Exercise 1. Reading: Chapter 4 - Counting Methods and Pigeonhole Principle

Chapter 5: Recurrence Relations

1 Examples of Recurrence Relation

Example 2 (The Fibonacci Sequence)**.** For $n \geq 3$,

$$f_n = f_{n-1} + f_{n-2},$$

$f_1 = 1$ and $f_2 = 2$.

Here, $f_1 = 1$ and $f_2 = 2$ are *initial conditions* of the recurrence relation.

Definition 3 (Recurrence Relation, Initial Conditions)**.** A recurrence relation for the sequence $a_0, a_1, \ldots$ is an equation that relates a_n to certain of its predecessors $a_0, a_1, \ldots, a_{n-1}$.
Initial conditions for the sequence $a_0, a_1, \ldots$ are explicitly given values for a finite number of the terms of the sequence.

Example 4 (Tower of Hanoi)**.**

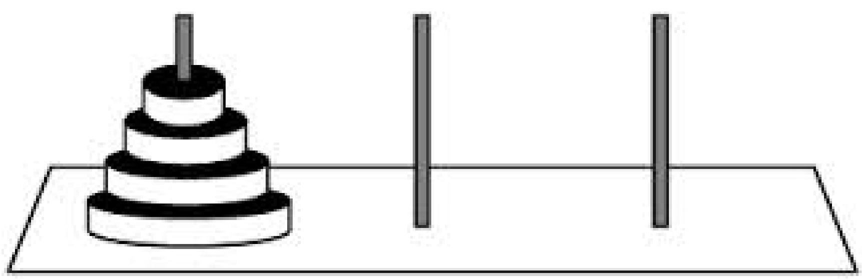

Figure 1: Tower of Hanoi

- There are n disks of various sizes with holes in their centers and three pegs.
- If a disk is on a peg, only a disk of smaller diameter can be placed on top of the first disk.
- Given all disks are stacked on one peg, the problem is to transfer the disks to another peg by moving one disk at a time.

Solution 5 (Tower of Hanoi)**.**
Let $c_1, c_2, \ldots$ be the sequence, where c_n is the number of moves our solution takes to solve the puzzle for n disks.

- If $n = 1$, we move it to the desired peg.
- If $n > 1$ disks are on peg-1, we begin by applying our algorithm recursively, that is, we first move the top $n-1$ disks to peg-2. (The bottom disk on peg-1 stays fixed)
- Next, we move the remaining disk on peg-1 to peg-3.
- Finally, we move the $n-1$ disks on peg-2 to peg-3 by applying our algorithm recursively

If $n > 1$, we solve the $(n-1)$-disk problem twice and we explicitly move one disk. Therefore,

$$c_n = 2c_{n-1} + 1$$

for $n > 1$.
The initial condition is

$$c_1 = 1.$$

(Later, we will solve the recurrence relation to show that $c_n = 2^n - 1$.)

Exercise 6. Write explicit solutions for the Tower of Hanoi puzzle for $n = 3, 4$.

Exercise 7. Let S_n denote the number of n-bit strings that do not contain the pattern 000. Find a recurrence relation and initial conditions for the sequence $\{S_n\}$.

Exercise 8. Let S_n denote the number of n-bit strings that do not contain the pattern 00.

1. Find a recurrence relation and initial conditions for the sequence $\{S_n\}$.
2. Show that $S_n = f_{n+1}$, $n = 1, 2, \ldots$, where f denotes the Fibonacci sequence.

Exercise 9. By considering the number of n-bit strings with exactly i 0's and the previous exercise, show that

$$f_{n+1} = \sum_{i=0}^{\lfloor (n+1)/2 \rfloor} C(n+1-i, i)$$

for $n = 1, 2, \ldots$.

Exercise 10. Derive a recurrence relation and initial condition for the number of ways to divide a convex $(n+2)$-sided polygon, $n \geq 1$, into triangles by drawing $n-1$ lines through the corners that do not intersect in the interior of the polygon. (A polygon is convex if any line joining two points in the polygon lies wholly in the polygon.)
For example, there are five ways to divide a convex pentagon into triangles by drawing two nonintersecting lines through the corners.
What is the number of ways to divide an n-sided polygon?

Exercise 11. Suppose that we have n Liras and that each day we buy either orange juice (1 Lira), milk (2 Liras) or cheese (2 Liras). If R_n is the number of ways of spending all the money, show that

$$R_n = R_{n-1} + 2R_{n-2}.$$

Order is taken into account. For example, there are 11 ways to spend 4 Liras:MC, CM, OOM, OOC, OMO, OCO, MOO, COO, OOOO, MM, CC.

Exercise 12. Derive a recurrence relation for $C(n,k) = \binom{n}{k}$, the number of k-element subsets of an n-element subset. Specifically, write $C(n+1,k)$ in terms of $C(n,i)$ for appropriate i.

Exercise 13. Derive a recurrence relation for $S(k,n)$, the number of ways of choosing k items, allowing repetitions from n available types. Specifically, write $S(k,n)$ in terms of $S(k-1,i)$ for appropriate i.

Exercise 14. Let $S(n,k)$ denote the number of functions from $\{1,\ldots,n\}$ onto $\{1,\ldots,k\}$. Show that $S(n,k)$ satisfies the recurrence relation

$$S(n,k) = k^n - \sum_{i=1}^{k-1} C(k,i)S(n,i).$$

2 Solving Recurrence Relations

Example 15. Solve the recurrence relation

$$a_n = a_{n-1} + 3,$$

subject to the initial condition $a_1 = 2$, by using iteration.

- Substituting $a_{n-1} = a_{n-2} + 3$ into a_n gives $a_n = (a_{n-2} + 3) + 3$.
- Substituting $a_{n-2} = a_{n-3} + 3$ into a_n gives
  $a_n = ((a_{n-3} + 3) + 3) + 3 = a_{n-3} + 3 \cdot 3$.
- Apply iteratively.
- Finally, $a_n = (a_1 + 3 \cdot (n-1)) = 2 + 3(n-1)$.

Example 16. Solve the recurrence relation

$$S_n = 2S_{n-1},$$

subject to the initial condition $S_0 = 1$.
Answer: $S_n = 2^n$.

Example 17. Find an explicit formula for c_n, the minimum number of moves in which the n-disk Tower of Hanoi puzzle can be solved.
Answer: $c_n = 2^n - 1$.

Definition 18 (Linear Homogeneous Recurrence Relation of Order k)**.** A linear homogeneous recurrence relation of order k with constant coefficients is a recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}, \quad c_k \neq 0.$$

A linear homogeneous recurrence relation of order k with constant coefficients, together with the k initial conditions

$$a_0 = C_0,\ a_1 = C_1,\ \ldots, a_{k-1} = C_{k-1},$$

uniquely defines a sequence $a_0, a_1, \ldots$.

Example 19 (Examples that are not linear)**.**

- $a_n = 3a_{n-1}a_{n-2}$

- $a_n - a_{n-1} = 2n$
- $a_n = 3na_{n-1}$

Definition 20 (Method of Solving Linear Homogeneous Recurrence Relation)**.** In general, the solution S_n has the form $S_n = t^n$ for some fixed number t.

Example 21. Consider the relation

$$a_n = 5a_{n-1} - 6a_{n-2}$$

with initial conditions $a_0 = 7$ and $a_1 = 16$.

Solution 22. We will look for a solution in the form $V_n = t^n$. From above, we know that a solution should satisfy

$$V_n = 5V_{n-1} - 6V_{n-2}.$$

This implies,

$$t^n = 5t^{n-1} - 6t^{n-2}.$$

Dividing by t^{n-2} and reorganizing the equality, we obtain

$$t^2 - 5t + 6 = 0.$$

This equation has two solutions as $t = 2$ and $t = 3$. Therefore, there are two solutions $S_n = 2^n$ and $T_n = 3^n$.

General Solution: More generally, the solution is $U_n = bS_n + dT_n$, where b and d are any real numbers.

Initial Conditions: The numbers b and d should satisfy the initial conditions $a_0 = 7$ and $a_1 = 16$. Therefore,

$$7 = U_0 = b2^0 + d3^0 = b + d \qquad 16 = U_1 = b2^1 + d3^1 = 2b + 3d.$$

Solving these equations give $b = 5$ and $d = 2$. The general solution is

$$U_n = 52^n + 23^n$$

and $a_n = U_n = 52^n + 23^n$ for $n \geq 0$.

Theorem 23. Let

$$a_n = c_1 a_{n-1} + c_2 a_{n-2}$$

be a second-order, linear homogeneous recurrence relation with constant coefficients and with initial conditions $a_0 = C_0$ and $a_1 = C_1$. If S and T are solutions, we will look for a solution in the form $U = bS + dT$. If r is a root of

$$t^2 - c_1 t - c_2 = 0$$

and r_1 and r_2 are roots of this equation with $r_1 \neq r_2$, then there exist constants b and d such that $a_n = br_1^n + dr_2^n$ for $n \geq 0$.

Example 24 (Explicit Formula of the Fibonacci Sequence)**.** For $n \geq 3$,

$$f_n = f_{n-1} + f_{n-2},$$

and initial conditions are $f_1 = 1$ and $f_2 = 2$.
By the quadratic formula, we have

$$t^2 - t - 1 = 0.$$

The solutions are $t = (1 \pm \sqrt{5})/2$. Therefore, the solution is of the form

$$f_n = b\left(\frac{1+\sqrt{5}}{2}\right)^n + d\left(\frac{1-\sqrt{5}}{2}\right)^n$$

To satisfy the initial condition, we need

$$b\left(\frac{1+\sqrt{5}}{2}\right) + d\left(\frac{1-\sqrt{5}}{2}\right) = 1$$

$$b\left(\frac{1+\sqrt{5}}{2}\right)^2 + d\left(\frac{1-\sqrt{5}}{2}\right)^2 = 2$$

By solving this system for b and d, we obtain

$$b = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right), \qquad d = -\frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right).$$

The solution for f_n is

$$f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}.$$

Theorem 25. Let

$$a_n = c_1 a_{n-1} + c_2 a_{n-2}$$

be a second-order, linear homogeneous recurrence relation with constant coefficients.

Let a be the sequence satisfying the above relation with $a_0 = C_0$ and $a_1 = C_1$. If both roots of $t^2 - c_1 t - c_2 = 0$ are equal to r, then there exist constants b and d such that

$$a_n = br^n + dnr^n$$

for $n \geq 1$.

Theorem 26. For the general linear homogeneous recurrence relation of *order k* with constant coefficients, if r is a root of

$$t^k - c_1 t^{k-1} - c_2 t^{k-2} - \cdots - c_k = 0$$

with multiplicity m, then $r^n, nr^n, \ldots, n^{m-1} r^n$ are solutions for this equation.

Exercise 27. Give the order of each linear homogeneous recurrence relation with constant coefficients.

1. $a_n = 2na_{n-2} - a_{n-1}$,
2. $a_n = 7a_{n-2} - 6a_{n-3}$,
3. $a_n = a_{n-1} + n$.

Exercise 28. Solve the recurrence relation $a_n = 2^n a_{n-1}$, $n > 0$, with the initial condition $a_0 = 1$.

Exercise 29. Solve the recurrence relations with the given initial conditions below.

1. $a_n = 6a_{n-1} - 8a_{n-2}$; $a_0 = 1$, $a_1 = 0$
2. $a_n = 7a_{n-1} - 10a_{n-2}$; $a_0 = 5$, $a_1 = 16$
3. $a_n = 2a_{n-1} + 8a_{n-2}$; $a_0 = 4$, $a_1 = 10$

Exercise 30. Solve the recurrence relations with the given initial conditions below.

1. $a_n = -3a_{n-1}$; $a_0 = 2$
2. $a_n = 2na_{n-1}$; $a_0 = 1$
3. $a_n = a_{n-1} + n$; $a_0 = 0$

Exercise 31. Solve the recurrence relation $\sqrt{a_n} = \sqrt{a_{n-1}} + 2\sqrt{a_{n-2}}$ with initial conditions $a_0 = a_1 = 1$ by making substitution $b_n = \sqrt{a_n}$.

Exercise 32. Solve the recurrence relation

$$a_n = \sqrt{\frac{a_{n-2}}{a_{n-1}}}$$

with initial conditions $a_0 = 8$, $a_1 = 1/(2\sqrt{2})$ by taking the logarithm of both sides and making the substitution $b_n = \log a_n$.