Numerical Schemes for the Euler Equations.

- Let us consider the one-dimensional (quasi) Euler equations in differential form,

$$\frac{\partial W}{\partial t} + \frac{1}{S}\frac{\partial}{\partial x}(FS) = Q$$

where $W = \begin{bmatrix} \rho \\ \rho u \\ e \end{bmatrix}$, $F = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ (e+p)u \end{bmatrix}$

and $Q = \begin{bmatrix} 0 \\ \frac{p}{S}\,\partial S/\partial x \\ 0 \end{bmatrix}$

while $p = (\gamma - 1)\rho\varepsilon$

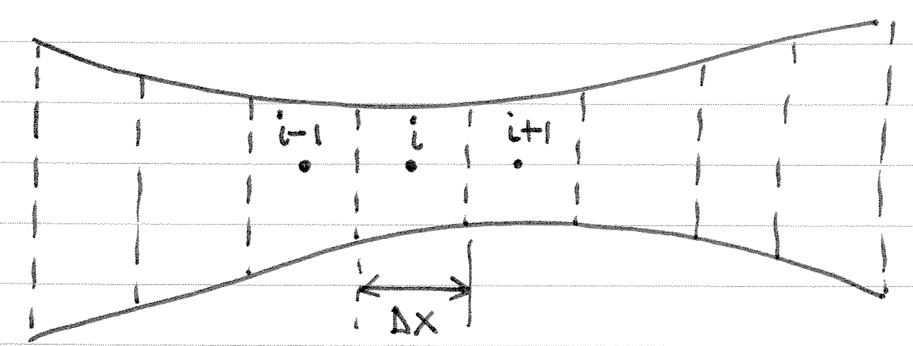
$\varepsilon = \frac{e}{\rho} - \frac{u^2}{2}$

$S = S(x)$.

Finite - Difference Approach

- Let us discretize the differential form using a finite-difference representation.

- The grid can be defined as,



- If a backward difference first-order equation is employed to describe the equation, then

$$\frac{\partial w}{\partial t} = -\frac{1}{S_i}\,\frac{F_i S_i - F_{i-1} S_{i-1}}{\Delta x} + \frac{1}{S_i}\begin{bmatrix} 0 \\ p_i \dfrac{S_i - S_{i-1}}{\Delta x} \\ 0 \end{bmatrix}$$

If the pseudo temporal derivative is discretized with an explicit first-order scheme, then.

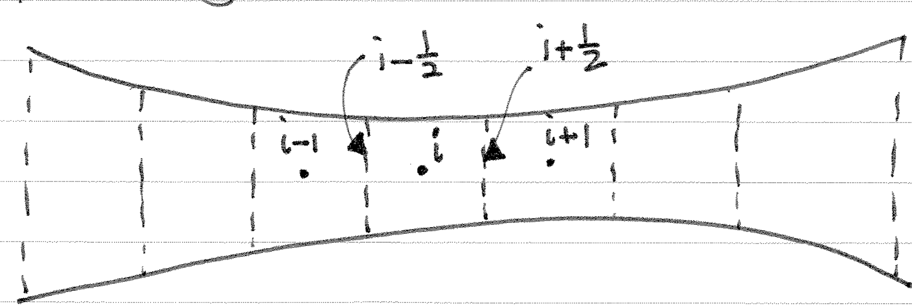
$$\frac{w_i^{n+1} - w_i^{n}}{\Delta t} = -\frac{1}{S_i}\,\frac{F_i^n S_i - F_{i-1}^n S_{i-1}}{\Delta x} + \frac{1}{S_i}\begin{bmatrix} 0 \\ p_i^n \dfrac{S_i - S_{i-1}}{\Delta x} \\ 0 \end{bmatrix}$$

The solution, w_i^{n+1}, at the new time step or iteration can be written as,

$$w_i^{n+1} = w_i^n - \frac{\Delta t}{S_i \Delta x}\left(F_i^n S_i - F_{i-1}^n S_{i-1}\right) + \frac{\Delta t}{S_i}\begin{bmatrix} 0 \\ p_i^n \dfrac{S_i - S_{i-1}}{\Delta x} \\ 0 \end{bmatrix}$$

In the Finite-Volume Approach

- Define the grid or control-volume as,



- In integral form, the equations are defined as,

$$\frac{\partial W}{\partial t} + \frac{1}{\forall} \int_s \vec{F}\, d\vec{s} = \frac{1}{\forall} \int_\forall Q \, dv$$

- A finite-volume discretization would reveal,

$$\frac{\partial W}{\partial t} = -\frac{1}{\forall_i}\left(F_{i+\frac{1}{2}} S_{i+\frac{1}{2}} - F_{i-\frac{1}{2}} S_{i-\frac{1}{2}}\right) + \frac{1}{\forall_i}\begin{bmatrix} 0 \\ p_i (S_{i+\frac{1}{2}} - S_{i-\frac{1}{2}}) \\ 0 \end{bmatrix}$$

- using the same explicit first-order scheme for the temporal derivative yields.

$$W_i^{n+1} = W_i^n - \frac{\Delta t}{\forall_i}\left(F^n_{i+\frac{1}{2}} S_{i+\frac{1}{2}} - F^n_{i-\frac{1}{2}} S_{i-\frac{1}{2}}\right) + \frac{\Delta t}{\forall_i}\begin{bmatrix} 0 \\ P_i^n (S_{i+\frac{1}{2}} - S_{i-\frac{1}{2}}) \\ 0 \end{bmatrix}$$

- In the case of the 1D Quasi Euler equations, the difference between the finite-difference and finite-volume schemes are minimal.

- MacCormack's Method for the Euler equations (69)

- An explicit two-step, predictor-corrector,

predictor : $\overline{W_i^{n+1}} = W_i^n - \Delta t \left(\frac{D_+}{\Delta x} F_i^n \right)$

corrector : $W_i^{n+1} = \frac{1}{2} \left[W_i^n + \overline{W_i^{n+1}} - \Delta t \left(\frac{D_-}{\Delta x} \overline{F_i^{n+1}} \right) \right]$

- use a forward-difference approximation in the predictor step and backward difference in the corrector step. The net result is a second-order accurate central difference approximation.

- The scheme is stable if $\Delta t \leq \frac{\Delta x}{|u| + c}$

where $|u| + c$ is the eigenvalue of $A = \frac{\partial F}{\partial W}$

In 2D, $\Delta t \leq \dfrac{1}{\frac{|u|}{\Delta x} + \frac{|v|}{\Delta y} + c\sqrt{\frac{1}{(\Delta x)^2} + \frac{1}{(\Delta y)^2}}}$

The extension in 3D follows from above.

- For 2D and 3D, the additional fluxes are simply added such as $\Delta t \left(\frac{D_+}{\Delta x} F_i^n + \frac{D_+}{\Delta y} G_i^n \right)$ in 2D.

- Beam and Warming Method for the Euler Eqns. (75-76)

- Employ a Crank–Nicolson method,

$$W_i^{n+1} = W_i^n - \frac{\Delta t}{2}\left[\frac{D_0}{\Delta x}F_i^n + \frac{D_0}{\Delta x}F_i^{n+1}\right]$$

where D_0 indicates central differencing in both space and time and hence its second-order accurate.

- To evaluate F_i^{n+1}, use a Taylor series approximatio

$$F_i^{n+1} = F_i^n + \frac{\partial F_i}{\partial W_i}^n\left(W_i^{n+1} - W_i^n\right) + \text{higher order terms.}$$

where $\frac{\partial F_i}{\partial W_i}^n = A_i^n$, the Jacobian matrix

- Then define $\delta W_i^{n+1} = W_i^{n+1} - W_i^n$,

- Subsequently, $\frac{D_0}{\Delta x}F_i^{n+1} = \frac{D_0}{\Delta x}F_i^n + \frac{D_0}{\Delta x}A_i^n\,\delta W_i^{n+1}$

- Substitute the above equation into the first equation above,

$$W_i^{n+1} = W_i^n - \frac{\Delta t}{2}\left[\frac{D_0}{\Delta x}F_i^n + \frac{D_0}{\Delta x}F_i^n + \frac{D_0}{\Delta x}A_i^n\,\delta W_i^{n+1}\right]$$

$$\delta W_i^{n+1} + \frac{\Delta t}{2}\frac{D_0}{\Delta x} A_i^n \delta W_i^{n+1} = -\Delta t\left(\frac{D_0}{\Delta x} F_i^n\right)$$

$$\left[I + \frac{\Delta t}{2}\frac{D_0}{\Delta x} A_i^n\right]\delta W_i^{n+1} = -\Delta t\left(\frac{D_0}{\Delta x} F_i^n\right)$$

However $-\Delta t\left(\frac{D_0}{\Delta x} F_i^n\right)$ is simply the residual

from $\frac{\Delta W_i^n}{\Delta t} + \frac{D_0}{\Delta x} F_i^n = 0$

$$\Delta W_i^n = -\Delta t\left(\frac{D_0}{\Delta x} F_i^n\right).$$

Hence

$$\left[I + \frac{\Delta t}{2}\frac{D_0}{\Delta x} A_i^n\right]\delta W_i^{n+1} = \Delta W_i^n$$

- In 2D,

$$\left[I + \frac{\Delta t}{2}\left(\frac{D_0}{\Delta x} A_{ij}^n + \frac{D_0}{\Delta y} B_{ij}^n\right)\right]\delta W_{ij}^{n+1} = \Delta W_{ij}^n$$

- In one-dimensional A_i^n is a 3x3 matrix and in 2D, A_{ij}^n and B_{ij}^n are 4x4 matrices.

- Employ approximate factorization,

$$\left[I + \frac{\Delta t}{2}\frac{D_0}{\Delta x} A_{ij}^n\right]\left[I + \frac{\Delta t}{2}\frac{D_0}{\Delta y} B_{ij}^n\right]\delta W_{ij}^{n+1} = \Delta W_{ij}^n$$

- Let $\delta W_{ij}^* = \left[I + \frac{\Delta t}{2} \frac{D_0}{\Delta y} B_{ij}^n \right] \delta W_{ij}^{n+1}$

and slove for

$$\left[I + \frac{\Delta t}{2} \frac{D_0}{\Delta x} A_{ij}^n \right] \delta W_{ij}^* = \Delta W_{ij}^n$$

Then solve for,

$$\left[I + \frac{\Delta t}{2} \frac{D_0}{\Delta y} B_{ij}^n \right] \delta W_{ij}^{n+1} = \delta W_{ij}^* .$$

- <u>Jameson's Method for the Euler Equations</u> (1981)

- Fourth-order Runge-Kutta scheme,

$$W_i^{(0)} = W_i^n$$
$$W_i^{(1)} = W_i^n - \Delta t/2 \, \frac{D_0}{\Delta x} F_i^{(0)}$$
$$W_i^{(2)} = W_i^n - \Delta t/2 \, \frac{D_0}{\Delta x} F_i^{(1)}$$
$$W_i^{(3)} = W_i^n - \Delta t/2 \, \frac{D_0}{\Delta x} F_i^{(2)}$$
$$W_i^{n+1} = \frac{1}{6} \left(W_i^{(0)} + 2W_i^{(1)} + 2W_i^{(2)} + W_i^{(3)} \right)$$

- Jameson's second 4th. order RK scheme,

$$W_i^{(0)} = W_i^{(n)}$$

$$W_i^{(k)} = W_i^{n} - \alpha_k \Delta t \left(\frac{D_0}{\Delta x} F_i^{(k-1)} \right)$$

$$W_i^{n+1} = W_i^{(4)}$$

where $\alpha_K = \dfrac{1}{5-K}$

and $k = 1, \ldots, 4$.

- Both methods are fourth-order in time and second order in space.

- The schemes are stable for

$$\Delta t \leq \frac{2\sqrt{2}}{\frac{|u|}{\Delta x} + \frac{|v|}{\Delta y} + c\sqrt{\frac{1}{(\Delta x)^2} + \frac{1}{(\Delta y)^2}}} \quad \text{in 2D}.$$

Evaluating Fluxes

We now will look at several different ways to evaluate the fluxes,

– Steger and Warming's Flux Split Method.

Extending for the flux splitting that was demonstrated for the wave equation, where either a backward or forward difference was employed depending on the direction of the characteristic, Steger and Warming introduced the following.

Let F equal the Jacobian times the solution vector, W,

$$F = AW = \frac{\partial F}{\partial W} W$$

Then the Jacobians can be diagonalized as

$$A = S^{-1} C^{-1} \begin{bmatrix} u & 0 & 0 \\ 0 & u+c & 0 \\ 0 & 0 & u-c \end{bmatrix} C S$$

where $S = \begin{bmatrix} 1 & 0 & 0 \\ -u/\rho & 1/\rho & 0 \\ \alpha\beta & -u\beta & \beta \end{bmatrix}$

$$S^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ u & \rho & 0 \\ \alpha & \rho u & 1/\beta \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & -1/c^2 \\ 0 & \rho c & 1 \\ 0 & -\rho c & 1 \end{bmatrix} \qquad C^{-1} = \begin{bmatrix} 1 & \frac{1}{2c^2} & \frac{1}{2c^2} \\ 0 & \frac{1}{2\rho c} & -\frac{1}{2\rho c} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

where $\alpha = \frac{1}{2}(u^2+v^2)$ and $\beta = \gamma - 1$

- If the eigenvalues are positive, then the characteristic travels in the positive x-direction, while the opposite is true for negative eigenvalues.

- Similar to the wave equation, for positive characteristics, backward-difference approximations should be used and forward-difference for negative characteristics.

- Thus the flux can be written as,

$$F_+ = A_+ W \quad \text{and} \quad F_- = A_- W$$

where
$$A_+ = S^{-1} C^{-1} \Lambda_+ C S$$
$$A_- = S^{-1} C^{-1} \Lambda_- C S$$

- To ensure that the split fluxes were continuous if the characteristic speeds were very close to zero, then the eigenvalues were replaced with

$$\lambda_+' = \frac{\lambda + \sqrt{\lambda^2+\varepsilon^2}}{2}$$

$$\lambda_-' = \frac{\lambda - \sqrt{\lambda^2+\varepsilon^2}}{2}$$

where ε is very small.

- Then the fluxes can be written across the control surface as,

$$F^n_{i+\frac{1}{2}} = F^n_{+i} + F^n_{-i+1}$$

$$= A^n_{+i} W_i^n + A^n_{-i+1} W^n_{i+1}$$

- The Steger-Warming scheme is very dissipative. To reduce the dissipation, a Modified Steger-Warming was proposed,

$$F^n_{i+\frac{1}{2}} = F^n_{+i+\frac{1}{2}} + F^n_{-i+\frac{1}{2}}$$

$$= \overline{A_+}^n_{i+\frac{1}{2}} W_i^n + \overline{A_-}^n_{i+\frac{1}{2}} W^n_{i+1}$$

where $\overline{A_+}^n_{i+\frac{1}{2}}$ or $\overline{A_-}^n_{i+\frac{1}{2}}$ are evaluated

with $\overline{W}^n_{i+\frac{1}{2}}$ data $= \dfrac{W_i^n + W^n_{i+1}}{2}$

In other words, the solution is evaluated at the surface and the Jacobian is evaluated based on the averaged informatio

- The modified - Steger - Warming is however less disipative and often does not contain sufficient dissipation in region of discontinuit

- This can be avoided by blending in the original Steger - Warming scheme in regions of high pressure gradients.

- Let $\left(\frac{\Delta p}{\Delta x}\right)_{i+\frac{1}{2}} = \frac{p_{i+1} - p_i}{\min(p_i, p_{i+1})}$

and $\omega_{i+\frac{1}{2}} = \frac{1}{1 + \left(\frac{\Delta p}{\Delta x}\right)^2_{i+\frac{1}{2}}}$.

Then the corrected - modified - Steger - Warming Flux is,

$$F_{i+\frac{1}{2}} = \omega_{i+\frac{1}{2}} F_{i+\frac{1}{2}}^{(MSW)} + (1-\omega_{i+\frac{1}{2}}) F_{i+\frac{1}{2}}^{(SW)}$$

where MSW stands for modified - Steger - Warming, and similarly for SW.

~~evaluated based on the averaged information.~~

- Roe Flux Difference Vector Splitting

  - The Roe scheme is derived such that

$$F_{i+1} - F_i = \hat{A}_{i+\frac{1}{2}} \left(\omega_{i+1} - \omega_i \right)$$

where the Jacobian matrix $\hat{A}_{i+\frac{1}{2}}$ are chosen to satisfy the above equation.

  - The Jacobian uses "geometrically" averaged data to approximately solve the Riemann problem

  - The method splits the 'flux difference' vector in contrast to the previous methods that only split the flux vector.

  - The elements of $\hat{A}_{i+\frac{1}{2}}$ for the Quasi 1D Euler Equations, are determined from,

$$\hat{\rho}_{i+\frac{1}{2}} = \sqrt{\rho_i \rho_{i+1}}$$

$$\hat{u}_{i+\frac{1}{2}} = \frac{\sqrt{\rho_i}\, u_i + \sqrt{\rho_{i+1}}\, u_{i+1}}{\sqrt{\rho_i} + \sqrt{\rho_{i+1}}}$$

$$\hat{H}_{i+\frac{1}{2}} = \frac{\sqrt{\rho_i} \left(\frac{e_i + p_i}{\rho_i} \right) + \sqrt{\rho_{i+1}} \left(\frac{e_{i+1} + p_{i+1}}{\rho_{i+1}} \right)}{\sqrt{\rho_i} + \sqrt{\rho_{i+1}}}$$

- The Roe flux is then written as,

$$F_{i+\frac{1}{2}} = \frac{1}{2}(F_i + F_{i+1}) - \frac{1}{2}\left|\hat{A}_{i+\frac{1}{2}}\right|(W_{i+1} - W_i)$$

where $|A| = A_+ - A_-$

- The method sometimes places shocks in expansion region. To avoid this an entropy correction is required

Let $\hat{\lambda}_{i+\frac{1}{2}}$ be an eigenvalue of $\hat{A}_{i+\frac{1}{2}}$,

$$\hat{\lambda}_{i+\frac{1}{2}} = \left(\hat{u}_{i+\frac{1}{2}},\ \hat{u}_{i+\frac{1}{2}} + \hat{c}_{i+\frac{1}{2}},\ \hat{u}_{i+\frac{1}{2}} - \hat{c}_{i+\frac{1}{2}}\right)$$

where $\hat{c}_{i+\frac{1}{2}} = \sqrt{(\gamma-1)\left(\hat{h}_{i+\frac{1}{2}} - \frac{1}{2}\hat{u}_{i+\frac{1}{2}}\right)}$

Let λ_i and λ_{i+1} be the values of this eigenvalue at i and $i+1$ and $\varepsilon_{i+\frac{1}{2}}$ be defined as follows

$$\varepsilon_{i+\frac{1}{2}} = \max\left\{0,\ \hat{\lambda}_{i+\frac{1}{2}} - \lambda_i,\ \lambda_{i+1} - \hat{\lambda}_{i+\frac{1}{2}}\right\}$$

Then

If $\left|\hat{\lambda}_{i+\frac{1}{2}}\right| \le \varepsilon_{i+\frac{1}{2}}$

then $$\hat{\lambda}_{i+\frac{1}{2}} = \frac{1}{2}\left(\frac{\hat{\lambda}^2_{i+\frac{1}{2}}}{\varepsilon_{i+\frac{1}{2}}} + \varepsilon_{i+\frac{1}{2}}\right)$$

- Scalar Dissipation.

  - The Roe flux is costly due to the evaluation of $\hat{A}$.
  - One approach is to replace $\hat{A}$ with the eigenvalues of the Jacobian matrix.

  - The new scheme can be written as,

$$F_{i+\frac{1}{2}} = \frac{1}{2}(F_i + F_{i+1}) - \frac{1}{2}\varepsilon\lambda_{i+\frac{1}{2}}(W_{i+1} - W_i)$$

where $\lambda_{i+\frac{1}{2}} = \max\left(u_{i+\frac{1}{2}},\ u_{i+\frac{1}{2}} + c_{i+\frac{1}{2}},\ u_{i+\frac{1}{2}} - c_{i+\frac{1}{2}}\right)$.

If $u_{i+\frac{1}{2}} > 0$, then $\lambda_{i+\frac{1}{2}} = u_{i+\frac{1}{2}} + c_{i+\frac{1}{2}}$

and ε is a constant coefficient, selected to ensure a stable scheme.

- Complete List of Schemes.

Steger-Warming (SW) =

$$F_{i+\frac{1}{2}}^{n\,(SW)} = A_{+_i} W_i^n + A_{-_{i+1}} W_{i+1}^n$$

Modified-Steger-Warming (MSW) =

$$F_{i+\frac{1}{2}}^{n\,(MSW)} = \overline{A}_{+_{i+\frac{1}{2}}} W_i^n + \overline{A}_{-_{i+\frac{1}{2}}} W_{i+1}^n$$

Corrected-Modified-Steger-Warming (CMSW)

$$F_{i+\frac{1}{2}}^{n\,(CMSW)} = \omega_{i+\frac{1}{2}} F_{i+\frac{1}{2}}^{n\,(MSW)} + (1-\omega_{i+\frac{1}{2}}) F_{i+\frac{1}{2}}^{n\,(SW)}$$

Roe

$$F_{i+\frac{1}{2}}^{n\,(ROE)} = \frac{1}{2}\left(F_i^n + F_{i+1}^n\right) - \frac{1}{2}\left|\hat{A}_{i+\frac{1}{2}}^n\right|\left(W_{i+1}^n - W_i^n\right)$$

Scalar Dissipation

$$F_{i+\frac{1}{2}}^{n\,(scalar)} = \frac{1}{2}\left(F_i^n + F_{i+1}^n\right) - \frac{1}{2}\,\varepsilon\,\lambda_{i+\frac{1}{2}}^n\left(W_{i+1}^n - W_i^n\right)$$