BBM205 Midterm Exam I

Time: 16:00-18:00

11 November 2020

Question:	1	2	3	4	5	6	7	Total
Points:	16	18	11	13	15	10	14	97
Score:								

1. (16 points) Assume that you have a statement P(x) for a number x and you aim to show the following:

$$\exists x (P(x) \implies Q(x))$$

Which of the proof ideas would be valid to prove the above statement?

Solution:

(a) (4 points) True: Show there exists an x such that if Q(x) is false, then P(x) is false.

(b) (4 points) False: Assume that there exists an x such that P(x) is true and Q(x) is false and derive a contradiction.

(c) (4 points) True: Find an x such that Q(x) is true or P(x) is false.

(d) (4 points) True: Assume that for all x, P(x) is true and Q(x) is false, and derive a contradiction.

2. Determine by using a truth table whether the following equivalences are true or false.

$$(x \implies y) \equiv (x \lor \neg y)$$

$$(x \implies y) \equiv (\neg x \implies \neg y)$$

(c) (6 points)
$$(x \implies y) \equiv ((y \land x) \lor \neg x)$$

Solution: Left as an exercise.

3. (11 points) Show that the number below is irrational.

$$\sqrt{x}$$

Take the value of x depending on your meeting number as follows:

(Meeting Number,
$$x$$
) = { $(1, 11), (2, 7), (3, 5), (4, 3)$ }

Say, if you are in meeting 1, then you will take x=11, and similarly for others.

Solution: The proof is exactly the same as for x = 2, which we covered in the lecture notes. The only difference is that instead of 2, there is x.

4. (a) (5 points) Determine the following numbers by showing all your work:

$$2^{2020} \mod 7$$
, $3^{2020} \mod 7$, $4^{2020} \mod 7$

(b) (5 points) Determine the following numbers by showing all your work:

$$2^{2020} \mod 8$$
, $3^{2020} \mod 8$, $4^{2020} \mod 8$

(c) (3 points) Write a necessary and sufficient condition for the multiplicative inverse of a number k to exist in modulo n.

Solution:

(a)
$$2^0 \equiv 1 \mod 7, \quad 3^0 \equiv 1 \mod 7, \quad 4^0 \equiv 1 \mod 7, \\ 2^1 \equiv 2 \mod 7, \quad 3^1 \equiv 3 \mod 7, \quad 4^1 \equiv 4 \mod 7, \\ 2^2 \equiv 4 \mod 7, \quad 3^2 \equiv 2 \mod 7, \quad 4^2 \equiv 2 \mod 7, \\ 2^3 \equiv 1 \mod 7, \dots \quad 3^3 \equiv 6 \mod 7, \quad 4^3 \equiv 1 \mod 7, \\ 3^4 \equiv 4 \mod 7, \quad 3^5 \equiv 5 \mod 7, \\ 3^6 \equiv 1 \mod 7, \dots$$

Since the pattern (1,2,4) repeats in $2^i \mod 7$ and $2020 \equiv 1 \mod 3$, we have $2^{2020} \mod 7 \equiv 2^1 \mod 7 \equiv 2$.

Since the pattern (1, 3, 2, 6, 4, 5) repeats in $3^i \mod 7$ and $2020 \equiv 4 \mod 6$, we have $3^{2020} \mod 7 \equiv 3^4 \mod 7 \equiv 4$.

Since the pattern (1,4,2) repeats in $4^i \mod 7$ and $2020 \equiv 1 \mod 3$, we have $4^{2020} \mod 7 \equiv 4^1 \mod 7 \equiv 4$.

(b) $2^0 \equiv 1 \mod 8, \quad 3^0 \equiv 1 \mod 8, \quad 4^0 \equiv 1 \mod 8, \\ 2^1 \equiv 2 \mod 8, \quad 3^1 \equiv 3 \mod 8, \quad 4^1 \equiv 4 \mod 8, \\ 2^2 \equiv 4 \mod 8, \quad 3^2 \equiv 1 \mod 8, \quad 4^2 \equiv 0 \mod 8, \\ 2^3 \equiv 0 \mod 8, \quad 3^3 \equiv 3 \mod 8, ... \quad 4^3 \equiv 0 \mod 8, ...$ $2^4 \equiv 0 \mod 8, ...$

So, $2^{2020} \mod 8 \equiv 0$, and $4^{2020} \mod 8 \equiv 0$.

Since the pattern (1,3) repeats and $2020 \equiv 0 \mod 2$, we have $3^{2020} \mod 8 \equiv 3^0 \mod 7 \equiv 1$.

- (c) The condition is that gcd(k, n) = 1.
- 5. The bipartite graph G has parts A and B with labelled vertices as below:

$$A = \{10a + 1, 10a + 2, 10a + 3, 10a + 4, 10a + 5\},\$$

$$B = \{10a + 6, 10a + 7, 10a + 8, 10a + 9, 10a + 10\},\$$

The (undirected) edge set is defined as

$$E = \{xy : x \in A, y \in B, \gcd(x, y) = 1\}$$

In this graph, find

- (a) (5 points) a maximal matching and explain why it is maximal.
- (b) (5 points) a maximum matching.
- (c) (5 points) a matching that is maximal but not maximum (if it exists).

In this question, take the value of a depending on your meeting number as below: (Meeting Number, a) = $\{(1,0),(2,1),(3,2),(4,3)\}$ Say, if you are in meeting 2, then you will take a=1, and similarly for others.

Solution: Say a=2. The solutions for a=0,1,3 can be solved using the same idea.

(a) (5 points) Then the edges below comprise a maximal matching:

$$\{(21,29),(23,30),(25,27)\}.$$

There are no edges between the vertices 22, 24, 26 and 28, because they have a common divisor 2. Hence, this matching is maximal.

(b) (5 points) The edges below comprise a maximum matching:

$$\{(21, 26), (22, 27), (23, 30), (24, 29), (25, 28)\}.$$

- (c) (5 points) Same as in part (a).
- 6. (10 points) Exhibit two nonisomorphic, connected graphs with x vertices and x edges. In this question, take the value of x depending on your meeting number as follows: (Meeting Number, x)=(1,4), (2,5), (3,6), (4,7) Say, if you are in meeting 3, then you will take x=6, and similarly for others.

Solution: For all parts, the cycle C_x and the cycle C_{x-1} with an edge attached to it are nonisomorphic graphs with x edges and x vertices.

7. In this question, take the value of "a" depending on your meeting number as follows:

(Meeting Number, a) =
$$\{(1, 2), (2, 3), (3, 2), (4, 3)\}$$

Say, if you are in meeting 4, then you will take a=3, and similarly for others.

(a) (5 points) Find the smallest positive integer n, for which the following is true.

$$n! > a^n$$

(b) (9 points) Call the number found in the first part k. Show by using induction that $n! \ge a^n$ for all $n \ge k$.

Solution: The solutions for a=2,3 use similar ideas, so only the solution for a=3 is provided below.

(a)

$$1! = 1$$
 $3^{1} = 3$
 $2! = 2$ $3^{2} = 9$
 $3! = 6$ $3^{3} = 27$
 $4! = 24$ $3^{4} = 81$
 $5! = 120$ $3^{5} = 243$
 $6! = 720$ $3^{6} = 729$
 $7! = 5040$ $3^{7} = 2187$

Hence, the smallest n=7.

(b) Base step: See part (a). Inductive Step: Use the hypothesis that $n! \geq 3^n$ to show that $(n+1)! \geq 3^{n+1}$. We have

$$(n+1)! = (n+1)n! \ge (n+1)3^n \ge 8 \cdot 3^n \ge 3^{n+1},$$

where the first inequality holds by the inductive hypothesis $n! \geq 3^n$.