BBM205 Final Exam Solutions

Time: 10:00-12:00

January, 12 2020

Question:	1	2	3	4	5	Total
Points:	20	20	20	20	20	100
Score:						

1. (20 points) Solve the recurrence relations below for $n \ge 0$, where in each part x denotes your group number:

$$a_n = a_{n-1} + n, \qquad a_0 = x$$

$$a_n = (x+3)a_{n-1} + 2^n, \qquad a_0 = 1$$

$$\sqrt{a_n} = \sqrt{a_{n-1}} + 2\sqrt{a_{n-2}}, \qquad a_0 = a_1 = 1$$

Solution:

- (a) By telescope method, we have $a_n = \sum_{i=1}^n i + a_0 = n(n+1)/2 + x$.
- (b) There are two parts in the solution, homogeneous and particular. So, let $a_n = a_n^h + a_n^p$. The homogeneous part is found by solving $a_n = (x+3)a_{n-1}$. Hence, $a_n^h = A(x+3)^n$. The particular solution is given by a guess as $a_n^p = B2^n$. To find B, we substitute this into the recursive formula: $B2^n = (x+3)B2^{n-1} + 2^n$, yielding 2B = (x+3)B + 2. Thus, B = -2/(x+1).

Finally, we solve for A in $a_0 = 1 = A(x+3)^0 + B2^0 = A(x+3)^0 - 2 \cdot 2^0/(x+1)$. So, A = 1 + 2/(x+1).

(c) We let $b_n = \sqrt{a_n}$. So, $b_n = b_{n-1} + 2b_{n-2}$. By substituting, $b_n = r^n$, we have $r^2 - r - 2 = (r - 2)(r + 1) = 0$. This gives $b_n = A2^n + B(-1)^n$. By using the initial conditions, we have $a_0 = 1 = A + B$ and $a_1 = 1 = 2A - B$. Therefore, A = 2/3 and B = 1/3. So, $a_n = b_n^2$, where $b_n = (2/3)2^n + (1/3)(-1)^n$.

- 2. (20 points) Prove the following for a simple graph on n vertices:
 - a) If for every pair of vertices x and y, $deg(x) + deg(y) \ge n 1$, then G is connected.
 - b) Show by finding an example that if the condition in part (a) is replaced with $deg(x) + deg(y) \ge n 2$, then G may not be connected.

Solution: See resources at Piazza page for CS70 (Fall 2017) Hw3 solutions.

- 3. (20 points) a) Prove that if p is a prime number with p > 3, then either $p \equiv 1 \mod 3$ or $p \equiv -1 \mod 3$.
 - b) We call two primes with difference 2 twin primes, such as 11 and 13. Use part (a) to show that all pairs of twin primes $\{p,q\}$ with p,q>5 are disjoint pairs.

Solution: See resources at Piazza page for CS70 (Fall 2019) Hw1 solutions.

4. (20 points) Show by using induction on n that the formula for A^n below is correct for the given matrix A and $n \ge 1$.

$$A = \begin{bmatrix} -2 & -9 \\ 1 & 4 \end{bmatrix}, \qquad A^n = \begin{bmatrix} 1 - 3n & -9n \\ n & 1 + 3n \end{bmatrix}.$$

Solution: Base case: n=1

$$A = \begin{bmatrix} -2 & -9 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 1 - 3 \cdot 1 & -9 \cdot 1 \\ 1 & 1 + 3 \cdot 1 \end{bmatrix}$$

Inductive case: We observe that

$$A^{n+1} = \begin{bmatrix} -2 & -9 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 1 - 3n & -9n \\ n & 1 + 3n \end{bmatrix} = \begin{bmatrix} -2 + 6n - 9n & 18n - 9 - 27n \\ 1 - 3n + 4n & -9n + 4 + 12n \end{bmatrix} = \begin{bmatrix} 1 - 3 \cdot (n+1) & -9 \cdot (n+1) \\ (n+1) & 1 + 3 \cdot (n+1) \end{bmatrix}$$

5. (20 points) In the following questions, pick the correct answer.

Solution:
(a) If for a graph with n vertices and e edges, $e \leq 3n-6$, then G is planar. () True (X) False
(b) Any graph that does not contain a triangle is 2-colorable.() True (X) False
(c) Every graph with a Hamilton cycle has an Euler tour.() True (X) False
(d) If all vertices in a simple graph have degree at least 2, then G is not connected. () True (X) False
(e) For two integers a and b, if a=bq+r ($q \neq 0$), then gcd(a,b) is equal to: () gcd(a,q) (X) gcd(b,r) () gcd(q,r) () gcd(a,r)