Project 1 Analysis

September 11, 2015

Turns out I was plotting my exact solution wrong... This took me a while to realize. Figure 1.

I know $CFL = \nu = 1$ is the limit of stability for these schemes. I tried with CFL = 1.05 and all the schemes start to blow up!

Now, I found it was weird that Lax-Wendroff, Upwind and Lax-Friedrichs gave exactly the same results, so I looked at the TE for these schemes. I didn't derive the Truncation Error for Leap Frog or McCormack.

1 Truncation Error

Performing a Taylor expansion on the Upwind, Lax-Friedrichs and Lax-Wendroff schemes leads to the truncation error. Moreover, replacing time derivatives with spatial derivatives allows us to notice the artificial dissipation and dispersion introduced by these schemes – dissipation in the case of Upwind and Lax-Friedrichs, dispersion in Lax-Wendroff.

If we rearrange such that the dissipation/dispersion terms are functions of ν, we can notice that these terms vanish for $\nu = 1$. **This explains why Upwind, Lax-Friedrichs, Lax-Wendroff have the same profile for $\nu = 1$.**

Moreover, in the case that $\nu > 1$, the dissipation is *negative*, i.e. the dissipation turns into an *amplification*. As for dispersion, the artificial dispersive term is this:

$$\frac{A\Delta x^2}{6}(\nu^2 - 1)u_{xxx} \tag{1}$$

which is *negative* for $\nu < 1$ and becomes positive for $\nu > 1$. I'm not sure what this physically means, but it clearly makes the solution unstable.

Thus, for $\nu = 1$, the linear advection equation is supposedly solved exactly as only the higher order derivative terms are left. However, this is only true for low enough Δt and Δx, especially in this case where exact solution is almost discontinuous. So, do we recover the exact solution for low $\Delta t, \Delta x$?

2 Lowering $\Delta t, \Delta x$

Consequently, I also tried decreasing both Δt and Δx, but such that I keep $\nu = 1$. In this case, I do recover the exact solution. See Figure 2.

3 Leap Frog and McCormack?

I re-checked my code for Leap Frog and McCormack multiple times, but I believe it to be correct since the results for a $\nu = 0.25$, as in Figure 3, make sense for these two schemes. However, doing a Taylor expansion of the Leap frog scheme gives me:

$$(u_t + Au_x) = \frac{A\Delta x^2}{6}(\nu^2 - 1)u_{xxx} + ... \tag{2}$$

which is the same dispersive term as in Lax-Wendroff, so I'm puzzled.

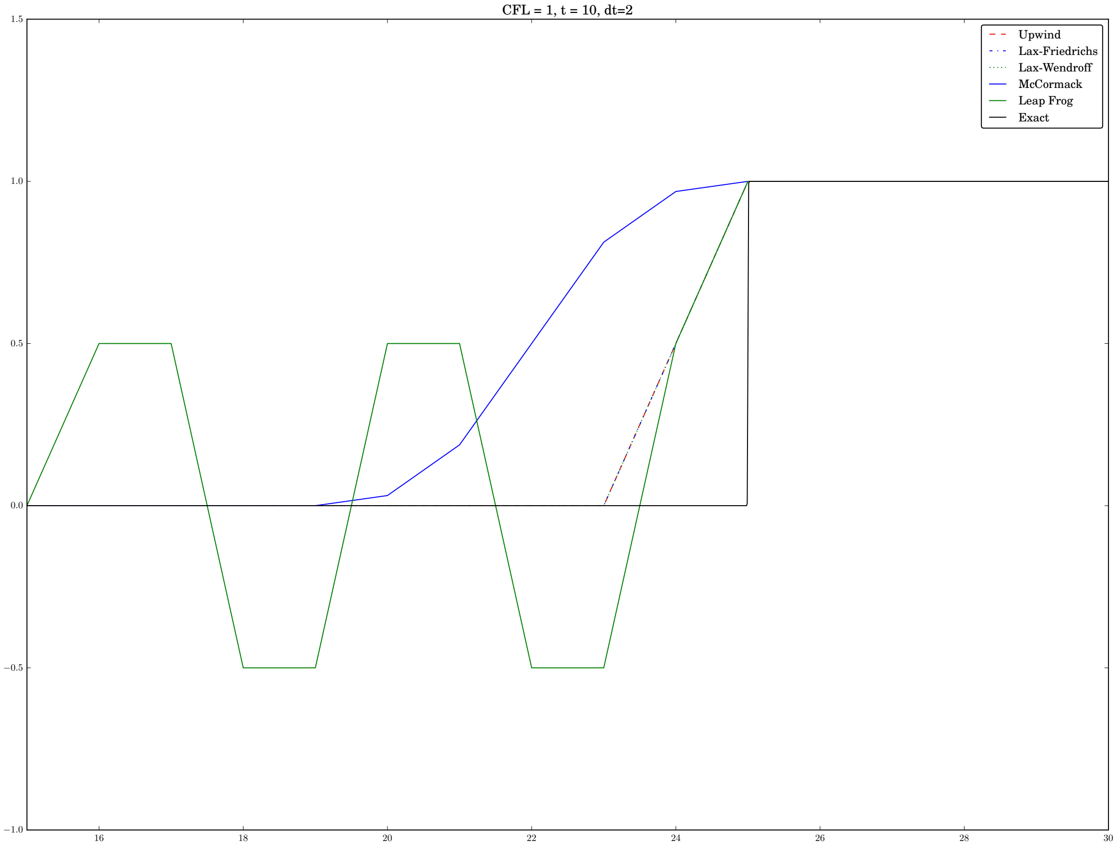

Figure 1: High Δt

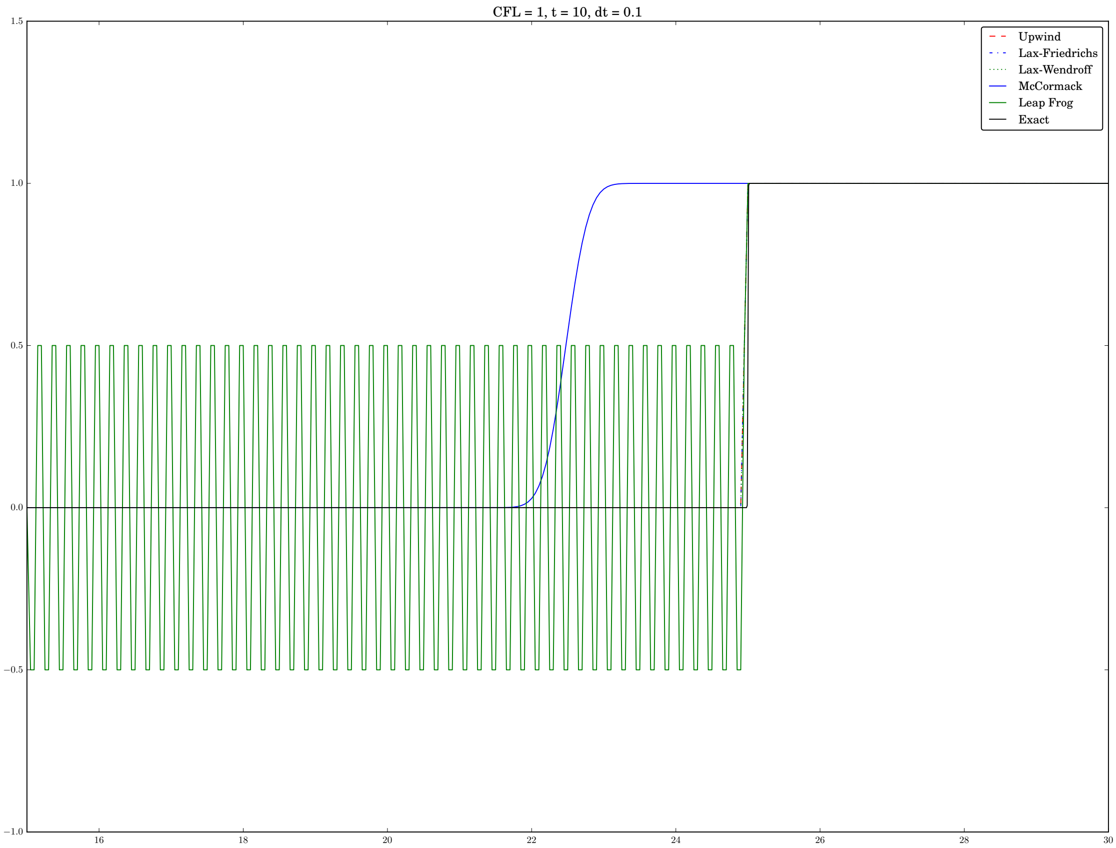

Figure 2: Low Δt. Results for Lax* and Upwind are exact.

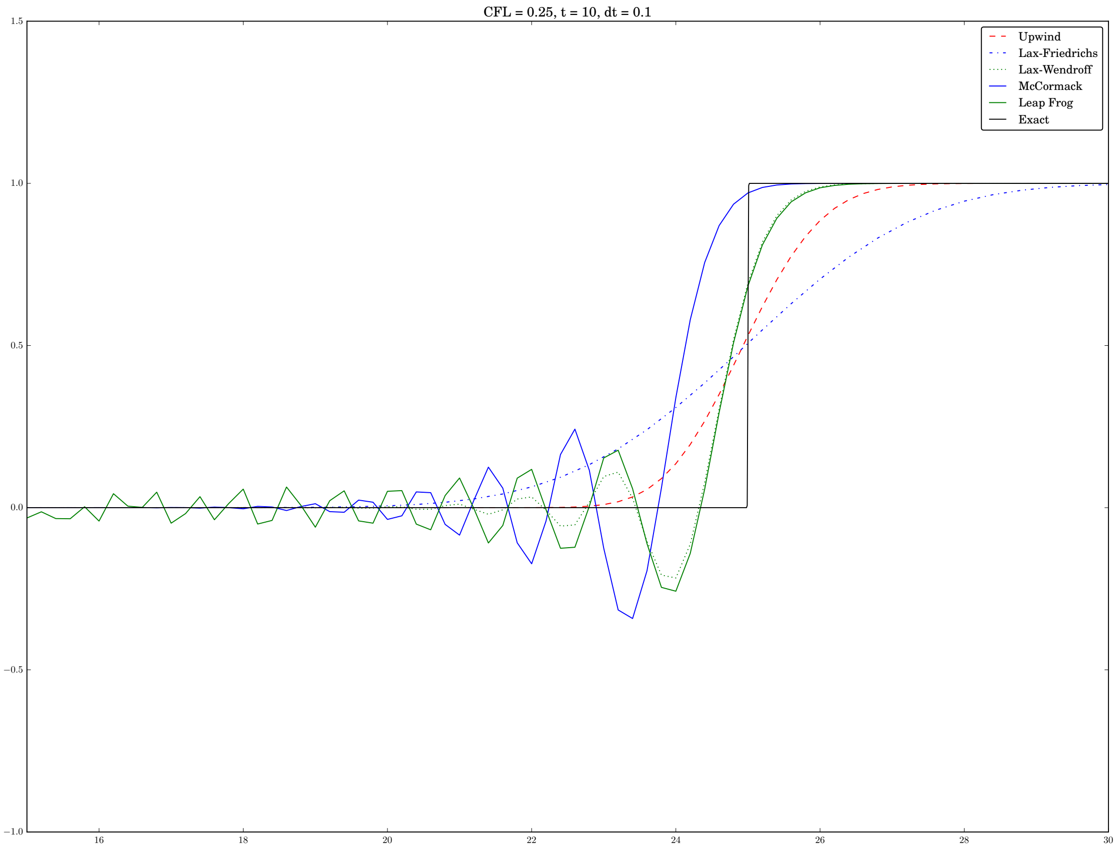

Figure 3: Lower ν. Results for Leap Frog and McCormack are realistic.