that however parentheses are inserted in $x_0 \cdot x_1 \cdot x_2 \cdot \cdots \cdot x_n$,

- one "·" operator remains outside all parentheses.
- This final operator appears between two of the n + 1 numbers, say x_k and x_{k+1}.
- Since there are C_k ways to insert parentheses in the product $x_0 \cdot x_1 \cdot x_2 \cdot \cdots \cdot x_k$ and
- C_{n-k-1} ways to insert parentheses in the product $x_{k+1} \cdot x_{k+2} \cdot \cdots \cdot x_n$, we have

Counting the Ways to Parenthesize a Product

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- C_{n-k-1} ways to insert parentheses in the product $x_{k+1} \cdot x_{k+2} \cdot \cdots \cdot x_n$, we have

$$C_n = C_0 C_{n-1} + C_1 C_{n-2} + \cdots + C_{n-2} C_1 + C_{n-1} C_0$$

$$= \sum_{k=0}^{n-1} C_k C_{n-k-1}$$

The initial conditions are $C_0 = 1$ and $C_1 = 1$.

Counting the Ways to Parenthesize a Product

$((x_0 \cdot x_1) \cdot x_2) \cdot x_3$,
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- C_{n-k-1} ways to insert parentheses in the product $x_{k+1} \cdot x_{k+2} \cdot \cdots \cdot x_n$, we have

$$C_n = C_0 C_{n-1} + C_1 C_{n-2} + \cdots + C_{n-2} C_1 + C_{n-1} C_0$$

$$= \sum_{k=0}^{n-1} C_k C_{n-k-1}$$

The sequence $\{C_n\}$ is the sequence of **Catalan Numbers**.

Solving Linear Recurrence Relations

Section 8.2

Section Summary

- Linear Homogeneous Recurrence Relations
- Solving Linear Homogeneous Recurrence Relations with Constant Coefficients.
- Solving Linear Nonhomogeneous Recurrence Relations with Constant Coefficients.

Linear Homogeneous Recurrence Relations

Definition: A *linear homogeneous recurrence relation of degree* k *with constant coefficients* is a recurrence relation of the form:

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots.. + c_k\, a_{n-k},$$

where c_1, c_2, ….,c_k are real numbers, and $c_k \neq 0$

- *linear* : the right-hand side is a sum of the previous terms of the sequence each multiplied by a function of n.
- *homogeneous* because no terms occur that are not multiples of the a_j's. Each coefficient is a constant.
- *degree* is k because a_n is expressed in terms of the previous k terms of the sequence.

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where c_1, c_2, ...,c_k are real numbers, and $c_k \neq 0$

By strong induction, a sequence satisfying such a recurrence relation is uniquely determined by the recurrence relation and the k initial conditions

$$a_0 = C_0, a_1 = C_1, \ldots, a_{k-1} = C_{k-1}.$$

Examples of Linear Homogeneous Recurrence Relations

- $P_n = (1.11)P_{n-1}$ linear homogeneous recurrence relation of degree one
- $f_n = f_{n-1} + f_{n-2}$ linear homogeneous recurrence relation of degree two
- $a_n = a_{n-1} + a_{n-2}^2$ not linear
- $H_n = 2H_{n-1} + 1$ not homogeneous
- $B_n = nB_{n-1}$ coefficients are not constants

Solving Linear Homogeneous Recurrence Relations

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- Note that $a_n = r^n$ is a solution to the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$ if and only if $r^n = c_1 r^{n-1} + c_2 r^{n-2} + \cdots + c_k r^{n-k}$.

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- Algebraic manipulation yields the *characteristic equation*:
  $r^k - c_1r^{k-1} - c_2r^{k-2} - \cdots - c_{k-1}r - c_k = 0$

Solving Linear Homogeneous Recurrence Relations

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- The solutions to the characteristic equation are called the *characteristic roots* of the recurrence relation.
- The roots are used to give an explicit formula for all the solutions of the recurrence relation.

Solving Linear Homogeneous Recurrence Relations of Degree Two

Theorem 1: Let c_1 and c_2 be real numbers.

Suppose that $r^2 - c_1 r - c_2 = 0$ has two distinct roots r_1 and r_2.

Then the sequence $\{a_n\}$ is a solution to the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if

$$a_n = \alpha r_1^n + \alpha_2 r_2^n$$

for $n = 0, 1, 2, \ldots$, where α_1 and α_2 are constants.

Using Theorem 1

Example: What is the solution to the recurrence relation

$a_n = a_{n-1} + 2a_{n-2}$ with $a_0 = 2$ and $a_1 = 7$?

Solution:

- The characteristic equation is $r^2 - r - 2 = 0$.
- Its roots are $r = 2$ and $r = -1$.
- $\{a_n\}$ is a solution to the recurrence relation if and *only if*
  - $a_n = \alpha_1 2^n + \alpha_2 (-1)^n$, *for some constants* α_1 *and* α_2.

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- $\{a_n\}$ is a solution to the recurrence relation if and *only if*
  - $a_n = \alpha_1 2^n + \alpha_2(-1)^n$, *for some constants* α_1 *and* α_2.

To find the constants α_1 and α_2:

$a_0 = 2 = \alpha_1 + \alpha_2$

$a_1 = 7 = \alpha_1 2 + \alpha_2(-1)$.

Solving these equations, we find that $\alpha_1 = 3$ and $\alpha_2 = -1$.

- Hence, the solution is the sequence $\{a_n\}$ with $a_n = 3 \cdot 2^n - (-1)^n$.

An Explicit Formula for the Fibonacci Numbers

We can use Theorem 1 to find an explicit formula for the Fibonacci numbers.

The sequence of Fibonacci numbers satisfies the recurrence relation $f_n = f_{n-1} + f_{n-2}$ with the initial conditions: $f_0 = 0$ and $f_1 = 1$.

Solution: The roots of the characteristic equation $r^2 - r - 1 = 0$ are

$$r_1 = \frac{1+\sqrt{5}}{2}$$

$$r_2 = \frac{1-\sqrt{5}}{2}$$

Fibonacci Numbers (*continued*)

Therefore by Theorem 1

$$f_n = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right)^n$$

for some constants α_1 and α_2.

Using the initial conditions $f_0 = 0$ and $f_1 = 1$, we have

$$f_0 = \alpha_1 + \alpha_2 = 0$$

$$f_1 = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right) + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right) = 1 .$$

Solving, we obtain

$$\alpha_1 = \frac{1}{\sqrt{5}} \quad , \quad \alpha_2 = -\frac{1}{\sqrt{5}}.$$

Hence,

$$f_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)^n$$

The Solution when there is a Repeated Root

Theorem 2: Let c_1 and c_2 be real numbers with $c_2 \neq 0$. Suppose that $r^2 - c_1 r - c_2 = 0$ has one *repeated root* r_0. Then the sequence $\{a_n\}$ is a solution to the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if

$$a_n = \alpha r_0^n + \alpha_2 n r_0^n$$

for $n = 0,1,2,\ldots$, where α_1 and α_2 are constants.

Using Theorem 2

Example: What is the solution to the recurrence relation $a_n = 6a_{n-1} - 9a_{n-2}$ with $a_0 = 1$ and $a_1 = 6$?

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Solution: The characteristic equation is $r^2 - 6r + 9 = 0$.
The only root is $r = 3$. Therefore, $\{a_n\}$ is a solution to the recurrence relation if and only if

$$a_n = \alpha_1 3^n + \alpha_2 n(3)^n$$

where α_1 and α_2 are constants.

To find the constants α_1 and α_2, note that

$a_0 = 1 = \alpha_1$ and
$a_1 = 6 = \alpha_1 \cdot 3 + \alpha_2 \cdot 3$.

Solving, we find that $\alpha_1 = 1$ and $\alpha_2 = 1$.
Hence, $a_n = 3^n + n3^n$.

Solving Linear Homogeneous Recurrence Relations of Arbitrary Degree

This theorem can be used to solve linear homogeneous recurrence relations with constan

Theorem 3: Let $c_1, c_2, \ldots, c_k$ be real numbers. Suppose that the characteristic equation

$$r^k - c_1 r^{k-1} - \cdots - c_k = 0$$

has k distinct roots $r_1, r_2, \ldots, r_k$. Then a sequence $\{a_n\}$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots.. + c_k a_{n-k}$$

if and only if

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \cdots + \alpha_k r_k^n$$

for $n = 0, 1, 2, \ldots$, where $\alpha_1, \alpha_2, \ldots, \alpha_k$ are constants.

The General Case with Repeated Roots Allowed

Theorem 4: Let $c_1, c_2, \ldots, c_k$ be real numbers. Suppose that the characteristic equation

$$r^k - c_1 r^{k-1} - \cdots - c_k = 0$$

has t distinct roots $r_1, r_2, \ldots, r_t$ with multiplicities $m_1, m_2, \ldots, m_t$, respectively so that $m_i \geq 1$ for $i = 0, 1, 2, \ldots, t$ and $m_1 + m_2 + \ldots + m_t = k$. Then a sequence $\{a_n\}$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$$

if and only if

$$a_n = (\alpha_{1,0} + \alpha_{1,1} n + \cdots + \alpha_{1,m_1-1} n^{m_1-1}) r_1^n + (\alpha_{2,0} + \alpha_{2,1} n + \cdots + \alpha_{2,m_2-1} n^{m_2-1}) r_2^n + \cdots + (\alpha_{t,0} + \alpha_{t,1} n + \cdots + \alpha_{t,m_t-1} n^{m_t-1}) r_t^n$$

for $n = 0, 1, 2, \ldots$, where $\alpha_{i,j}$ are constants for $1 \leq i \leq t$ and $0 \leq j \leq m_i - 1$.

multiplicity of root r_1

multiplicity of root r_2

multiplicity of root r_t

Linear Nonhomogeneous Recurrence Relations with Constant Coefficients

Definition: A *linear nonhomogeneous recurrence relation with constant coefficients* is a recurrence relation of the form:

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots.. + c_k\, a_{n-k} + F(n)\,,$$

where c_1, c_2, ...,c_k are real numbers, and $F(n)$ is a function not identically zero depending only on n.

The recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots.. + c_k\, a_{n-k}\,,$$

is called the *associated homogeneous recurrence relation*.

Linear Nonhomogeneous Recurrence Relations with Constant Coefficients (*cont.*)

The following are linear nonhomogeneous recurrence relations with constant coefficients:

$a_n = a_{n-1} + 2^n$,

$a_n = a_{n-1} + a_{n-2} + n^2 + n + 1$,

$a_n = 3a_{n-1} + n3^n$,

$a_n = a_{n-1} + a_{n-2} + a_{n-3} + n!$

where the following are the associated linear homogeneous recurrence relations, respectively:

$a_n = a_{n-1}$,

$a_n = a_{n-1} + a_{n-2}$,

$a_n = 3a_{n-1}$,

$a_n = a_{n-1} + a_{n-2} + a_{n-3}$

Solving Linear Nonhomogeneous Recurrence Relations with Constant Coefficients

Theorem 5: If $\{a_n^{(p)}\}$ is a particular solution of the nonhomogeneous linear recurrence relation with constant coefficients

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n),$$

then every solution is of the form $\{a_n^{(p)} + a_n^{(h)}\}$, where $\{a_n^{(h)}\}$: solution of the associated homogeneous recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}.$$

Solving Linear Nonhomogeneous Recurrence Relations with Constant Coefficients (*continued*)

Example: Find all solutions of the recurrence relation $a_n = 3a_{n-1} + 2n$. What is the solution with $a_1 = 3$?

Solution: The associated linear homogeneous equation is $a_n = 3a_{n-1}$. Its solutions are $a_n^{(h)} = \alpha 3^n$, where α is a constant.

Solving Linear Nonhomogeneous Recurrence Relations with Constant Coefficients (*continued*)

Example: Find all solutions of the recurrence relation $a_n = 3a_{n-1} + 2n$.
What is the solution with $a_1 = 3$?

Solution: The associated linear homogeneous equation is $a_n = 3a_{n-1}$.
Its solutions are $a_n^{(h)} = \alpha 3^n$, where α is a constant.

Because $F(n)= 2n$ is a polynomial in n of degree one, to find a particular solution we might try a linear function in n, say $p_n = cn + d$, where c and d are constants.

Solving Linear Nonhomogeneous Recurrence Relations with Constant Coefficients (*continued*)

Example: Find all solutions of the recurrence relation $a_n = 3a_{n-1} + 2n$.
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Because $F(n) = 2n$ is a polynomial in n of degree one, to find a particular solution we might try a linear function in n, say $p_n = cn + d$, where c and d are constants.

Suppose that $p_n = cn + d$ is such a solution.
Then $a_n = 3a_{n-1} + 2n$ becomes $cn + d = 3(c(n-1) + d) + 2n$.

Solving Linear Nonhomogeneous Recurrence Relations with Constant Coefficients (*continued*)

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Then $a_n = 3a_{n-1} + 2n$ becomes $cn + d = 3(c(n-1) + d)+ 2n$.

Simplifying yields $(2 + 2c)n + (2d - 3c) = 0$. It follows that $cn + d$ is a solution if and only if $2 + 2c = 0$ and $2d - 3c = 0$. Therefore, $cn + d$ is a solution if and only if $c = -1$ and $d = -3/2$.
Consequently, $a_n^{(p)} = -n - 3/2$ is a particular solution.

Solving Linear Nonhomogeneous Recurrence Relations with Constant Coefficients (*continued*)

Example: Find all solutions of the recurrence relation $a_n = 3a_{n-1} + 2n$. What is the solution with $a_1 = 3$?

Solution: The associated linear homogeneous equation is $a_n = 3a_{n-1}$.
$a_n^{(h)} = \alpha 3^n$, homogeneous solution.

$a_n^{(p)} = -n - 3/2$ is a particular solution.

By Theorem 5, all solutions are of the form

$a_n = a_n^{(p)} + a_n^{(h)} = -n - 3/2 + \alpha 3^n$, where α is a constant.

Solving Linear Nonhomogeneous Recurrence Relations with Constant Coefficients (*continued*)

Example: Find all solutions of the recurrence relation $a_n = 3a_{n-1} + 2n$.
What is the solution with $a_1 = 3$?

Solution: The associated linear homogeneous equation is $a_n = 3a_{n-1}$.
$a_n^{(h)} = \alpha 3^n$, homogeneous solution.

$a_n^{(p)} = -n - 3/2$ is a particular solution.

By Theorem 5, all solutions are of the form

$a_n = a_n^{(p)} + a_n^{(h)} = -n - 3/2 + \alpha 3^n$, where α is a constant.

To find the solution with $a_1 = 3$,
let $n = 1$ in the above formula for the general solution.

Then $3 = -1 - 3/2 + 3\alpha$, and $\alpha = 11/6$.

Hence, the solution is $a_n = -n - 3/2 + (11/6)3^n$.