Introduction to QFT: Kinematics & Perturbative Dynamics

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What are we doing here?

Learning vs understanding vs what we do here

- Learning means changing behavior: acquiring a skill.
- Understanding is about gaining insight: this happens through relating the unknown to the already known.

Here, we have time for neither. We aim to

1. Expose you to some of the ideas of QFT.
2. Make said ideas plausible.

There's a lot of ideas in Quantum Field Theory

Quantum Field Theory is the outstanding achievement of 20th century physics. Different people think of QFT in different ways, but we can say that it is a theoretical framework for calculating correlation functions that describe the quantum physics of a very large number of degrees of freedom.

The input of the theory: The geometry of spacetime, the particle content, the gauge (or internal) symmetries, and massess and coupling constants.

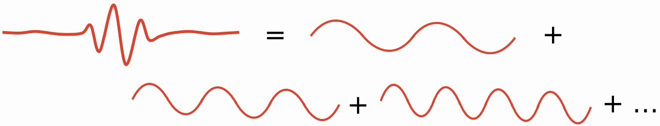
…and there are many quantum field theories

- Nonrelativistic QFTs are used to calculate the properties of some of the most interesting and useful materials available to us, like superconductors.
- Relativistic QFTs that use extra dimensions or include supersymmetry charges are used by theorist to attempt an explanation of some puzzling properties of our universe.
- Exotic matter with higher derivative kinetical terms or higher spin is another possibility.

This lectures will be somewhat historical in that we will have in mind QFT as it was developed: as a relativistic, $3+1$ dimensional theory of fermion matter and photons.

QFT is quantum mechanics for a special space of states

We want to be able to calculate “amplitudes” that go from a configuration (a set of values for a field defined in spacetime) to another.
How do we treat this theory where each state is in some sense infinite? Easy: we descompose it in a base:



So how do we build a base for our states?

Outline

- Some Kinematics of Quantum Field Theory
- Young diagrams and operator spaces
- A couple of ideas about the Dirac field
- Feynman diagrams and graphical languages
- Some perturbative Dynamics of Quantum Field Theory
- A brief intro to the renormalization group

Wave equations in particle physics

Wave equations are somewhat old-fashioned

Schrödinger, 1927: In analogy with optics, classical Newtonian trajectories are like the straight rays of geometrical optics, the zero-wavelength limit of a proper undulatory mechanics. Therefore “material points consist of, or are nothing but, wave-systems”. So wave equations are the fundamental rules of physics.

In the modern perspective, the machinery of quantum mechanics –Hilbert spaces, C^*-algebras, etcetera– allow us to talk about the quantum properties of free systems without any reference to wave systems. Why, then, do we need wave equations?

Free wave equations are kinematical shorthand

«The purely kinematical properties of an isolated quantum mechanical system, i.e. its behaviour under translations, rotations, uniform motions, are completely described by the associated representation of the kinematical group. At this stage, the notion of wave equation for particles is to be considered as a particular method, in general not the most convenient one, of specifying the representation corresponding to the particle: the invariance of the wave equation under the group operation means that its solutions span a representation space for this group.»

Lèvy-Leblond
in Nonrelativistic particles and wave equations

Wave equations are crucial for interacting theories.

«But suppose we now wish to study how the particle behaves in external fields, for instance in order to know its intrinsic electromagnetic properties. Then we cannot use the representation of the kinematical group which only describes the free particle. This is where the notion of wave equations recovers all of its usefulness. Indeed, wave equations constitute the only tool we know of, which enables us to describe interactions in ordinary (first quantized) QM, via the trick of gauge invariance (of the second kind). »

Lèvy-Leblond
in Nonrelativistic particles and wave equations

Elementary particles are Poincaré irreps...

The Poincaré algebra is the semidirect product $\mathfrak{t}_4 \rtimes \mathfrak{so}(1,3)$ with Lie brackets

$$[M_{\mu\nu}, P_\rho] = -i(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu), \qquad [P_\mu, P_\nu] = 0,$$
$$[M_{\mu\nu}, M_{\rho\sigma}] = -i(\eta_{\mu\rho}M_{\nu\sigma} - \eta_{\mu\sigma}M_{\nu\rho} - \eta_{\nu\rho}M_{\mu\sigma} + \eta_{\nu\sigma}M_{\mu\rho}).$$

It has two algebraic invariants

$$\mathcal{C}_2 = P^\mu P_\mu, \qquad \mathcal{C}_4 = W^\mu W_\mu.$$

W_μ is the Pauli-Lubanski four-vector $W_\mu = (1/2)\varepsilon_{\mu\sigma\tau\rho}M^{\sigma\tau}P^\rho$. The one-particle states satisfy

$$\mathcal{C}_2 \left|\Psi\right\rangle = m^2 \left|\Psi\right\rangle, \qquad \mathcal{C}_4 \left|\Psi\right\rangle = -m^2 j(j+1) \left|\Psi\right\rangle$$

We call m the mass and j the spin of Ψ. We write the states of the theory as

$$\left|\Psi\right\rangle = \left|m^2, j, \mathbb{p}, \sigma\right\rangle.$$

…but fields are built in Lorentz representations

The $\mathfrak{so}(1,3)$ subalgebra generated by the $M_{\mu\nu}$ is called the Lorentz algebra. Is is traditionally written in terms of rotations $\mathbb{J}$ and boosts $\mathbb{K}$ operators

$$M^{0i} = K_i, \qquad M^{ij} = \epsilon_{ijk} J_k.$$

It is isomorphic to the $\mathfrak{su}(2)_A \oplus \mathfrak{su}(2)_B$ generated by the commuting sets

$$\mathbb{A} = \frac{1}{2}(\mathbb{J} - i\mathbb{K}), \qquad \mathbb{B} = \frac{1}{2}(\mathbb{J} + i\mathbb{K}).$$

We label the Lorentz irreps with the $\mathfrak{su}(2)$ numbers (a,b). The two Casimir operators for the Lorentz algebra are

$$\mathcal{C}_2 = M_{\mu\nu} M^{\mu\nu} \quad \text{and} \quad \mathcal{C}_2' = M_{\mu\nu} \widetilde{M}^{\mu\nu}$$

where $\widetilde{M}^{\mu\nu} \equiv 1/2\, \varepsilon^{\mu\nu\alpha\beta} M_{\alpha\beta}$. In terms of the $\mathfrak{su}(2)$ operators $\mathbb{A}^2$ and $\mathbb{B}^2$,

$$M_{\mu\nu} M^{\mu\nu} = 4(\mathbb{A}^2 + \mathbb{B}^2), \quad M_{\mu\nu} \widetilde{M}^{\mu\nu} = -4i(\mathbb{A}^2 - \mathbb{B}^2).$$

Wave equations mediate between Lorentz and Poincaré

Quantum fields are built from operators that create or destroy these states:

$$\Psi_l(x) = \int d\Gamma[\kappa e^{ip\cdot x}\omega_l(\Gamma)a^\dagger(\Gamma) + \lambda e^{-ip\cdot x}\omega_l^c(\Gamma)a(\Gamma)].$$

(Here $\Gamma = \mathbb{p}, \sigma$.) But field coefficients like $\omega_l(\Gamma)$ transform in representations of the Lorentz algebra.

This is the meaning of wave equations: they pick the degrees of freedom in Lorentz representations with good Poincaré quantum numbers. An (a, b) Lorentz representations contains $j = |a-b|, |a-b|+1, \ldots, a+b$.

A field theory can be Lorentz invariant and not describe "elementary systems" in the Wigner sense. Consider the vector Lagrangian

$$\mathcal{L} = \frac{\alpha}{2}(\partial_\rho A^\rho)^2 + \frac{\beta}{2}\partial_\rho A_\sigma \partial^\rho A^\sigma - V(A^\mu).$$

Inspecting the Hessian

$$H^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_0 A_\mu)\partial(\partial_0 A_\mu)} = \alpha g^{0\mu} g^{0\nu} + \beta g^{\mu\nu},$$

we can see that

$$\det H = (\alpha + \beta)\beta^3.$$

Unless $\alpha + \beta = 0$ o $\beta = 0$ (which fixes $j = 1$ o $j = 0$) there are no constraints, and the theory mixes two irreducible representations. (It has four degrees of freedom).

There are three massive wave equations in the SM

- The Klein-Gordon equation enforces the mass Poincaré quantum number. Since it describes a massless field, there can be no constraints because the degrees of freedom match (i.e., the $(0,0)$ Lorentz representation contains only $j=0$).
- The massive Dirac equation is a constraint that removes two degrees of freedom from the four-dimensional Dirac representation

$$\left(\frac{1}{2},0\right)\oplus\left(0,\frac{1}{2}\right).$$

  This representation is also single spin, so the Dirac constraint is just enforcing well-defined parity.
- The Proca equation for a massive vector field selects the spin Poincaré number, picking up the physical three degrees of freedom corresponding to $j=1$ contained in the representation

$$\left(\frac{1}{2},\frac{1}{2}\right).$$

We don't know why the SM only includes some spins in some representations

$$(0,0)$$

$$\left(\tfrac{1}{2},0\right) \qquad \left(0,\tfrac{1}{2}\right)$$

$$(1,0) \qquad \left(\tfrac{1}{2},\tfrac{1}{2}\right) \qquad (0,1)$$

$$\left(\tfrac{3}{2},0\right) \qquad \left(1,\tfrac{1}{2}\right) \qquad \left(\tfrac{1}{2},1\right) \qquad \left(0,\tfrac{3}{2}\right)$$

$$(2,0) \qquad \left(\tfrac{3}{2},\tfrac{1}{2}\right) \qquad (1,1) \qquad \left(\tfrac{1}{2},\tfrac{3}{2}\right) \qquad (0,2)$$

We don't know why the SM only includes some spins in some representations

$$\boxed{(0,0)}$$

Higgs field

$$\left(\tfrac{1}{2},0\right) \qquad \left(0,\tfrac{1}{2}\right)$$

$$(1,0) \qquad \left(\tfrac{1}{2},\tfrac{1}{2}\right) \qquad (0,1)$$

$$\left(\tfrac{3}{2},0\right) \qquad \left(1,\tfrac{1}{2}\right) \qquad \left(\tfrac{1}{2},1\right) \qquad \left(0,\tfrac{3}{2}\right)$$

$$(2,0) \qquad \left(\tfrac{3}{2},\tfrac{1}{2}\right) \qquad (1,1) \qquad \left(\tfrac{1}{2},\tfrac{3}{2}\right) \qquad (0,2)$$

We don't know why the SM only includes some spins in some representations

$$(0,0)$$

$$\boxed{(\tfrac{1}{2},0) \qquad (0,\tfrac{1}{2})}$$

Quarks & Leptons

$$(1,0) \qquad (\tfrac{1}{2},\tfrac{1}{2}) \qquad (0,1)$$

$$(\tfrac{3}{2},0) \qquad (1,\tfrac{1}{2}) \qquad (\tfrac{1}{2},1) \qquad (0,\tfrac{3}{2})$$

$$(2,0) \qquad (\tfrac{3}{2},\tfrac{1}{2}) \qquad (1,1) \qquad (\tfrac{1}{2},\tfrac{3}{2}) \qquad (0,2)$$

We don't know why the SM only includes some spins in some representations

$$(0,0)$$

$$\left(\tfrac{1}{2},0\right) \qquad \left(0,\tfrac{1}{2}\right)$$

$$(1,0) \qquad \boxed{\left(\tfrac{1}{2},\tfrac{1}{2}\right)} \qquad (0,1)$$

Gauge bosons

$$\left(\tfrac{3}{2},0\right) \qquad \left(1,\tfrac{1}{2}\right) \qquad \left(\tfrac{1}{2},1\right) \qquad \left(0,\tfrac{3}{2}\right)$$

$$(2,0) \qquad \left(\tfrac{3}{2},\tfrac{1}{2}\right) \qquad (1,1) \qquad \left(\tfrac{1}{2},\tfrac{3}{2}\right) \qquad (0,2)$$

We don't know why the SM only includes some spins in some representations

$$(0,0)$$

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$$\left(\tfrac{3}{2},0\right) \qquad \left(1,\tfrac{1}{2}\right) \qquad \left(\tfrac{1}{2},1\right) \qquad \left(0,\tfrac{3}{2}\right)$$

$$(2,0) \qquad \left(\tfrac{3}{2},\tfrac{1}{2}\right) \qquad \boxed{(1,1)} \qquad \left(\tfrac{1}{2},\tfrac{3}{2}\right) \qquad (0,2)$$ Graviton?

We don’t know why the SM only includes some spins in some representations

$$
\begin{array}{ccccccccc}
 & & & & (0,0) & & & & \\
 & & & (\frac{1}{2},0) & & (0,\frac{1}{2}) & & & \\
 & & \boxed{(1,0)} & & (\frac{1}{2},\frac{1}{2}) & & \boxed{(0,1)} & & \\
 & \boxed{(\frac{3}{2},0)} & & \boxed{(1,\frac{1}{2})} & & \boxed{(\frac{1}{2},1)} & & \boxed{(0,\frac{3}{2})} & \\
\boxed{(2,0)} & & \boxed{(\frac{3}{2},\frac{1}{2})} & & (1,1) & & \boxed{(\frac{1}{2},\frac{3}{2})} & & \boxed{(0,2)}
\end{array}
$$

And these?

So far, we neither understand, nor need them.

What about massless wave equations?

Massless one-particle states satisfy

$$P^2 \left|\Psi\right\rangle = 0 \quad W^2 \left|\Psi\right\rangle = 0.$$

Also, $W \cdot P = 0$. But two null vectors which are orthogonal have to be proportional to each other. Therefore, for massless states

$$\left(P^\mu \pm \lambda W^\mu\right) \left|\Psi\right\rangle = 0.$$

λ is known as the helicity. In general, every massless Poincaré representation is two-dimensional.

We take a pragmatic approach: treat massless fields as massive, take the mass to zero at the end of the calculation, and if it doesn't blow up we'll say it is alright. ;-)

Observables in quantum mechanics obey Jordan algebras

A Jordan algebra is a non associative abelian algebra whose bilinear product $\bullet$ satisfies the axioms [1]

$$x \bullet y = y \bullet x$$
$$(x \bullet y) \bullet (x \bullet x) = x \bullet [y \bullet (x \bullet x)].$$

Jordan algebras were invented to describe quantum observables, which satisfy these axioms[2] . They determine the Hilbert space of physical states.

Alfsen-Shultz theorem: two C*-algebras have isomorphic state spaces if they are isomorphic as Jordan algebras

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[1] N. Jacobson. Structure and Representations of Jordan Algebras , 1968.

[2] Jordan, Wigner & von Neumann, On an algebraic generalization of the quantum mechanical formalism, 1934

Lie algebras describe symmetries

A Lie algebra, on the other hand, has a bilinear product $\diamond$ satisfying [3]

$$\begin{aligned} x \diamond y &= -y \diamond x, \\ x \diamond (y \diamond z) &= -y \diamond (z \diamond x) - z \diamond (x \diamond y), \end{aligned}$$

the second axiom being called the Jacobi identity. Lie algebras are very important for the description of symmetry and dynamics.

[3] Fuchs, Symmetries, Lie Algebras and Representations: A Graduate Course for Physicists, 2003

Jordan-Lie algebras combine both structures

If we have an algebra where both products $\bullet, \diamond$ are defined, we require as consistency conditions the Leibnitz rule

$$(a \bullet b) \diamond c = a \bullet (b \diamond c) + (a \diamond c) \bullet b,$$

and the associator rule

$$(a \bullet b) \bullet c - a \bullet (b \bullet c) = \frac{\hbar^2}{4} (a \diamond c) \diamond b,$$

for some $\hbar$. This kind of structure is called a Jordan-Lie algebra, or a Poisson algebra for $\hbar = 0$.

Jordan-Lie algebras matter naturally appear in QFT

If we have an associative product, we can define Lie and Jordan products as commutators and anticommutators

$$A \bullet B = \frac{1}{2}\{A, B\} = \frac{1}{2}(AB + BA), \quad A \diamond B = [A, B] = AB - BA,$$

This guarantees that the axioms of a Jordan-Lie algebra are satisfied.

The Clifford-$\mathfrak{u}(2)$ algebra satisfied by the matrix identity plus the Pauli matrices is a Jordan-Lie algebra. We will soon meet a different Clifford algebra for the Dirac γ^μ matrices.

Lie$\otimes$Lie = Lie, but Jordan algebras are unique

If we have two representation of a Lie algebra $\mathfrak{g}$

$$D_1(x) \circ D_1(y) = D_1(x \diamond y), \quad D_2(x) \circ D_2(y) = D_2(x \diamond y).$$

then the tensor product

$$D_1 \otimes D_2(x) = D_1(x) \otimes \mathbb{1}_2 + \mathbb{1}_1 \otimes D_2(x).$$

is also a representation of $\mathfrak{g}$. However, the tensor product of two representations of a Jordan algebra do not satisfy the original Jordan algebra.

We build the representations of a Lie algebra $\mathfrak{g}$ as the tensor product of a small number of fundamental representations, but different representations of a Lie algebra possess a different Jordan structure.

So this is why Jordan-Lie algebras matter

- Lie algebras characterize the symmetries of physical systems.
- Different representations of the same Lie algebra possess different Jordan algebras.
- The full Jordan-Lie algebras are properties of elementary systems, and related to their possible interactions.

So, what wave equations you can build in a given Lorentz representation depends on what the operators available for building a kinetical operator are.

A familiar example is the $\mathfrak{su}(2)$ $j = 1/2$ algebra

$$J_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad J_2 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad J_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The Lie generators close under the Lie product $\diamond$

$$J_m \diamond J_n = [J_m, J_n] = i\,\varepsilon_{mno} J_o,$$

but with the Jordan product $\bullet$ produce

$$J_i \bullet J_j = \frac{1}{4}\delta_{ij} e, \quad e \bullet J_i = J_i, \quad e \bullet e = e,$$

with

$$e = \frac{1}{j(j+1)} \sum_{i=1}^{3} J_i \bullet J_i = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad e \diamond J_i = [e, J_i] = 0.$$

An element of the $j = 1/2$ algebra has a natural decomposition in terms of operators transforming as $j = 0, 1$ objects.

Young diagrams classify the Lie representations

The use of Young diagrams for the classification of irreducible representations of Lie algebras stems from the following three facts:

1. The Young diagrams of n boxes are in 1-to-1 correspondence with the irreducible representations of the permutation group $\mathscr{S}_n$.
2. An arbitrary representation of a Lie algebra can be built from the tensorial product of a small number of fundamental representations.
3. The tensor product of a representation of a Lie algebra commutes with the action of the permutation group $\mathscr{S}_n$.

The dual role of Young diagrams

Imagine a basis $\{v_i\}$ for the vector space V on which $D(\mathfrak{g})$ acts. We define a basis for the tensor product $D^{\otimes n}(\mathfrak{g})$

$$v_{i1i2\ldots in} \equiv v_{i1} \otimes v_{i2} \otimes \ldots v_{in},$$

with indexes running from 1 to $\dim V$. We define an action of $\mathscr{S}_n$ as follows: the permutation $\sigma \in \mathscr{S}_n$ acts on the basis vectors as

$$v_{i1i2\ldots in} \to v_{\sigma(i1)\sigma(i2)\ldots\sigma(in)}.$$

Since the $\mathfrak{g}$ algebra acts on this space through

$$D^{\otimes n} = (D \otimes \mathbb{1} \otimes \ldots \mathbb{1}) + (\mathbb{1} \otimes D \otimes \ldots \mathbb{1}) + \ldots (\mathbb{1} \otimes \mathbb{1} \otimes \ldots D),$$

the actions of $\mathfrak{g}$ and $\mathscr{S}_n$ commute. In the $\mathscr{S}_n$ decomposition of $V^{\otimes n}$

$$V^{\otimes n} = \bigoplus_{\lambda} W_\lambda,$$

with the sum over n partition λ, every subspace W_λ will be a representation of $\mathfrak{g}$, and a basis for the spaces W_λ can be obtained by acting on the tensor basis with the Young projector Y^λ.

The spin operators are written in $\mathfrak{su}(2)$ Young diagrams

Since $\mathfrak{su}(2)$ has rank $r = 1$, we need only one-row diagrams. The Dynkin label corresponds to the integers $[2j]$. To the spin j representation of $\mathfrak{su}(2)$ we assign the Young diagram with $2j$ boxes in a single row

$$[2j] \Leftrightarrow \underbrace{\square\cdots\square}_{2j}\,.$$

We replace the $\{1, 1\}$ diagram with the singlet in our previous results:

$$j = 1/2 \Leftrightarrow [1] \Leftrightarrow \square \qquad \square^{\otimes 2} = \begin{smallmatrix}\square\\ \square\end{smallmatrix} \oplus \square\square = \mathbb{1} \oplus \square\square$$

$$j = 1 \Leftrightarrow [2] \Leftrightarrow \square\square \qquad (\square\square)^{\otimes 2} = \mathbb{1} \oplus \square\square \oplus \square\square\square\square\,.$$

In general, for the spin j representation, we will have the operator space

$$\left(\underbrace{\square\cdots\square}_{2j}\right)^{\otimes 2} = \mathbb{1} \oplus \square\square \oplus \square\square\square\square \oplus \cdots \oplus \underbrace{\square\cdots\square}_{4j}\,.$$

The $\mathfrak{so}(1,3)$ Young diagrams follow from $\mathfrak{su}(2)$

The Lorentz algebra $\mathfrak{so}(1,3)$ breaks down as the sum of simple subalgebras

$$\mathfrak{so}(1,3) = \mathfrak{su}(2) \oplus \mathfrak{su}(2).$$

By following the $\mathfrak{su}(2)$ example, irreps of $\mathfrak{so}(1,3)$ can be labeled by the Dynkin labels $[2a][2b]$, and associated with the composite Young diagrams

$$[2a][2b] \Leftrightarrow \underbrace{{\color{red}\blacksquare}\cdots{\color{red}\blacksquare}}_{2a}\underbrace{{\color{blue}\blacksquare}\cdots{\color{blue}\blacksquare}}_{2b}\ .$$

There are two fundamental representations, out of which every representation can be built:

$${\color{red}\blacksquare} \quad \text{and} \quad {\color{blue}\blacksquare}\ .$$

For the Jordan-Lie algebra for different representations of this algebra only it is enough to consider the products of the composite box

$$\square = {\color{red}\blacksquare}{\color{blue}\blacksquare}\ ,$$

corresponding to the vector representation $(1/2, 1/2)$.

We only need the characterization of the Young diagrams with one and two rows [4]. The dimension of completely traceless tensors in $\mathfrak{so}(1,3)$ is given by

$$d_{[n]} = (n+1)^2 \qquad \overbrace{\square\square\cdots\square\square}^{n}\,,$$

$$d_{[n,m]} = 2(n-m+1)(n+m+1) \qquad \begin{array}{l} \overbrace{\square\square\cdots\square\square}^{n} \\ \underbrace{\square\square\cdots\square}_{m} \end{array}\,.$$

These diagrams describe either a symmetrical tensor of rank n, or a mixed-symmetry tensor of rank $n+m$. The factor 2 comes about because we are considering both the self-dual and anti-self dual parts.

[4] Zhong-Qi Ma, Group Theory for Physicists, 2007

There are two kinds of parity-invariant representations

Parity exchanges $\mathbb{A}$ and $\mathbb{B}$:

$$\Pi \mathbb{A} \Pi^{-1} = \mathbb{B}.$$

The parity-invariant Lorentz representations fall in two groups:

- The (a, a) integer-spin Fierz-Pauli irreducible representations. All the boson fields in the Standard Model transform in this class.
- The reducible $(a, b) \oplus (b, a)$ representations with $a \neq b$. These include fermionic Fierz-Pauli representations as well as the Joos-Weinberg family. The Dirac Field belongs to this class.

We call these non-chiral and chiral representations, according with the vanishing of the chirality operator

$$\chi = \frac{i\, M_{\mu\nu} \widetilde{M}^{\mu\nu}}{4a(a+1) - 4b(b+1)}.$$

Today, we want to study the Dirac and the Proca (vector) fields.

Parity is not a scalar

In the chiral basis, the parity operator that swaps $(j,0)$ and $(0,j)$ is

$$\Pi = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}.$$

This operator satisfies the Lie rules

$$\Pi \diamond \mathbb{J} = 0 \quad \Pi \diamond \mathbb{K} = 2\Pi\mathbb{K},$$

Parity rotates as a scalar, but transforms under boosts into $\mathbb{V} = 2i\Pi\mathbb{K}$. In turn,

$$J_i \diamond V_j = i\varepsilon_{ijk} V_k, \quad K_i \diamond V_j = -4\Pi K_i \bullet K_j.$$

These rules involve the Jordan algebra of the Lorentz generators.

The covariant properties of Π will depend on the objects

$$\Pi\{\mathbb{K}, \mathbb{K}\bullet\mathbb{K}, (\mathbb{K}\bullet\mathbb{K})\bullet\mathbb{K}, \dots\},$$

that form a symmetric tensor $S_{\mu_1\cdots\mu_{2j}}$ with time component $S_{0\cdots 0} = \Pi$.

How do we characterize operators?

Let us say we have some Hilbert space, with a basis

$$\{|\phi_i\rangle\}.$$

Then, a basis for the linear operator acting on this Hilbert space is given by the projection operators:

$$\{|\phi_i\rangle\langle\phi_j|\}.$$

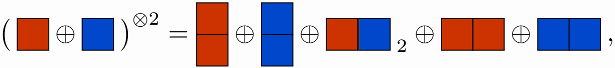
Therefore, the properties of the vector space of operators acting on some Hilbert space follow from the properties of the states.

Operator space of the Dirac representation

The first example is the $(1/2, 0) \oplus (0, 1/2)$ space. The Lorentz decomposition of the operator space is

$$\left[\left(\frac{1}{2}, 0\right) \oplus \left(0, \frac{1}{2}\right)\right]^2 = (0,0)_2 \oplus (1,0) \oplus (0,1) \oplus \left(\frac{1}{2}, \frac{1}{2}\right)_2 .$$

This corresponds to a pair of scalars, an antisymmetric tensor, and a pair of four-vectors:



The scalars corresponds to the Lorentz Casimirs $\mathcal{C}_2$ and $\mathcal{C}_2'$, that we normalize to the identity $\mathbb{1}$ and chirality γ_5. The asymmetrical tensor in the $(1,0) \oplus (0,1)$ corresponds to the Lorentz generators $\sigma_{\mu\nu}$. As for the vectors object

$$\gamma_\mu = \eta_{0\mu} \Pi - 2i \Pi M_{0\mu} ,$$

transforms as a four-vector

$$[M_{\rho\sigma} , \gamma_\mu] = i\eta_{\mu\rho}\gamma_\sigma - i\eta_{\mu\sigma}\gamma_\rho .$$

We can check that $\gamma_5 \gamma^\mu$ is an independent four-vector operator. Then, the covariant basis is $\{\mathbb{1}, \gamma_5, \gamma_\mu, \gamma_5\gamma_\mu, \sigma_{\mu\nu}\}$

How to build a Lagrangian for the Dirac field?

$$\mathcal{L} = \text{kinetical} \; - \text{potential}$$

The kinetical term is obtained by contracting the spacetime derivative of the field (or the momentum, in the Fourier transform representation) with the operators acting on the Hilbert space. There are two possibilities:

$$\{\mathbb{1}\Box, i\gamma_\mu\partial^\mu\}.$$

The first one produces a Klein-Gordon-like theory; the second one leads to a Dirac equation:

$$i\gamma_\mu\partial^\mu\psi - V(\psi, \bar{\psi}) = 0.$$

For the free particle,

$$(i\gamma_\mu\partial^\mu - m)\psi = 0.$$

The properties of γ^μ are fixed

In the covariant basis $\{\mathbb{1}, \gamma_5, \gamma_\mu, \gamma_5\gamma_\mu, \sigma_{\mu\nu}\}$ The tensor $\sigma_{\mu\nu}$ the Lorentz algebra

$$[\sigma_{\mu\nu}, \sigma_{\rho\tau}] = -i(\eta_{\mu\rho}\sigma_{\nu\tau} - \eta_{\mu\tau}\sigma_{\nu\rho} - \eta_{\nu\rho}\sigma_{\mu\tau} + \eta_{\nu\tau}\sigma_{\mu\rho}).$$

Since they are scalars,

$$[\sigma_{\mu\nu}, \mathbb{1}] = [\sigma_{\mu\nu}, \gamma_5] = 0.$$

On the other hand, γ_μ rotates as a vector:

$$[\sigma_{\mu\nu}, \gamma_\rho] = i\eta_{\nu\rho}\gamma_\mu - i\eta_{\mu\rho}\gamma_\nu.$$

Last, we get the commutator and anticommutator rules

$$[\gamma_\mu, \gamma_\nu] = -2i\sigma_{\mu\nu} \quad \{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}.$$

The meaning of Dirac's equation

Acting on the Dirac equation with $(i\gamma_\nu \partial^\nu + m)$ we get

$$(i\gamma_\nu \partial^\nu + m)(i\gamma_\nu \partial^\nu - m)\psi = \left(-\frac{1}{2}\{\gamma_\mu, \gamma_\nu\}\partial^\mu \partial^\nu + m\right)\psi = 0.$$

But since

$$\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu},$$

then

$$(i\gamma_\nu \partial^\nu + m)(i\gamma_\nu \partial^\nu - m)\psi = \left(\partial^2 + m\right)\psi = 0.$$

This justifies the name we've given the constant m: it is indeed the mass of the Dirac field.

If we Fourier transform the Dirac equation we get

$$(\gamma_\mu p^\mu - m)\psi(p) = 0.$$

Going to the rest frame where the components of the momentum are $(m, \mathbb{0})$ we obtain

$$(\gamma_0 - 1)\psi = 0,$$

which is an eigenvalue equation for $\gamma_0 = \Pi$, the parity operator.
So Dirac equation is the covariant statement that the Dirac field has well-defined parity.

We can give an explicit representation for the gamma matrices (by picking a particular basis):

$$\gamma_0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} = \mathbb{1} \otimes \sigma_3, \gamma_i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} = \sigma_i \otimes i\sigma_2,$$

where each entry is understood to be a 2×2 matrix.
Here, it is clear that

$$(\gamma_0 - 1)\psi = 2 \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1} \end{pmatrix} \psi = 0,$$

so that only two components of the Dirac field in the rest frame are nonzero.

It is also useful to know that $\gamma_5 = -i\gamma_0\gamma_1\gamma_2\gamma_3 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$.

From the algebra of the gamma matrices,

$$\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu},$$

we can see that

$$(\gamma_0)^2 = \mathbb{1}, \quad (\gamma_i)^2 = -\mathbb{1}.$$

The matrix γ_0 is hermitean, while γ_i is antihermitean, which is often expressed as

$$(\gamma_\mu)^\dagger = \gamma_0 \gamma_\mu \gamma_0.$$

For this reason, the bilinear $\psi^\dagger \gamma_\mu \psi$ is not hermitean. Rather,

$$(\psi^\dagger \gamma_0 \gamma_\mu \psi)^\dagger = \psi^\dagger \gamma_0 \gamma_\mu \psi.$$

We define $\bar{\psi} = \psi^\dagger \gamma_0$. It is $\bar{\psi}\psi$ and not $\psi^\dagger \psi$ that transforms as a scalar.

We can now write a Lagrangian for Dirac

The Dirac Lagrangian is

$$\mathcal{L} = \bar{\psi}(\ \gamma^\mu \partial_\mu - m)\psi$$

(Please check that the Euler-Lagrange equations of this Lagrangian give back the Dirac equation.)

Given a representation of the gamma matrices, solving the Fourier-transformed Dirac equation

$$(\gamma^\mu p_\mu - m)\psi$$

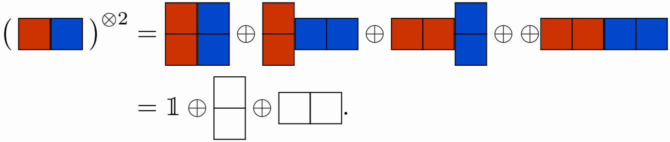
is a linear algebra problem.

What about the Proca field?

For the representation $(1/2, 1/2)$, the Lorentz decomposition is

$$\left(\frac{1}{2}, \frac{1}{2}\right)^{\otimes 2} = (0,0) \oplus (1,0) \oplus (0,1) \oplus (1,1),$$

or in Young diagram form,



We identify the $(0,0)$ operator with $\mathbb{1}$ and the $(1,0) \oplus (0,1)$ operators with the Lorentz generators $M_{\mu\nu}$.

The symmetrical operator $(1,1)$ is

$$T_{\mu\nu}{}^{\alpha}{}_{\beta} = \eta_{\mu}{}^{\alpha}\eta_{\nu\beta} + \eta_{\nu}{}^{\alpha}\eta_{\mu\beta}$$

We want $j = 1$

The most general kinetic operator for the vector spin one representation is

$$\mathcal{L}_{\pm} = (\partial_\mu A^\alpha)^\dagger \left(\Sigma^{\mu\nu}{}_{\alpha\beta}\right)\partial_\nu A^\beta$$

where

$$\Sigma^{\mu\nu}{}_{\alpha\beta} = A\mathbb{1}\eta^{\mu\nu}\eta_{\alpha\beta} + BT^{\mu\nu}{}_{\alpha\beta}\,.$$

The tensor $T^{\mu\nu}{}_{\alpha\beta}$ is given by

$$T^{\mu\nu}{}_{\alpha\beta} = \eta^\mu{}_\alpha\eta^\nu{}_\beta + \eta^\nu{}_\alpha\eta^\mu{}_\beta.$$

This describes particles with spin one only if we choose the coefficients so that $A = -2B$. Then, the wave equation is

$$\partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) + mA^\nu = 0.$$

So where are we now?

So far, we have built the wave equations for the two systems that we are interested in:

- Massive spin $j = 1/2$ fields transforming in the $\left(\frac{1}{2}, 0\right) \oplus \left(0, \frac{1}{2}\right)$ obey the Dirac equation

$$(i\gamma^\mu \partial_\mu - m)\psi(x) = 0.$$

- Spin $j = 1$ fields transforming in the $\left(\frac{1}{2}, \frac{1}{2}\right)$ obey the Proca equation

$$\partial_\rho(\partial^\rho A^\mu(x) - \partial^\mu A^\rho(x)) + m^2 A^\mu(x) = 0.$$

Let us talk a bit about the spinor fields that we have built, discuss quantum mechanical scattering to give a flavor of the origin of Feynman diagrams, and then go on to study the interacting theory.

Spin and statistics are related in QFT

In Quantum Mechanics, there is an additional postulate that is needed when the number of particles is larger than one, known as the Pauli exclusion principle.

Multi-particle states of indistinguishable half-integer spin particles are always found in the completely antisymmetrical sector (fermions); those of indistinguishable integer spin particles are always in the completely symmetrical sector (bosons).

This is an axiom that has no justification within nonrelativistic quantum mechanics. However, it can be proved in relativistic QFT, where it is known as the spin-statistics theorem. We will not prove it here!

Why do we care? In 1928 Jordan and Wigner showed how this is equivalent to requiring that the creation and annihilation operators for half-integer fields satisfy anticommutation rather than commutation rules.

Then, the Dirac field is a fermion

We will require that the one-particle harmonic operators

$$b^\dagger(\mathbb{p}, \sigma)\,|0\rangle = |\mathbb{p}, \sigma\rangle$$

(where the $m^2, j = 1/2$ labels are implicit), obey

$$\{b(\mathbb{p}, \sigma), b^\dagger(\mathbb{p}', \sigma')\} = \delta(\mathbb{p} - \mathbb{p}')\delta_{\sigma,\sigma'},$$

and

$$\{b^\dagger(\mathbb{p}, \sigma), b^\dagger(\mathbb{p}', \sigma')\} = \{b(\mathbb{p}, \sigma), b(\mathbb{p}', \sigma')\} = 0.$$

This allows us to define $N(\mathbb{p}, \sigma) = b^\dagger(\mathbb{p}, \sigma)b(\mathbb{p}, \sigma)$. Please check that

$$\{N(\mathbb{p}, \sigma), b^\dagger(\mathbb{p}', \sigma')\} = b^\dagger(\mathbb{p}', \sigma')$$

and

$$\{N(\mathbb{p}, \sigma), b(\mathbb{p}', \sigma')\} = -b(\mathbb{p}', \sigma').$$

We have analogous rules for $a(\mathbb{p}, \sigma)$, the antiparticle creation operator.

We can guess at the form of the field anticommutators

From our Lagrangian

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi$$

we find the conjugate momenta

$$\pi = \frac{\delta \mathcal{L}}{\delta \partial_0 \psi} = i\psi^\dagger.$$

With it, we can calculate the classical Hamiltonian:

$$\mathcal{H} = \pi \partial_0 \psi - \mathcal{L} = i\bar{\psi}\gamma^0 \partial_0 \psi.$$

It appears reasonable to expect that

$$\{\psi(\mathbb{x}, t), i\psi^\dagger(\mathbb{y}, t)\} = \mathbb{1}\delta(\mathbb{x} - \mathbb{y})$$

We will derive this result.

We fix the coefficients u and v

Consider plane wave solutions $u(\mathbb{p},\sigma)e^{-ipx}$ and $v(\mathbb{p},\sigma)e^{ipx}$. Plugin them in Dirac's equation we get

$$(\not{p}-m)u(\mathbb{p},\sigma)=0, \qquad (\not{p}+m)v(\mathbb{p},\sigma)=0.$$

In the rest frame the solutions are

$$u=\begin{pmatrix}\xi_\pm\\ 0\end{pmatrix} \qquad v=\begin{pmatrix}0\\ \xi_\pm\end{pmatrix},$$

where

$$\xi_+=\begin{pmatrix}1\\ 0\end{pmatrix} \quad \xi_-=\begin{pmatrix}0\\ 1\end{pmatrix}$$

This means that $\bar{u}u=1$, and $\bar{v}v=-1$. Since this are invariant products, this is valid in any frame. (This is not the only normalization possible, of course).

At $E \approx m$, we have small and large components

In a boosted frame

$$u(\mathbb{p}, \sigma) = \exp(-\frac{i}{2}\mathbb{b}\cdot\mathbb{K})\, u(\mathbb{0}, \sigma) = \cosh\frac{|b|}{2}\mathbb{1}\begin{pmatrix}\xi_\pm \\ 0\end{pmatrix} + i\sinh\frac{|b|}{2}\mathbb{b}\cdot\mathbb{K}\begin{pmatrix}\xi_\pm \\ 0\end{pmatrix}.$$

where $|\mathbb{b}| = |\mathbb{p}|/E$ and $\cosh|\mathbb{b}| = E/m$. This can be rewritten using

$$\cosh\frac{|b|}{2} = \left(\frac{\cosh|\mathbb{b}| + 1}{2}\right)^{1/2} \qquad \sinh\frac{|b|}{2} = \left(\frac{\cosh|\mathbb{b}| - 1}{2}\right)^{1/2}$$

as

$$u(\mathbb{p}, \pm 1/2) = \sqrt{\frac{E+m}{2m}}\begin{pmatrix}\xi_\pm \\ \dfrac{1}{E+m}\mathbb{p}\cdot\sigma\xi_\pm\end{pmatrix}$$

For 'slow' Dirac particles, $E \approx m$ and

$$u(\mathbb{p}, \pm 1/2) = \begin{pmatrix}\xi_\pm \\ \dfrac{1}{2m}\mathbb{p}\cdot\sigma\xi_\pm\end{pmatrix}.$$

In the rest frame,

$$\sum_\sigma u(\mathbb{p},\sigma)\bar{u}(\mathbb{p},\sigma) = \begin{pmatrix} \xi_+\xi_+ + \xi_-\xi_- & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & 0 \end{pmatrix} = \frac{1}{2}\left(\gamma^0 + \mathbb{1}\right)$$

and

$$\sum_\sigma v(\mathbb{p},\sigma)\bar{v}(\mathbb{p},\sigma) = \begin{pmatrix} 0 & 0 \\ 0 & -\xi_+\xi_+ - \xi_-\xi_- \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -\mathbb{1} \end{pmatrix} = \frac{1}{2}\left(\gamma^0 - \mathbb{1}\right).$$

The covariant versions of these are

$$\sum_\sigma u(\mathbb{p},\sigma)\bar{u}(\mathbb{p},\sigma) = \frac{1}{2m}\left(\not{p} + m\mathbb{1}\right)$$

and

$$\sum_\sigma v(\mathbb{p},\sigma)\bar{v}(\mathbb{p},\sigma) = \frac{1}{2m}\left(\not{p} - m\mathbb{1}\right)$$

We know enough to do some calculations

We have all the ingredients for writing our quantum Dirac field

$$\psi(x) = \sum_\sigma \int \frac{d^3p}{(2\pi)^3\sqrt{2E}} \left(u(\mathbb{p},\sigma)e^{-ipx}a(\mathbb{p},\sigma) + v(\mathbb{p},\sigma)e^{ipx}b^\dagger(\mathbb{p},\sigma)\right)$$

and

$$\bar{\psi}(x) = \sum_\sigma \int \frac{d^3p}{(2\pi)^3\sqrt{2E}} \left(\bar{u}(\mathbb{p},\sigma)e^{ipx}a^\dagger(\mathbb{p},\sigma) + \bar{v}(\mathbb{p},\sigma)e^{-ipx}b(\mathbb{p},\sigma)\right)$$

Important point: time ordering here has to take into account the anticommutativity of harmonic operators.

$$T\{\psi(x)\xi(y)\} = \psi(x)\xi(y)\theta(x_0 - y_0) - \xi(y)\psi(x)\theta(y_0 - x_0)$$

What we would like to calculate? (Blackboard exercises)

1. The energy of the system.
2. The momentum.
3. The charge .
4. The expectation value $\langle 0| \psi(0)\bar{\psi}(x) |0\rangle$.

The flavor of perturbative expansions

Let us recall quantum mechanical elastic scattering by a potential. Say we have a Hamiltonian that can be split in a free part plus a potential:

$$\mathcal{H} = \mathcal{H}_0 + V.$$

At early times, suppose the system is in an eigenstate $\mathcal{H}_0 \left|\phi\right\rangle = E \left|\phi\right\rangle$. We aim to find a solution of the full Hamiltonian with the same eigenvalue, $\mathcal{H} \left|\psi\right\rangle = E \left|\psi\right\rangle$. We can formally write

$$\left|\psi\right\rangle = \left|\phi\right\rangle + \frac{1}{E - \mathcal{H}_0 + i\epsilon} V \left|\psi\right\rangle .$$

Define an operator T by

$$V \left|\psi\right\rangle = T \left|\phi\right\rangle .$$

We get an operator equation

$$T = V + V \frac{1}{E - \mathcal{H}_0 + i\epsilon} T.$$

The flavor of perturbative expansions

To solve this, we iterate this equation replacing T by the rhs:

$$T = V + V \frac{1}{E - \mathcal{H}_0 + i\epsilon} T = V + V \frac{1}{E - \mathcal{H}_0 + i\epsilon} \left(V + V \frac{1}{E - \mathcal{H}_0 + i\epsilon} T \right) \ldots$$

Defining $\Pi_{LS} = \dfrac{1}{E - \mathcal{H}_0 + i\epsilon}$ we see that this has the structure

$$T = V + V\Pi_{LS}V + V\Pi_{LS}V\Pi_{LS}V + V\Pi_{LS}V\Pi_{LS}V\Pi_{LS}V + \ldots$$

The first term in this expansion is called the Born approximation.

This expression can be put in a graphical form, where we put

$V = \bullet$ and $\Pi_{LS} =$ 

Then, the perturbative solution is

Notation matters because it helps up thinking

$$
\begin{aligned}
\frac{1}{c}\frac{\partial \mathrm{X}}{\partial t} &= \frac{\partial \mathrm{N}}{\partial y} - \frac{\partial \mathrm{M}}{\partial z}, & \frac{1}{c}\frac{\partial \mathrm{L}}{\partial t} &= \frac{\partial \mathrm{Y}}{\partial z} - \frac{\partial \mathrm{Z}}{\partial y}, \\
\frac{1}{c}\frac{\partial \mathrm{Y}}{\partial t} &= \frac{\partial \mathrm{L}}{\partial z} - \frac{\partial \mathrm{N}}{\partial x}, & \frac{1}{c}\frac{\partial \mathrm{M}}{\partial t} &= \frac{\partial \mathrm{Z}}{\partial x} - \frac{\partial \mathrm{X}}{\partial z}, \\
\frac{1}{c}\frac{\partial \mathrm{Z}}{\partial t} &= \frac{\partial \mathrm{M}}{\partial x} - \frac{\partial \mathrm{L}}{\partial y}, & \frac{1}{c}\frac{\partial \mathrm{N}}{\partial t} &= \frac{\partial \mathrm{X}}{\partial y} - \frac{\partial \mathrm{Y}}{\partial x},
\end{aligned}
$$

where (X, Y, Z) denotes the vector of the electric force, and (L, M, N) that of the magnetic force.

These are Maxwell's equation, fully equivalent to

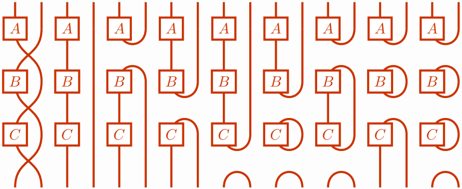
$$\nabla \times \mathbb{E} = -\frac{\partial \mathbb{H}}{\partial t} \qquad \nabla \times \mathbb{H} = \frac{\partial \mathbb{E}}{\partial t}$$

However, in the second version it is easier to see that when we do

$$(\mathbb{E}, \mathbb{H}) \to (\mathbb{H}, -\mathbb{E})$$

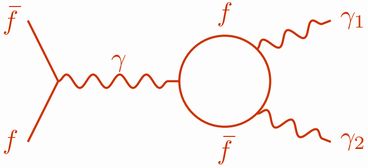
we get the same equations. Hurray for notation!

Graphical languages are powerful notation



The mathematical formalism of categories may be expressed through a graphical language. This is an auxiliar both to calculation and reasoning.

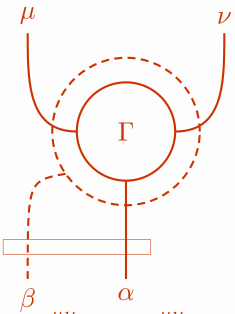
Feynman gave QFT to the masses



A Feynman diagram corresponds to a particular contribution to some transition amplitude between asymptotically free states in a QFT. It is easier to classify and list Feynman diagrams than algebraic structures.

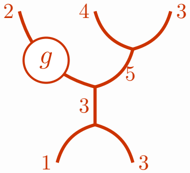
Some other neat examples are Penrose tensor diagrams...

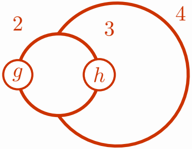
This tensor notation allows to represent arbitrary tensor and show their symmetries in a compact way.



$$\Gamma^{\mu\nu}{}_{\alpha;\beta} - \Gamma^{\mu\nu}{}_{\beta;\alpha}$$

…and the birdtrackss of Pedrag Cvitanovic





Birdtracks are tools for the description of the coupling of group representations and for the calculation of invariant quantities.

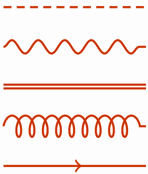
All these are graphical languages

Two classes of elements:

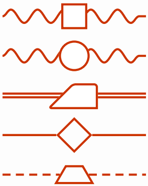

Wires


Boxes

The extreme points of wires represent objects; the wires are transformations between objects. Different objects, different lines:



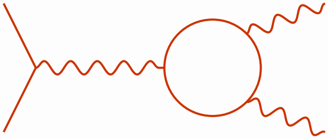
We can also draw different transformations as different shaped boxes:



It is natural to compose transformations by joining the wires coming out of boxes:

Complex mathematical structures can be put in graphical language

A Feynman diagram is the graphical representation of some operator where the wires represent the physical states and the boxes are vertices or propagators.



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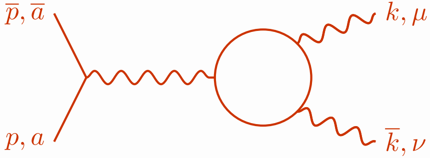

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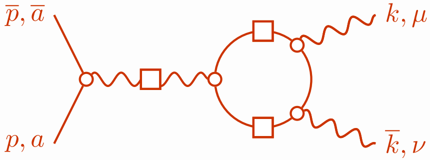

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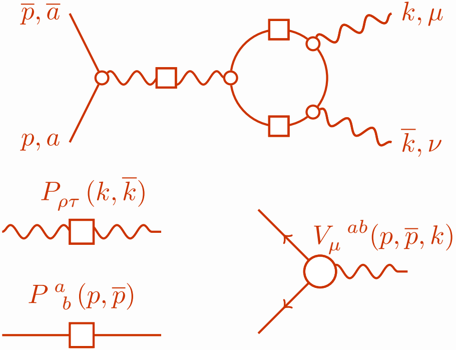


Categories are the mathematical objects that unify graphical languages

A category $\mathbb{C}$ has the following elements

- A collection of objects $A, B, C, D \ldots$.
- A collection $\mathbb{C}(A, B)$ of morphisms $f, g, h, \ldots$ between every pair of objects A, B.
- A composition rule between morphisms that satisfies

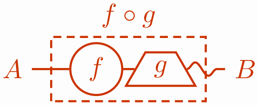
$$h \circ (g \circ f\ ) = (h \circ g) \circ f$$

- An identity map for every object $1_A \in \mathbb{C}(A, A)$

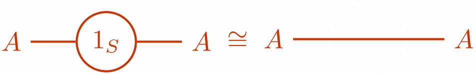
We may think of $A, B, C, D \ldots$ as physical systems (our free states) and in $f, g, h \ldots$ as the physical interactions. Objects are kinematics and maps are dynamics; they are like our wires and boxes.

Our graphical languages obey the categorical rules

We represent the composition as simply the union of wires:



Identity maps are boxes the join a wire with itself:



With this definition, all categorical axioms are fulfilled. This property is called solidity of the graphical language, and it guarantees that every categorial equation has a graphical version. If the opposite happens, the language is complete. Feynman diagrams are solid and complete.

Some further reading on diagrammatic languages

1. Physics, Topology, Logic and Computation: A Rosetta Stone Baez & Stay (2009)
2. Applications of Negative Dimension Tensors, Penrose (1971)
3. Diagram Techniques in Group Theory, Stedman (1990)
4. New Structures in Physics, Coecke (2011)
5. Drawing theories apart, Kaiser (2005)

How do we get the Feynman rules of a theory?

It fundamentally relies in the separation between a free Hamiltonian and the interaction terms. In QM, if we have

$$\mathcal{H}(t) = \mathcal{H}_0 + V(t)$$

we can go to the interaction picture where operators evolve like

$$\mathcal{O}_0(\mathbb{x}, t) = \exp(i\mathcal{H}_0(t - t_0))\mathcal{O}(\mathbb{x}) \exp(-i\mathcal{H}_0(t - t_0))$$

If the full time evolution is given by the full solution to

$$i\partial_0 S(t, t_0) = \mathcal{H}(t) S(t, t_0)$$

then the Heisenberg picture operators are

$$\mathcal{O}(\mathbb{x}, t) = U^\dagger(t, t_0) \exp(i\mathcal{H}_0(t - t_0))\mathcal{O}_0(\mathbb{x}, t) U(t, t_0)$$

given by

$$U(t, t_0) = \exp(i\mathcal{H}_0(t - t_0)) S(t, t_0).$$

The solution for the time evolution of the system is

$$U(t,t_0) = \exp T\left\{-i\int_{t_0}^{t} d\tau V(\tau))\right\}.$$

The series of this exponential gives the Dyson series, a perturbative expansion. Now, what we really want is to consider times in the far past and the far future, so that

$$U(\infty,-\infty) = \exp\left(i\int_{-\infty}^{\infty} d^4x\mathcal{L}_I[\mathcal{O}_0]\right).$$

The time ordered field product evaluated in the vaccum of the interacting theory is

$$\langle\Omega| T\{\mathcal{O}(x_1)\ldots\mathcal{O}(x_n)\}|\Omega\rangle = \frac{\langle 0| U(\infty,t_1)\mathcal{O}(x_1)\ldots\mathcal{O}(x_n)U(\infty,t_n)|0\rangle}{\langle 0| U(\infty,-\infty)|0\rangle}$$

which can be rewritten as

$$\langle\Omega| T\{\mathcal{O}(x_1)\ldots\mathcal{O}(x_n)\}|\Omega\rangle = \frac{\langle 0| T\{\mathcal{O}_0(x_1)\ldots\mathcal{O}_0(x_n)e^{i\int d^4x\mathcal{L}_I[\mathcal{O}_0]}\}|0\rangle}{\langle 0| T\{e^{i\int d^4x\mathcal{L}_I[\mathcal{O}_0]}\}|0\rangle}$$

The interacting theory is written in terms of the free fields

For example, the two-point correlation function will be

$$\langle\Omega|\, T\{\mathcal{O}(x_1)\mathcal{O}(x_2)\}\,|0\rangle = \frac{\langle 0|\, T\{\mathcal{O}_0(x_1)\mathcal{O}_0(x_2)e^{i\int d^4x\mathcal{L}_I[\mathcal{O}_0]}\}\,|\Omega\rangle}{\langle 0|\, T\{e^{i\int d^4x\mathcal{L}_I[\mathcal{O}_0]}\}\,|0\rangle}.$$

Consider the numerator:

$$\sum_n \frac{i^n}{n!}\langle 0|\, T\{\mathcal{O}_0(x_1)\mathcal{O}_0(x_2)\left(\int d^4x\mathcal{L}_I[\mathcal{O}_0]\right)^n\}\,|0\rangle$$

For clarity, we call y_i those coordinates that we integrate over. Then, we get

$$\langle 0|\, T\{\mathcal{O}_0(x_1)\mathcal{O}_0(x_2)\}\,|0\rangle + i\int dy_1\,\langle 0|\, T\{\mathcal{O}_0(x_1)\mathcal{O}_0(x_2)\mathcal{L}_I[\mathcal{O}_0(y_1)]\}\,|0\rangle$$
$$+\frac{i^2}{2!}\int dy_1 dy_2\,\langle 0|\, T\{\mathcal{O}_0(x_1)\mathcal{O}_0(x_2)\mathcal{L}_I[\mathcal{O}_0(y_1)]\mathcal{L}_I[\mathcal{O}_0(y_2)]\}\,|0\rangle + \dots$$

In the end, this means calculating vevs of time-ordered free fields!

Here is where the harmonic algebra is essential

Please remember that our free fields can be written as

$$\mathcal{O}_0(x) = \mathcal{O}_{0+}(x) + \mathcal{O}_{0-}(x)$$

where $\mathcal{O}_{0\pm}(x)$ contains only creation (annihilation) operators. Consider for example the product

$$\langle 0| T\{\psi(x)\psi(y)\} |0\rangle = \langle 0| [\psi(x)\psi(y)\theta(x_0 - y_0) - \psi(y)\psi(x)\theta(y_0 - x_0)] |0\rangle$$

where $\psi(x)$ is the free Dirac field. We only need to consider the contributions

$$\langle 0| [\psi_-(x)\psi_+(y)\theta(x_0 - y_0) - \psi_-(y)\psi_+(x)\theta(y_0 - x_0)] |0\rangle \, ;$$

every other contribution has to vanish. In general, the only nonzero terms are those with the same number of creation an annihilation operators, with the creation operators to the right of the corresponding annihilation operator. Each pair of operators creating and then destroying the same particle is called a contraction.

A simple example

Let us consider the interaction Lagrangian

$$\mathcal{L}_I[\psi(x), A_\mu(x)] = -ie\bar{\psi}(x)\gamma^\mu\psi(x)A_\mu(x).$$

The two-point function for the Dirac field has the following contributions. The zero-order contribution is the free propagator

$$D_F(x-y) \equiv \langle 0|\, T\{\psi(x_1)\psi(x_2)\}\, |0\rangle$$

The next term is

$$-i^2 e \int dy\, \langle 0|\, T\{\psi(x_1)\psi(x_2)\bar{\psi}(y)\gamma^\mu\psi(y)A_\mu(y)\}\, |0\rangle$$

This has to vanish because it will either annihilate a photon from the vacuum or create a photon that is not annihilated.

Our first quantum correction

The first nonzero correction will be

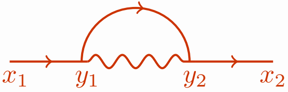
$$-\frac{ie^2}{2!}\int dy_1 dy_2 \, \langle 0|\, T\{\psi_{x_1}\psi_{x_2}\bar{\psi}_{y_1}\gamma_\mu\psi_{y_1}A^\mu_{y_1}\psi_{x_2}\bar{\psi}_{y_2}\gamma_\nu\psi_{y_2}A^\nu_{y_2}\}\,|0\rangle$$

where we have slightly condensed the notation showing variable dependence. There are a number of nonzero contributions. For example

$$-\frac{ie^2}{2!}\int dy_1 dy_2 \, \langle 0|\, T\{\psi^-_{x_1}\psi^+_{x_2}\bar{\psi}_{y_1}\gamma_\mu\psi_{y_1}A^{\mu-}_{y_1}\psi_{x_2}\bar{\psi}_{y_2}\gamma_\nu\psi_{y_2}A^{\nu+}_{y_2}\}\,|0\rangle\,,$$

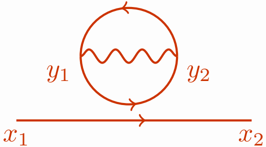
which can be recast as

$$-\frac{ie^2}{2!}\int dy_1 dy_2 D_F(x_1-y_1)\gamma_\mu D_F(y_1-y_2)\gamma_\nu D^{\mu\nu}_\gamma(y_2-y_1)D_F(y_2-x_2),$$

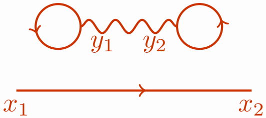


Other terms appear, like for example…

$$-\frac{ie^2}{2!}D_F(x_1-x_2)\int dy_1dy_2\gamma_\mu D_F(y_1-y_2)\gamma_\nu D_\gamma^{\mu\nu}(y_2-y_1)D_F(y_2-y_1),$$



$$-\frac{ie^2}{2!}D_F(x_1-x_2)\int dy_1dy_2\gamma_\mu D_F(y_1-y_2)\gamma_\nu D_\gamma^{\mu\nu}(y_2-y_1)D_F(y_2-y_1),$$



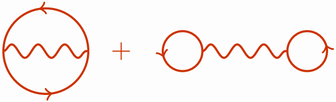
As we discussed, concatenation means composition, while yuxtaposition means multiplication.

But what about the denominator?

Remember that in the definition of the interacting vev we also had to multiply by the inverse of

$$\langle 0|\, T\{e^{i\int d^4x \mathcal{L}_I[\mathcal{O}_0]}\}\, |0\rangle$$

The first nonzero terms are exactly those we have associated with



Order by order, the effect then is to eliminate all the diagrams that contain bubbles unattached to the lines with free endpoints.

We can draw some conclusions

1. We get the full result if we calculate only the nonzero contributions which contain no bubbles.
2. There is a contribution for every contraction of the fields in our vev. (This result is known as Wick's theorem.)
3. Only contributions with every field contracted can be nonzero.
4. Feynman diagrams correspond to all different terms after contraction, not to all contractions. In this sense they are a more economical description.

Some modern books on QFT and Feynman diagrams

1. Quantum Field Theory and the Standard Model, Schwartz (2014).
2. Quantum Field Theory in a Nutshell, Zee (2010).
3. Advanced Topics in Quantum Field Theory, Shifman (2012)
4. Modern Quantum Field Theory, Banks (2008)

We have briefly sketched a path to perturbative QFT calculations

1. First, we identify the asymptotically free states, classifying the kinematics of our theory.
2. Then we write interaction terms for the Lagrangian (equivalently, in the wave equation) that are analogous to the potential in quantum mechanics.
3. Transition amplitudes can be calculated in terms of the free states at some fixed order in an expansion on the interaction Lagrangian.
4. Connected Feynman diagrams represent different contributions to some amplitude.

We can bypass this processe by building directly the Feynman diagrams as compositions of a small number of elementary diagrams, together with a prescription for relating them to particular field contractions (i.e., the Feynman rules).

Our field theory still includes many parameters that must be fixed somehow

Beside fundamental constants (for which an explanation probably needs to come from beyond QFT) we have several quantities which are operationally defined at some scale.

We compare our predictions with experimental data

- We have discussed how, starting from some Lagrangian compatible with our principles and symmetries (causality, relativity, locality, gauge invariance) we can construct transition amplitudes as vacuum expectation values of field products.
- These amplitudes can be related to physical measurements in the form of cross sections, decay lifetimes, etc. We can use some of these measurements to find out the value of our free parameters.
- Once we have fixed the free parameters, we get a predictive theory.

(This discussion follow A hint of renormalization, Bertrand Delamotte, Am. J. Phys. 72, 170, 2004.)

Imagine that from the Lagrangian

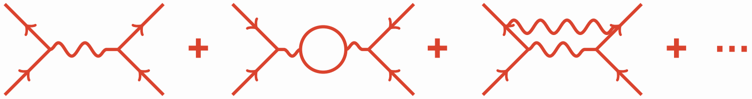
$$\mathcal{L} = \bar{\psi}(\not{\partial} - ie\not{A})\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}$$

we can calculate some amplitude that goes as

$$\sigma = \frac{e^4}{4\pi^2 s}\left(\frac{4}{3} + \frac{1}{\sin^2\phi/2} + \log\sin^2\phi/2\right) + \mathcal{O}(e^6).$$

In the perturbative expansion, this is represented by some simple diagram like

But the experiment is not measuring that; it is really measuring



This approach runs into trouble when some terms are divergent

One of the second order contributions to the scattering amplitude of two spin $1/2$ partices goes like

$$\mathcal{M} = -\frac{1}{2}\alpha^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{\not{k} - m + i\epsilon} \frac{1}{(\not{K} - \not{k}) - m + i\epsilon}$$

with $K = k_1 + k_2$. If we think for a moment, we can see that something is wrong here; the integral goes as

$$\int \frac{d^4k}{k^2} \sim (\infty)^2.$$

Similarly, in the scattering of a pair of scalars we get a term

$$\mathcal{M} = -\frac{1}{2}(-i\lambda^2) \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 - m^2 + i\epsilon} \frac{1}{(K - k)^2 - m^2 + i\epsilon},$$

which goes as

$$\int \frac{d^4k}{k^4} \sim \log(\infty).$$

The origin of the divergences is the sum over intermediate states

An heuristic argument goes like this: when we sum over states with large energies, we are probing the short distance structure of the theory. If the distances are short enough, there is enough energy for the production of more particles.

We can give a technical argument: divergences appear because some contributions involve propagators at the same spacetime points (closed loops), and propagators are actually distributions with singularities. k“This shift comes out infinite in all existing theories and has therefore always been ignored.

But also: we don’t actually know the theory at arbitrarily high energies!
Hans Bethe, 1947: “This shift comes out infinite in all existing theories and has therefore always been ignored.”

We can try to understand this problem through a toy model

Imagine a theory that enables us to calculate some physical quantity F in terms of another physical quantity x, depending on a parameter g_0:

$$F(x) = g_0 + g_0^2 F_1(x) + g_0^3 F_2(x) + \dots$$

For example, we can think of some scattering amplitude in QED; F can be the cross section, and x the square of the center of mass energy.

Since we have one free parameter, one measurement of F at some value $x = \mu$ specifies completely the theory.

Our measuremente defines a renormalized parameter

$$g_\mu = F(\mu).$$

We would like to calculate the bare parameter g_0 in terms of g_μ. The problem occurs when some of the coeficients involve quantities like

$$F_1(x) = \alpha \int_0^\infty \frac{dt}{t+x}$$

The relation between g_0 and g_μ is complcated because our expansion is singular!

We would like a way to cut out the infinities

The apparition of divergences puts at risk the predictability of our theory. We have devised a method to reveal the finite form of our theory; we call it renormalization.

The idea is that everything is fine if we eliminate g_0 in favor of g_μ

The hypothesis or renormalizability is that the problem is not the singularity of the expansion, but rather that g_0 is the wrong parameter.
The scheme is:

1. Rewrite a version of $F(x)$ for which the coefficients $F_i(x)$ are well-defined.
2. Replace g_0 with g_μ
3. Take a limit that gives back the original expansion.

(See supplementary slides).

Thanks!