1.2. Inversion Method I

design Algorithm that samples RN according to $\rho(x) = c \cdot x^n$, $x \in [0,1]$

1) find normalisation constant $\int_0^1 c x^n \, dx = 1$

$$\Leftrightarrow c\left[\frac{1}{n+1} x^{n+1}\right]_0^1 = 1$$

$$\Leftrightarrow c \cdot \left[\frac{1}{n+1}\right] = 1 \Leftrightarrow c = (n+1)$$

2) find cumulative Function $F(x)$:

$$F(x) = \int_0^x c x^n \, dx = (n+1)\left[\frac{1}{n+1} x^{n+1}\right]_0^x = x^{n+1}$$

3) inverse $\to F^{-1}(\xi) = \sqrt[n+1]{\xi}$ with $\xi \sim$ uniform $[0\ 1)$

1.3. Inversionmethod II for $\rho(x) = c x^2$ with $x \in [0,2]$

- normalisation constant: $\int_0^2 c x^2 \, dx \overset{!}{=} 1 \Leftrightarrow \left[\frac{c}{3} x^3\right]_0^2 \overset{!}{=} 1$

$$\frac{c}{3} 8 = 1 \Rightarrow c = \frac{3}{8}$$

- cumulative distribution $F(x) = \int_0^x \frac{3}{8} x'^2 \, dx' = \frac{3}{8}\left[\frac{1}{3} x'^3\right]_0^x = \frac{1}{8} x^3$

~ inverse Function: $F^{-1}(\xi) = \sqrt[3]{8\xi}$

1.4. Additional exercises

(1) $\rho(x) = \mu e^{-\mu x}$ $\qquad \int_0^\infty \rho(x) \, dx = 1$ for $x \geq 0$

$$F(x) = \int_0^x \mu e^{-\mu x} \, dx = \left[-e^{-\mu x}\right]_0^x = 1 - e^{-\mu x}$$

get inverse: $F(x) = \xi = 1 - e^{-\mu x}$

$$\Leftrightarrow 1 - \xi = e^{-\mu x} \Leftrightarrow -\frac{\ln(1-\xi)}{\mu} = x = F^{-1}(\xi)$$

$$\hookrightarrow F^{-1}(\xi) = \frac{-\ln(1-\xi)}{\mu}$$

(2.) $\rho(x) = 2x e^{-x^2}$ for $x \geq 0$ already normalized

$u = x^2$, $du = 2x \, dx$, $dx = \frac{1}{2x} du$

$$F(x) = \int_0^x 2x e^{-x^2} \, dx = \int_0^{x^2} e^{-u} \, du = 1 - e^{-x^2}$$

find $F^{-1}(\xi)$: $1 - e^{-x^2} = \xi \Leftrightarrow 1 - \xi = e^{-x^2} \Leftrightarrow -\ln(1-\xi) = x^2$

$$x = \sqrt{-\ln(1-\xi)}$$

$$\hookrightarrow F^{-1}(\xi) = \sqrt{-\ln(1-\xi)}$$

(3.) $\rho(x) = \frac{1}{(a+bx)^n}$ normalisation: $\int_0^\infty \rho(x) \, dx = c \frac{a^{1-n}}{b(n-1)} \overset{!}{=} 1 \Rightarrow c = \frac{b(n-1)}{a^{(1-n)}} = a^{n-1} \cdot b \cdot (n-1)$

$$F(x) = \int_0^x \frac{1}{(a+bx')^n} \, dx' = \int_a^{bx+a} \frac{1}{b} \frac{1}{u^n} \, du = \left[\frac{1}{-b(n-1)} \frac{1}{u^{n-1}}\right]_a^{ax+b} = \frac{1}{c}\left(1 - \frac{a^{n-1}}{(a+bx)^{n-1}}\right) = \xi$$

$$1 - \xi \cdot c = \frac{a^{n-1}}{(a+bx)^{n-1}} \Leftrightarrow \frac{a^{n-1}}{(1-\xi c)} = (a+bx)^{n-1} \Leftrightarrow \frac{a}{\sqrt[n-1]{1-\xi c}} = a + bx$$

$$F^{-1}(\xi) = \frac{a}{b}\left(\frac{1}{\sqrt[n-1]{1-\xi c}} - 1\right)$$