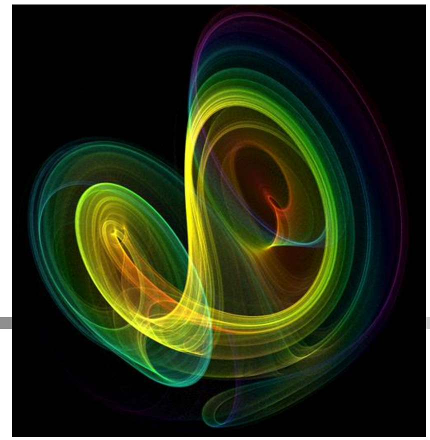
DYNAMICAL SYSTEMS



Prof. Dr. Mehmet Önder Efe

E-Mail : **onderefe@gmail.com**
URL : **http://web.cs.hacettepe.edu.tr/~onderefe/**
Textbook : **M.Ö. Efe, *Otomatik Kontrol Sistemleri*, Seçkin Y.**

Course Outline

PART 1

- **Introduction to Control Engineering**
- **Review of Complex Variables & Functions**
- **Review of Laplace Transform**
- **Review of Linear Algebra**

PART 2

- **Linear Differential Equations**
- **Obtaining Transfer Functions (TFs)**
- **Block Diagrams**
- **An Introduction to Stability for TFs**
- **Concept of Feedback and Closed Loop**
- **Basic Control Actions, P-I-D Effects**

PART 3

- **Concept of Stability**
- **Stability Analysis of the Closed Loop System by Routh Criterion**
- **State Space Representation and Stability**

PART 4

- **Transient Response Analysis**
  - **First Order Systems**
  - **Second Order Systems**
  - **Using MATLAB with Simulink**
- **Steady State Errors**

PARTS 5-6

- Root Locus Analysis
- Design Based on Root Locus
- Midterm

PART 7

- Frequency Response Analysis
  - Bode Plots
  - Gain Margin and Phase Margin
  - Polar Plots and Margins
  - Nyquist Stability Criterion

PARTS 8-9

- **Design of Control Systems in State Space**
  - **Canonical Realizations**
  - **Controllability and Observability**
  - **Linear State Feedback**
  - **Pole Placement**
  - **Bass-Gura and Ackermann Formulations**
  - **Properties of State Feedback**
  - **Observer Design and Observer Based Compensators**

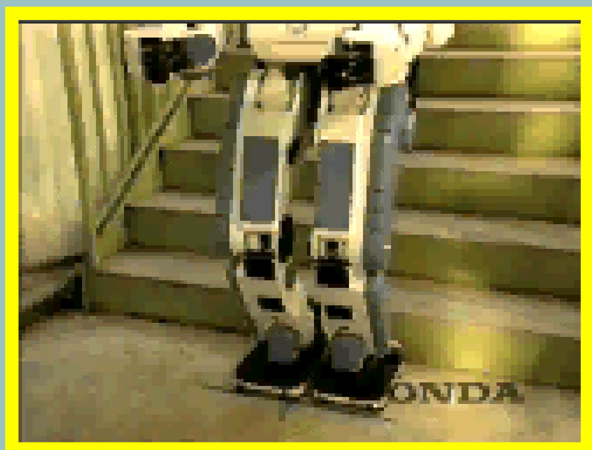
PART 10

- Concept of Robustness
- Concept of Optimality
- Concept of Adaptive Systems
- Concept of Intelligence in Control

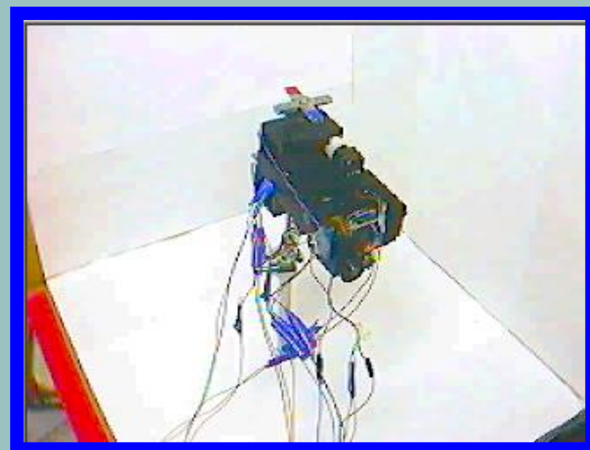
PART 11

- Final Exam

P-1 Introduction to Dynamical Systems

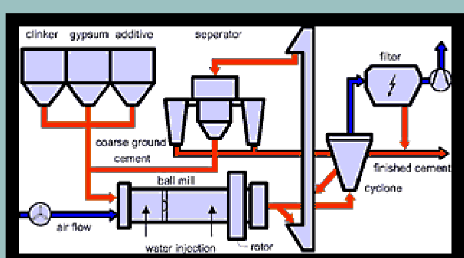


© Honda, Humanoid Robot



Courtesy: Efe, Acay, Unsal,
Vande Weghe, Khosla,
Carnegie Mellon University, 2001

Courtesy: Acay
Carnegie Mellon University, 2001



Industrial process control

Control of unmanned aerial vehicles

Chemical process control

What is a dynamic system?

A dynamical system is a concept in mathematics where a fixed rule describes the time dependence of a point in a geometrical space.

http://en.wikipedia.org/wiki/Dynamical_system

What is control theory?

The mathematical study of how to manipulate the parameters affecting the behavior of a system to produce the desired or optimal outcome.

http://mathworld.wolfram.com/ControlTheory.html

How to classify in terms of time?

- Continuous time systems
  - -Differential equations
    - Laplace transform
- Discrete time systems
  - -Difference equations
    - Issues of sampling
    - z Transform

How to classify in terms of representation?

- **Linear systems**
  - **-Differential equations**
  - **-Difference equations**
- **Nonlinear systems**
  - **-Differential equations**
  - **-Difference equations**

How to classify in terms of representation?

- **Ordinary Differential Equations**
- **Partial Differential Equations**

What common alternatives do we have?

- **Proportional Integral Derivative**
- **Classical control**
- **State space methods**
- **Optimal control**
- **Robust control**
- **Nonlinear control**
- **Stochastic control**
- **Adaptive control**
- **Intelligent control**
- **…**

What engineering aspects should we consider?

- Disturbance rejection
- Insensitivity to parameter variations
- Stability
- Rise time
- Overshoot
- Settling time
- Steady state error
- ...

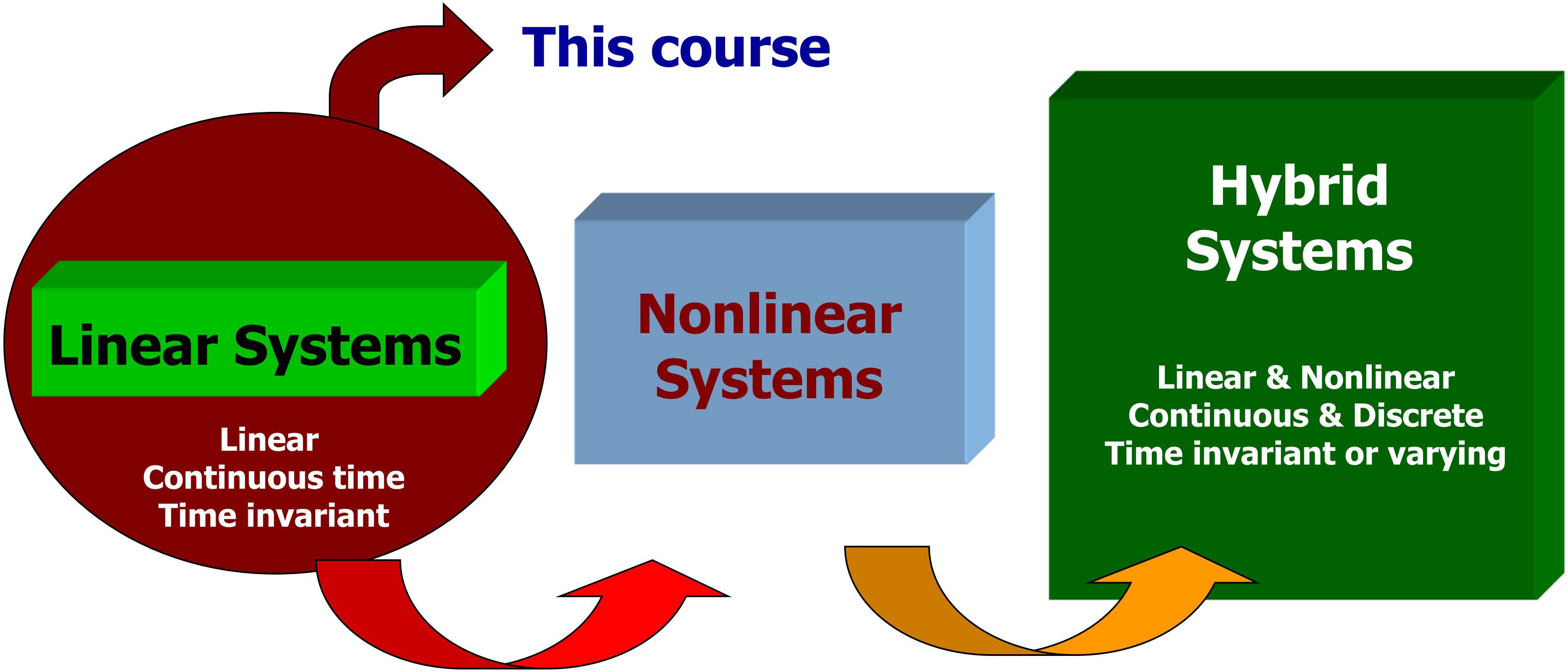
What else should we think about?

- Cost (money/time)
- Computational complexity
- Manufacturability (any extraordinary requirements?)
- Reliability (mean time between failures)
- Adaptability (with low cost for similar applications)
- Understandability
- Politics (opinions of your boss and distance from standard practice)

What mathematical tools shall we use?

- Calculus & Linear Algebra
- Laplace Transform
- Fourier Transform
- Complex Variables and Functions
- Ordinary Differential Equations (ODE)
- ...

What sort of systems shall we cover?



A natural way to follow is to start with Linear Time Invariant (LTI) Systems

P-1 Review of Complex Variables & Functions

$s = \sigma + j\omega$ → Complex variable

$F(s) = F_x + jF_y$ → Function of the complex variable s

$|F(s)| = \left(F_x^2 + F_y^2\right)^{1/2}$ → Magnitude of the function F(s)

$\angle F(s) = \tan^{-1} \dfrac{F_y}{F_x}$ → Angle of the function F(s)

$\overline{F}(s) = F_x - jF_y$ → Complex conjugate of the function F(s)

$$\frac{d}{ds}G(s) = \lim_{\Delta\sigma\to 0}\left(\frac{\Delta G_x}{\Delta\sigma} + j\frac{\Delta G_y}{\Delta\sigma}\right)$$

$$= \frac{\partial G_x}{\partial\sigma} + j\frac{\partial G_y}{\partial\sigma}$$

$$\frac{d}{ds}G(s) = \lim_{\Delta s\to 0}\frac{G(s+\Delta s)-G(s)}{\Delta s}$$

$$= \lim_{\Delta s\to 0}\frac{\Delta G}{\Delta s}$$

$$\frac{d}{ds}G(s) = \lim_{j\Delta\omega\to 0}\left(\frac{\Delta G_x}{j\Delta\omega} + j\frac{\Delta G_y}{j\Delta\omega}\right)$$

$$= -j\frac{\partial G_x}{\partial\omega} + \frac{\partial G_y}{\partial\omega}$$

If the derivative along these two directions give the same value

$$\frac{\partial G_x}{\partial \sigma} + j\frac{\partial G_y}{\partial \sigma} = -j\frac{\partial G_x}{\partial \omega} + \frac{\partial G_y}{\partial \omega} \quad \text{or}$$

Cauchy-Riemann conditions

$$\frac{\partial G_x}{\partial \sigma} = \frac{\partial G_y}{\partial \omega} \quad \text{and} \quad \frac{\partial G_y}{\partial \sigma} = -\frac{\partial G_x}{\partial \omega}$$

Then the derivative dG(s)/ds can uniquely be determined

$$G(s)=\frac{1}{s+1} \text{ with } s=\sigma+j\omega$$

$$\mathrm{G}(\sigma+j\omega)=\frac{1}{\sigma+j\omega+1}=G_x+jG_y$$

$$G_x=\frac{\sigma+1}{(\sigma+1)^2+\omega^2} \text{ and } G_y=\frac{-\omega}{(\sigma+1)^2+\omega^2}$$

Except at $s=-1$ (i.e. $\sigma=-1$ and $\omega=0$)

$$\frac{\partial G_x}{\partial\sigma}=\frac{\partial G_y}{\partial\omega}=\frac{\omega^2-(\sigma+1)^2}{[(\sigma+1)^2+\omega^2]^2}$$

$$\frac{\partial G_y}{\partial\sigma}=-\frac{\partial G_x}{\partial\omega}=\frac{2\omega(\sigma+1)}{[(\sigma+1)^2+\omega^2]^2}$$

Hence the derivative dG(s)/ds is analytic in the entire s-plane except at s=-1; the derivative is as follows:

$$\boxed{\frac{dG(s)}{ds} = -\frac{1}{(s+1)^2}}$$

The derivative of an analytic function can be obtained by differentiating G(s) simply with respect to (w.r.t) s.

- The points at which the function G(s) is analytic are called **ordinary points**
- The points at which the function G(s) is **not** analytic are called **singular points**
- At singular points the function G(s) or its derivatives approach infinity, and these points are called **poles**
- The function $G(s)=1/(s+1)$ has a pole at $s=-1$, and this pole is single. $G(s)=1/(s+1)^p$ has p poles all at $s=-1$.
- The function $G(s)=(s+3)/[(s+1)(s+2)]$ has two zeros at $s=-3$ and $\boldsymbol{s=\infty}$; and two poles at $s_1=-1$ and $s_2=-2$

Euler's Theorem

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots$$

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots$$

$$\cos\theta + j\sin\theta = 1 + (j\theta) + \frac{(j\theta)^2}{2!} + \frac{(j\theta)^3}{3!} + \frac{(j\theta)^4}{4!} + \cdots$$

$$\text{since } e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$\cos\theta + j\sin\theta = e^{j\theta}$$

$$\cos\theta = \frac{1}{2}(e^{j\theta} + e^{-j\theta}) \text{ and } \sin\theta = \frac{1}{2j}(e^{j\theta} - e^{-j\theta})$$

P-1 Review of Laplace Transform

$f(t)$	A function of time such that $f(t)=0$ for $t<0$
s	A complex variable
L	Laplace operator
$F(s)$	Laplace transform of $f(t)$

The **Laplace transform** is given by

$$\boxed{L\{f(t)\} = F(s) = \int_0^{\infty} f(t)e^{-st}dt}$$

The **inverse Laplace transform** is given by

$$L^{-1}\{F(s)\} = f(t) = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} F(s)e^{st}\,ds\,, \text{for } t > 0$$

Where c, the abscissa of convergence, is a real constant and is chosen larger than the real parts of all singular points of $F(s)$. Thus, the path of integration is parallel to the $j\omega$ axis and is displaced by the amount c from it. This path of integration is to the right of all singular points.

We will utilize simpler methods for inversion

When does the Laplace transform exist?

The Laplace transform exists if the Laplace integral converges, more explicitly

IF $f(t)$ is sectionally continuous on every finite interval on the range $t > 0$

AND

IF $f(t)$ is of exponential order as $t \to \infty$

Which functions are of exponential order?

A function $f(t)$ is said to be of exponential order if a real positive σ exists such that

$$\boxed{\lim_{t\to\infty} e^{-\sigma t}\left|f(t)\right| \to 0}$$

If this limit approaches zero for $\sigma > \sigma_c$, then σ_c is said to be the abscissa of convergence

For example

$$\boxed{\lim_{t\to\infty} e^{-\sigma t}\left|Ae^{-\alpha t}\right|}$$

This is $f(t)$

- This limit approaches zero for $\sigma > -\alpha$.
- The abscissa of convergence is therefore $\sigma_c = -\alpha$
- The Laplace integral will converge only if s, the real part of s, is greater than the abscissa of convergence

$$\boxed{L\{f(t)\} = F(s) = \int_0^\infty f(t)e^{-st}\,dt}$$

The Laplace integral

What is the abscissa of convergence of

$$F(s)=\frac{K(s+a)}{(s+b)(s+c)}$$

Hint: Find partial fractions, and take inverse Laplace transform, find $f(t)$, and check if

$$\lim_{t\to\infty} e^{-\sigma t}|f(t)| \to 0$$

The answer is $\sigma_c > \max(-b, -c)$. This will be clear after we see how to perform the inversion.

The first conclusion by *Analytic Extension Theorem*

If $L\{f(t)\}=F(s)$ is obtained, and σ_c is determined, $F(s)$ is valid on the entire s-plane except at the poles of $F(s)$.

The second conclusion by *Physical Realizability*

Functions like $f(t)=e^{t^2}$ or $f(t)=te^{t^2}$, which increase faster than the exponential function, do not have Laplace transform, however, on finite time intervals they do.

Laplace transform of *Exponential Function*

$$f(t) = 0 \qquad \text{for } t < 0$$
$$f(t) = Ae^{-\alpha t} \qquad \text{for } t \geq 0$$
where α and A are constants

$$L\{Ae^{-\alpha t}\} = \int_0^\infty Ae^{-\alpha t} e^{-st} dt = \int_0^\infty Ae^{-(s+\alpha)t} dt = \frac{A}{s+\alpha}$$

The abscissa of convergence: $s > -\alpha$

Exponential function produces a pole in the complex plane

Laplace transform of *Step Function,* 1(t)

$$f(t) = 0 \quad \text{for } t < 0$$
$$f(t) = A \quad \text{for } t > 0$$
where A is a constant

$$\boxed{L\{A\} = \int_0^{\infty} Ae^{-st}\,dt = \frac{A}{s}}$$

The abscissa of convergence: $s > 0$

Step function produces a pole at the origin of the complex plane

Laplace transform of *Ramp Function*

$$f(t) = 0 \quad \text{for } t < 0$$
$$f(t) = At \quad \text{for } t \geq 0$$

$$L\{At\} = \int_0^\infty Ate^{-st}dt = At\frac{e^{-st}}{-s}\bigg|_0^\infty - \int_0^\infty A\frac{e^{-st}}{-s}dt$$

$$= \frac{A}{s}\int_0^\infty e^{-st}dt = -\frac{A}{s^2}e^{-st}\bigg|_0^\infty = \frac{A}{s^2}$$

Ramp function produces double poles at the origin of the complex plane

Laplace transform of *Sinusoidal Function*

$$f(t) = 0 \qquad \text{for } t < 0$$
$$f(t) = A\sin\omega t \qquad \text{for } t \geq 0$$
where A and ω are constants

$$L\{A\sin\omega t\} = \int_0^\infty A\sin\omega t e^{-st}dt = \int_0^\infty \frac{A}{2j}(e^{j\omega t} - e^{-j\omega t})e^{-st}dt$$

$$= \frac{A}{2j}\frac{1}{s-j\omega} - \frac{A}{2j}\frac{1}{s+j\omega} = \frac{A\omega}{s^2+\omega^2} \quad \text{Similarly}$$

$$L\{A\cos\omega t\} = \frac{As}{s^2+\omega^2}$$

Sinusoidal functions produce poles on the imaginary ($j\omega$) axis

Several Properties of Laplace Transform
&
Laplace Transforms of Important Functions

$$\boxed{L\{f(t)\} = F(s) = \int_0^{\infty} f(t)e^{-st}\,dt}$$

1. Linearity

$$\boxed{L\{a_1 f_1(t) + a_2 f_2(t)\} = a_1 L\{f_1(t)\} + a_2 L\{f_2(t)\}}$$

$$\begin{aligned}
\mathcal{L}\{\alpha f(t) + \beta g(t)\} &= \int_0^\infty (\alpha f(t) + \beta g(t)) e^{-st} \mathrm{d}t \\
&= \int_0^\infty \alpha f(t) e^{-st} \mathrm{d}t + \int_0^\infty \beta g(t) e^{-st} \mathrm{d}t \\
&= \alpha \int_0^\infty f(t) e^{-st} \mathrm{d}t + \beta \int_0^\infty g(t) e^{-st} \mathrm{d}t \\
&= \alpha \mathcal{L}\{f(t)\} + \beta \mathcal{L}\{g(t)\}
\end{aligned}$$

2. Time Shift (Delay, Advance)

$$\boxed{L\{f(t-\tau)\} = e^{-\tau s} F(s)}$$

$$\boxed{L\{f(t)\} = F(s)}$$

$$\begin{aligned}
\mathcal{L}\{f(t-a)\} &= \int_0^\infty f(t-a)e^{-st}\mathrm{d}t \\
&= \int_{-a}^\infty f(y)e^{-s(a+y)}\mathrm{d}y \\
&= \int_0^\infty f(y)e^{-s(a+y)}\mathrm{d}y \\
&= e^{-as}\int_0^\infty f(y)e^{-sy}\mathrm{d}y
\end{aligned}$$

3. Multiplication by e^{-at}

$$\boxed{L\{e^{-at}f(t)\} = F(s+a)}$$

$$\boxed{L\{f(t)\} = F(s)}$$

$$\mathcal{L}\{e^{-at}f(t)\} = \int_0^\infty e^{-at}f(t)e^{-st}\mathrm{d}t$$
$$= \int_0^\infty f(t)e^{-(s+a)t}\mathrm{d}t$$

4. Change of Time Scale $a > 0$

$$\boxed{L\{f(t/a)\} = aF(as)}$$

$$\boxed{L\{f(t)\} = F(s)}$$

$$\begin{aligned}
\mathcal{L}\left\{f\left(\frac{t}{a}\right)\right\} &= \int_0^\infty f\left(\frac{t}{a}\right)e^{-st}\mathrm{d}t \\
&= \int_0^\infty f(y)e^{-say}a\mathrm{d}y \\
&= a\int_0^\infty f(y)e^{-(as)y}\mathrm{d}y \\
&= aF(as)
\end{aligned}$$

5. Real Differentiation

$$\boxed{L\{\frac{d}{dt}f(t)\} = sF(s) - f(0)}$$

$$\boxed{L\{f(t)\} = F(s)}$$

$$\int_0^\infty f(t)e^{-st}\mathrm{d}t = f(t)\frac{e^{-st}}{-s}\bigg|_0^\infty - \int_0^\infty \frac{\mathrm{d}f(t)}{\mathrm{d}t}\frac{e^{-st}}{-s}\mathrm{d}t$$

$$F(s) = \frac{f(0)}{s} + \frac{1}{s}\int_0^\infty \frac{\mathrm{d}f(t)}{\mathrm{d}t}e^{-st}\mathrm{d}t$$

$$\mathcal{L}\left\{\frac{\mathrm{d}f(t)}{\mathrm{d}t}\right\} = sF(s) - f(0)$$

6. Real Integration

$$L\left\{\int_0^t f(\tau)d\tau\right\} = \frac{1}{s}F(s)$$

$$\boxed{L\{f(t)\} = F(s)}$$

If $f(t)$ is of exp. order

$$\mathcal{L}\left\{\int_0^t f(y)\mathrm{d}y\right\} = \int_0^\infty \left(\int_0^t f(y)\mathrm{d}y\right) e^{-st}\mathrm{d}t$$

Apply integration by parts

$$\int_0^\infty \left(\int_0^t f(y)\mathrm{d}y\right) e^{-st}\mathrm{d}t = \left(\int_0^t f(y)\mathrm{d}y\right) \frac{e^{-st}}{-s}\Bigg|_0^\infty - \int_0^\infty f(t)\frac{e^{-st}}{-s}\mathrm{d}t$$

$$= \frac{1}{s}\int_0^\infty f(t)e^{-st}\mathrm{d}t$$

$$\mathcal{L}\left\{\int_0^t f(y)\mathrm{d}y\right\} = \frac{1}{s}F(s)$$

7. Multiplication by t

$$\boxed{L\{tf(t)\} = \frac{d}{ds}F(s)}$$

$$\boxed{L\{f(t)\} = F(s)}$$

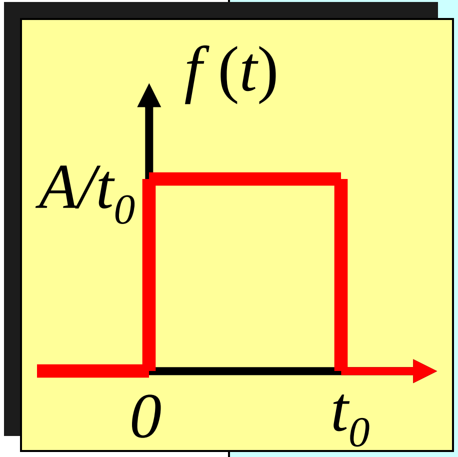
$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}s^n} F(s)$$

8. Laplace Transform of a Pulse

$$f(t) = \frac{A}{t_0} \quad \text{for } 0 < t < t_0$$

$$f(t) = 0 \quad \text{for } t < 0, t_0 < t$$

$$f(t) = \frac{A}{t_0}1(t) - \frac{A}{t_0}1(t - t_0)$$



$$L\{f(t)\} = L\left\{\frac{A}{t_0}1(t) - \frac{A}{t_0}1(t - t_0)\right\}$$

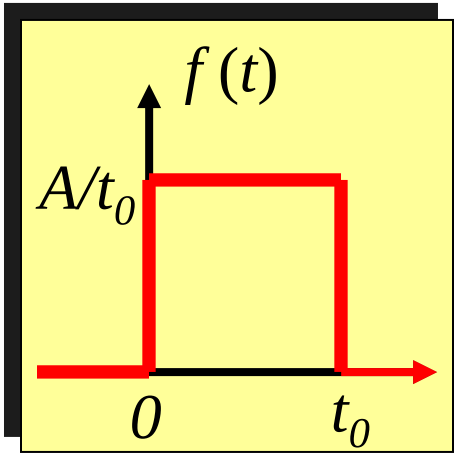
$$= \frac{A}{t_0}\left(L\{1(t)\} - L\{1(t - t_0)\}\right)$$

$$= \frac{A}{t_0}L\{1(t)\}\left(1 - e^{-st_0}\right)$$

$$= \frac{A}{t_0}\frac{1}{s}\left(1 - e^{-st_0}\right)$$

9. Laplace Transform of Impulse Function

$$L\left\{\lim_{t_0\to 0} f(t)\right\} = \lim_{t_0\to 0}\left(\frac{A}{t_0}\frac{1}{s}\left(1-e^{-st_0}\right)\right) = A$$



10. Final Value Theorem

$$\boxed{\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)}$$

$$\boxed{L\{f(t)\} = F(s)}$$

This theorem can be applicable if

- $f(t)$ settles down to a constant limit
- sF(s) has no poles on the imaginary axis, this obviously means oscillations in $f(t)$
- sF(s) has no poles on the right half s-plane

11. Initial Value Theorem

$$\boxed{f(0+) = \lim_{s\to\infty} sF(s)}$$

$$\boxed{L\{f(t)\} = F(s)}$$

This Theorem can be applicable if

- $f(t)$ and $df(t)/dt$ are both Laplace transformable
- The limit on the right hand side exists

12. Laplace Transform of Convolution

$$\boxed{L\{f(t) * g(t)\} = F(s)G(s)}$$

where

$$\boxed{f(t) * g(t) = \int_0^t f(\tau)g(t-\tau)d\tau}$$

and by duality

$$\boxed{L\{f(t)g(t)\} = F(s) * G(s)}$$

$$t - y := p$$

$$\begin{aligned}
\int_0^\infty \int_0^\infty f(y)g(t-y)e^{-st}\mathrm{d}y\mathrm{d}t &= \int_0^\infty \int_0^\infty f(y)g(t-y)e^{-st}\mathrm{d}t\mathrm{d}y \\
&= \int_0^\infty \int_{-y}^\infty f(y)g(p)e^{-s(p+y)}\mathrm{d}p\mathrm{d}y \\
&= \int_0^\infty \int_0^\infty f(y)g(p)e^{-s(p+y)}\mathrm{d}y\mathrm{d}p \\
&= \int_0^\infty f(y)e^{-sy}\mathrm{d}y \int_0^\infty g(p)e^{-sp}\mathrm{d}p \\
&= \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\}
\end{aligned}$$

Inverse Laplace Transform

Typical Inversion Methods

- Use of inversion integral
  Complicated and generally takes long time

- Use of table (Textbook pp.22-23)
  Easiest way but you may not always be able to find what you are looking for in the table explicitly

We will take a look at Partial Fraction Expansion

Partial Fraction Expansion

Consider
$$\boxed{F(s)=\frac{B(s)}{A(s)}=\frac{K(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)}}$$

where m<n.

- If m=n, then find out the constant term and separately write in the expansion, then invert.
- If m>n, then find out the polynomial in s, and write and invert it separately.
- $-z_i$'s are zeros and $-p_i$'s are poles.
- Poles and zeros may be complex numbers as well

If m<n, the expression

$$F(s)=\frac{B(s)}{A(s)}=\frac{K(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)}$$

can be expanded as

$$F(s)=\frac{B(s)}{A(s)}=\frac{a_1}{s+p_1}+\frac{a_2}{s+p_2}+\cdots+\frac{a_n}{s+p_n}$$

where

$$a_i=\left[(s+p_i)F(s)\right]_{s=-p_i}$$

Consider

$$F(s)=\frac{B(s)}{A(s)}=\frac{B(s)}{\left(\prod_{i=1}^{m}(s+p_i)\right)(s+\alpha)^q}$$

where, deg A<q+m. This expression can be expanded as

$$F(s)=\frac{B(s)}{A(s)}=\sum_{i=1}^{m}\frac{a_i}{s+p_i}+\sum_{j=1}^{q}\frac{c_j}{(s+\alpha)^j}$$

with

$$c_{q-k}=\left[\frac{1}{k!}\frac{d^k}{ds^k}\left((s+\alpha)^q F(s)\right)\right]_{s=-\alpha}$$

$k=0,1,\ldots,q-1$

Example (2.3 from book)

Find the inverse Laplace transform of

$$\frac{1}{(s+1)(s+2)} = \frac{R_1}{s+1} + \frac{R_2}{s+2}$$

Example (2.4 from book)

Find the inverse Laplace transform of

$$\frac{1}{(s^2+1)(s+2)} = \frac{R_1}{s+j} + \frac{R_2}{s-j} + \frac{R_3}{s+2}$$

$$R_1 = \left((s+j)F(s)\right)\big|_{s=-j} = -\frac{1}{10} + j\frac{1}{5}$$

$$R_2 = \left((s-j)F(s)\right)\big|_{s=j} = -\frac{1}{10} - j\frac{1}{5} = \overline{R}_1$$

$$R_3 = \left((s+2)F(s)\right)\big|_{s=-2} = \frac{1}{5}$$

$$f(t) = \left(\frac{1}{5}e^{-2t} + \left(-\frac{1}{10} + j\frac{1}{5}\right)e^{-jt} + \left(-\frac{1}{10} - j\frac{1}{5}\right)e^{jt}\right)1(t)$$

$$e^{jt} + e^{-jt} = 2\cos(t)$$

$$e^{jt} - e^{-jt} = 2j\sin(t)$$

$$f(t) = \frac{1}{5}(e^{-2t} - \cos(t) + 2\sin(t))1(t)$$

An Example

Find the inverse Laplace transform of

$$F(s) = \frac{B(s)}{A(s)} = \frac{s+3}{(s+1)(s+2)^3}$$

Solution: Rewrite it as

$$F(s) = \frac{a_1}{s+1} + \frac{c_1}{s+2} + \frac{c_2}{(s+2)^2} + \frac{c_3}{(s+2)^3}$$

$$\boxed{a_1 = \left[(s+1)F(s)\right]_{s=-1} = 2}$$

$$\boxed{c_3 = \left[(s+2)^3 F(s)\right]_{s=-2} = -1}$$

$$\boxed{c_2 = \frac{d}{ds}\left[(s+2)^3 F(s)\right]_{s=-2} = -2}$$

$$\boxed{c_1 = \frac{1}{2!}\frac{d^2}{ds^2}\left[(s+2)^3 F(s)\right]_{s=-2} = -2}$$

$$F(s)=\frac{2}{s+1}+\frac{-2}{s+2}+\frac{-2}{(s+2)^2}+\frac{-1}{(s+2)^3}$$

$$L\left\{\frac{t^{n-1}e^{-at}}{(n-1)!}\right\}=\frac{1}{(s+a)^n}\text{ where } n=1,2,3,\ldots$$

$$f(t)=2e^{-t}-2e^{-2t}-2te^{-2t}-\frac{1}{2}t^2e^{-2t}$$

$$f(t)=2e^{-t}-(1/2)(4+4t+t^2)e^{-2t}$$

Example (2.5 from book)

Find the inverse Laplace transform of

$$F(s) = \frac{(s+1)^3}{(s+2)^4}$$

$$F(s) = \frac{R_1}{s+2} + \frac{R_2}{(s+2)^2} + \frac{R_3}{(s+2)^3} + \frac{R_4}{(s+2)^4}$$

$$R_4 = \left((s+2)^4 F(s)\right)|_{s=-2} = -1$$

$$R_3 = \left(\frac{\mathrm{d}}{\mathrm{d}s}(s+2)^4 F(s)\right)\Big|_{s=-2} = 3$$

$$R_2 = \frac{1}{2!}\left(\frac{\mathrm{d}^2}{\mathrm{d}s^2}(s+2)^4 F(s)\right)\Big|_{s=-2} = -3$$

$$R_1 = \frac{1}{3!}\left(\frac{\mathrm{d}^3}{\mathrm{d}s^3}(s+2)^4 F(s)\right)\Big|_{s=-2} = 1$$

$$F(s) = \frac{1}{s+2} + \frac{-3}{(s+2)^2} + \frac{3}{(s+2)^3} + \frac{-1}{(s+2)^4}$$

$$\mathcal{L}\{e^{-2t}1(t)\} = \frac{1}{s+2} = Q(s)$$

$$\mathcal{L}\{te^{-2t}1(t)\} = -\frac{\mathrm{d}}{\mathrm{d}s}Q(s) = \frac{1}{(s+2)^2}$$

$$\mathcal{L}\{t^2e^{-2t}1(t)\} = \frac{\mathrm{d}^2}{\mathrm{d}s^2}Q(s) = \frac{2}{(s+2)^3}$$

$$\mathcal{L}\{t^3e^{-2t}1(t)\} = -\frac{\mathrm{d}^3}{\mathrm{d}s^3}Q(s) = \frac{6}{(s+2)^4}$$

$$f(t) = -\frac{1}{6}\left(t^3 - 9t^2 + 18t - 6\right)e^{-2t}1(t)$$

Example

$$\frac{s+2}{(s+1)^3}=\frac{a_1}{s+1}+\frac{a_2}{(s+1)^2}+\frac{a_3}{(s+1)^3}$$

$$\frac{s+2}{(s+1)^3}(s+1)^3=\frac{a_1}{s+1}(s+1)^3+\frac{a_2}{(s+1)^2}(s+1)^3+\frac{a_3}{(s+1)^3}(s+1)^3$$

$$s+2=a_1(s+1)^2+a_2(s+1)^1+a_3 \Rightarrow a_3=1$$

$$1=2a_1(s+1)+a_2 \Rightarrow a_2=1$$

$$0=2a_1 \Rightarrow a_1=0$$

$$\frac{s+2}{(s+1)^3}=\frac{s+1}{(s+1)^3}+\frac{1}{(s+1)^3}=\frac{0}{s+1}+\frac{1}{(s+1)^2}+\frac{1}{(s+1)^3}$$

Example 1/5

$$\frac{s^2+1}{(s+2)(s+1)^3}=\frac{a}{s+2}+\frac{c_1}{s+1}+\frac{c_2}{(s+1)^2}+\frac{c_3}{(s+1)^3}$$

$$\frac{s^2+1}{(s+2)(s+1)^3}(s+2)=\frac{a}{s+2}(s+2)+\frac{c_1}{s+1}(s+2)+\frac{c_2}{(s+1)^2}(s+2)+\frac{c_3}{(s+1)^3}(s+2)$$

$$\frac{s^2+1}{(s+1)^3}=a+\frac{c_1}{s+1}(s+2)+\frac{c_2}{(s+1)^2}(s+2)+\frac{c_3}{(s+1)^3}(s+2),\quad \text{Insert } s=-2$$

$$\frac{(-2)^2+1}{(-2+1)^3}=a=-5$$

Example 2/5

$$\frac{s^2+1}{(s+2)(s+1)^3}=\frac{a}{s+2}+\frac{c_1}{s+1}+\frac{c_2}{(s+1)^2}+\frac{c_3}{(s+1)^3}$$

$$\frac{s^2+1}{(s+2)(s+1)^3}(s+1)^3=\frac{a}{s+2}(s+1)^3+\frac{c_1}{s+1}(s+1)^3+\frac{c_2}{(s+1)^2}(s+1)^3+\frac{c_3}{(s+1)^3}(s+1)^3$$

$$\frac{s^2+1}{(s+2)}=\frac{a}{s+2}(s+1)^3+c_1(s+1)^2+c_2(s+1)^1+c_3,\quad \text{Insert } s=-1$$

$$c_3=\frac{(-1)^2+1}{-1+2}=2$$

Example 3/5

$$\frac{s^2+1}{(s+2)} = \frac{a}{s+2}(s+1)^3 + c_1(s+1)^2 + c_2(s+1)^1 + c_3$$

Take derivative then insert $s=-1$

$$\frac{2s(s+2)-(s^2+1)}{(s+2)^2} = a\left(\frac{3(s+1)^2(s+2)-(s+1)^3}{(s+2)^2}\right) + \left(2c_1(s+1)\right) + c_2$$

$$\frac{s^2+4s-1}{(s+2)^2} = \left\{a\frac{(s+1)^2\,(2s+5)}{(s+2)^2} + \left(2c_1(s+1)\right)\right\} + c_2$$

$$c_2 = -4$$

Example 4/5

$$\frac{s^2+4s-1}{(s+2)^2}=\left\{a\frac{(s+1)^2(2s+5)}{(s+2)^2}+\left(2c_1(s+1)\right)\right\}+c_2$$

Take derivative then insert $s=-1$

$$\frac{(2s+4)(s+2)^2-2(s+2)(s^2+4s-1)}{(s+2)^4}=a\frac{d}{ds}\left\{\frac{(s+1)^2(2s+5)}{(s+2)^2}\right\}+2c_1$$

$$10=2c_1\Rightarrow c_1=5$$

Example 5/5

$$\frac{s^2+1}{(s+2)(s+1)^3}=\frac{5}{s+1}-\frac{4}{(s+1)^2}+\frac{2}{(s+1)^3}-\frac{5}{s+2}$$

$$\boxed{L\left\{\frac{t^{n-1}e^{-at}}{(n-1)!}\right\}=\frac{1}{(s+a)^n}\ \text{where } n=1,2,3,\ldots}$$

$f(t)$	$F(s)$
$\delta(t)$	1
$1(t)$	$\frac{1}{s}$
t	$\frac{1}{s^2}$
$\frac{t^{n-1}}{(n-1)!} \quad (n = 1, 2, 3, \ldots)$	$\frac{1}{s^n}$
e^{-at}	$\frac{1}{s+a}$
te^{-at}	$\frac{1}{(s+a)^2}$
$\frac{t^{n-1}e^{-at}}{(n-1)!} \quad (n = 1, 2, 3, \ldots)$	$\frac{1}{(s+a)^n}$

$f(t)$	$F(s)$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
$\sinh(\omega t)$	$\frac{\omega}{s^2-\omega^2}$
$\cosh(\omega t)$	$\frac{s}{s^2-\omega^2}$
$\frac{1-e^{-at}}{a}$	$\frac{1}{s(s+a)}$
$\frac{e^{-at}-e^{-bt}}{b-a}$	$\frac{1}{(s+a)(s+b)}$

$f(t)$	$F(s)$
$\frac{be^{-bt}-ae^{-at}}{b-a}$	$\frac{s}{(s+a)(s+b)}$
$\frac{1}{ab}\left(1+\frac{1}{a-b}(be^{-at}-ae^{-bt})\right)$	$\frac{s}{s(s+a)(s+b)}$
$\frac{1}{a^2}(1-e^{-at}-ate^{-at})$	$\frac{1}{s(s+a)^2}$
$\frac{1}{a^2}(at-1+e^{-at})$	$\frac{1}{s^2(s+a)}$
$e^{-at}\sin(\omega t)$	$\frac{\omega}{(s+a)^2+\omega^2}$
$e^{-at}\cos(\omega t)$	$\frac{(s+a)}{(s+a)^2+\omega^2}$

P-1 Review of Linear Algebra

Inner (Dot) Product of Vectors

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \text{ then } x \cdot y = x^{\mathrm{T}} y = \sum_{i=1}^{n} x_i y_i$$

Multiplication of Two Matrices

$$\boxed{C = AB}$$

$$\boxed{A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \qquad B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix}}$$

$$\boxed{c_{ij} = \sum_{k} a_{ik} b_{kj}}$$

Determinant

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$|A| = (-1)^{1+1} a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (-1)^{1+2} a_{12} \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} +$$
$$(-1)^{1+3} a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$|A| = a_{11}(a_{22}a_{33} - a_{32}a_{32}) - a_{12}(a_{11}a_{33} - a_{13}a_{31}) +$$
$$a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

Determinant

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11}+a_{21} & a_{12}+a_{22} & a_{13}+a_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11}+a_{12} & a_{12} & a_{13} \\ a_{21}+a_{22} & a_{22} & a_{23} \\ a_{31}+a_{32} & a_{32} & a_{33} \end{vmatrix}$$

Given a determinant, summing two rows and writing the result as one of those rows do not change the value of the determinant.

Similarly, summing two columns and using the result as one of those columns do not change the value of the determinant.

Eigenvalues and Eigenvectors

$$\boxed{|\lambda I - A| = 0}$$

$$Av = \lambda v$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \lambda \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

$$\lambda I - A = \begin{bmatrix} \lambda - a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & \lambda - a_{22} & \cdots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & \lambda - a_{nn} \end{bmatrix}$$

Characteristic Polynomial

$$\boxed{\left|\lambda I - A\right| = \lambda^n + \alpha_1 \lambda^{n-1} + \cdots + \alpha_n = 0}$$

*Note that a polynomial is said to be **monic** if the coefficient of the highest order term is equal to unity*

Cayley-Hamilton Theorem

Every square matrix satisfies its characteristic polynomial

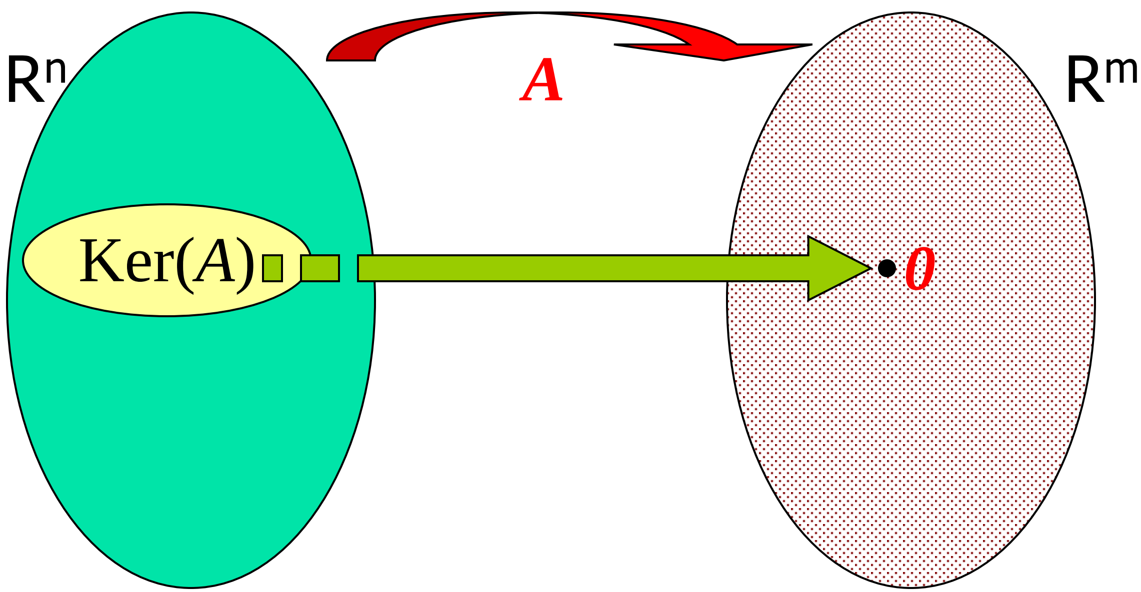
$$\boxed{A^n + \alpha_1 A^{n-1} + \cdots + \alpha_n I = 0}$$

Kernel and Image

$$A : R^n \to R^m$$

$$\mathrm{Ker}(A) = \mathrm{Null}(A) := \{x \in R^n : Ax = 0\}$$

$$\mathrm{Im}(A) = \mathrm{Range}(A) := \{y \in R^m : y = Ax, x \in R^n\}$$

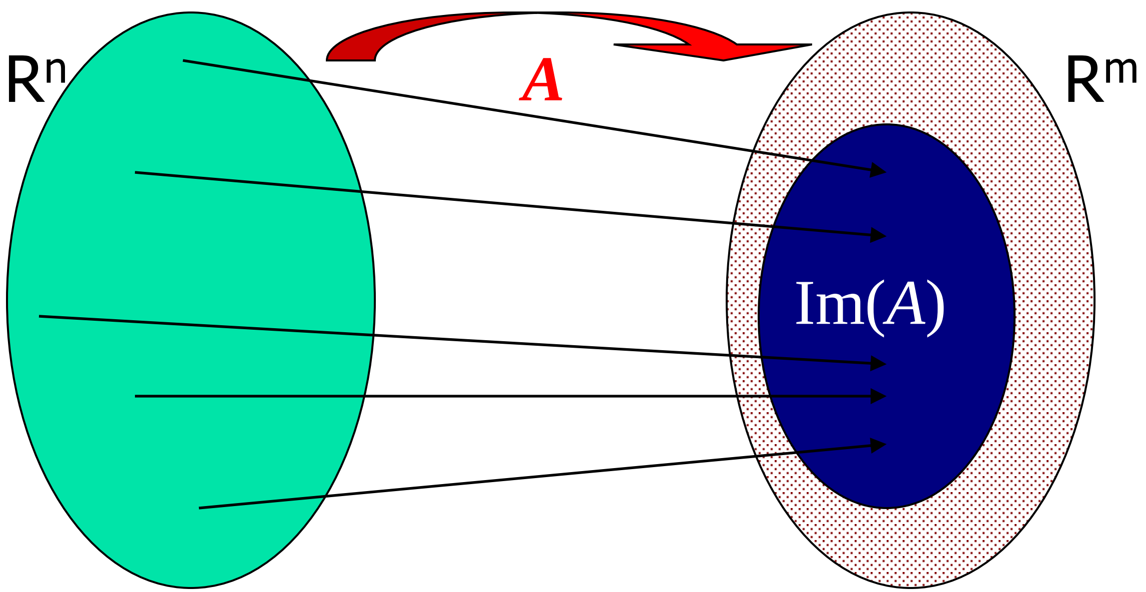


Kernel and Image

$$A : R^n \to R^m$$

$$\mathrm{Ker}(A) = \mathrm{Null}(A) := \{x \in R^n : Ax = 0\}$$

$$\mathrm{Im}(A) = \mathrm{Range}(A) := \{y \in R^m : y = Ax, x \in R^n\}$$



Linear Dependence/Independence

Let $\boxed{x_i \in R^n}$

Set $\boxed{x = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_k x_k}$

If a set of α_j (**other than all zero**) yields $x=0$, then $\{x_1, x_2, \ldots, x_k\}$ set is said to be linearly dependent otherwise $x_{1\ldots k}$ are linearly independent