Graph Algorithms

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3. Februar 2020

Outline

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1. Directed Graphs
2. Graph Traversal Algorithms
3. Shortest Path in Weighted Graphs
   - Single-Source Shortest Path
   - Bellman Ford Algorithm

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Dijkstra's Algorithm

- *Relax node* = relax outgoing edges

procedure $\textsc{relax}(u)$
    for all v with $w(u, v) \neq \infty$ **do** $\textsc{relax}(u, v)$
procedure $\textsc{dijkstra}(s)$
    $F \leftarrow \emptyset$, $D \leftarrow \textsc{initEstimate}(\mathsf{s})$
    while $V \setminus F \neq \emptyset$ **do**
        $u \leftarrow \text{Some } u \in V \setminus F,\ D(u) \text{ minimal}$
        $P \leftarrow F \cup \{u\}$
        $\textsc{relax}(u)$
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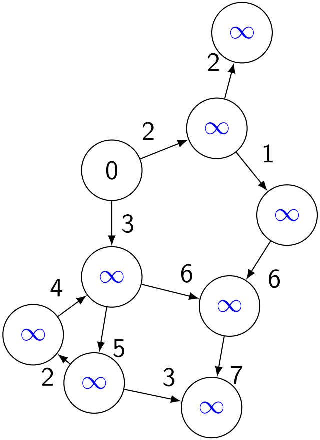

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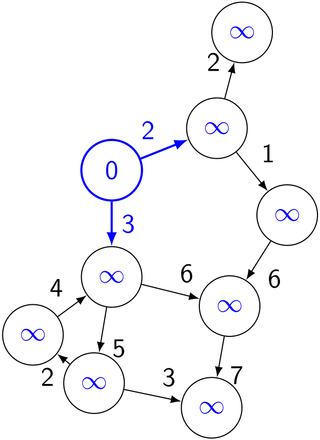

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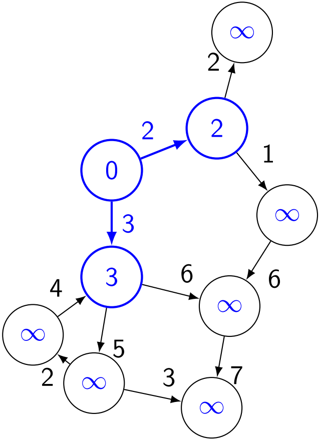

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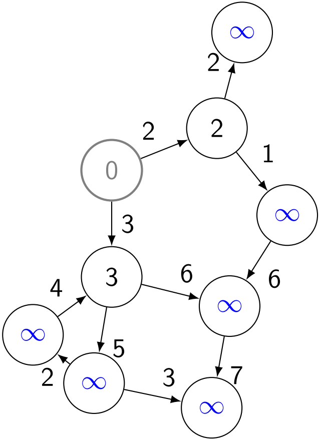

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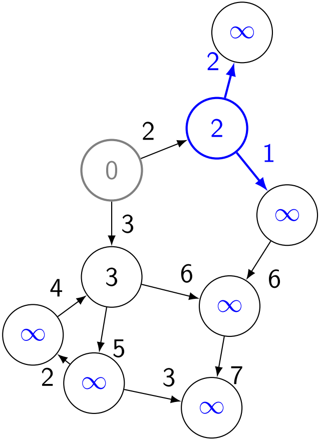

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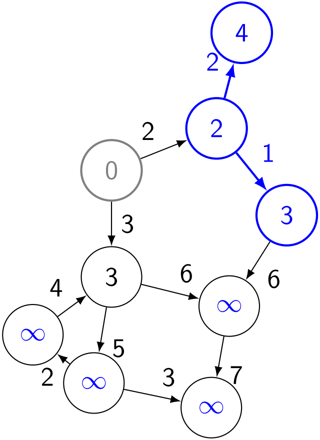

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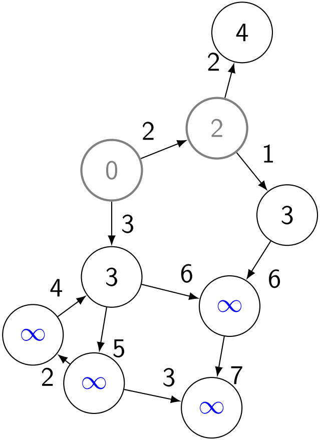

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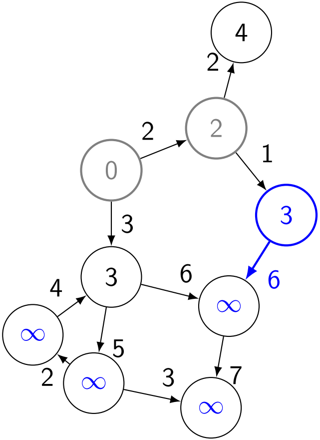

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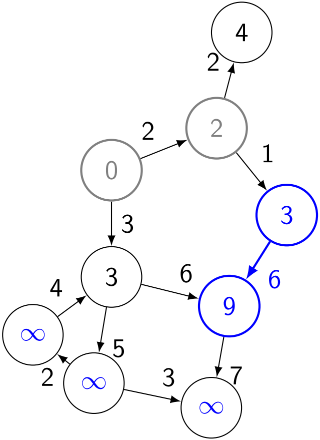

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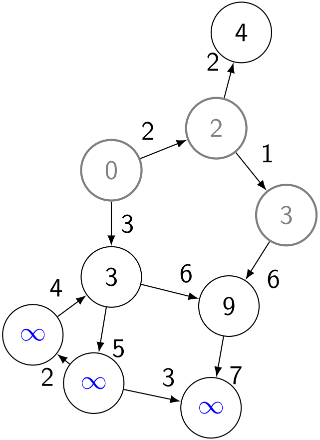

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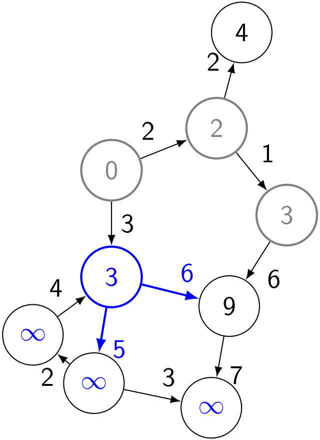

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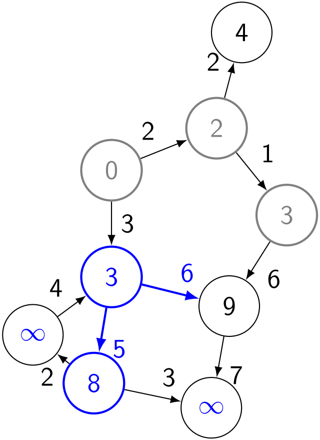

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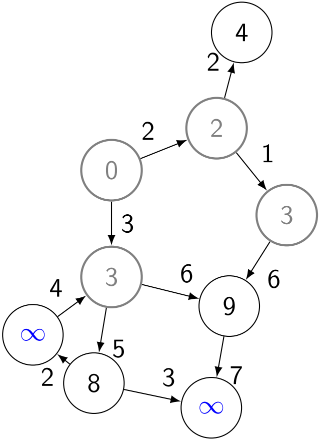

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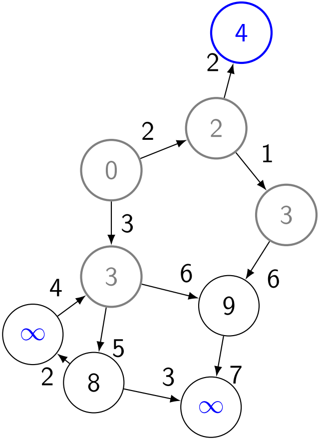

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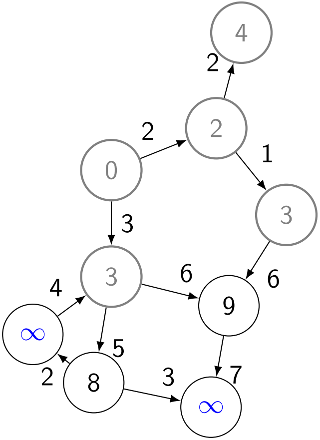

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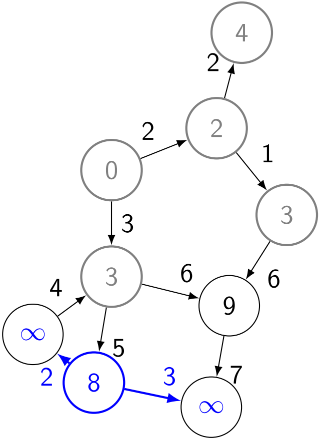

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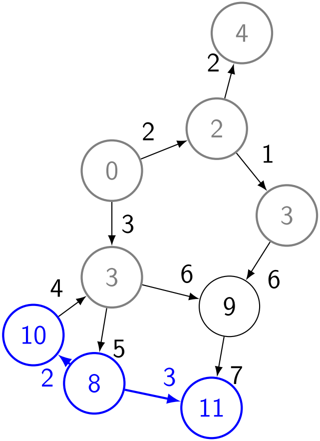

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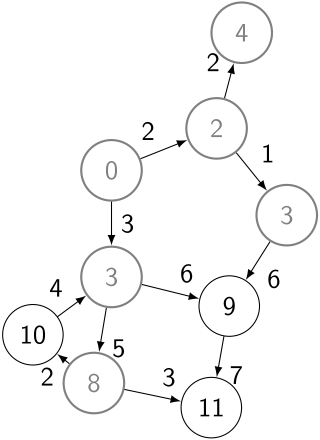

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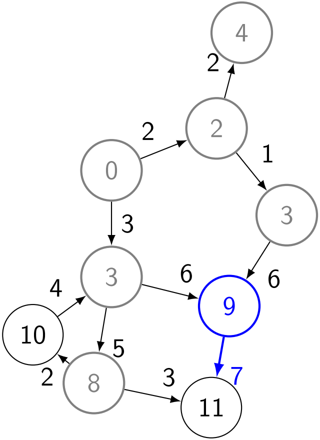

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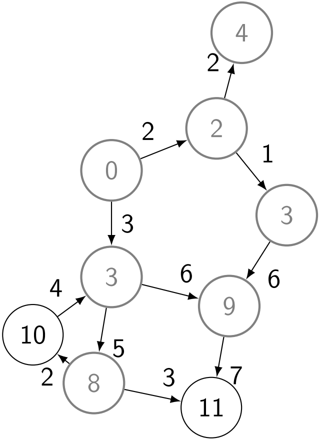

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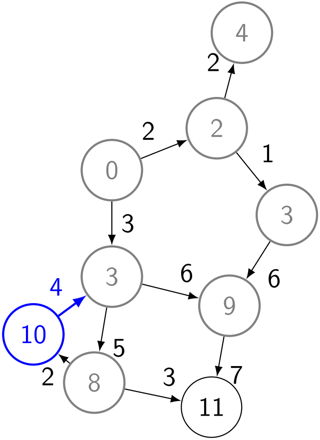

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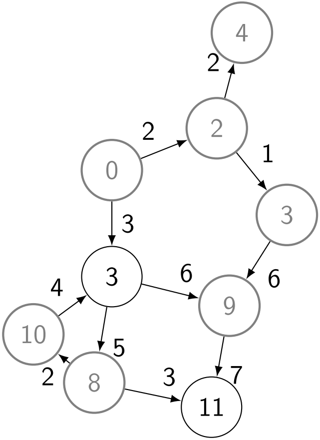

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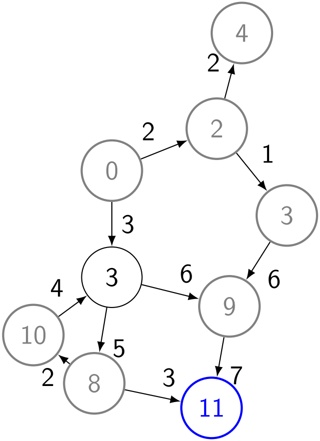

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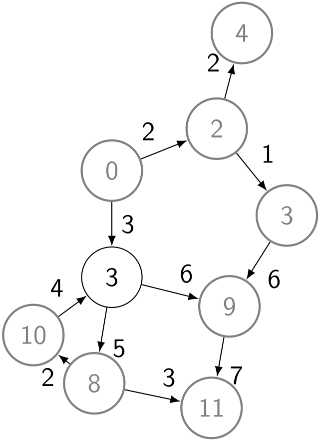

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Dijkstra's Algorithm: Correctness

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- Idea: Relaxed nodes have precise estimate
- As invariant: For all $u \in F$
  - $D(u) = \delta(u)$ (precise)
  - $\forall v.\ D(v) \leq \delta(u) + w(u, v)$ (relaxed with precise $D(u)$)
  - $\forall u \in V.\ D(u) \geq \delta(u)$ (over-estimate)
- Initially: $F = \emptyset$, holds trivially!
- First iteration: Relaxes s, and $D(s) = 0$ is precise
- Further iterations: See next slide!
- Finally: $F \supseteq V$, thus D precise for all nodes!

Dijkstra Algorithm: Invariant preservation

Assume $s \in F$ and for all $u \in F$

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Thus, we would have picked v' instead of v!

Implementing Dijkstra

- Use priority queue for nodes not yet relaxed
  - relaxation: priority of node already in queue may decrease
  - requires decrease-key operation
- Instead of adding all nodes to PQ initially, add nodes as they are discovered
  - unreachable nodes won't be explored
  - no ∞ in PQ required
- Use predecessor map to compute actual paths

Heaps with Decrease-Key

- Recall min-heaps.
- *sift-up* restores heap-property for element with too small priority
    - e.g. after we decreased its priority
    - needs index of element in heap!
- To find index of node in heap:
    - maintain map from node names to index in heap!

Complexity

- Operations:
  - every edge relaxed at most once,
  - every node added and extracted from PQ at most once
- Cost of relaxation, addition, and extraction: $O(\log |V|)$.
- Thus: $O((|E| + |V|) \log |V|)$

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  - recall: Dijkstra relaxes node with minimal $D(u)$

Pseudocode

procedure $\textsc{astar}(s, t)$
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  while $V \setminus F \neq \emptyset$ **do**
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    if $u = t$ **then return** $D(t)$
    $\textsc{relax}(u)$

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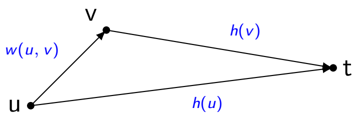

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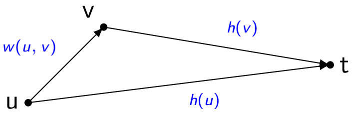

  - monotonicity implies admissibility

Heuristics

- Let s be source, and t be target node
- *Admissible*: distance never over-estimated
  - for all nodes u. $h(u) \leq \delta(u, t)$
- *Monotone*: estimate can't get better when taking edge
  - for all nodes u, v. $h(u) \leq w(u, v) + h(v)$
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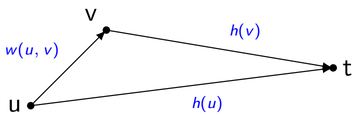

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    - proof by induction over shortest path from u to t.

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- But, for practical problems, typically much better!