Lectures 14 and 15 (Week 9)
Advanced Reasoning with FOL

COMP24412: Symbolic AI

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This Week

Going from formulas to clauses

Organisation saturation using the Given Clause Algorithm

A more optimal Ordered Resolution

Soundness and Completeness of (Ordered) Resolution

More optimisations in preprocessing and proof search

How to use Vampire

Aim and Learning Outcomes

The aim of these two lectures is to:

Understand how we can efficiently check entailments for general first-order knowledge bases

Learning Outcomes

By the end you will be able to:

1. Translate an arbitrary first-order formulas to a set of clauses
2. Recall and describe key terminology (e.g. clause, saturation, ...)
3. Describe (with examples) the given clause algorithm for saturation
4. Explain and apply <u>ordered</u> resolution
5. Describe why ordered resolution is sound and complete
6. Use Vampire to check entailments

Quick Recap

A literal is an an atom or its negation. A clause is a disjunction of literals. Clauses are implicitly universally quantified.

Resolution works on clauses

$$\frac{\neg A \vee B \qquad C \vee D}{(B \vee D)\theta} \quad \theta = \mathsf{mgu}(A, C)$$

Is sound as the conclusion is true in any model of the premises.

Before applying resolution we need to *name-apart* the premises

We will use resolution for refutational based reasoning.

Recall $\Gamma \models \phi$ if and only if $\Gamma \cup \{\neg\phi\} \models$ *false*. We will saturate $\Gamma \cup \{\neg\phi\}$ until there is nothing left to add or we have derived *false*.
(this doesn't necessarily terminate as FOL is semi-decidable).

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$$p(x, a) \qquad \neg p(b, x)$$

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Reminder: Naming Apart

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Equivalent to $(\forall x.p(x, a)) \wedge (\forall y.p(b, y))$ i.e. clauses $p(x, a)$ and $\neg p(b, y)$

Hidden rule: Naming Apart.
When unifying literals from two different clauses you should first rename variables so that the literals do not share literals.

Sometimes I do this implicitly e.g. in $p(x)$ and $\neg p(x)$ we should rename to $p(x)$ and $\neg p(y)$ and then apply $\{x \mapsto y\}$ but that's tedious.

# An aside about terminology: Theorem Proving

What is this Theorem Proving?

It is a subfield of automated reasoning that focuses on finding proofs of entailment.

The terminology is a bit different due to roots in mathematical logic.

What we are referring to as a knowledge base is referred to as a <u>theory</u> and any statement entailed by a theory is a <u>theorem</u> of that theory, hence theorem proving.

Architecture of a (Saturation-Based) Theorem Prover
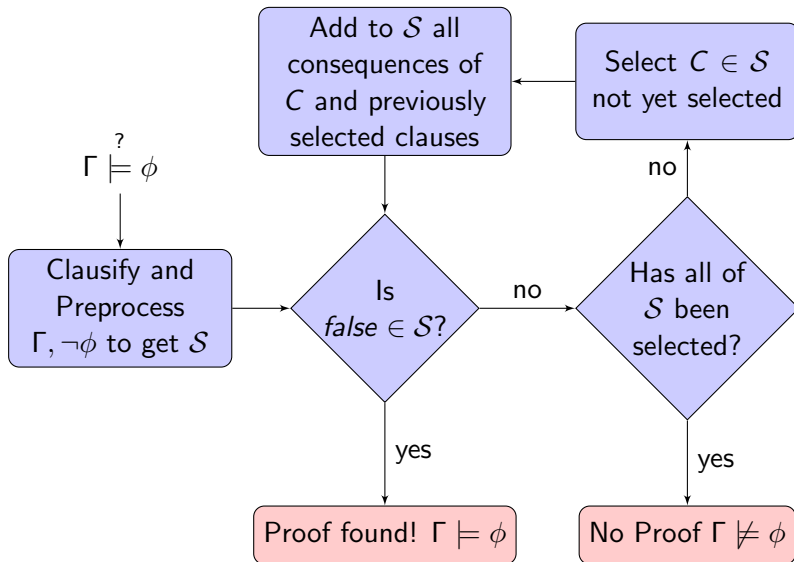
Saturation Based Reasoning

To decide $\Gamma \models \varphi$ we are going to follow the below steps

We split $\mathcal{S}$ into two parts to keep track of selected and not selected.

1. Let $NotDone = \Gamma \cup \{\neg\varphi\}$ and $Done$ be an empty set of clauses
2. Transform $NotDone$ into clauses
3. If $NotDone$ contains *false* then **return valid**
4. Select a clause C from $NotDone$
5. Perform all inferences (e.g. resolution) between C and all clauses in $Done$ putting any (useful) *new* children in $NotDone$
6. Move C to $Done$
7. If $NotDone$ is not empty stop go to 3
8. **return not valid**

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Transformation to Clausal Form

To go from general formula to set of clauses we're going to go through the following steps

1. Rectify the formula
2. Transform to *Negation Normal Form*
3. Eliminate quantifiers
4. Transform into conjunctive normal form
5. Eliminate equality

At any stage we might simplify using rules about *true* and *false*, e.g. *false* $\wedge \phi =$ *false*, and remove tautologies, e.g. $p \vee p$.

Rectification

(Recall point in Week 7 about renaming variables in quantified formulas)

A formula is rectified if each quantifier binds a different variable and all bound variables are distinct from free ones.

To rectify a formula we identify any name clashes, pick one and rename it consistently. It is important to work out which variables are in the scope of a quantifier e.g.

$$\forall x.(p(x) \vee \exists x.r(x)) \qquad becomes \qquad \forall x(p(x) \vee \exists y.r(y))$$

To automate this we typically rename bound variables starting with x_0 etc.

Negation Normal Form

Apply rules in a completely deterministic syntactically-guided way

$$
\begin{aligned}
\neg(F_1 \wedge \ldots \wedge F_n) &\Rightarrow & \neg F_1 \vee \ldots \vee \neg F_n \\
\neg(F_1 \vee \ldots \vee F_n) &\Rightarrow & \neg F_1 \wedge \ldots \wedge \neg F_n \\
F_1 \rightarrow F_2 &\Rightarrow & \neg F_1 \vee F_2 \\
\neg\neg F &\Rightarrow & F \\
\neg \forall x_1, \ldots, x_n F &\Rightarrow & \exists x_1, \ldots, x_n \neg F \\
\neg \exists x_1, \ldots, x_n F &\Rightarrow & \forall x_1, \ldots, x_n \neg F \\
\neg(F_1 \leftrightarrow F_2) &\Rightarrow & F_1 \otimes F_2 \\
\neg(F_1 \otimes F_2) &\Rightarrow & F_1 \leftrightarrow F_2 \\
F_1 \leftrightarrow F_2 &\Rightarrow & (F_1 \rightarrow F_2) \wedge (F_2 \rightarrow F_1); \\
F_1 \otimes F_2 &\Rightarrow & (F_1 \vee F_2) \wedge (\neg F_1 \vee \neg F_2).
\end{aligned}
$$

Where $\otimes$ is exclusive or. Can get an exponential increase in size.

Let's do these by hand and then later see what Vampire does.

$$(\forall x.p(x)) \rightarrow (\exists x.p(x))$$

$$(\forall x.(p(x) \lor q(x)) \leftrightarrow (\neg \exists x.(\neg p(x) \land \neg q(x)))$$

$$\forall x, y, z.((f(x) = y \land f(x) = z) \rightarrow y = z)$$

$$\forall x.((\exists y.p(x, y)) \rightarrow q(x)) \land p(a, b) \land \neg \exists x.q(x)$$

Dealing with Existential Quantifiers

There is a best kind of pizza
$\exists x. \forall y. ((pizza(x) \land pizza(y) \land x \neq y) \rightarrow better(x, y))$

There are two different people who live in the same house
$\exists x. \exists y. \exists z. (x \neq y \land lives\_in(x, z) \land lives\_in(y, z))$

Everybody loves somebody
$\forall x. \exists y. loves(x, y)$

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Let $\mathcal{I}$ be some interpretation. If $\mathcal{I}(\exists x. \phi[x])$ is true then there must be some domain constant d such that $\mathcal{I}(\phi[d])$ is true.

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Let a be a fresh constant symbol (a new name for the object that has to exist). As it is fresh we can freely let $\mathcal{I}(a) = d$.
(This requires the Axiom of Choice)

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Let $\mathcal{I}$ be some interpretation. If $\mathcal{I}(\forall x.\exists y.\phi[x, y])$ is true then for every domain constant d_1 there must be some other domain constant d_2 such that $\mathcal{I}(\phi[d_1, d_2])$ is true. But how do we find d_2 given d_1?

Let f be a fresh function symbol whose interpretation can be made to *select* the necessary domain constant.

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Forget about them

Skolemisation

This process is called Skolemisation and the new symbols we introduce are called Skolem constants or Skolem functions.

The rules are simply

$$\forall x_1, \ldots, x_n F \;\Rightarrow\; F$$
$$\exists x_1, \ldots, x_n F \;\Rightarrow\; F\{x_1 \mapsto f_1(y_1, \ldots, y_m), \ldots, x_n \mapsto f_n(y_1, \ldots, y_m)\},$$

where f_i are fresh function symbols of the correct arrity and $y_1, \ldots y_m$ are the free variables of F e.g. they are the things universally quantified in the larger scope.

Remember that Skolem constants/functions act as witnesses for something that we know has to exist for the formula to be true.

Do these two formulas have the same models?

$$\exists x. \forall y. p(x, y) \qquad \forall y. p(a, y)$$

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No - the first one does not need to interpret a

Validity or Satisfiability Preserving?

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Validity or Satisfiability Preserving?

Do these two formulas have the same models?

$$\exists x.\forall y.p(x, y) \qquad \forall y.p(a, y)$$

No - the first one does not need to interpret *a*

Do the formulas both have models?

Yes - there's no negation, just make everything true

When we introduce new symbols we preserve satisfiability (the existence of models) not validity (the same models).

However, most transformations are stronger than this and preserve models on the initial signature.

CNF Transformation

CNF stands for Clausal Normal Form (or Conjunctive Normal Form, similar but different)

Now the only connectives left should be $\wedge$ and $\vee$ and we need to push the $\vee$ symbols under the $\wedge$ symbols

The associated rule is

$$(A_1 \wedge \ldots \wedge A_m) \vee B_1 \vee \ldots \vee B_n \quad \Rightarrow \quad \begin{array}{ll} (A_1 \vee B_1 \vee \ldots \vee B_n) & \wedge \\ \qquad \cdots & \wedge \\ (A_m \vee B_1 \vee \ldots \vee B_n). & \end{array}$$

This can lead to exponential growth (see optimisation in a bit).

Eliminating Equality

At the moment we're just dealing with the Resolution Calculus. Next week I'll show you how to deal with equality natively but before we do this we need to get rid of equality.

As outlined in Week 7, we add a new equality predicate eq and add clauses capturing reflexivity, symmetry, and transitivity of equality:

$$eq(x, x) \qquad \neg eq(x, y) \vee eq(y, x) \qquad \neg eq(x, y) \vee \neg eq(y, z)) \vee eq(x, z)$$

And clauses representing congruence of functions and predicates:

$$\neg eq(x_1, y_1) \vee \ldots \vee \neg eq(x_n, y_n) \vee eq(f(x_1, \ldots x_n), f(y_1, \ldots, y_n)))$$
$$\neg eq(x_1, y_1) \vee \ldots \vee \neg eq(x_n, y_n) \vee \neg p(x_1, \ldots x_n) \vee p(y_1, \ldots, y_n))$$

for all functions f and predicates p of arity n

Vampire

Automated theorem prover for first-order logic

Is a refutational saturation-based theorem prover implementing resolution

Very efficient/powerful (wins lots of competitions)

Used as a back-box solver in lots of other things (in academia and industry)

Available from `https://vprover.github.io` (and I'll provide a direct download link)

You will get to use Vampire in Coursework 3.

TPTP

Language for describing first-order formulas in ASCII format.

See http://tptp.cs.miami.edu/~tptp/

For example

```
fof(one,axiom, ![X] : (happy(X) <=> (?[Y] : loves(X,Y)))).
fof(two,axiom, ![X] : (rich(X) => loves(X,money))).
fof(three,axiom, rich(giles)).
fof(goal,conjecture, happy(giles)).
```

Important - axiom for knowledge base, conjecture for goal and Vampire will do the negation for you.

Looking at Vampire's clausification

Run

```
./vampire --mode clausify --equality_proxy RSTC problem
```

on problem file `problem`

It will sometimes do a lot more than what we've discussed above.

Vampire performs lots of optimisations e.g. naming subformulas, removing pure symbols, or unused definitions.

Let's do these by hand and then see what Vampire does.

$$(\forall x. p(x)) \rightarrow (\exists x. p(x))$$

$$(\forall x. (p(x) \lor q(x)) \leftrightarrow (\neg \exists x. (\neg p(x) \land \neg q(x))))$$

$$\forall x, y, z. ((f(x) = y \land f(x) = z) \rightarrow y = z)$$

$$\forall x. ((\exists y. p(x, y)) \rightarrow q(x)) \land p(a, b) \land \neg \exists x. q(x)$$

Also, which of the above statements are valid or inconsistent? How do we ask Vampire?

Optimisation (subformula naming)

To go from general formula to set of clauses we're going to go through the following steps

1. Rectify the formula
2a. Transform to *Equivalence Negation Form*
2b. (Optimisation) Apply *naming* of subformulas
2c. Transform to *Negation Normal Form*
3. Eliminate quantifiers
4. Transform into conjunctive normal form
5. Eliminate equality

We first transform into ENF (halfway to NNF) to allow us to perform the naming optimisation

Equivalence Negation Form

Push negations in but preserve equivalences e.g. drop last two rules of NNF transformation.

$$\neg(F_1 \wedge \ldots \wedge F_n) \quad \Rightarrow \quad \neg F_1 \vee \ldots \vee \neg F_n$$
$$\neg(F_1 \vee \ldots \vee F_n) \quad \Rightarrow \quad \neg F_1 \wedge \ldots \wedge \neg F_n$$
$$F_1 \rightarrow F_2 \quad \Rightarrow \quad \neg F_1 \vee F_2$$
$$\neg\neg F \quad \Rightarrow \quad F$$
$$\neg \forall x_1, \ldots, x_n F \quad \Rightarrow \quad \exists x_1, \ldots, x_n \neg F$$
$$\neg \exists x_1, \ldots, x_n F \quad \Rightarrow \quad \forall x_1, \ldots, x_n \neg F$$
$$\neg(F_1 \leftrightarrow F_2) \quad \Rightarrow \quad F_1 \otimes F_2$$
$$\neg(F_1 \otimes F_2) \quad \Rightarrow \quad F_1 \leftrightarrow F_2$$

Only get a linear increase in size.

Subformula Naming

We want to get to a conjunction of disjunctions but this process can 'blow up' in general e.g.

$$p(x, y) \leftrightarrow (q(x) \leftrightarrow (p(y, y) \leftrightarrow q(y))),$$

is equivalent to

$$p(x, y) \lor \neg q(y) \lor \neg p(y, y) \lor \neg q(x))$$
$$p(x, y) \lor q(y) \lor p(y, y) \lor \neg q(x))$$
$$p(x, y) \lor p(y, y) \lor \neg q(y) \lor q(x))$$
$$p(x, y) \lor q(y) \lor \neg p(y, y) \lor q(x))$$
$$q(x) \lor \neg q(y) \lor \neg p(y, y) \lor \neg p(x, y))$$
$$q(x) \lor q(y) \lor p(y, y) \lor \neg p(x, y))$$
$$p(y, y) \lor \neg q(y) \lor \neg q(x) \lor \neg p(x, y))$$
$$q(y) \lor \neg p(y, y) \lor \neg q(x) \lor \neg p(x, y))$$

We can replace

$$p(x, y) \leftrightarrow (q(x) \leftrightarrow (p(y, y) \leftrightarrow q(y))),$$

by

$$p(x, y) \leftrightarrow (q(x) \leftrightarrow n(y));$$
$$n(y) \leftrightarrow (p(y, y) \leftrightarrow q(y)).$$

to get the same number of clauses but each clause is simpler (better for reasoning).

In the case when the subformula $F(x_1, \ldots, x_k)$ has only positive occurrences in G, one can use the axiom $n(x_1, \ldots, x_k) \rightarrow F(x_1, \ldots, x_k)$ instead of $n(x_1, \ldots, x_k) \leftrightarrow F(x_1, \ldots, x_k)$. This will lead to fewer clauses.

Subformula Naming

Assigning a name n to $F_2 \leftrightarrow F_3$ yields two formulas

$$F_1 \leftrightarrow n;$$
$$n \leftrightarrow (F_2 \leftrightarrow F_3),$$

where the second formula has the same structure as the original formula $F_1 \leftrightarrow (F_2 \leftrightarrow F_3)$.

When to Name Subformulas?

Vampire uses a heuristic that that estimates how many clauses a subformula will produce and names that subformula if that number is above a certain threshold.

Naming can not increase the number of clauses introduced but does not always reduce.

For those of you who took Logic and Modelling. The idea is the same as the optimised structural transformation in the propositional case but we don't always apply it as the cost on reasoning is much higher here.

Is this Knowledge Base consistent?

```
fof(a1, axiom, ![X] : (winner(X) <=> ?[Y] : wins(Y) = X)).
fof(a2, axiom, ![X] : (wins(X) = position(X,first))).
fof(a3, axiom, ![X] : (takes_part(X,Y) | ~takes_part(X,Y))).
fof(a4, axiom, ![X,Y] : ((wins(X) = Y) => enters(X,Y))).
```

Removing Unnecessary Things

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```

We can remove

- Tautologies - formulas that are true in every model
- Definitions of things that are unused
- References to predicates that are used with a single polarity as we know how to make this predicate true.

Try running `vampire -explain updr` and `vampire -explain fde`

Now we have our problem as a set of clauses we need to see if it is consistent or not.

Recall, we do this by saturation.

This is similar to the forward-chaining algorithm but we want to avoid the problems with forward-chaining.

One way of doing this has already been to add the negation of the goal/query into the clause set, thus making the reasoning more goal-directed

The next thing to do is organise things as a best-first rather than breadth-first search.

Given Clause Algorithm

I've renamed things again! I use the term *active* and *passive* as these are the terms used within Vampire (for historic reasons).

input: *Init*: set of clauses;
var *active*, *passive*, *unprocessed*: set of clauses;
var *given*, *new*: clause;
active := ∅; *unprocessed* := *Init*;
loop
   while *unprocessed* ≠ ∅
     new :=*pop*(*unprocessed*);
     if *new* = □ **then** **return** *unsatisfiable*;
     add *new* to *passive*
   if *passive* = ∅ **then** **return** *satisfiable* or *unknown*
   given := *select*(*passive*);   (* clause selection *)
   move *given* from *passive* to *active*;
   unprocessed :=*infer*(*given*, *active*);   (* generating inferences *)

Clause Selection and Fairness

What if we infinitely delay performing an inference?

We lose the partial decidability

A saturation process is fair if no clause is delayed infinitely often

Two fair clause selection strategies:

- First-in first-out. This is the same as forward-chaining and whilst fair is not very efficient
- Smallest (in number of symbols) first

  (there are a finite number of terms with at most k symbols)

  The intuition is that smaller clauses are 'closer' to the empty clause

Vampire uses an age-weight ratio, alternately selecting based on age (first-in first-out) and weight (number of symbols).

Complete Example

$$\left\{ \begin{array}{c} \forall x.(happy(x) \leftrightarrow \exists y.(loves(x,y))) \\ \forall x.(rich(x) \rightarrow loves(X, money)) \\ rich(giles) \end{array} \right\} \vDash happy(giles)$$