This week's agenda

- PID Controllers
  - Structure of the PID Controller
  - Proportional, Integral and Derivative Terms
  - Setpoint Weighting
- Process Models
  - 2 Parameter Models
  - 3 Parameter Models
  - 4 Parameter Models
  - Comparison of Models
- Parameter Tuning
- Integrator Antiwindup
- Performance Measures

PID Controllers

- A very widely accepted industrial approach
- Few parameters to tune
- Physical meaning for each term
- Low cost in manufacturing

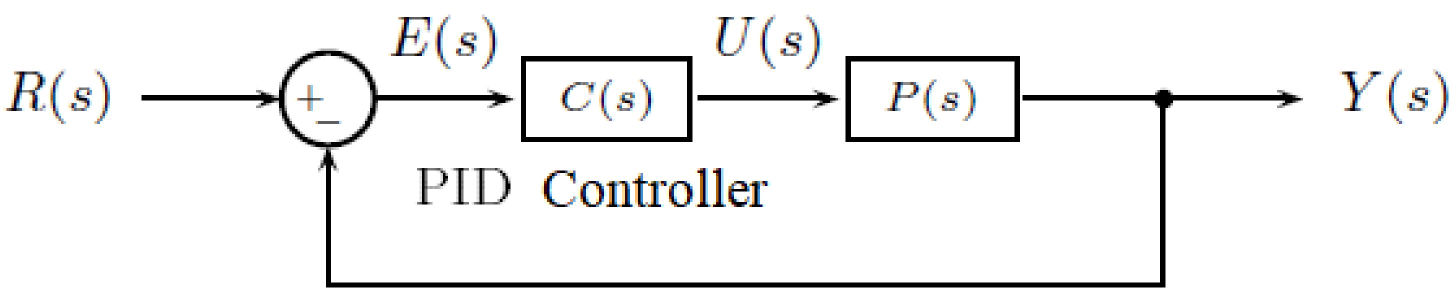
Structure of the PID Controllers

E(s): Input to PID controller, i.e. the error signal

U(s): Controller output, or control signal

R(s): Command signal

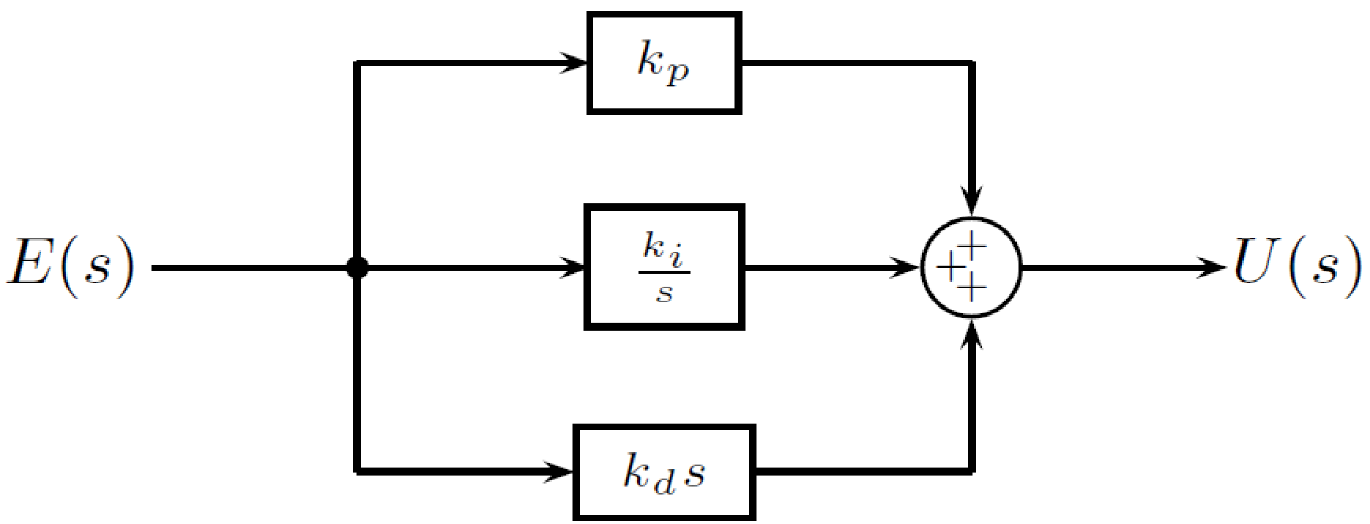
Y(s): Closed loop system response



Structure of the PID Controllers
Parallel Form

$$C(s) = k_p + \frac{k_i}{s} + k_d s$$

$$u(t) = k_p e(t) + k_i \int_0^t e(\tau)\mathrm{d}\tau + k_d \frac{\mathrm{d}e(t)}{\mathrm{d}t}$$



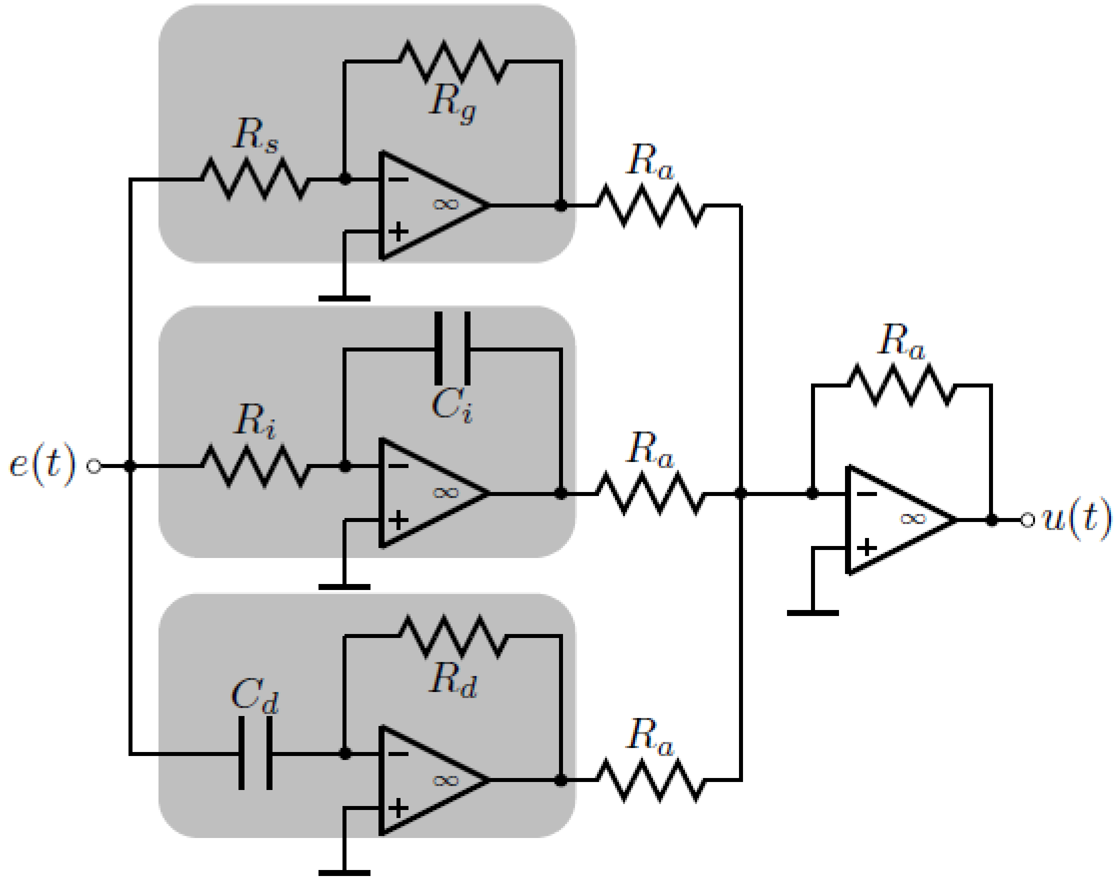
Structure of the PID Controllers Noninteracting Form

$$C(s) = K\left(1 + \frac{1}{sT_i} + sT_d\right)$$

$$k_p = K,\ k_i = \frac{K}{T_i},\ k_d = KT_d$$

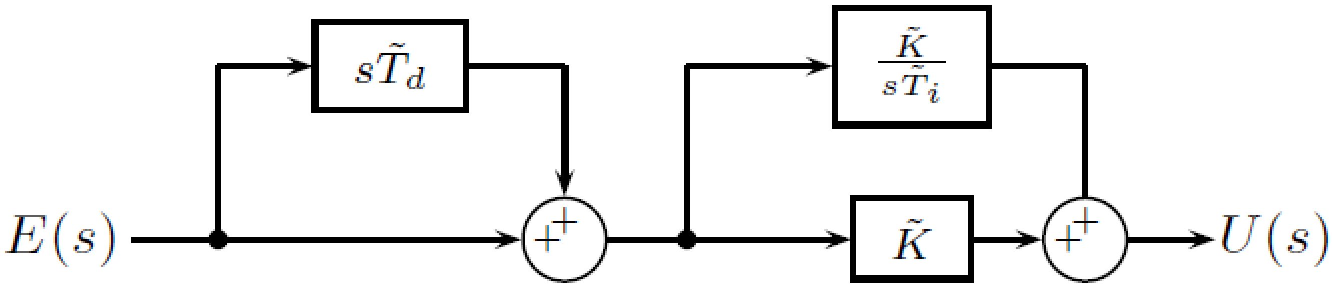
$$C(s) = k_p + \frac{k_i}{s} + k_d s$$

Structure of the PID Controllers
Circuit Realization of Parallel Form



$$\frac{Y(s)}{U(s)} = \frac{R_g}{R_s} + R_d C_d s + \frac{1}{R_i C_i s}$$

Structure of the PID Controllers Interacting Form



$$C(s) = \tilde{K}\left(1 + s\tilde{T}_d\right)\left(1 + \frac{1}{s\tilde{T}_i}\right) = \frac{U(s)}{E(s)}$$

Relations Between the Parameters

$$K = \tilde{K}\frac{\tilde{T}_i + \tilde{T}_d}{\tilde{T}_i}$$

$$T_i = \tilde{T}_i + \tilde{T}_d$$

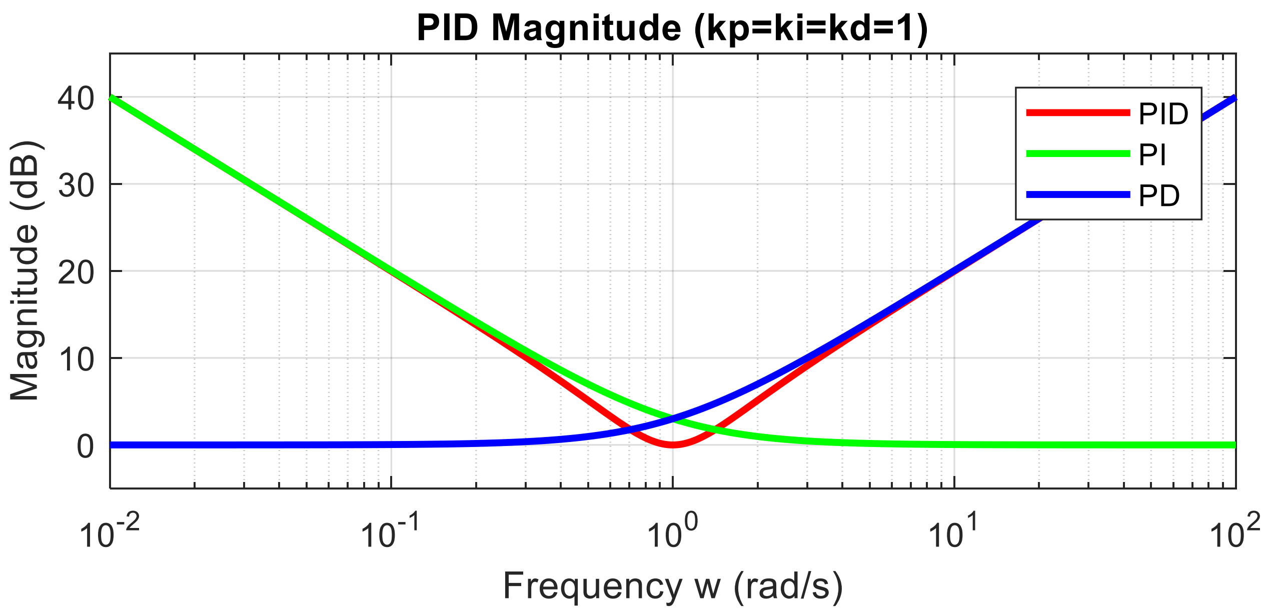
$$T_d = \frac{\tilde{T}_i\tilde{T}_d}{\tilde{T}_i + \tilde{T}_d}$$

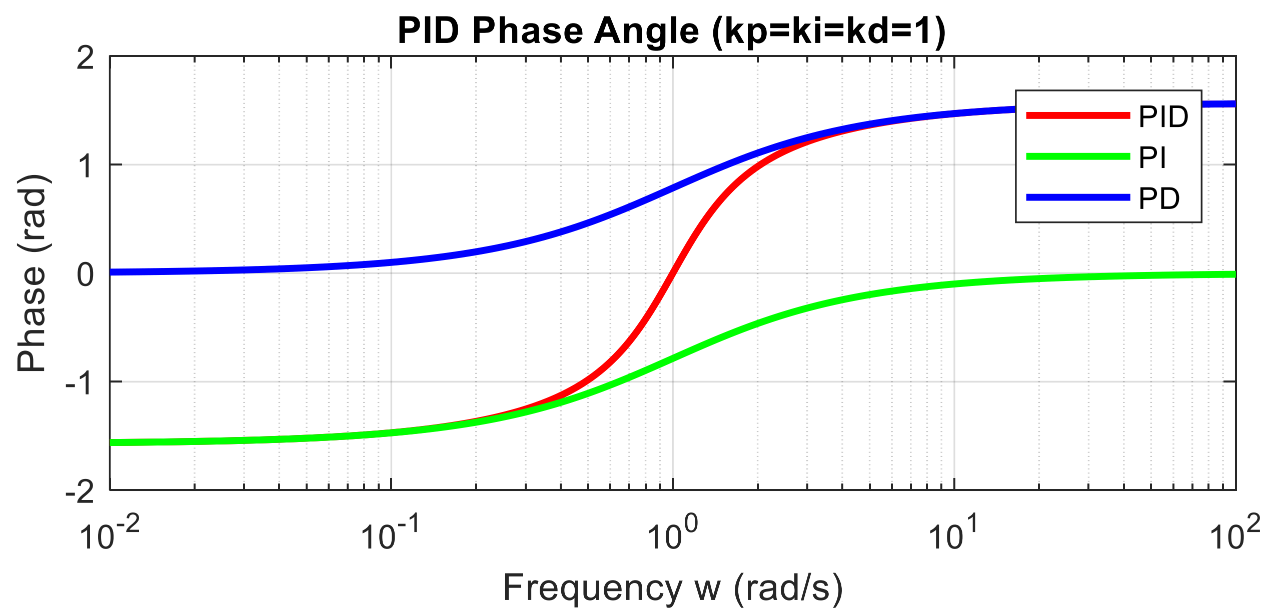
$$\tilde{K} = \frac{K}{2}\left(1 + \sqrt{1 - \frac{4T_d}{T_i}}\right)$$

$$\tilde{T}_i = \frac{T_i}{2}\left(1 + \sqrt{1 - \frac{4T_d}{T_i}}\right)$$

$$\tilde{T}_d = \frac{T_i}{2}\left(1 - \sqrt{1 - \frac{4T_d}{T_i}}\right)$$

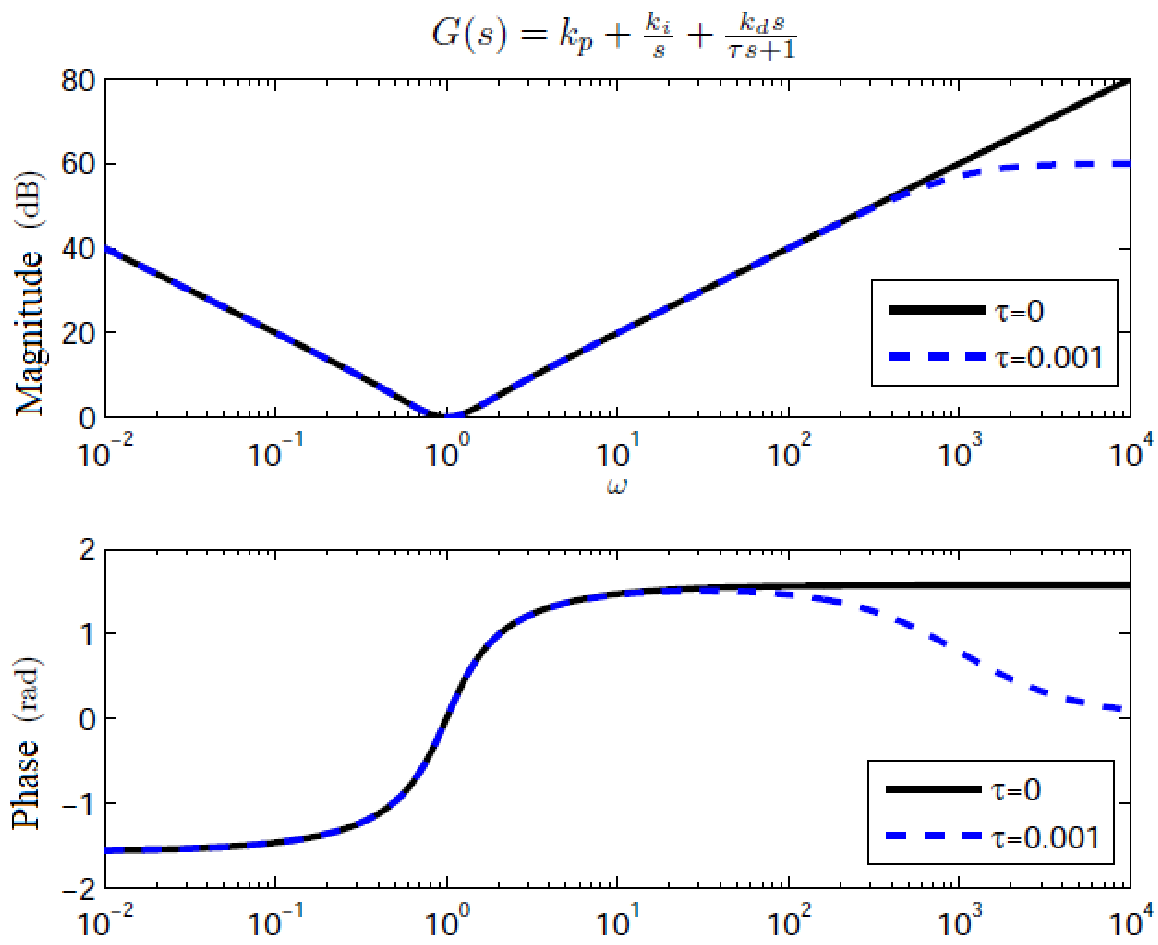
$$\tilde{T}_i \geq 4\tilde{T}_d$$


PID Magnitude (kp=ki=kd=1)
Magnitude (dB)
40
30
20
10
0
PID
PI
PD
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
Frequency w (rad/s)


PID Phase Angle (kp=ki=kd=1)
Phase (rad)
2
1
0
-1
-2
PID
PI
PD
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
Frequency w (rad/s)

Finite Bandwidth Derivative Realization $k_p=k_i=k_d=1$ and; $\tau=0$ and $\tau=0.001$

$$C(s) = k_p + \frac{k_i}{s} + \frac{k_d s}{\tau s + 1}, \quad 0 < \tau \ll 1$$



Variants of PID: P, PI, PD

$$u(t) = k_p e(t)$$

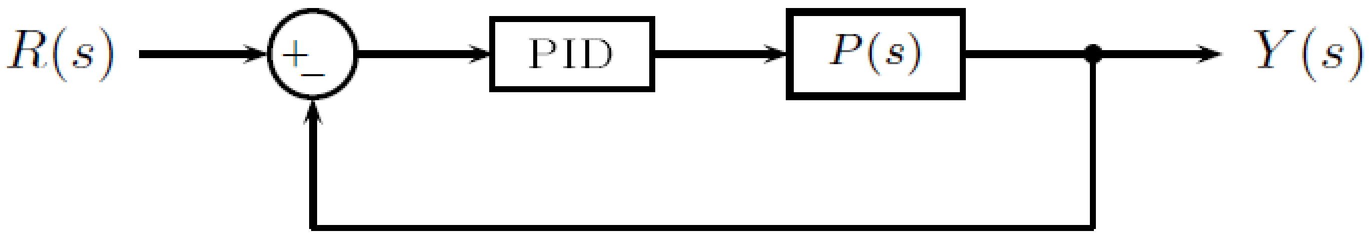
$$u(t) = k_p e(t) + k_i \int_0^t e(\tau)\mathrm{d}\tau$$

$$u(t) = k_p e(t) + k_d \frac{\mathrm{d}e(t)}{\mathrm{d}t}$$

A Simple MATLAB Code

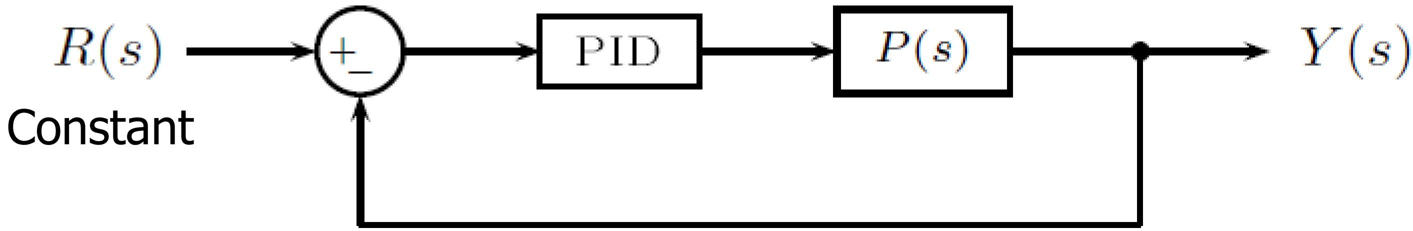
```
kp=1;ki=1;kd=0.01;
P=tf(1,[1 1])
PIDcontroller=tf([kd kp ki],[1 0]);
ClosedLoopTF=feedback(P*PIDcontroller,1);
t=0:0.001:20;
step(ClosedLoopTF,t)
```

Setpoint Weighting



- If the refererrnce signal changes suddenly
  - $k_p e(t)$ changes suddenly
  - $k_d\, de(t)/dt$ changes suddenly
- Setpoint weighting approach aims at reducing the adverse effects of these sudden changes.

Setpoint Weighting



$$
\begin{aligned}
u(t) &= k_p e(t) + k_i \int_0^t e(\tau)\mathrm{d}\tau + k_d \frac{\mathrm{d}e(t)}{\mathrm{d}t} \\
&= k_p(r(t) - y(t)) + k_i \int_0^t e(\tau)\mathrm{d}\tau + k_d(\cancelto{0}{\dot{r}(t)} - \dot{y}(t)) \\
&= k_p(r(t) - y(t)) + k_i \int_0^t e(\tau)\mathrm{d}\tau - k_d\dot{y}(t)
\end{aligned}
$$

This approach prevents sudden jumps in the control signal

Setpoint Weighting

$$u(t) = k_p e_p(t) + k_i \int_0^t e(\tau)\mathrm{d}\tau + k_d \frac{\mathrm{d}e_d(t)}{\mathrm{d}t}$$

$$e_p(t) := \alpha_p r(t) - y(t) \qquad 0 < \alpha_p < 1$$

$$e_d(t) := \alpha_d r(t) - y(t) \qquad 0 < \alpha_d < 1$$

$$r(t) = c_1 1(t) + c_2 1(t - T), \quad c_1 > 0, c_2 > 0$$

$$r(t) = \begin{cases} 0 & t < 0 \\ c_1 & 0 < t < T \\ c_1 + c_2 & t > T \end{cases}$$

A Simple MATLAB Code

$$t = T \qquad k_p(c_1 + c_2 - y(T+\varepsilon)) - k_p(c_1 - y(T-\varepsilon)) \approx k_p c_2$$

$$y(T+\varepsilon) \approx y(T-\varepsilon)$$

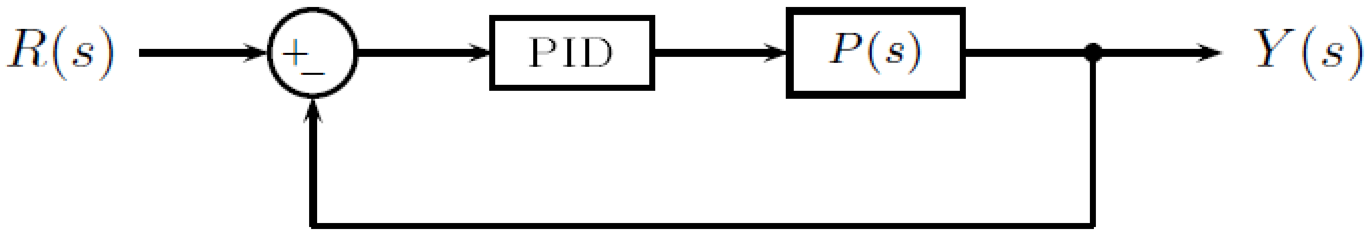
$$k_p(\alpha_p(c_1 + c_2) - y(T+\varepsilon)) - k_p(\alpha_p c_1 - y(T-\varepsilon))\alpha_p \approx k_p c_2 \alpha_p$$

$$0 < \alpha_p < 1$$

An Example

$$k_p = 1.8 \qquad k_i = 1.7 \qquad k_d = 0.1$$

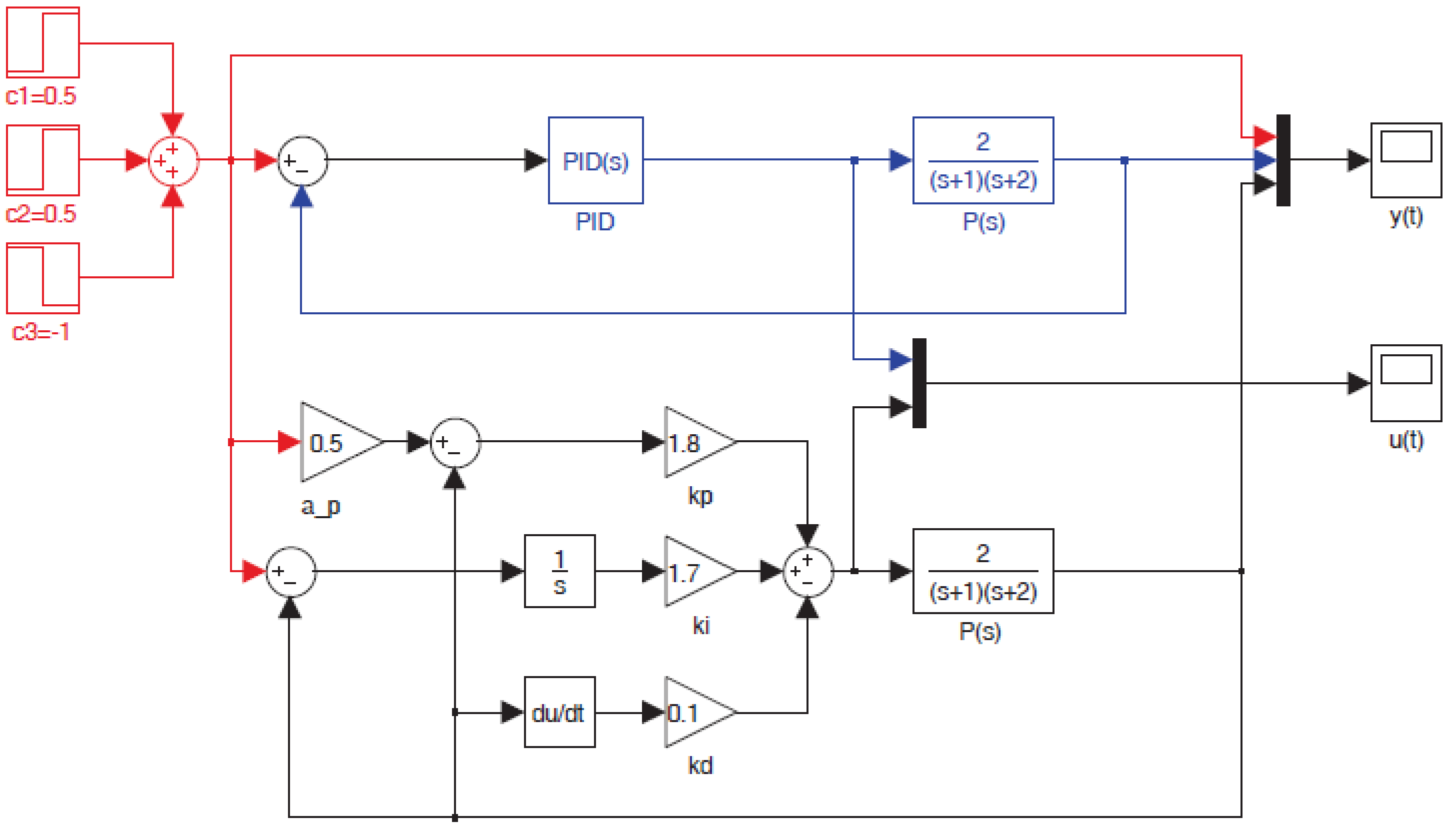
$$P(s) = \frac{2}{(s+1)(s+2)}$$



$$r(t) = \begin{cases} 0 & t < 0 \\ 0.5 & 0 < t < 10 \\ 1 & 10 < t < 20 \\ -1 & 20 < t < 30 \end{cases}$$

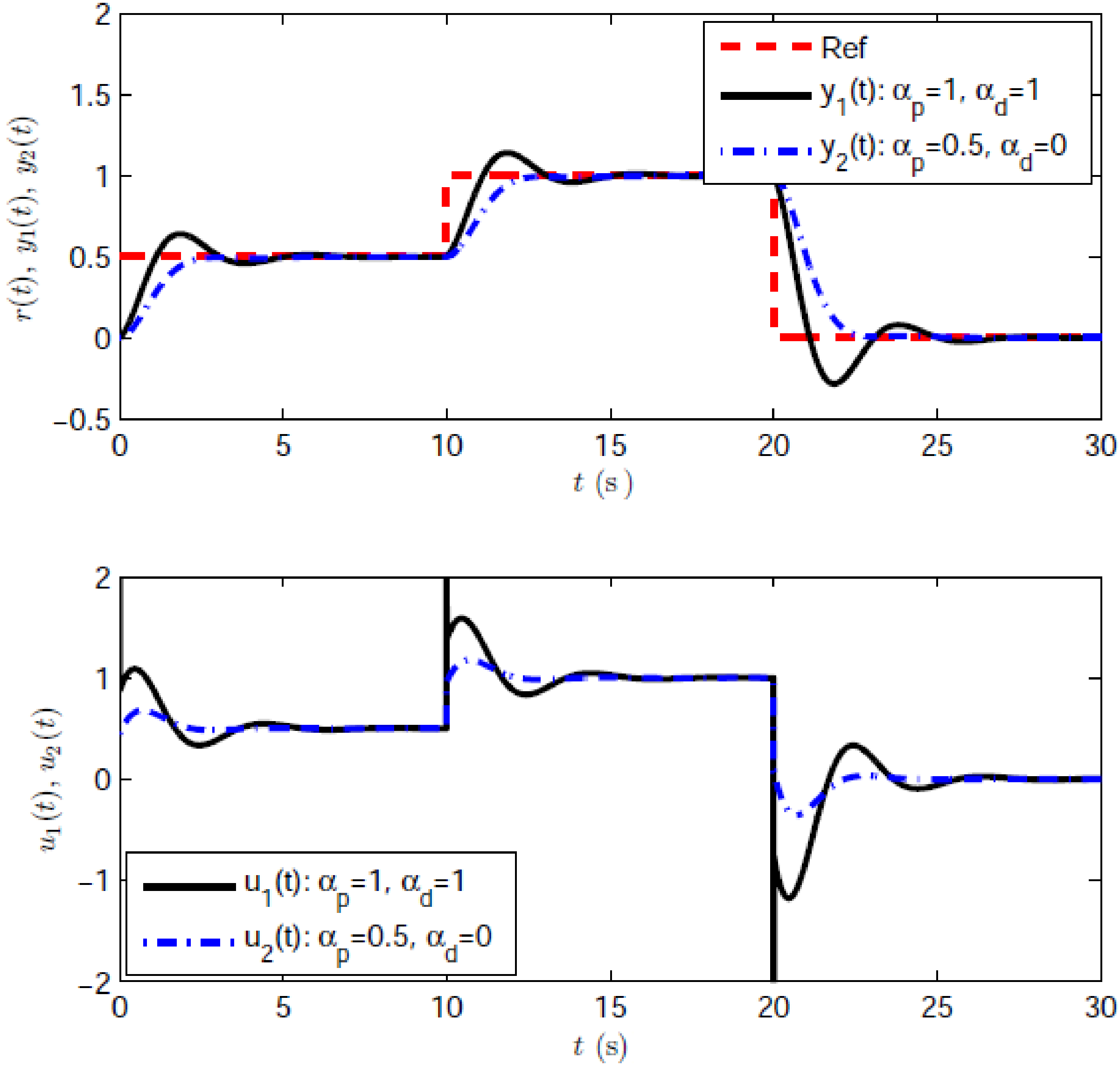
Compare the responses with α_p=0.5 and α_d=0 with the plain PID.

An Example



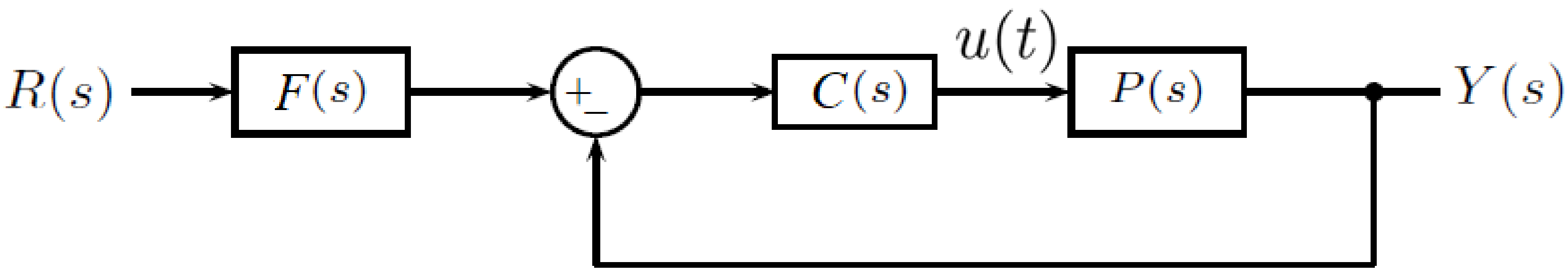
c1=0.5
c2=0.5
c3=-1
PID(s)
PID
2
(s+1)(s+2)
P(s)
y(t)
u(t)
0.5
a_p
1.8
kp
1
s
1.7
ki
du/dt
0.1
kd
2
(s+1)(s+2)
P(s)

An Example



Ref
y1(t): αp=1, αd=1
y2(t): αp=0.5, αd=0
r(t), y1(t), y2(t)
t (s)
u1(t), u2(t)
u1(t): αp=1, αd=1
u2(t): αp=0.5, αd=0

Setpoint Weighting



$$u(t) = K_p\left(\beta r(t) - y(t) + \frac{1}{T_i}\int_0^t e(\tau)d\tau - T_d\frac{dy(t)}{dt}\right)$$

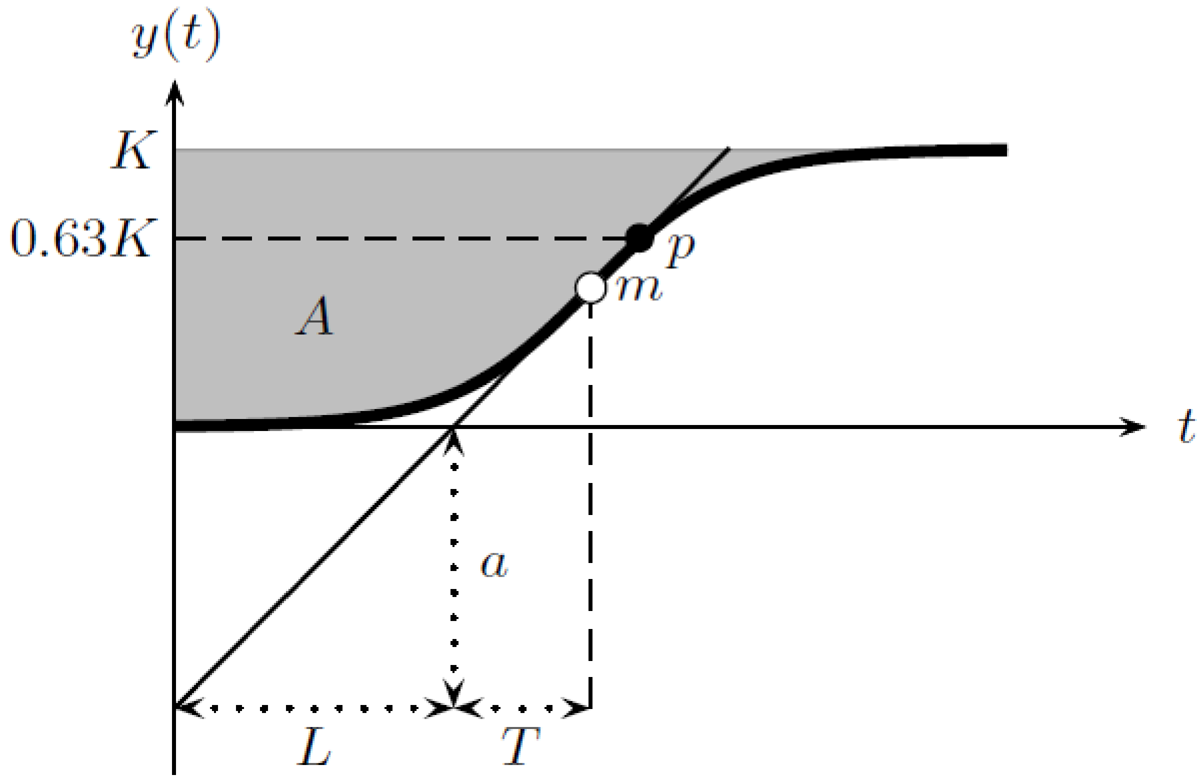
$$C(s) = K_p\left(1 + \frac{1}{T_i s} + T_d s\right)$$

$$F(s) = \frac{1 + \beta T_i s}{1 + T_i s + T_i T_d s^2}$$

$\beta < 1$ reduces the overshoot.

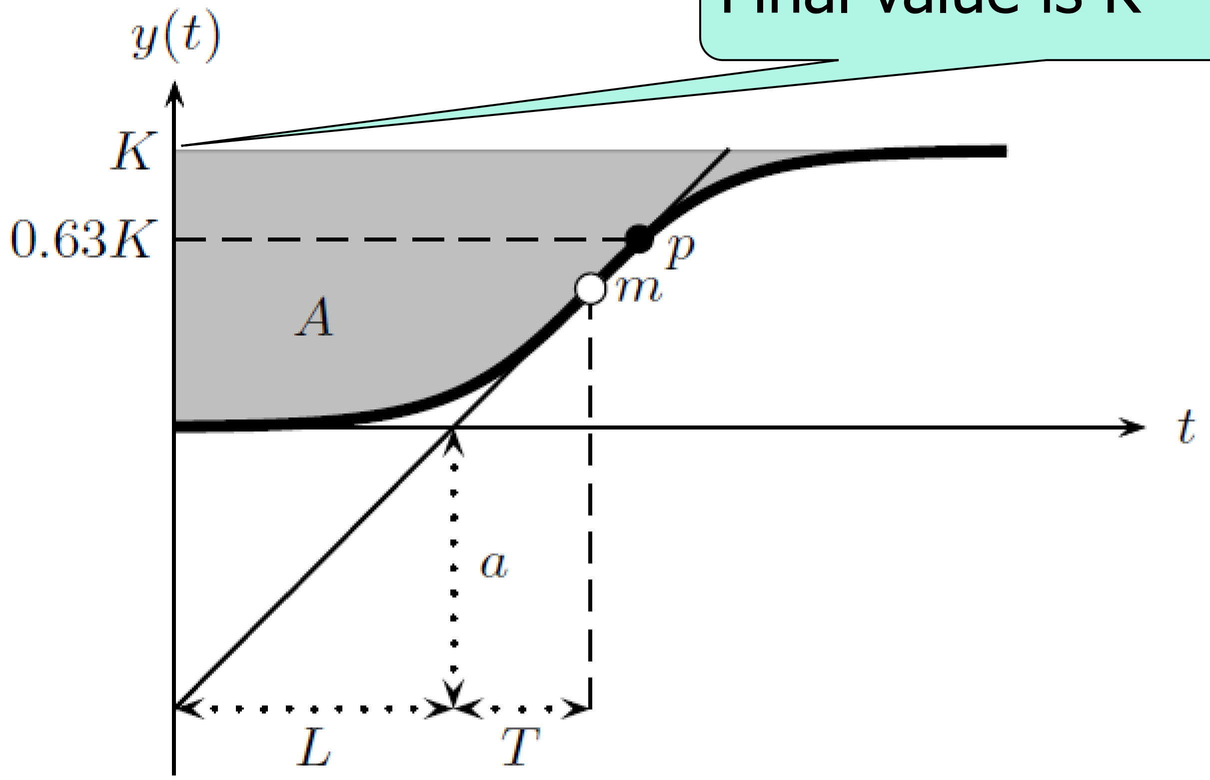
Process Models

We apply a step command to a system and obtain its response as shown below.

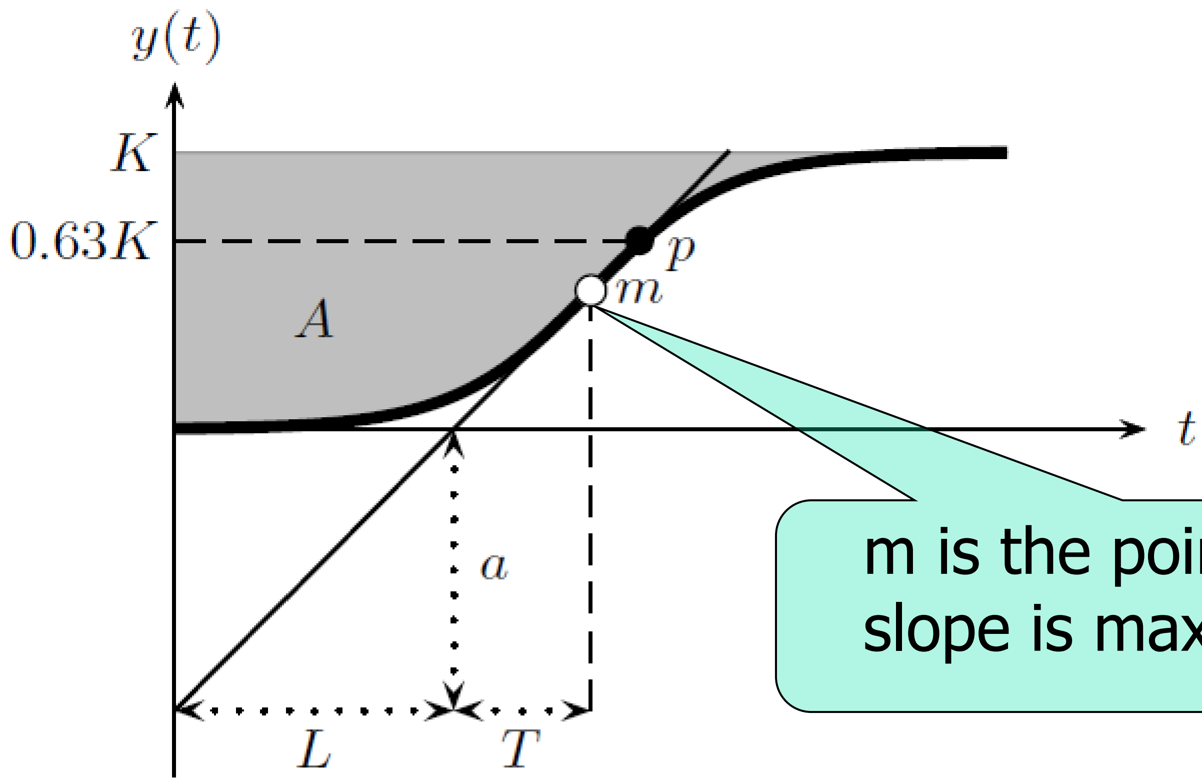


Process Models

Final value is K



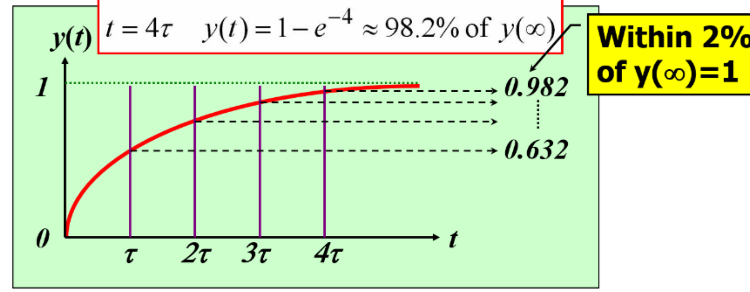
Process Models

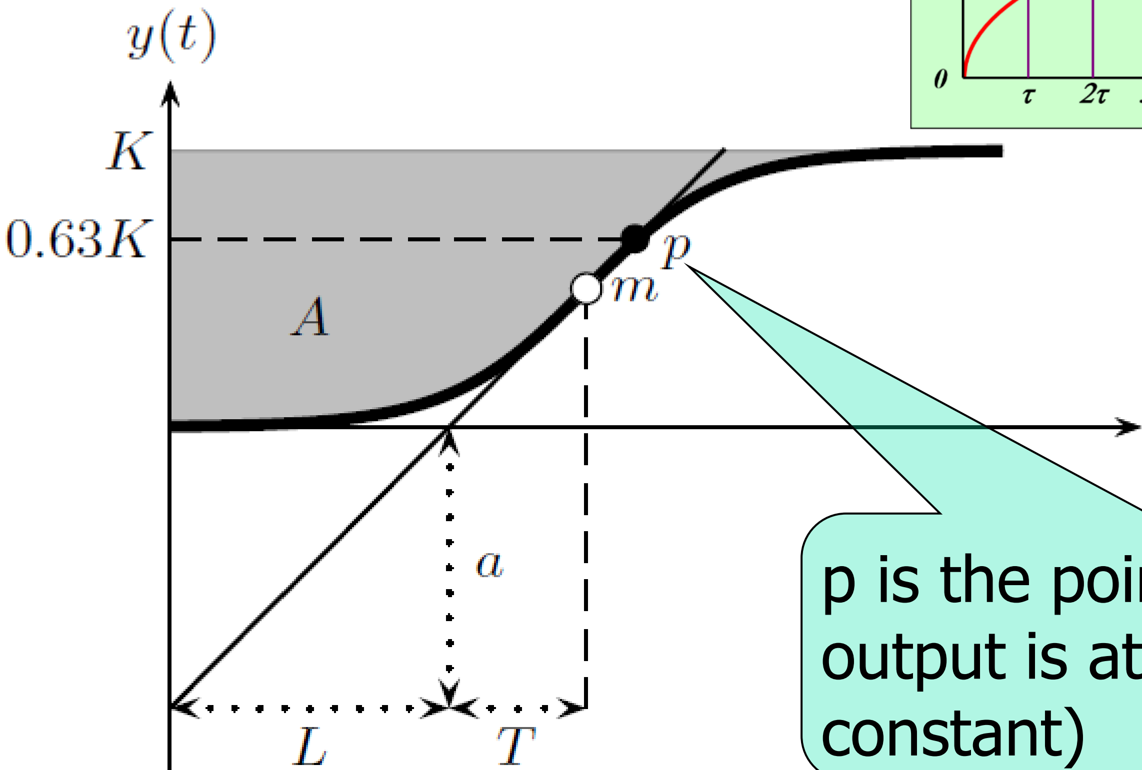


m is the point where the slope is maximum

Process Models

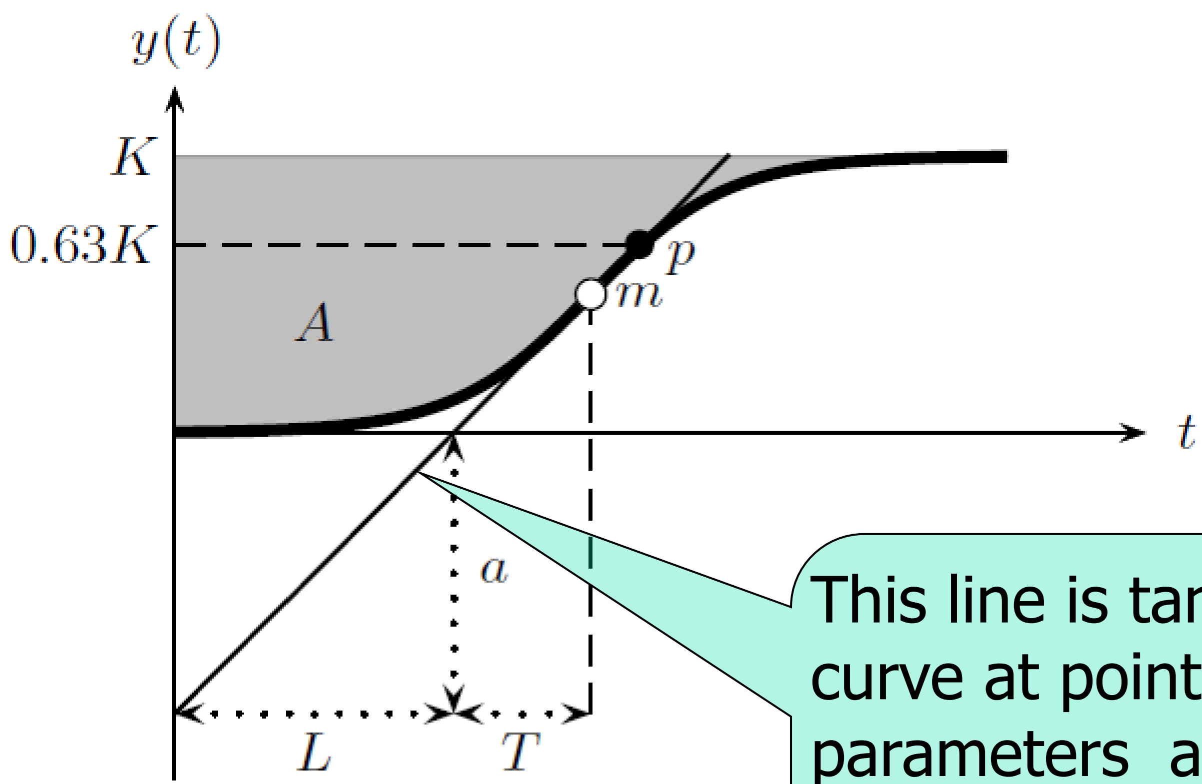
$$
\begin{array}{lll}
t=0 & y(t)=1-e^{-0} & =0\% \text{ of } y(\infty) \\
t=\tau & y(t)=1-e^{-1} & \approx 63.2\% \text{ of } y(\infty) \\
t=2\tau & y(t)=1-e^{-2} & \approx 86.5\% \text{ of } y(\infty) \\
t=3\tau & y(t)=1-e^{-3} & \approx 95.0\% \text{ of } y(\infty) \\
t=4\tau & y(t)=1-e^{-4} & \approx 98.2\% \text{ of } y(\infty)
\end{array}
$$





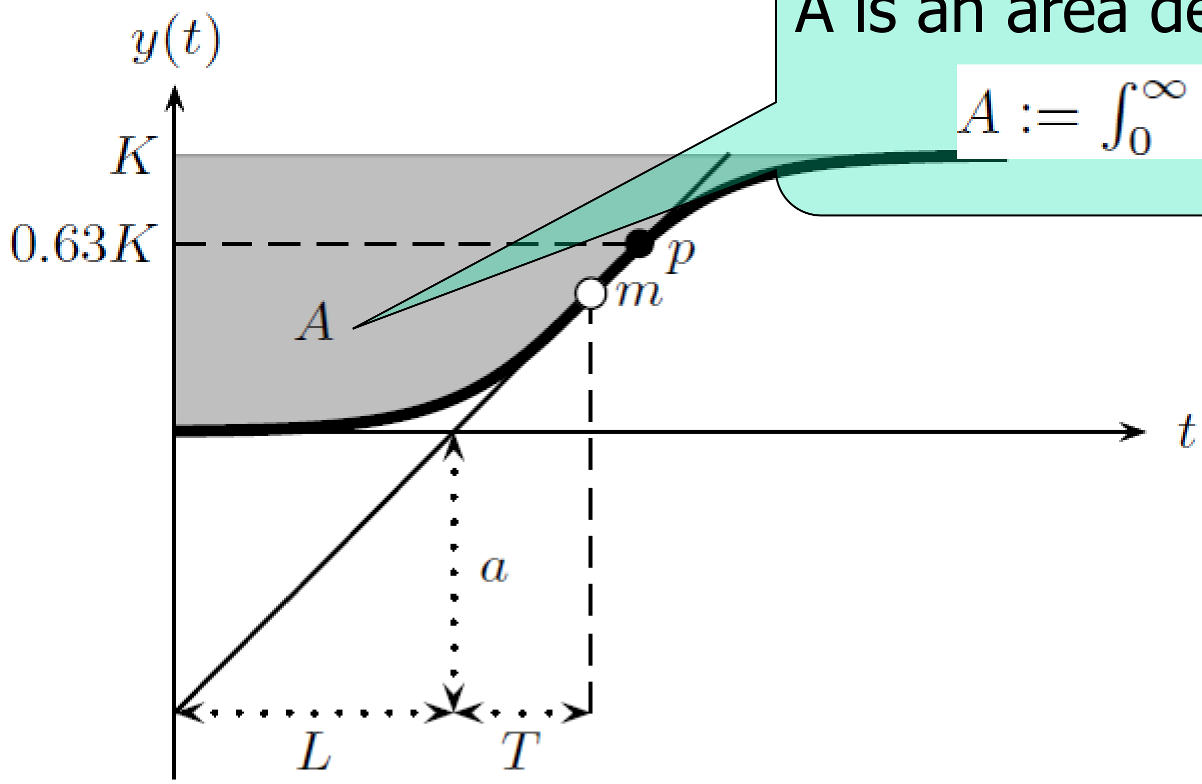
p is the point where the output is at 0.63K (time constant)

Process Models



This line is tangent to the curve at point p. The parameters a, L and T are defined according to this line.

Process Models

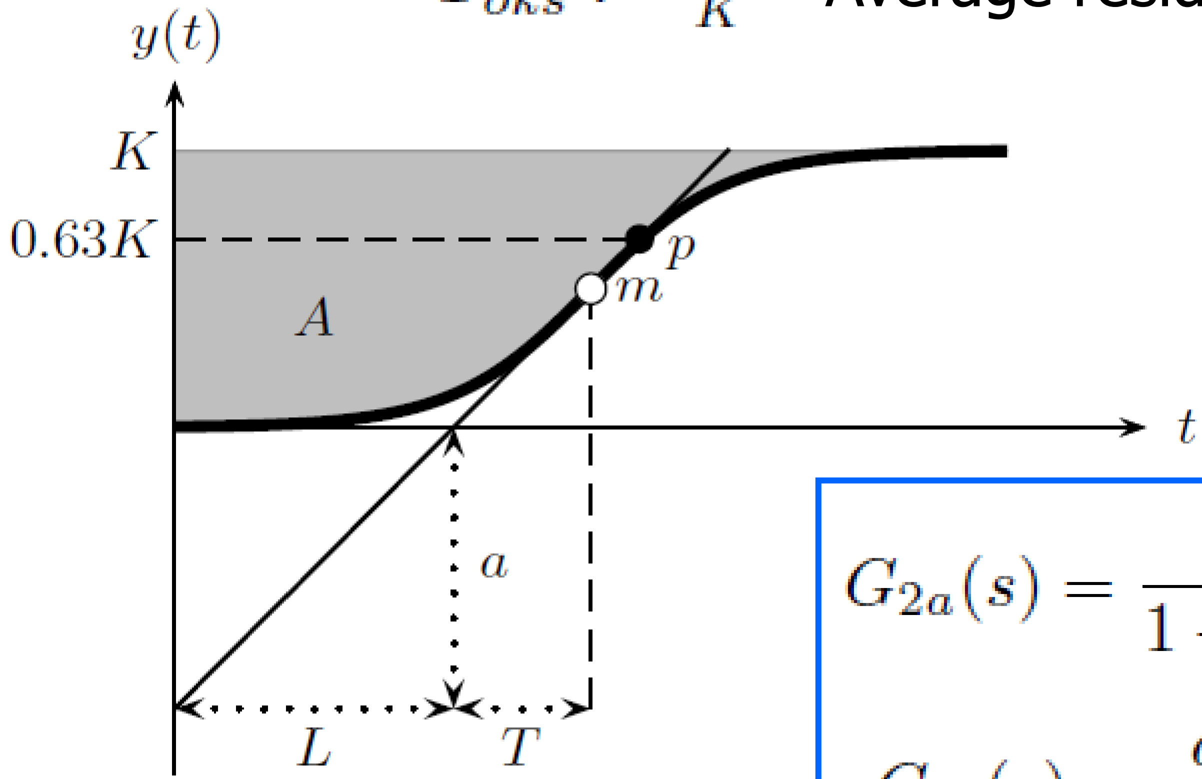


A is an area defined as

$$A := \int_0^\infty (K - y(t))\,\mathrm{d}t$$

2 Parameter Models

$$T_{oks} := \frac{A}{K}$$ Average residence time

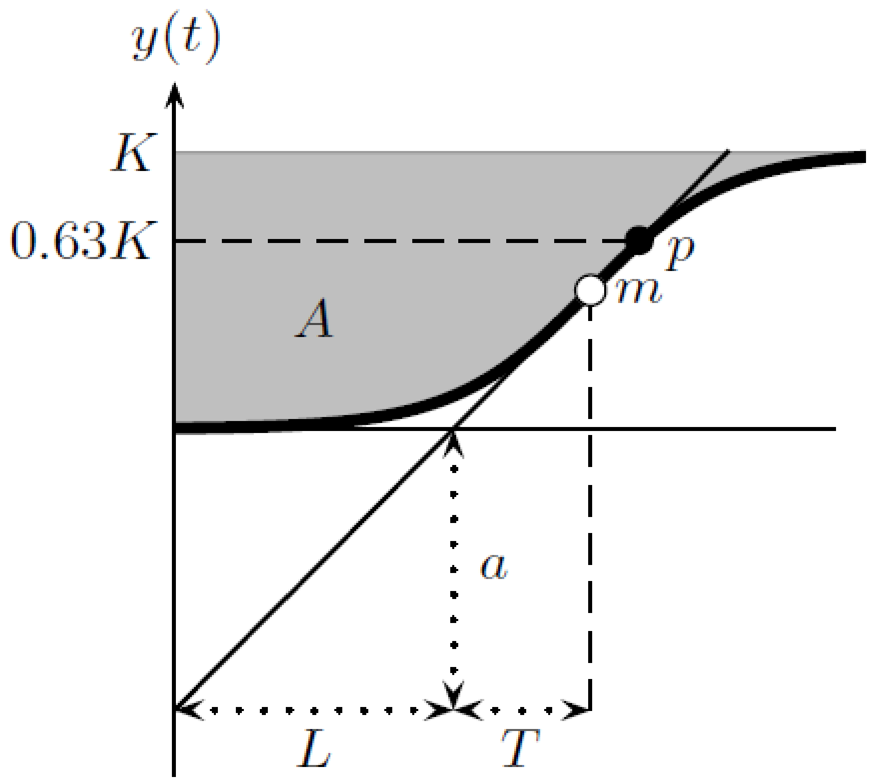


$$G_{2a}(s) = \frac{K}{1 + sT_{oks}}$$

$$G_{2b}(s) = \frac{a}{sL} e^{-sL}$$

3 Parameter Models

$$G_{3a}(s) = \frac{K}{1+sT}e^{-sL}$$

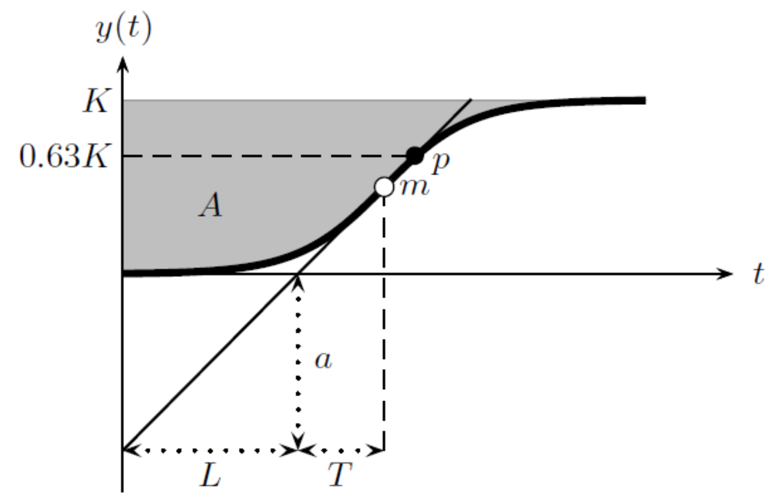


$$G_{3b}(s) = \frac{K}{(1+sT_a)^2}e^{-sL}$$

Solve below equation for any time instant, find T_a and insert into model G_{3b}.

$$y(t) = K\left(1-\left(1+\frac{t-L}{T_a}\right)e^{-(t-L)/T_a}\right)1(t)$$

4 Parameter Models



To find T_1 and T_2, choose two points from the solution and solve the below solution of y(t).

$$G_4(s) = \frac{K}{(1+sT_1)(1+sT_2)}e^{-sL}$$

$$y(t) = K\left(1 + \frac{T_2 e^{-(t-L)/T_2} - T_1 e^{-(t-L)/T_1}}{T_1 - T_2}\right)1(t), \quad T_1 \neq T_2$$

Parameter Tuning: Ziegler Nichols

$$C(s) = K\left(1 + \frac{1}{sT_i} + sT_d\right)$$

Controller	K	T_i	T_d	T_p
P	$\frac{1}{a}$			$4L$
PI	$\frac{0.9}{a}$	$3L$		$5.7L$
PID	$\frac{1.2}{a}$	$2L$	$\frac{L}{2}$	$3.4L$

T_p: Period of possible damped oscillations

Parameter Tuning: Chein-Hrones-Reswick (No Overshoot)

Controller	K	T_i	T_d
P	$\frac{0.3}{a}$		
PI	$\frac{0.6}{a}$	$4L$	
PID	$\frac{0.95}{a}$	$2.4L$	$0.42L$

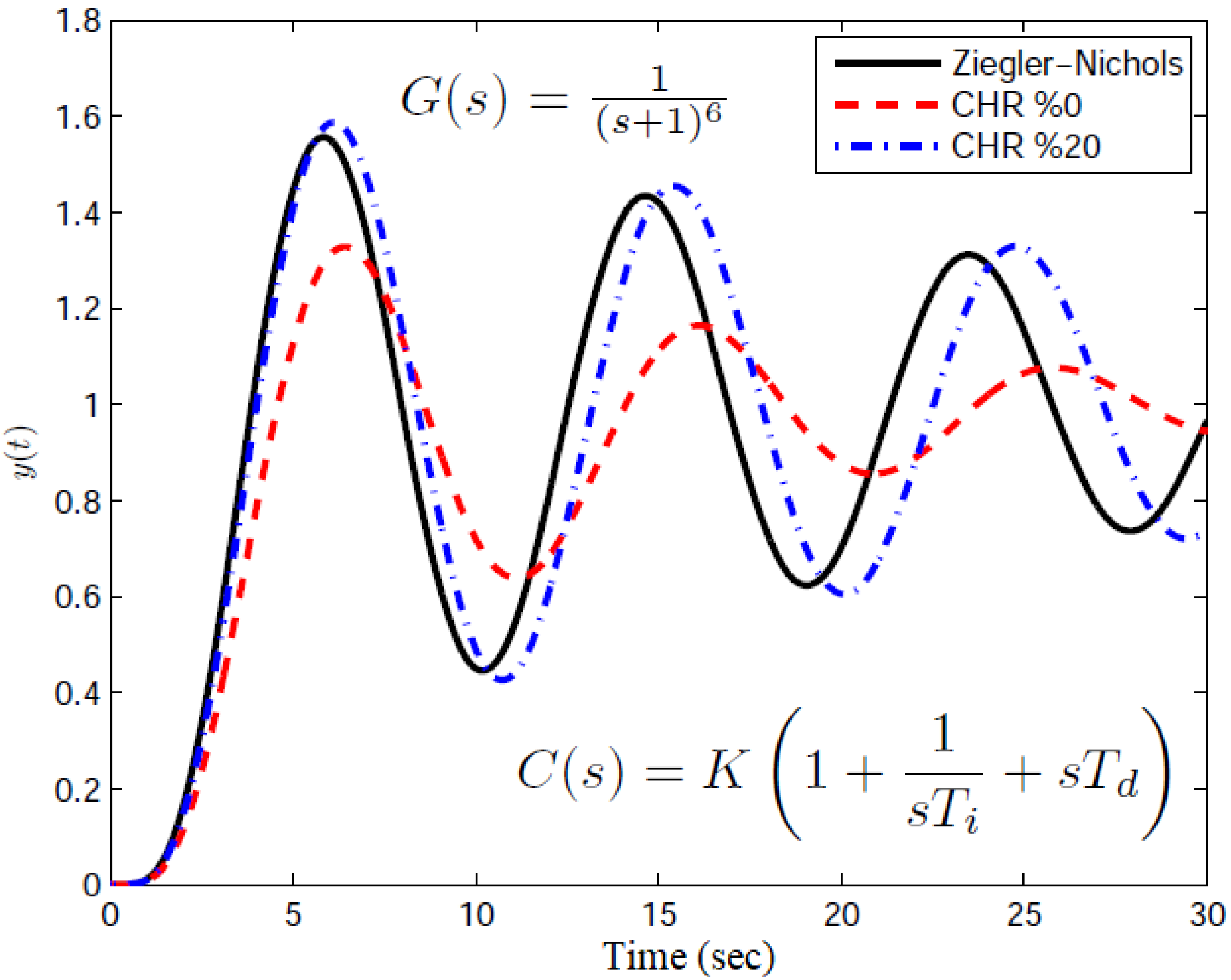
$$C(s) = K\left(1 + \frac{1}{sT_i} + sT_d\right)$$

Parameter Tuning: Chein-Hrones-Reswick (20% Overshoot)

Controller	K	T_i	T_d
P	$\frac{0.7}{a}$		
PI	$\frac{0.7}{a}$	$2.3L$	
PID	$\frac{1.2}{a}$	$2L$	$0.42L$

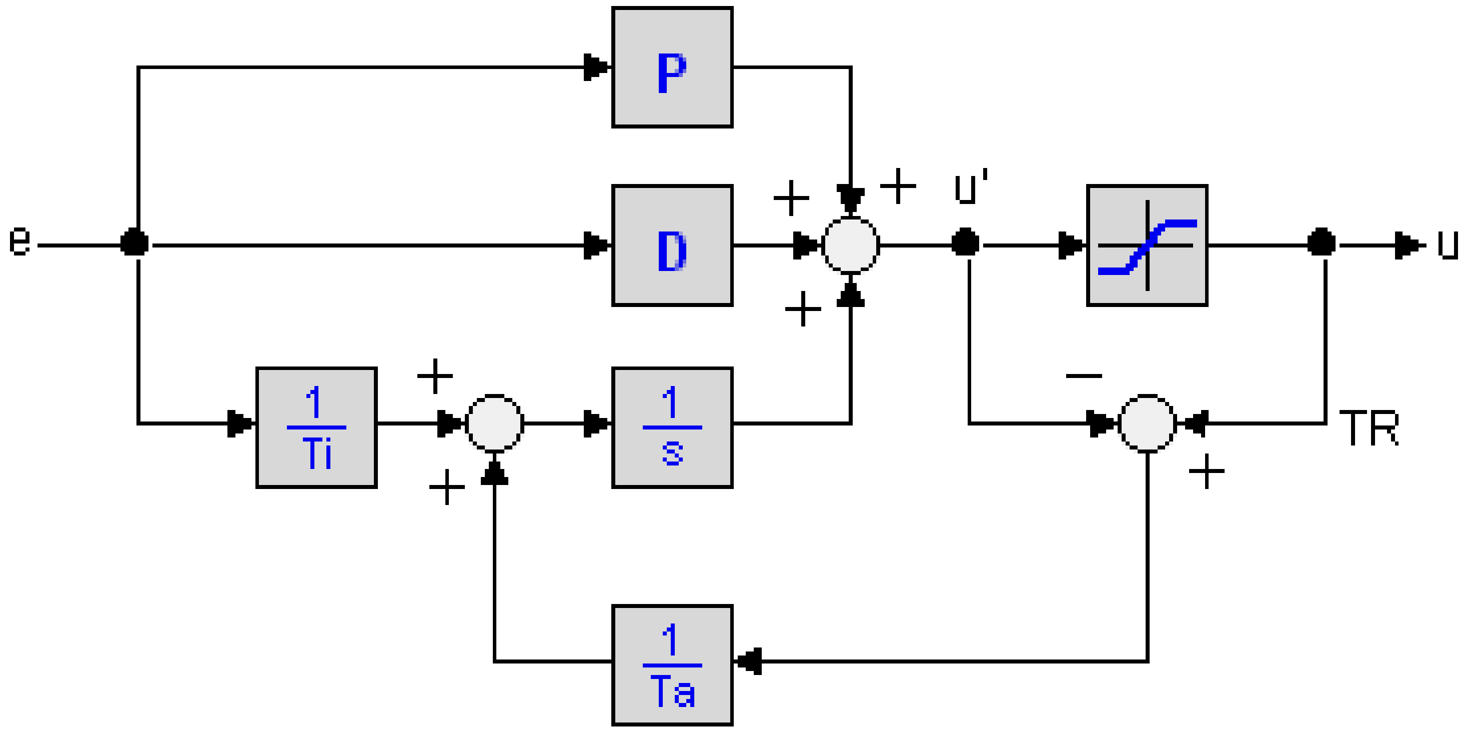
$$C(s) = K\left(1 + \frac{1}{sT_i} + sT_d\right)$$

Example

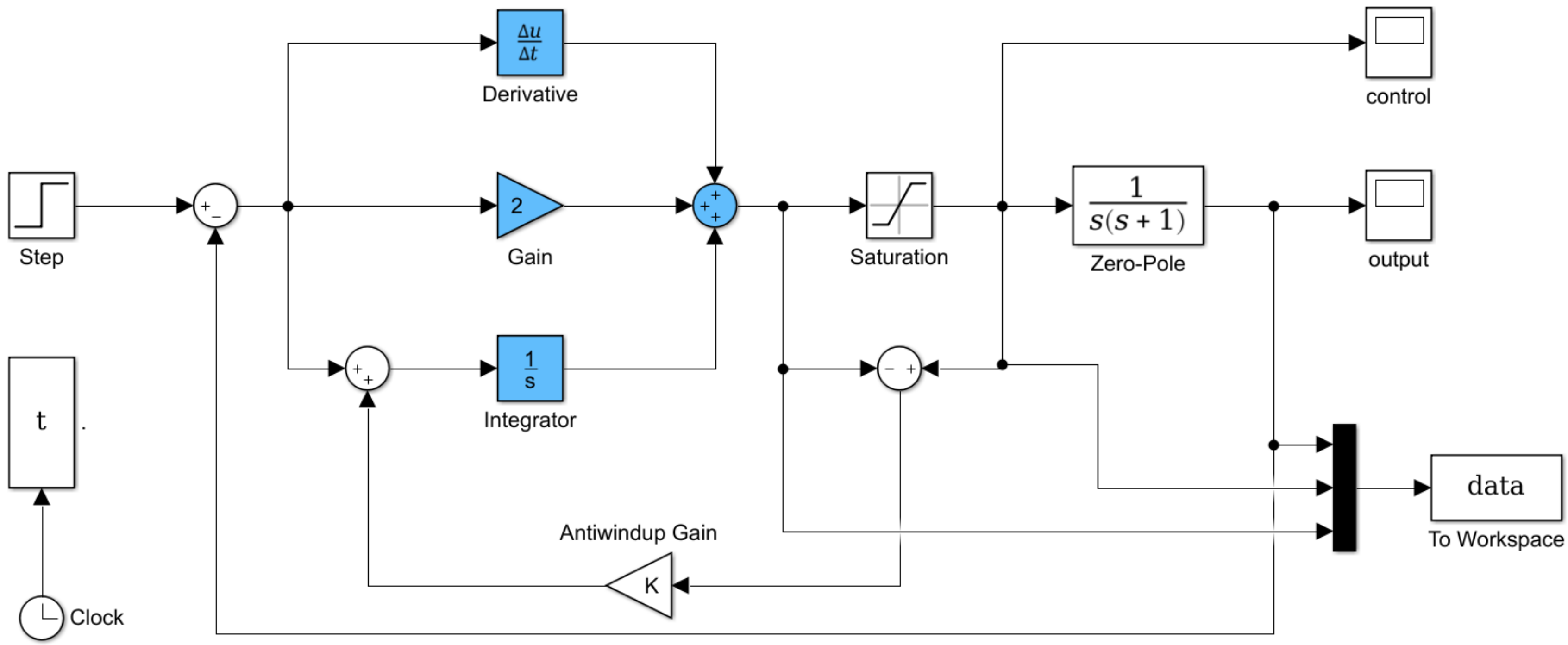


$G(s) = \frac{1}{(s+1)^6}$
Ziegler–Nichols
CHR %0
CHR %20
$C(s) = K\left(1 + \frac{1}{sT_i} + sT_d\right)$
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
$y(t)$
0
5
10
15
20
25
30
Time (sec)

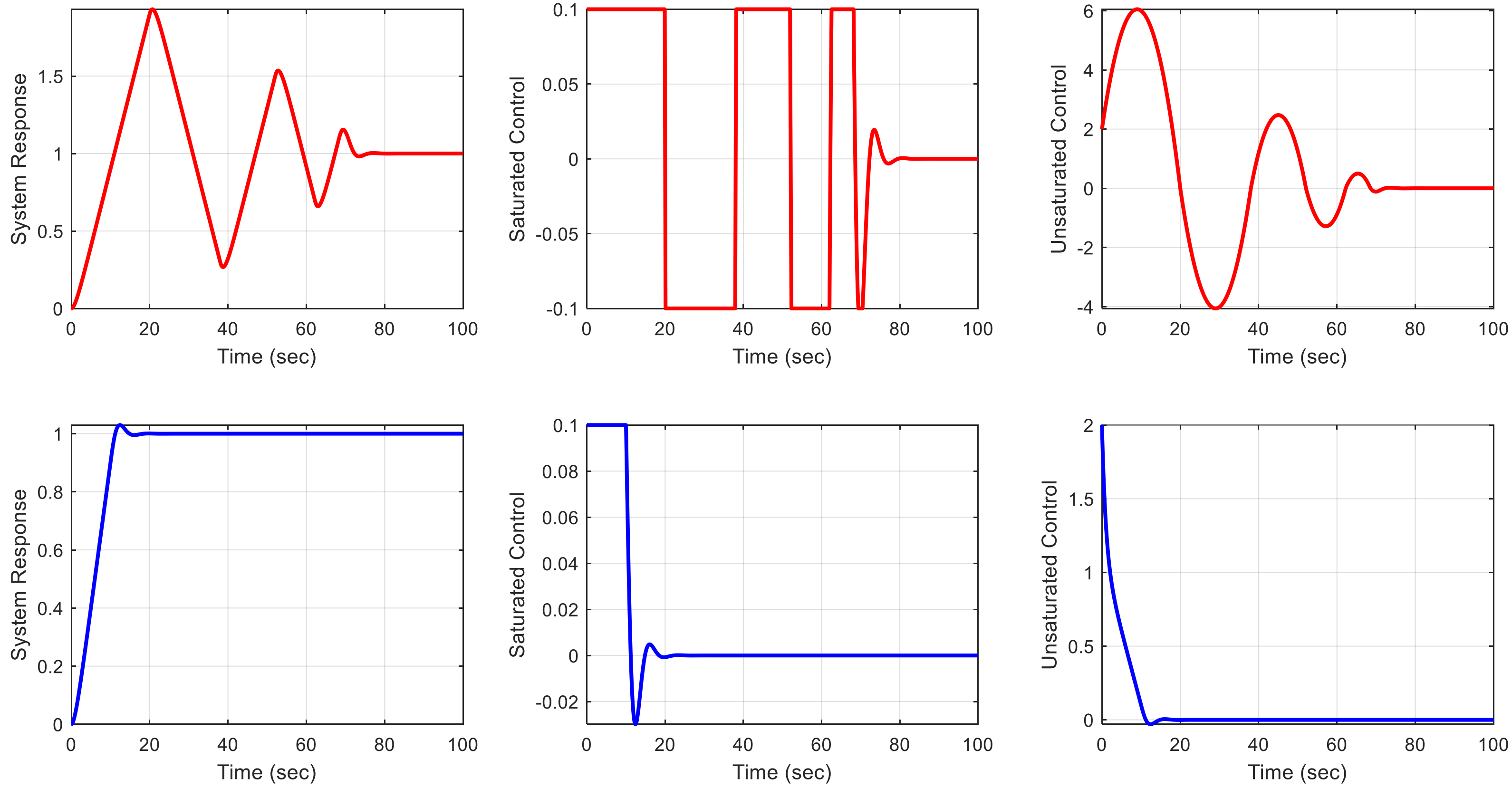
Integrator Antiwindup



Integrator Antiwindup



Integrator Antiwindup



Performance Measures

$$ISE := \int_0^T e(t)^2 \mathrm{d}t$$

Integral of the Squared Error

$$IAE := \int_0^T |e(t)| \mathrm{d}t$$

Integral of the Absolute value of Error

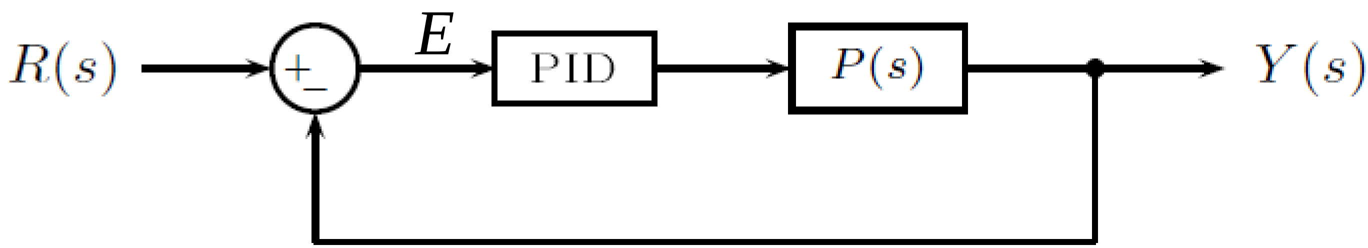
$$ITAE := \int_0^T t|e(t)| \mathrm{d}t$$

Integral of Time multiplied by Absolute Error

$$ITSE := \int_0^T te(t)^2 \mathrm{d}t$$

Integral of Time multiplied by Squared Error

Performance Measures

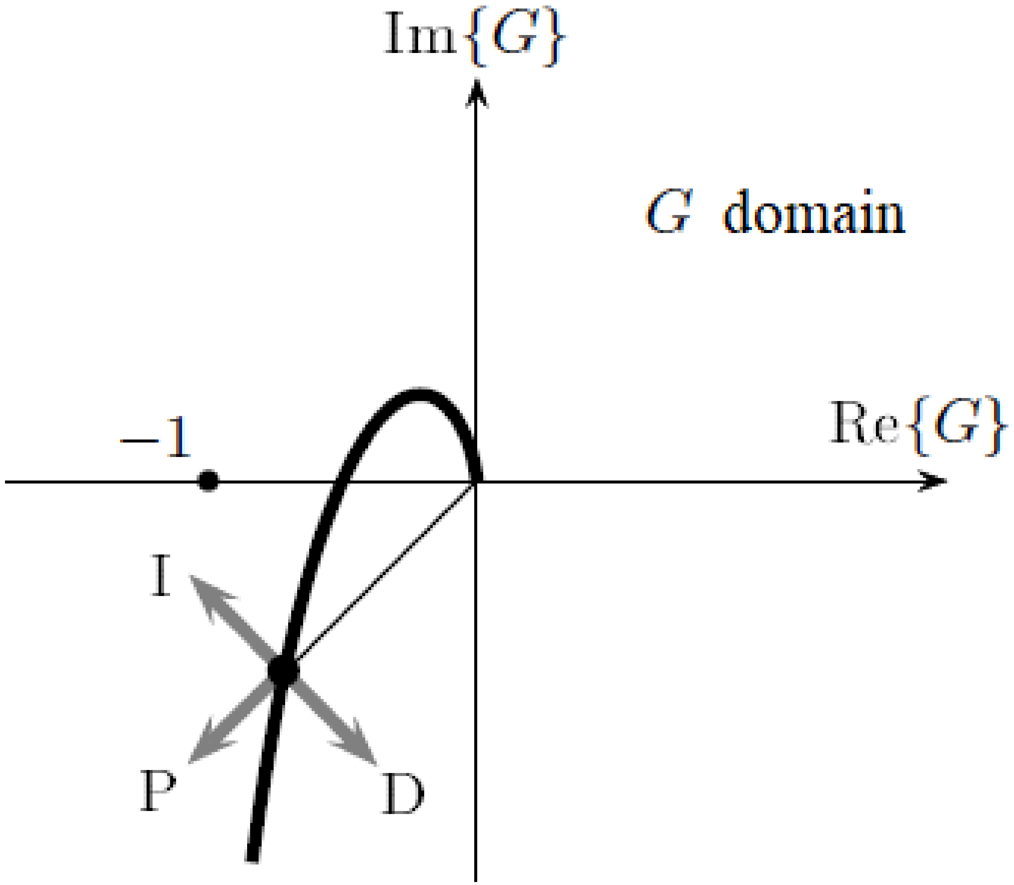


$$C = k_p + \frac{k_i}{s} + k_d s = \frac{k_d s^2 + k_p s + k_i}{s}, P = \frac{B(s)}{A(s)}$$

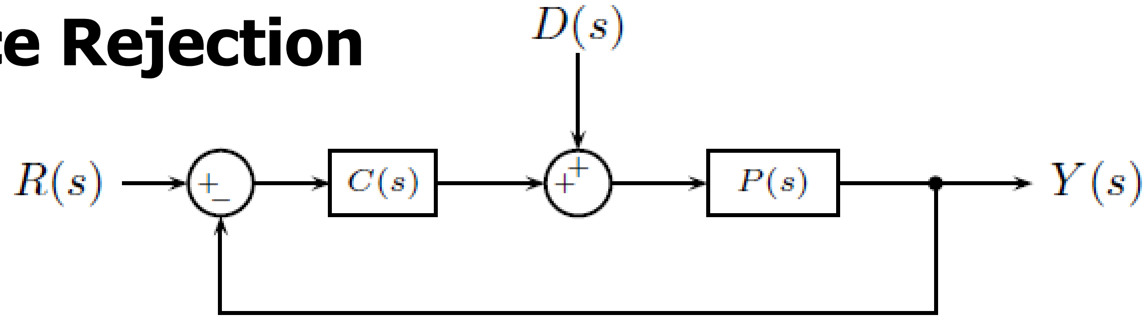
$$E = \frac{1}{1+PC} R = \frac{1}{1 + \frac{B(s)}{A(s)} \frac{k_d s^2 + k_p s + k_i}{s}} R = \frac{sA(s)}{sA(s) + (k_d s^2 + k_p s + k_i) B(s)} R$$

$e(k_p, k_i, k_d, t)$ needs to be computed and optimized to obtain minimum ISE, IAE, ITAE or ITSE.

Effect of P, I and D Actions on Nyquist Plot



Disturbance Rejection



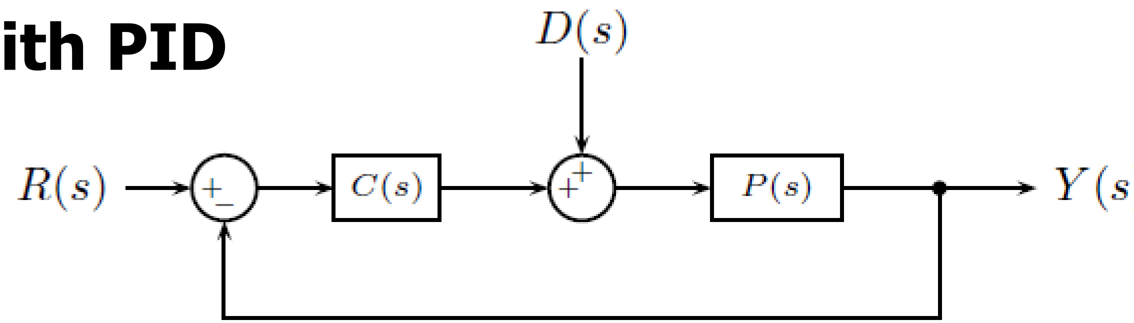
$$Y=\frac{PC}{1+PC}R+\frac{P}{1+PC}D,\qquad R=\frac{1}{s},D=\frac{a}{s}e^{-bs},C=k_p+\frac{k_i}{s}=\frac{k_ps+k_i}{s}$$

$$Y=\frac{PC}{1+PC}\frac{1}{s}+\frac{P}{1+PC}\frac{a}{s}e^{-bs}=\frac{P}{1+PC}\frac{1}{s}\left(C+ae^{-bs}\right)=\frac{C+ae^{-bs}}{P^{-1}+C}\frac{1}{s}$$

$$Y=\frac{\dfrac{k_ps+k_i}{s}+ae^{-bs}}{P^{-1}+\dfrac{k_ps+k_i}{s}}\frac{1}{s}=\frac{k_ps+k_i+sae^{-bs}}{sP^{-1}+k_ps+k_i}\frac{1}{s},\qquad P^{-1}=\frac{D(s)}{N(s)}$$

$$y(\infty)=\lim_{s\to 0}sY(s)=\lim_{s\to 0}\frac{k_ps+k_i+sae^{-bs}}{sP^{-1}+k_ps+k_i}=1$$

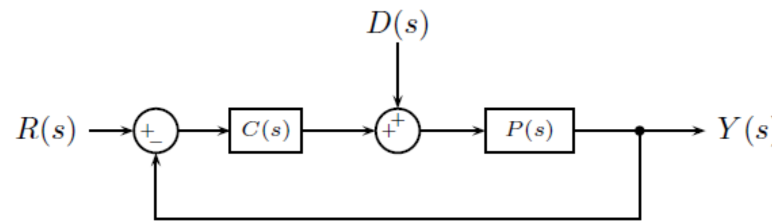
Stability with PID



$$T=\frac{PC}{1+PC},\quad P=\frac{N(s)}{D(s)},\quad C=k_p+\frac{k_i}{s}+k_d s=\frac{k_d s^2+k_p s+k_i}{s}$$

$$T=\frac{\dfrac{N(s)}{D(s)}\dfrac{k_d s^2+k_p s+k_i}{s}}{1+\dfrac{N(s)}{D(s)}\dfrac{k_d s^2+k_p s+k_i}{s}}=\frac{N(s)\left(k_d s^2+k_p s+k_i\right)}{sD(s)+N(s)\left(k_d s^2+k_p s+k_i\right)}$$

Stability with PID



$$P = \frac{a}{s(s+b)}$$

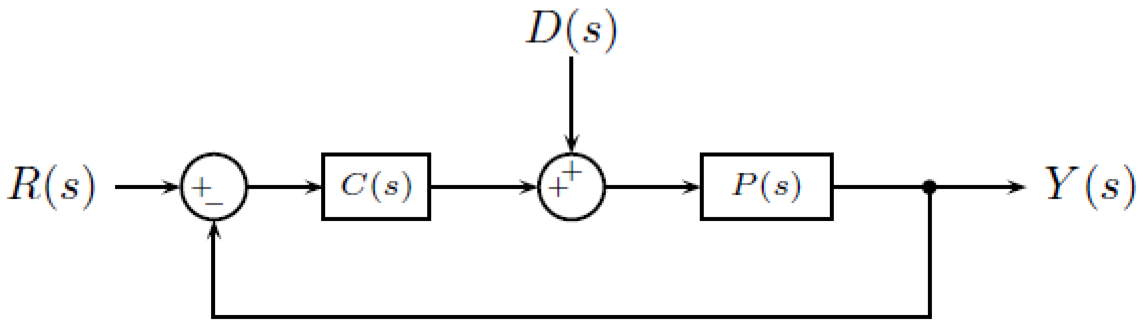
$$T = \frac{a\left(k_d s^2 + k_p s + k_i\right)}{s^2(s+b) + a\left(k_d s^2 + k_p s + k_i\right)} = \frac{a\left(k_d s^2 + k_p s + k_i\right)}{s^3 + \left(b + a k_d\right)s^2 + a k_p s + a k_i}$$

$$\begin{array}{c|cc}
s^3 & 1 & a k_p \\
s^2 & \left(b + a k_d\right) & a k_i \\
s^1 & a\dfrac{\left(b + a k_d\right)k_p - k_i}{\left(b + a k_d\right)} & 0 \\
s^0 & a k_i &
\end{array}$$

$$\begin{array}{c|cc}
s^3 & 1 & k_p \\
s^2 & \left(1 + k_d\right) & k_i \\
s^1 & \dfrac{\left(1 + k_d\right)k_p - k_i}{\left(1 + k_d\right)} & 0 \\
s^0 & k_i &
\end{array}$$

Stability with PD Case

$$P = \frac{a}{s(s+b)}$$



$$T = \frac{a\left(k_d s + k_p\right)}{s^2 + \left(b + ak_d\right)s + ak_p}$$

$$\begin{array}{c|cc} s^2 & 1 & ak_p \\ s^1 & \left(b+ak_d\right) & \\ s^0 & ak_p & \end{array}$$

$$b + ak_d > 0 \Rightarrow k_d > -\frac{b}{a}$$

$$ak_p > 0 \Rightarrow k_p > 0$$