Solving Lotka-Volterra Model with Runge-Kutta 4th Order Method

Selina Kim
McMaster University
Project Final Write-Up, DATASCI 2G03
kim391@mcmaster.ca
December 9, 2023

I. Introduction

The Lotka-Volterra predator-prey model[1] (further on referred to as the "predator-prey model") predicts the dynamic between a predator and prey—specifically, how the population of each species interact with each other over time.

While the basic premise of the model is to predict the dynamic of the population of two species, the model can be modified and applied to predict different biological systems as well as market-share competition in economics.

The predator-prey model is based on a number of assumptions. The predator and prey are each represented by one variable, implying that any variety (e.g. age, physicality, health, etc) within the population of the predator or prey does not affect its ability to survive and/or hunt. The predator population's growth rate is entirely dependent on the population of the prey (food supply) and its natural birth rate. The prey population's death rate is entirely dependent on the population of the predator and its natural death rate.

II. Method

A. Equations

The predator-prey model is expressed with the following system of ODEs:

$$\begin{cases} \frac{dx}{dt} = \alpha x - \beta xy \\ \frac{dy}{dt} = \delta xy - \gamma y \end{cases} \tag{1}$$

where each of the variables represent the following:

- x represents the population of the prey
- y represents the population of predators
- t represents time
- $\frac{dx}{dt}$ represents the instantaneous rates of the prey
- $\frac{dy}{dt}$ represents the instantaneous rates of the prey
- α represents a real constant that describes the prey's per capita growth rate
- β represents a real constant that describes the effect of the presence of predators on the prey growth rate
- δ represents a real constant that describes the effect of the presence of prey on the predator's growth rate
- γ represents a real constant that describes the predator's per capita death rate

[1]The basic Lokta-Volterra model and equations were acquired from [1]

B. Solver

To solve the predator-prey model, we use the Runge-Kutta 4th Order Method [2]. This method evaluates the derivative four times for each step n. We modify the solver to fit the basic predator-prey model; making it evaluate for 2 derivatives ($\frac{dx}{dt}$ and $\frac{dy}{dt}$):

$$\begin{aligned} x_{i+1} &= x_i + \frac{1}{6}(k_{x1} + 2 \cdot k_{x2} + 2 \cdot k_{x3} + k_{x4}) \\ y_{i+1} &= y_i + \frac{1}{6}(k_{y1} + 2 \cdot k_{y2} + 2 \cdot k_{y3} + k_{y4}) \end{aligned} \tag{2}$$

where

$$\begin{cases} k_{x1} = \Delta t \cdot \frac{dx}{dt}(x_i, y_i) \\ k_{y1} = \Delta t \cdot \frac{dy}{dt}(x_i, y_i) \\ k_{x2} = \Delta t \cdot \frac{dx}{dt}(x_i + \frac{1}{2}k_{x1}, y_i + \frac{1}{2}k_{y1}) \\ k_{y2} = \Delta t \cdot \frac{dy}{dt}(x_i + \frac{1}{2}k_{x1}, y_i + \frac{1}{2}k_{y1}) \\ k_{x3} = \Delta t \cdot \frac{dx}{dt}(x_i + \frac{1}{2}k_{x2}, y_i + \frac{1}{2}k_{y2}) \\ k_{y3} = \Delta t \cdot \frac{dy}{dt}(x_i + \frac{1}{2}k_{x2}, y_i + \frac{1}{2}k_{y2}) \\ k_{x4} = \Delta t \cdot \frac{dx}{dt}(x_i + k_{x3}, y_i + k_{y3}) \\ k_{y4} = \Delta t \cdot \frac{dy}{dt}(x_i + k_{x3}, y_i + k_{y3}), \end{cases}$$

for $0 \leq i \leq n$ and $i, n \in \mathbb{W}$.

Assuming we are given the initial populations of the prey and predator (x_0 and y_0 respectively), the total number of steps n, the time interval $[t_{\text{start}}, t_{\text{end}}]$, and thus the timestep $\Delta t = \frac{1}{n}(t_{\text{end}} - t_{\text{start}})$.

III. Testing

All tests of this section were tested on the code in the directory `project-demo`.

We test the basic implementation of the model by comparing the test results of the program using the same values as the implementation from an original paper that solves the same model with the Runge-Kutta 4th Order Method.

A. Test 1

1) Parameters

The first original paper [3] we are referencing uses the following values:

$$\alpha = 0.4, \qquad x_0 = 100,$$
$$\beta = 0.4, \qquad y_0 = 8,$$
$$\delta = 0.09, \qquad t = 50.$$
$$\gamma = 2.0,$$

Since the exact implementation details for how the number of steps n and timesteps Δt was calculated is unknown, we set $n = 500000$ for higher accuracy of the Runge-Kutta method.

2) Expected Result

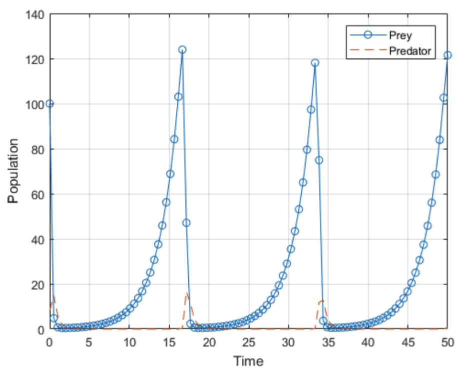

Fig. 1. Expected plot results of Test 1 (from original paper [3])

3) Actual Result

Based on Fig. 2. and TABLE I, we can find that the prey population x first peaks at a population near 100. After the first peak, x repeatedly peaks at a population near 127. The predator population y continuously peaks at a population near 19. Thus, we can confirm that the program is outputting accurate results as the population and time in which each x and y peak are relatively equivalent to the expected results in [3].

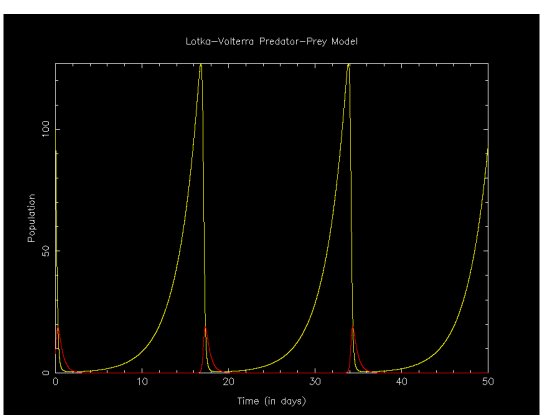

Fig. 2. Actual plot results of Test 1 (yellow curve represents prey population x, red curve represents predator population y)

Δt	x	y
0.002	99.4371	
0.272		18.8354
16.806	127.218	
17.314		18.8355
33.848	127.218	
34.356		18.8355

TABLE I
RAW DATA OF PEAK POINTS IN THE CURVE OF PREY POPULATION x AND PREDATOR POPULATION y OF FIG. 2

B. Test 2

1) Parameters

The second original paper [4] we are referencing uses the following values:

$$\alpha = 0.1, \qquad \gamma = 0.1,$$
$$\beta = \{0.125, 0.15, 0.175\}, \qquad x_0 = 2,$$
$$\delta = 0.1, \qquad y_0 = 2,$$

i.e., the value of β varies while the other parameters remain the same.

Since the exact implementation details for how the number of steps n and timesteps Δt was calculated is unknown, we set $n = 5500$ and $t = 550$ to demonstrate a roughly similar time interval as the original paper for easier comparison.

2) Expected Result

The expected plot results shown in the original paper [4] indicate that as the death rate of the prey β increases, the prey population curve peaks at a higher population and the predator population curve peaks at a lower population. The lowest dip that occurs between each peak for both the prey population and predator population curve also gets lower as β increases.

3) Actual Result

Based on Fig. 3., Fig. 4. and Fig. 5., we can confirm that the program is outputting accurate results as the curves for both the prey population and predator population are showing the same behaviour as the expected results when the death rate of the prey β increases.

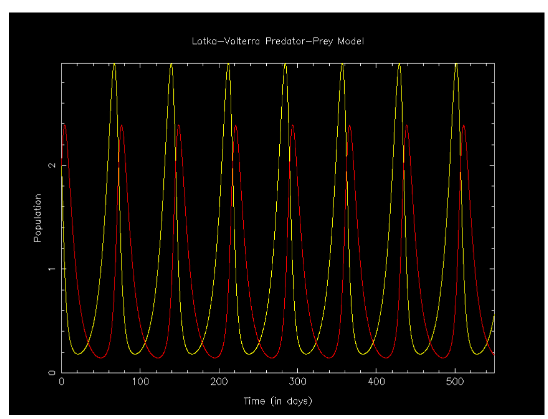

Fig. 3. Actual plot results of Test 2 with $\beta = 0.125$ (yellow curve represents prey population x, red curve represents predator population y)

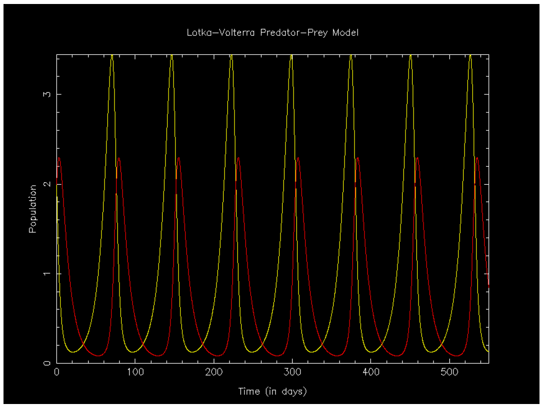

Fig. 4. Actual plot results of Test 2 with $\beta = 0.15$ (yellow curve represents prey population x, red curve represents predator population y)

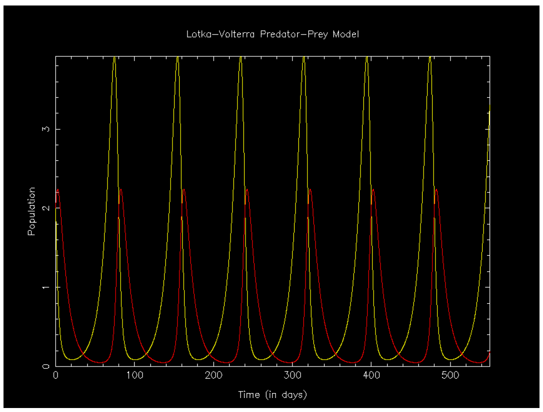

Fig. 5. Actual plot results of Test 2 with $\beta = 0.175$ (yellow curve represents prey population x, red curve represents predator population y)

IV. Results

A. Extension: Two Preys and One Predator

All tests and evaluations of this section were based on the code in the directory `project-extended-1`.

We extend the demo program to include an additional prey, using the variables $x1$, $x2$ and y to denote the first prey, the second prey, and the predator respectively. The growth and death rate of the first prey is denoted by α_{x1} and β_{x1} respectively. Similarly, the growth and death rate of the second prey is denoted by α_{x2} and β_{x2} respectively.

Looking at the results from Test 1 to Test 7—which plot the system of the same variables with the exception of increasing the growth or death rate of one of the preys or predator by 0.05—we can note that there are generally 3 types of results.

The first type is seen in Test 2 (Fig. 7.) and Test 5 (Fig. 10.), where each of the populations gradually lead into a state of equilibrium—each population oscillating between a certain range of populations in the shape of a sine wave.

The second type is seen in Test 1 (Fig. 6.), Test 6 (Fig. 11.), and Test 7 (Fig. 12.) where all 3 populations behave as expected from a basic predator-prey model with one prey and one predator—with repeating peaks and dips. This behaviour is implies that a 0.5 increase in the death rate of the first prey and a 0.5 increase in the birth rate of the second prey result in the same outcome of the first prey's population dying out. This is fairly expected as the initial population of the first prey is smallest of the 3, and the initial population of the second prey is the largest of the 3.

The third type is seen in Test 3 (Fig. 8.) and Test 4 (Fig. 9.) where the population of the first prey $x1$ dies out while the population of the predator and second prey behave the same as a basic predator-prey model with one prey and one predator.

1) Test 1

$$\begin{array}{cc}
\alpha_{x1} = 0.1, & x1_0 = 1, \\
\beta_{x1} = 0.1, & x2_0 = 3, \\
\alpha_{x2} = 0.1, & y_0 = 2, \\
\beta_{x2} = 0.1, & t = 1000, \\
\delta = 0.1, & n = 500000. \\
\gamma = 0.1, &
\end{array}$$

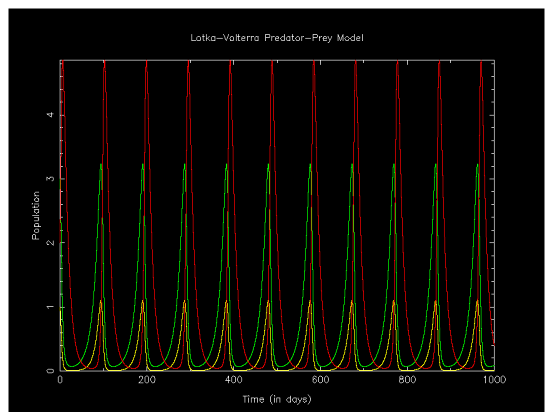

Fig. 6. Actual plot results of Extension Test 1 (yellow curve represents prey 1 population $x1$, green curve represents prey 2 population $x2$, red curve represents predator population y)

2) Test 2

$$\begin{array}{cc}
\alpha_{x1} = 0.15, & x1_0 = 1, \\
\beta_{x1} = 0.1, & x2_0 = 3, \\
\alpha_{x2} = 0.1, & y_0 = 2, \\
\beta_{x2} = 0.1, & t = 1000, \\
\delta = 0.1, & n = 500000. \\
\gamma = 0.1, &
\end{array}$$

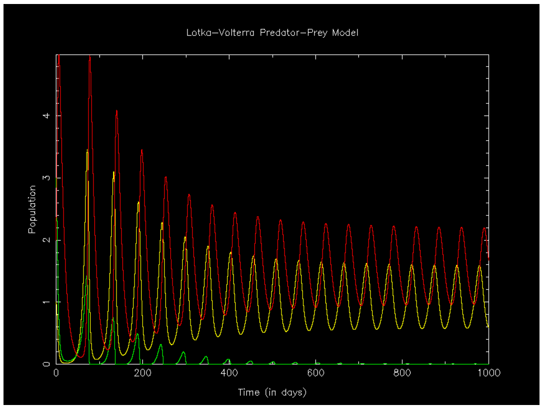

Fig. 7. Actual plot results of Extension Test 2 (yellow curve represents prey 1 population $x1$, green curve represents prey 2 population $x2$, red curve represents predator population y)

3) Test 3

$$
\begin{array}{cc}
\alpha_{x1} = 0.1, & x1_0 = 1, \\
\beta_{x1} = 0.15, & x2_0 = 3, \\
\alpha_{x2} = 0.1, & y_0 = 2, \\
\beta_{x2} = 0.1, & t = 1000, \\
\delta = 0.1, & n = 500000. \\
\gamma = 0.1, &
\end{array}
$$

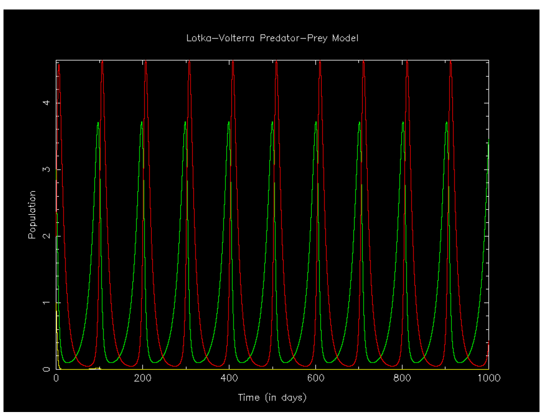

Fig. 8. Actual plot results of Extension Test 3 (yellow curve represents prey 1 population $x1$, green curve represents prey 2 population $x2$, red curve represents predator population y)

4) Test 4

$$
\begin{array}{cc}
\alpha_{x1} = 0.1, & x1_0 = 1, \\
\beta_{x1} = 0.1, & x2_0 = 3, \\
\alpha_{x2} = 0.15, & y_0 = 2, \\
\beta_{x2} = 0.1, & t = 1000, \\
\delta = 0.1, & n = 500000. \\
\gamma = 0.1, &
\end{array}
$$

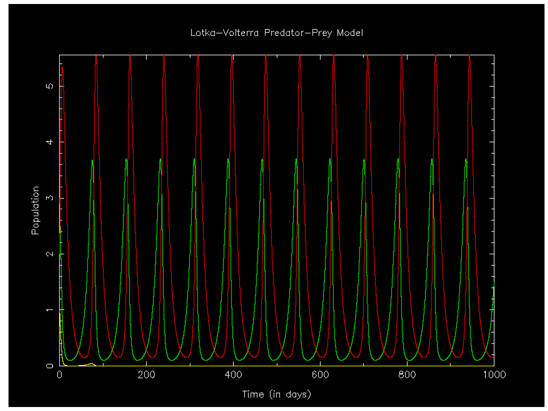

Fig. 9. Actual plot results of Extension Test 4 (yellow curve represents prey 1 population $x1$, green curve represents prey 2 population $x2$, red curve represents predator population y)

5) Test 5

$$
\begin{array}{cc}
\alpha_{x1} = 0.1, & x1_0 = 1, \\
\beta_{x1} = 0.1, & x2_0 = 3, \\
\alpha_{x2} = 0.1, & y_0 = 2, \\
\beta_{x2} = 0.15, & t = 1000, \\
\delta = 0.1, & n = 500000. \\
\gamma = 0.1, &
\end{array}
$$

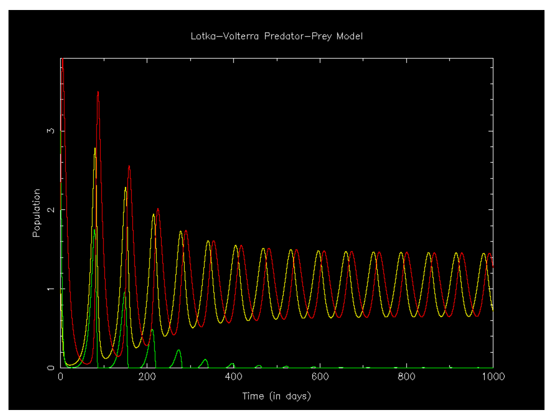

Fig. 10. Actual plot results of Extension Test 5 (yellow curve represents prey 1 population $x1$, green curve represents prey 2 population $x2$, red curve represents predator population y)

6) Test 6

$$
\begin{array}{cc}
\alpha_{x1} = 0.1, & x1_0 = 1, \\
\beta_{x1} = 0.1, & x2_0 = 3, \\
\alpha_{x2} = 0.1, & y_0 = 2, \\
\beta_{x2} = 0.1, & t = 1000, \\
\delta = 0.15, & n = 500000. \\
\gamma = 0.1, &
\end{array}
$$

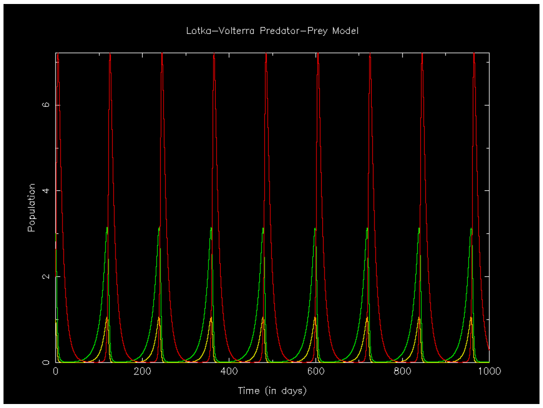

Fig. 11. Actual plot results of Extension Test 6 (yellow curve represents prey 1 population $x1$, green curve represents prey 2 population $x2$, red curve represents predator population y)

7) Test 7

$$\begin{aligned} \alpha_{x1} &= 0.1, & x1_0 &= 1, \\ \beta_{x1} &= 0.1, & x2_0 &= 3, \\ \alpha_{x2} &= 0.1, & y_0 &= 2, \\ \beta_{x2} &= 0.1, & t &= 1000, \\ \delta &= 0.1, & n &= 500000. \\ \gamma &= 0.15, \end{aligned}$$

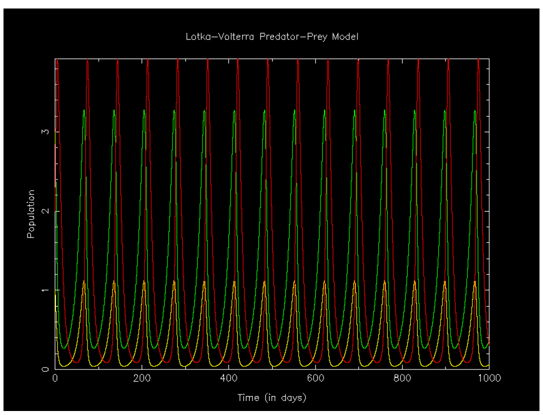

Fig. 12. Actual plot results of Extension Test 7 (yellow curve represents prey 1 population $x1$, green curve represents prey 2 population $x2$, red curve represents predator population y)

V. Conclusion

The predator-prey model demonstrates an interesting aspect of how populations of certain species interact with each other. In the case of the basic predator-prey model with 1 prey and 1 predator, it can be clearly noted how the decrease of the population in one species eventually causes the resurgence in population for the other. However, when the model becomes more complicated, such as adding more preys or predators to the system, more factors need to be taken into account to be able to demonstrate a more accurate and realistic behaviour of multiple populations interacting with each other. While my implementation of the model with 2 preys and 1 predator definitely captured interesting cases, there is room to suspect that there could be more modifications to the equations to make the outcomes more accurate (e.g. the first prey could be also be a predator to the second prey or vice versa).

References

[1] S. Forrest. Predator-prey models. [Online]. Available: https://www.cs.unm.edu/~forrest/classes/cs365/lectures/Lotka-Volterra.pdf

[2] E. Süli and D. F. Mayers, "Initial value problems for odes," in *An Introduction to Numerical Analysis*. Cambridge University Press, 2003, p. 310–360.

[3] T. Blaszak. Lotka-volterra models of predator-prey relationships. [Online]. Available: https://web.mst.edu/~huwen/teaching_Predator_Prey_Tyler_Blaszak.pdf

[4] E. Juarlin, "Solution of simple prey-predator model by runge kutta method," *Journal of Physics: Conference Series*, vol. 1341, no. 6, p. 062024, oct 2019. [Online]. Available: https://dx.doi.org/10.1088/1742-6596/1341/6/062024