This week's agenda

- **Concept of Stability**
- **Stability Analysis of the Closed Loop System by Routh Criterion**
- **State Space Representation and Stability**

P-3 Concept of Stability

What is stability?

- Stability is a property of the system regardless of the signals at the inputs and outputs
- Stability is an underlying requirement in every control system

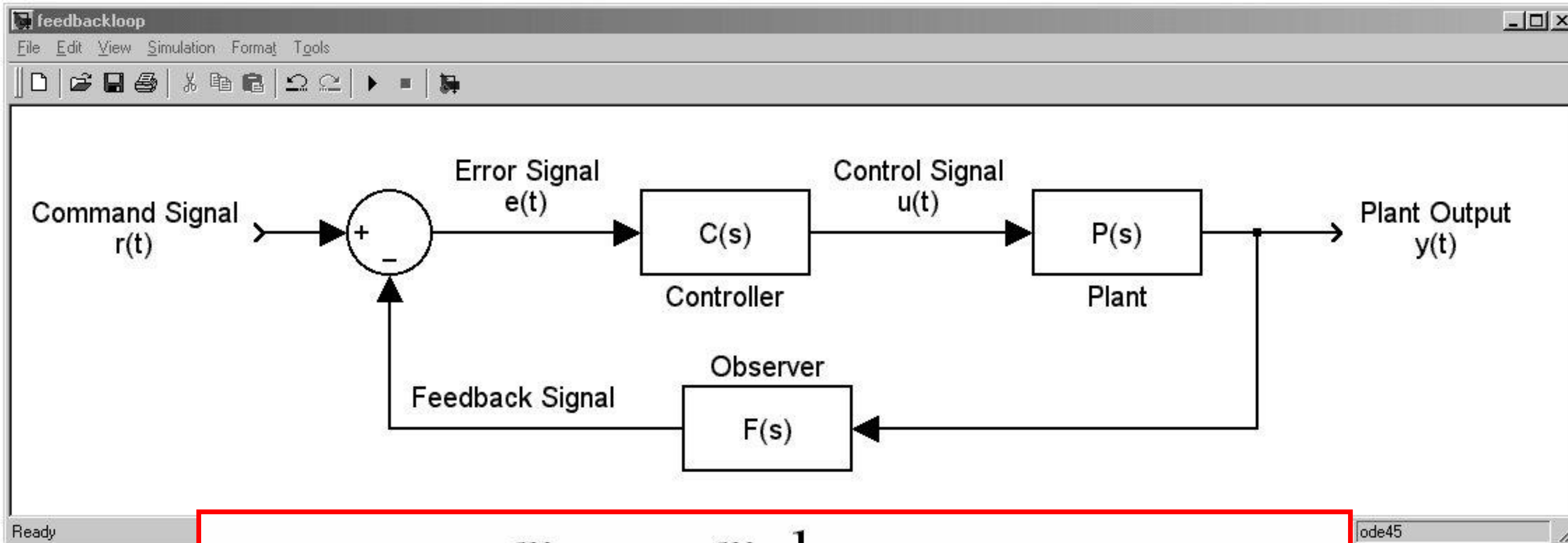
Why do we need to analyze stability?

- An unstable system is potentially dangerous!
- When the power is turned on, the output will increase (decrease/oscillate) indefinitely…
- Eventually this will damage the physical setup

P-3 Stability Analysis of the Closed Loop System by Routh Criterion

Consider the feedback loop

$$\frac{Y(s)}{R(s)} = \frac{P(s)C(s)}{1+P(s)C(s)F(s)} = T(s)$$



$$T(s) = \frac{b'_0 s^m + b'_1 s^{m-1} + \cdots + b'_{m-1} s + b'_m}{a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n}$$

$$
\begin{array}{c|ccccccc}
s^n & a_0 & a_2 & a_4 & a_6 & \cdot & \cdot & \cdot \\
s^{n-1} & a_1 & a_3 & a_5 & a_7 & \cdot & \cdot & \cdot \\
s^{n-2} & b_1 & b_2 & b_3 & b_4 & \cdot & \cdot & \cdot \\
s^{n-3} & c_1 & c_2 & c_3 & c_4 & \cdot & \cdot & \cdot \\
s^{n-4} & d_1 & d_2 & d_3 & d_4 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & & & & & \\
\cdot & \cdot & \cdot & & & & & \\
\cdot & \cdot & \cdot & & & & & \\
s^2 & e_1 & e_2 & & & & & \\
s^1 & f_1 & & & & & & \\
s^0 & g_1 & & & & & &
\end{array}
$$

T(s) will let you first place these terms

$$\frac{Y(s)}{R(s)} = \frac{P(s)C(s)}{1+P(s)C(s)F(s)} = T(s)$$

$$T(s) = \frac{b'_0 s^m + b'_1 s^{m-1} + \cdots + b'_{m-1} s + b'_m}{a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n}$$

ROW #3

Evaluate till the remaining bs are all zero

$$b_1 = \frac{a_1 a_2 - a_0 a_3}{a_1}$$

$$b_2 = \frac{a_1 a_4 - a_0 a_5}{a_1}$$

$$b_3 = \frac{a_1 a_6 - a_0 a_7}{a_1}$$

$$\vdots$$

$$\begin{array}{ccccccccc}
s^n & a_0 & a_2 & a_4 & a_6 & \cdot & \cdot & \cdot \\
s^{n-1} & a_1 & a_3 & a_5 & a_7 & \cdot & \cdot & \cdot \\
s^{n-2} & \boxed{b_1} & \boxed{b_2} & \boxed{b_3} & \boxed{b_4} & \cdot & \cdot & \cdot \\
s^{n-3} & c_1 & c_2 & c_3 & c_4 & \cdot & \cdot & \cdot \\
s^{n-4} & d_1 & d_2 & d_3 & d_4 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & & & & & \\
\cdot & \cdot & \cdot & & & & & \\
\cdot & \cdot & \cdot & & & & & \\
s^2 & e_1 & e_2 & & & & & \\
s^1 & f_1 & & & & & & \\
s^0 & g_1 & & & & & &
\end{array}$$

ROW #4

Evaluate till the remaining cs are all zero

$$c_1 = \frac{b_1 a_3 - a_1 b_2}{b_1}$$

$$c_2 = \frac{b_1 a_5 - a_1 b_3}{b_1}$$

$$c_3 = \frac{b_1 a_7 - a_1 b_4}{b_1}$$

$$\vdots$$

$$\begin{array}{c|ccccccc}
s^n & a_0 & a_2 & a_4 & a_6 & \cdot & \cdot & \cdot \\
s^{n-1} & a_1 & a_3 & a_5 & a_7 & \cdot & \cdot & \cdot \\
s^{n-2} & b_1 & b_2 & b_3 & b_4 & \cdot & \cdot & \cdot \\
s^{n-3} & c_1 & c_2 & c_3 & c_4 & \cdot & \cdot & \cdot \\
s^{n-4} & d_1 & d_2 & d_3 & d_4 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & & & & & \\
\cdot & \cdot & \cdot & & & & & \\
\cdot & \cdot & \cdot & & & & & \\
s^2 & e_1 & e_2 & & & & & \\
s^1 & f_1 & & & & & & \\
s^0 & g_1 & & & & & &
\end{array}$$

ROW #5

Evaluate till the remaining bs are all zero

$$d_1 = \frac{c_1 b_2 - b_1 c_2}{c_1}$$

$$d_2 = \frac{c_1 b_3 - b_1 c_3}{c_1}$$

$$d_3 = \frac{c_1 b_4 - b_1 c_4}{c_1}$$

$$\vdots$$

$$\begin{array}{cccccccc}
s^n & a_0 & a_2 & a_4 & a_6 & \cdot & \cdot & \cdot \\
s^{n-1} & a_1 & a_3 & a_5 & a_7 & \cdot & \cdot & \cdot \\
s^{n-2} & b_1 & b_2 & b_3 & b_4 & \cdot & \cdot & \cdot \\
s^{n-3} & c_1 & c_2 & c_3 & c_4 & \cdot & \cdot & \cdot \\
s^{n-4} & \boxed{d_1} & \boxed{d_2} & \boxed{d_3} & \boxed{d_4} & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & & & & & \\
\cdot & \cdot & \cdot & & & & & \\
\cdot & \cdot & \cdot & & & & & \\
s^2 & e_1 & e_2 & & & & & \\
s^1 & f_1 & & & & & & \\
s^0 & g_1 & & & & & &
\end{array}$$

Routh table

$$
\begin{array}{c|ccccc}
s^n & a_n & a_{n-2} & a_{n-4} & a_{n-6} & \cdots \\
s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & a_{n-7} & \cdots \\
s^{n-2} & c_n = -\frac{1}{a_{n-1}}\begin{vmatrix} a_n & a_{n-2} \\ a_{n-1} & a_{n-3} \end{vmatrix} & c_{n-1} = -\frac{1}{a_{n-1}}\begin{vmatrix} a_n & a_{n-4} \\ a_{n-1} & a_{n-5} \end{vmatrix} & c_{n-2} = -\frac{1}{a_{n-1}}\begin{vmatrix} a_n & a_{n-6} \\ a_{n-1} & a_{n-7} \end{vmatrix} & c_{n-3} = -\frac{1}{a_{n-1}}\begin{vmatrix} a_n & a_{n-8} \\ a_{n-1} & a_{n-9} \end{vmatrix} & \cdots \\
s^{n-3} & d_n = -\frac{1}{c_n}\begin{vmatrix} a_{n-1} & a_{n-3} \\ c_n & c_{n-1} \end{vmatrix} & d_{n-1} = -\frac{1}{c_n}\begin{vmatrix} a_{n-1} & a_{n-5} \\ c_n & c_{n-2} \end{vmatrix} & d_{n-2} = -\frac{1}{c_n}\begin{vmatrix} a_{n-1} & a_{n-7} \\ c_n & c_{n-3} \end{vmatrix} & & \\
s^{n-4} & e_n = -\frac{1}{d_n}\begin{vmatrix} c_n & c_{n-1} \\ d_n & d_{n-1} \end{vmatrix} & e_{n-1} = -\frac{1}{d_n}\begin{vmatrix} c_n & c_{n-2} \\ d_n & d_{n-2} \end{vmatrix} & e_{n-2} = -\frac{1}{d_n}\begin{vmatrix} c_n & c_{n-3} \\ d_n & d_{n-3} \end{vmatrix} & & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \\
s^2 & f_n & f_{n-1} & & & \\
s^1 & g_n & & & & \\
s^0 & h_n & & & &
\end{array}
$$

Remarks

- Repeat the same pattern till you reach the end i.e. g_1
- The complete array of coefficients is triangular
- Dividing or multiplying any row by a positive number can simplify the calculation without altering the stability conclusion

Routh's stability criterion states that

For

$$T(s)=\frac{b'_0 s^{m}+b'_1 s^{m-1}+\cdots+b'_{m-1}s+b'_m}{a_0 s^{n}+a_1 s^{n-1}+\cdots+a_{n-1}s+a_n}$$

The number of poles on the right hand s-plane is equal to the number of sign changes in the first column of the table

- **Note that, we only need the signs of the numbers in the first column**

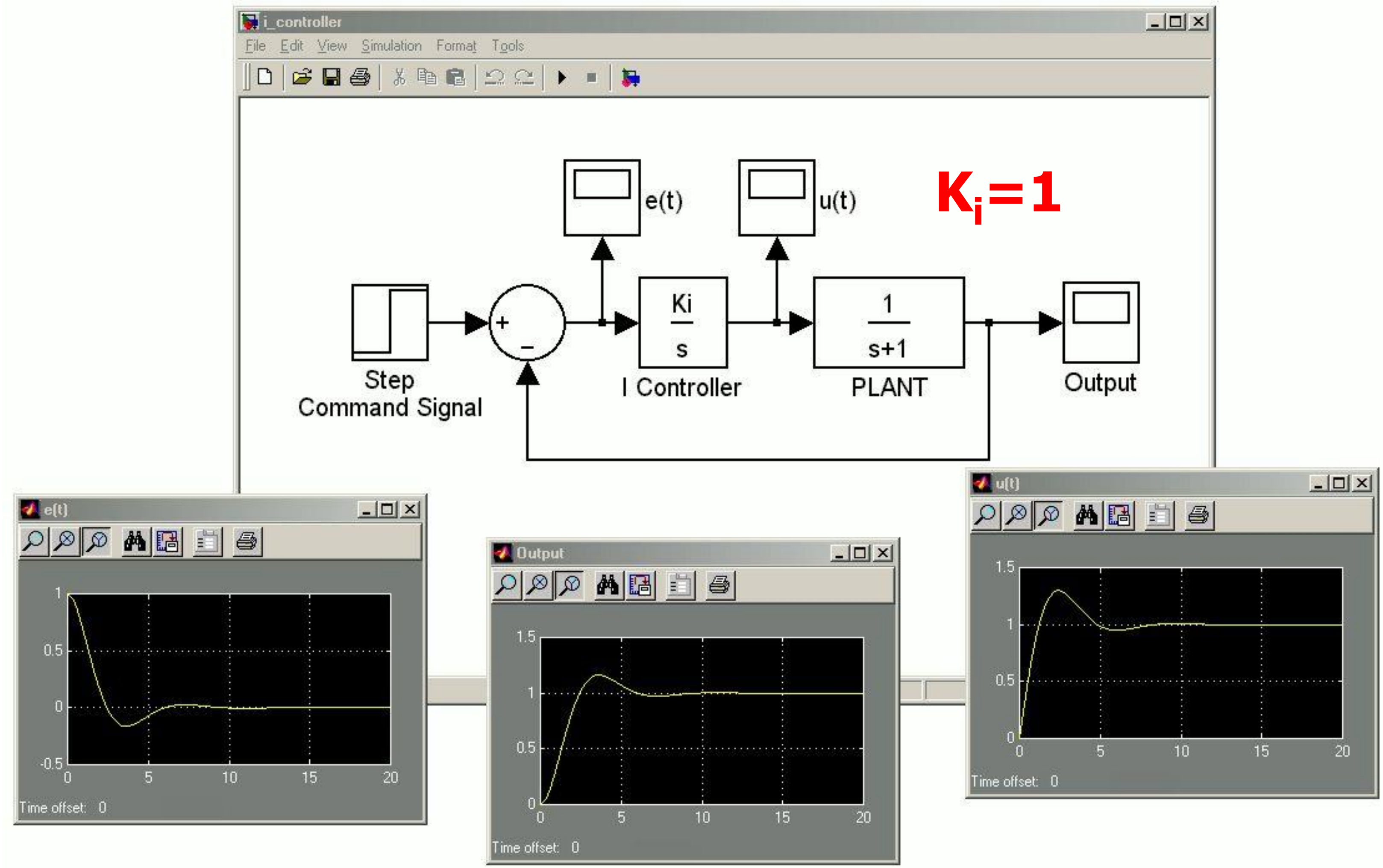
In other words…

$$
\begin{array}{c|cccccccc}
s^n & a_0 & a_2 & a_4 & a_6 & \cdot & \cdot & \cdot \\
s^{n-1} & a_1 & a_3 & a_5 & a_7 & \cdot & \cdot & \cdot \\
s^{n-2} & b_1 & b_2 & b_3 & b_4 & \cdot & \cdot & \cdot \\
s^{n-3} & c_1 & c_2 & c_3 & c_4 & \cdot & \cdot & \cdot \\
s^{n-4} & d_1 & d_2 & d_3 & d_4 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & & & & & \\
\cdot & \cdot & & & & & & \\
\cdot & \cdot & \cdot & & & & & \\
s^2 & e_1 & e_2 & & & & & \\
s^1 & f_1 & & & & & & \\
s^0 & g_1 & & & & & &
\end{array}
$$

These terms must have the same signs for stability

First Example

Recall that we analyzed the following diagram in I-Controller



First Example

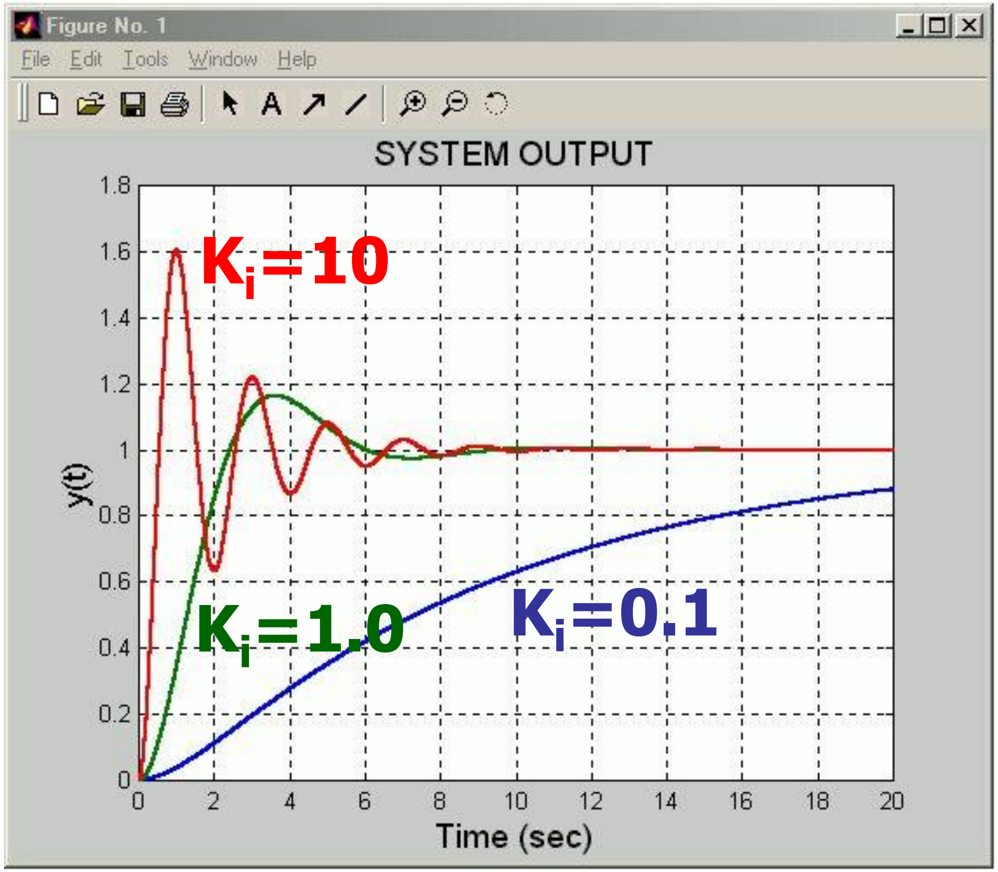
Did we have to choose $K_i=1$?
NO!

$$T(s)=\frac{K_i}{s^2+s+K_i}$$

$$\begin{array}{c|c|c} s^2 & 1 & K_i \\ s^1 & 1 & \\ s^0 & K_i & \end{array}$$

For no sign change in the first column, $K_i>0$ is required. Any positive integral gain would work fine

First Example - System Output



Notice that what they do ultimately are the same, but how they do differ.

Small $K_i \Rightarrow$ Overdamped (Approaches very slowly)

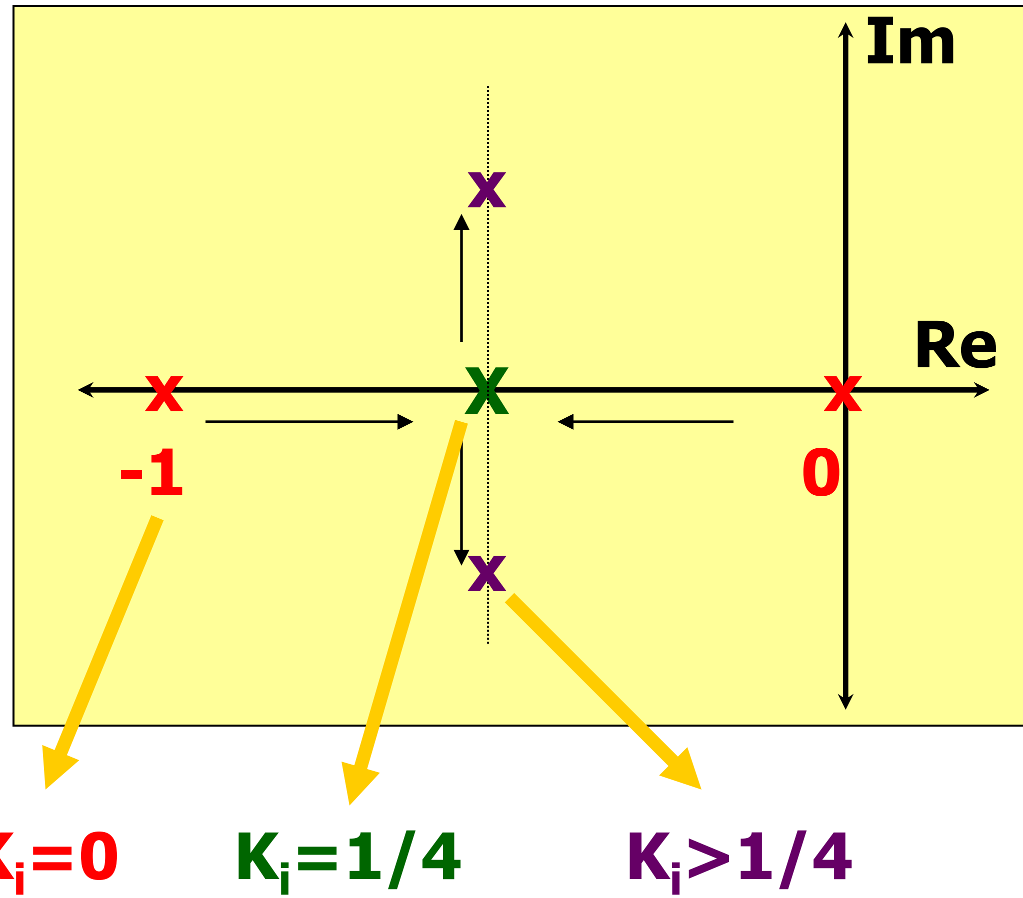
Large $K_i \Rightarrow$ Underdamped (More quickly but with oscillations)

First Example
Where do the oscillations come from?

$$T(s) = \frac{K_i}{s^2 + s + K_i}$$

$$\Delta = 1 - 4K_i$$

$$s_{1,2} = -\frac{1}{2} \pm \sqrt{\frac{1}{4} - K_i}$$

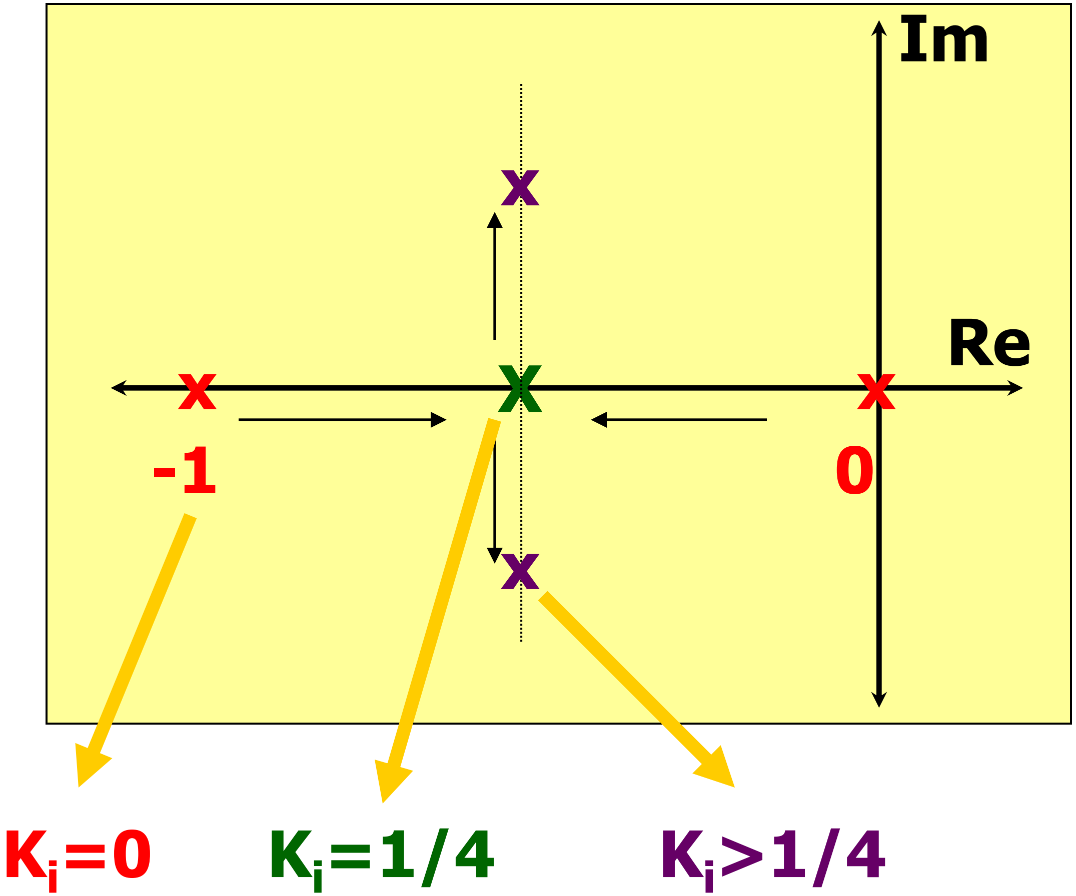


First Example
Where do the oscillations come from?

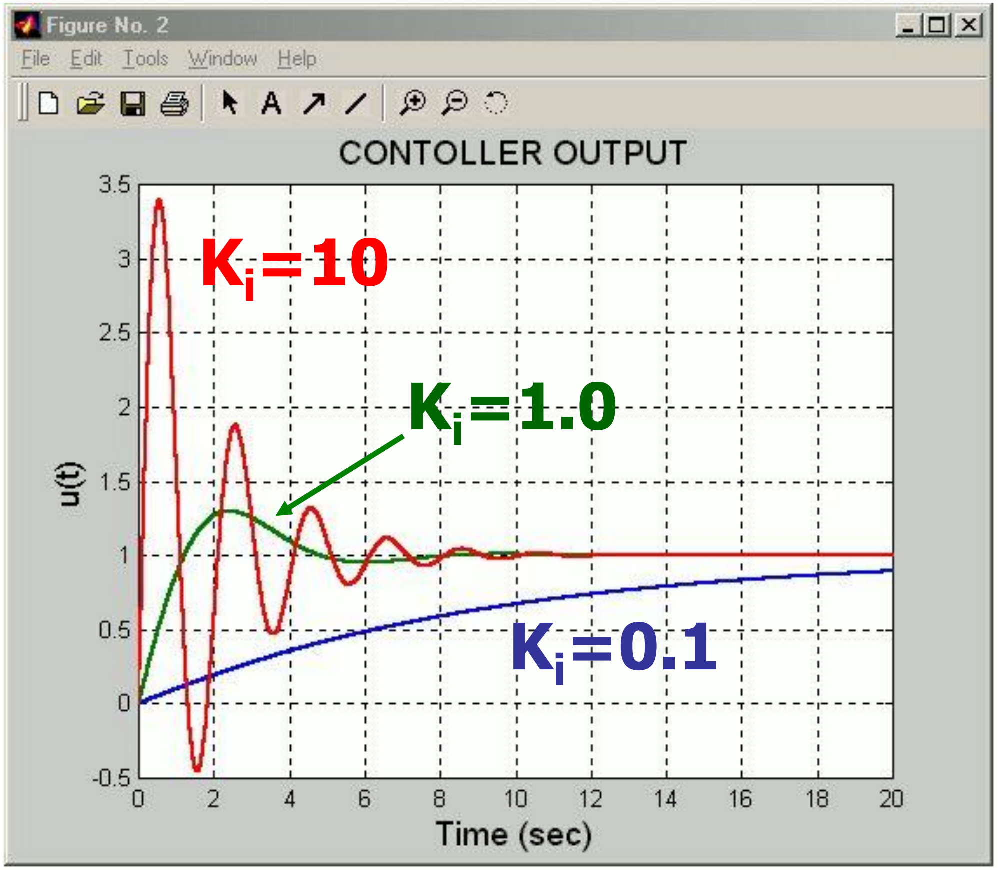
$0<K_i<1/4$
Distinct real poles

$K_i=1/4$
Double poles at $s=-1/2$

$K_i>1/4$
Complex conjugate poles with real parts $-1/2$



First Example - Controller Output



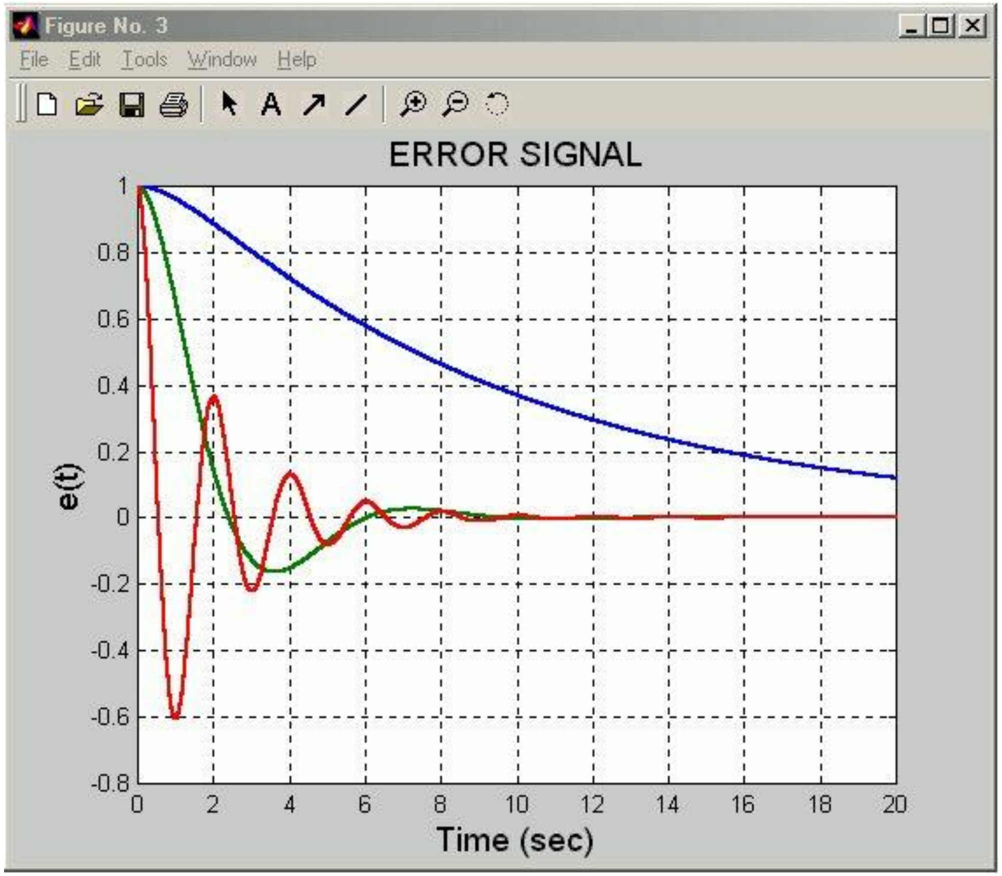
$0<u(t)<1$ for $K_i=0.1$
$0<u(t)<1.3$ for $K_i=1$
$-0.45<u(t)<3.4$ for $K_i=10$

As the controller gain is increased, the range of control signal expands.

- **Can your physical controller provide it?**
- **Is that control signal applicable?**

Small $K_i \Rightarrow$ Overdamped (Approaches very slowly)
Large $K_i \Rightarrow$ Underdamped (More quickly but with oscillations)

First Example - Error Signals



How fast you want the error signal come down to zero?

This signal is the input to the controller. Is that physically applicable to your controller?

First Example - Remarks

- We learned how to check stability of the closed loop (CL) TF
- A set of controller gains (K_i for this example) can result in stable CL. We analyzed what happens with different values
- We learned what questions to ask in the design phase

Example-2

$$H(s) = 7\frac{s+6}{3s^2+5s+1}$$

$$\begin{array}{c|cc|} s^2 & 3 & 1 \\ s^1 & 5 & \\ s^0 & * & \end{array} \Rightarrow \begin{array}{c|cc|} s^2 & 3 & 1 \\ s^1 & 5 & 0 \\ s^0 & -\frac{1}{5}(3\times 0 - 1\times 5) & \end{array} \Rightarrow \begin{array}{c|cc|} s^2 & 3 & 1 \\ s^1 & 5 & 0 \\ s^0 & 1 & \end{array}$$

Example-3

$$H(s) = 2\frac{s^2}{s^3-6s^2+11s-6}$$

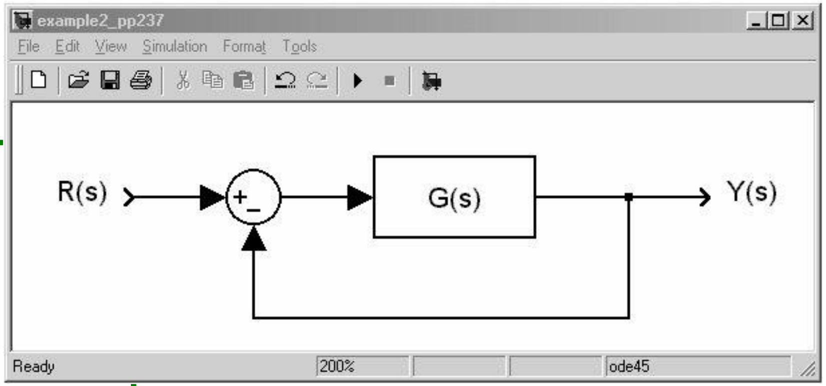
$$\begin{array}{l|rr|}
s^3 & 1 & 11 \\
s^2 & -6 & -6 \\
s^1 & 10 & 0 \\
s^0 & -6 &
\end{array}$$

$$D(s) = (s-1)(s-2)(s-3)$$

Example-4

$$G(s) = \frac{K}{s\left(s^2+s+1\right)(s+2)}$$

$$T(s) = \frac{G}{1+G} = \frac{K}{s\left(s^2+s+1\right)(s+2)+K}$$



Determine the range of K for stability

The characteristic equation is

$$\boxed{s^4+3s^3+3s^2+2s+K=0}$$

Example-4 (Textbook Ogata 3rd Ed. p.237)

$$s^4+3s^3+3s^2+2s+K=0$$

$$\begin{array}{llll} s^4 & 1 & 3 & K \\ s^3 & 3 & 2 & 0 \\ s^2 & 7/3 & K & \\ s^1 & 2-(9/7)K & & \\ s^0 & K & & \end{array}$$

$$2-(9/7)K>0$$

$$K>0$$

$$0<K<\frac{14}{9}$$

Handling the special cases - Example 1
A zero in the first column

Consider $\boxed{s^3 + 2s^2 + s + 2 = 0}$

s^3	1	1
s^2	2	2
s^1	0	
s^0	?	

Insert ε for 0

ε>0

s^3	1	1
s^2	2	2
s^1	$0 \approx \varepsilon$	
s^0	$2\varepsilon/\varepsilon = 2$	

Handling the special cases - Example 1
A zero in the first column

- **No sign change means no roots on the right half s-plane**
- **In this example, two roots were at s=±j**

$$\boxed{s^3 + 2s^2 + s + 2 = 0}$$

$$\begin{array}{lcc} s^3 & 1 & 1 \\ s^2 & 2 & 2 \\ s^1 & 0 \approx \varepsilon & \\ s^0 & 2\varepsilon/\varepsilon = 2 & \end{array}$$

Handling the special cases - Example 2
A zero in the first column

$$\boxed{s^3 - 3s + 2 = 0}$$

$$\begin{array}{lcc}
s^3 & 1 & -3 \\
s^2 & 0 \approx \varepsilon & 2 \\
s^1 & -3 - 2/\varepsilon & \\
s^0 & 2 &
\end{array}$$

One sign change

One sign change

- **Two sign changes mean two roots on the right half s-plane**

$$\boxed{s^3 - 3s + 2 = (s-1)^2(s+2) = 0}$$

Handling the special cases - Remarks

$$\begin{array}{ll} \cdot & \\ \cdot & \\ s^k & \text{positive} \\ s^{k-1} & 0 \approx \varepsilon \\ s^{k-2} & \text{positive} \\ \cdot & \\ \cdot & \end{array}$$

No sign change, i.e. no roots on the right half s-plane

But, there are a pair of imaginary roots

Handling the special cases - Remarks

$\cdot$	
$\cdot$	
s^k	positive
s^{k-1}	$0 \approx \varepsilon$
s^{k-2}	negative
$\cdot$	
$\cdot$	

or

$\cdot$	
$\cdot$	
s^k	negative
s^{k-1}	$0 \approx \varepsilon$
s^{k-2}	positive
$\cdot$	
$\cdot$	

One sign change, i.e. there is one root on the right half s-plane from this change

Example-3 and Example 4: Use of Epsilon

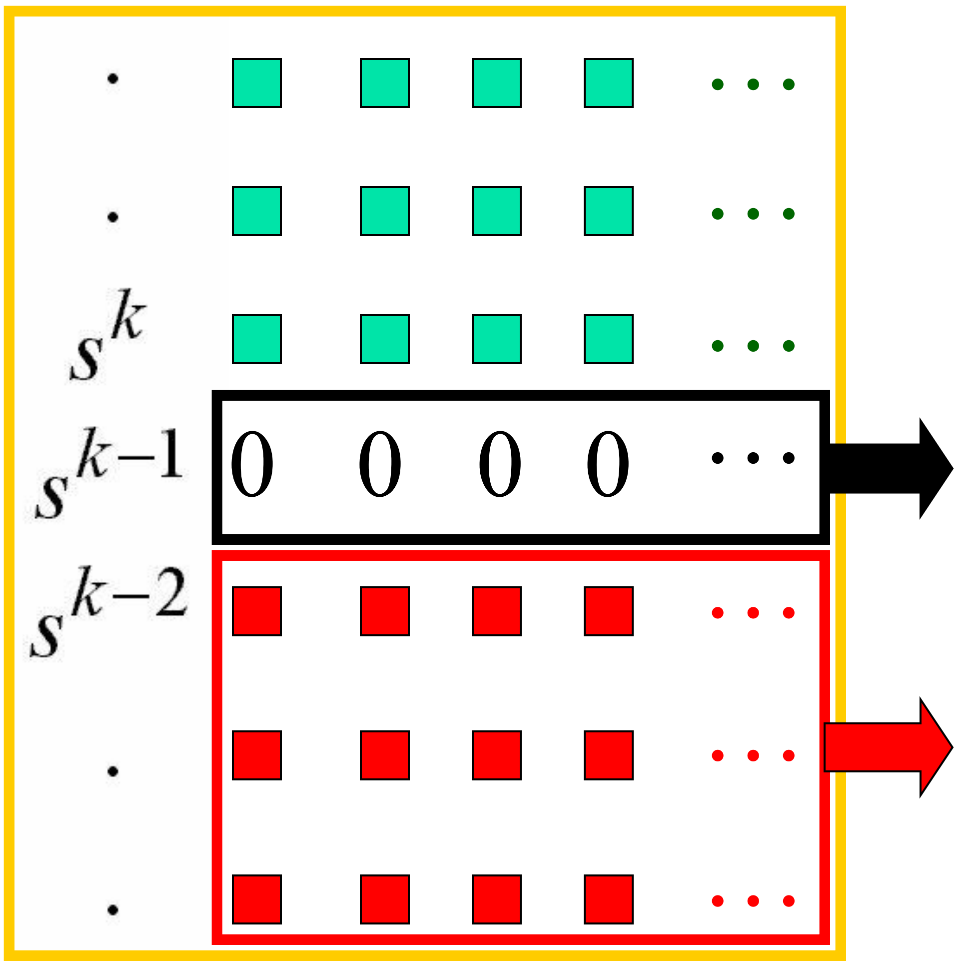
$$\boxed{D(s) = (s^2+4)(s+1) = s^3 + s^2 + 4s + 4}$$

$$\begin{array}{l|cc|} s^3 & 1 & 4 \\ s^2 & 1 & 4 \\ s^1 & 0 & \\ s^0 & * & \end{array} \Rightarrow \begin{array}{l|cc|} s^3 & 1 & 4 \\ s^2 & 1 & 4 \\ s^1 & \epsilon & \\ s^0 & -\frac{1}{\epsilon}(1\times 0 - 4\times\epsilon) & \end{array} \Rightarrow \begin{array}{l|cc|} s^3 & 1 & 4 \\ s^2 & 1 & 4 \\ s^1 & \epsilon & \\ s^0 & 4 & \end{array}$$

$$\boxed{D(s) = (s^2+4)(s-1) = s^3 - s^2 + 4s - 4}$$

$$\begin{array}{l|cc|} s^3 & 1 & 4 \\ s^2 & -1 & -4 \\ s^1 & 0 & \\ s^0 & * & \end{array} \Rightarrow \begin{array}{l|cc|} s^3 & 1 & 4 \\ s^2 & -1 & -4 \\ s^1 & -\epsilon & \\ s^0 & -\frac{-1\times 0-(-4)\times(-\epsilon)}{-\epsilon} & \end{array} \Rightarrow \begin{array}{l|cc|} s^3 & 1 & 4 \\ s^2 & -1 & -4 \\ s^1 & -\epsilon & \\ s^0 & -4 & \end{array}$$

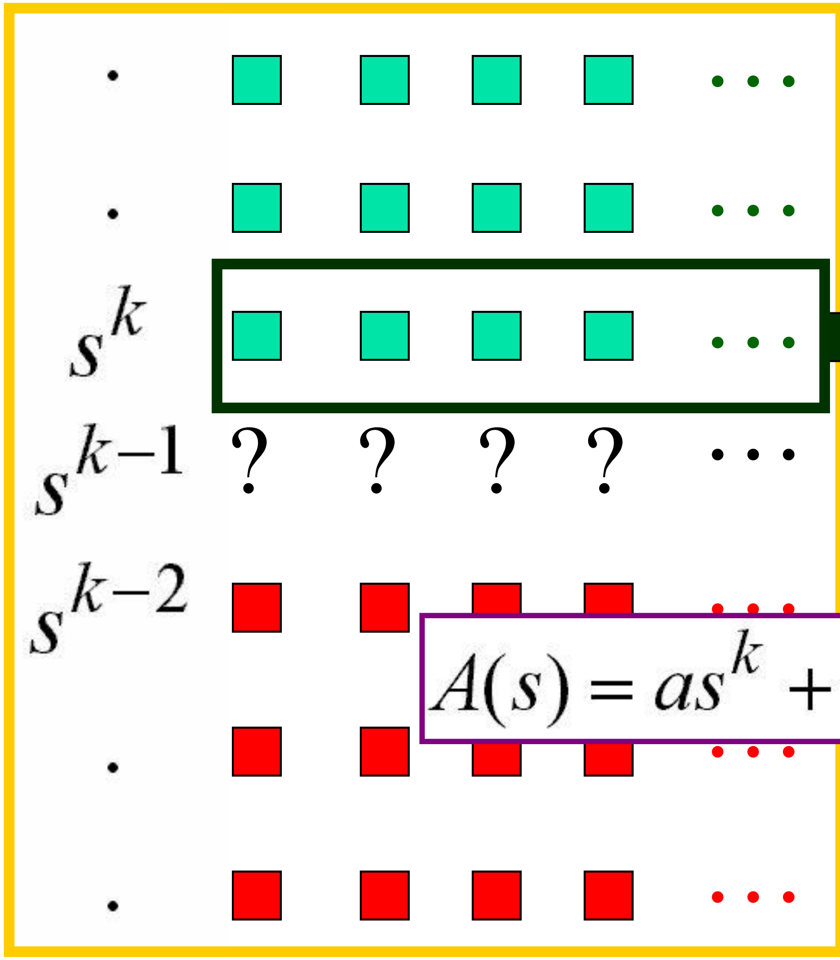
Handling the special cases
A row is entirely zero



This row is entirely zero!

You cannot proceed to calculate these terms!

Handling the special cases
A row is entirely zero



Determine the auxiliary polynomial A(s) from this row

$$A(s) = as^k + bs^{k-2} + cs^{k-4} + ds^{k-6} + \cdots$$

Handling the special cases
A row is entirely zero

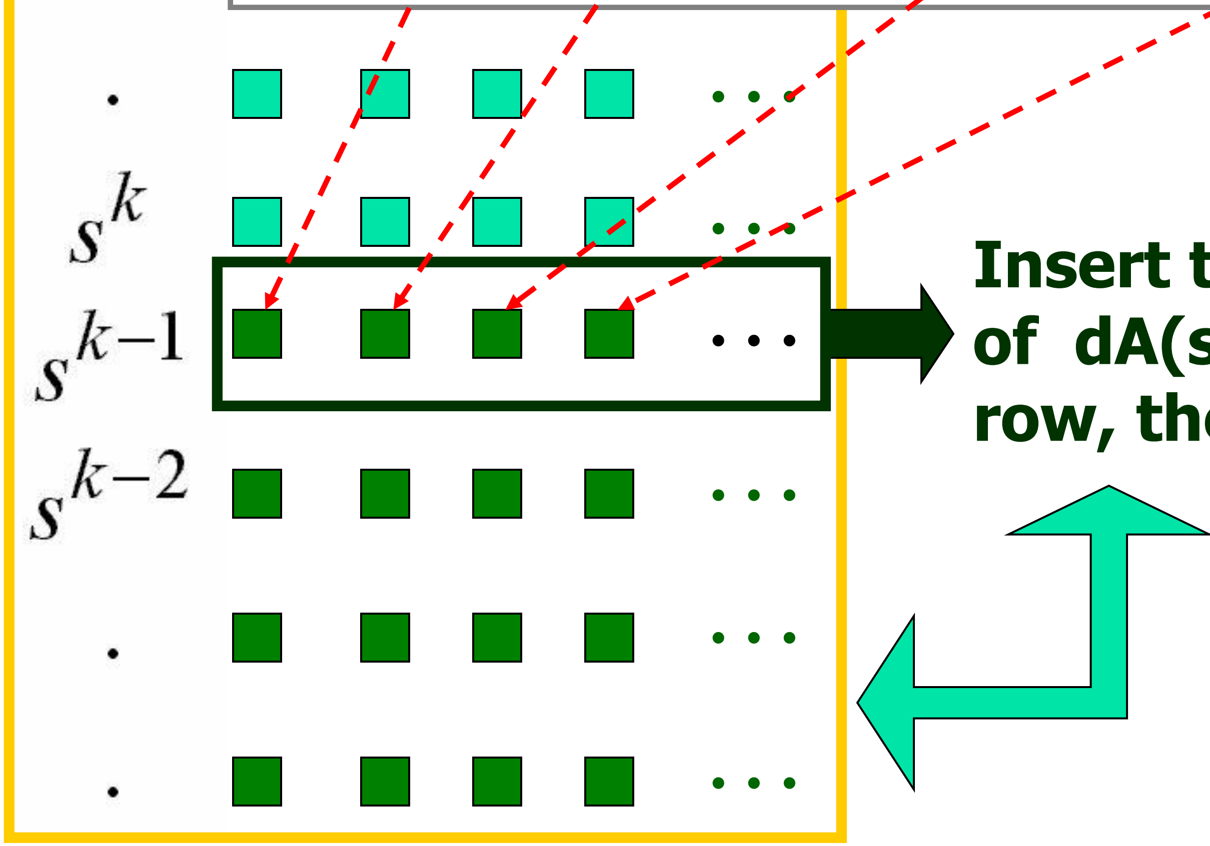
$$\begin{array}{cccccc}
\cdot & \blacksquare & \blacksquare & \blacksquare & \blacksquare & \cdots \\
\cdot & \blacksquare & \blacksquare & \blacksquare & \blacksquare & \cdots \\
s^{k} & \boxed{a} & \boxed{b} & \boxed{c} & \boxed{d} & \cdots \\
s^{k-1} & ? & ? & ? & ? & \cdots \\
s^{k-2} & \blacksquare & \blacksquare & \blacksquare & \blacksquare & \cdots \\
\cdot & \blacksquare & \blacksquare & \blacksquare & \blacksquare & \cdots \\
\cdot & \blacksquare & \blacksquare & \blacksquare & \blacksquare & \cdots
\end{array}$$

Determine the auxiliary polynomial A(s) from this row

$$\boxed{A(s) = as^{k} + bs^{k-2} + cs^{k-4} + ds^{k-6} + \cdots}$$

Handling the special cases
A row is entirely zero

$$\frac{dA(s)}{ds} = \boxed{ak}s^{k-1} + \boxed{b(k-2)}s^{k-3} + \boxed{c(k-4)}s^{k-5} + \boxed{d(k-6)}s^{k-7} + \cdots$$



Insert the coefficients of dA(s)/ds into this row, then continue...

Handling the special cases - An Example
A row is entirely zero

$$s^5 + 2s^4 + 24s^3 + 48s^2 - 25s - 50 = 0$$

$$\begin{array}{llll} s^5 & 1 & 24 & -25 \\ s^4 & 2 & 48 & -50 \\ s^3 & \boxed{0 \quad 0} & & \end{array}$$

$$A(s) = 2s^4 + 48s^2 - 50$$

$$\frac{dA(s)}{ds} = \boxed{8s^3 + 96s}$$

Handling the special cases - An Example
A row is entirely zero

$$s^5 + 2s^4 + 24s^3 + 48s^2 - 25s - 50 = 0$$

s^5	1	24	-25
s^4	2	48	-50
s^3	$\boxed{8}$	$\boxed{96}$	
s^2	24	-50	
s^1	112.6666	0	
s^0	-50		

$$\frac{dA(s)}{ds} = \boxed{8s^3 + 96s}$$

One sign change: One of the roots is in the right half s-plane

An Example

$$D(s) = s^5 + 2s^4 + 2s^3 + 4s^2 + 3s + 6$$

$$\begin{array}{c|ccc|}
s^5 & 1 & 2 & 3 \\
s^4 & 2 & 4 & 6 \\
s^3 & 0 & 0 & 0 \\
s^2 & * & * & \\
s^1 & * & & \\
s^0 & * & &
\end{array}$$

$$A(s) = 2s^4 + 4s^2 + 6$$

$$\frac{\mathrm{d}A(s)}{\mathrm{d}s} = 8s^3 + 8s$$

$$\begin{array}{c|rcc|}
s^5 & 1 & 2 & 3 \\
s^4 & 2 & 4 & 6 \\
s^3 & 8 & 8 & \\
s^2 & 2 & 6 & \\
s^1 & -16 & & \\
s^0 & 6 & &
\end{array}$$

Another Example

$$\left|\begin{array}{l|cc} s^3 & 1 & 6 \\ s^2 & 3 & 1 \\ s^1 & -\frac{1}{3}(1\times 1-6\times 3) & 0 \\ s^0 & * & \end{array}\right| \Rightarrow \left|\begin{array}{l|cc} s^3 & 1 & 6 \\ s^2 & 3 & 1 \\ s^1 & \frac{17}{3} & 0 \\ s^0 & -\frac{1}{\frac{17}{3}}(3\times 0-1\times \frac{17}{3}) & \end{array}\right| \Rightarrow \left|\begin{array}{l|cc} s^3 & 1 & 6 \\ s^2 & 3 & 1 \\ s^1 & \frac{17}{3} & 0 \\ s^0 & 1 & \end{array}\right|$$

Yet Another Example

$$
\begin{array}{c|ccc|}
s^4 & 1 & 1 & 1 \\
s^3 & -5 & -6 & 0 \\
s^2 & -\frac{1}{5} & 1 & \\
s^1 & -31 & 0 & \\
s^0 & 1 & &
\end{array}
$$

Handling the special cases – Example 3
A row is entirely zero

$$\begin{array}{l|rrr|}
s^5 & 1 & 1 & 1 \\
s^4 & 1 & -1 & -2 \\
s^3 & 2 & 3 & \\
s^2 & -\frac{5}{2} & -2 & \\
s^1 & \frac{7}{5} & & \\
s^0 & -2 & &
\end{array}$$

A Final Example

$$\begin{array}{c|cccc|}
s^6 & 1 & 4 & 1 & 1 \\
s^5 & 2 & -1 & 6 & \\
s^4 & \frac{9}{2} & -2 & 1 & \\
s^3 & -\frac{1}{9} & \frac{50}{9} & & \\
s^2 & 223 & 1 & & \\
s^1 & \frac{11\,151}{2007} & & & \\
s^0 & 1 & & &
\end{array}$$

Build the Routh Table and Find Proper K

$$(a)\quad H_1(s) = \frac{s+1}{s^2 + Ks + 1}$$

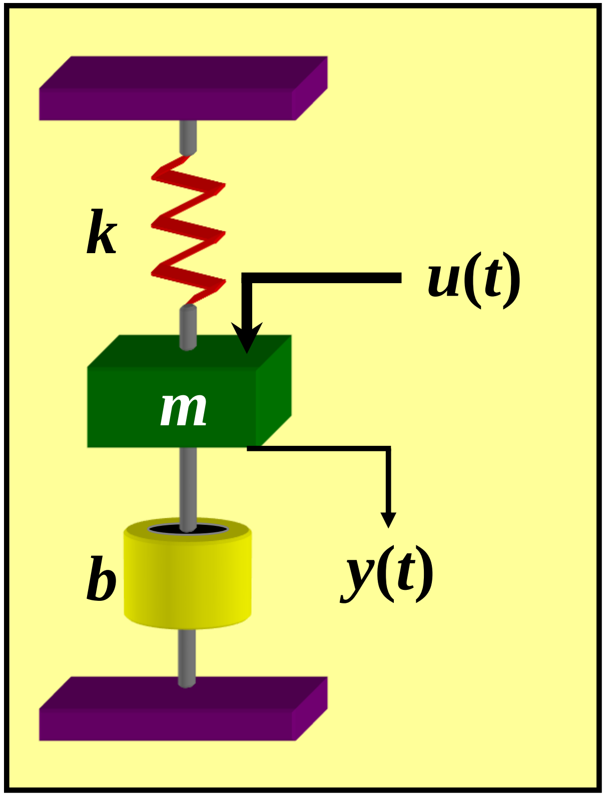
$$(b)\quad H_2(s) = \frac{s^2+1}{s^3 + 7s^2 + Ks + K}$$

$$(c)\quad H_3(s) = \frac{K(s-1)}{s^2 + (K-3)s + K + 2}$$

Final Remarks on Routh Criterion

- **The goal of using Routh stability criterion is to explain whether the characteristic equation has roots on the right half s-plane.**
- **A parameter (e.g. a gain) may change the locations of the CL poles, and Routh criterion lets us know for which range the CL system is stable.**

P-3 State Space Representation and Stability



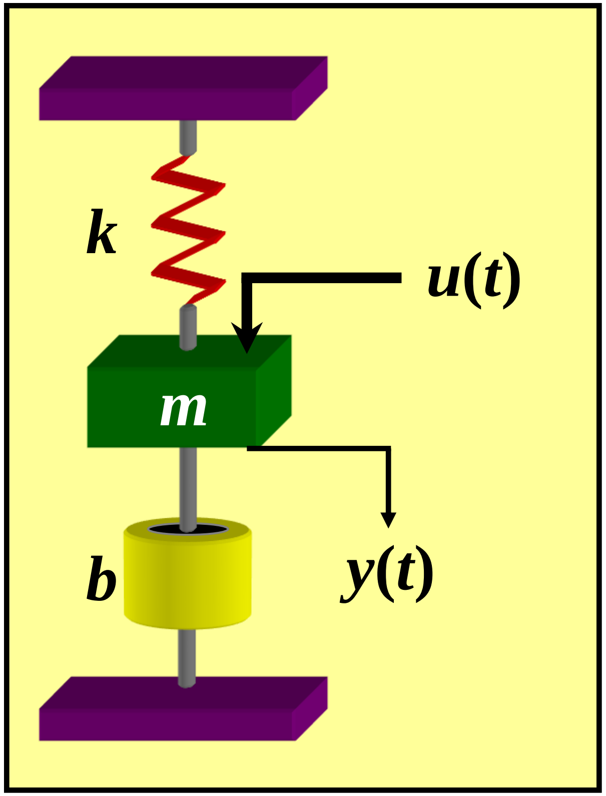
Consider the mass-spring-damper system. Laws of physics lead us to

$$\boxed{m\ddot{y} + b\dot{y} + ky = u}$$

Let us define the state as

$$x_1(t) = y(t)$$
$$x_2(t) = \dot{y}(t)$$

State Space Representation



Dynamics

$$\boxed{m\ddot{y} + b\dot{y} + ky = u}$$

State

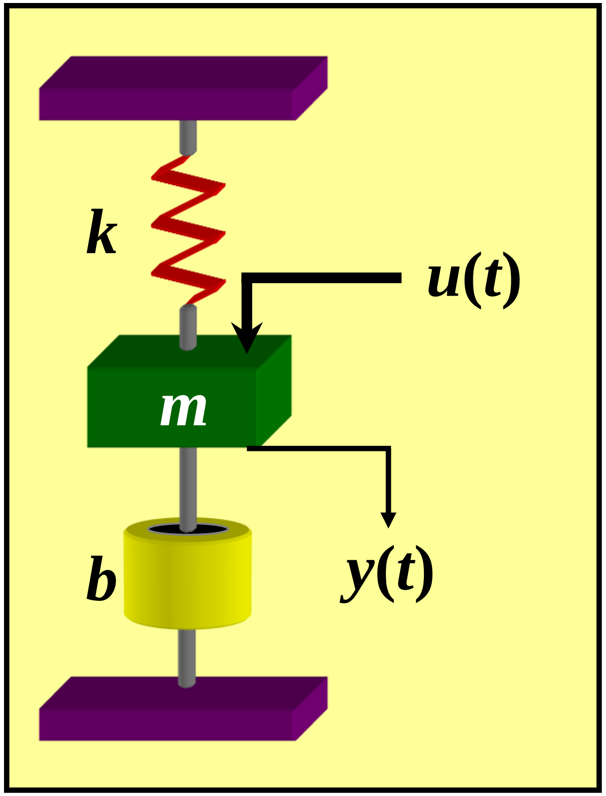
$$x_1(t) = y(t)$$
$$x_2(t) = \dot{y}(t)$$

State equation

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{k}{m}x_1 - \frac{b}{m}x_2 + \frac{1}{m}u$$

Output equation $\boxed{y = x_1}$

State Space Representation

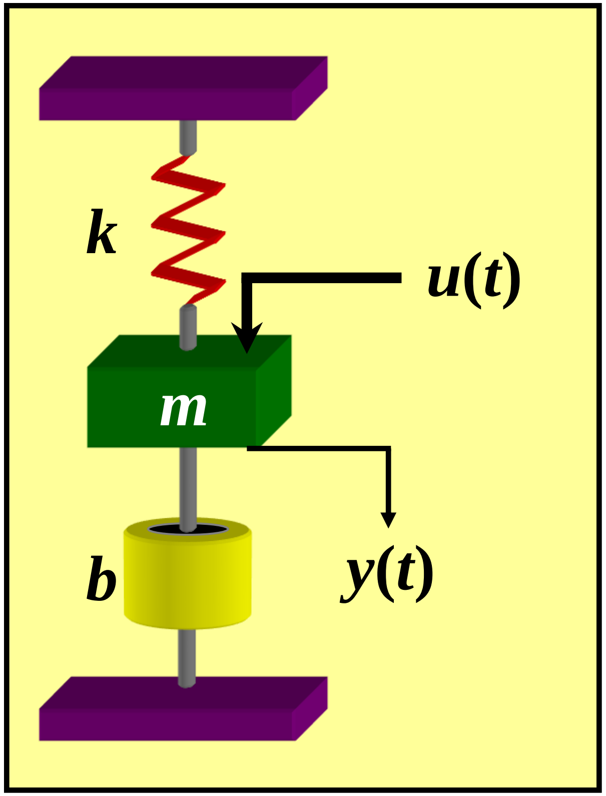


$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k/m & -b/m \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/m \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

Correlation between State Space Representations and Transfer Functions



$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

$$sX(s) - x(0) = AX(s) + BU(s)$$
$$Y(s) = CX(s) + DU(s)$$

$$(sI - A)X(s) = \cancel{x(0)} + BU(s)$$
$$Y(s) = CX(s) + DU(s)$$

$$X(s) = (sI - A)^{-1} BU(s)$$
$$Y(s) = CX(s) + DU(s)$$

Correlation between State Space Representations and Transfer Functions

$$X(s) = (sI - A)^{-1} BU(s)$$
$$Y(s) = CX(s) + DU(s)$$

$$\frac{Y(s)}{U(s)} = C(sI - A)^{-1} B + D$$

$$\frac{Y(s)}{U(s)} = \begin{bmatrix} 1 & 0 \end{bmatrix} \left\{ \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -k/m & -b/m \end{bmatrix} \right\}^{-1} \begin{bmatrix} 0 \\ 1/m \end{bmatrix} + 0$$

Transfer function

$$\frac{Y(s)}{U(s)} = \frac{1}{ms^2 + bs + k}$$

$\Longleftrightarrow$

Time Domain Dynamics

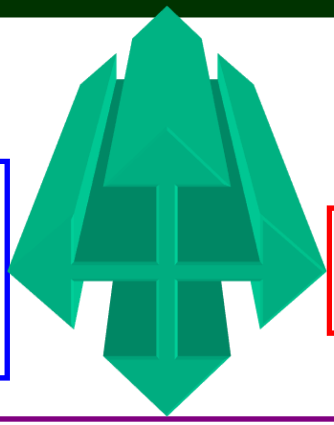
$$m\ddot{y} + b\dot{y} + ky = u$$

Relation between State Space Representations and Transfer Functions

What does this tell us?

Transfer Function

$$\frac{Y(s)}{U(s)} = \frac{1}{ms^2 + bs + k}$$



Time Domain Dynamics

$$m\ddot{y} + b\dot{y} + ky = u$$

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

State Space Representation

Relation between State Space Representations and Transfer Functions

- The dynamics of a linear system can be expressed in any of the forms
  - Differential equations
  - Transfer functions
  - State space representation

One has to note that given the TF for a system, state space representation is not unique. Different realizations can be performed.

State Space Representation

State: **The essence of past that influences the future. State is the smallest set of variables to describe the dynamics of a system**

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k/m & -b/m \end{bmatrix} \boxed{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} + \begin{bmatrix} 0 \\ 1/m \end{bmatrix} u$$

State Variables
The dimension of the state vector is fixed for a given system

State Space Representation

The dynamics of the system can uniquely be determined with the knowledge of $x_1(t_0)$, $x_2(t_0)$ and $u(t)$ for $t \geq t_0$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k/m & -b/m \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/m \end{bmatrix} u$$

The state space is a space whose axes are the states. For the above example, axes are x_1 axis and x_2 axis.

State Space Representation

In general we have a set of differential equations

$$\dot{\underline{x}} = \underline{f}(\underline{x},\underline{u},t)$$
$$\underline{y} = \underline{g}(\underline{x},\underline{u},t)$$

We linearize them and get

$$\dot{\underline{x}}(t) = A(t)\underline{x}(t) + B(t)\underline{u}(t)$$
$$\underline{y}(t) = C(t)\underline{x}(t) + D(t)\underline{u}(t)$$

The elements of the matrices may be time-varying

State Space Representation

We simply dropped the underlines. Clearly the state will be a vector if its dimension is larger than one.

$$\begin{aligned}\dot{\underline{x}} &= A\underline{x} + B\underline{u}\\ \underline{y} &= C\underline{x} + D\underline{u}\end{aligned} \quad \text{or} \quad \begin{aligned}\dot{x} &= Ax + Bu\\ y &= Cx + Du\end{aligned}$$

Or may be time invariant

State Space Representation and Stability

Assume you are given the system

$$\boxed{\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}}$$

The stability of this system can be determined by checking the eigenvalues of the matrix A

Those eigenvalues are the poles of the transfer function

$$\boxed{\frac{Y(s)}{U(s)} = C(sI - A)^{-1}B + D}$$

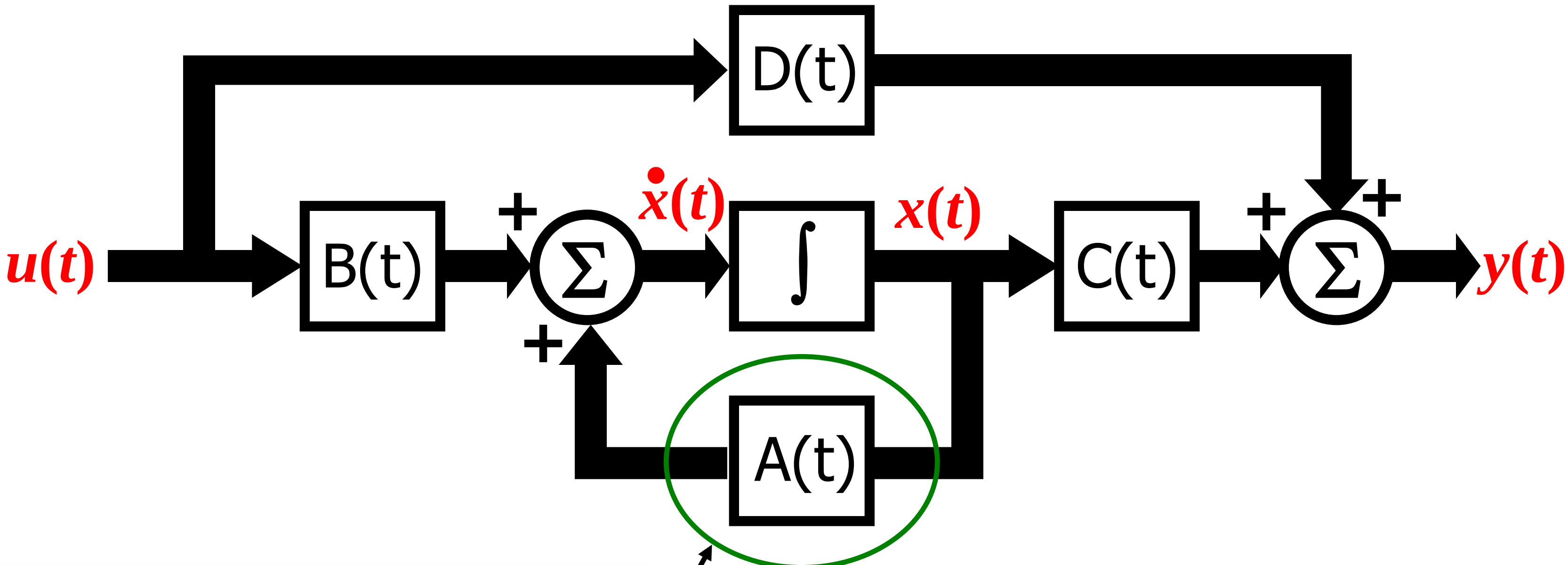
State Space Representation and Stability

$$\boxed{\mathrm{eig}\{A\} = \{\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n\}}$$

$$\boxed{|\lambda I - A| = 0}$$

- **If Re$\{\lambda_i\}<0$ for $i=1,2,\ldots,n$**
  Then the system is stable
- **If Re$\{\lambda_i\}>0$ for some i**
  Then the system is unstable
- **If Re$\{\lambda_i\}=0$ for some i**
  Then the system has poles on the imaginary axis

State Space Representation and Stability
In summary...



Check the real parts of the eigenvalues of A(t)

An Example on Stability

$$\dot{x}_1 = x_2 - x_3$$
$$\dot{x}_2 = -x_1 + x_2 + x_3$$
$$\dot{x}_3 = ax_1 + x_2 - x_3 + u$$

$$y = x_1$$
$$T(s) = \frac{Y(s)}{U(s)} = \frac{X_1(s)}{U(s)}$$

Determine the range of a for stability

An Example on Stability

$$\dot{x}_1 = x_2 - x_3$$
$$\dot{x}_2 = -x_1 + x_2 + x_3$$
$$\dot{x}_3 = ax_1 + x_2 - x_3 + u$$

$$y = x_1$$
$$T(s) = \frac{Y(s)}{U(s)} = \frac{X_1(s)}{U(s)}$$

$$sX_1(s) = X_2(s) - X_3(s)$$
$$sX_2(s) = -X_1(s) + X_2(s) + X_3(s)$$
$$sX_3(s) = aX_1(s) + X_2(s) - X_3(s) + U(s)$$

An Example on Stability

$$sX_1(s) = X_2(s) - X_3(s)$$

$$sX_2(s) = -X_1(s) + X_2(s) + X_3(s)$$

$$sX_3(s) = aX_1(s) + X_2(s) - X_3(s) + U(s)$$

$$\boxed{X_3(s) = X_2(s) - sX_1(s)}$$

$$\boxed{X_2(s) = -\frac{s+1}{s-2}X_1(s)}$$

An Example on Stability

$$X_3(s) = X_2(s) - sX_1(s)$$

$$X_2(s) = -\frac{s+1}{s-2}X_1(s)$$

$$X_3(s) = -\frac{s^2-s+1}{s-2}X_1(s)$$

An Example on Stability

$$sX_1(s) = X_2(s) - X_3(s)$$

$$sX_2(s) = -X_1(s) + X_2(s) + X_3(s)$$

$$sX_3(s) = aX_1(s) + X_2(s) - X_3(s) + U(s)$$

$$X_2(s) = -\frac{s+1}{s-2}X_1(s)$$

$$X_3(s) = -\frac{s^2-s+1}{s-2}X_1(s)$$

An Example on Stability

$$T(s) = \frac{2-s}{s^3 + (a-1)s - 2a}$$

$$\begin{array}{lcc}
s^3 & 1 & a-1 \\
s^2 & 0 & -2a \\
s^1 & & \\
s^0 & &
\end{array}$$

Remember what to do now!

An Example on Stability

$$\boxed{T(s) = \frac{2-s}{s^3 + (a-1)s - 2a}}$$

$$\boxed{\begin{array}{lcc}
s^3 & 1 & a-1 \\
s^2 & \varepsilon & -2a \\
s^1 & \left[\varepsilon(a-1)+2a\right]/\varepsilon & 0 \\
s^0 & -2a &
\end{array}}$$

An Example on Stability

$$\boxed{T(s)=\frac{2-s}{s^3+(a-1)s-2a}}$$

$$\begin{array}{l|cc}
s^3 & 1 & a-1 \\
s^2 & \varepsilon & -2a \\
s^1 & a(1+2/\varepsilon)-1 & 0 \\
s^0 & -2a &
\end{array}$$

This term becomes negative

$a < 0$

An Example on Stability

$$T(s)=\frac{2-s}{s^3+(a-1)s-2a}$$

$$\begin{array}{lcc} s^3 & 1 & a-1 \\ s^2 & \varepsilon & -2a \\ s^1 & a(1+2/\varepsilon)-1 & 0 \\ s^0 & -2a & \end{array}$$

a > 1/[1+2/ε]

ε > 0 and ε ≈ 0

a > 0

This term becomes negative

An Example on Stability

$$T(s) = \frac{2-s}{s^3 + (a-1)s - 2a}$$

$$A = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 1 & 1 \\ a & 1 & -1 \end{bmatrix}$$

The system is unstable regardless of the value of a. In other words, A has at least one eigenvalue in the right half s-plane

Can this system have poles on the imaginary axis?

$$T(s)=\frac{2-s}{s^3+(a-1)s-2a}$$

Assume the answer is yes… Then for $s=j\alpha$ the denominator must be zero, i.e.

$$(j\alpha)^3+(a-1)(j\alpha)-2a=0$$

$$j\left(-\alpha^3+(a-1)\alpha\right)-2a=0$$

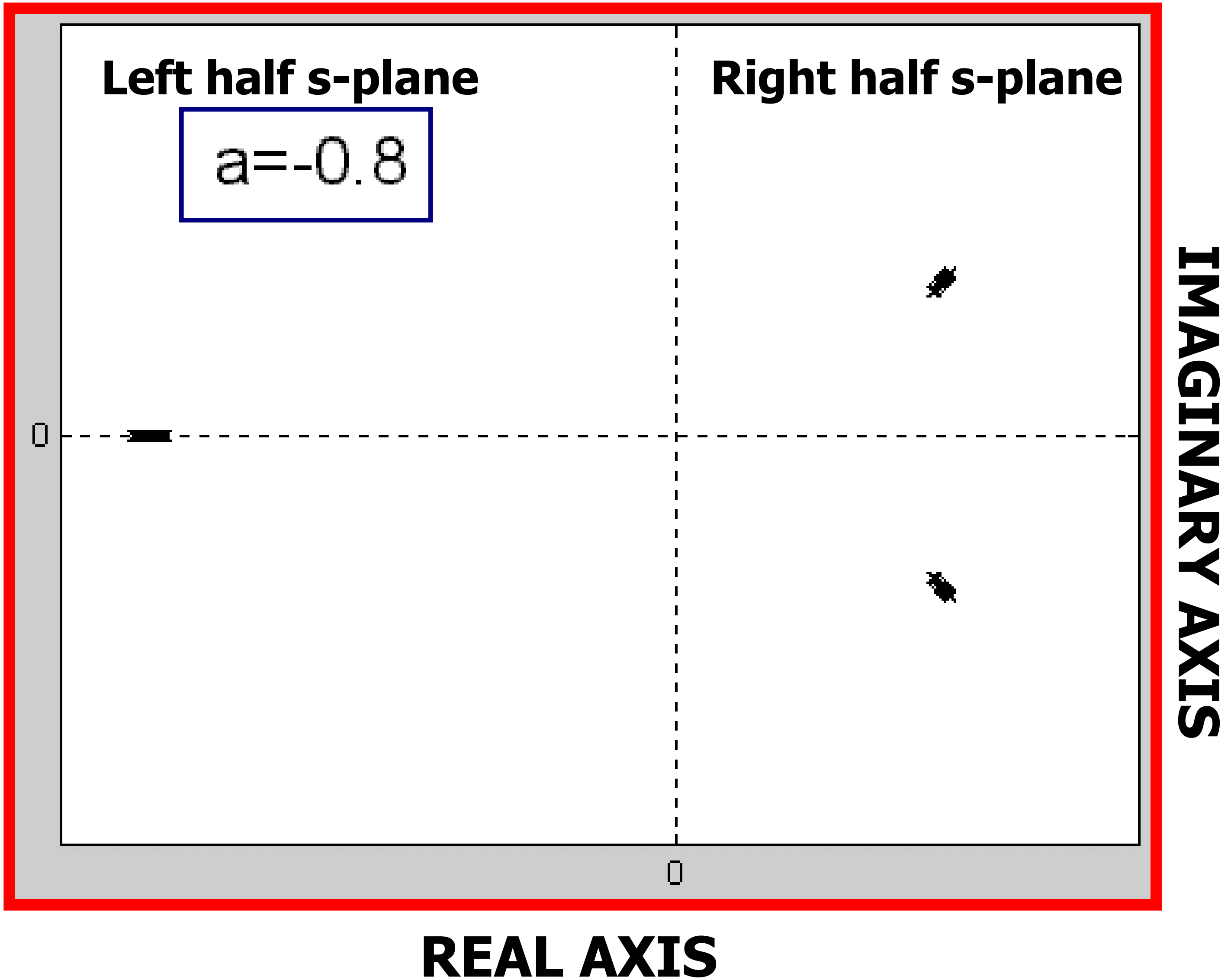
No value of a can lead to zero real and imaginary parts simultaneously

Can this system have complex conjugate poles on the imaginary axis?

$$T(s) = \frac{2-s}{s^3 + (a-1)s - 2a}$$

The answer is no. Only one pole passes through the origin when a=0.

Watch now...



Left half s-plane
Right half s-plane
a=-0.8
0
0
IMAGINARY AXIS
REAL AXIS