CS 70 Discrete Mathematics and Probability Theory
Fall 2019 Alistair Sinclair and Yun S. Song **Quiz 2**

1. [True or False?] For each of the questions below, answer TRUE or FALSE. [No need to justify.]

(a) ☐ Suppose you proved the inductive step for a statement $P(n)$ but then discovered that $P(29)$ is false. Thus, $P(1)$ has to be false.

(b) ☐ Suppose you proved the inductive step for a statement $P(n)$ but then discovered that $P(29)$ is false. Then, we cannot say anything about P(50).

(c) ☐ In a stable marriage instance where there is a man at the bottom of each woman's preference list, the man is paired with his least favorite woman in every stable pairing.

(d) ☐ In a stable marriage instance where there is a man at the top of each woman's preference list, the man is paired with his favorite woman in every stable pairing.

(e) Suppose that, on day k of some execution of a stable marriage algorithm, Alice likes the boy who she currently has on a string more than the boy who Betty has on a string.

☐ It's guaranteed that on every subsequent day, this will continue to be true.

Solution:

(a) TRUE. Suppose $P(1)$ is true. That provides the base case of the induction, and together with the inductive step this implies that $P(n)$ is true for all n, including for $n = 29$. But this is a contradiction, as $P(29)$ is false, so $P(1)$ must be false.

(b) TRUE. Since $P(29)$ is false, and there is no base case to start the induction. So $P(50)$ is not necessarily true. But there is nothing preventing $P(50)$ from being true, so it could either betrue or false.

(c) FALSE. Consider lists M_1: $W_1 > W_2$, M_2: $W_2 > W_1$, W_1: $M_1 > M_2$, W_2: $M_1 > M_2$

(d) TRUE. if M is the man at the top and his favorite woman is W, then if (M,W') and (W,M') are in a pairing, (M,W) are a rogue couple because they mutually prefer each other.

(e) FALSE. Tomorrow, Betty might receive a proposal from some third boy who Alice has a mad crush on.

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2. [Inequality.] Prove by induction on n that if n is a natural number and $x > 0$, then $(1+x)^n \geq 1+nx$.

Solution:

- Base case: When $n = 0$ the claim holds since $(1+x)^0 \geq 1 + 0 \cdot x$.
- Inductive hypothesis: Now, assume as our inductive hypothesis that $(1+x)^k \geq 1 + kx$ for some value of $n = k$ where $k > 0$.
- Inductive step: For $n = k+1$, we can show the following chain of inequalities:

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\begin{aligned}
(1+x)^{k+1} &= (1+x)^k(1+x) && (1)\\
&\geq (1+kx)(1+x) \qquad \text{(by the inductive hypothesis)} && (2)\\
&\geq 1 + kx + x + kx^2 && (3)\\
&\geq 1 + (k+1)x + kx^2 && (4)\\
&\geq 1 + (k+1)x, \qquad (kx^2 > 0 \text{ since } k > 0, x > 0) && (5)
\end{aligned}
$$

By induction, we have shown that $\forall n \in \mathbb{N}, (1+x)n \geq 1 + nx$

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3. [Stable Marriage.] Suppose that after running the Stable Marriage Algorithm with n men and n women, the pairing that results includes the couple $(1, A)$. Suppose that after a few days man 1 changes his mind, and decides that he does not like woman A as much as he thought he did (i.e. he put her the last person on his preference list). What is the maximum number of rogue couples that result in the existing pairing from such a change to 1's preference list? Give a one or two sentence justification for why the number of rogue couples can be as large as you claim. Also give a one or two sentence justification for why the remaining couples cannot be rogue couples.

Solution: There can be $n-1$ rogue couples; this happens when all of the women prefer 1 to all the other men, and 1 at first preferred A to all other women but now puts her as his least preferred. This situation makes $(1, X)$ a rogue couple for all $X \neq A$. (i.e. one rogue couple for every woman who is not A).

Moreover, there cannot be more than $n-1$ rogue couples. To see this, notice that whether or not (i, X) is a rogue couple can be determined just by looking at the preference lists of i and X (and the names of their partners). Since 1 is the only person whose preference list changes, he must be part of any rogue couple, and so there can be at most $n-1$ rogue couples (since $(1, A)$ clearly is not a rogue couple). So $n-1$ is the maximum number of rogue couples.