

Lecture 4: Review of Discrete Probability
BBM205

Exercises (from Rosen's book)

$E_1 \cap E_2$ is the event that it is divisible by both 2 and 5, or equivalently, that it is divisible by 10. Because $|E_1| = 50$, $|E_2| = 20$, and $|E_1 \cap E_2| = 10$, it follows that

$$\begin{aligned} p(E_1 \cup E_2) &= p(E_1) + p(E_2) - p(E_1 \cap E_2) \\ &= \frac{50}{100} + \frac{20}{100} - \frac{10}{100} = \frac{3}{5}. \end{aligned}$$

Probabilistic Reasoning

A common problem is determining which of two events is more likely. Analyzing the probabilities of such events can be tricky. Example 10 describes a problem of this type. It discusses a famous problem originating with the television game show *Let's Make a Deal*.

EXAMPLE 10



The Monty Hall Three-Door Puzzle Suppose you are a game show contestant. You have a chance to win a large prize. You are asked to select one of three doors to open; the large prize is behind one of the three doors and the other two doors are losers. Once you select a door, the game show host, who knows what is behind each door, does the following. First, whether or not you selected the winning door, he opens one of the other two doors that he knows is a losing door (selecting at random if both are losing doors). Then he asks you whether you would like to switch doors. Which strategy should you use? Should you change doors or keep your original selection, or does it not matter?

Solution: The probability you select the correct door (before the host opens a door and asks you whether you want to change) is $1/3$, because the three doors are equally likely to be the correct door. The probability this is the correct door does not change once the game show host opens one of the other doors, because he will always open a door that the prize is not behind.

The probability that you selected incorrectly is the probability the prize is behind one of the two doors you did not select. Consequently, the probability that you selected incorrectly is $2/3$. If you selected incorrectly, when the game show host opens a door to show you that the prize is not behind it, the prize is behind the other door. You will always win if your initial choice was incorrect and you change doors. So, by changing doors, the probability you win is $2/3$. In other words, you should always change doors when given the chance to do so by the game show host. This doubles the probability that you will win. (A more rigorous treatment of this puzzle can be found in Exercise 15 of Section 6.3.)

Exercises

- What is the probability that a card selected from a deck is an ace?
- What is the probability that a die comes up six when it is rolled?
- What is the probability that a randomly selected integer chosen from the first 100 positive integers is odd?
- What is the probability that a randomly selected day of the year (from the 366 possible days) is in April?
- What is the probability that the sum of the numbers on two dice is even when they are rolled?
- What is the probability that a card selected from a deck is an ace or a heart?
- What is the probability that when a coin is flipped six times in a row, it lands heads up every time?
- What is the probability that a five-card poker hand contains the ace of hearts?
- What is the probability that a five-card poker hand does not contain the queen of hearts?
- What is the probability that a five-card poker hand contains the two of diamonds and the three of spades?
- What is the probability that a five-card poker hand contains the two of diamonds, the three of spades, the six of hearts, the ten of clubs, and the king of hearts?

12. What is the probability that a five-card poker hand contains exactly one ace?
13. What is the probability that a five-card poker hand contains at least one ace?
14. What is the probability that a five-card poker hand contains cards of five different kinds?
15. What is the probability that a five-card poker hand contains two pairs (that is, two of each of two different kinds and a fifth card of a third kind)?
16. What is the probability that a five-card poker hand contains a flush, that is, five cards of the same suit?
17. What is the probability that a five-card poker hand contains a straight, that is, five cards that have consecutive kinds? (Note that an ace can be considered either the lowest card of an A-2-3-4-5 straight or the highest card of a 10-J-Q-K-A straight.)
18. What is the probability that a five-card poker hand contains a straight flush, that is, five cards of the same suit of consecutive kinds?
- *19. What is the probability that a five-card poker hand contains cards of five different kinds and does not contain a flush or a straight?
20. What is the probability that a five-card poker hand contains a royal flush, that is, the 10, jack, queen, king, and ace of one suit?
21. What is the probability that a die never comes up an even number when it is rolled six times?
22. What is the probability that a positive integer not exceeding 100 selected at random is divisible by 3?
23. What is the probability that a positive integer not exceeding 100 selected at random is divisible by 5 or 7?
24. Find the probability of winning the lottery by selecting the correct six integers, where the order in which these integers are selected does not matter, from the positive integers not exceeding
 - a) 30. b) 36. c) 42. d) 48.
25. Find the probability of winning the lottery by selecting the correct six integers, where the order in which these integers are selected does not matter, from the positive integers not exceeding
 - a) 50. b) 52. c) 56. d) 60.
26. Find the probability of selecting none of the correct six integers, where the order in which these integers are selected does not matter, from the positive integers not exceeding
 - a) 40. b) 48. c) 56. d) 64.
27. Find the probability of selecting exactly one of the correct six integers, where the order in which these integers are selected does not matter, from the positive integers not exceeding
 - a) 40. b) 48. c) 56. d) 64.
28. In a superlottery, a player selects 7 numbers out of the first 80 positive integers. What is the probability that a person wins the grand prize by picking 7 numbers that are among the 11 numbers selected at random by a computer.
29. In a superlottery, players win a fortune if they choose the eight numbers selected by a computer from the positive integers not exceeding 100. What is the probability that a player wins this superlottery?
30. What is the probability that a player wins the prize offered for correctly choosing five (but not six) numbers out of six integers chosen at random from the integers between 1 and 40, inclusive?
31. Suppose that 100 people enter a contest and that different winners are selected at random for first, second, and third prizes. What is the probability that Michelle wins one of these prizes if she is one of the contestants?
32. Suppose that 100 people enter a contest and that different winners are selected at random for first, second, and third prizes. What is the probability that Kumar, Janice, and Pedro each win a prize if each has entered the contest?
33. What is the probability that Abby, Barry, and Sylvia win the first, second, and third prizes, respectively, in a drawing if 200 people enter a contest and
 - a) no one can win more than one prize.
 - b) winning more than one prize is allowed.
34. What is the probability that Bo, Colleen, Jeff, and Rohini win the first, second, third, and fourth prizes, respectively, in a drawing if 50 people enter a contest and
 - a) no one can win more than one prize.
 - b) winning more than one prize is allowed.
35. In roulette, a wheel with 38 numbers is spun. Of these, 18 are red, and 18 are black. The other two numbers, which are neither black nor red, are 0 and 00. The probability that when the wheel is spun it lands on any particular number is $1/38$.
 - a) What is the probability that the wheel lands on a red number?
 - b) What is the probability that the wheel lands on a black number twice in a row?
 - c) What is the probability that the wheel lands on 0 or 00?
 - d) What is the probability that in five spins the wheel never lands on either 0 or 00?
 - e) What is the probability that the wheel lands on a number between 1 and 6, inclusive, on one spin, but does not land between them on the next spin?
36. Which is more likely: rolling a total of 8 when two dice are rolled or rolling a total of 8 when three dice are rolled?
37. Which is more likely: rolling a total of 9 when two dice are rolled or rolling a total of 9 when three dice are rolled?
38. Two events E_1 and E_2 are called **independent** if $p(E_1 \cap E_2) = p(E_1)p(E_2)$. For each of the following pairs of events, which are subsets of the set of all possible outcomes when a coin is tossed three times, determine whether or not they are independent.
 - a) E_1 : the first coin comes up tails; E_2 : the second coin comes up heads.

- b) E_1 : the first coin comes up tails; E_2 : two, and not three, heads come up in a row.
- c) E_1 : the second coin comes up tails; E_2 : two, and not three, heads come up in a row.

(We will study independence of events in more depth in Section 6.2.)

39. Explain what is wrong with the statement that in the Monty Hall Three-Door Puzzle the probability that the prize is behind the first door you select and the probability that the prize is behind the other of the two doors that Monty does not open are both $1/2$, because there are two doors left.
40. Suppose that instead of three doors, there are four doors in the Monty Hall puzzle. What is the probability that you win by not changing once the host, who knows what is

behind each door, opens a losing door and gives you the chance to change doors? What is the probability that you win by changing the door you select to one of the two remaining doors among the three that you did not select?

41. This problem was posed by the Chevalier de Méré and was solved by Blaise Pascal and Pierre de Fermat.
 - a) Find the probability of rolling at least one six when a die is rolled four times.
 - b) Find the probability that a double six comes up at least once when a pair of dice is rolled 24 times. Answer the query the Chevalier de Méré made to Pascal asking whether this probability was greater than $1/2$.
 - c) Is it more likely that a six comes up at least once when a die is rolled four times or that a double six comes up at least once when a pair of dice is rolled 24 times?

6.2 Probability Theory

Introduction

Links



In Section 6.1 we introduced the notion of the probability of an event. (Recall that an event is a subset of the possible outcomes of an experiment.) We defined the probability of an event E as Laplace did, that is,

$$p(E) = \frac{|E|}{|S|},$$

the number of outcomes in E divided by the total number of outcomes. This definition assumes that all outcomes are equally likely. However, many experiments have outcomes that are not equally likely. For instance, a coin may be biased so that it comes up heads twice as often as tails. Similarly, the likelihood that the input of a linear search is a particular element in a list, or is not in the list, depends on how the input is generated. How can we model the likelihood of events in such situations? In this section we will show how to define probabilities of outcomes to study probabilities of experiments where outcomes may not be equally likely.

Suppose that a fair coin is flipped four times, and the first time it comes up heads. Given this information, what is the probability that heads comes up three times? To answer this and similar questions, we will introduce the concept of *conditional probability*. Does knowing that the first flip comes up heads change the probability that heads comes up three times? If not, these two events are called *independent*, a concept studied later in this section.

Many questions address a particular numerical value associated with the outcome of an experiment. For instance, when we flip a coin 100 times, what is the probability that exactly 40 heads appear? How many heads should we expect to appear? In this section we will introduce *random variables*, which are functions that associate numerical values to the outcomes of experiments.

Links



HISTORICAL NOTE The Chevalier de Méré was a French nobleman, a famous gambler, and a bon vivant. He was successful at making bets with odds slightly greater than $1/2$ (such as having at least one six come up in four tosses of a die). His correspondence with Pascal asking about the probability of having at least one double six come up when a pair of dice is rolled 24 times led to the development of probability theory. According to one account, Pascal wrote to Fermat about the Chevalier saying something like “He’s a good guy but, alas, he’s no mathematician.”

or mutual enemies. The probability that there are either k mutual friends or k mutual enemies among the n people equals $p(\bigcup_{i=1}^{\binom{n}{k}} E_i)$.

According to our assumption it is equally likely for two people to be friends or enemies. The probability that two people are friends equals the probability that they are enemies; both probabilities equal $1/2$. Furthermore, there are $\binom{k}{2} = k(k-1)/2$ pairs of people in S_i because there are k people in S_i . Hence, the probability that all k people in S_i are mutual friends and the probability that all k people in S_i are mutual enemies both equal $(1/2)^{k(k-1)/2}$. It follows that $p(E_i) = 2(1/2)^{k(k-1)/2}$.

The probability that there are either k mutual friends or k mutual enemies in the group of n people equals $p(\bigcup_{i=1}^{\binom{n}{k}} E_i)$. Using Boole's Inequality (Exercise 15), it follows that

$$p\left(\bigcup_{i=1}^{\binom{n}{k}} E_i\right) \leq \sum_{i=1}^{\binom{n}{k}} p(E_i) = \binom{n}{k} \cdot 2\left(\frac{1}{2}\right)^{k(k-1)/2}.$$

By Exercise 17 in Section 5.4, we have $\binom{n}{k} \leq n^k / 2^{k-1}$. Hence,

$$\binom{n}{k} 2\left(\frac{1}{2}\right)^{k(k-1)/2} \leq \frac{n^k}{2^{k-1}} 2\left(\frac{1}{2}\right)^{k(k-1)/2}.$$

Now if $n < 2^{k/2}$, we have

$$\frac{n^k}{2^{k-1}} 2\left(\frac{1}{2}\right)^{k(k-1)/2} < \frac{2^{k(k/2)}}{2^{k-1}} 2\left(\frac{1}{2}\right)^{k(k-1)/2} = 2^{2-(k/2)} \leq 1$$

where the last step follows because $k \geq 4$.

We can now conclude that $p(\bigcup_{i=1}^{\binom{n}{k}} E_i) < 1$ when $k \geq 4$. Hence, the probability of the complementary event, that there is no set of either k mutual friends or mutual enemies at the party, is greater than 0. It follows that if $n < 2^{k/2}$, there is at least one set such that no subset of k people are mutual friends or mutual enemies. \triangleleft

Exercises

- What probability should be assigned to the outcome of heads when a biased coin is tossed, if heads is three times as likely to come up as tails? What probability should be assigned to the outcome of tails?
- Find the probability of each outcome when a loaded die is rolled, if a 3 is twice as likely to appear as each of the other five numbers on the die.
- Find the probability of each outcome when a biased die is rolled, if rolling a 2 or rolling a 4 is three times as likely as rolling each of the other four numbers on the die and it is equally likely to roll a 2 or a 4.
- Show that conditions (i) and (ii) are met under Laplace's definition of probability, when outcomes are equally likely.
- A pair of dice is loaded. The probability that a 4 appears on the first die is $2/7$, and the probability that a 3 appears on the second die is $2/7$. Other outcomes for each die appear with probability $1/7$. What is the probability of 7 appearing as the sum of the numbers when the two dice are rolled?
- What is the probability of these events when we randomly select a permutation of $\{1, 2, 3\}$?
 - 1 precedes 3.
 - 3 precedes 1.
 - 3 precedes 1 and 3 precedes 2.
- What is the probability of these events when we randomly select a permutation of $\{1, 2, 3, 4\}$?
 - 1 precedes 4.
 - 4 precedes 1.
 - 4 precedes 1 and 4 precedes 2.
 - 4 precedes 1, 4 precedes 2, and 4 precedes 3.
 - 4 precedes 3 and 2 precedes 1.

8. What is the probability of these events when we randomly select a permutation of $\{1, 2, \dots, n\}$ where $n \geq 4$?
- 1 precedes 2.
 - 2 precedes 1.
 - 1 immediately precedes 2.
 - n precedes 1 and $n - 1$ precedes 2.
 - n precedes 1 and n precedes 2.
9. What is the probability of these events when we randomly select a permutation of the 26 lowercase letters of the English alphabet?
- The permutation consists of the letters in reverse alphabetic order.
 - z is the first letter of the permutation.
 - z precedes a in the permutation.
 - a immediately precedes z in the permutation.
 - a immediately precedes m , which immediately precedes z in the permutation.
 - m , n , and o are in their original places in the permutation.
10. What is the probability of these events when we randomly select a permutation of the 26 lowercase letters of the English alphabet?
- The first 13 letters of the permutation are in alphabetical order.
 - a is the first letter of the permutation and z is the last letter.
 - a and z are next to each other in the permutation.
 - a and b are not next to each other in the permutation.
 - a and z are separated by at least 23 letters in the permutation.
 - z precedes both a and b in the permutation.
11. Suppose that E and F are events such that $p(E) = 0.7$ and $p(F) = 0.5$. Show that $p(E \cup F) \geq 0.7$ and $p(E \cap F) \geq 0.2$.
12. Suppose that E and F are events such that $p(E) = 0.8$ and $p(F) = 0.6$. Show that $p(E \cup F) \geq 0.8$ and $p(E \cap F) \geq 0.4$.
13. Show that if E and F are events, then $p(E \cap F) \geq p(E) + p(F) - 1$. This is known as **Bonferroni's Inequality**.
14. Use mathematical induction to prove the following generalization of Bonferroni's Inequality:

$$p(E_1 \cap E_2 \cap \dots \cap E_n) \geq p(E_1) + p(E_2) + \dots + p(E_n) - (n - 1),$$

where E_1, E_2, \dots, E_n are n events.

15. Show that if E_1, E_2, \dots, E_n are events from a finite sample space, then

$$p(E_1 \cup E_2 \cup \dots \cup E_n) \leq p(E_1) + p(E_2) + \dots + p(E_n).$$

This is known as **Boole's Inequality**.

16. Show that if E and F are independent events, then \bar{E} and \bar{F} are also independent events.

17. If E and F are independent events, prove or disprove that \bar{E} and F are necessarily independent events.

In Exercises 18, 20, and 21 assume that the year has 366 days and all birthdays are equally likely. In Exercise 19 assume it is equally likely that a person is born in any given month of the year.

18. a) What is the probability that two people chosen at random were born on the same day of the week?
 b) What is the probability that in a group of n people chosen at random, there are at least two born on the same day of the week?
 c) How many people chosen at random are needed to make the probability greater than $1/2$ that there are at least two people born on the same day of the week?
19. a) What is the probability that two people chosen at random were born during the same month of the year?
 b) What is the probability that in a group of n people chosen at random, there are at least two born in the same month of the year?
 c) How many people chosen at random are needed to make the probability greater than $1/2$ that there are at least two people born in the same month of the year?
20. Find the smallest number of people you need to choose at random so that the probability that at least one of them has a birthday today exceeds $1/2$.
21. Find the smallest number of people you need to choose at random so that the probability that at least two of them were both born on April 1 exceeds $1/2$.
- *22. February 29 occurs only in leap years. Years divisible by 4, but not by 100, are always leap years. Years divisible by 100, but not by 400, are not leap years, but years divisible by 400 are leap years.
- What probability distribution for birthdays should be used to reflect how often February 29 occurs?
 - Using the probability distribution from part (a), what is the probability that in a group of n people at least two have the same birthday?
23. What is the conditional probability that exactly four heads appear when a fair coin is flipped five times, given that the first flip came up heads?
24. What is the conditional probability that exactly four heads appear when a fair coin is flipped five times, given that the first flip came up tails?
25. What is the conditional probability that a randomly generated bit string of length four contains at least two consecutive 0s, given that the first bit is a 1? (Assume the probabilities of a 0 and a 1 are the same.)
26. Let E be the event that a randomly generated bit string of length three contains an odd number of 1s, and let F be the event that the string starts with 1. Are E and F independent?
27. Let E and F be the events that a family of n children has children of both sexes and has at most one boy, respectively. Are E and F independent if
- $n = 2$
 - $n = 4$
 - $n = 5$

28. Assume that the probability a child is a boy is 0.51 and that the sexes of children born into a family are independent. What is the probability that a family of five children has
- exactly three boys?
 - at least one boy?
 - at least one girl?
 - all children of the same sex?
29. A group of six people play the game of “odd person out” to determine who will buy refreshments. Each person flips a fair coin. If there is a person whose outcome is not the same as that of any other member of the group, this person has to buy the refreshments. What is the probability that there is an odd person out after the coins are flipped once?
30. Find the probability that a randomly generated bit string of length 10 does not contain a 0 if bits are independent and if
- a 0 bit and a 1 bit are equally likely.
 - the probability that a bit is a 1 is 0.6.
 - the probability that the i th bit is a 1 is $1/2^i$ for $i = 1, 2, 3, \dots, 10$.
31. Find the probability that a family with five children does not have a boy, if the sexes of children are independent and if
- a boy and a girl are equally likely.
 - the probability of a boy is 0.51.
 - the probability that the i th child is a boy is $0.51 - (i/100)$.
32. Find the probability that a randomly generated bit string of length 10 begins with a 1 or ends with a 00 for the same conditions as in parts (a), (b), and (c) of Exercise 30, if bits are generated independently.
33. Find the probability that the first child of a family with five children is a boy or that the last two children of the family are girls, for the same conditions as in parts (a), (b), and (c) of Exercise 31.
34. Find each of the following probabilities when n independent Bernoulli trials are carried out with probability of success p .
- the probability of no successes
 - the probability of at least one success
 - the probability of at most one success
 - the probability of at least two successes
35. Find each of the following probabilities when n independent Bernoulli trials are carried out with probability of success p .
- the probability of no failures
 - the probability of at least one failure
 - the probability of at most one failure
 - the probability of at least two failures
36. Use mathematical induction to prove that if E_1, E_2, \dots, E_n is a sequence of n pairwise disjoint events in a sample space S , where n is a positive integer, then $p(\bigcup_{i=1}^n E_i) = \sum_{i=1}^n p(E_i)$.
- *37. (Requires calculus) Show that if E_1, E_2, \dots is an infinite sequence of pairwise disjoint events in a sample space S , then $p(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} p(E_i)$. [Hint: Use Exercise 36 and take limits.]
38. A pair of dice is rolled in a remote location and when you ask an honest observer whether at least one die came up six, this honest observer answers in the affirmative.
- What is the probability that the sum of the numbers that came up on the two dice is seven, given the information provided by the honest observer?
 - Suppose that the honest observer tells us that at least one die came up five. What is the probability the sum of the numbers that came up on the dice is seven, given this information?
- *39. This exercise employs the probabilistic method to prove a result about round-robin tournaments. In a **round-robin tournament** with m players, every two players play one game in which one player wins and the other loses.
- We want to find conditions on positive integers m and k with $k < m$ such that it is possible for the outcomes of the tournament to have the property that for every set of k players, there is a player who beats every member in this set. So that we can use probabilistic reasoning to draw conclusions about round-robin tournaments, we assume that when two players compete it is equally likely that either player wins the game and we assume that the outcomes of different games are independent. Let E be the event that for every set S with k players, where k is a positive integer less than m , there is a player who has beaten all k players in S .
- Show that $p(\overline{E}) \leq \sum_{j=1}^m p(F_j)$, where F_j is the event that there is no player who beats all k players from the j th set in a list of the $\binom{m}{k}$ sets of k players.
 - Show that the probability of F_j is $(1 - 2^{-k})^{m-k}$.
 - Conclude from parts (a) and (b) that $p(\overline{E}) \leq \binom{m}{k}(1 - 2^{-k})^{m-k}$ and, therefore, that there must be a tournament with the described property if $\binom{m}{k}(1 - 2^{-k})^{m-k} < 1$.
 - Use part (c) to find values of m such that there is a tournament with m players such that for every set S of two players, there is a player who has beaten both players in S . Repeat for sets of three players.
- *40. Devise a Monte Carlo algorithm that determines whether a permutation of the integers 1 through n has already been sorted (that is, it is in increasing order), or instead, is a random permutation. A step of the algorithm should answer “true” if it determines the list is not sorted and “unknown” otherwise. After k steps, the algorithm decides that the integers are sorted if the answer is “unknown” in each step. Show that as the number of steps increases, the probability that the algorithm produces an incorrect answer is extremely small. [Hint: For each step, test whether certain elements are in the correct order. Make sure these tests are independent.]
41. Use pseudocode to write out the probabilistic primality test described in Example 16.