3.2 Estimate the Integral $\int_0^{\pi/2} \cos x \, dx$ with Importance sampling

- use importance sampling with weight function $g(x) = a + bx^2$
- sample from $g(x)$: needs to be normalised

$$\int_0^{\pi/2} g(x)\,dx = \left[ax + \frac{b}{3}x^3\right]_0^{\pi/2} = a\frac{\pi}{2} + \frac{b\pi^3}{3\cdot 2^3} = a\frac{\pi}{2} + b\frac{\pi^3}{24} \overset{!}{=} 1$$

$$a = \frac{2}{\pi} - \frac{b\pi^2}{12}$$

- optimal parameters for $g(x)$:

$\sigma^2\left(\frac{\cos(x)}{g(x)}\right)$ is at a minimum

with importance sampling, using $g(x) = a + bx^2$
Taylor expanding $\cos x$ around $x=0$

$$\cos x \approx 1 - \frac{1}{2}x^2 + \frac{x^4}{4!} \dots$$

↳ to make $\frac{\cos(x)}{g(x)}$ as constant as possible,
choose $a = 1$ and $b = -0.4$ to keep $g(x) > 0$, but close to Taylor approx.

sampling from $g(x)$ is easiest with rejection sampling, as inversion of the (cubic) cumulative is tricky

↳ sample from normalized $g^*(x) = \frac{1}{Z}(1 - 0.4x^2)$

$$Z = \int_0^{\pi/2} (1 - 0.4x^2)\,dx = \frac{\pi}{2} - \frac{0.4 \cdot \pi^3}{3 \cdot 8}$$

Rejection sampling with $h(x) = \frac{1}{c}\cdot(1 - x)$ w. $x \in [0, \pi/2]$

with $c = \int_0^{\pi/2} 1 - \frac{1}{2}x\,dx = \left[x - \frac{1}{4}x^2\right]_0^{\pi/2} = \frac{\pi}{2} - \frac{\pi^2}{16}$

Inversion: $H(x) = \int_0^x h(x')\,dx' = \frac{1}{c}\left(x - \frac{1}{4}x^2\right)$

$H^{-1}(y)$: $\frac{1}{c}\left(x - \frac{1}{4}x^2\right) = y \Leftrightarrow x - \frac{1}{4}x^2 = cy$

$\Leftrightarrow x_{1/2} = 2 \pm \sqrt{4 - 4cy} =$

↳ $H^{-1}(y) = 2 - \sqrt{4 - 4cy}$ (pick solution in appropriate range)

Exercise 4.3: Effectiveness of Importance sampling

$$f(x) = \begin{cases} 0 & \text{for } x < T \\ 1 & \text{for } x \geq T \end{cases} \quad ; \; g(x) = e^{-x} \text{ for } x > 0$$

find estimate for $\langle f \rangle_g = \int_{\mathbb{R}} f(x) g(x) dx$ with importance sampling

with proposal: $g(a,x) = a \exp(-ax)$ with $0 < a < 1$

- $\langle f \rangle_g = \int_0^\infty e^{-x} f(x)\, dx = \int_0^T e^{-x} \cdot 0\, dx + \int_T^\infty e^{-x}\, dx = \left[-e^{-x}\right]_T^\infty = \underline{\underline{e^{-T}}}$ □

- $\sigma^2(f) = \underbrace{\int_0^\infty e^{-x} f(x)^2 dx}_{= \int_0^\infty e^{-x} f(x) dx = \langle f \rangle} - \underbrace{\left| \int_0^\infty e^{-x} f(x) dx \right|^2}_{= \langle f \rangle^2} = \underline{\langle f \rangle - \langle f \rangle^2}$ □

- $\sigma^2\left(a, \frac{f(x) g(x)}{g(a,x)}\right) = \int_0^\infty \left(\frac{f(x) g(x)}{g(a,x)}\right)^2 g(a,x) dx - \left| \int_0^\infty \frac{f(x) g(x)}{\cancel{g(a,x)}} \cancel{g(a,x)} dx \right|^2$

$$= \int_0^\infty \frac{f(x)^2 g(x)^2}{g(a,x)} dx - e^{-2T} = \int_T^\infty \frac{1}{a} \frac{e^{-2x}}{\exp(-ax)} dx - e^{-2T} = \frac{1}{a} \int_T \exp(-x(2-a))\, dx - e^{-2T}$$

$$= \frac{1}{a} \left[\frac{1}{-(2-a)} \exp(-x(2-a))\right]_0^\infty - e^{-2T}$$

$$= 0 + \frac{\exp(-T(2-a))}{a(2-a)} - e^{-2T}$$

- Find parameter a_* that minimizes Variance:

$$\frac{\partial}{\partial a} \sigma^2\left(\frac{f(x) g(x)}{g(a,x)}\right) = \frac{+T \exp(-T(2-a))}{(2-a) a} + \frac{\exp(-T(2-a))}{(2-a)^2 a} - \frac{e^{-T(2-a)}}{(2-a) a^2} \stackrel{!}{=} 0$$

$$\cdot \frac{1}{\exp(\ldots)} \Big| \iff T + \frac{1}{(2-a)} - \frac{1}{a} \stackrel{!}{=} 0$$

$$\iff T(a^2 - 2a) = 2a - 2$$

$$\iff T a^2 - 2(T+1) a - 2 = 0$$

from abc-Formula:

$$a_1 = \frac{\sqrt{T^2 + 1} + T + 1}{T}, \quad a_2 = \frac{-\sqrt{T^2 + 1} + T + 1}{T}$$

in appropriate range