SA Discretization

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1 Governing Equation

The Spalart-Allmaras model solves for a variable $\hat{\nu}$ related to the eddy viscosity through

$$\mu_T = \rho \hat{\nu} f_{v_1}$$

where

$$f_{v_1} = \frac{\chi^3}{\chi^3 + C_{v_1}^3}$$
$$\chi = \frac{\hat{\nu}}{\nu}$$

The governing transport equation is then:

$$\frac{\partial \hat{\nu}}{\partial t} + \mathcal{A} = \mathcal{P} + \mathcal{D} + \mathcal{SD} + \mathcal{FD}$$

where

$$\begin{aligned}
\text{Advection}: \quad & \mathcal{A} = u_j \frac{\partial \hat{\nu}}{\partial x_j} \\
\text{Production}: \quad & \mathcal{P} = C_{b_1}(1 - f_{t_2})\Omega\hat{\nu} \\
\text{Destruction}: \quad & \mathcal{D} = \sqrt{\gamma}\frac{M_\infty}{Re}\left\{C_{b_1}\left[(1 - f_{t_2})f_{v_2} + f_{t_2}\right]\frac{1}{\kappa^2} - C_{w_1} f_w\right\}\left(\frac{\hat{\nu}}{d}\right)^2 \\
\text{Second-Order Diffusion}: \quad & \mathcal{SD} = \sqrt{\gamma}\frac{M_\infty}{Re}\frac{1}{\sigma}\frac{\partial}{\partial x_j}\left[(\hat{\nu} + \nu)\frac{\partial \hat{\nu}}{\partial x_j}\right] \\
\text{First-Order Diffusion}: \quad & \mathcal{FD} = \sqrt{\gamma}\frac{M_\infty}{Re}\frac{C_{b_2}}{\sigma}\frac{\partial \hat{\nu}}{\partial x_i}\frac{\partial \hat{\nu}}{\partial x_i}
\end{aligned}$$

The terms are grouped slightly differently than in the original reference because of the common nondimensionalization factors.

2 Solving Strategy and Backward Euler Implicit Method

See discretization of KW-SST for detailed explanations. The only difference is there is only equation, thus no assumptions need to be made in the construction of the Jacobian. However, the notation is described below.

Let $\mathbf{B}_\alpha$ be an operator representing the contribution to the lower diagonal term of the factorized Jacobian in the α direction. For example, let

$$\phi = a\hat{\nu}_{i-1} + b\hat{\nu}_i + c\hat{\nu}_{i+1}$$

Then,

$$\mathbf{B}_\xi(\phi) = \frac{\partial\left(\phi\right)_{i-1,j,k}}{\partial\hat{\nu}} = a$$

Similary, let $\mathbf{D}_\alpha$ and $\mathbf{S}$ represent the contributions to the upper diagonal and diagonal, respectively. Using the same example:

$$\mathbf{D}_\xi(\phi) = \frac{\partial\left(\phi\right)_{i+1,j,k}}{\partial\hat{\nu}} = c$$
$$\mathbf{S}(\phi) = \frac{\partial\left(\phi\right)_{i,j,k}}{\partial\hat{\nu}} = b$$

These variable names have been chosen because that is what is used in `Syn3D`.

3 Discretization

All expansions are done in two dimensions instead of three for the sake of brevity.

3.1 Advection

It should be noted that the advection term appears on the left-hand side of the governing equation, thus it needs to be *subtracted* from the residual and Jacobian.

$$\mathcal{A} = u_j\frac{\partial\hat{\nu}}{\partial x_j} = u_1\frac{\partial\hat{\nu}}{\partial x} + u_2\frac{\partial\hat{\nu}}{\partial y}$$

Transforming to computational space:

$$\mathcal{A} = u_1 \frac{\partial \hat{\nu}}{\partial \xi} \frac{\partial \xi}{\partial x} + u_1 \frac{\partial \hat{\nu}}{\partial \eta} \frac{\partial \eta}{\partial x} + u_2 \frac{\partial \hat{\nu}}{\partial \xi} \frac{\partial \xi}{\partial y} + u_2 \frac{\partial \hat{\nu}}{\partial \eta} \frac{\partial \eta}{\partial y}$$

Collecting all the ξ (i direction) terms:

$$\mathcal{A}_\xi = u_1 \frac{\partial \hat{\nu}}{\partial \xi} \frac{\partial \xi}{\partial x} + u_2 \frac{\partial \hat{\nu}}{\partial \xi} \frac{\partial \xi}{\partial y}$$

The advection is discretized using a first-order upwinding scheme where q determines the flow direction.

$$q = u \frac{\partial \xi}{\partial x} + v \frac{\partial \xi}{\partial y}$$

Then, the advection term in the i direction can be written as:

$$\mathcal{A}_\xi = q^+_{i,j} (\hat{\nu}_{i,j} - \hat{\nu}_{i-1,j}) + q^-_{i,j} (\hat{\nu}_{i+1,j} - \hat{\nu}_{i,j})$$

where

$$\begin{aligned} q^+ &= \frac{1}{2} (q + |q|) \\ q^- &= \frac{1}{2} (q - |q|) \end{aligned}$$

3.1.1 Jacobian

$$\begin{aligned} \mathbf{B}_\xi(\mathcal{A}_\xi) &= -q^+_{i,j} \\ \mathbf{D}_\xi(\mathcal{A}_\xi) &= q^-_{i,j} \\ \mathbf{S}_\xi(\mathcal{A}_\xi) &= \left(q^+ - q^-\right)_{i,j} \end{aligned}$$

3.2 Production

$$\mathcal{P} = C_{b_1} (1 - f_{t_2}) \Omega \hat{\nu}$$

No discretization is required.

3.2.1 Jacobian

$$\mathbf{B}_\alpha(\mathcal{P}) = \mathbf{D}_\alpha(\mathcal{P}) = 0 \quad \forall\ \alpha$$
$$\mathbf{S}(\mathcal{P}) = C_{b_1}(1 - f_{t_2})\Omega$$

3.3 Destruction

$$\mathcal{D} = K_D \cdot \hat{\nu}^2$$
$$K_D = \sqrt{\gamma}\frac{M_\infty}{Re}\left\{C_{b_1}\left[(1 - f_{t_2})f_{v_2} + f_{t_2}\right]\frac{1}{\kappa^2} - C_{w_1}f_w\right\}\left(\frac{1}{d}\right)^2$$

Again, no discretization is required.

3.3.1 Jacobian

$$\mathbf{B}_\alpha(\mathcal{P}) = \mathbf{D}_\alpha(\mathcal{P}) = 0 \quad \forall\ \alpha$$
$$\mathbf{S}(\mathcal{P}) = 2K_D \cdot \hat{\nu}_{i,j,k}$$

3.4 Second-Order Diffusion

$$\begin{aligned}\mathcal{SD} &= \sqrt{\gamma}\frac{M_\infty}{Re}\frac{1}{\sigma}\frac{\partial}{\partial x_j}\left[(\hat{\nu} + \nu)\frac{\partial \hat{\nu}}{\partial x_j}\right] \\ &= \sqrt{\gamma}\frac{M_\infty}{Re}\frac{1}{\sigma}\left\{\frac{\partial}{\partial x}\left[(\hat{\nu} + \nu)\frac{\partial \hat{\nu}}{\partial x}\right] + \frac{\partial}{\partial y}\left[(\hat{\nu} + \nu)\frac{\partial \hat{\nu}}{\partial y}\right] + \frac{\partial}{\partial z}\left[(\hat{\nu} + \nu)\frac{\partial \hat{\nu}}{\partial z}\right]\right\}\end{aligned}$$

For brevity, the constant factor $\sqrt{\gamma}\frac{M_\infty}{Re}\frac{1}{\sigma}$ won't be written anymore throughout this section.

The terms inside square brackets expand into:

$$(\hat{\nu} + \nu)\frac{\partial \hat{\nu}}{\partial x_j} = (\hat{\nu} + \nu)\left[\frac{\partial \hat{\nu}}{\partial \xi}\frac{\partial \xi}{\partial x_j} + \frac{\partial \hat{\nu}}{\partial \eta}\frac{\partial \eta}{\partial x_j}\right]$$

Then:

$$\frac{\partial}{\partial x}\left[(\hat{\nu}+\nu)\frac{\partial\hat{\nu}}{\partial x}\right] = \overbrace{\frac{\partial\xi}{\partial x}\frac{\partial}{\partial\xi}\left[(\hat{\nu}+\nu)\frac{\partial\hat{\nu}}{\partial\xi}\frac{\partial\xi}{\partial x}\right] + \frac{\partial\eta}{\partial x}\frac{\partial}{\partial\eta}\left[(\hat{\nu}+\nu)\frac{\partial\hat{\nu}}{\partial\eta}\frac{\partial\eta}{\partial x}\right]}^{\text{retained}}$$
$$+\underbrace{\frac{\partial\eta}{\partial x}\frac{\partial}{\partial\eta}\left[(\hat{\nu}+\nu)\frac{\partial\hat{\nu}}{\partial\xi}\frac{\partial\xi}{\partial x}\right] + \frac{\partial\xi}{\partial x}\frac{\partial}{\partial\xi}\left[(\hat{\nu}+\nu)\frac{\partial\hat{\nu}}{\partial\eta}\frac{\partial\eta}{\partial x}\right]}_{\text{ignored}}$$

As shown above, terms involving derivatives both in ξ and η are neglected. *My guess is that this is because it makes the ADI formulation impossible, since it would involve a larger stencil – like in* `nsflux.f` *– than permitted* It can also be said that these cross terms have a smaller contribution to the overall diffusion term.

Repeating this above for directions x, y, z and collecting the ξ terms gives:

$$\mathcal{SD}_\xi = \frac{\partial\xi}{\partial x}\frac{\partial}{\partial\xi}\left[(\hat{\nu}+\nu)\frac{\partial\hat{\nu}}{\partial\xi}\frac{\partial\xi}{\partial x}\right] + \frac{\partial\xi}{\partial y}\frac{\partial}{\partial\xi}\left[(\hat{\nu}+\nu)\frac{\partial\hat{\nu}}{\partial\xi}\frac{\partial\xi}{\partial y}\right] + \frac{\partial\xi}{\partial z}\frac{\partial}{\partial\xi}\left[(\hat{\nu}+\nu)\frac{\partial\hat{\nu}}{\partial\xi}\frac{\partial\xi}{\partial z}\right]$$

Then, let:

$$\mathcal{SD}_{\xi,x} = \frac{\partial\xi}{\partial x}\frac{\partial}{\partial\xi}\left[(\hat{\nu}+\nu)\frac{\partial\hat{\nu}}{\partial\xi}\frac{\partial\xi}{\partial x}\right]$$

We use a flux-like discretization such that:

$$\mathcal{SD}_{xi,x} = \left(\frac{\partial\xi}{\partial x}\right)_i \left[F_{i+1/2} - F_{i-1/2}\right]$$
$$F_{i+1/2} = \left((\hat{\nu}+\nu)\frac{\partial\hat{\nu}}{\partial\xi}\frac{\partial\xi}{\partial x}\right)_{i+1/2}$$

and compute quantities at the face $i+1/2$ using the following:

$$\phi_{i+1/2} = \frac{1}{2}\left(\phi_{i+1}+\phi_i\right)$$
$$\left(\frac{\partial\phi}{\partial\xi}\right)_{i+1/2} = \phi_{i+1}-\phi_i$$

Plugging back into $\mathcal{SD}_{\xi,x}$:

$$\begin{aligned}\mathcal{SD}_{\xi,x} = \left(\frac{\partial\xi}{\partial x}\right)_i \Bigg\{ \left(\frac{\partial\xi}{\partial x}\right)_{i+1/2} &\left[\nu_{i+1/2}\cdot(\hat{\nu}_{i+1}-\hat{\nu}_i)+\frac{1}{2}(\hat{\nu}_{i+1}+\hat{\nu}_i)(\hat{\nu}_{i+1}-\hat{\nu}_i)\right] \\ -\left(\frac{\partial\xi}{\partial x}\right)_{i-1/2} &\left[\nu_{i-1/2}\cdot(\hat{\nu}_{i}-\hat{\nu}_{i-1})+\frac{1}{2}(\hat{\nu}_{i}+\hat{\nu}_{i-1})(\hat{\nu}_{i}-\hat{\nu}_{i-1})\right]\Bigg\}\end{aligned}$$

3.4.1 Jacobian

One can notice that the term $(\hat{\nu}_i+\hat{\nu}_{i-1})(\hat{\nu}_i-\hat{\nu}_{i-1})$ is a difference of squares factorization. Thus, $\mathcal{SD}_{\xi,x}$ can be rewritten as:

$$\begin{aligned}\mathcal{SD}_{\xi,x} = \left(\frac{\partial\xi}{\partial x}\right)_i \Bigg\{ \left(\frac{\partial\xi}{\partial x}\right)_{i+1/2} &\left[\nu_{i+1/2}\cdot(\hat{\nu}_{i+1}-\hat{\nu}_i)+\frac{1}{2}(\hat{\nu}_{i+1}^2-\hat{\nu}_i^2)\right] \\ -\left(\frac{\partial\xi}{\partial x}\right)_{i-1/2} &\left[\nu_{i-1/2}\cdot(\hat{\nu}_{i}-\hat{\nu}_{i-1})+\frac{1}{2}(\hat{\nu}_{i}^2-\hat{\nu}_{i-1}^2)\right]\Bigg\}\end{aligned}$$

Consequently, the Jacobian contributions of $\mathcal{SD}_{\xi,x}$ are:

$$\begin{aligned}\mathbf{B}_\xi(\mathcal{SD}_\xi,x) &= \left(\frac{\partial\xi}{\partial x}\right)_i\left(\frac{\partial\xi}{\partial x}\right)_{i-1/2}\left\{\nu_{i-1/2}+\hat{\nu}_{i-1}\right\} \\ \mathbf{D}_\xi(\mathcal{SD}_\xi,x) &= \left(\frac{\partial\xi}{\partial x}\right)_i\left(\frac{\partial\xi}{\partial x}\right)_{i+1/2}\left\{\nu_{i+1/2}+\hat{\nu}_{i+1}\right\} \\ \mathbf{S}_\xi(\mathcal{SD}_\xi,x) &= \left(\frac{\partial\xi}{\partial x}\right)_i\left\{\left(\frac{\partial\xi}{\partial x}\right)_{i+1/2}(-\nu_{i+1/2}-\hat{\nu}_i)+\left(\frac{\partial\xi}{\partial x}\right)_{i-1/2}(-\nu_{i-1/2}-\hat{\nu}_i)\right\}\end{aligned}$$

3.5 First-Order Diffusion

$$\begin{aligned}\mathcal{FD} &= \sqrt{\gamma}\frac{M_\infty}{Re}\frac{C_{b_2}}{\sigma}\frac{\partial\hat{\nu}}{\partial x_i}\frac{\partial\hat{\nu}}{\partial x_i} \\ &= \sqrt{\gamma}\frac{M_\infty}{Re}\frac{C_{b_2}}{\sigma}\left[\left(\frac{\partial\hat{\nu}}{\partial x}\right)^2+\left(\frac{\partial\hat{\nu}}{\partial y}\right)^2+\left(\frac{\partial\hat{\nu}}{\partial z}\right)^2\right]\end{aligned}$$

For brevity, the constant factor $\sqrt{\gamma}\frac{M_\infty}{Re}\frac{C_{b_2}}{\sigma}$ won't be written anymore throughout this section.

Similar to the other terms, transforming to computational space and collecting the ξ terms yields:

$$\mathcal{FD}_\xi = \left(\frac{\partial\hat{\nu}}{\partial\xi}\right)_i^2\cdot\underbrace{\left[\left(\frac{\partial\xi}{\partial x}\right)^2+\left(\frac{\partial\xi}{\partial y}\right)^2+\left(\frac{\partial\xi}{\partial z}\right)^2\right]_i}_{COEF}$$

We then approximate the derivative with a central difference:

$$\left(\frac{\partial \hat{\nu}}{\partial \xi}\right)_i = \frac{1}{2}(\hat{\nu}_{i+1} - \hat{\nu}_{i-1})$$

Thus:

$$\mathcal{FD}_\xi = COEF \cdot \frac{1}{4}\left(\hat{\nu}_{i+1}^2 - 2\hat{\nu}_{i+1}\hat{\nu}_{i-1} + \hat{\nu}_{i-1}^2\right)$$

3.5.1 Jacobian

$$\begin{aligned}
\mathbf{B}_\xi(\mathcal{FD}_\xi) &= COEF \cdot \frac{1}{2}(\hat{\nu}_{i-1} - \hat{\nu}_{i+1}) \\
\mathbf{D}_\xi(\mathcal{FD}_\xi) &= COEF \cdot \frac{1}{2}(\hat{\nu}_{i+1} - \hat{\nu}_{i-1}) \\
\mathbf{S}_\alpha(\mathcal{FD}_\alpha) &= 0 \quad \forall\ \alpha
\end{aligned}$$