BBM 205 Discrete Mathematics
Hacettepe University
http://web.cs.hacettepe.edu.tr/~bbm205

Lecture 3: Method of Induction

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Resources:
Kenneth Rosen, "Discrete Mathematics and App."
cs.colostate.edu/ cs122/.Spring15/home_resources.php
http://www.cs.nthu.edu.tw/ wkhon/math16.html

The principle of (ordinary) induction

Let $P(n)$ be a predicate. If

1. $P(0)$ is true, and
2. $P(n)$ IMPLIES $P(n+1)$ for all non-negative integers n

then

- ▷ $P(m)$ is true for all non-negative integers m

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1. The first item says that $P(0)$ holds
2. The second item says that $P(0) \to P(1)$, and $P(1) \to P(2)$, and $P(2) \to P(3)$, *etc.*

- ▷ Intuitively, there is a domino effect that eventually shows that $\forall n \in \mathbb{N}.\ P(n)$

Proof by induction

To prove by induction $\forall k \in \mathbb{N}.\ P(k)$ is true, follow these three steps:

Base Case: Prove that $P(0)$ is true

Inductive Hypothesis: Let $k \geq 0$. We assume that $P(k)$ is true

Inductive Step: Prove that $P(k+1)$ is true

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Remark

Proofs by mathematical induction do not always start at the integer 0. In such a case, the base case begins at a starting point $b \in \mathbb{Z}$. In this case we prove the property only for integers $\geq b$ instead of for all $n \in \mathbb{N}$

$$\forall k \in \;. \;\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$$

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(By induction) Let $P(k)$ be the predicate "$\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$"

Base Case: $\sum_{i=1}^{0} i = 0 = \frac{0(0+1)}{2}$, thus $P(0)$ is true

Inductive Hypothesis: Let $k \geq 0$. We assume that $P(k)$ is true, *i.e.* $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$

Inductive Step:

$$\begin{aligned} \sum_{i=1}^{k+1} i &= \left[\sum_{i=1}^{k} i\right] + (k+1) \\ &= \frac{k(k+1)}{2} + (k+1) \qquad \text{(by I.H.)} \\ &= \frac{k(k+1)+2(k+1)}{2} \\ &= \frac{(k+1)(k+2)}{2} \end{aligned}$$

Thus $P(k+1)$ is true $\square$

$\forall k \in \mathbb{N}.\ k^3 - k$ **is divisible by** 3

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(By induction) Let $P(k)$ be the predicate "$k^3 - k$ is divisible by 3"

Base Case: Since $0 = 3 \cdot 0$, it is the case that 3 divides $0 = 0^3 - 0$, thus $P(0)$ is true

Inductive Hypothesis: Let $k \geq 0$. We assume that $P(k)$ is true, *i.e.* $k^3 - k$ is divisible by 3

Inductive Step:

$$\begin{aligned}
(k+1)^3 - (k+1) &= (k^3 + 3k^2 + 3k + 1) - (k+1) \\
&= k^3 + 3k^2 + 2k \\
&= (k^3 - k) + 3k^2 + 3k \\
&= 3(\ell + k^2 + k) \text{ for some } \ell \qquad \text{(by I.H.)}
\end{aligned}$$

Thus $(k+1)^3 - (k+1)$ is divisible by 3. So we can conclude that $P(k+1)$ is true $\square$

$\forall k \geq 4.\ 2^k < k!$

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(By induction) Let $P(k)$ be the predicate "$2^k < k!$"

Base Case: $2^4 = 16 < 24 = 4!$, thus $P(4)$ is true

Inductive Hypothesis: Let $k \geq 4$. We assume that $P(k)$ is true, *i.e.* $2^k < k!$

Inductive Step:

$$\begin{aligned} 2^{k+1} &= 2 \cdot 2^k \\ &< 2 \cdot k! && \text{(by I.H.)} \\ &< (k+1) \cdot k! && (k \geq 4) \\ &= (k+1)! \end{aligned}$$

Thus $P(k+1)$ is true $\square$

Mathematical Induction

- Next, we are going to show the following two statements to be true:

1. $P(1)$, called basic step
2. $\forall n\ (P(n) \rightarrow P(n+1))$, called inductive step, where domain of n is all positive integers

- If both can be shown true, then we can conclude that $\forall n\ P(n)$ is true [why?]

Correctness of Mathematical Induction

- The correctness is based on the following axiom on positive integers:

> Well-Ordering Property :
>
> Every non-empty collection of non-negative integers has a smallest element

- Using well-ordering property, we can prove that mathematical induction is correct

Correctness of Mathematical Induction

- Proof :

  Suppose on the contrary that the two statements are true, but the conclusion $\forall n\, P(n)$ is not true.

  Then $\exists\, n \neg P(n)$, so that by the well-ordering property, there is a smallest k with $\neg P(k)$ is true.

  This k cannot be 1 (by basic step). Then, $k - 1$ is positive, so that $P(k - 1)$ is true (by the choice of k).

  Thus $P(k)$ is true (by $P(k - 1)$ and inductive step), and a contradiction occurs.

Back to the Example

- We let

  $P(n)$:= "The sum of first n positive odd integers is n^2"

  and we hope to use mathematical induction to show $\forall n\, P(n)$ is true

- Can we show the basic step to be true?
- Can we show the inductive step to be true?

Back to the Example

- Can we show the basic step to be true?

- The basic step is P(1), which is :

  P(1) := “The sum of first 1 positive odd integers is 1^2”

  This is obviously true.

Back to the Example

- Can we show the inductive step to be true?
- The inductive step is $\forall n\, (P(n) \rightarrow P(n+1))$
- To show it is true, we focus on an arbitrary chosen k, and see if $P(k) \rightarrow P(k+1)$ is true
  - If so, by universal generalization,

$$\forall n\, (P(n) \rightarrow P(n+1)) \text{ is true}$$

Back to the Example

- Suppose that P(k) is true. That is,

  P(k) := "The sum of first k positive odd integers is k^2"

  This implies $1 + 3 + \ldots + (2k - 1) = k^2$.

  Then, we have

  $$1 + 3 + \ldots + (2k - 1) + (2k + 1) = k^2 + (2k + 1) = (k + 1)^2,$$

  so that P(k+1) is true if P(k) is true.

Remark

- Note : When we show that the inductive step is true, we do not show $P(k+1)$ is true.

  Instead, we show the conditional statement

  $$P(k) \rightarrow P(k+1) \text{ is true.}$$

  This allows us to use $P(k)$ as the premise, and gives us an easier way to show $P(k+1)$

- Once basic step and inductive step are proven, by mathematical induction, $\forall n\, P(n)$ is true

Remark

- Mathematical induction is a very powerful technique, because we show just two statements, but this can imply infinite number of cases to be correct
- However, the technique does not help us find new theorems. In fact, we have to obtain the theorem (by guessing) in the first place, and induction is then used to formally confirm the theorem is correct

More Examples

- Ex 1: Show that for all positive integer n,

$$n < 2^n$$

- Ex 2: Show that for all positive integer n,

$$n^3 - n \text{ is divisible by } 3$$

- Ex 3: Show that for all positive integer n,

$$1^2 + 2^2 + 3^2 + \ldots + n^2 = n(n+1)(2n+1) / 6$$

Using a Different Basic Step

- When we apply the induction technique, it is not necessary to have $P(1)$ as the basic step

- We may replace the basic step by $P(k)$ for some fixed k. If both basic step and inductive step are true, this will imply that

$$\forall n \geq k \, (P(n))$$

More Examples

- Ex 4: Show that for all positive integer $n \geq 4$,

$$2^n < n!$$

- Ex 5: Show that for all non-negative integer n,

$$1 + 2 + \ldots + 2^n = 2^{n+1} - 1$$

- Ex 6: Show that for non-negative integer n,

$$7^{n+2} + 8^{2n+1} \text{ is divisible by } 57$$

Interesting Examples

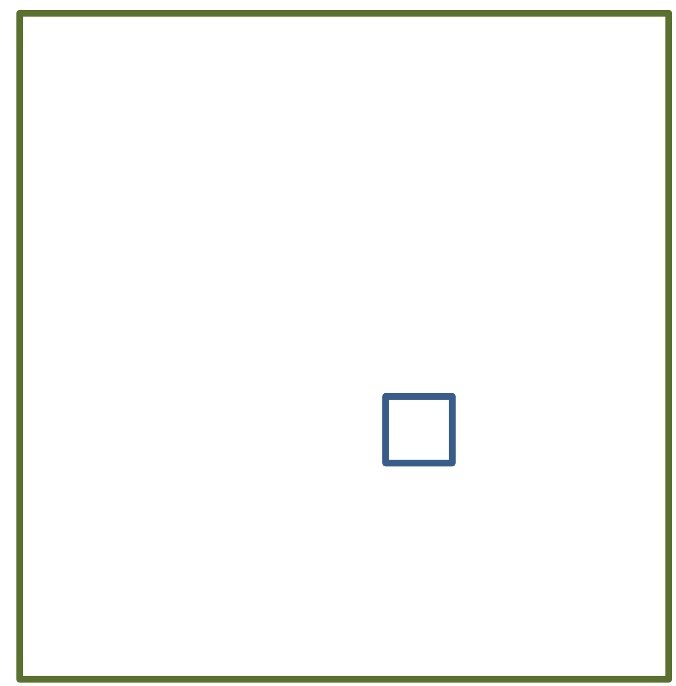
Snowball Fight

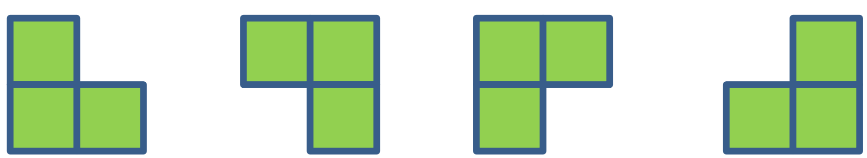
- There are $2n + 1$ people
- Each must throw to the nearest
- All with distinct distance apart
- Show that at least one is not hit by any snowball

Interesting Examples

Tiling (Again!)

- A big square of size $2^n \times 2^n$
- Somewhere inside, a 1×1 small square is removed
- Show that the remaining board can always be tiled by L-shaped dominoes :





each consists of three 1×1 squares

Strong Induction

- An alternative form of induction, called **strong induction**, uses a different inductive step:

$$\forall n \; ((P(1) \wedge P(2) \wedge \ldots \wedge P(n)) \rightarrow P(n+1))$$

- The basic step is still to prove $P(1)$ to be true
- Again, if both the basic and inductive steps are true, then we can conclude that $\forall n \; P(n)$ is true [how?]

Examples

- Ex 1:

  Define the n^{th} Fibonacci number, F_n, as follows:

  $$F_0 = 1, \quad F_1 = 1,$$

  $$F_n = F_{n-1} + F_{n-2}, \quad \text{when } n \geq 2$$

  By the above recursive definition, we get the first few Fibonacci numbers :

  1, 1, 2, 3, 5, 8, 13, 21, 34, 55, …

Examples

- Ex 1 (continued):

  Show that F_n can be computed by the formula

$$F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

Examples

- Ex 2: Quicksort is a recursive algorithm for sorting a collection of distinct numbers :
  1. If there is at most 1 number to sort, done
  2. Else, pick any number x from the collection, and use x to divide the remaining numbers into two groups:

     those smaller than x, those larger than x.

     Next, apply Quicksort to sort each group (putting x in-between afterwards)

Examples

- For instance, suppose the input collection of numbers contains 1, 4, 3, 10, 7, 2
- First round, say we pick $x = 3$
- Then we will form two groups S and L:

$$S = \{ 1, 2 \} \text{ and } L = \{ 4, 10, 7 \}$$

- After that, we apply Quicksort on each group, and in the end, we report

Quicksort(S), x, Quicksort(L)

Examples

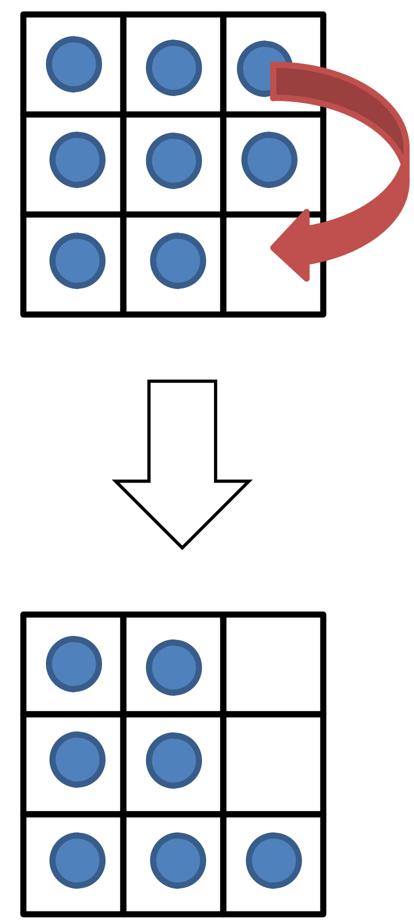
- Ex 2 (continued):

  Show that Quicksort can correctly sort any collection of n distinct numbers

Interesting Example

Peg Solitaire

- There are pegs on a board
- A peg can jump over another one into an adjacent empty square, so that the jumped-over peg is eliminated
- Target: Can we eliminate all but one peg ?



Interesting Example

- Show that if we start with $n \times n$ pegs (arranged as a square) on a board with infinite size, and n is not divisible by 3, then we can eliminate all but one peg
- Hint: Let $P(n)$ denote the above proposition. Show that $P(1)$ and $P(2)$ are true, and for all n, $P(3n+1) \rightarrow P(3n+5)$, $P(3n+2) \rightarrow P(3n+4)$ are true

Common Mistakes

- Show that

  P(n) = "any n cats will have the same color"

  is true for all positive integer n.

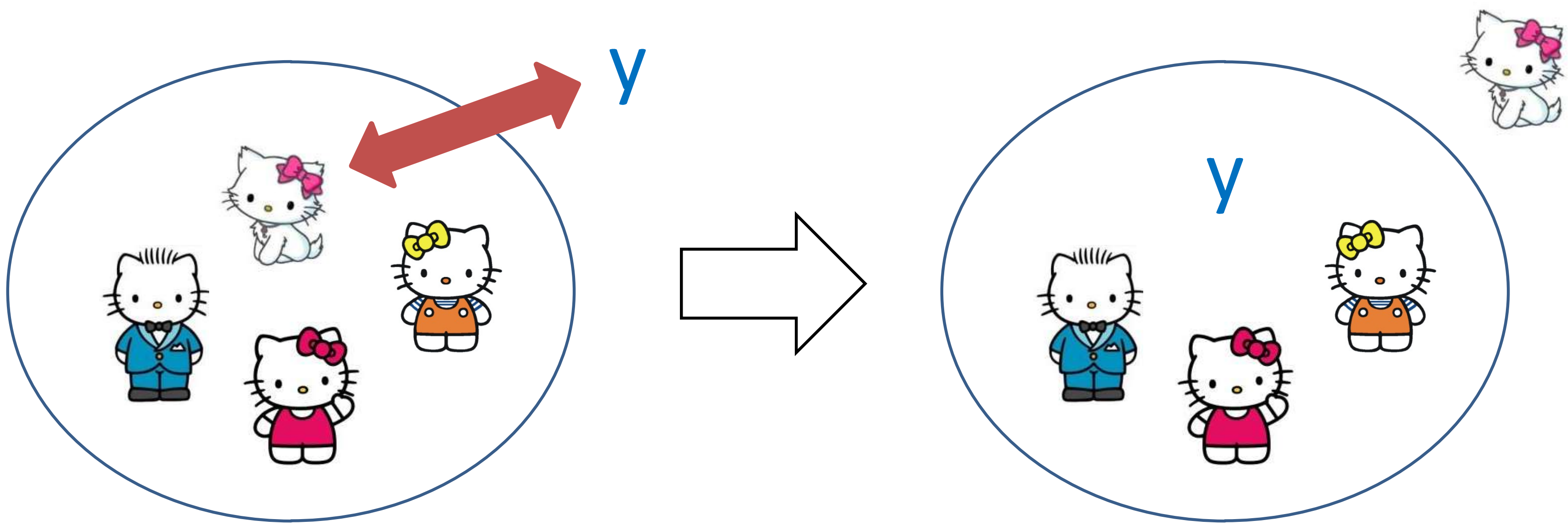
- Proof: The basic step P(1) is obviously true.

  Next, assume P(k) is true. Then, when we have k + 1 cats, we can remove one of them, say y, so that by P(k), they will have the same color

Common Mistakes

- Proof (continued):

  Now, we exchange the removed cat with one of the other k cats :



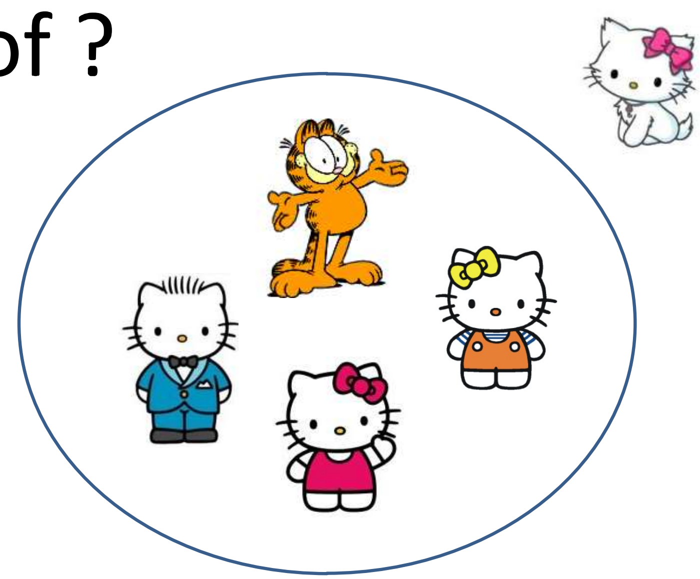
Common Mistakes

- Proof (continued):

  Then, by P(k) again, y must have the same color as the other k – 1 cats.

  This implies all the cats are of the same color!

- What's wrong with the proof ?



Every natural number $k > 1$ can be written as a product of primes

(By induction) Let $P(k)$ be the predicate "k can be written as a product of primes"

Base Case: Since 2 is a prime number, $P(2)$ is true

Inductive Hypothesis: Let $k \geq 1$. We assume that $P(k)$ is true, *i.e.* "k can be written as a product of primes"

Inductive Step: We distinguish two cases: *(i)* Case $k+1$ is a prime, then $P(k+1)$ is true; *(ii)* Case $k+1$ is not a prime. Then by definition of primality, there must exist $1 < n, m < k+1$ such that $k+1 = n \cdot m$. But then we know by I.H. that n and m can be written as a product of primes (since $n, m \leq k$). Therefore, $k+1$ can also be written as a product of primes. Thus, $P(k+1)$ is true

$\square$

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$\square$

$\longrightarrow$ If we had only assumed $P(k)$ to be true, then we could not apply our I.H. to n and m