NP-Completeness

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- NP: problems with poly-time certificates
  - NP-hard: harder than any problem in NP
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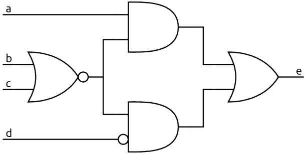
- NP: problems with poly-time certificates
  - NP-hard: harder than any problem in NP
  - NP-complete = in NP + NP-hard
- How to show that problem is ...
  - ... in NP: show it has poly-time certificates
  - ... NP-hard: reduce another NP-hard problem to it

Circuit-SAT

- Given a combinational circuit with n gates, $m \leq 2n$ inputs, and one output
  - Let's restrict gate types to AND, OR, NOT



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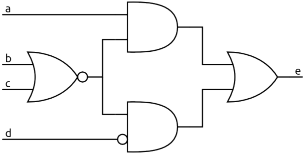

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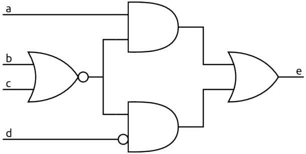

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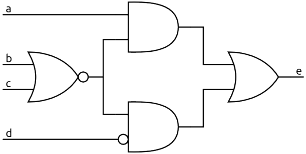

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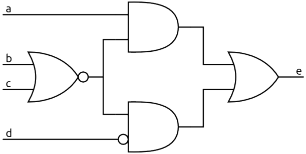

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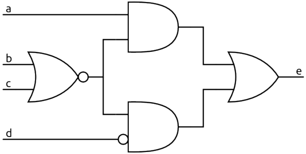

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  - precise proof: lot's of subtle technical details

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- Now: How to construct such a circuit

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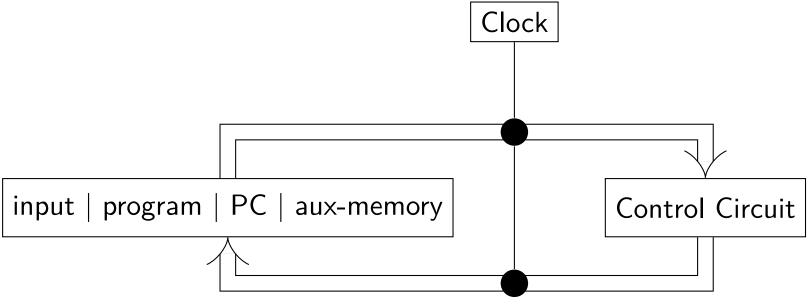
  - easy enough to be simulated by standard computer
    - in $poly(n)$ time per cycle!

Our Computer



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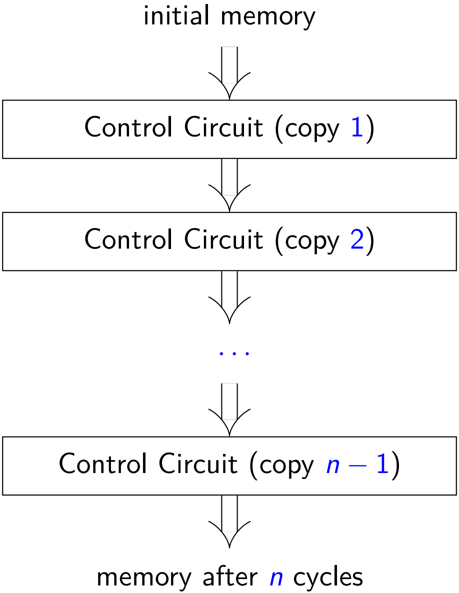
  - here: keep only the single bit that contains result

Circuit for n cycles



Outlook

- We have an initial NP-complete problem
- We now reduce it to other problems, to show that they are NP-complete, too!
- Circuit-SAT $\leq_p$ SAT $\leq_p$ 3SAT $\leq_p$ CLIQUE $\ldots$

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  - can obviously be done in polynomial time
    - constant work per gate and variable!

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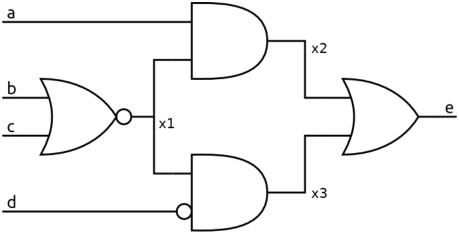

$$
\begin{array}{rc}
 & e \\
\wedge & \neg x_1 \leftrightarrow b \vee c \\
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     - similar for OR, NOT
  2. convert to CNF
     - Convert clauses $x_i \leftrightarrow \dots$ to CNF
     - e.g. $x_i \leftrightarrow x_j \wedge x_k$ to $(\neg x_1 \vee x_2) \wedge (\neg x_1 \vee x_3) \wedge (x_1 \vee \neg x_2 \vee \neg x_3)$

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     - for literals: use $y_i = x_i$
     - for node $t_i = AND(t_j, t_k)$, add clause $y_i \leftrightarrow y_j \wedge y_k$
     - similar for OR, NOT
  2. convert to CNF
     - Convert clauses $x_i \leftrightarrow \ldots$ to CNF
     - e.g. $x_i \leftrightarrow x_j \wedge x_k$ to $(\neg x_1 \vee x_2) \wedge (\neg x_1 \vee x_3) \wedge (x_1 \vee \neg x_2 \vee \neg x_3)$
  3. eliminate duplicates:
     - remove clauses that contain both, x and $\neg x$
     - remove duplicate literals from clause

3SAT

- Boolean formula in CNF, exactly 3 literals over different variables per clause
  - E.g. $(x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_1 \vee x_2 \vee x_3) \wedge (\neg x_1 \vee \neg x_2 \vee x_3)$
- Obviously 3SAT is in NP.
- To prove it is NP-Hard: Show SAT $\leq_p$ 3SAT
- Converting SAT formula to 3SAT
  1. encode parse tree (same idea as for Circuit-SAT $\leq_p$ SAT)
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  4. fill clauses to 3 variables
     a $l_1 \vee l_2$ to $(l_1 \vee l_2 \vee p) \wedge (l_1 \vee l_2 \vee \neg p)$ for fresh variable p
     b l_1 to $(l_1 \vee p) \wedge (l_1 \vee \neg p)$ for fresh variable p, then a)

SAT $\leq_p$ 3SAT — Example

SAT Formula: $\neg(x_1 \land \neg x_2) \lor x_3$

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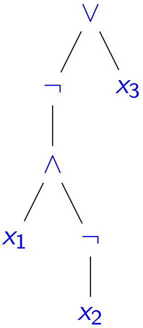
SAT Formula: $\neg(x_1 \wedge \neg x_2) \vee x_3$

Parse tree:



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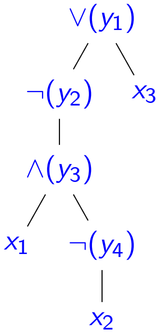

New variables $y_1, \ldots, y_4$.

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Parse tree:

- $\vee(y_1)$
  - $\neg(y_2)$
    - $\wedge(y_3)$
      - x_1
      - $\neg(y_4)$
        - x_2
  - x_3

New variables $y_1, \ldots, y_4$.

Formula for parse-tree:

$$y_1 \wedge (y_1 \leftrightarrow y_2 \vee x_3) \wedge (y_2 \leftrightarrow \neg y_3)$$
$$\wedge\, (y_3 \leftrightarrow x_1 \wedge y_4) \wedge y_4 \leftrightarrow \neg x_2$$

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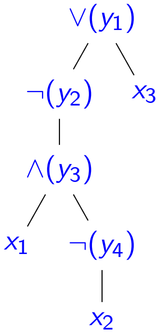

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convert to CNF: $y_1 \wedge (\neg y_1 \vee y_2 \vee x_3) \wedge$
$(y_1 \vee \neg y_2) \wedge (y_1 \vee \neg x_3) \wedge \ldots$

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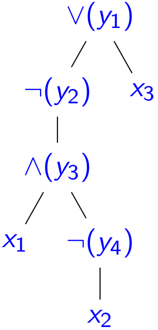

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Formula for parse-tree:

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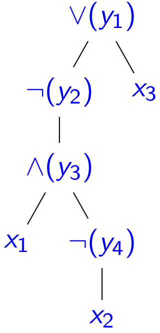

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fill clauses to 3 variables

$(y_1 \vee p_1 \vee p_2) \wedge (y_1 \vee p_1 \vee \neg p_2)$
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- Good, if we want to reduce 3SAT to a problem
  - Only need to encode very special clauses to problem

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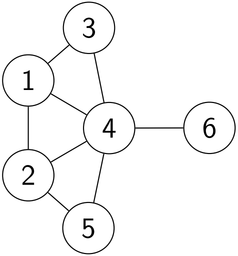
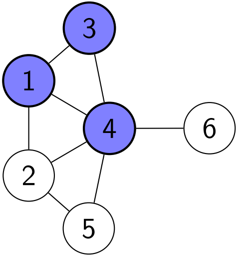
- Decision problem: is there a clique of size $\geq k$?

CLIQUE examples



CLIQUE examples



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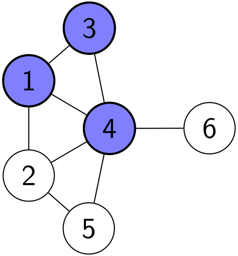

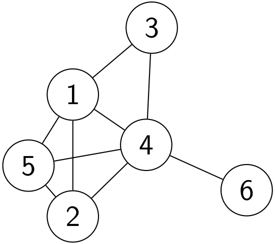

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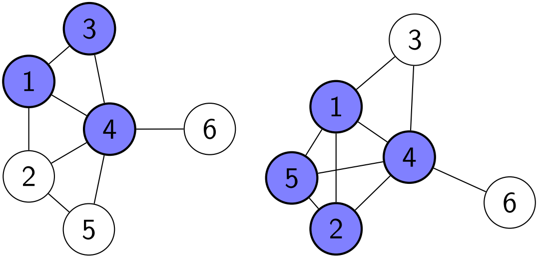

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    - edges between non-contradicting literals of different clauses: $E = \{(u_i^j, u_{i'}^{j'}) \mid i \neq i' \wedge j \leq 3 \wedge (u_i^j, u_{i'}^{j'}) \text{ not contradicting}\}$
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  - claim: graph has n-clique, iff formula satisfiable!

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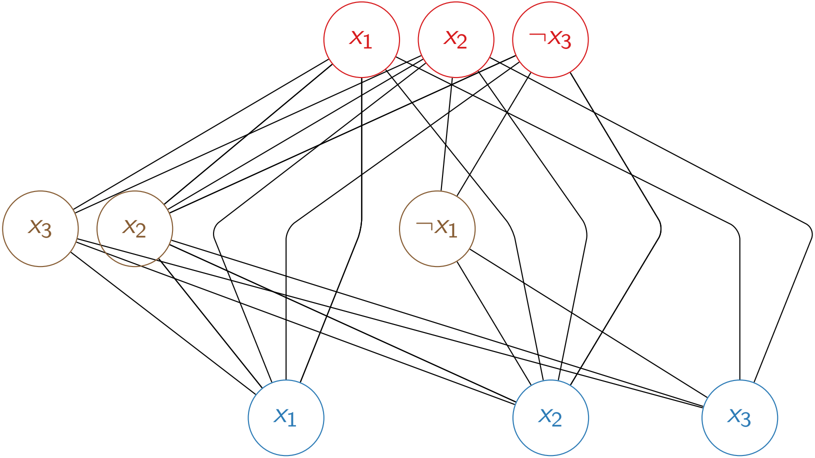
Assume clique of size n.

- must involve literals of different clauses
- which are non-contradictory
- setting them to true yields solution

3SAT $\leq_p$ CLIQUE Example

Formula: $(x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee x_2 \vee x_3)$



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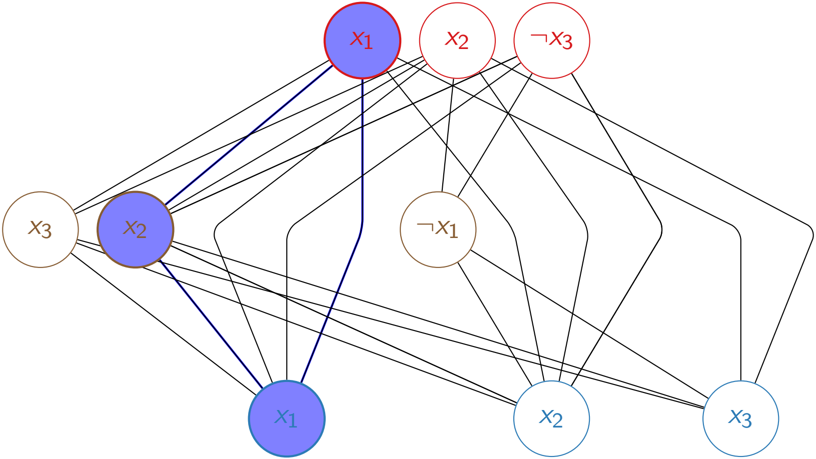

Solution: $x_1 = \top, x_2 = \top, x_3 = ?$

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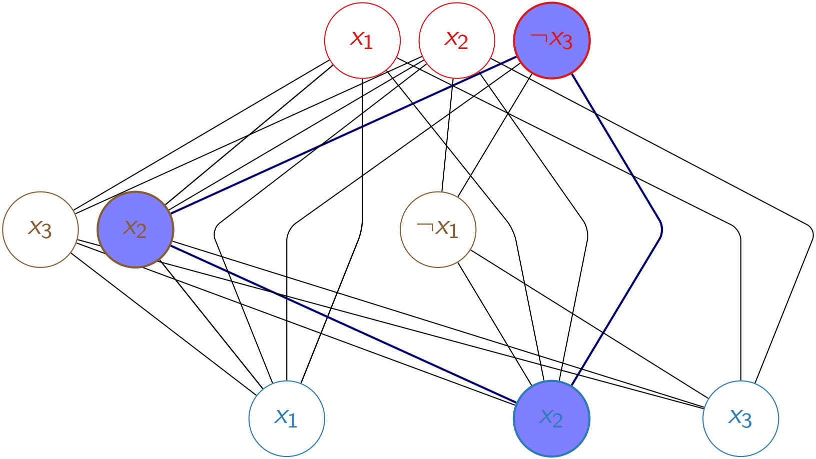

Solution: $x_1 = ?, x_2 = \top, x_3 = \bot$

Conclusions

- NP and NP-complete problems
  - no poly-time algorithms known for NP-hard problems
  - if you encounter one: special case?, approximation?
- Prove that problem is NP-complete:
  - in NP: show poly-time certification
  - NP-hard: reduce other NP-hard problem to it
- Many realistic problems NP-hard
  - You'll come across a few more in this lecture
  - Knapsack, Integer Linear Programming, ...