Aerospace Engineering – AE464

Finite Element Analysis

Triangular vs Quadrilateral vs Analytical

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Introduction

This project is done to compare and contrast the two main 2-D meshes learned in the course and compare them up to an exact analytical solution to measure their error and to comment on their accuracy. The problem consists of a twelve-node 2-D rectangle with thermal boundary conditions. The objective is to calculate the temperature distribution on this rectangle. The left and bottom sides are insulated. The left side has a temperature of zero. The top side has a temperature distribution such that T=100cos(πx/6). The temperature distribution must be calculated using a triangular and a rectangular mesh. These solutions will be compared with each other and the analytical solution given by T(x, y) = 100cosh(πy/6)cos(πx/6)/cosh(π/3).

Method

1. Elemental Stiffness Matrix

There is a basic relation to find the elemental stiffness matrix of any 2D element given the derivative of the shape functions.

$$K^e = \int_{\Omega} B^{e^T} D B^e d\Omega$$

D being the product of the thermal conductivity k = 1.0 and a 3x3 identity matrix.

Triangular

Finding the stiffness matrix for a three-node triangular element is relatively simple because the partial derivatives of the shape functions in terms of both x and y result in a constant value. Therefore, the integration step can be skipped. The B^e for a three-node triangular element can be found using the following:

$$B^e = \frac{1}{2A^e}\begin{bmatrix} y_2 - y_3 & y_3 - y_1 & y_1 - y_2 \\ x_3 - x_2 & x_1 - x_3 & x_2 - x_1 \end{bmatrix}$$

Quadrilateral

The quadrilateral stiffness matrix requires a bit more effort because the shape functions of a 2D rectangle (or square in this case) can be found by multiplying the linear shape functions of the sides. This results in an xy value to be present in the shape function equation which means that the partial derivative will not be a constant and the integration step cannot be avoided.

2. Global Stiffness Matrix

The global stiffness matrix is a synthesis of the individual elemental matrixes combined into a matrix that represents the physical relation between the individual elements. To do this we need a connectivity matrix that relates the elemental node numbering to the global node numbering. In this project, this matrix was written by hand and inserted into the code but for bigger meshes, various algorithms or meshing programs can also be used to obtain it. Each node in an element coincides with a global node. Using this we can loop through every value in the element stiffness matrix and use the connectivity matrix to determine where in the global stiffness matrix the value needs to be added into.

3. Finding Temperature

Once the boundary conditions are defined and the stiffness matrix is calculated, we are left with ‘n’ unknowns and ‘n’ equations, n being the number of nodes. The boundary conditions of course define either the temperature or the net external heat flux acting on a node. This is done using a loop and defining the equations then lugging them into MATLAB’s equation solver. The boundary conditions defined:

$$f = [0 \;\; 0 \;\; 0 \;\; f_4 \;\; 0 \;\; 0 \;\; 0 \;\; f_4 \;\; f_9 \;\; f_{10} \;\; f_{11} \;\; f_{12}]$$

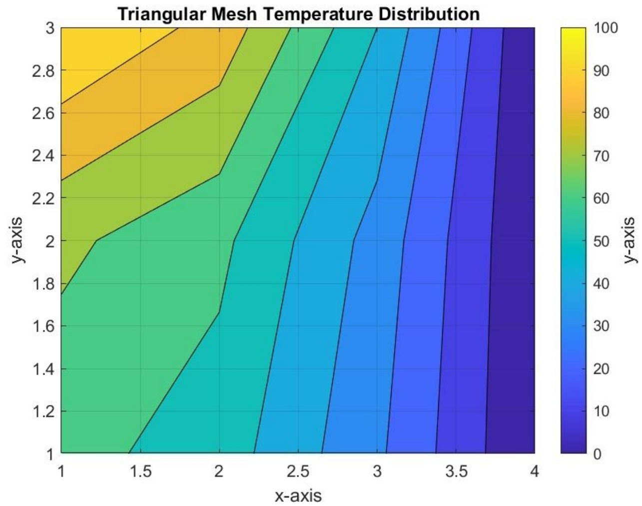
$$T = [T_1 \;\; T_2 \;\; T_3 \;\; 0 \;\; T_5 \;\; T_6 \;\; T_7 \;\; 0 \;\; 100 \;\; 86.6 \;\; 50 \;\; 0]$$

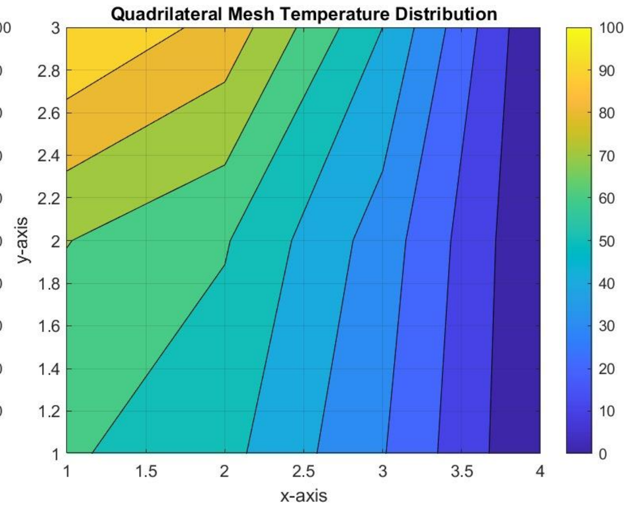
Results

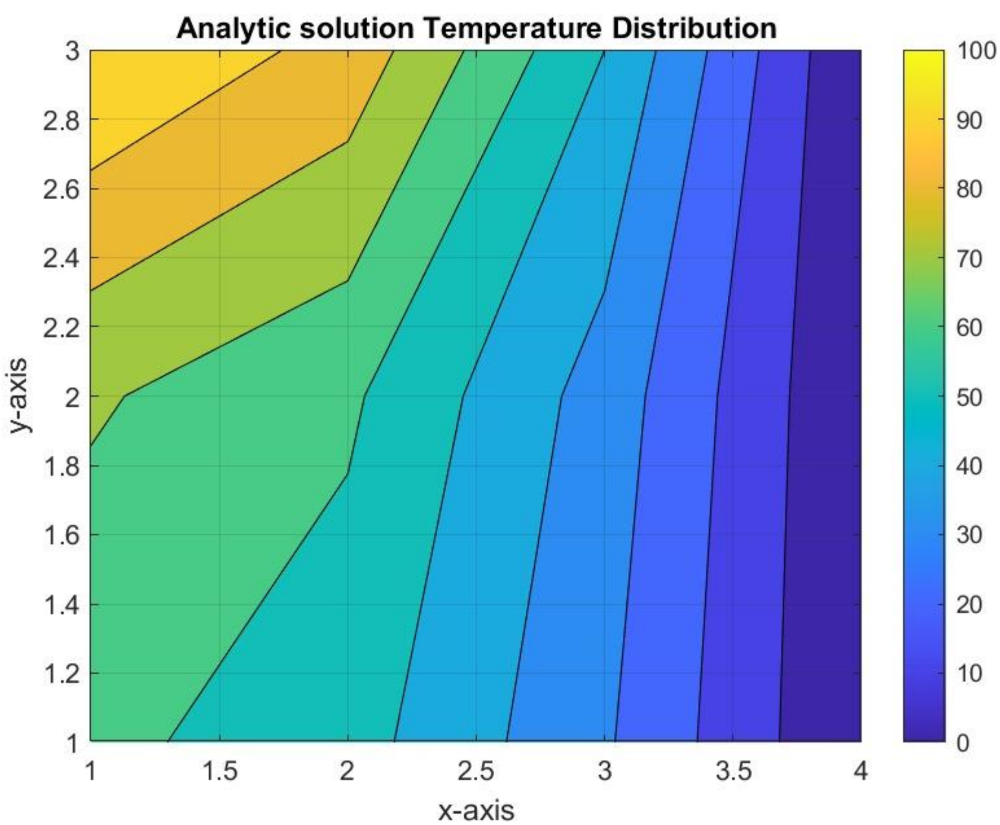
Node ID	Triangular Mesh	Quadrilateral Mesh	Analytic Solution
1	63.621	61.287	62.489
2	55.098	53.076	54.117
3	31.811	30.643	31.244
4	0	0	0
5	72.145	70.302	71.252
6	62.48	60.883	61.706
7	36.073	35.151	35.626
8	0	0	0
9	100	100	100
10	86.603	86.603	86.603
11	50	50	50
12	0	0	0

In the above table it is evident that both of the meshing methods are fairly accurate. The errors are about one percent at most. The triangular mesh solution seems to be slightly closer to the analytical solution. Something that cannot be seen in the table above is the fact that due to the integration process, the quadrilateral solution took significantly longer to calculate. Even though the triangular mesh had twice as many elements it calculated much faster due to the simplicity of the calculations of triangular stiffness matrixes.

Now, let’s visualize the temperature distribution on a graph:

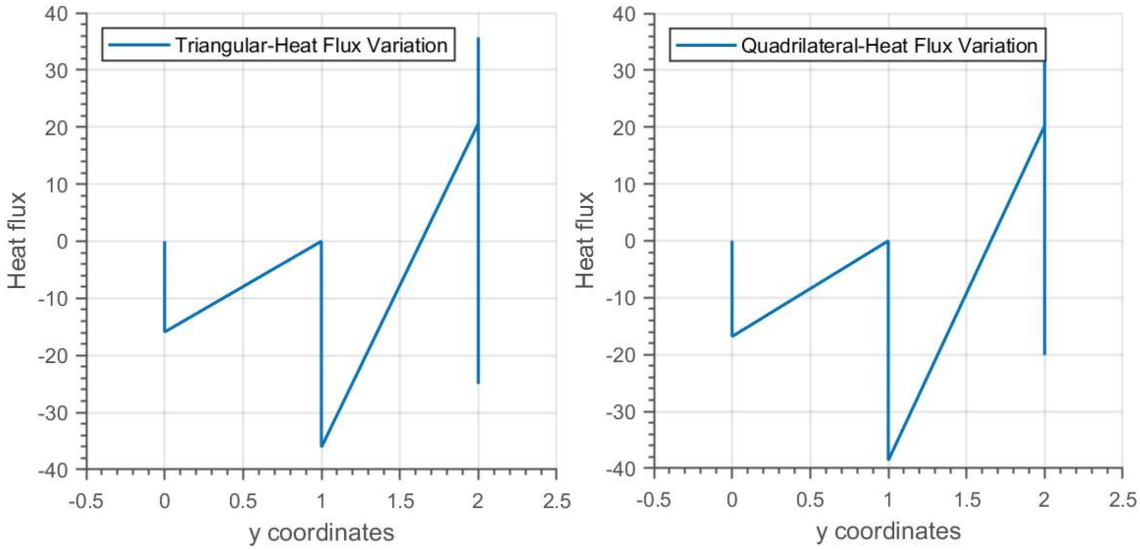






Looking at the graphs above, it is clear that both of the methods get a correct general solution to the temperature distribution problem. When inspected in more detail however it becomes clear that the triangular meshing solution is closer to the analytical one than the quadrilateral solution. This can be observed especially at point 2 on the y-axis where both analytic and triangular solutions have a sharp upward slope but the quadrilateral one has a much lower slope.

Now, let's look at the heat flux variation on the right edge:



These are the results of the heat flux distributions on the right edge of the rectangle. There seems to be discontinuities and sharp corners. Since the linear assumption was made on the shape functions as a whole, the results of the heat fluxes are also linear. There are minor differences between the two but in general the two meshing methods, triangular and quadrilateral, produced the same result.

Conclusion

In conclusion, even though the rectangle was divided into elements using only twelve nodes, the solutions were accurate to about one percent. In class, the theoretical outlook was that the triangular meshes were not as accurate or effective as quadrilateral meshes. Even so, the fact that the derivatives of the shape functions of three node triangles made the calculations much easier. Furthermore, the triangle mesh was just as accurate as the quadrilateral one if not more accurate. If more elements were used then the error could be brought down to negligible values. This shows how close to reality FEM solutions can be.