MECH 539: Computational Aerodynamics
Department of Mechanical Engineering, McGill University

Notes for Final Project : Solve the Quasi One-Dimensional Euler Equations for Various Artificial Dissipation Schemes
Due 30th April, 2012

Pseudo Code.

1. **Initialize.** Setup the grid, and initialize the state vector (density, momentum, and energy as well as static pressure) flow using the specified flow conditions. Impose the exit static pressure.

2. **Iteration Loop**

   - Compute the time step, Δt_i for each control volume based on the chosen temporal discretization scheme.
   - Compute the flux, $F_{i+1/2}$ across each edge based on the chosen scheme.
   - Compute the residual, $R_i^n = F^n_{i+\frac{1}{2}} S_{i+\frac{1}{2}} - F^n_{i+\frac{1}{2}} S_{i+\frac{1}{2}} - Q_i^n$, for all control volumes, $i = 2, ..., i_{\text{max}} - 1$.
   - Update the state vector based on the chosen temporal discretization. For the explicit Euler, the equation would be $w_i^{n+1} = w_i^n - \frac{\Delta t_i}{V_i} R_i^n$, for all control volumes, $i = 2, ..., i_{\text{max}} - 1$.
   - Update the static pressure, speed of sound, and Mach number for each control volume, $i = 2, ..., i_{\text{max}} - 1$.
   - **Update Inlet and Exit Boundary Conditions using Characteristic Boundary Conditions**
     * Update the inlet boundary condition, by solving for the $u - c$ characteristic if inlet is subsonic. If inlet is supersonic, then do not update $i = 1$. (see page 12 of *NavierStokes-BoundaryConditions.pdf*)
     * Update the exit boundary condition, by solving all three characteristics for both subsonic and supersonic exit boundary conditions. Place a conditional statement on how the change in the static pressure, δp, is computed. (see pages 13 and 14 of *NavierStokes-BoundaryConditions.pdf*)
   - **Check for convergence** by monitoring, R_i, which should converge to machine zero.
   - Repeat the **iteration loop** until convergence.

Inlet and Exit Boundary Conditions.

1. **Inlet Boundary Conditions**

   – *Supersonic Inlet.* Specify the total pressure, p_t, total temperature, T_t, and Mach number, M at cell $i = 1$. Static pressure, p, static temperature, T, speed of sound, c, velocity, u, and energy, e can be initialized from these three values using the isentropic relations as stated in page 11 of *NavierStokes-BoundaryConditions.pdf.* As the solution iterates, then you do not update $i = 1$ since all three characteristics are running right.

   – *Subsonic Inlet.* Solve for the $u - c$ characteristic as follows.

     * Compute $\frac{\partial p}{\partial u}$ from taking the derivative of p with respect to u in the isentropic relations.

$$\frac{\partial p}{\partial u} = p_t \left(\frac{\gamma}{\gamma - 1}\right)\left[1 - \frac{\gamma - 1}{\gamma + 1}\frac{(u_1^n)^2}{a_*^2}\right]^{1/(\gamma-1)} \cdot \left(-2\frac{\gamma - 1}{\gamma + 1}\frac{u_1^n}{a_*^2}\right)$$

       where $a_*^2 = 2\gamma\left(\frac{\gamma-1}{\gamma+1}\right)c_v T_t$, and $c_v = R/(\gamma - 1)$.

     * Compute δu

$$\lambda = \left(\frac{u_2^n + u_1^n}{2} - \frac{c_2^n + c_1^n}{2}\right)\frac{(\Delta t)_1}{\Delta x} \quad \text{where,} \quad (\Delta t)_1 = \frac{\text{CFL}\Delta x}{u_1^n + c_1^n}$$

$$\delta u = \frac{-\lambda\left[p_2^n - p_1^n - \rho_1^n c_1^n (u_2^n - u_1^n)\right]}{\frac{\partial p}{\partial u} - \rho_1^n c_1^n}$$

     * Update flow properties.

$$\begin{aligned}
u_1^{n+1} &= u_1^n + \delta u \\
T_1^{n+1} &= T_t\left[1 - \frac{\gamma - 1}{\gamma + 1}\frac{(u_1^n)^2}{a_*^2}\right] \\
p_1^{n+1} &= p_t\left[\frac{T_1^{n+1}}{T_t}\right]^{\gamma/(\gamma-1)} \\
\rho_1^{n+1} &= p_1^{n+1}/(RT_1^{n+1}) \\
e_1^{n+1} &= \rho_1^{n+1}\left[c_v T_1^{n+1} + \frac{1}{2}(u_1^{n+1})^2\right] \\
c_1^{n+1} &= \sqrt{\frac{\gamma p_1^{n+1}}{\rho_1^{n+1}}} \\
\text{Mach}_1^{n+1} &= u_1^{n+1}/c_1^{n+1}
\end{aligned}$$

2. **Exit Boundary Conditions**

– *Supersonic and Subsonic Exit.*

* Compute eigenvalues.

$$\lambda_1 = \left(\frac{u^n_{imax} + u^n_{imax-1}}{2} \right) \frac{(\Delta t)_{imax}}{\Delta x}$$

$$\lambda_2 = \left(\frac{u^n_{imax} + u^n_{imax-1}}{2} + \frac{c^n_{imax} + c^n_{imax-1}}{2} \right) \frac{(\Delta t)_{imax}}{\Delta x}$$

$$\lambda_3 = \left(\frac{u^n_{imax} + u^n_{imax-1}}{2} - \frac{c^n_{imax} + c^n_{imax-1}}{2} \right) \frac{(\Delta t)_{imax}}{\Delta x}$$

$$\text{where, } (\Delta t)_{imax} = \frac{\text{CFL}\Delta x}{u^n_{imax} + c^n_{imax}}$$

* Compute characteristic relations.

$$R_1 = -\lambda_1 \left[\rho^n_{imax} - \rho^n_{imax-1} - \frac{1}{(c^n_{imax})^2} (p^n_{imax} - p^n_{imax-1}) \right]$$

$$R_2 = -\lambda_2 \left[p^n_{imax} - p^n_{imax-1} + \rho^n_{imax} c^n_{imax} (u^n_{imax} - u^n_{imax-1}) \right]$$

$$R_3 = -\lambda_3 \left[p^n_{imax} - p^n_{imax-1} - \rho^n_{imax} c^n_{imax} (u^n_{imax} - u^n_{imax-1}) \right]$$

* Compute exit Mach number

$$\text{Mach}^n_{imax} = \frac{(u^n_{imax} + u^n_{imax-1})/2}{(c^n_{imax} + c^n_{imax-1})/2}$$

* Compute δp based on either a subsonic or supersonic exit.

  if $\text{Mach}^n_{imax} > 1$ **then**

  $\delta p = \frac{1}{2}(R_2 + R_3)$

  else

  $\delta p = 0$

  end if

* Update $\delta \rho$ and δu

$$\delta \rho = R_1 + \frac{\delta p}{(c^n_{imax})^2}$$

$$\delta u = \frac{R_2 - \delta p}{\rho^n_{imax} c^n_{imax}}$$

* Update flow properties.

$$
\begin{aligned}
\rho_{imax}^{n+1} &= \rho_{imax}^{n} + \delta\rho \\
u_{imax}^{n+1} &= u_{imax}^{n} + \delta u \\
p_{imax}^{n+1} &= p_{imax}^{n} + \delta p \\
T_{imax}^{n+1} &= \frac{p_{imax}^{n+1}}{\rho_{imax}^{n+1} R} \\
e_{imax}^{n+1} &= \rho_{imax}^{n+1} \left[c_v T_{imax}^{n+1} + \frac{1}{2}(u_{imax}^{n+1})^2 \right] \\
c_{imax}^{n+1} &= \sqrt{\frac{\gamma p_{imax}^{n+1}}{\rho_{imax}^{n+1}}} \\
\mathrm{Mach}_{imax}^{n+1} &= u_{imax}^{n+1} / c_{imax}^{n+1}
\end{aligned}
$$