BBM 205 Discrete Mathematics
Hacettepe University

Lecture 11b: Stable Matchings

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Resources:
http://www.cs.cmu.edu/ ./15251/schedule.html
http://www.cs.princeton.edu/courses/archive/spring13/cos423

From Last Time

Bipartite maximum matching problem

Bipartite maximum matching problem

Input: A *bipartite* graph $G = (X, Y, E)$.

Output: A maximum matching in G.

Important Definition: Augmenting paths

Let **M** be some matching.

An ***augmenting path*** with respect to **M** is an alternating path such that:

- the first and last vertices are **not** matched by **M**

Algorithm to find maximum matching

Theorem:

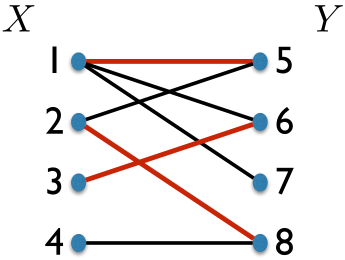
A matching **M** is maximum **if and only if** there is **no** augmenting path with respect to **M**.

Algorithm:

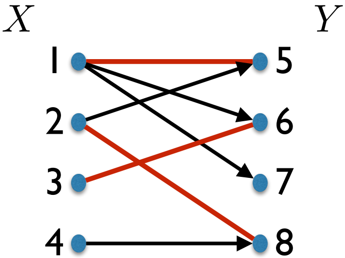
- Start with a single edge as your matching **M**.
- Repeat until there is no augmenting path w.r.t. **M**:
  - Find an augmenting path with respect to **M**.
  - Update **M** according to the augmenting path.

OK, but how do you find an augmenting path?

Algorithm to find augmenting path

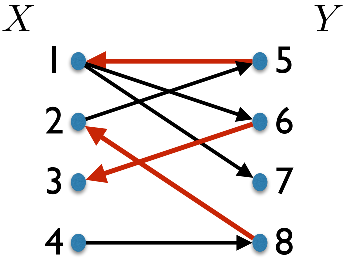


Algorithm to find augmenting path



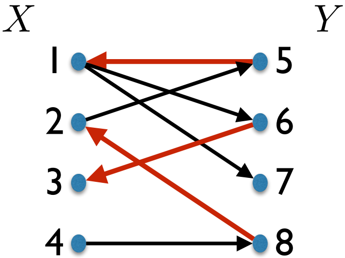
- direct edges **<u>not</u>** in **M** from left to right (X to Y).

Algorithm to find augmenting path



- direct edges **<u>not</u>** in **M** from left to right (X to Y).
- direct edges in **M** from right to left (Y to X).

Algorithm to find augmenting path

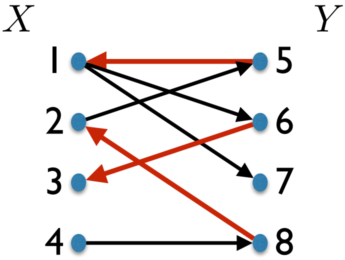


- direct edges **not** in **M** from left to right $(X$ to $Y)$.
- direct edges in **M** from right to left $(Y$ to $X)$.

Observation:

There is an augmenting path **iff**
there is a directed path from an *unmatched* $x \in X$
to an *unmatched* $y \in Y$.

Algorithm to find augmenting path



Algorithm:

- for each *unmatched* $x \in X$:
    - do DFS(x), stop when you find *unmatched* $y \in Y$.

Running time: $O(n+m)$

Important Note

Theorem:

A matching **M** is maximum **if and only if** there is **no** augmenting path with respect to **M**.

This theorem holds for <u>all</u> graphs.

The algorithm works for <u>bipartite</u> graphs.

How do you solve a problem like this?

1. Formulate the problem

2. **Ask**: Is there a trivial algorithm?

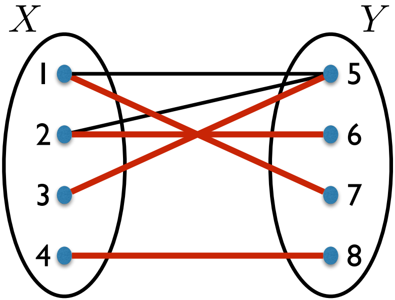
3. **Ask**: Is there a better algorithm?

4. Find and analyze

Hall’s Theorem

Characterization for perfect matchings

Often we are interested in perfect matchings.

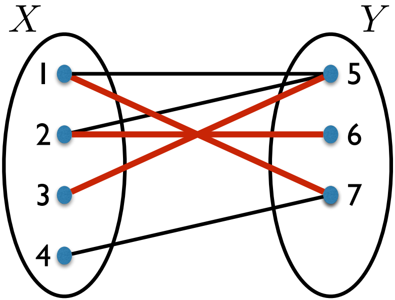


<u>An obstruction:</u>

$$|X| \neq |Y|$$

Characterization for perfect matchings

Often we are interested in perfect matchings.



An obstruction:

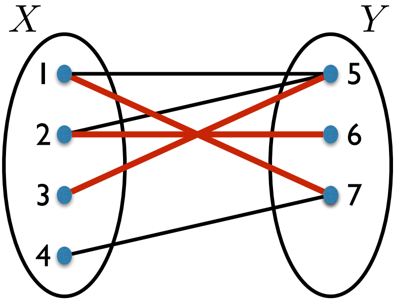
If $|X| > |Y|$, we cannot "cover" all the nodes in X.

If $|X| > |N(X)|$, we cannot "cover" all the nodes in X.

Characterization for perfect matchings

Often we are interested in perfect matchings.

$$S = \{1, 3, 4\}$$
$$N(S) = \{5, 7\}$$



<u>An obstruction:</u>

For $S \subseteq X$:

if $|S| > |N(S)|$, we cannot "cover" all the nodes in S.

Characterization for perfect matchings

Is this the only type of obstruction?

Theorem [Hall's Theorem]:

Let $G = (X, Y, E)$ be a bipartite graph.

There is a matching covering all vertices in X **iff**

$$\forall S \subseteq X \ : \ \ |S| \leq |N(S)| \, .$$

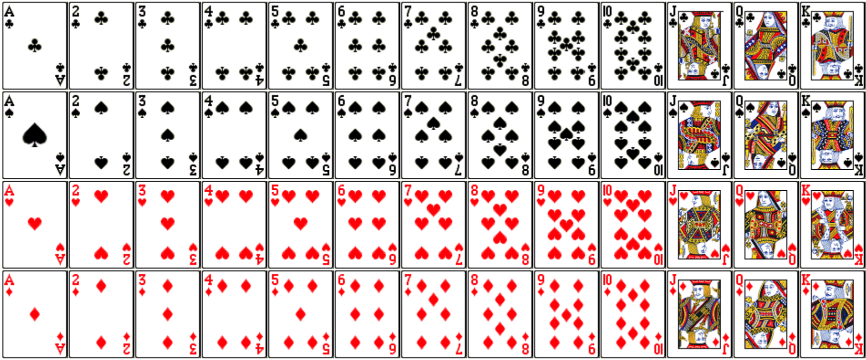
Corollary:

$G = (X, Y, E)$ has a perfect matching **iff**

$|X| = |Y|$ and $\forall S \subseteq X, \ \ |S| \leq |N(S)| \, .$

An application of Hall's Theorem

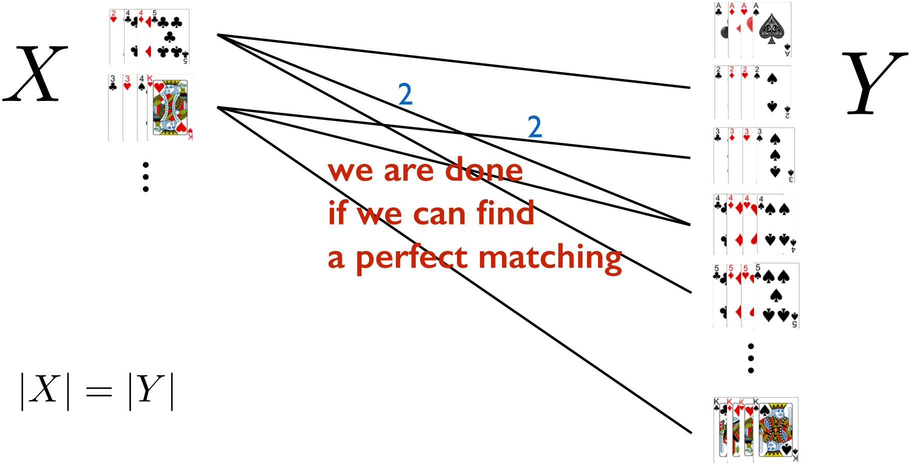
Rank: 1 2 3 4 5 6 7 8 9 10 J Q K



Suppose a deck of cards is dealt into 13 piles of 4 cards each.

<u>**Claim**</u>: there is a way to select one card from each pile so that you have one card from each rank.

An application of Hall's Theorem

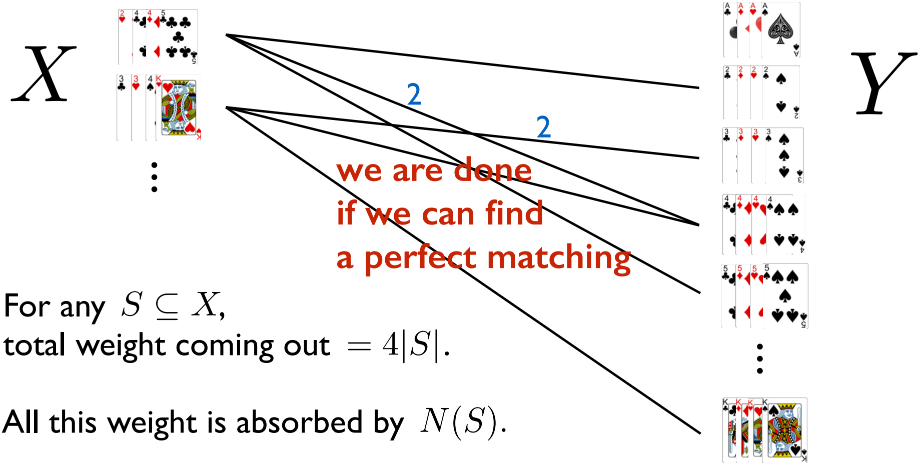


$$|X| = |Y|$$

__Want to show:__

For any $S \subseteq X$, $|S| \leq |N(S)|$.

An application of Hall's Theorem



For any $S \subseteq X$,
total weight coming out $= 4|S|$.

All this weight is absorbed by $N(S)$.

Each $y \in N(S)$ absorbs ≤ 4 units of this weight.

$\Longrightarrow$ $N(S)$ absorbs ≤ $4|N(S)|$ units. $\Longrightarrow$ $\cancel{4}|S| \leq \cancel{4}|N(S)|$

Stable matching problem

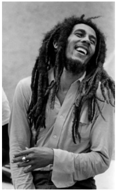
2-Sided Markets

A market with 2 distinct groups of participants each with their own preferences.

2-Sided Markets

1. Moogle
2. Macrosoft
3. 

4. UMBRELLA CORPORATION "Our business is life itself..."



Macrosoft

1. Alice
2. Bob
3. Charlie
4. David

Moogle

UMBRELLA CORPORATION

"Our business is life itself..."

•

•

•

1. Bob
2. David
3. Alice
4. Charlie

Other examples:

medical residents - hospitals

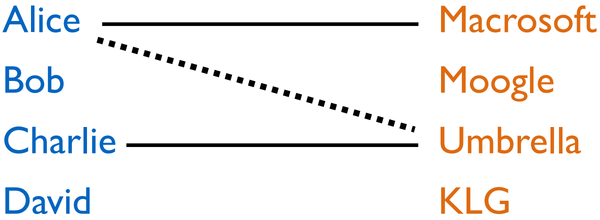
students - colleges

professors - colleges

⋮

Aspiration: A Good Centeralized System

What can go wrong?

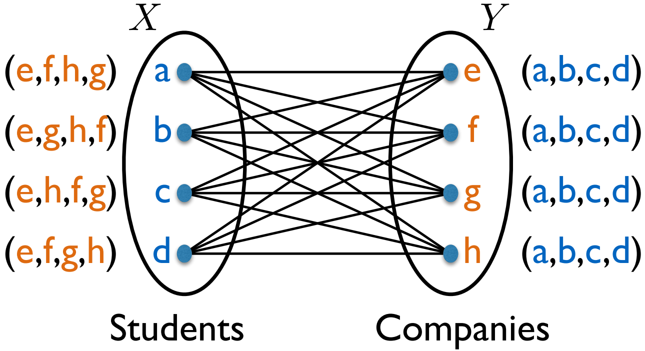


Suppose Alice gets “matched” with Macrosoft.
Charlie gets “matched” with Umbrella.

But, say, Alice prefers Umbrella over Macrosoft
and Umbrella prefers Alice over Charlie.

Formalizing the problem

An instance of the problem can be represented as a
complete bipartite graph + *preference list of each node.*

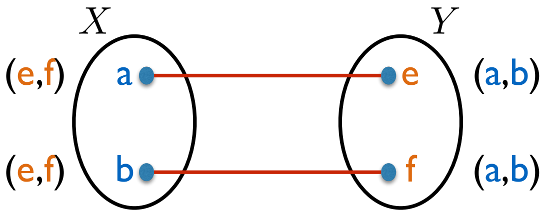


$$|X| = |Y| = n$$

Goal: Find a stable matching.

Formalizing the problem

What is a stable matching?



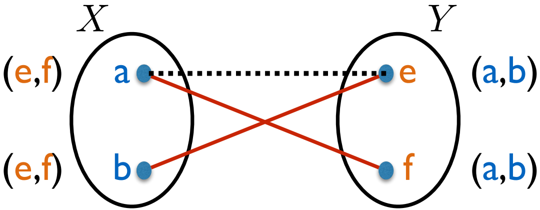
1. It has to be a perfect matching.

2. Cannot contain an unstable pair:

   A pair (x, y) unmatched
   but they prefer each other over their current partners.

Formalizing the problem

What is a stable matching?



(a, e) is an unstable pair.

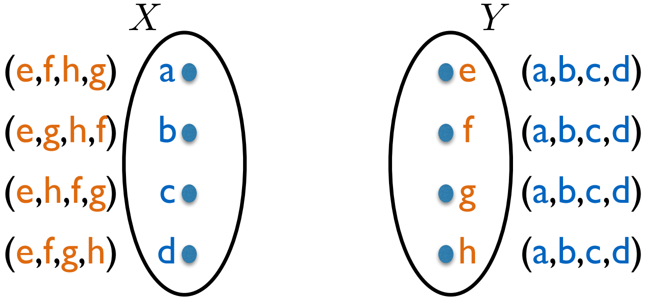
1. It has to be a perfect matching.

2. Cannot contain an unstable pair:

   A pair (x, y) <u>unmatched</u>
   but they prefer each other over their current partners.

Formalizing the problem

An instance of the problem can be represented as a *complete bipartite graph* + *preference list of each node.*



$$|X| = |Y| = n$$

Goal: Find a stable matching.

(Is it guaranteed to always exist?)

A variant: Roommate problem

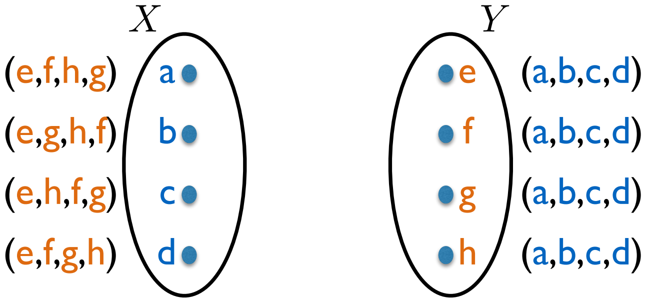
A non-bipartite version

(c,b,d) a • • c (b,a,d)

(a,c,d) b • • d (a,c,b)

Does this have a stable matching?

Stable matching: Is there a trivial algorithm?

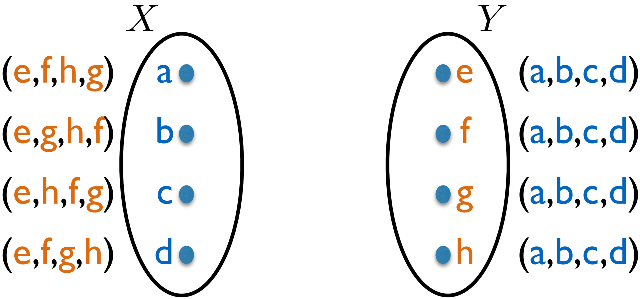


Trivial algorithm:

Try all possible perfect matchings,
and check if it is stable.

perfect matchings in terms $n = |X|$:

Stable matching: Is there a trivial algorithm?



Trivial algorithm:

Try all possible perfect matchings,

and check if it is stable.

perfect matchings in terms $n = |X|$: $n!$

Gale-Shapley demo

hospitals' preference lists

	1st	2nd	3rd	4th	5th
Atlanta	Wayne	Val	Yolanda	Zeus	Xavier
Boston	Yolanda	Wayne	Val	Xavier	Zeus
Chicago	Wayne	Zeus	Xavier	Yolanda	Val
Dallas	Val	Yolanda	Xavier	Wayne	Zeus
Eugene	Wayne	Yolanda	Val	Zeus	Xavier

students' preference lists

	1st	2nd	3rd	4th	5th
Val	Eugene	Atlanta	Boston	Dallas	Chicago
Wayne	Chicago	Boston	Dallas	Atlanta	Eugene
Xavier	Boston	Chicago	Dallas	Eugene	Atlanta
Yolanda	Atlanta	Eugene	Dallas	Chicago	Boston
Zeus	Dallas	Boston	Eugene	Chicago	Atlanta

Gale-Shapley demo

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Wayne	Chicago	Boston	Dallas	Atlanta	Eugene
Xavier	Boston	Chicago	Dallas	Eugene	Atlanta
Yolanda	Atlanta	Eugene	Dallas	Chicago	Boston
Zeus	Dallas	Boston	Eugene	Chicago	Atlanta

Atlanta proposes to Wayne

Gale-Shapley demo

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Atlanta	Wayne	Val	Yolanda	Zeus	Xavier
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Dallas	Val	Yolanda	Xavier	Wayne	Zeus
Eugene	Wayne	Yolanda	Val	Zeus	Xavier

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Wayne	Chicago	Boston	Dallas	Atlanta	Eugene
Xavier	Boston	Chicago	Dallas	Eugene	Atlanta
Yolanda	Atlanta	Eugene	Dallas	Chicago	Boston
Zeus	Dallas	Boston	Eugene	Chicago	Atlanta

Atlanta proposes to Wayne

Wayne accepts
(since previously unmatched)

Gale–Shapley demo

hospitals' preference lists

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Chicago	Wayne	Zeus	Xavier	Yolanda	Val
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Xavier	Boston	Chicago	Dallas	Eugene	Atlanta
Yolanda	Atlanta	Eugene	Dallas	Chicago	Boston
Zeus	Dallas	Boston	Eugene	Chicago	Atlanta

Boston proposes to Yolanda

Gale-Shapley demo

hospitals' preference lists

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Atlanta	Wayne	Val	Yolanda	Zeus	Xavier
Boston	Yolanda	Wayne	Val	Xavier	Zeus
Chicago	Wayne	Zeus	Xavier	Yolanda	Val
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Eugene	Wayne	Yolanda	Val	Zeus	Xavier

students' preference lists

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Wayne	Chicago	Boston	Dallas	Atlanta	Eugene
Xavier	Boston	Chicago	Dallas	Eugene	Atlanta
Yolanda	Atlanta	Eugene	Dallas	Chicago	Boston
Zeus	Dallas	Boston	Eugene	Chicago	Atlanta

Boston proposes to Yolanda

Yolanda accepts (since previously unmatched)

Gale-Shapley demo

hospitals' preference lists

	1st	2nd	3rd	4th	5th
Atlanta	Wayne	Val	Yolanda	Zeus	Xavier
Boston	Yolanda	Wayne	Val	Xavier	Zeus
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Wayne	Chicago	Boston	Dallas	Atlanta	Eugene
Xavier	Boston	Chicago	Dallas	Eugene	Atlanta
Yolanda	Atlanta	Eugene	Dallas	Chicago	Boston
Zeus	Dallas	Boston	Eugene	Chicago	Atlanta

Chicago proposes to Wayne

Gale–Shapley demo

hospitals' preference lists

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Xavier	Boston	Chicago	Dallas	Eugene	Atlanta
Yolanda	Atlanta	Eugene	Dallas	Chicago	Boston
Zeus	Dallas	Boston	Eugene	Chicago	Atlanta

Chicago proposes to Wayne

Wayne accepts (and renounces Atlanta)

Gale-Shapley demo

hospitals' preference lists

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Xavier	Boston	Chicago	Dallas	Eugene	Atlanta
Yolanda	Atlanta	Eugene	Dallas	Chicago	Boston
Zeus	Dallas	Boston	Eugene	Chicago	Atlanta

Atlanta proposes to Val

Gale–Shapley demo

hospitals' preference lists

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Wayne	Chicago	Boston	Dallas	Atlanta	Eugene
Xavier	Boston	Chicago	Dallas	Eugene	Atlanta
Yolanda	Atlanta	Eugene	Dallas	Chicago	Boston
Zeus	Dallas	Boston	Eugene	Chicago	Atlanta

Atlanta proposes to Val

Val accepts (since previously unmatched)

Gale–Shapley demo

hospitals' preference lists

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Zeus	Dallas	Boston	Eugene	Chicago	Atlanta

Dallas proposes to Val

Gale–Shapley demo

hospitals' preference lists

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Xavier	Boston	Chicago	Dallas	Eugene	Atlanta
Yolanda	Atlanta	Eugene	Dallas	Chicago	Boston
Zeus	Dallas	Boston	Eugene	Chicago	Atlanta

Dallas proposes to Val

Val rejects (since she prefers Atlanta)

Gale-Shapley demo

hospitals' preference lists

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Zeus	Dallas	Boston	Eugene	Chicago	Atlanta

Dallas proposes to Yolanda

Gale–Shapley demo

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Yolanda	Atlanta	Eugene	Dallas	Chicago	Boston
Zeus	Dallas	Boston	Eugene	Chicago	Atlanta

Dallas proposes to Yolanda

Dianne accepts
(and renounces Boston)

Gale–Shapley demo

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Boston proposes to Wayne

Gale–Shapley demo

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Yolanda	Atlanta	Eugene	Dallas	Chicago	Boston
Zeus	Dallas	Boston	Eugene	Chicago	Atlanta

Boston proposes to Wayne

Wayne rejects
(since he prefers Chicago)

Gale-Shapley demo

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Boston proposes to Val

Gale–Shapley demo

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Val	Eugene	Atlanta	Boston	Dallas	Chicago
Wayne	Chicago	Boston	Dallas	Atlanta	Eugene
Xavier	Boston	Chicago	Dallas	Eugene	Atlanta
Yolanda	Atlanta	Eugene	Dallas	Chicago	Boston
Zeus	Dallas	Boston	Eugene	Chicago	Atlanta

Boston proposes to Val

**Val rejects
(since she prefers Atlanta)**

Gale–Shapley demo

hospitals' preference lists

	1st	2nd	3rd	4th	5th
Atlanta	Wayne	Val	Yolanda	Zeus	Xavier
Boston	Yolanda	Wayne	Val	Xavier	Zeus
Chicago	Wayne	Zeus	Xavier	Yolanda	Val
Dallas	Val	Yolanda	Xavier	Wayne	Zeus
Eugene	Wayne	Yolanda	Val	Zeus	Xavier

students' preference lists

	1st	2nd	3rd	4th	5th
Val	Eugene	Atlanta	Boston	Dallas	Chicago
Wayne	Chicago	Boston	Dallas	Atlanta	Eugene
Xavier	Boston	Chicago	Dallas	Eugene	Atlanta
Yolanda	Atlanta	Eugene	Dallas	Chicago	Boston
Zeus	Dallas	Boston	Eugene	Chicago	Atlanta

Boston proposes to Xavier

Gale-Shapley demo

hospitals' preference lists

	1st	2nd	3rd	4th	5th
Atlanta	Wayne	Val	Yolanda	Zeus	Xavier
Boston	Yolanda	Wayne	Val	Xavier	Zeus
Chicago	Wayne	Zeus	Xavier	Yolanda	Val
Dallas	Val	Yolanda	Xavier	Wayne	Zeus
Eugene	Wayne	Yolanda	Val	Zeus	Xavier

students' preference lists

	1st	2nd	3rd	4th	5th
Val	Eugene	Atlanta	Boston	Dallas	Chicago
Wayne	Chicago	Boston	Dallas	Atlanta	Eugene
Xavier	Boston	Chicago	Dallas	Eugene	Atlanta
Yolanda	Atlanta	Eugene	Dallas	Chicago	Boston
Zeus	Dallas	Boston	Eugene	Chicago	Atlanta

Boston proposes to Xavier

Xavier accepts (since previously unmatched)

Gale-Shapley demo

hospitals' preference lists

	1st	2nd	3rd	4th	5th
Atlanta	Wayne	Val	Yolanda	Zeus	Xavier
Boston	Yolanda	Wayne	Val	Xavier	Zeus
Chicago	Wayne	Zeus	Xavier	Yolanda	Val
Dallas	Val	Yolanda	Xavier	Wayne	Zeus
Eugene	Wayne	Yolanda	Val	Zeus	Xavier

students' preference lists

	1st	2nd	3rd	4th	5th
Val	Eugene	Atlanta	Boston	Dallas	Chicago
Wayne	Chicago	Boston	Dallas	Atlanta	Eugene
Xavier	Boston	Chicago	Dallas	Eugene	Atlanta
Yolanda	Atlanta	Eugene	Dallas	Chicago	Boston
Zeus	Dallas	Boston	Eugene	Chicago	Atlanta

Eugene proposes to Wayne

Gale–Shapley demo

hospitals' preference lists

	1st	2nd	3rd	4th	5th
Atlanta	Wayne	Val	Yolanda	Zeus	Xavier
Boston	Yolanda	Wayne	Val	Xavier	Zeus
Chicago	Wayne	Zeus	Xavier	Yolanda	Val
Dallas	Val	Yolanda	Xavier	Wayne	Zeus
Eugene	Wayne	Yolanda	Val	Zeus	Xavier

students' preference lists

	1st	2nd	3rd	4th	5th
Val	Eugene	Atlanta	Boston	Dallas	Chicago
Wayne	Chicago	Boston	Dallas	Atlanta	Eugene
Xavier	Boston	Chicago	Dallas	Eugene	Atlanta
Yolanda	Atlanta	Eugene	Dallas	Chicago	Boston
Zeus	Dallas	Boston	Eugene	Chicago	Atlanta

Eugene proposes to Wayne

Wayne rejects (since he prefers Chicago)

Gale–Shapley demo

hospitals' preference lists

	1st	2nd	3rd	4th	5th
Atlanta	Wayne	Val	Yolanda	Zeus	Xavier
Boston	Yolanda	Wayne	Val	Xavier	Zeus
Chicago	Wayne	Zeus	Xavier	Yolanda	Val
Dallas	Val	Yolanda	Xavier	Wayne	Zeus
Eugene	Wayne	Yolanda	Val	Zeus	Xavier

students' preference lists

	1st	2nd	3rd	4th	5th
Val	Eugene	Atlanta	Boston	Dallas	Chicago
Wayne	Chicago	Boston	Dallas	Atlanta	Eugene
Xavier	Boston	Chicago	Dallas	Eugene	Atlanta
Yolanda	Atlanta	Eugene	Dallas	Chicago	Boston
Zeus	Dallas	Boston	Eugene	Chicago	Atlanta

Eugene proposes to Yolanda

Gale-Shapley demo

hospitals' preference lists

	1st	2nd	3rd	4th	5th
Atlanta	Wayne	Val	Yolanda	Zeus	Xavier
Boston	Yolanda	Wayne	Val	Xavier	Zeus
Chicago	Wayne	Zeus	Xavier	Yolanda	Val
Dallas	Val	Yolanda	Xavier	Wayne	Zeus
Eugene	Wayne	Yolanda	Val	Zeus	Xavier

students' preference lists

	1st	2nd	3rd	4th	5th
Val	Eugene	Atlanta	Boston	Dallas	Chicago
Wayne	Chicago	Boston	Dallas	Atlanta	Eugene
Xavier	Boston	Chicago	Dallas	Eugene	Atlanta
Yolanda	Atlanta	Eugene	Dallas	Chicago	Boston
Zeus	Dallas	Boston	Eugene	Chicago	Atlanta

Eugene proposes to Yolanda

Yolanda accepts (and renounces Dallas)

Gale-Shapley demo

hospitals' preference lists

	1st	2nd	3rd	4th	5th
Atlanta	Wayne	Val	Yolanda	Zeus	Xavier
Boston	Yolanda	Wayne	Val	Xavier	Zeus
Chicago	Wayne	Zeus	Xavier	Yolanda	Val
Dallas	Val	Yolanda	Xavier	Wayne	Zeus
Eugene	Wayne	Yolanda	Val	Zeus	Xavier

students' preference lists

	1st	2nd	3rd	4th	5th
Val	Eugene	Atlanta	Boston	Dallas	Chicago
Wayne	Chicago	Boston	Dallas	Atlanta	Eugene
Xavier	Boston	Chicago	Dallas	Eugene	Atlanta
Yolanda	Atlanta	Eugene	Dallas	Chicago	Boston
Zeus	Dallas	Boston	Eugene	Chicago	Atlanta

Dallas proposes to Xavier

Gale–Shapley demo

hospitals' preference lists

	1st	2nd	3rd	4th	5th
Atlanta	Wayne	Val	Yolanda	Zeus	Xavier
Boston	Yolanda	Wayne	Val	Xavier	Zeus
Chicago	Wayne	Zeus	Xavier	Yolanda	Val
Dallas	Val	Yolanda	Xavier	Wayne	Zeus
Eugene	Wayne	Yolanda	Val	Zeus	Xavier

students' preference lists

	1st	2nd	3rd	4th	5th
Val	Eugene	Atlanta	Boston	Dallas	Chicago
Wayne	Chicago	Boston	Dallas	Atlanta	Eugene
Xavier	Boston	Chicago	Dallas	Eugene	Atlanta
Yolanda	Atlanta	Eugene	Dallas	Chicago	Boston
Zeus	Dallas	Boston	Eugene	Chicago	Atlanta

Dallas proposes to Xavier

Xavier rejects (since he prefers Boston)

Gale–Shapley demo

hospitals' preference lists

	1st	2nd	3rd	4th	5th
Atlanta	Wayne	Val	Yolanda	Zeus	Xavier
Boston	Yolanda	Wayne	Val	Xavier	Zeus
Chicago	Wayne	Zeus	Xavier	Yolanda	Val
Dallas	Val	Yolanda	Xavier	Wayne	Zeus
Eugene	Wayne	Yolanda	Val	Zeus	Xavier

students' preference lists

	1st	2nd	3rd	4th	5th
Val	Eugene	Atlanta	Boston	Dallas	Chicago
Wayne	Chicago	Boston	Dallas	Atlanta	Eugene
Xavier	Boston	Chicago	Dallas	Eugene	Atlanta
Yolanda	Atlanta	Eugene	Dallas	Chicago	Boston
Zeus	Dallas	Boston	Eugene	Chicago	Atlanta

Dallas proposes to Wayne

Gale–Shapley demo

hospitals' preference lists

	1st	2nd	3rd	4th	5th
Atlanta	Wayne	Val	Yolanda	Zeus	Xavier
Boston	Yolanda	Wayne	Val	Xavier	Zeus
Chicago	Wayne	Zeus	Xavier	Yolanda	Val
Dallas	Val	Yolanda	Xavier	Wayne	Zeus
Eugene	Wayne	Yolanda	Val	Zeus	Xavier

students' preference lists

	1st	2nd	3rd	4th	5th
Val	Eugene	Atlanta	Boston	Dallas	Chicago
Wayne	Chicago	Boston	Dallas	Atlanta	Eugene
Xavier	Boston	Chicago	Dallas	Eugene	Atlanta
Yolanda	Atlanta	Eugene	Dallas	Chicago	Boston
Zeus	Dallas	Boston	Eugene	Chicago	Atlanta

Dallas proposes to Wayne

Wayne rejects (since he prefers Chicago)

Gale–Shapley demo

hospitals' preference lists

	1st	2nd	3rd	4th	5th
Atlanta	Wayne	Val	Yolanda	Zeus	Xavier
Boston	Yolanda	Wayne	Val	Xavier	Zeus
Chicago	Wayne	Zeus	Xavier	Yolanda	Val
Dallas	Val	Yolanda	Xavier	Wayne	Zeus
Eugene	Wayne	Yolanda	Val	Zeus	Xavier

students' preference lists

	1st	2nd	3rd	4th	5th
Val	Eugene	Atlanta	Boston	Dallas	Chicago
Wayne	Chicago	Boston	Dallas	Atlanta	Eugene
Xavier	Boston	Chicago	Dallas	Eugene	Atlanta
Yolanda	Atlanta	Eugene	Dallas	Chicago	Boston
Zeus	Dallas	Boston	Eugene	Chicago	Atlanta

Dallas proposes to Zeus

Gale-Shapley demo

hospitals' preference lists

	1st	2nd	3rd	4th	5th
Atlanta	Wayne	Val	Yolanda	Zeus	Xavier
Boston	Yolanda	Wayne	Val	Xavier	Zeus
Chicago	Wayne	Zeus	Xavier	Yolanda	Val
Dallas	Val	Yolanda	Xavier	Wayne	Zeus
Eugene	Wayne	Yolanda	Val	Zeus	Xavier

students' preference lists

	1st	2nd	3rd	4th	5th
Val	Eugene	Atlanta	Boston	Dallas	Chicago
Wayne	Chicago	Boston	Dallas	Atlanta	Eugene
Xavier	Boston	Chicago	Dallas	Eugene	Atlanta
Yolanda	Atlanta	Eugene	Dallas	Chicago	Boston
Zeus	Dallas	Boston	Eugene	Chicago	Atlanta

Dallas proposes to Zeus

Zeus accepts
(since previously unmatched)

Gale–Shapley demo

hospitals' preference lists

	1st	2nd	3rd	4th	5th
Atlanta	Wayne	Val	Yolanda	Zeus	Xavier
Boston	Yolanda	Wayne	Val	Xavier	Zeus
Chicago	Wayne	Zeus	Xavier	Yolanda	Val
Dallas	Val	Yolanda	Xavier	Wayne	Zeus
Eugene	Wayne	Yolanda	Val	Zeus	Xavier

students' preference lists

	1st	2nd	3rd	4th	5th
Val	Eugene	Atlanta	Boston	Dallas	Chicago
Wayne	Chicago	Boston	Dallas	Atlanta	Eugene
Xavier	Boston	Chicago	Dallas	Eugene	Atlanta
Yolanda	Atlanta	Eugene	Dallas	Chicago	Boston
Zeus	Dallas	Boston	Eugene	Chicago	Atlanta

STOP
(stable matching)

The Gale-Shapley proposal algorithm

While there is a man m who is not matched:

- Let w be the highest ranked woman in m's list to whom m has not proposed yet.
- If w is unmatched, or w prefers m over her current match:
  - Match m and w. (The previous match of w is now unmatched.)

Cool, but does it work correctly?

- Does it always terminate?
- Does it always find a stable matching?

(Does a stable matching always exist?)

Gale-Shapley algorithm analysis

Theorem:

The *Gale-Shapley proposal algorithm* always terminates with a stable matching after at most n^2 iterations.

A *constructive* proof that a stable matching always exists.

3 things to show:

1. Number of iterations is at most n^2.
2. The algorithm terminates with a perfect matching.
3. The matching has no unstable pairs.

Gale-Shapley algorithm analysis

1. Number of iterations is at most n^2.

iterations = # proposals

No man proposes to a woman more than once.

So each man makes at most n proposals.

There are n men in total.

$\implies$ # proposals $\leq n^2$.

$\implies$ # iterations $\leq n^2$.

Gale-Shapley algorithm analysis

2. The algorithm terminates with a perfect matching.

If we don't have a perfect matching:

A man is not matched

$\implies$ All women must be matched

$\implies$ All men must be matched.

Contradiction

<u>Second implication:</u>

There are an equal number of men and women.

Gale-Shapley algorithm analysis

2. The algorithm terminates with a perfect matching.

If we don't have a perfect matching:

A man is not matched

$\Longrightarrow$ All women must be matched

$\Longrightarrow$ All men must be matched.

Contradiction

First implication:

Observe: once a woman is matched, she stays matched.

A man got rejected by every woman:

case1: she was already matched, or

case2: she got a better offer

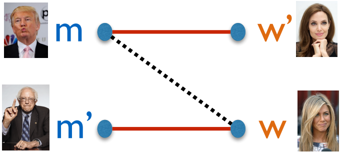
Either way, she was matched at some point.

Gale-Shapley algorithm analysis

3. The matching has <u>no</u> unstable pairs.

> “<u>Improvement” Lemma</u>:
> **(i)** A man can only go down in his preference list.
> **(ii)** A woman can only go up in her preference list.

Unstable pair:
(m,w) <u>unmatched</u>
but they prefer each other.



Consider any <u>unmatched</u> (m,w). **WTS**: it cannot be unstable.

Case 1: m never proposed to w

by (i), m prefers w’ over w

Case 2: m proposed to w

w rejected m $\Longrightarrow$ by (ii), w prefers m’ over m ☐

Further questions

Theorem:

The *Gale-Shapley proposal algorithm* always terminates with a stable matching after at most n^2 iterations.

Does the order of how we pick men matter?

Would it lead to different matchings?

Is the algorithm "fair"?

Does this algorithm favor men or women or neither?

Further questions

m and **w** are *valid partners* if there is a stable matching in which they are matched.

best(**m**) = highest ranked valid partner of **m**

Theorem:
Gale-Shapley algorithm returns $\{(\mathbf{m}, \text{best}(\mathbf{m})) : \mathbf{m} \in X\}$.

Not at all obvious this would be a matching, let alone a stable matching!

Further questions

worst(**w**) = lowest ranked valid partner of **w**

Theorem:
Gale-Shapley algorithm returns {(worst(**w**), **w**) : **w** $\in$ Y }.

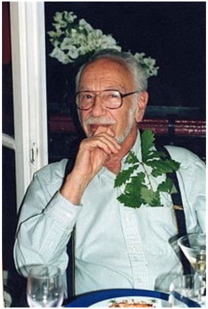
Real-world applications

Variants of the Gale-Shapley algorithm
is used for:

- matching medical students and hospitals
- matching students to high schools (e.g. in New York)
- matching students to universities (e.g. in Hungary)
- matching users to servers

⋮

The Gale-Shapley Proposal Algorithm (1962)



Nobel Prize in Economics   2012

"for the theory of stable allocations and the practice of market design."