Syn3D nondimensionalization

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1 Dimensionless Variables

1.1 Current State

The Navier-Stokes momentum equations in `Syn3D` are currently non-dimensionalized as follows, where $\overline{\phi}$ and ϕ are dimensional and dimensionless, respectively:

$$\rho = \frac{\overline{\rho}}{\rho_\infty} \qquad u = \frac{\overline{u}}{\sqrt{P_\infty/\rho_\infty}} \qquad P = \frac{\overline{P}}{P_\infty}$$
$$\mu_V = \frac{\overline{\mu_V}}{\mu_\infty} \qquad t = \frac{\overline{t}\cdot\sqrt{P_\infty/\rho_\infty}}{c} \qquad x = \frac{\overline{x}}{c}$$

where c is the chord length, also often referred to as the reference length L.

1.2 Turbulent Quantities

The $k-\omega$equations also have to be non-dimensionalized and it has to be consistent with the Navier-Stokes equations. Both the NASA Turbulence Modeling Resource and `CFL3D` use the following:

$$\mu_T = \frac{\overline{\mu}_T}{\mu_\infty} \qquad k = \frac{\overline{k}}{a_\infty^2} \qquad \omega = \frac{\overline{\omega}\mu_\infty}{\rho_\infty a_\infty^2} \qquad \Omega = \frac{\overline{\Omega}c}{a_\infty}$$

where a is the speed of sound.

The above needs to be slightly tweaked to match `Syn3D`. In fact, replacing occurences of a with $\sqrt{P_\infty/\rho_\infty}$ is all that is necessary, since those two are related by a factor of $\sqrt{\gamma}$. Consequently, we can expect the nondimensionalized equations to be very similar, only varying by factors of $\sqrt{\gamma}$.

The following nondimensionalization is then used:

$$\mu_T = \frac{\overline{\mu}_T}{\mu_\infty} \qquad k = \frac{\overline{k}}{P_\infty/\rho_\infty} \qquad \omega = \frac{\overline{\omega}\mu_\infty}{\rho_\infty P_\infty/\rho_\infty}$$
$$d = \frac{\overline{d}}{c} \qquad \Omega = \frac{\overline{\Omega}c}{\sqrt{P_\infty/\rho_\infty}}$$

2 Nondimensionalized Equations

In order for the equations to take unitless quantities as inputs and also return unitless quantities, the dimensionless variables defined above need to be substituted into the governing equations. In other words, one cannot solve dimensional equations using dimensionless variables. The derivation is not shown here.

2.1 Navier-Stokes

The equations are slightly simplified to remain concise. While this is not technically part of the nondimensionalization, it should be noted that the Boussinesq approximation is used and the terms associated with molecular diffusion and turbulent transport in the energy equation are neglected (see NASA's website).

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} = 0 \tag{2.1}$$
$$\frac{\partial(\rho u)}{\partial t} + \frac{\partial(\rho u u)}{\partial x} = -\frac{\partial P}{\partial x} + \sqrt{\gamma}\frac{M_\infty}{Re}\frac{\partial}{\partial x}\left((\mu_V + \mu_T)\frac{\partial u}{\partial x}\right) \tag{2.2}$$

$$\begin{aligned}\frac{\partial(\rho E)}{\partial t} + \frac{\partial(\rho u E + uP)}{\partial x} =& \sqrt{\gamma}\frac{M_\infty}{Re}\frac{\partial}{\partial x}\left[(\mu_V + \mu_T)u\frac{\partial u}{\partial x}\right] \\ &- \sqrt{\gamma}\frac{M_\infty}{Re}\frac{\partial}{\partial x}\left[\left(\frac{\mu_V}{Pr} + \frac{\mu_T}{Pr_T}\right)\frac{\gamma}{\gamma - 1}\frac{\partial(P/\rho)}{\partial x}\right]\end{aligned} \tag{2.3}$$

The term $\sqrt{\gamma}\frac{M_\infty}{Re}$ comes from the chosen nondimensionalization.

2.2 K-ω SST

Not only must one nondimensionalize the transport equations, but it is necessary to perform the same for the boundary conditions and additional functions.

2.2.1 Transport Equations

$$\rho\frac{\partial k}{\partial t}+\rho u\frac{\partial k}{\partial x}=\sqrt{\gamma}\frac{M_\infty}{Re}\mu_T\Omega^2--\left(\sqrt{\gamma}\frac{M_\infty}{Re}\right)^{-1}\beta^*\rho k\omega+\sqrt{\gamma}\frac{M_\infty}{Re}\frac{\partial}{\partial x}\left[\left(\mu_V+\sigma_k\mu_T\right)\frac{\partial k}{\partial x}\right] \tag{2.4}$$

$$\begin{aligned}\rho\frac{\partial \omega}{\partial t}+\rho u\frac{\partial \omega}{\partial x}=&\sqrt{\gamma}\frac{M_\infty}{Re}\rho\gamma\Omega^2-\left(\sqrt{\gamma}\frac{M_\infty}{Re}\right)^{-1}\beta^*\rho\omega^2+\sqrt{\gamma}\frac{M_\infty}{Re}\frac{\partial}{\partial x}\left[\left(\mu_V+\sigma_k\mu_T\right)\frac{\partial \omega}{\partial x}\right]\\&+\sqrt{\gamma}\frac{M_\infty}{Re}2(1-F_1)\frac{\rho\sigma}{\omega}\frac{\partial k}{\partial x}\frac{\partial \omega}{\partial x}\end{aligned} \tag{2.5}$$

2.2.2 Additional Functions

The turbulent eddy viscosity is computed from:

$$\mu_T=\min\left(\frac{\rho k}{\omega},\left(\sqrt{\gamma}\frac{M_\infty}{Re}\right)^{-1}\frac{\rho a_1 k}{\Omega F_2}\right) \tag{2.6}$$

Blending functions are given by:

$$\begin{aligned}F_1&=\tanh(arg_1^4)\quad;\quad arg_1=\min\left[\max(\Gamma_1,\Gamma_2),\Gamma_3\right]\\F_2&=\tanh(arg_2^2)\quad;\quad arg_2=\max\left(2\Gamma_1,\Gamma_1\right)\end{aligned}$$

where

$$\Gamma_1=\sqrt{\gamma}\frac{M_\infty}{Re}\frac{\sqrt{k}}{\omega d} \tag{2.7}$$

$$\Gamma_2=\left(\sqrt{\gamma}\frac{M_\infty}{Re}\right)^2\frac{500\mu_V}{\rho d^2\omega} \tag{2.8}$$

$$\Gamma_3=\frac{P_\infty}{c^2}\frac{4\rho k}{\mathrm{CD}d^2} \tag{2.9}$$

$$\mathrm{CD}=\max\left(\frac{P_\infty}{c^2}2\rho\sigma\frac{1}{\omega}\frac{\partial k}{\partial x}\frac{\partial \omega}{\partial x},10^{-20}\right) \tag{2.10}$$

The factors in CD and Γ_3 are neglected since the solver sets $P_\infty$and c to 1. A note of this is also present in the code.

2.2.3 Boundary Conditions

Freestream BCs for the flat plate case are

$$k_{farfield} = 9 \cdot 10^{-9} \tag{2.11}$$

$$\omega_{farfield} = 10^{-6} \tag{2.12}$$

Viscous wall BCs are given as:

$$\omega_{wall} = \left(\sqrt{\gamma}\frac{M_\infty}{Re}\right)^2 60\frac{\mu_V}{\beta_1 \rho d^2} \tag{2.13}$$

$$k_{wall} = 0 \tag{2.14}$$

2.2.4 Implementation

The implementation is straight forward. However, it is relevant to note that both μ_V and μ_T are injected with $\sqrt{\gamma}\frac{M_\infty}{Re}$. This was originally done with μ_V in `viscf.f`.

In other words, one can let:

$$\hat{\mu}_V = \sqrt{\gamma}\frac{M_\infty}{Re}\mu_V \tag{2.15}$$

$$\hat{\mu}_T = \sqrt{\gamma}\frac{M_\infty}{Re}\mu_T \tag{2.16}$$

These can be substituted in the nondimensionalized equations/functions/BCs above.

2.3 Spalart-Allmaras

The Spalart-Allmaras model solves for a variable $\hat{\nu}$ related to the eddy viscosity through

$$\mu_T = \rho\hat{\nu} f_{v_1} \tag{2.17}$$

where

$$f_{v_1} = \frac{\chi^3}{\chi^3 + C_{v_1}^3} \tag{2.18}$$

$$\chi = \frac{\hat{\nu}}{\nu} \tag{2.19}$$

2.3.1 Transport Equation

$$\begin{aligned}\frac{\partial\hat{\nu}}{\partial t}+u\frac{\partial\hat{\nu}}{\partial x}= & \ C_{b_1}(1-f_{t_2})\Omega\hat{\nu}\\ & +\sqrt{\gamma}\frac{M_\infty}{Re}\left\{C_{b_1}\left[(1-f_{t_2})f_{v_2}+f_{t_2}\right]\frac{1}{\kappa^2}-C_{w_1}f_w\right\}\left(\frac{\hat{\nu}}{d}\right)^2\\ & +\sqrt{\gamma}\frac{M_\infty}{Re}\frac{1}{\sigma}\frac{\partial}{\partial x}\left[\left(\hat{\nu}+(1+C_{b_2})\hat{\nu}\right)\frac{\partial\hat{\nu}}{\partial x}\right]\\ & +\sqrt{\gamma}\frac{M_\infty}{Re}\frac{C_{b_2}}{\sigma}\left(\frac{\partial\hat{\nu}}{\partial x}\right)^2\end{aligned}\tag{2.20}$$

The terms are grouped slightly differently than in the original reference because of the common factors. This is done in CFL3D.

2.3.2 Additional Functions

Only the functions that end up with an additional factor are shown.

$$\begin{aligned}r&=\min(r_1,10)\\ r_1&=\frac{1}{\left(\sqrt{\gamma}\frac{M_\infty}{Re}\right)^{-1}\frac{\Omega d^2\kappa^2}{\hat{\nu}}+f_{v_2}}\end{aligned}\tag{2.21}$$

To look more like the original definition of r, this can be written as:

$$r=\min\left(\frac{\hat{\nu}}{\left(\sqrt{\gamma}\frac{M_\infty}{Re}\right)^{-1}\hat{S}\kappa^2d^2},10\right)\tag{2.22}$$

$$\hat{S}=\Omega+\sqrt{\gamma}\frac{M_\infty}{Re}\frac{\hat{\nu}f_{v_2}}{\kappa^2d^2}\tag{2.23}$$

2.3.3 Boundary Conditions

The boundary conditions are:

$$\hat{\nu}_{wall}=0\tag{2.24}$$

$$\hat{\nu}_{far}=3\text{ to }5\tag{2.25}$$