SA Discretization

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February 2, 2016

1 Governing Equation

The Spalart-Allmaras model solves for a variable $\hat{\nu}$ related to the eddy viscosity through

$$\mu_T = \rho \hat{\nu} f_{v_1} \tag{1.1}$$

where

$$f_{v_1} = \frac{\chi^3}{\chi^3 + C_{v_1}^3} \tag{1.2}$$

$$\chi = \frac{\hat{\nu}}{\nu} \tag{1.3}$$

The governing transport equation is then:

$$\frac{\partial \hat{\nu}}{\partial t} + \mathcal{A} = \mathcal{P} + \mathcal{D} + \mathcal{SD} + \mathcal{FD} \tag{1.4}$$

where

$$\text{Advection :} \quad \mathcal{A} = u_j \frac{\partial \hat{\nu}}{\partial x_j} \tag{1.5}$$

$$\text{Production :} \quad \mathcal{P} = C_{b_1}(1 - f_{t_2})\Omega\hat{\nu} \tag{1.6}$$

$$\text{Destruction :} \quad \mathcal{D} = \sqrt{\gamma}\frac{M_\infty}{Re}\left\{C_{b_1}\left[(1 - f_{t_2})f_{v_2} + f_{t_2}\right]\frac{1}{\kappa^2} - C_{w_1} f_w\right\}\left(\frac{\hat{\nu}}{d}\right)^2 \tag{1.7}$$

$$\text{Second-Order Diffusion :} \quad \mathcal{SD} = \sqrt{\gamma}\frac{M_\infty}{Re}\frac{1}{\sigma}\frac{\partial}{\partial x_j}\left[(\hat{\nu} + \nu)\frac{\partial \hat{\nu}}{\partial x_j}\right] \tag{1.8}$$

$$\text{First-Order Diffusion :} \quad \mathcal{FD} = \sqrt{\gamma}\frac{M_\infty}{Re}\frac{C_{b_2}}{\sigma}\frac{\partial \hat{\nu}}{\partial x_i}\frac{\partial \hat{\nu}}{\partial x_i} \tag{1.9}$$

The terms are grouped slightly differently than in the original reference because of the common nondimensionalization factors.

2 Solving Strategy and Backward Euler Implicit Method

See discretization of KW-SST for detailed explanations. The only difference is there is only equation, thus no assumptions need to be made in the construction of the Jacobian. However, the notation is described below.

Let $\mathbf{B}_\alpha$ be an operator representing the contribution to the lower diagonal term of the factorized Jacobian in the α direction. For example, let

$$\phi = a\hat{\nu}_{i-1} + b\hat{\nu}_i + c\hat{\nu}_{i+1}$$

Then,

$$\mathbf{B}_\xi(\phi) = \frac{\partial\left(\phi\right)_{i-1,j,k}}{\partial\hat{\nu}} = a$$

Similary, let $\mathbf{D}_\alpha$ and $\mathbf{S}$ represent the contributions to the upper diagonal and diagonal, respectively. Using the same example:

$$\mathbf{D}_\xi(\phi) = \frac{\partial\left(\phi\right)_{i+1,j,k}}{\partial\hat{\nu}} = c \tag{2.1}$$

$$\mathbf{S}(\phi) = \frac{\partial\left(\phi\right)_{i,j,k}}{\partial\hat{\nu}} = b \tag{2.2}$$

These variable names have been chosen because that is what is used in `Syn3D`.

3 Discretization

All expansions are done in two dimensions instead of three for the sake of brevity.

3.1 Advection

It should be noted that the advection term appears on the left-hand side of the governing equation, thus it needs to be *subtracted* from the residual and Jacobian.

$$\mathcal{A} = u_j\frac{\partial\hat{\nu}}{\partial x_j} = u_1\frac{\partial\hat{\nu}}{\partial x} + u_2\frac{\partial\hat{\nu}}{\partial y} \tag{3.1}$$

Transforming to computational space:

$$\mathcal{A} = u_1 \frac{\partial \hat{\nu}}{\partial \xi}\frac{\partial \xi}{\partial x} + u_1 \frac{\partial \hat{\nu}}{\partial \eta}\frac{\partial \eta}{\partial x} + u_2 \frac{\partial \hat{\nu}}{\partial \xi}\frac{\partial \xi}{\partial y} + u_2 \frac{\partial \hat{\nu}}{\partial \eta}\frac{\partial \eta}{\partial y} \tag{3.2}$$

Collecting all the ξ (i direction) terms:

$$\mathcal{A}_\xi = u_1 \frac{\partial \hat{\nu}}{\partial \xi}\frac{\partial \xi}{\partial x} + u_2 \frac{\partial \hat{\nu}}{\partial \xi}\frac{\partial \xi}{\partial y} \tag{3.3}$$

The advection is discretized using a first-order upwinding scheme where q determines the flow direction.

$$q = u\frac{\partial \xi}{\partial x} + v\frac{\partial \xi}{\partial y}$$

Then, the advection term in the i direction can be written as:

$$\mathcal{A}_\xi = q^+_{i,j}(\hat{\nu}_{i,j} - \hat{\nu}_{i-1,j}) + q^-_{i,j}(\hat{\nu}_{i+1,j} - \hat{\nu}_{i,j})$$

where

$$q^+ = \frac{1}{2}\left(q + |q|\right) \tag{3.4}$$

$$q^- = \frac{1}{2}\left(q - |q|\right) \tag{3.5}$$

3.1.1 Jacobian

$$\mathbf{B}_\xi(\mathcal{A}_\xi) = -q^+_{i,j} \tag{3.6}$$

$$\mathbf{D}_\xi(\mathcal{A}_\xi) = q^-_{i,j} \tag{3.7}$$

$$\mathbf{S}_\xi(\mathcal{A}_\xi) = \left(q^+ - q^-\right)_{i,j} \tag{3.8}$$

3.2 Production

$$\mathcal{P} = C_{b_1}(1 - f_{t_2})\Omega\hat{\nu}$$

No discretization is required.

3.2.1 Jacobian

$$\mathbf{B}_\alpha(\mathcal{P}) = \mathbf{D}_\alpha(\mathcal{P}) = 0 \quad \forall\ \alpha \tag{3.9}$$

$$\mathbf{S}(\mathcal{P}) = C_{b_1}(1 - f_{t_2})\Omega \tag{3.10}$$

3.3 Destruction

$$\mathcal{D} = K_D \cdot \hat{\nu}^2 \tag{3.11}$$

$$K_D = \sqrt{\gamma}\frac{M_\infty}{Re}\left\{C_{b_1}\left[(1 - f_{t_2})f_{v_2} + f_{t_2}\right]\frac{1}{\kappa^2} - C_{w_1}f_w\right\}\left(\frac{1}{d}\right)^2 \tag{3.12}$$

Again, no discretization is required.

3.3.1 Jacobian

$$\mathbf{B}_\alpha(\mathcal{P}) = \mathbf{D}_\alpha(\mathcal{P}) = 0 \quad \forall\ \alpha \tag{3.13}$$

$$\mathbf{S}(\mathcal{P}) = 2K_D \cdot \hat{\nu}_{i,j,k} \tag{3.14}$$

3.4 Second-Order Diffusion

$$\mathcal{SD} = \sqrt{\gamma}\frac{M_\infty}{Re}\frac{1}{\sigma}\frac{\partial}{\partial x_j}\left[(\hat{\nu} + (1 + C_{b_2})\hat{\nu})\frac{\partial \hat{\nu}}{\partial x_j}\right]$$

For brevity, the constant factor $\sqrt{\gamma}\frac{M_\infty}{Re}\frac{1}{\sigma}$ won't be written anymore throughout this section.

The term inside square brackets expands into: