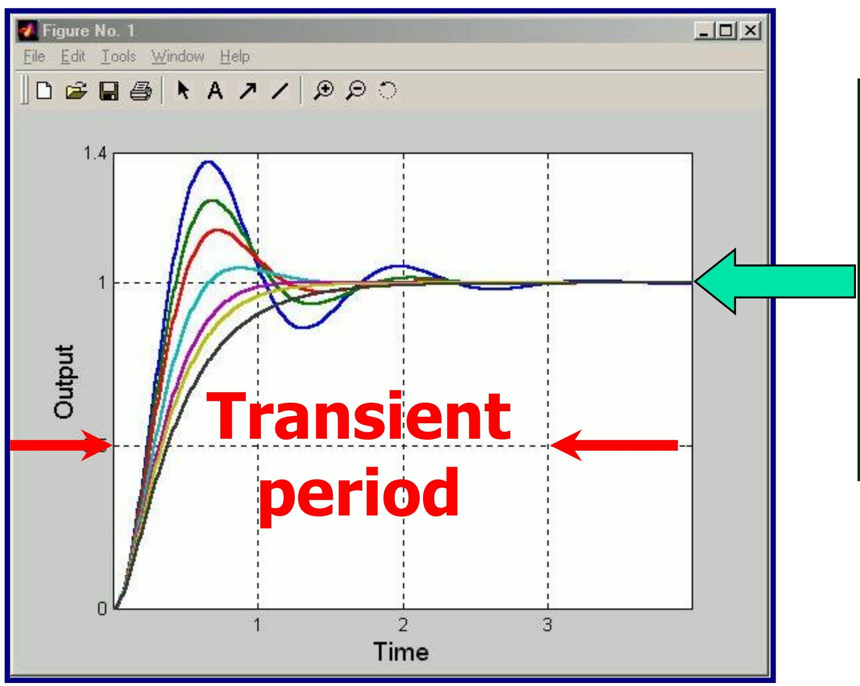
This week's agenda

- **Transient Response Analysis**
  - **First order systems**
  - **Second Order Systems**
  - **Using Matlab with Simulink**
- **Steady State Errors**

P-4 Transient Response Analysis

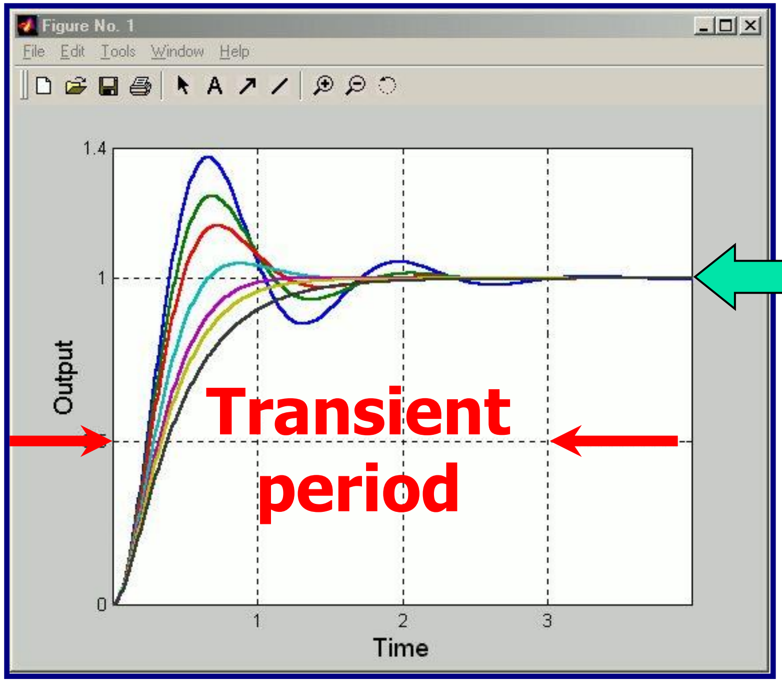
- **Transient response is the evolution of the signals in a control system until the final behavior is reached.**



The final values for all curves are the same but the way they converge differ

Transient Response Analysis

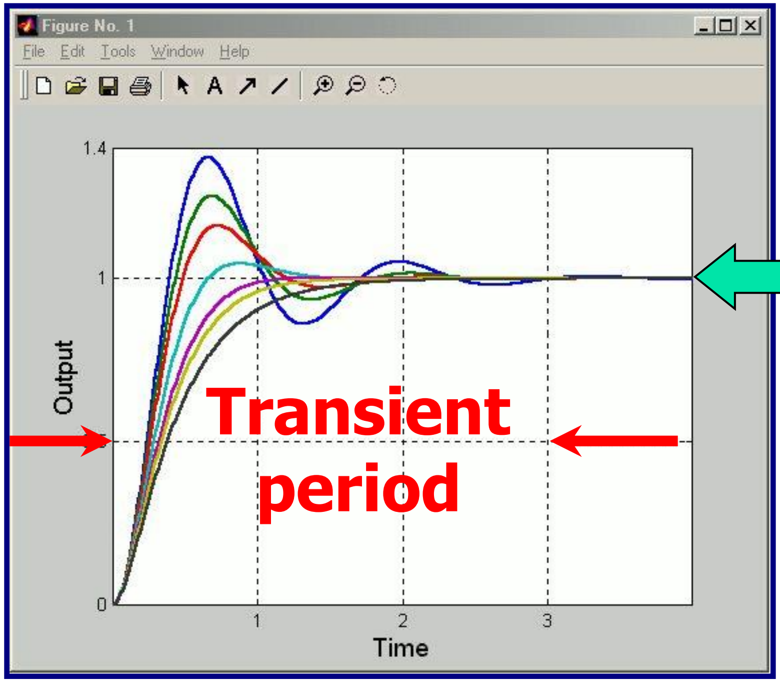
- **Transient response is the evolution of the signals in a control system until the final behavior is reached.**



Which one best suits your needs?

Transient Response Analysis

- What are our needs?
- We have to quantify the result with a set of performance specifications



Which one best suits your needs?

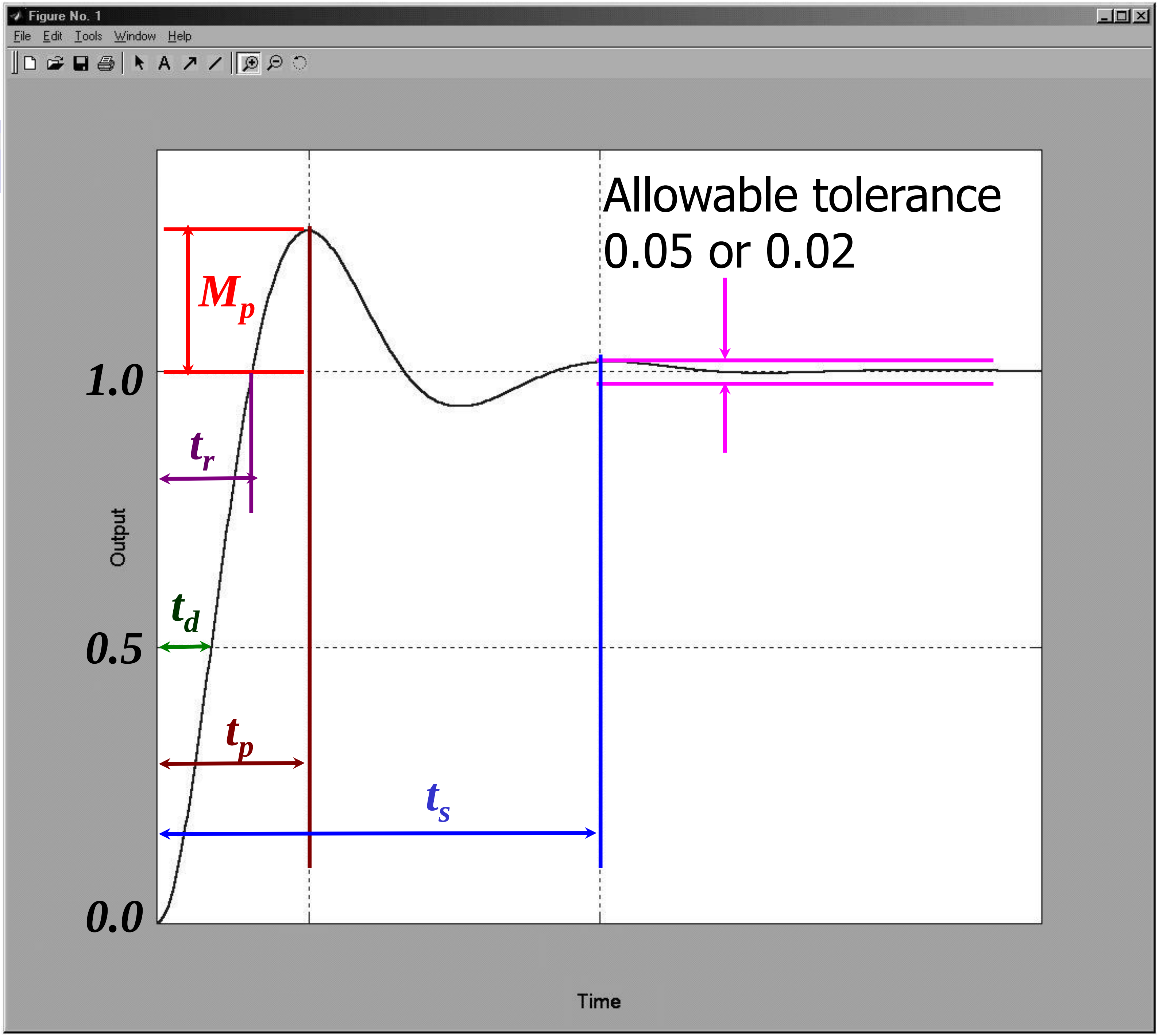


Figure No. 1
File Edit Tools Window Help
Allowable tolerance
0.05 or 0.02
M_p
1.0
t_r
Output
t_d
0.5
t_p
t_s
0.0
Time

Transient Response Analysis

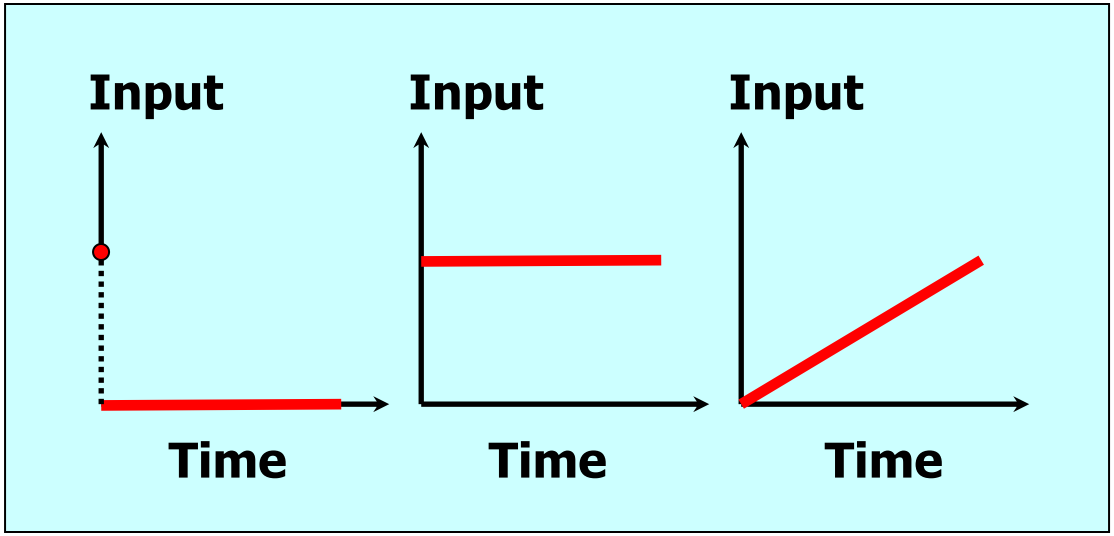
- **Did it have to be the response to a step input?**
- **The answer is no. We select several reasonable test signals to study/improve the transient response.**

Transient Response Analysis

- **What inputs are reasonable?**
- **Those you may encounter in the practical implementation of your control system are reasonable to study**

Transient Response Analysis

- **More explicitly**
  - **Impulse function to study the effects of shock inputs**
  - **Step input to study sudden disturbances**
  - **Ramp input to study gradually changing inputs**



Transient Response Analysis
First Order Systems

$$R(s) \longrightarrow \boxed{T(s)} \longrightarrow Y(s)$$

$$T(s) = \frac{1}{\tau s + 1}$$

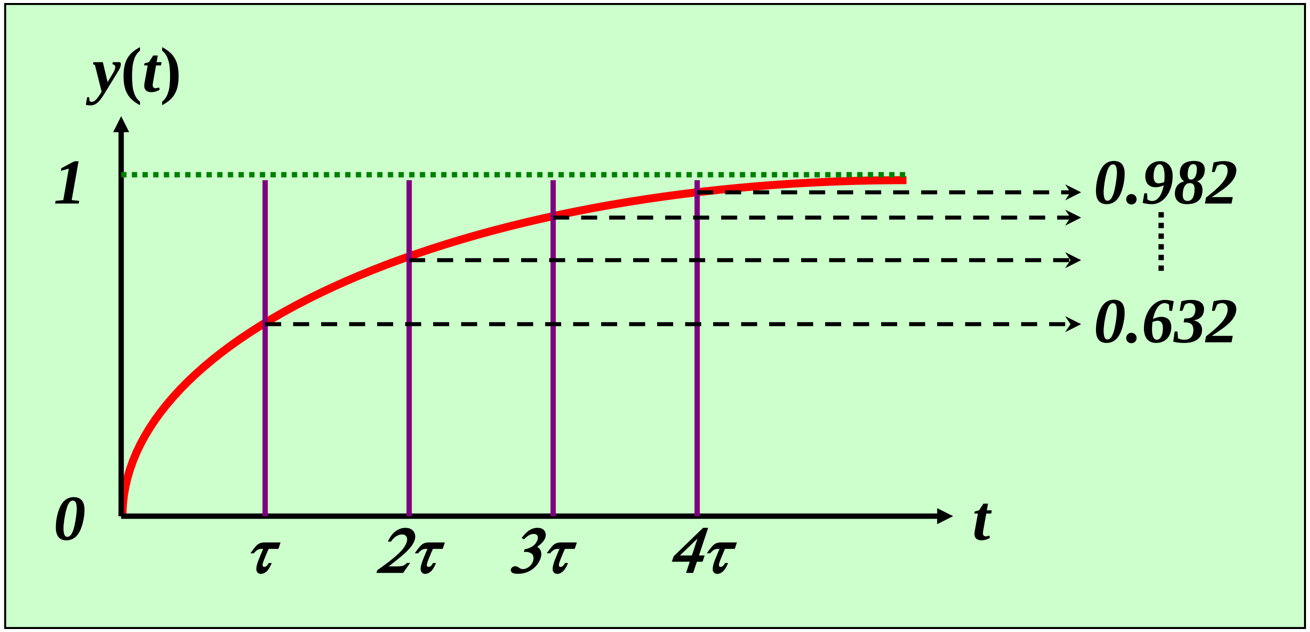
- **We will study**
  - **The unit step response, R(s)=1/s**
  - **The unit ramp response, R(s)=1/s^2**
  - **The unit impulse response, R(s)=1**
- **Clearly, Y(s)=T(s)R(s)**

Transient Response Analysis
First Order Systems, R(s)=1/s

$$Y(s)=\frac{1}{\tau s+1}\frac{1}{s}=\frac{1}{s}-\frac{1}{s+(1/\tau)}$$

$$y(t)=1-e^{-t/\tau},\quad \text{for } t\geq 0$$

Unit step response of a first order system



Transient Response Analysis
First Order Systems, R(s)=1/s

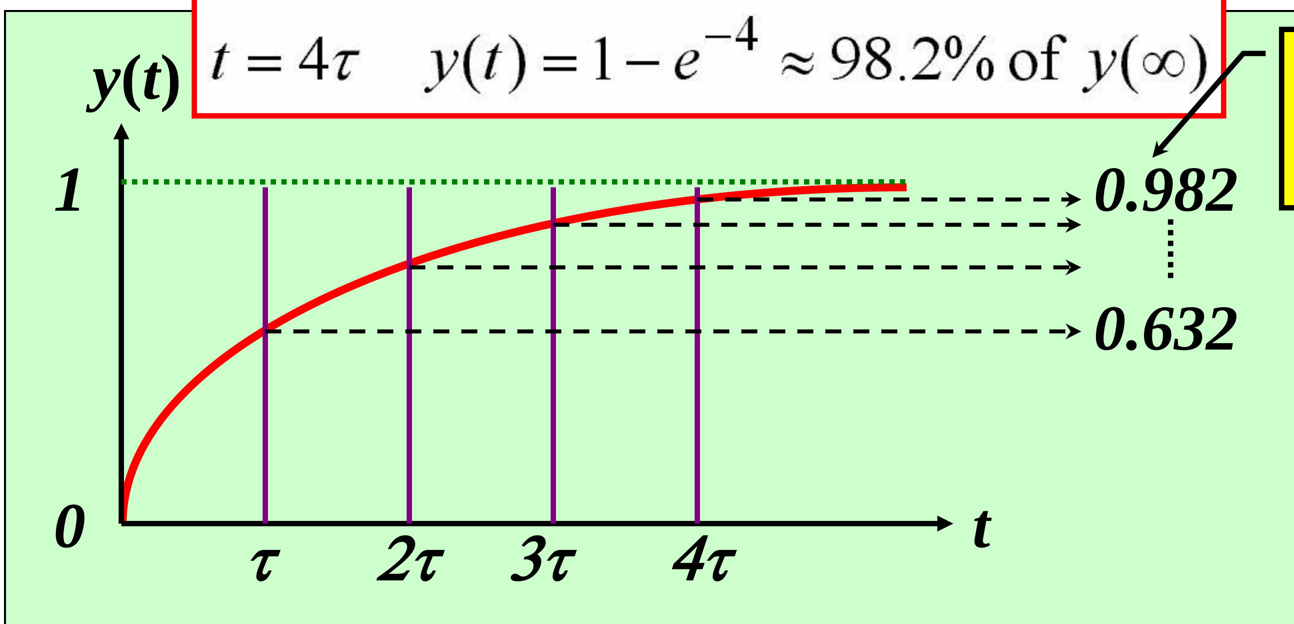
$$t=0 \qquad y(t)=1-e^{-0} \qquad = 0\% \text{ of } y(\infty)$$

$$t=\tau \qquad y(t)=1-e^{-1} \approx 63.2\% \text{ of } y(\infty)$$

$$t=2\tau \qquad y(t)=1-e^{-2} \approx 86.5\% \text{ of } y(\infty)$$

$$t=3\tau \qquad y(t)=1-e^{-3} \approx 95.0\% \text{ of } y(\infty)$$

$$t=4\tau \qquad y(t)=1-e^{-4} \approx 98.2\% \text{ of } y(\infty)$$



Within 2% of y(∞)=1

Transient Response Analysis
First Order Systems, $R(s)=1/s^2$

Unit ramp response of a first order system

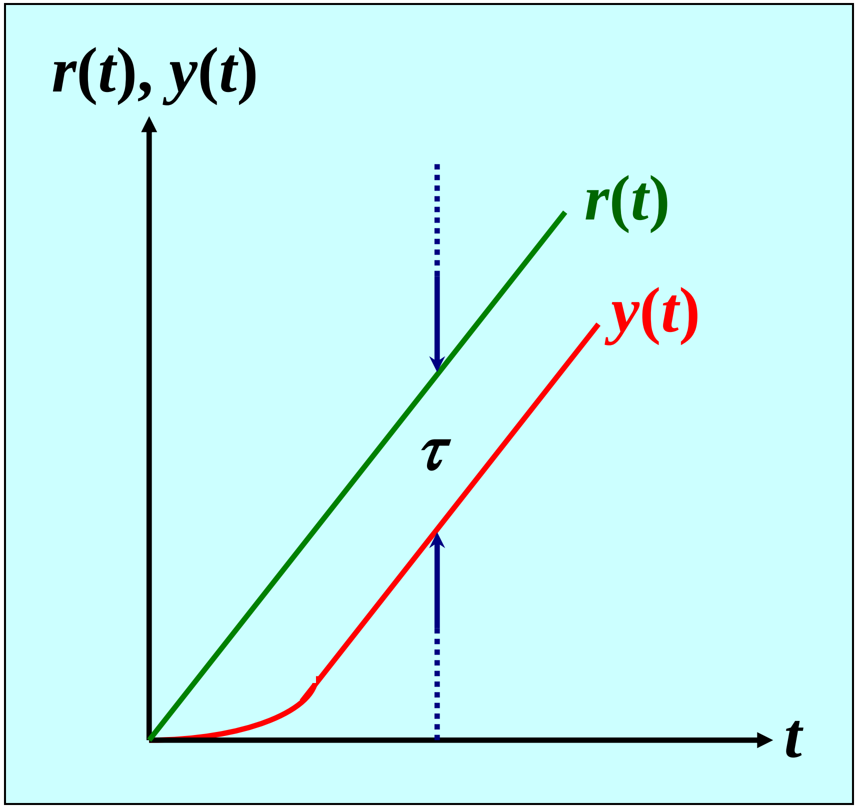
$$Y(s)=\frac{1}{\tau s+1}\frac{1}{s^2}=\frac{1}{s^2}-\frac{\tau}{s}+\frac{\tau^2}{\tau s+1}$$

$$y(t)=t-\tau+\tau e^{-t/\tau},\quad \text{for } t\geq 0$$

$$e(t)=r(t)-y(t)=\tau\left(1-e^{-t/\tau}\right)$$

$$\lim_{t\to\infty} e(t)=\tau=e(\infty)$$

Transient Response Analysis
First Order Systems, $R(s)=1/s^2$



Unit ramp response of a first order system

Transient Response Analysis
First Order Systems, R(s)=1

Unit impulse response of a first order system

$$T(s) = \frac{1}{\tau s + 1}$$

$$Y(s)=T(s)$$

$$y(t) = \frac{1}{\tau} e^{-t/\tau}$$

for $t \geq 0$

Transient Response Analysis
Second Order Systems

$$R(s) \longrightarrow \boxed{T(s)} \longrightarrow Y(s)$$

$$T(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

- **We will study**
  - **The unit step response, R(s)=1/s**
  - **The unit ramp response, R(s)=1/s^2**
  - **The unit impulse response, R(s)=1**
- **Clearly, Y(s)=T(s)R(s)**

Transient Response Analysis
Second Order Systems

Note that

$$T(s)=\frac{K}{Js^2+Bs+K}$$

$$T(s)=\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

$$T(s)=\frac{K/J}{s^2+(B/J)s+K/J}$$

$$\omega_n=\sqrt{K/J}$$

$$\zeta=(B/J)/\sqrt{4K/J}$$

Transient Response Analysis
Second Order Systems

$$T(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$\Delta = 4\zeta^2\omega_n^2 - 4\omega_n^2$$
$$= 4\omega_n^2\left(\zeta^2 - 1\right)$$
$$s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

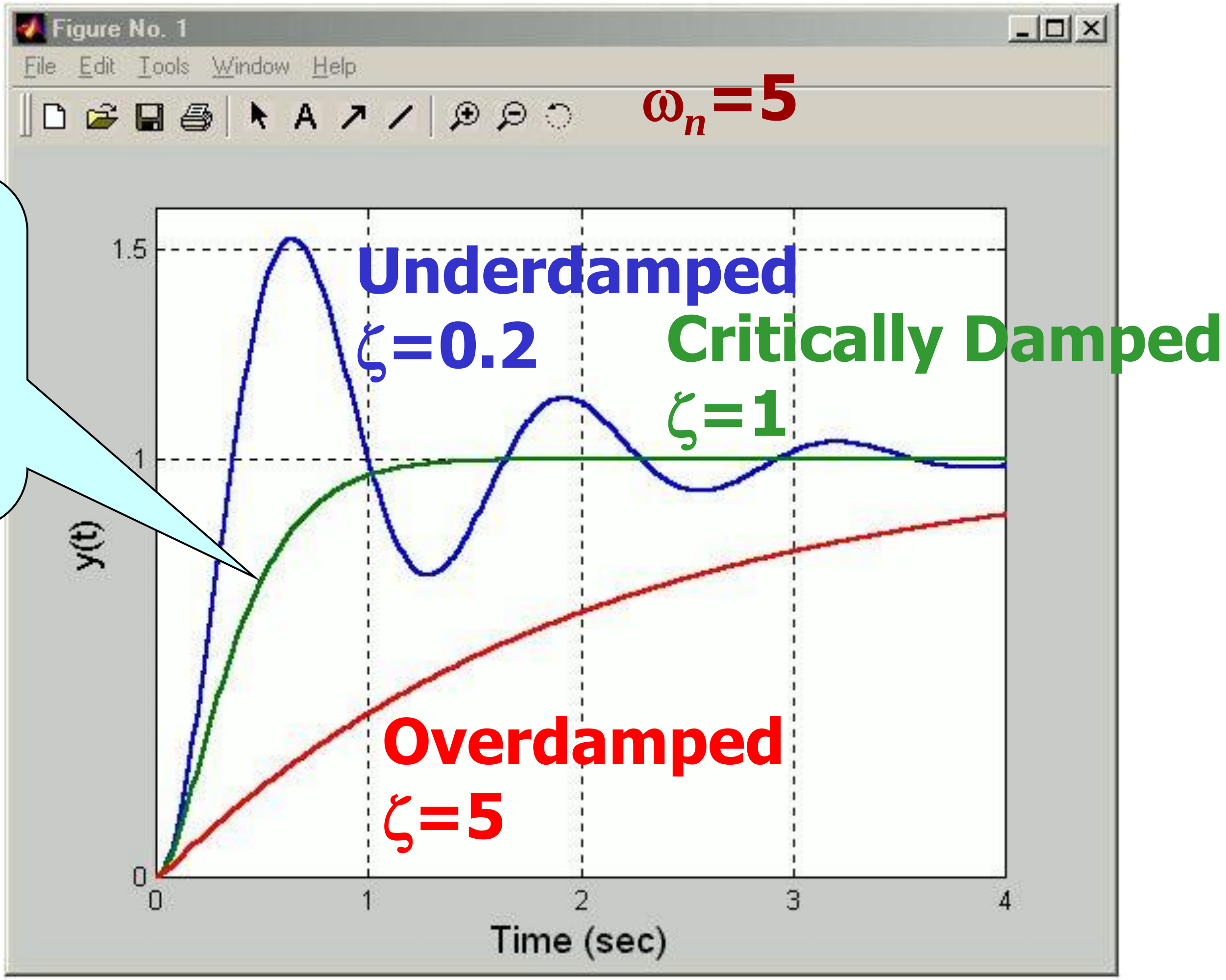
$0 < \zeta < 1$ → Underdamped

$\zeta = 1$ → Critically Damped

$\zeta > 1$ → Overdamped

Transient Response Analysis
Second Order Systems, R(s)=1/s

Suspension system in a car needs to be critically damped



Transient Response Analysis
Second Order Systems, R(s)=1/s
Underdamped Case (0<ζ<1)

$$T(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$s_{1,2} = -\zeta\omega_n \pm j\omega_d$$

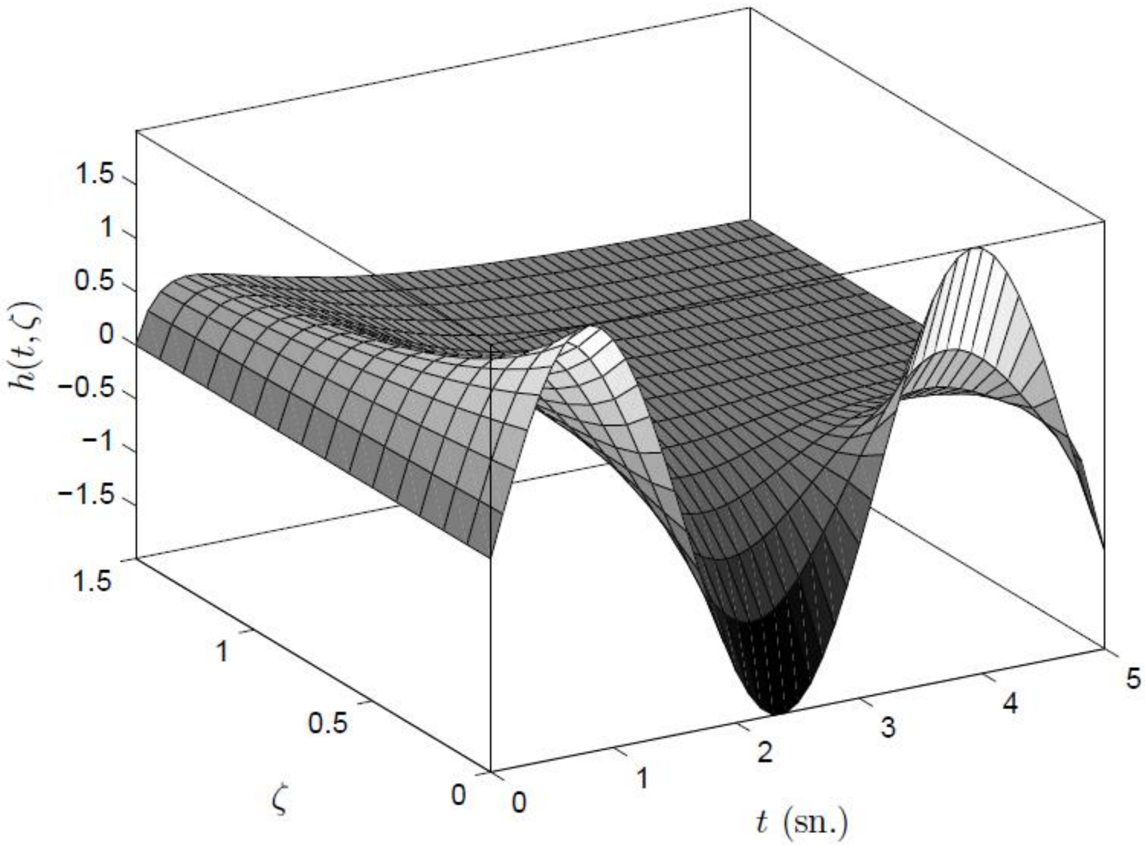
$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

Damping ratio

Natural frequency

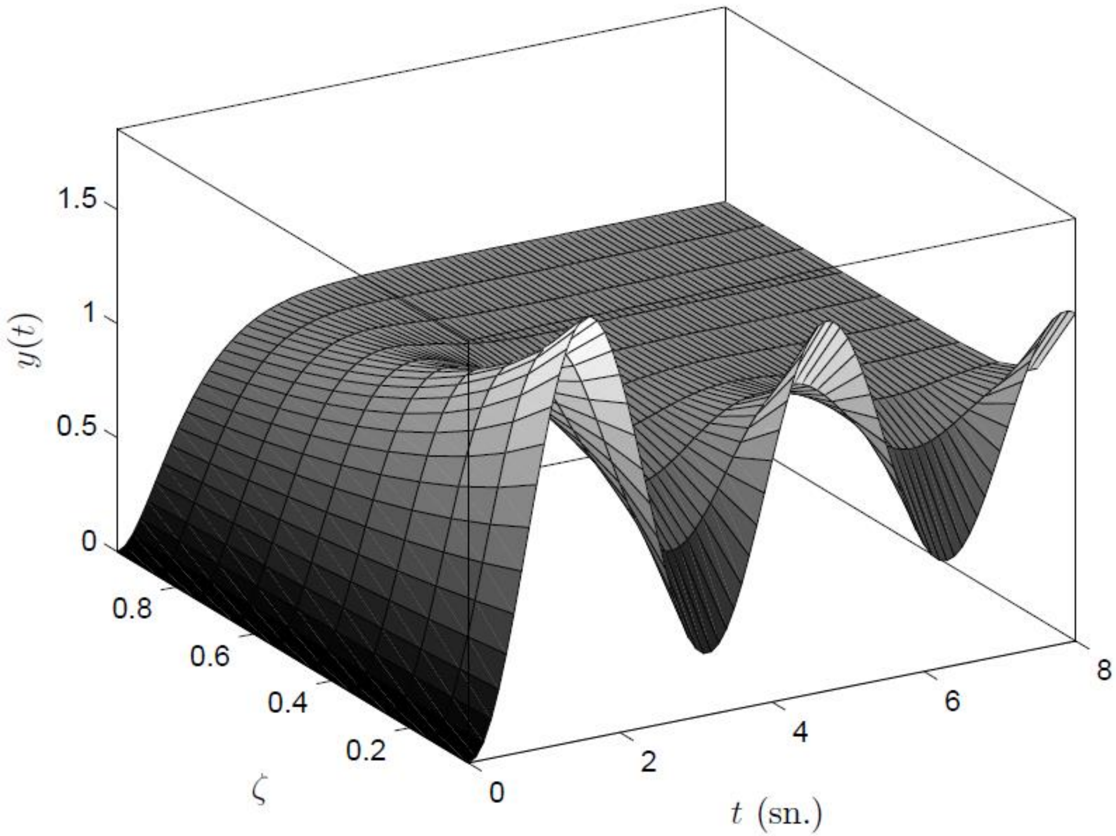
Damped natural frequency

Unit impulse reponse of a system, with w_n=2 rad/s and 0≤ ζ ≤1.5 and 0≤ t ≤ 5 sec.



$$T(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Unit step response of a system, with w_n=2 rad/s and 0.05≤ ζ ≤0.95 and 0 ≤ t ≤ 8 sec.



$$T(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Transient Response Analysis
Second Order Systems, R(s)=1/s
Underdamped Case (0<ζ<1)

$$Y(s)=\frac{\omega_n^2}{(s^2+2\zeta\omega_n s+\omega_n^2)s}=\frac{1}{s}-\frac{s+2\zeta\omega_n}{s^2+2\zeta\omega_n s+\omega_n^2}$$

$$Y(s)=\frac{1}{s}-\frac{s+\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d^2}-\frac{\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d^2}$$

$$L^{-1}\left\{\frac{s+\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d^2}\right\}=e^{-\zeta\omega_n t}\cos(\omega_d t)$$

$$L^{-1}\left\{\frac{\omega_d}{(s+\zeta\omega_n)^2+\omega_d^2}\right\}=e^{-\zeta\omega_n t}\sin(\omega_d t)$$

Transient Response Analysis
Second Order Systems, R(s)=1/s
Underdamped Case (0<ζ<1)

$$y(t) = 1 - e^{-\zeta\omega_n t}\left(\cos(\omega_d t) + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin(\omega_d t)\right)$$

$$y(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin\left(\omega_d t + \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right) \quad \text{for } t \geq 0$$

Transient Response Analysis
Second Order Systems, R(s)=1/s
Underdamped Case (0<ζ<1) - □ Digression

$$y(t) = 1 - e^{-\zeta\omega_n t}\left(\cos(\omega_d t) + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin(\omega_d t)\right)$$

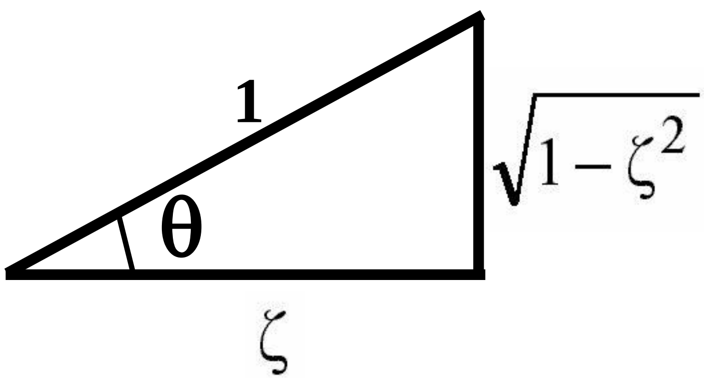
$$y(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left(\sqrt{1-\zeta^2}\cos(\omega_d t) + \zeta\sin(\omega_d t)\right)$$

$\sqrt{1-\zeta^2} \Rightarrow$ **sin(θ)** $\zeta \Rightarrow$ **cos(θ)**

Transient Response Analysis
Second Order Systems, R(s)=1/s
Underdamped Case (0<ζ<1) - Digression

$$\boxed{\sin(\theta)\cos(\omega_d t)+\cos(\theta)\sin(\omega_d t)=\sin(\omega_d t+\theta)}$$

$$\boxed{\sin(\theta)=\sqrt{1-\zeta^2}\ \text{and}\ \cos(\theta)=\zeta}$$



$$\boxed{\theta=\tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}}$$

Transient Response Analysis
Second Order Systems, R(s)=1/s
Underdamped Case (0<ζ<1) - Digression

$$y(t)=1-e^{-\zeta\omega_n t}\left(\cos(\omega_d t)+\frac{\zeta}{\sqrt{1-\zeta^2}}\sin(\omega_d t)\right)$$

$$y(t)=1-\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin\left(\omega_d t+\tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right)\quad \text{for } t\geq 0$$

- **End of digression**

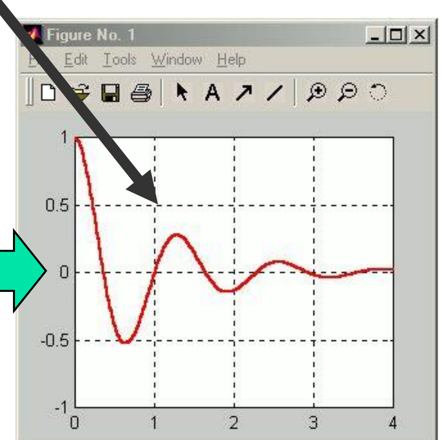
Transient Response Analysis
Second Order Systems, R(s)=1/s
Underdamped Case (0<ζ<1)

$$e(t) = r(t) - y(t)$$

$$e(t) = \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin\left(\omega_d t + \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right) \quad \text{for } t \geq 0$$

Oscillation frequency is ω_d

Damped sinusoidal oscillation converges to zero, $e(t)\rightarrow 0$



ω_n=5
ζ=0.2

Transient Response Analysis
Second Order Systems, R(s)=1/s
Extreme Case (ζ=0, Undamped)

$$e(t)=\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin\left(\omega_d t+\tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right)\ \text{for } t\geq 0$$

$$e(t)=\sin\left(\omega_d t+\tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right)\ \text{for } t\geq 0$$

Oscillations continue indefinitely

Transient Response Analysis
Second Order Systems, R(s)=1/s
Critically Damped Case (ζ=1)

$$Y(s)=\frac{\omega_n^2}{(s+\omega_n)^2 s}=\frac{1}{s}-\frac{s+2\omega_n}{(s+\omega_n)^2}$$

$$Y(s)=\frac{1}{s}-\frac{1}{s+\omega_n}-\frac{\omega_n}{(s+\omega_n)^2}$$

$$y(t)=1-e^{-\omega_n t}-\omega_n t e^{-\omega_n t} \quad \text{for } t\geq 0$$

$$y(t)=1-e^{-\omega_n t}\left(1+\omega_n t\right) \quad \text{for } t\geq 0$$

Transient Response Analysis
Second Order Systems, R(s)=1/s
Overdamped Case (ζ>1)

$$T(s)=\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

Two distinct poles on the negative real axis

$$\Delta = 4\zeta^2\omega_n^2 - 4\omega_n^2$$
$$= 4\omega_n^2\left(\zeta^2 - 1\right)$$
$$s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

$$s_1 = -\left(\zeta + \sqrt{\zeta^2 - 1}\right)\omega_n$$
$$s_2 = -\left(\zeta - \sqrt{\zeta^2 - 1}\right)\omega_n$$

$$Y(s)=\frac{\omega_n^2}{\left(s-s_1\right)\left(s-s_2\right)s}$$

Transient Response Analysis
Second Order Systems, R(s)=1/s
Overdamped Case ($\zeta>1$)

$$Y(s)=\frac{\omega_n^2}{(s-s_1)(s-s_2)s}$$

$$s_1=-\left(\zeta+\sqrt{\zeta^2-1}\right)\omega_n$$

$$s_2=-\left(\zeta-\sqrt{\zeta^2-1}\right)\omega_n$$

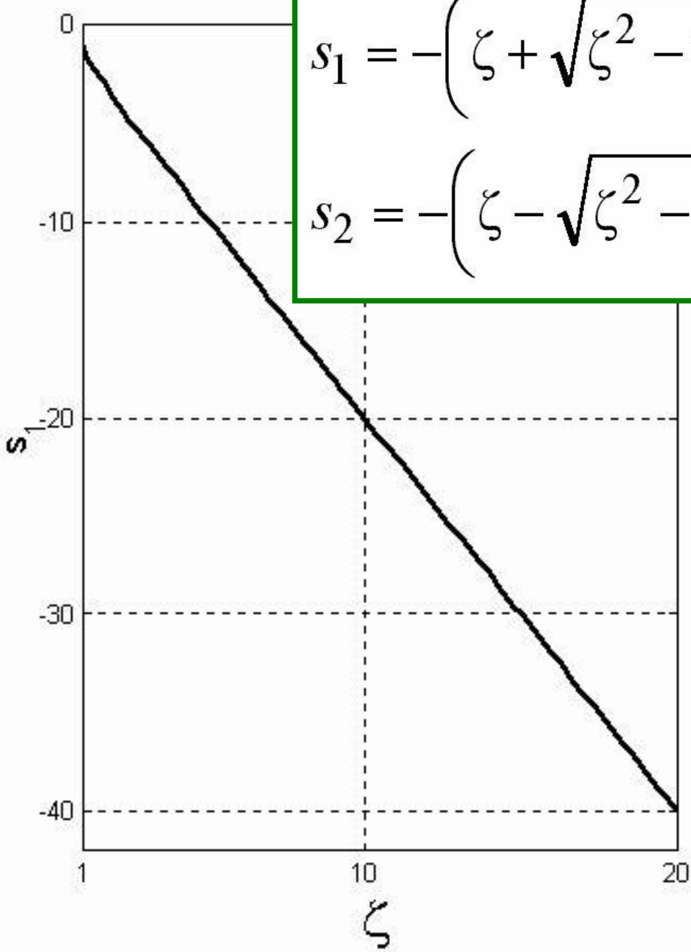
$$Y(s)=\frac{1}{s}+\frac{\omega_n}{2\sqrt{\zeta^2-1}}\left(\frac{1/s_2}{s-s_2}-\frac{1/s_1}{s-s_1}\right)$$

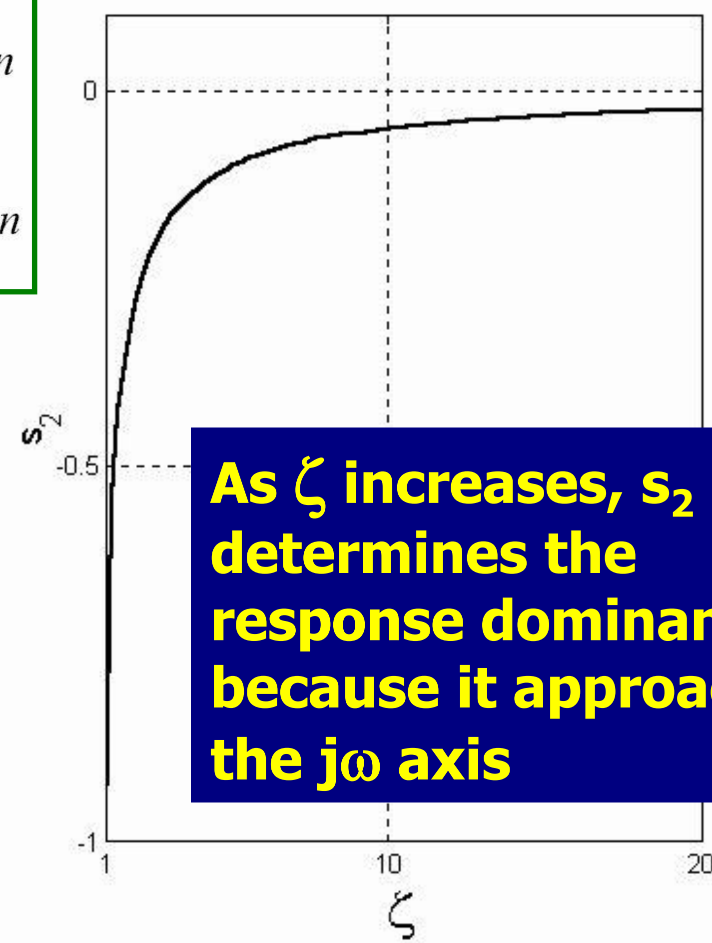
$$y(t)=1+\frac{\omega_n}{2\sqrt{\zeta^2-1}}\left(\frac{e^{s_2t}}{s_2}-\frac{e^{s_1t}}{s_1}\right)$$

Transient Response Analysis
Second Order Systems, R(s)=1/s
Overdamped Case (ζ>1). See $s_{1,2}$ for ω_n=1

$$s_1 = -\left(\zeta + \sqrt{\zeta^2 - 1}\right)\omega_n$$

$$s_2 = -\left(\zeta - \sqrt{\zeta^2 - 1}\right)\omega_n$$





As ζ increases, s_2 determines the response dominantly, because it approaches the jω axis

Transient Response Analysis
Second Order Systems, R(s)=1/s
Overdamped Case (ζ>>1)

$$T(s) = \frac{\left(\dfrac{s_1 s_2}{s_1 - s_2}\right)}{s - s_1} + \frac{\left(\dfrac{s_1 s_2}{s_2 - s_1}\right)}{s - s_2}$$

When ζ>>1

$\mathbf{s_1} \rightarrow -2\zeta\omega_n$

$\mathbf{s_2} \rightarrow 0$

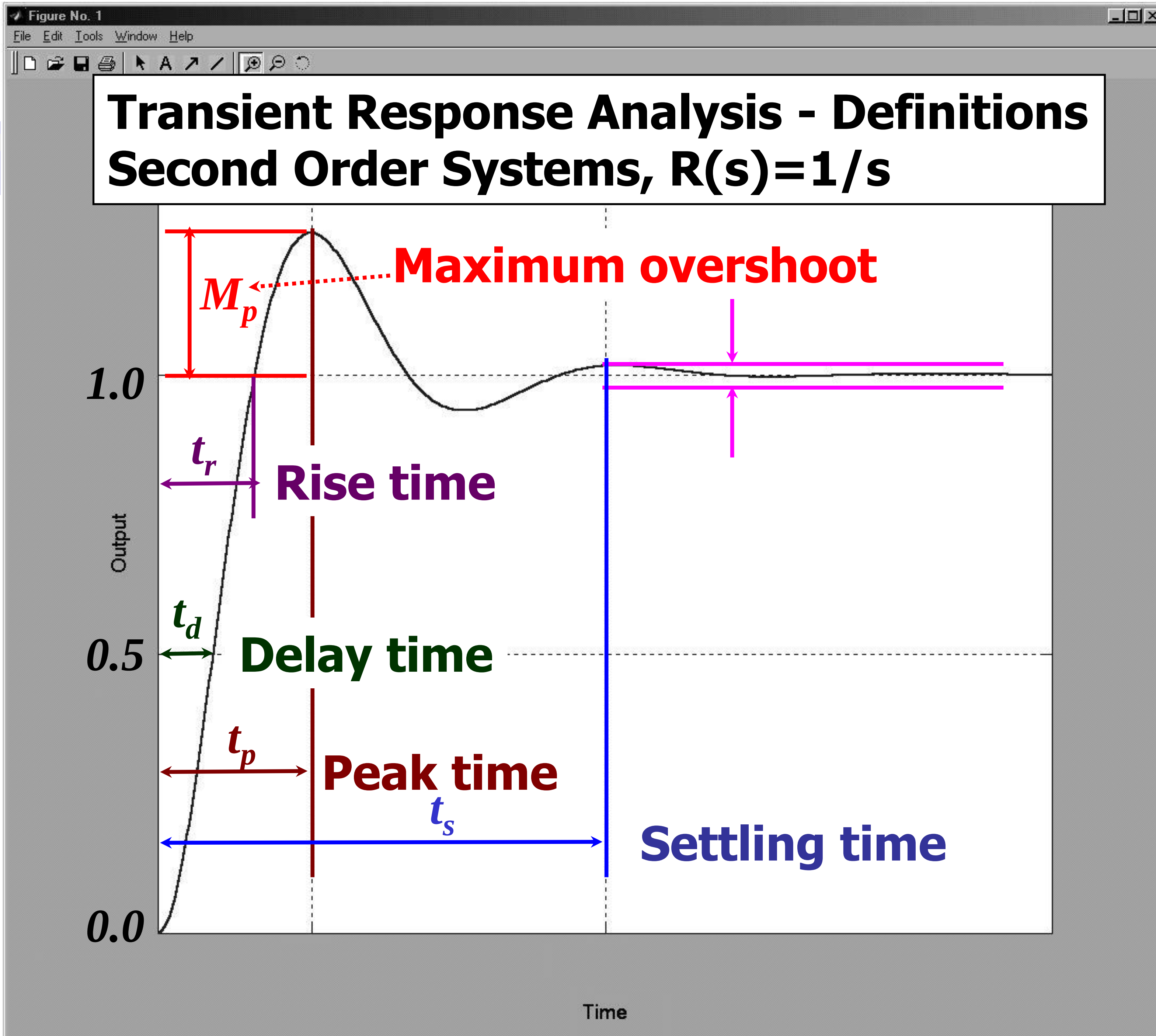
$$T(s) \cong \frac{\left(\dfrac{s_2}{s_2 / s_1 - 1}\right)}{s - s_2} \cong \frac{-s_2}{s - s_2}$$

≈ 0

$$y(t) = 1 - e^{s_2 t}$$

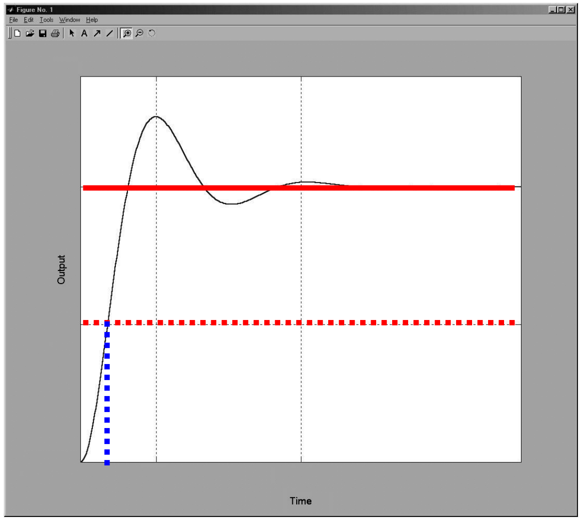
y(0)=0, y(∞)=1 are satisfied by an approximate dominant first order dynamics

Transient Response Analysis - Definitions
Second Order Systems, R(s)=1/s



Transient Response Analysis - Definitions Second Order Systems, R(s)=1/s

Delay Time (t_d): The time required to reach the half of the final value. Note that delay time is the time till first reach is observed.



Transient Response Analysis - Definitions
Second Order Systems, R(s)=1/s

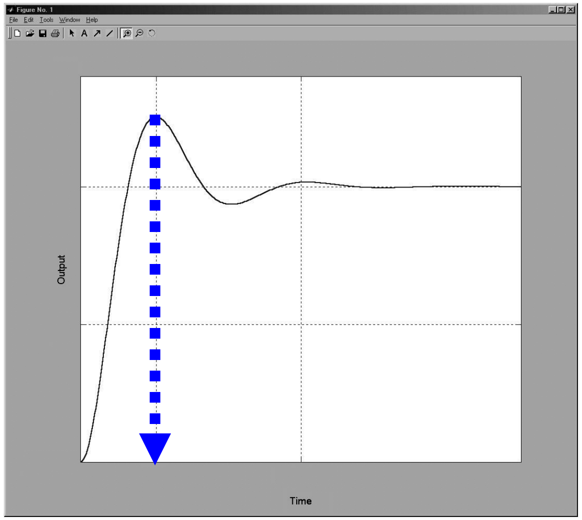
Rise Time (t_r): The time required to rise from 10% to 90% or 5% to 95% or 0% to 100% of the final value.

Generally for underdamped 2nd order systems

Generally for overdamped systems

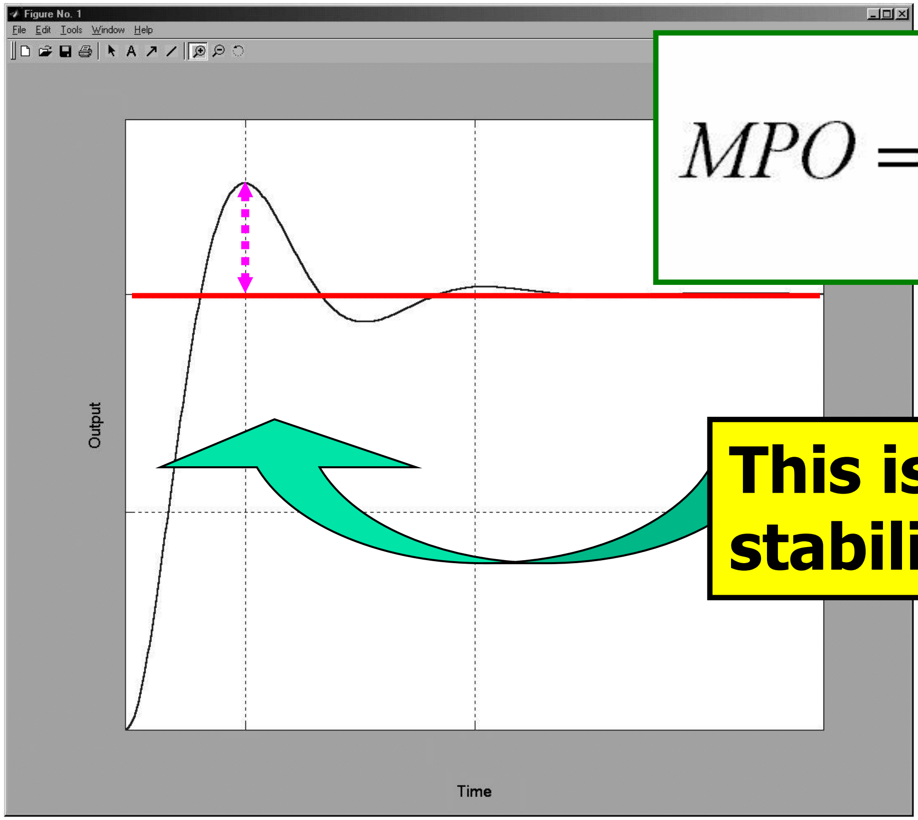
Transient Response Analysis - Definitions
Second Order Systems, R(s)=1/s

Peak Time (t_p): The time required for the response to reach the first peak of the overshoot.



Transient Response Analysis - Definitions
Second Order Systems, R(s)=1/s

Maximum (percent) Overshoot (M_p): The maximum peak value measured from the steady state value.

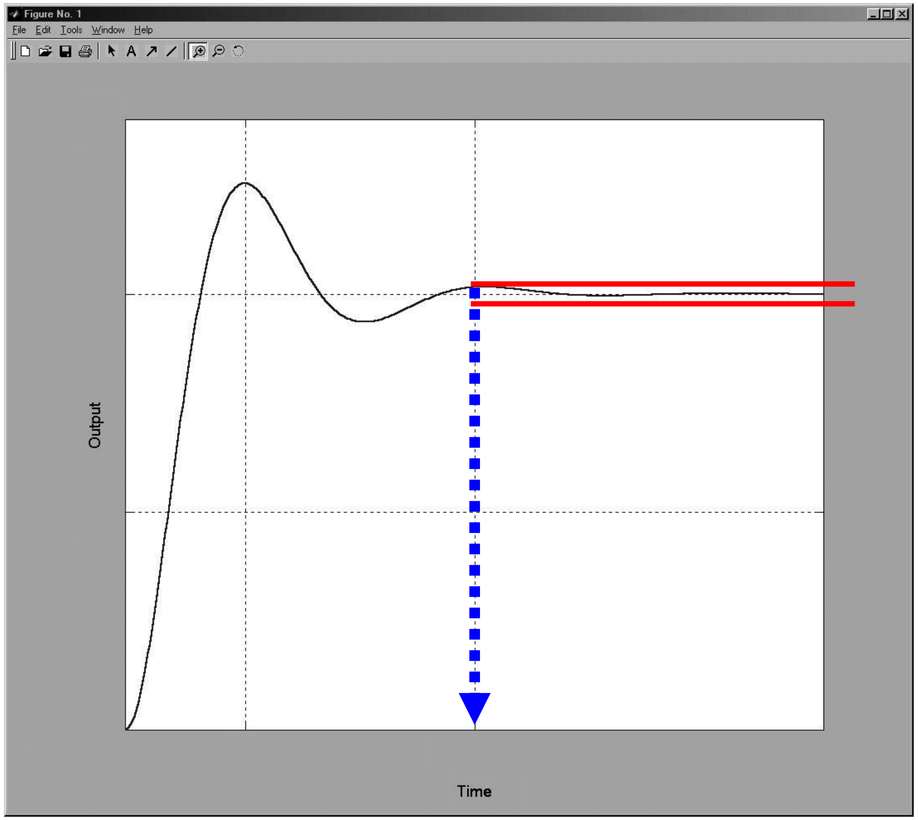


$$MPO = \frac{y(t_p) - y(\infty)}{y(\infty)} \times 100\%$$

This is a measure of relative stability of the system

Transient Response Analysis - Definitions
Second Order Systems, R(s)=1/s

Settling Time (t_s): The time required for the response to remain within a desired percentage (2% or 5%) of the final value.



Transient Response Specifications
Second Order Systems, R(s)=1/s

In a control system, the designer may want to observe some set of predefined transient response characteristics. This section focuses on the computation of the variables of transient response and their relevance to closed loop transfer function. Ultimately, this relevance will bring a set of constraints for the design of the controller.

Transient Response Specifications
Second Order Systems, R(s)=1/s
Calculation of Rise Time (t_r)

$$y(t_r)=1=1-e^{-\zeta\omega_n t_r}\left(\cos(\omega_d t_r)+\frac{\zeta}{\sqrt{1-\zeta^2}}\sin(\omega_d t_r)\right)$$

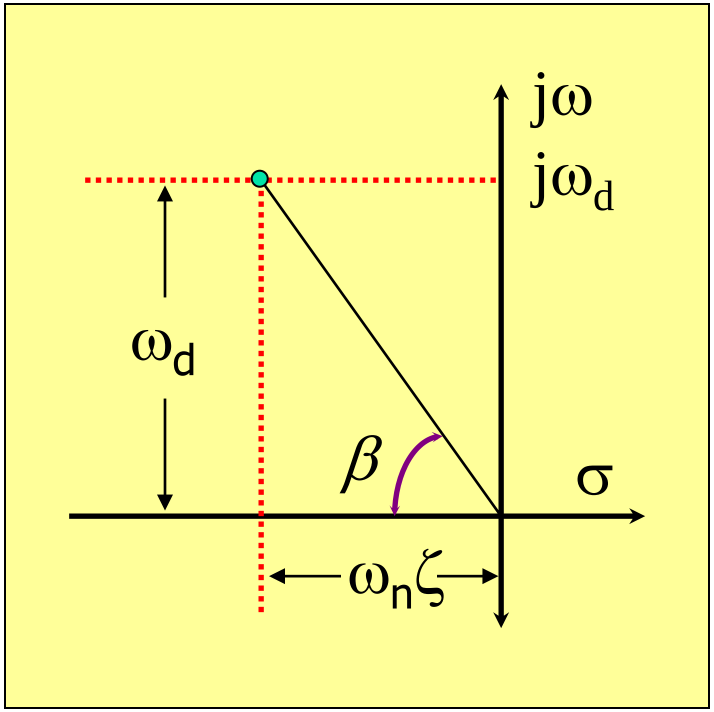
$$e^{-\zeta\omega_n t_r}\left(\cos(\omega_d t_r)+\frac{\zeta}{\sqrt{1-\zeta^2}}\sin(\omega_d t_r)\right)=0$$

$$\cos(\omega_d t_r)+\frac{\zeta}{\sqrt{1-\zeta^2}}\sin(\omega_d t_r)=0\Rightarrow \tan(\omega_d t_r)=-\frac{\sqrt{1-\zeta^2}}{\zeta}$$

Transient Response Specifications
Second Order Systems, R(s)=1/s
Calculation of Rise Time (t_r)

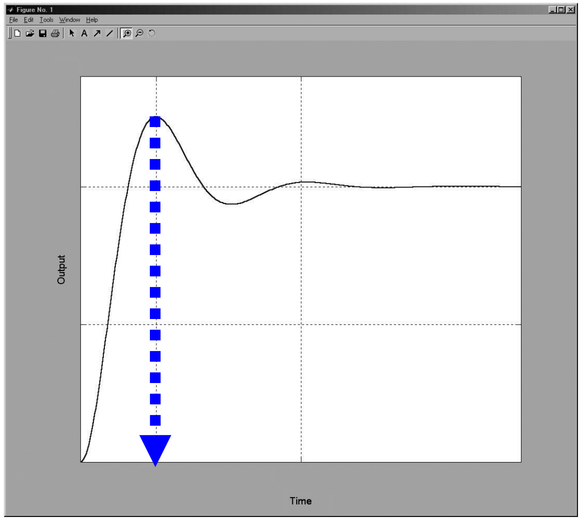
$$\tan(\omega_d t_r) = -\frac{\omega_n\sqrt{1-\zeta^2}}{\omega_n\zeta} = -\frac{\omega_d}{\omega_n\zeta}$$

$$t_r = \frac{1}{\omega_d}\arctan\left(-\frac{\omega_d}{\omega_n\zeta}\right) = \frac{\pi-\beta}{\omega_d}$$



Transient Response Specifications
Second Order Systems, R(s)=1/s
Calculation of Peak Time (t_p)

At $t=t_p$, $dy/dt=0$



$$\frac{dy(t)}{dt}=\zeta\omega_n e^{-\zeta\omega_n t}\left(\cancel{\cos(\omega_d t)}+\frac{\zeta}{\sqrt{1-\zeta^2}}\sin(\omega_d t)\right)$$

$$+e^{-\zeta\omega_n t}\left(\omega_d\sin(\omega_d t)-\cancel{\frac{\zeta\omega_d}{\sqrt{1-\zeta^2}}\cos(\omega_d t)}\right)$$

$$\frac{dy(t_p)}{dt}=\sin(\omega_d t_p)\frac{\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t_p}=0$$

Transient Response Specifications
Second Order Systems, R(s)=1/s
Calculation of Peak Time (t_p)

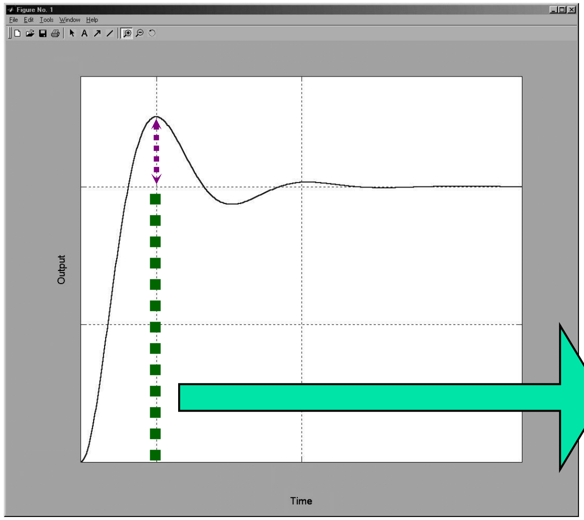
$$\boxed{\frac{dy(t_p)}{dt} = \sin\left(\omega_d t_p\right)\frac{\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t_p} = 0}$$

$$\boxed{\sin\left(\omega_d t_p\right) = 0 \ \text{ or } \ \omega_d t_p = 0, \pi, 2\pi, 3\pi, \ldots}$$

$$\boxed{\omega_d = \omega_n\sqrt{1-\zeta^2}}$$

$$\boxed{t_p = \frac{\pi}{\omega_d}}$$

Transient Response Specifications
Second Order Systems, R(s)=1/s
Calculation of Maximum Overshoot (M_p)



$$M_p = y(t_p) - 1 = e^{\left(-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}\right)}$$

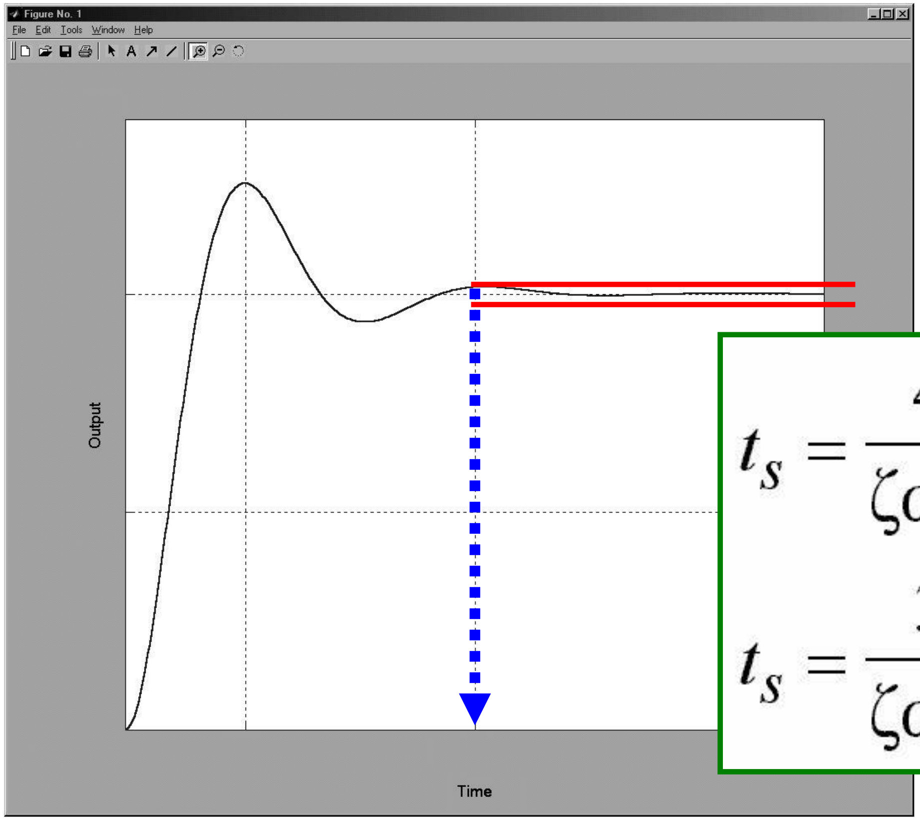
Note that maximum overshoot occurs at $t=t_p$

Transient Response Analysis - Definitions Damping Ratio and Mp

Mp	ln(Mp)	ln(Mp)^2	Zeta
0,05	-3,00	8,97	0,6901
0,10	-2,30	5,30	0,5912
0,15	-1,90	3,60	0,5169
0,20	-1,61	2,59	0,4560
0,25	-1,39	1,92	0,4037
0,30	-1,20	1,45	0,3579
0,35	-1,05	1,10	0,3169
0,40	-0,92	0,84	0,2800
0,45	-0,80	0,64	0,2463
0,50	-0,69	0,48	0,2155
0,55	-0,60	0,36	0,1869
0,60	-0,51	0,26	0,1605
0,65	-0,43	0,19	0,1359
0,70	-0,36	0,13	0,1128
0,75	-0,29	0,08	0,0912
0,80	-0,22	0,05	0,0709
0,85	-0,16	0,03	0,0517
0,90	-0,11	0,01	0,0335
0,95	-0,05	0,00	0,0163

Zeta	Mp
0,05	0,85
0,10	0,73
0,15	0,62
0,20	0,53
0,25	0,44
0,30	0,37
0,35	0,31
0,40	0,25
0,45	0,21
0,50	0,16
0,55	0,13
0,60	0,09
0,65	0,07
0,70	0,05
0,75	0,03
0,80	0,02
0,85	0,01
0,90	0,00
0,95	0,00

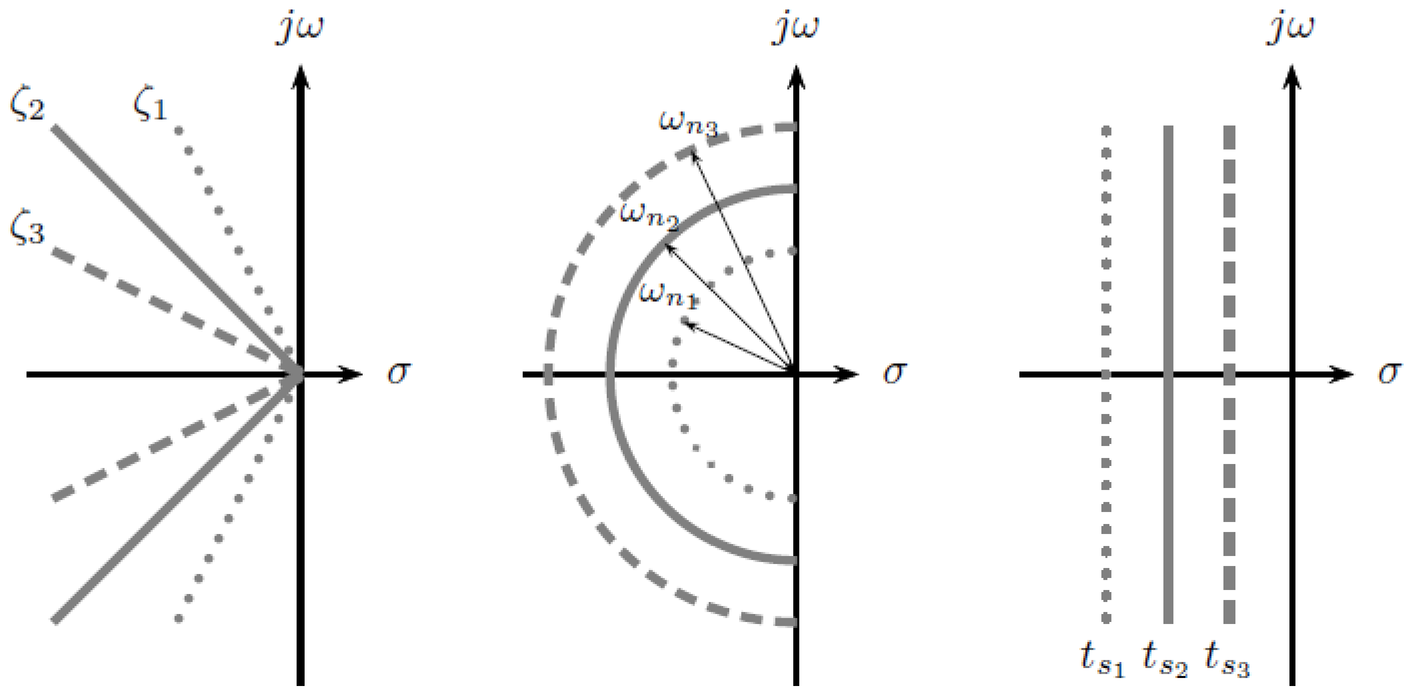
Transient Response Specifications
Second Order Systems, R(s)=1/s
Calculation of Settling Time (t_s)



$$t_s = \frac{4}{\zeta\omega_n}$$ → **2% Criterion**

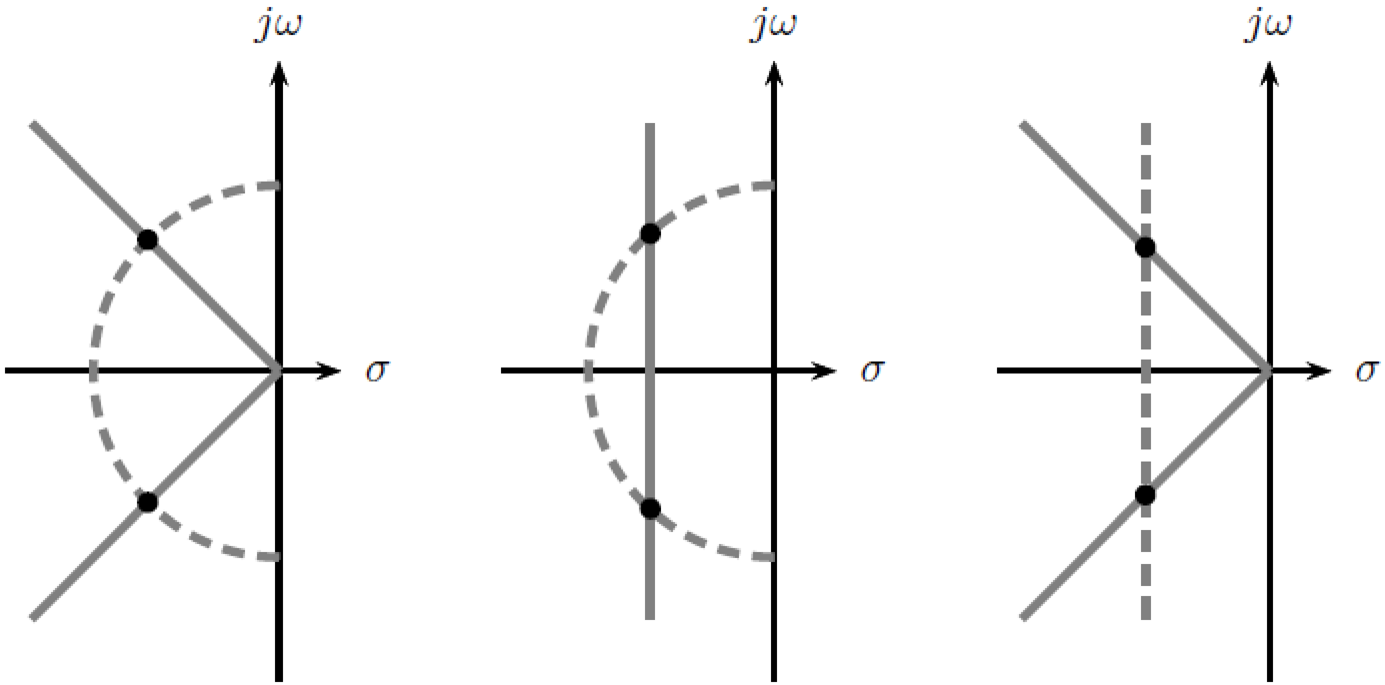
$$t_s = \frac{3}{\zeta\omega_n}$$ → **5% Criterion**

Transient Response Specifications Implications on the complex plane



$\zeta_1 < \zeta_2 < \zeta_3$ $\omega_{n_1} < \omega_{n_2} < \omega_{n_3}$ $t_{s_1} < t_{s_2} < t_{s_3}$

Transient Response Specifications Implications on the complex plane



w_n and ζ specified w_n and t_s specified t_s and ζ specified

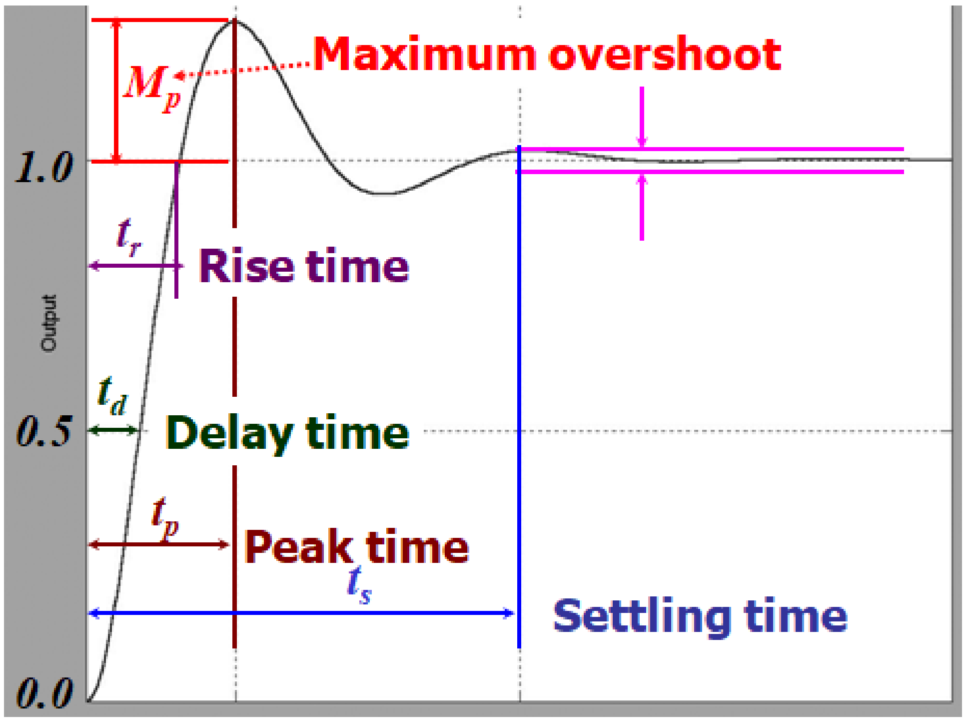
Example

$$H(s) = \frac{25}{s^2+8s+25}$$

$$H(s) = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

```
wn=5;
z=8/(2*wn);
wd=wn*sqrt(1-z*z);
beta=atan(wd/(z*wn));
tr=(pi-beta)/wd ts=3/(z*wn)
tp=pi/wd
Mp=exp(-z*pi/sqrt(1-z*z))
```

$t_s<t_p$! What is the meaning of this?



$$\omega_n = 5 \text{ rad/s},$$

$$t_r = 0.8327 \text{ s}$$

$$t_p = 1.0472 \text{ s}$$

$$t_s = 0.75 \text{ s}$$

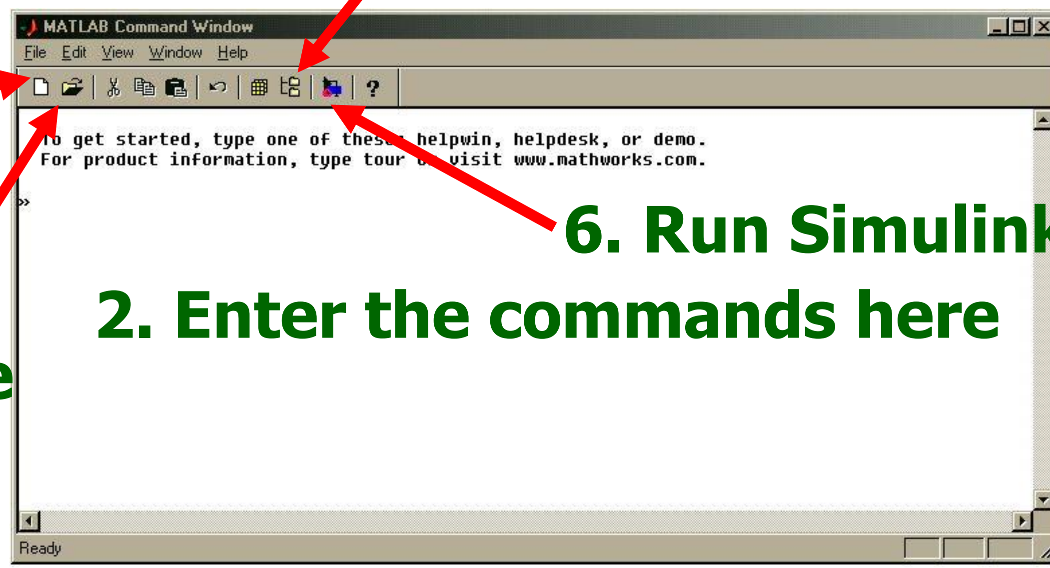
$$\omega_d = 3 \text{ rad/s}$$

$$M_p = 0.0152$$

Using Matlab with Simulink

1. Set your path

3. Create a new m-file



6. Run Simulink

4. Open an existing m-file

2. Enter the commands here

Using Matlab with Simulink

Try these first, see the results

MATLAB Command Window

File Edit View Window Help

```
To get started, type one of these: helpwin, helpdesk, or demo.
For product information, type tour or visit www.mathworks.com.

>> A=[3 7;8 14]

A =

     3     7
     8    14

>>
```

```
»inv(A)          »A'
»det(A)          »A(:,1)
»eig(A)          »A(2,:)
»A.^2            »diag(A)
»A^2             »A(1,1)*A(1,2)
»A*A             »A^3
»sum(A)          »i*A
»sum(sum(A))     »A+eye(2,2)
```

Ready

Using Matlab with Simulink
Useful commands/examples

- » clc
- » clear
- » figure
- » help {keyword}
- » close all
- » size(A)
- » rand(3,2)
- » real(a)
- » imag(a)
- » grid
- » zoom
- » clf

- » max(A)
- » min(A)
- » flops
- » who
- » whos
- » sin(pi/2)
- » cos(1.34)
- » atan(1.34)
- » abs(-2)
- » log(3)
- » log10(3)
- » sign(-2)

- » save
- » zeros(3,1)
- » ones(2,4)
- » ceil(1.34)
- » floor(1.34)
- » ezplot('sin(x)',[0,2])
- » helpdesk
- » roots([1 7 10])
- » ltiview
- » rlocus
- » nyquist
- » bode
- » margin

Using Matlab with Simulink
A command line demo - Step Response

MATLAB Command Window

File Edit View Window Help

```
>> wn = 1

wn =

     1

>> zeta = 0.9

zeta =

    0.9000

>> num=[wn^2]

num =

     1

>> den = [1 2*zeta*wn wn^2]

den =

    1.0000    1.8000    1.0000

>> sys = tf(num,den)

Transfer function:
        1
---------------
s^2 + 1.8 s + 1

>> step(sys)
>>
```

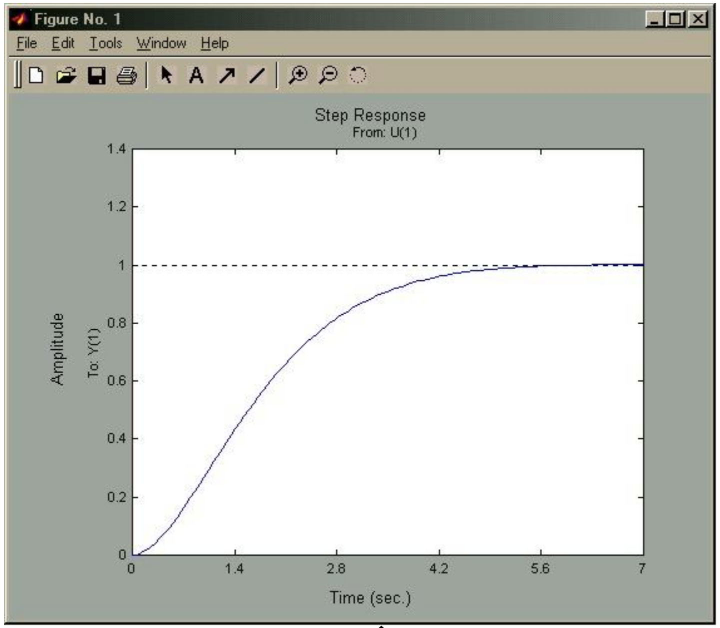
Ready

Numerator

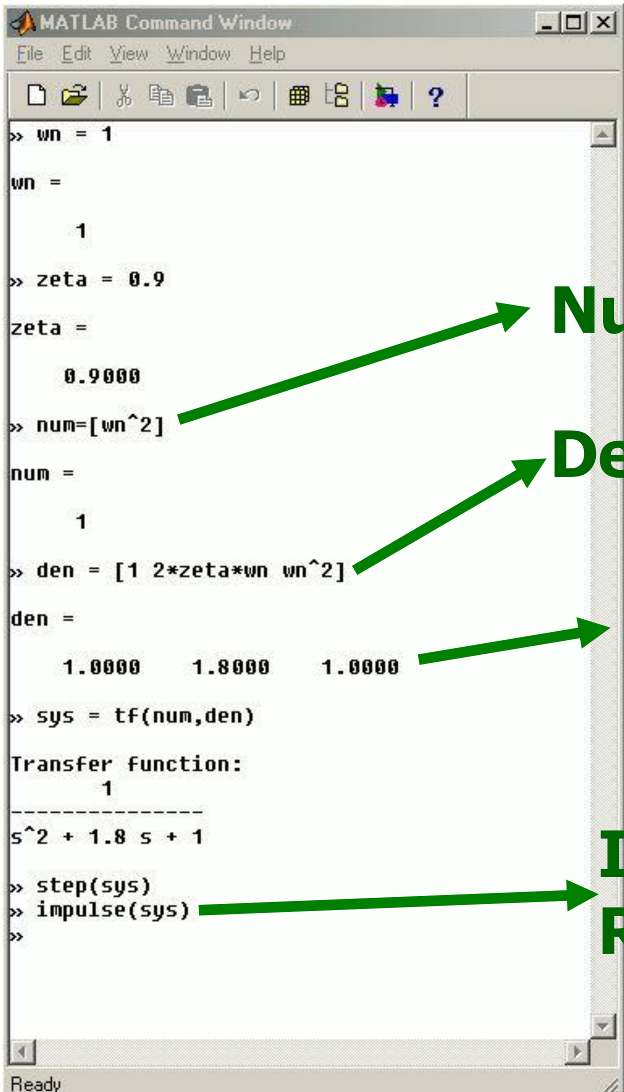
Denominator

Transfer Function

Step Response



Using Matlab with Simulink
A command line demo - Impulse Response

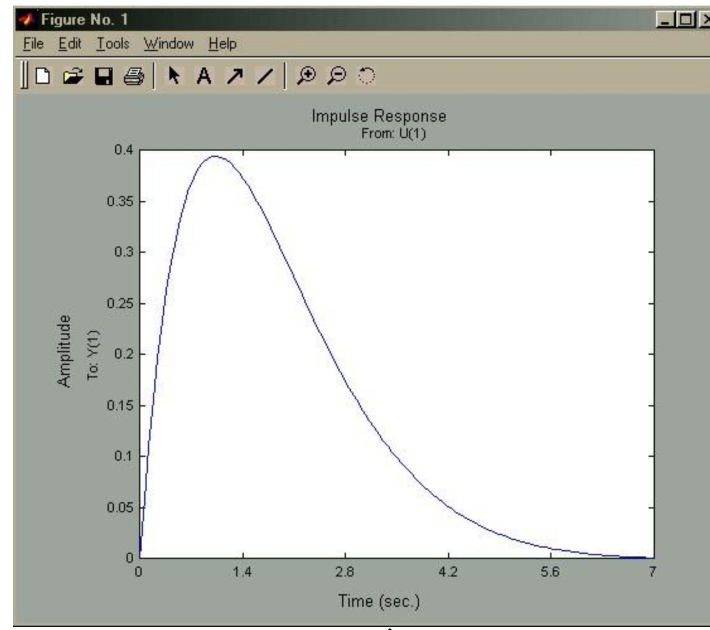


Numerator

Denominator

Transfer Function

Impulse Response



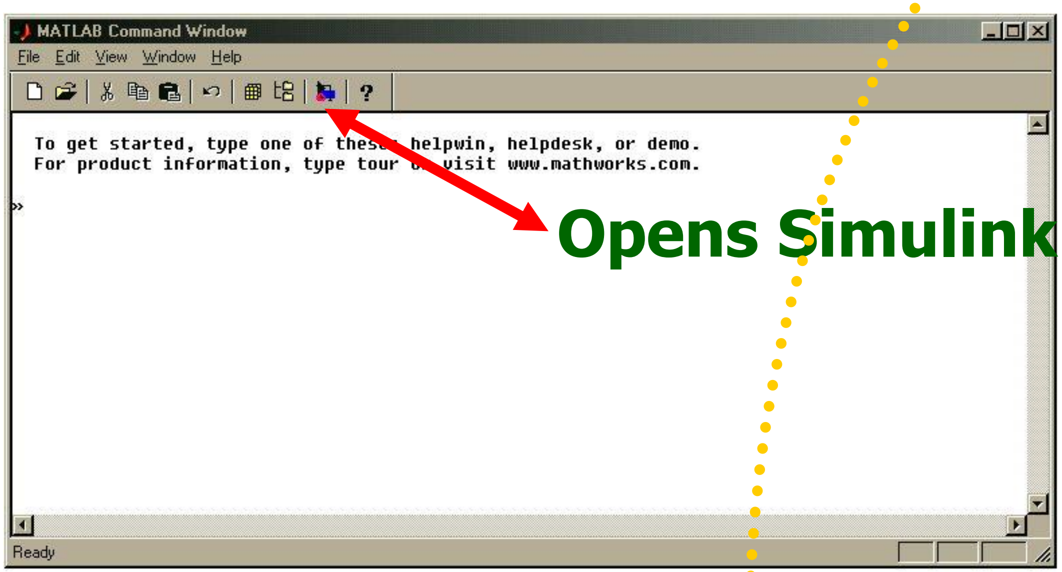
Using Matlab with Simulink

Type »help toolbox/control
To see all *control systems* related functions and library tools

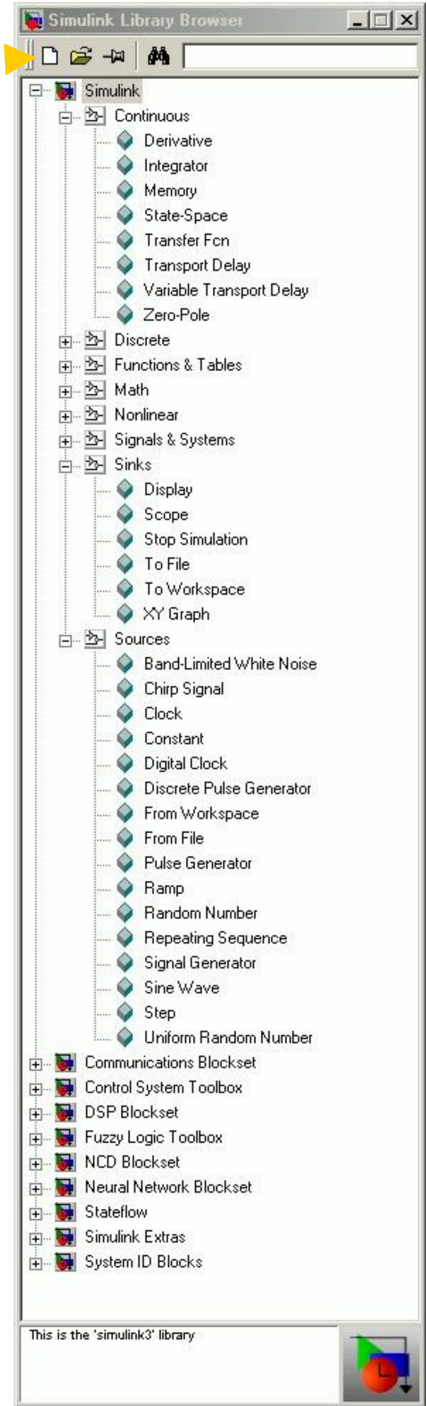
Type »help elmat
To see *elementary matrix* operators and related tools

Using Matlab with Simulink

Simulink



Opens Simulink

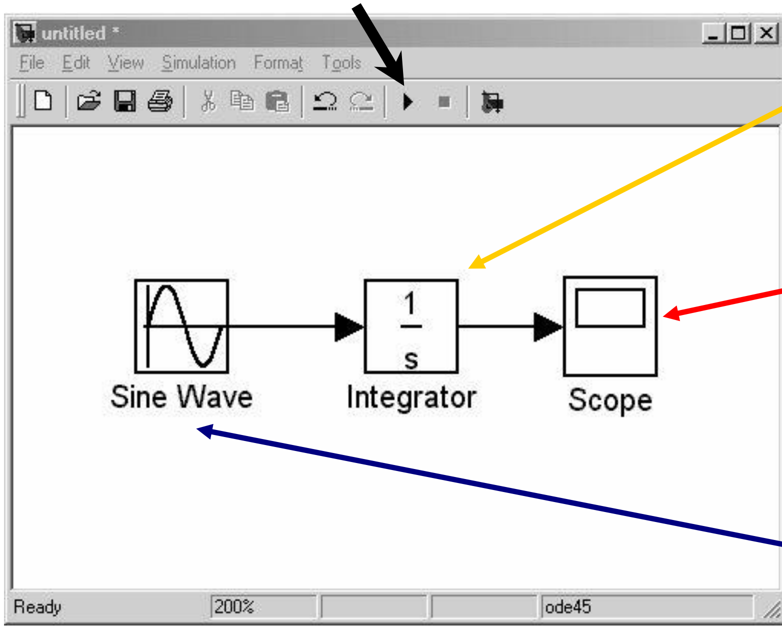


Creates a new model

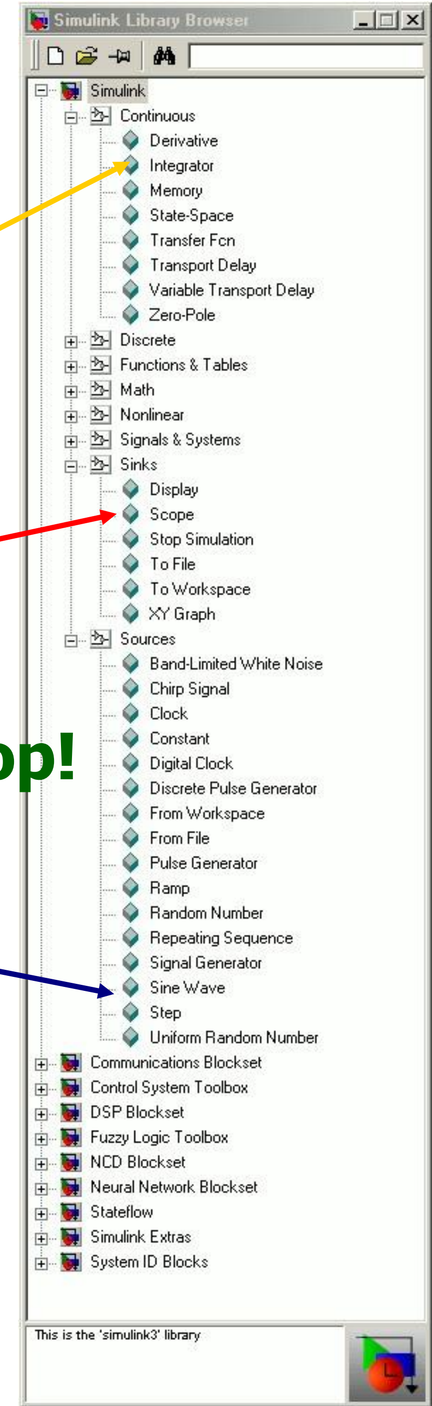
Using Matlab with Simulink

Simulink

4. Run the model



1. Drag & Drop!

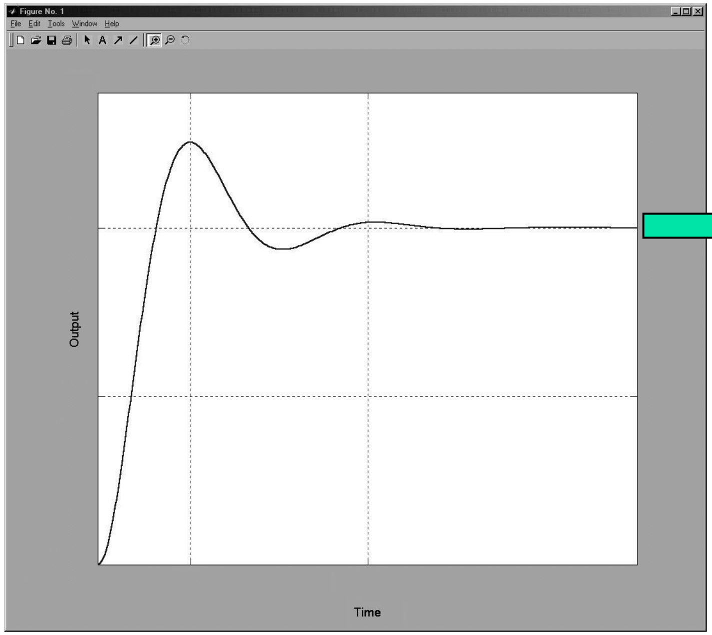


2. Connect the components

3. Double click to set the internal parameters (e.g. magnitude or phase of sine wave, initial value of the integrator etc.)

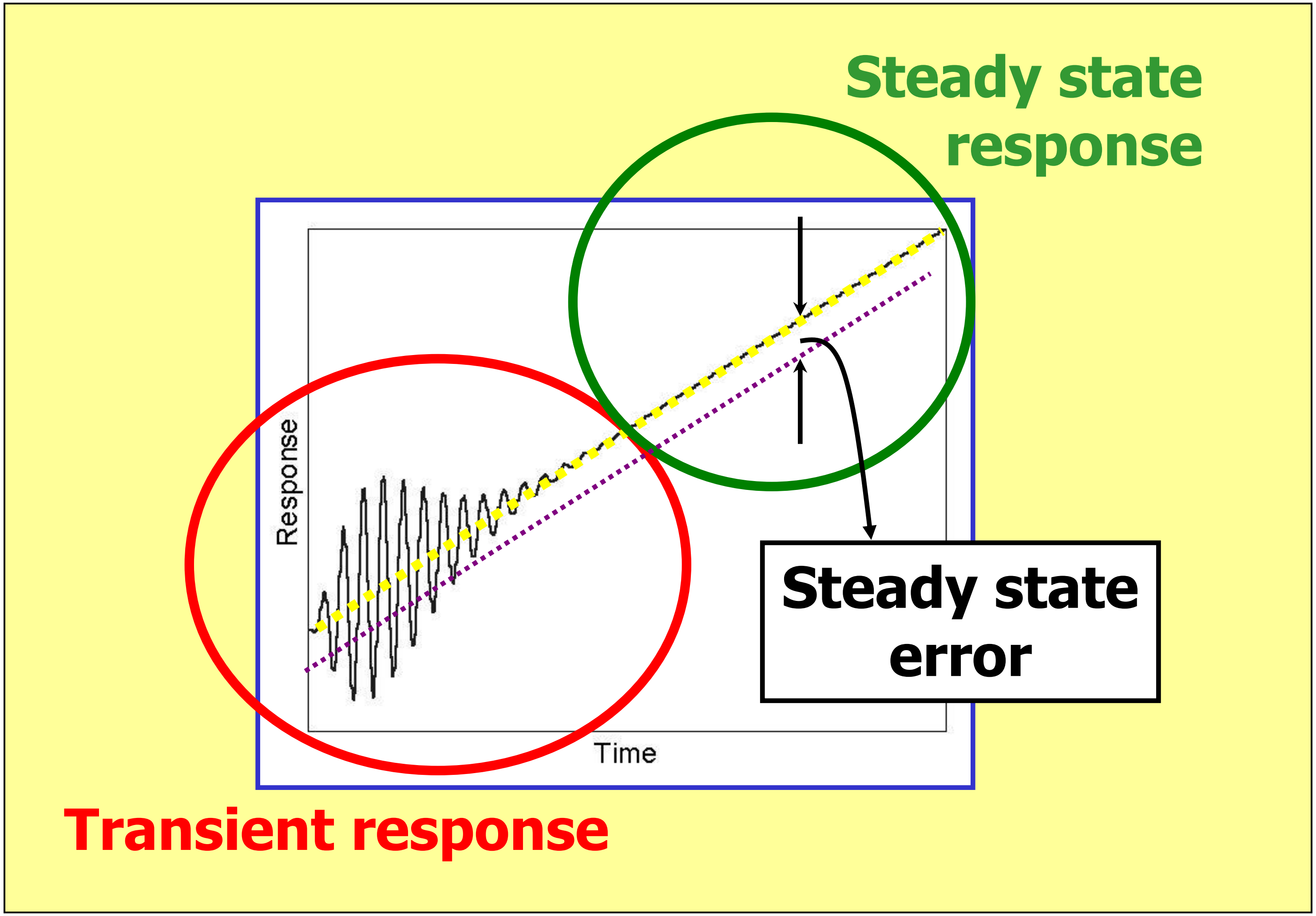
P-4 Steady State Errors

- **Steady state response is the manner in which the system output behaves as time approaches infinity**



This is the steady state value

Steady State Errors



Steady State Errors

- **Control systems can be classified according to their ability to follow several test inputs.**
- **We will analyze the steady state error for certain types of inputs, such as step, ramp or parabolic commands.**
- **Most input signals can be written as combinations of these signals, so the classification is reasonable.**

Steady State Errors

- **Whether a given control system will exhibit steady state error for a given type of input depends on the type of open loop transfer function of the system.**

- **Type of open loop transfer function is the number of integrators contained.**

$$\boxed{G(s) = \frac{K(s+b_1)(s+b_2)\cdots(s+b_m)}{s^N(s+a_1)(s+a_2)\cdots(s+a_n)}}$$

Steady State Errors

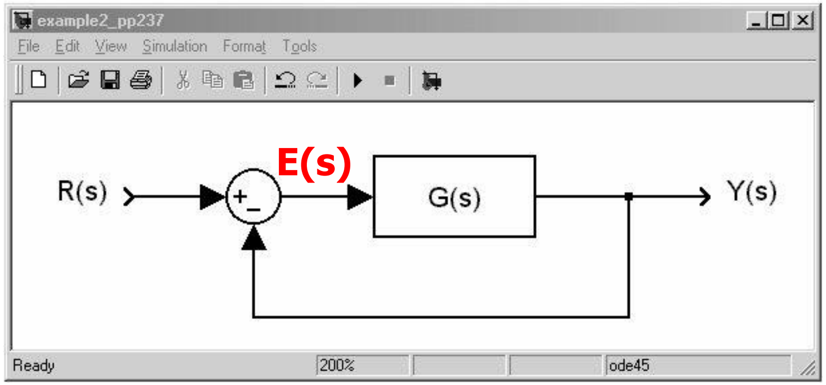
$$G(s)=\frac{K(s+b_1)(s+b_2)\cdots(s+b_m)}{s^N(s+a_1)(s+a_2)\cdots(s+a_n)}$$

We will consider only

N=0 ⇨ Type 0

N=1 ⇨ Type 1

N=2 ⇨ Type 2

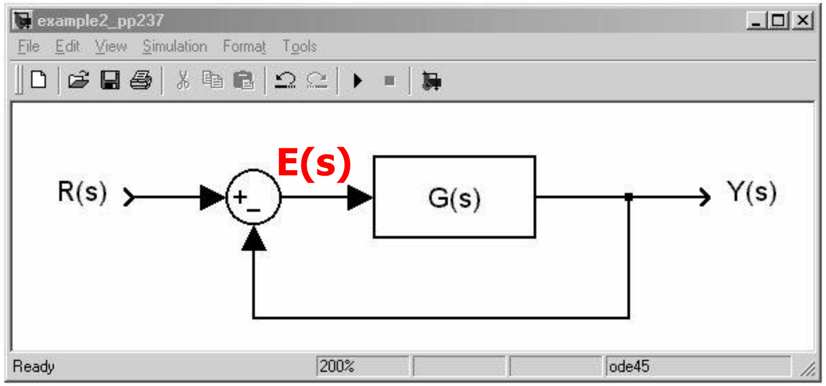


$$T(s)=\frac{Y(s)}{R(s)}=\frac{G(s)}{1+G(s)}$$

$$E(s)=\frac{1}{1+G(s)}R(s)$$

Steady State Errors

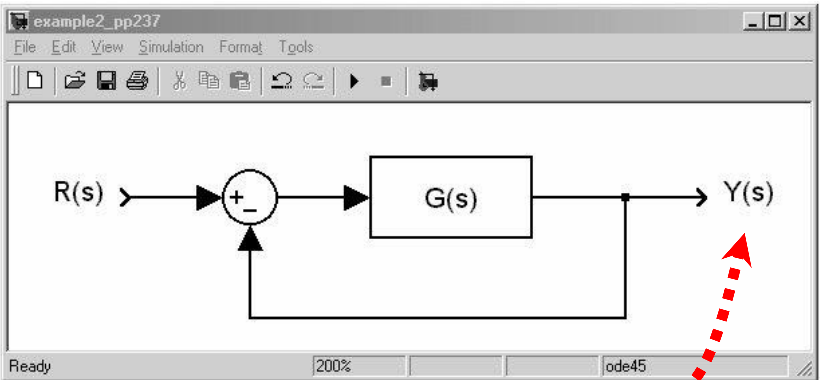
$$E(s) = \frac{1}{1+G(s)} R(s)$$



$$e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{sR(s)}{1+G(s)}$$

Final Value Theorem

Steady State Errors



Regardless of the corresponding physics, we will consider position, velocity and acceleration outputs

Position output		Velocity output		Acceleration output
Position	$\rightarrow \dfrac{d}{dt} \rightarrow$	Velocity	$\rightarrow \dfrac{d}{dt} \rightarrow$	Acceleration
Pressure		Pressure Change		Change in Pressure Change
Temperature		Temperature Change		Change in Temperature Change

Steady State Errors
Static Position/Velocity/Acceleration Error Constants

$$e_{ss} = \lim_{s\to 0} \frac{s}{1+G(s)}\frac{1}{s} \qquad K_p = \lim_{s\to 0} G(s) = G(0) \qquad e_{ss} = \frac{1}{1+K_p}$$

$$e_{ss} = \lim_{s\to 0} \frac{s}{1+G(s)}\frac{1}{s^2} \qquad K_v = \lim_{s\to 0} sG(s) \qquad e_{ss} = \frac{1}{K_v}$$

$$e_{ss} = \lim_{s\to 0} \frac{s}{1+G(s)}\frac{1}{s^3} \qquad K_a = \lim_{s\to 0} s^2G(s) \qquad e_{ss} = \frac{1}{K_a}$$

The larger the constants, the smaller the e_{ss}

Steady State Errors
Static Position/Velocity/Acceleration Error Constants

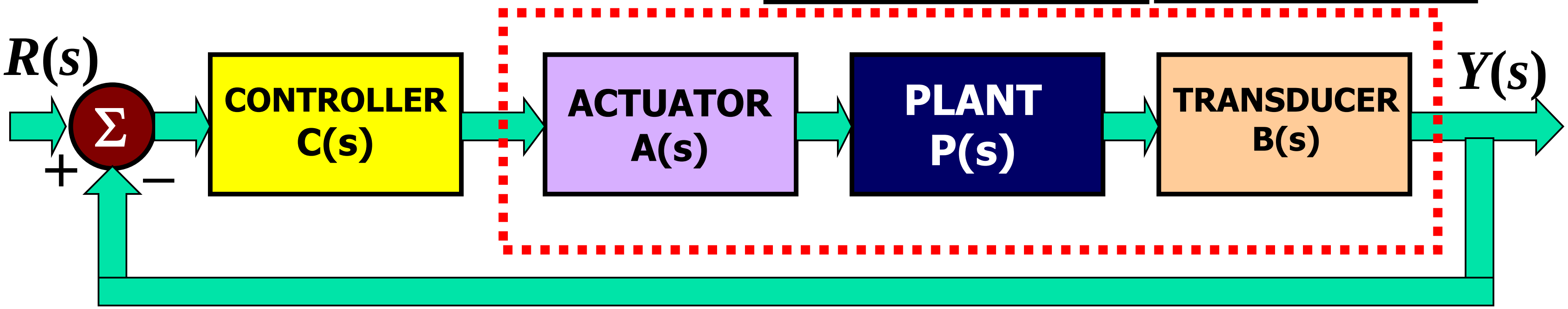
System Type \ Input Type	Step Input $r(t)=1$	Ramp Input $r(t)=t$	Acceleration Input $r(t)=t^2/2$
Type 0	$e_{ss}=\dfrac{1}{1+K_p}$	∞	∞
Type 1	0	$e_{ss}=\dfrac{1}{K_v}$	∞
Type 2	0	0	$e_{ss}=\dfrac{1}{K_a}$

Transient Response
Steady State Response

We analyzed the characteristics of the response of the closed loop system. In any practical design, you will have a number of design specifications, which may impose penalties on transient or steady state characteristics.

An Example

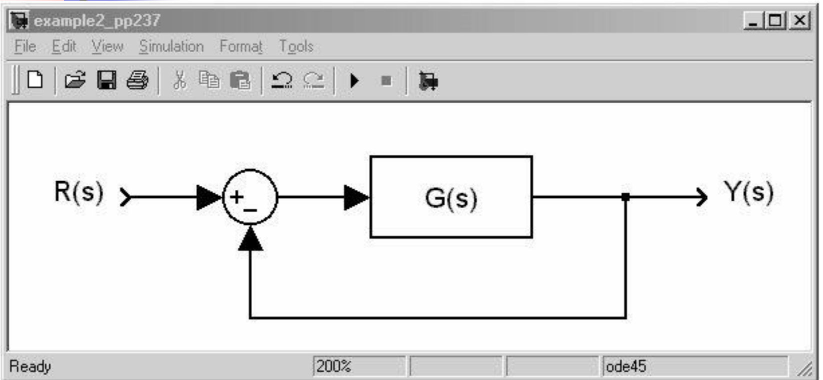
$$A(s) = K_1 \qquad P(s) = \frac{a}{s(s+b)} \qquad B(s) = \frac{K_2}{s+K_2}$$



$$G(s) = K_1 \frac{a}{s(s+b)} \frac{K_2}{s+K_2} C(s)$$

Open Loop Transfer Function

An Example



$$G(s)=\frac{K_1K_2a}{s(s+b)(s+K_2)}C(s)$$

$$K_1=10,\ K_2=20,\ a=1,\ b=4$$

Design a PD controller such that

- **The closed loop system becomes stable**
- **The closed loop system follows the unit ramp with minimum possible steady state error**
- **Response of the closed loop for unit step input exhibits maximum overshoot M_p=0.1**

These are the specifications of the design...

An Example
Stability Requirement

Choose controller as $\Rightarrow$ $\boxed{C(s) = K(s+20)}$

Open Loop TF $\Rightarrow$ $\boxed{G(s) = \dfrac{200K}{s(s+4)}}$

Closed Loop TF

$$\boxed{T(s) = \frac{G(s)}{1+G(s)} = \frac{200K}{s^2 + 4s + 200K}}$$

s^2	1	$200K$
s^1	4	0
s^0	$200K$	

$\Rightarrow$ $\boxed{K > 0}$

An Example
Steady State Error Requirement
Obtain minimum e_{ss} for ramp input

$$E(s)=\frac{1}{1+\dfrac{200K}{s(s+4)}}\frac{1}{s^2}=\frac{s^2+4s}{s^2+4s+200K}\frac{1}{s^2}$$

$$e_{ss}=\lim_{s\to 0} sE(s)=\lim_{s\to 0}\frac{s+4}{s^2+4s+200K}=\frac{1}{50K}$$

Should you choose K as large as possible?

An Example
Maximum Overshoot Requirement

Closed Loop TF

$$T(s) = \frac{G(s)}{1+G(s)} = \frac{200K}{s^2 + 4s + 200K}$$

$$M_p = e^{\left(-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}\right)} = 0.1$$

$$2\zeta\omega_n = 4$$

$$\omega_n^2 = 200K$$

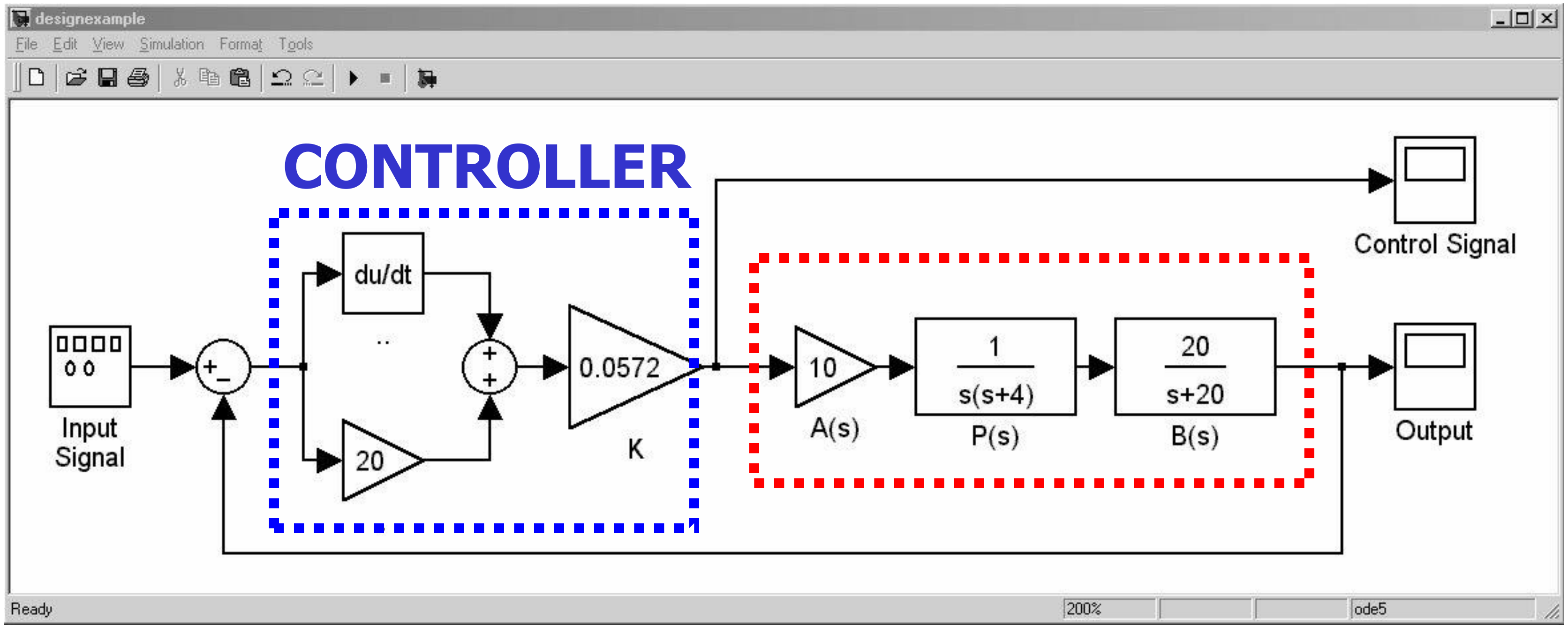
$$\zeta \cong 0.5912$$

$$\omega_n \cong 3.3832$$

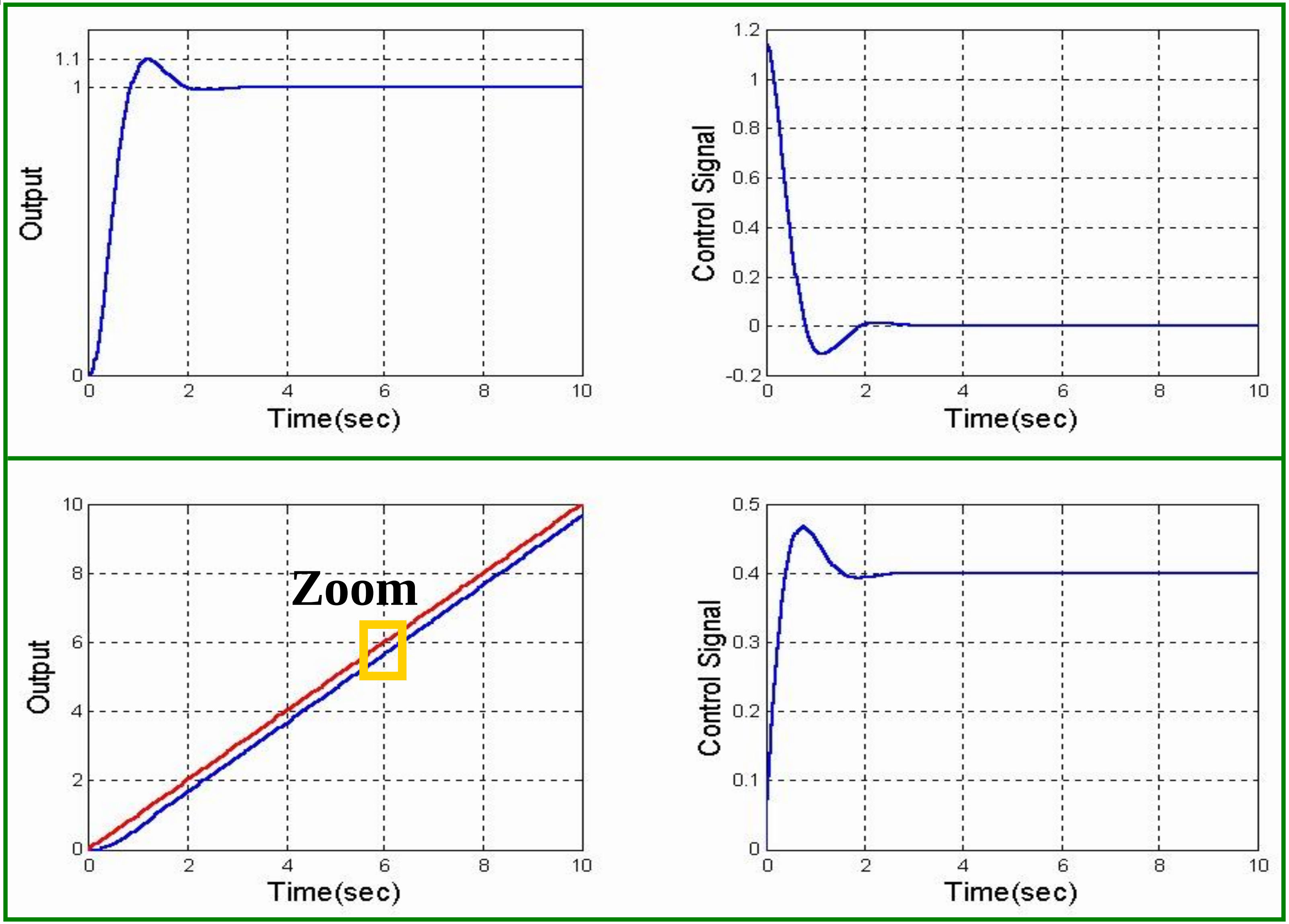
$$K \cong 0.0572$$

$$e_{ss} = \frac{1}{50K} \cong 0.3495$$

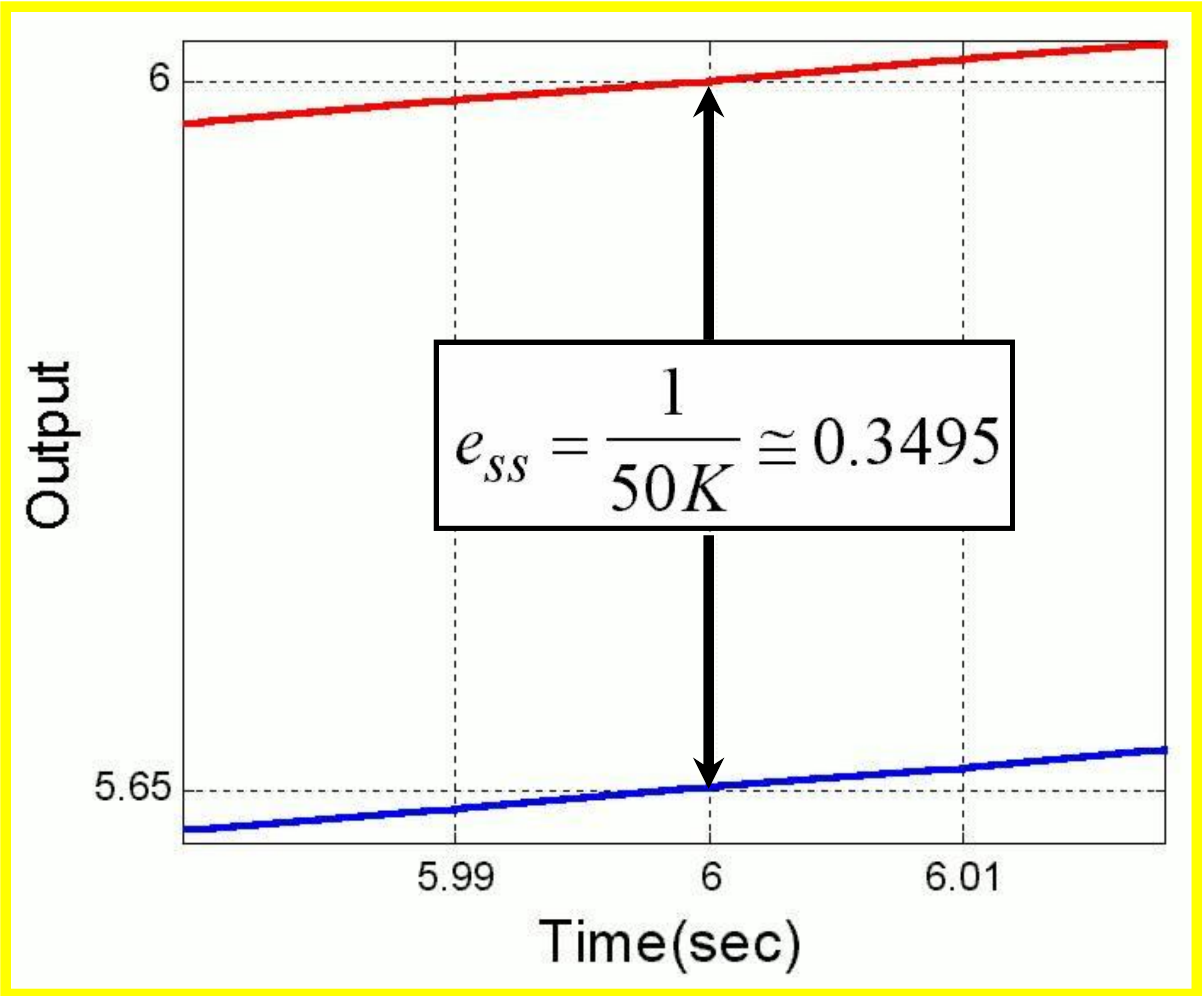
An Example
Justification of the Design



An Example
Justification of the Design



An Example
Justification of the Design



An Example
Remarks

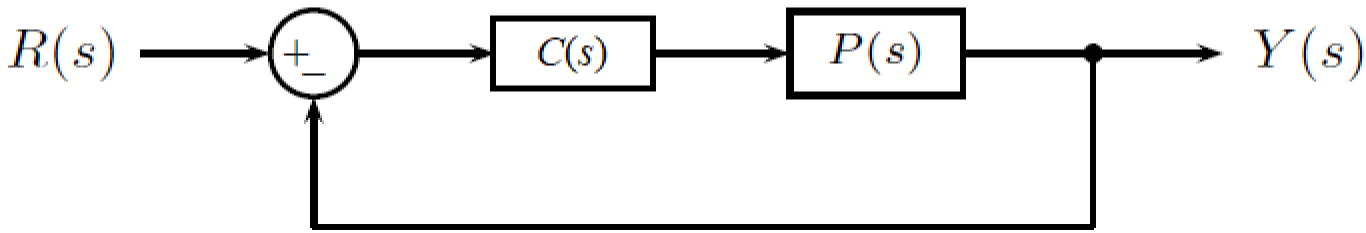
Controller is → $$\boxed{C(s) = K(s+20)}$$

Open Loop TF is

$$\boxed{G(s) = \frac{K_1 K_2 a}{s(s+b)(s+K_2)} C(s)}$$

The product of them cancels out the pole at $s=-K_2$. Never cancel an unstable pole!
Since $K_2>0$, we could do it. If K_2 were negative, an imperfect cancellation would result in instabilities in the long run; and in practice, we are always faced to imperfections!

Unstable Pole-Zero Cancellation

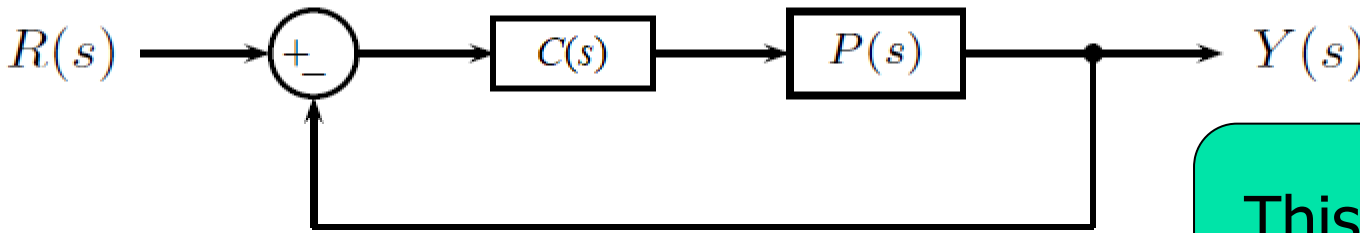


Looks fine but remember P(s) is a model, not the truth!

$$P = \frac{1}{s-1},\ C = K\frac{s-1}{s+a}$$

$$T = \frac{PC}{1+PC} = \frac{\dfrac{K}{s+a}}{1+\dfrac{K}{s+a}} = \frac{K}{s+a+K}$$

Unstable Pole-Zero Cancellation



This is what you expect

Let a=1, K=1

Looks fine but remember P(s) is a model, not the truth!

$$P = \frac{1}{s-1}, \; C = \frac{s-1}{s+1}$$

$$T = \frac{PC}{1+PC} = \frac{\frac{1}{s+1}}{1+\frac{1}{s+1}} = \frac{1}{s+2}$$

$$Y = \frac{1}{s(s+2)} = \frac{0.5}{s} - \frac{0.5}{s+2}, \; y(t) = \frac{1}{2}(1-e^{-2t})1(t)$$

Unstable Pole-Zero Cancellation

Reality is a little different

$$P = \frac{1}{s-1+\Delta},\ C = K\frac{s-1}{s+a}$$

$$T = \frac{PC}{1+PC} = \frac{\dfrac{K(s-1)}{(s-1+\Delta)(s+a)}}{1+\dfrac{K(s-1)}{(s-1+\Delta)(s+a)}} = \frac{K(s-1)}{(s-1+\Delta)(s+a)+K(s-1)}$$

$$T = \frac{K(s-1)}{s^2+(a-1+\Delta+K)s+(\Delta a-a-K)}$$

Denominator is not first order!

Unstable Pole-Zero Cancellation

Let a=1, K=1 and Δ=0.001

$$P=\frac{1}{s-0.999},\ C=\frac{s-1}{s+1}$$

$$T=\frac{PC}{1+PC}=\frac{\dfrac{s-1}{(s-0.999)(s+1)}}{1+\dfrac{s-1}{(s-0.999)(s+1)}}=\frac{s-1}{(s-0.999)(s+1)+(s-1)}$$

$$T=\frac{s-1}{s^2+1.001s-1.999}=\frac{s-1}{(s+2.00033340741564)(s-0.99933407415637)}$$

$$Y=\frac{s-1}{s(s+2.00033340741564)(s-0.99933407415637)}$$

$$=\frac{c_1}{s}+\frac{c_2}{s+2.00033340741564}-\frac{0.00022237043898794 5}{s-0.99933407415637}$$

Unstable Pole-Zero Cancellation

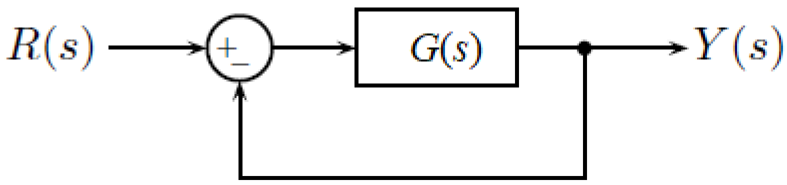
Let a=1, K=1 and Δ=0.001

$$Y=\frac{c_1}{s}+\frac{c_2}{s+2.00033340741564}-\frac{0.000222370438987945}{s-0.999333407415637}$$

$$y(t)=c_1 1(t)+c_2 e^{-2.00033340741564t}1(t)-0.000222370438987945e^{0.999333407415637t}1(t)$$

- **Unstable pole is cancelled by an unstable zero and the result is unstable due to the model imperfections**

Steady State Error Examples



Input Type / System Type	Step Input $r(t)=1$	Ramp Input $r(t)=t$	Acceleration Input $r(t)=t^2/2$
Type 0	$e_{ss}=\dfrac{1}{1+K_p}$	∞	∞
Type 1	0	$e_{ss}=\dfrac{1}{K_v}$	∞
Type 2	0	0	$e_{ss}=\dfrac{1}{K_a}$

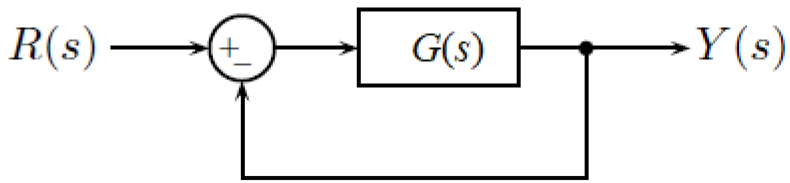
$$G(s)=\frac{K}{s+1},\quad K>0$$

$$K_p=\lim_{s\to 0}G(s)=K,\ e_{ss}=\frac{1}{1+K_p}=\frac{1}{1+K}$$

$$K_v=\lim_{s\to 0}sG(s)=0,\ e_{ss}=\frac{1}{K_v}=\infty$$

$$K_a=\lim_{s\to 0}s^2G(s)=0,\ e_{ss}=\frac{1}{K_a}=\infty$$

Steady State Error Examples



System Type \ Input Type	Step Input $r(t)=1$	Ramp Input $r(t)=t$	Acceleration Input $r(t)=t^2/2$
Type 0	$e_{ss}=\dfrac{1}{1+K_p}$	∞	∞
Type 1	0	$e_{ss}=\dfrac{1}{K_v}$	∞
Type 2	0	0	$e_{ss}=\dfrac{1}{K_a}$

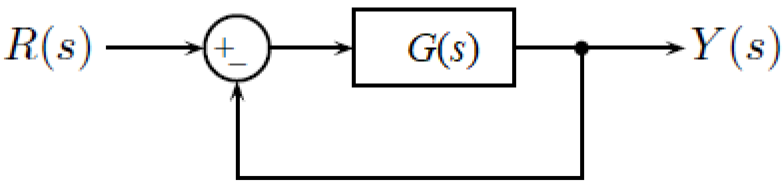
$$G(s)=\frac{K}{s},\quad K>0$$

$$K_p=\lim_{s\to 0}G(s)=\infty,\ e_{ss}=\frac{1}{1+K_p}=0$$

$$K_v=\lim_{s\to 0}sG(s)=K,\ e_{ss}=\frac{1}{K_v}=\frac{1}{K}$$

$$K_a=\lim_{s\to 0}s^2G(s)=0,\ e_{ss}=\frac{1}{K_a}=\infty$$

Steady State Error Examples



Input Type / System Type	Step Input $r(t)=1$	Ramp Input $r(t)=t$	Acceleration Input $r(t)=t^2/2$
Type 0	$e_{ss}=\dfrac{1}{1+K_p}$	∞	∞
Type 1	0	$e_{ss}=\dfrac{1}{K_v}$	∞
Type 2	0	0	$e_{ss}=\dfrac{1}{K_a}$

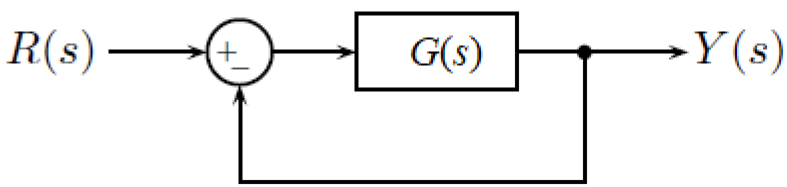
$$G(s)=\frac{K}{s(s+a)},\quad K>0,\ a>0$$

$$K_p=\lim_{s\to 0}G(s)=\infty,\ e_{ss}=\frac{1}{1+K_p}=0$$

$$K_v=\lim_{s\to 0}sG(s)=\frac{K}{a},\ e_{ss}=\frac{1}{K_v}=\frac{a}{K}$$

$$K_a=\lim_{s\to 0}s^2G(s)=0,\ e_{ss}=\frac{1}{K_a}=\infty$$

Steady State Error Examples



Input Type / System Type	Step Input $r(t)=1$	Ramp Input $r(t)=t$	Acceleration Input $r(t)=t^2/2$
Type 0	$e_{ss}=\dfrac{1}{1+K_p}$	∞	∞
Type 1	0	$e_{ss}=\dfrac{1}{K_v}$	∞
Type 2	0	0	$e_{ss}=\dfrac{1}{K_a}$

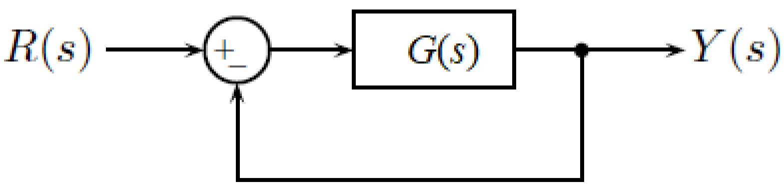
$$G(s)=\frac{K}{s^2(s+a)},\quad K>0,\ a>0$$

$$K_p=\lim_{s\to 0}G(s)=\infty,\ e_{ss}=\frac{1}{1+K_p}=0$$

$$K_v=\lim_{s\to 0}sG(s)=\infty,\ e_{ss}=\frac{1}{K_v}=0$$

$$K_a=\lim_{s\to 0}s^2G(s)=\frac{K}{a},\ e_{ss}=\frac{1}{K_a}=\frac{a}{K}$$

Steady State Error Examples
Pole zero cancellation



Input Type / System Type	Step Input $r(t)=1$	Ramp Input $r(t)=t$	Acceleration Input $r(t)=t^2/2$
Type 0	$e_{ss}=\dfrac{1}{1+K_p}$	∞	∞
Type 1	0	$e_{ss}=\dfrac{1}{K_v}$	∞
Type 2	0	0	$e_{ss}=\dfrac{1}{K_a}$

$$G(s)=P(s)C(s)=\frac{K_1}{s(s+a_1)}\,\frac{K_2(s+a_1)}{s+a_2}$$

$$K_i>0,\ a_i>0$$

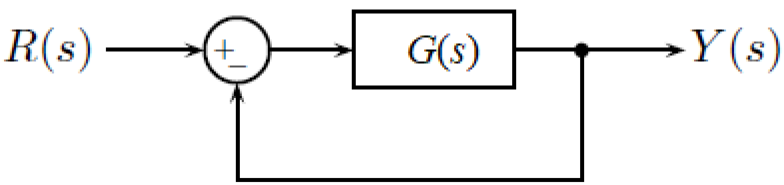
$$K_p=\lim_{s\to 0}G(s)=\infty,\ e_{ss}=\frac{1}{1+K_p}=0$$

$$K_v=\lim_{s\to 0}sG(s)=\frac{K_1K_2}{a_2},\ e_{ss}=\frac{1}{K_v}=\frac{a_2}{K_1K_2}$$

$$K_a=\lim_{s\to 0}s^2G(s)=0,\ e_{ss}=\frac{1}{K_a}=\infty$$

Note that we changed K_v from K_1/a_1 to K_1K_2/a_2.

Steady State Error Examples
Addition of a special term



Input Type / System Type	Step Input $r(t)=1$	Ramp Input $r(t)=t$	Acceleration Input $r(t)=t^2/2$
Type 0	$e_{ss}=\frac{1}{1+K_p}$	∞	∞
pe 1	0	$e_{ss}=\frac{1}{K_v}$	∞
pe 2	0	0	$e_{ss}=\frac{1}{K_a}$

$$G(s)=P(s)C(s)=\left(\frac{K}{s(s+a)}\right)\left(\frac{s+0.1}{s+0.01}\right)$$

$$K>0,\ a>0$$

$$K_p=\lim_{s\to 0}G(s)=\infty,\ e_{ss}=\frac{1}{1+K_p}=0$$

$$K_v=\lim_{s\to 0}sG(s)=\frac{10K}{a},\ e_{ss}=\frac{1}{K_v}=\frac{a}{10K}$$

$$K_a=\lim_{s\to 0}s^2G(s)=0,\ e_{ss}=\frac{1}{K_a}=\infty$$

Note that we changed K_v from K/a to 10K/a, which is better!

Steady State Error Examples

Typical design steps require

- Check stability
- Meet desired transient characteristics
- Meet desired steady state characteristics
- Validate your design