BBM 205 Discrete Mathematics
Hacettepe University
http://web.cs.hacettepe.edu.tr/∼bbm205

Lecture 8: Connectivity, Euler and Hamilton Paths,
Graph Coloring, Planar Graphs

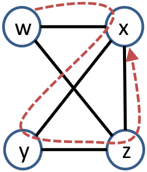
Lecturer: Lale Özkahya

Resources:
Kenneth Rosen, "Discrete Mathematics and App."
http://www.inf.ed.ac.uk/teaching/courses/dmmr
http://www.cs.nthu.edu.tw/ wkhon/math16.html

What is a Path ?

A path is a sequence of edges that begins at a vertex, and travels from vertex to vertex along edges of the graph. The number of edges on the path is called the length of the path.

- Ex : Consider the graph on the right.
  w → x → y → z → x corresponds to a path of length 4



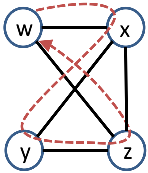
What is a Path ?

If a path begins and ends at the same vertex, the path is also called a circuit.

- Ex : Consider the graph on the right.

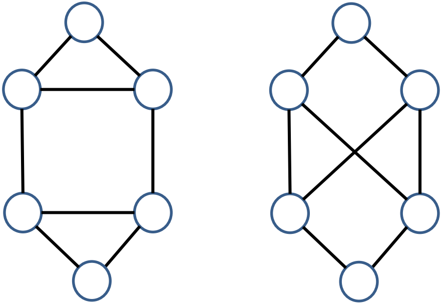
  w → x → y → z → w

  gives to a circuit of length 4



Paths and Isomorphism

Q: How to show that the following graphs are not isomorphic ?

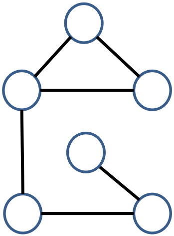


A: One contains a circuit of length 3 (a triangle), while the other does not

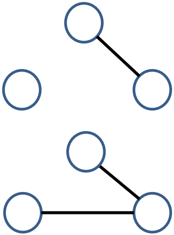
Paths and Connected Components

An undirected graph is connected if there is a path between any pair of vertices. Otherwise, it is disconnected.

- Ex :



Connected



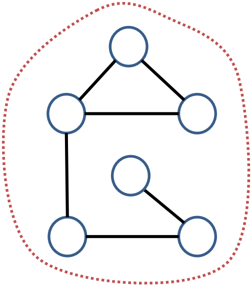
Disconnected

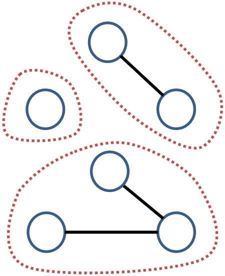
Paths and Connected Components

A connected subgraph is a connected component if it is not contained in any other connected subgraphs.

- Ex :

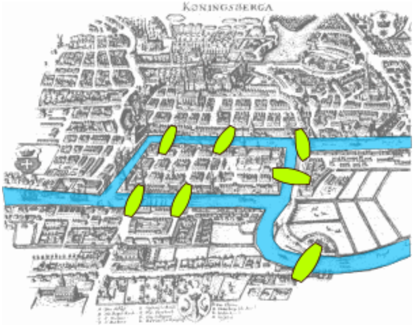
1 connected component





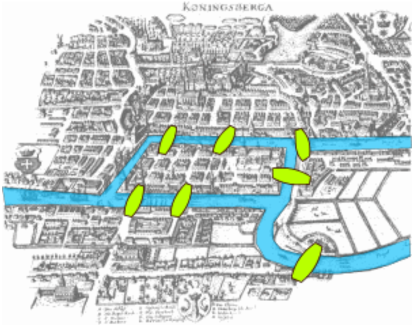
3 connected components

Euler Paths and Circuits



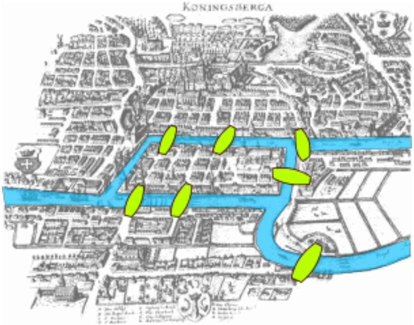
- The above is a map of a Prussian city called Königsberg during the 18th century

Euler Paths and Circuits



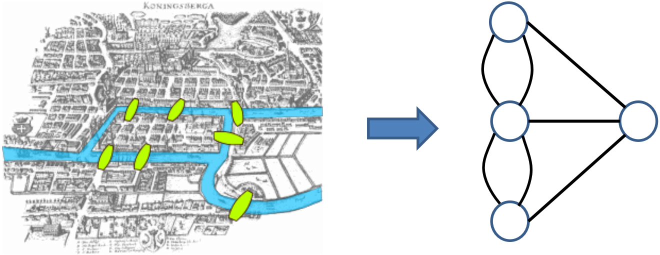
- The Pregel River (blue part) divides the city into 4 parts : Two sides and two large islands

Euler Paths and Circuits



- Seven bridges connect the sides with the islands
- Can we start at some location, travel each bridge exactly once, and go back to the same location ?

Euler Paths and Circuits



- Euler first represents the four parts and the seven bridges by a graph shown on the right

  ➔ The problem will be equivalent to :

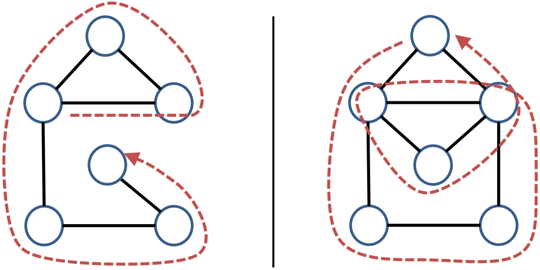
  Find a circuit that travels each edge exactly once

- Euler shows that there is NO such circuit

Euler Paths and Circuits

Definition : An Euler path in a graph is a path that contains each edge exactly once. If such a path is also a circuit, it is called an Euler circuit.

- Ex :



Euler path　　　　Euler circuit

Euler Paths and Circuits

Theorem : A connected graph G has an Euler circuit ⇔ each vertex of G has even degree.

- Proof : [The "only if" case]

  If the graph has an Euler circuit, then when we walk along the edges according to this circuit, each vertex must be entered and exited the same number of times.

  Thus, the degree of each vertex must be even.

Euler Paths and Circuits

- Proof : [The “if” case]

  If each vertex has an even degree, we shall use induction (on the number of edges) to show that an Euler circuit exists.

  (Basis) When there is one edge, it must be a self-loop ➔ An Euler circuit exists.

  (Inductive) We start at a vertex x, and obtain a path without using any edge twice, until we end at a vertex without any more unused edge to travel ➔ This vertex must be x (why?)

Euler Paths and Circuits

- Proof : [The “if” case (continued)]

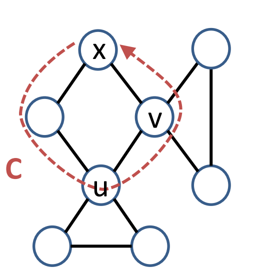
  Let C denote the above circuit.

  If we remove C from the graph, the degree of each vertex must still be even (why?). Further, each connected component with edges must share some vertex u with C, and has an Euler circuit C’ (why?)

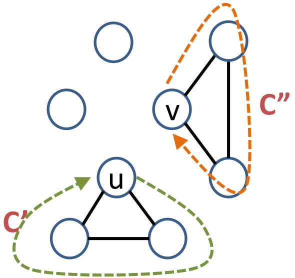
  ➔ We get an Euler circuit of the original graph, by walking on C until vertex u, then edges on C’, then back to u, and the remaining edges on C

Euler Paths and Circuits

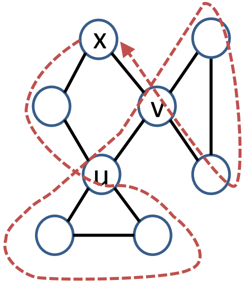
- Example on obtaining an Euler circuit :



Step 1:
Getting a circuit C by starting from a vertex x



Step 2:
Getting C' for each remaining component



Step 3:
Combining C and the C' of each component

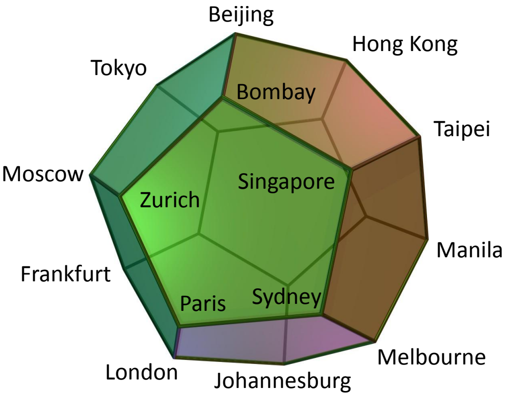
Euler Paths and Circuits

Corollary : A connected graph G has an Euler path, but no Euler circuits ⇔ exactly two vertices of G has odd degree.

- Proof : [The "only if" case] The degree of the starting and ending vertices of the Euler path must be odd, and all the others must be even.

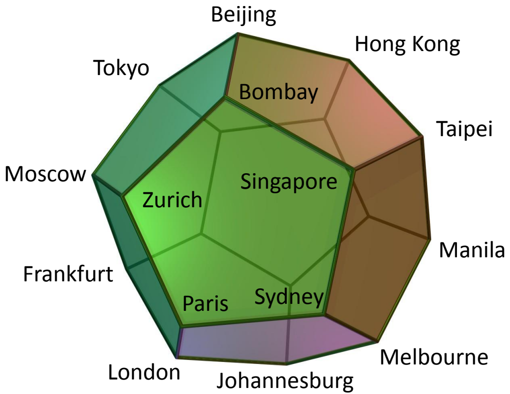
  [The "if" case] Let u and v be the vertices with odd degrees. Adding an edge between u and v will produce an Euler circuit → Removal of this edge thus implies an Euler path in the graph

Hamilton Paths and Circuits



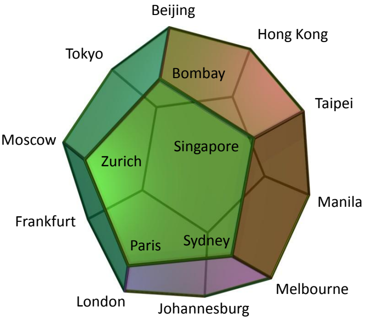
- The above is a regular dodecahedron (12-faced) with each vertex labeled with the name of a city

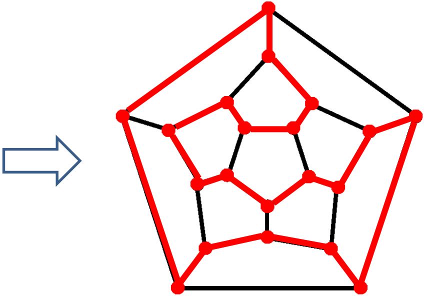
Hamilton Paths and Circuits



- Can we find a circuit (travelling along the edges) so that each city is visited exactly once ?

Hamilton Paths and Circuits



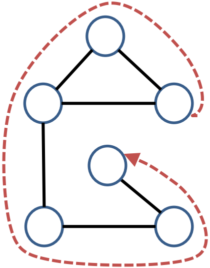


- The right graph is isomorphic to the dodecahedron, and it shows a possible way (in red) to travel

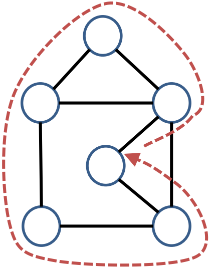
Hamilton Paths and Circuits

Definition : A Hamilton path in a graph is a path that visits each vertex exactly once. If such a path is also a circuit, it is called a Hamilton circuit.

- Ex :



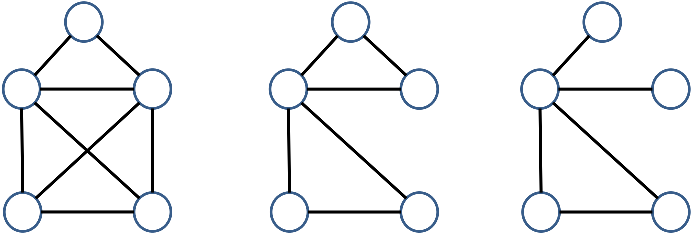
Hamilton path



Hamilton circuit

Hamilton Paths and Circuits

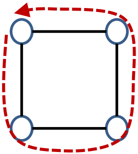
- Which of the following have a Hamilton circuit or, if not, a Hamilton path ?



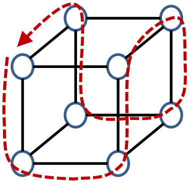
Hamilton Paths and Circuits

- Show that the n-dimensional cube Q_n has a Hamilton circuit, whenever $n \geq 2$

- Ex :



Q_2



Q_3

Hamilton Paths and Circuits

- Unlike Euler circuit or Euler path, there is no efficient way to determine if a graph contains a Hamilton circuit or a Hamilton path

  ➔ The best algorithm so far requires exponential time in the worst case

- However, it is shown that when the degree of the vertices are sufficiently large, the graph will always contain a Hamilton circuit

  ➔ We shall discuss two theorems in this form

Hamilton Paths and Circuits

- Before we give the two theorems, we show an interesting theorem by Bondy and Chvátal (1976)

  → The two theorems will then become corollaries of Bondy-Chvátal theorem

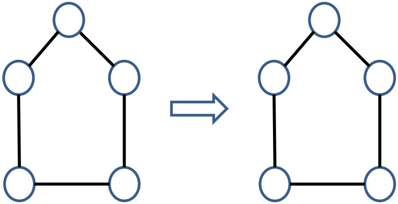
- Let G be a graph with n vertices

Definition : The Hamilton closure of G is a simple graph obtained by recursively adding an edge between two vertices u and v, whenever

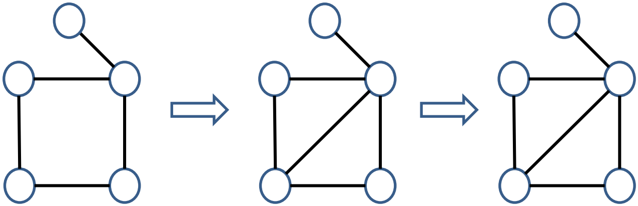
$$\deg(u) + \deg(v) \geq n$$

Hamilton Paths and Circuits

- Ex :

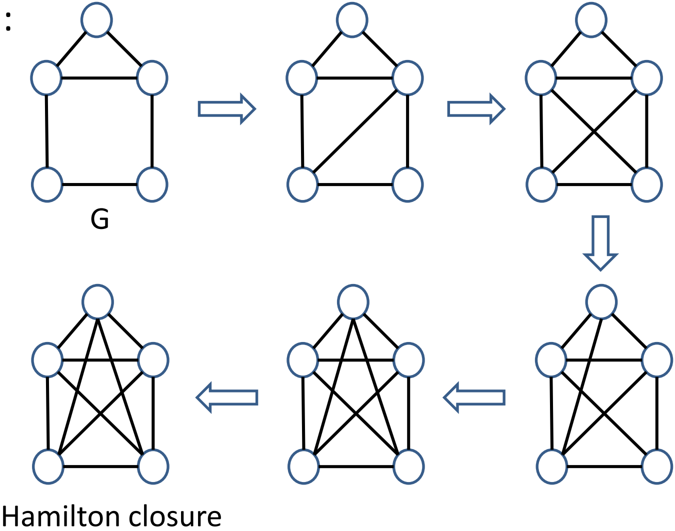


- Ex :



Hamilton Paths and Circuits

- Ex :



Hamilton Paths and Circuits

Theorem [Bondy and Chvátal (1976)] :

A graph G contains a Hamilton circuit ⇔ its Hamilton closure contains a Hamilton circuit

- The "only if" case is trivial
- For the "if" case, we can prove it by contradiction
- However, we shall give the proof a bit later, as we are now ready to talk about the two corollaries

Hamilton Paths and Circuits

- Let G be a simple graph with $n \geq 3$ vertices

Corollary [Dirac (1952)] :

If the degree of each vertex in G is at least n/2, then G contains a Hamilton circuit

Corollary [Ore (1960)] :

If for any pair of non-adjacent vertices u and v,

$$\deg(u) + \deg(v) \geq n,$$

then G contains a Hamilton circuit

Hamilton Paths and Circuits

- Proof of Dirac's and Ore's Theorems :

  It is easy to verify that

  (i) if the degree of each vertex is at least $n/2$, or

  (ii) if for any pair of non-adjacent vertices u and v,

$$\deg(u) + \deg(v) \geq n,$$

  → G's Hamilton closure is a complete graph K_n

  → When $n \geq 3$, K_n has a Hamilton circuit

  → Bondy-Chvátal implies that there will be a Hamilton circuit in G

Hamilton Paths and Circuits

- Next, we shall give the proof of the "if case" of Bondy-Chvátal's Theorem
- Proof ("if case"):

  Suppose on the contrary that

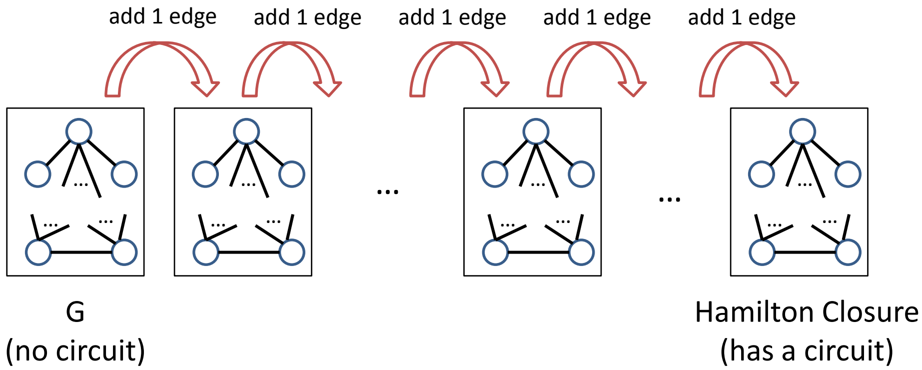
  (i) G does not have a Hamilton circuit, but

  (ii) G's Hamilton closure has a Hamilton circuit.

  Then, consider the sequence of graphs obtained by adding one edge each time when we produce the Hamilton closure from G

Hamilton Paths and Circuits

- Proof ("if case" continued):

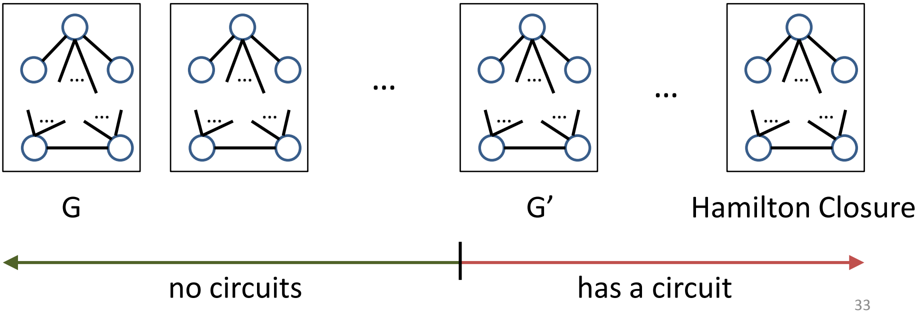


Hamilton Paths and Circuits

- Proof ("if case" continued):

  Let G' be the first graph in the sequence that contains a Hamilton circuit

  Let { u, v } be the edge added to produce G'



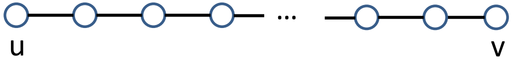
Hamilton Paths and Circuits

- Proof ("if case" continued):

  Now, we show that the graph before G' must also contain a Hamilton circuit, which immediately will cause a contradiction.

  Consider the graph before adding { u, v } to G'.

  It must contain a Hamilton path from u to v (why?)



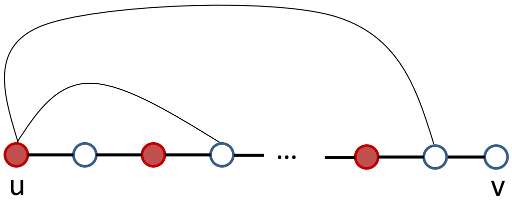
Hamilton Paths and Circuits

- Proof ("if case" continued):

  Also, since we are connecting u and v in G',

  $$\deg(u) + \deg(v) \geq n$$

  Consider all the nodes connected by u, and we mark their 'left' neighbors in red



Hamilton Paths and Circuits

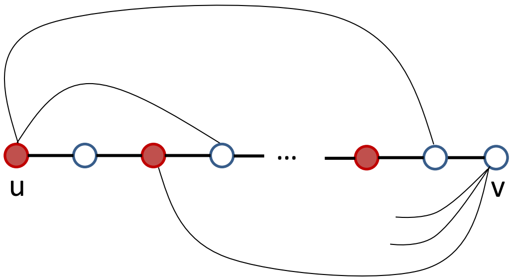
- Proof ("if case" continued):

  Since

  (i) v does not connect to u nor itself, and

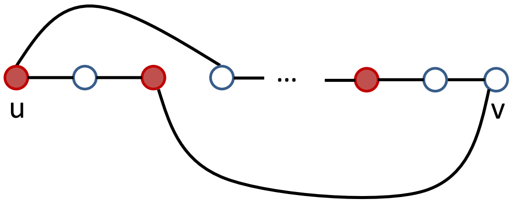
  (ii) deg(u) + deg(v) $\geq$ n

  ➔ v must connect to some red node (why?)



Hamilton Paths and Circuits

- Proof ("if case" continued):
  - ➔ We get a Hamilton circuit, even without connecting u and v !



  - ➔ This contradicts with the choice of G', and the theorem is thus correct

Graph Colouring

Suppose we have k distinct colours with which to colour the vertices of a graph. Let $[k] = \{1, \ldots, k\}$. For an undirected graph, $G = (V, E)$, an admissible vertex **k-colouring** of G is a function $c : V \rightarrow [k]$, such that for all $u, v \in V$, if $\{u, v\} \in E$ then $\quad c(u) \neq c(v)$.

For an integer $k \geq 1$, we say an undirected graph $G = (V, E)$ is **k-colourable** if there exists a k-colouring of G.

The **chromatic number** of G, denoted $\chi(G)$, is the *smallest positive integer* k, such that G is k-colourable.

Some observations about Graph colouring

- Note that any graph G with n vertices in n-colourable.
- The ***n*-Clique**, K_n, i.e., the complete graph on n vertices, has chromatic number $\chi(K_n) = n$. All its vertices must get assigned different colours in any admissible colouring.
- The clique number, $\omega(G)$, of a graph G is the maximum positive integer $r \geq 1$, such that K_r is a subgraph of G.
- Note that for all graphs G, $\omega(G) \leq \chi(G)$: if G has an r-clique then it is not $(r-1)$-colorable.
- However, in general, $\omega(G) \neq \chi(G)$. For instance, The 5-cycle, C_5, has $\omega(C_5) = 2 < \chi(C_5) = 3$.

More observations about colouring

- As already mentioned, any bipartite graph is 2-colourable. Indeed, that is an equivalent definition of being bipartite.
- More generally, a graph G is k-colourable precisely if it is k-partite, meaning its vertices can be partitioned into k disjoint sets such that all edges of the graph are between nodes in different parts.

Algorithms/complexity of colouring graphs

To determine whether a n-vertex graph $G = (V, E)$ is k-colourable by "*brute force*", we could try all possible colourings of n nodes with k colours.

Difficulty: There are k^n such k-colouring functions $c : V \to [k]$.

Question: Is there an efficient (polynomial time) algorithm for determining whether a given graph G is k-colourable?

Algorithms/complexity of colouring graphs

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Difficulty: There are k^n such k-colouring functions $c : V \rightarrow [k]$.

Question: Is there an efficient (polynomial time) algorithm for determining whether a given graph G is k-colourable?

Answer: No, no generally efficient (polynomial time) algorithm is known, and even the problem of determining whether a given graph is 3-colourable is **NP-complete**. (Even approximating the chromatic number of a given graph is NP-hard.)

In practice, there are hueristic algorithms that do obtain good colourings for many classes of graphs.

Applications of Graph Colouring (many)

Final Exam Scheduling

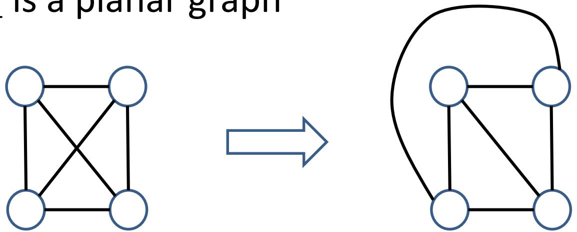
- There are n courses, $\{1, \ldots, n\}$.
- Some courses have the same students registered for both, so their exams can't be scheduled at the same time.
- Let $G = (\{1, \ldots, n\}, E)$ be a graph such that $\{i, j\} \in E$ if and only if $i \neq j$ and courses i and j have a student in common.
- **Question:** What is the minimum number of exam time slots needed to schedule all n exams?
- **Answer:** This is precisely the chromatic number $\chi(G)$ of G.

  Furthermore, a k-colouring of G yields an *admissible schedule* of exams into k time slots, allowing all students to attend all their exams, as long as different "colors" are scheduled in disjoint time slots.

What is a Planar Graph ?

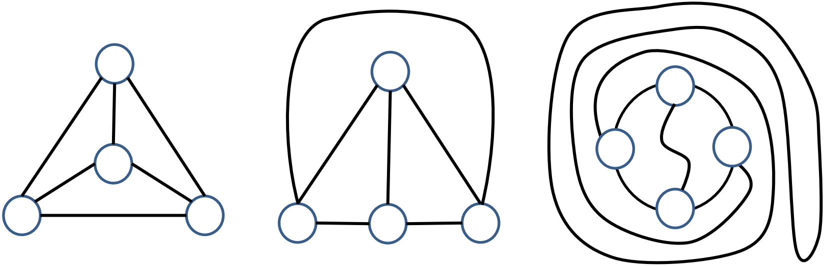
Definition : A planar graph is an undirected graph that can be drawn on a plane without any edges crossing. Such a drawing is called a planar representation of the graph in the plane.

- Ex : K_4 is a planar graph



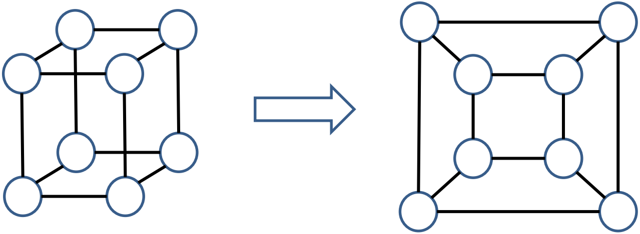
Examples of Planar Graphs

- Ex : Other planar representations of K_4



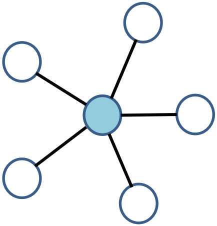
Examples of Planar Graphs

- Ex : Q_3 is a planar graph

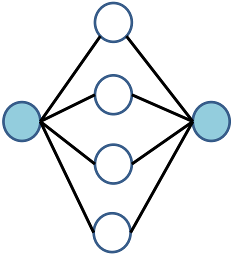


Examples of Planar Graphs

- Ex : $K_{1,n}$ and $K_{2,n}$ are planar graphs for all n



$K_{1,5}$

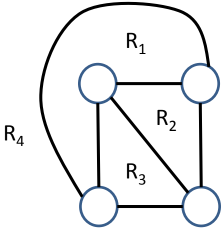


$K_{2,4}$

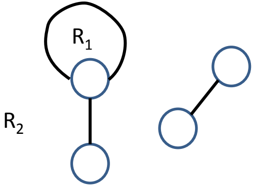
Euler's Planar Formula

Definition : A planar representation of a graph splits the plane into regions, where one of them has infinite area and is called the infinite region.

- Ex :



4 regions
(R_4 = infinite region)



2 regions
(R_2 = infinite region)

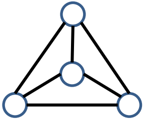
Euler's Planar Formula

- Let G be a connected planar graph, and consider a planar representation of G. Let

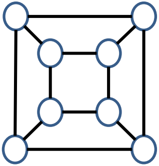
  V = # vertices, E = # edges, F = # regions.

Theorem : $V + F = E + 2.$

- Ex :



V = 4, F = 4, E = 6



V = 8, F = 6, E = 12

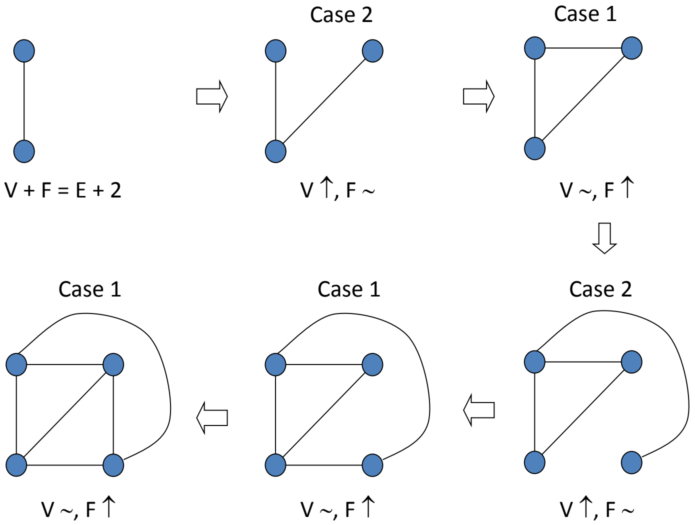
Euler's Planar Formula

- Proof Idea :
  - Add edges one by one, so that in each step, the subgraph is always connected
  - Use induction to show that the formula is always satisfied for each subgraph
  - For the new edge that is added, it either joins :

    (1) two existing vertices $\Rightarrow$ $V\sim$, $F\uparrow$

    (2) one existing + one new vertex $\Rightarrow$ $V\sim$, $F\uparrow$

Euler’s Planar Formula



Euler's Planar Formula

- Let G be a connected simple planar graph with

  V = # vertices, E = # edges.

$$\boxed{\text{Corollary : If } V \geq 3, \text{ then } E \leq 3V - 6.}$$

- Proof : Each region is surrounded by at least 3 edges (how about the infinite region?)

  ➔ $3F \leq$ total edges $= 2E$

  ➔ $E + 2 = V + F \leq V + 2E/3$

  ➔ $E \leq 3V - 6$

Euler's Planar Formula

Theorem : K_5 and $K_{3,3}$ are non-planar.

- Proof :

  (1) For K_5, $V = 5$ and $E = 10$

  $\Rightarrow$ $E > 3V - 6$ $\Rightarrow$ non-planar

  (2) For $K_{3,3}$, $V = 6$ and $E = 9$.

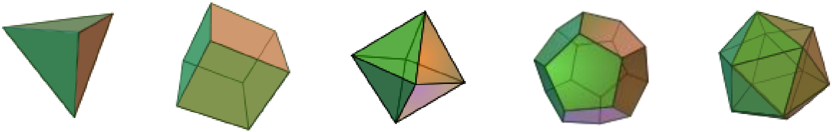
  $\Rightarrow$ If it is planar, each region is surrounded by at least 4 edges (why?)

  $\Rightarrow$ $F \leq \lfloor 2E/4 \rfloor = 4$

  $\Rightarrow$ $V + F \leq 10 < E + 2$ $\Rightarrow$ non-planar

Platonic Solids

Definition : A Platonic solid is a convex 3D shape that all faces are the same, and each face is a regular polygon



Platonic Solids

Theorem: There are exactly 5 Platonic solids

- Proof:

  Let n = # vertices of each polygon

  m = degree of each vertex

  For a platonic solid, we must have

  $$n F = 2E \qquad \text{and} \qquad V m = 2E$$

Platonic Solids

- Proof (continued):

  By Euler's planar formula,

  $$2E/m + 2E/n = V + F = E + 2$$

  ➔ $1/m + 1/n = 1/2 + 1/E$ …… (*)

  Also, we need to have

  $n \geq 3$ and $m \geq 3$ [from 3D shape]

  but one of them must be $= 3$ [from (*)]

Platonic Solids

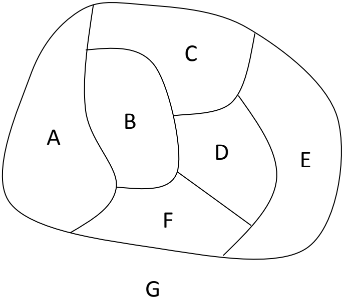
- Proof (continued):

  ➔ Either

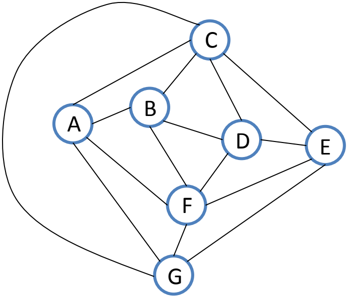
  (i) $n = 3$ (with $m = 3, 4,$ or 5)

  (ii) $m = 3$ (with $n = 3, 4,$ or 5)

Map Coloring and Dual Graph



A Map M

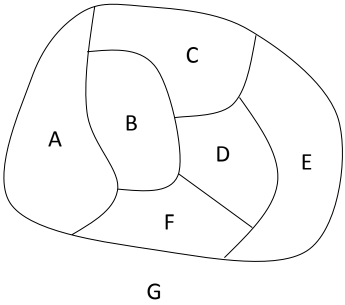


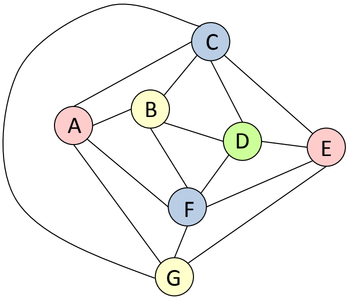
Dual Graph of M

Map Coloring and Dual Graph

Observation: A proper color of M

⇔ A proper vertex color the dual graph





Proper coloring : Adjacent regions (or vertices) have to be colored in different colors

Five Color Theorem

- Appel and Haken (1976) showed that every planar graph can be 4 colored

  (Proof is tedious, has 1955 cases and many subcases)

- Here, we shall show that :

> Theorem : Every planar graph can be 5 colored.

- The above theorem implies that every map can be 5 colored (as its dual is planar)

Five Color Theorem

- Proof :

  We assume the graph has at least 5 vertices. Else, the theorem will immediately follow.

  Next, in a planar graph, we see that there must be a vertex with degree at most 5.

  Else,

  $$2E = \text{total degree} \geq 3V$$

  which contradicts with the fact $E \leq 3V - 6$.

Five Color Theorem

- Proof (continued) :

  Let v be a vertex whose degree is at most 5.

  Now, assume inductively that all planar graphs with n – 1 vertices can be colored in 5 colors

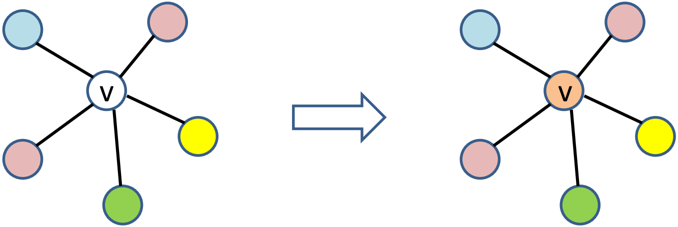
  ➔ Thus if v is removed, we can color the graph properly in 5 colors

  What if we add back v to the graph now ??

Five Color Theorem

- Proof (continued) :

  Case 1 : Neighbors of v uses at most 4 colors

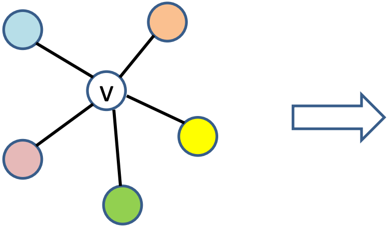


there is a 5th color for v

Five Color Theorem

- Proof (continued) :

  Case 2 : Neighbors of v uses up all 5 colors

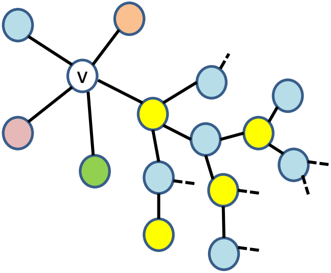


Can we save 1 color,
by coloring the yellow
neighbor in blue ?

Five Color Theorem

- Proof ("Case 2" continued):

Can we color the yellow neighbor in blue ?



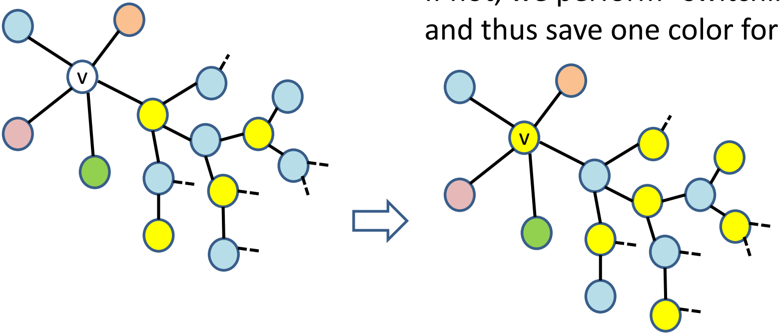
First, we check if the yellow neighbor can connect to the blue neighbor by a "switching" yellow-blue path

Five Color Theorem

- Proof ("Case 2" continued):

  Can we color the yellow neighbor in blue ?

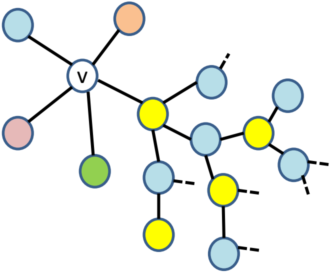
If not, we perform "switching" and thus save one color for v



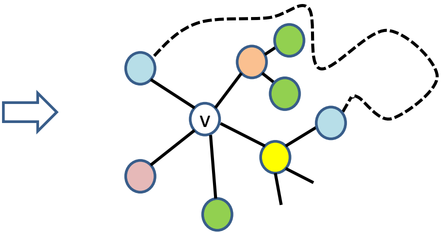
Five Color Theorem

- Proof (“Case 2” continued):

Can we color the yellow neighbor in blue ?



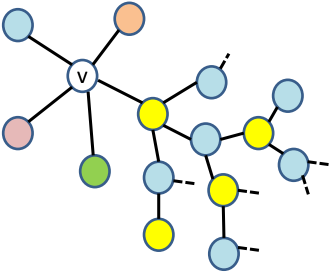
Else, they are connected
➔ orange and green cannot be connected by “switching path



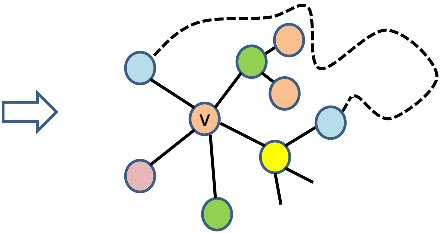
Five Color Theorem

- Proof ("Case 2" continued):

We color the orange neighbor in green !



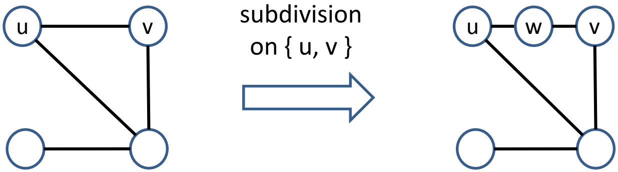
➔ we can perform "switching" (orange and green) to save one color for v



Kuratowski's Theorem

Definition : A subdivision operation on an edge { u, v } is to create a new vertex w, and replace the edge by two new edges { u, w } and { w, v }.

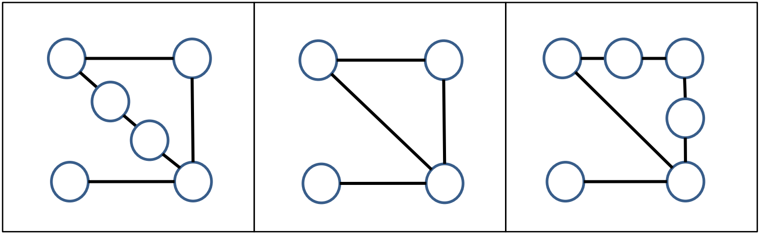
- Ex :



Kuratowski's Theorem

Definition : Graphs G and H are homeomorphic if both can be obtained from the same graph by a sequence of subdivision operations.

- Ex : The following graphs are all homeomorphic :



Kuratowski's Theorem

- In 1930, the Polish mathematician Kuratowski proved the following theorem :

Theorem :

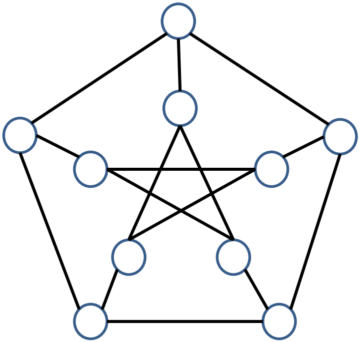
Graph G is non-planar

$\Leftrightarrow$ G has a subgraph homeomorphic to K_5 or $K_{3,3}$

- The "if" case is easy to show (how?)
- The "only if" case is hard (I don't know either …)

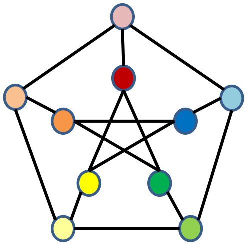
Kuratowski's Theorem

- Ex :   Show that the Petersen graph is non-planar.

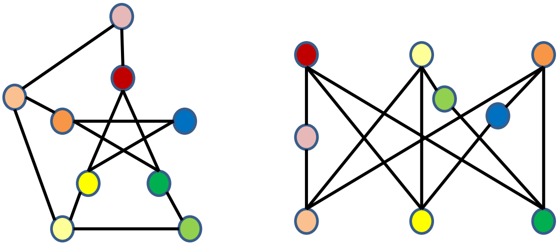


Kuratowski's Theorem

- Proof :



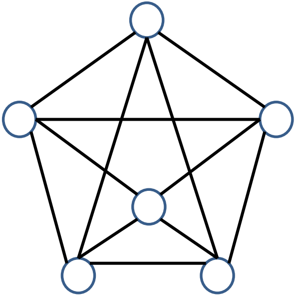
Petersen Graph



Subgraph homeomorphic to $K_{3,3}$

Kuratowski's Theorem

- Ex : Is the following graph planar or non-planar ?



Kuratowski's Theorem

- Ans : Planar