Ellipsoidal Toolbox

TCC Workshop

Alex A. Kurzhanskiy and Pravin Varaiya

(UC Berkeley)

March 27, 2006

Outline

- Problem setting and basic definitions
- Overview of existing methods and tools
- Ellipsoidal approach
- Systems with disturbances
- Hybrid systems
- Summary and outlook

Outline

- Problem setting and basic definitions
- Overview of existing methods and tools
- Ellipsoidal approach
- Systems with disturbances
- Hybrid systems
- Summary and outlook

System Equations

The controlled system:

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{t}, \boldsymbol{x}, \boldsymbol{u}), \quad \boldsymbol{t} \geq \boldsymbol{t}_0$$

state variable: $\boldsymbol{x} \in \mathbb{R}^{\mathrm{n}}$

Control:

- Open-loop: $\boldsymbol{u}(\boldsymbol{t}) \in \mathcal{P}(\boldsymbol{t}),\ \boldsymbol{t} \geq \boldsymbol{t}_0$
- Closed-loop: $\boldsymbol{u}(\boldsymbol{t}, \boldsymbol{x}) \in \mathcal{P}(\boldsymbol{t})\ (\boldsymbol{u}(\boldsymbol{t}, \boldsymbol{x}) \in \mathcal{P}(\boldsymbol{t}, \boldsymbol{x})),\ \boldsymbol{t} \geq \boldsymbol{t}_0$
- $\mathcal{P}(\boldsymbol{t})$ compact subset of $\mathbb{R}^{\mathrm{m}}$

Reachability (definitions)

- Reach set $X(t, t_0, X^0)$ at $t > t_0$ from $\{t_0, X^0\}$:

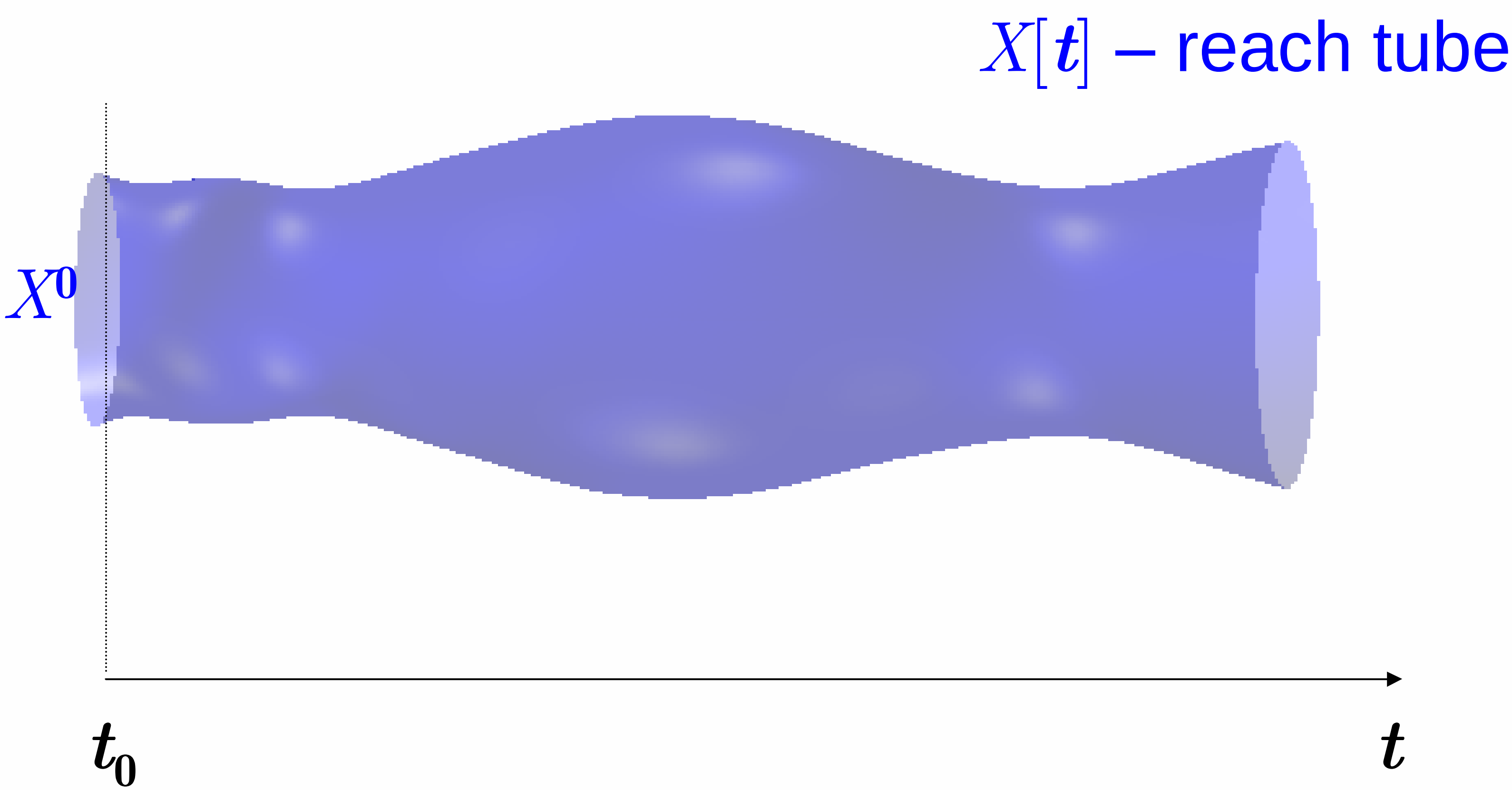
$$X(t, t_0, X^0) = \bigcup_{u(\cdot)\in\mathcal{P}(\cdot), x^0\in X^0} \{x(t, t_0, x^0 | u(\cdot))\}$$

- Reach tube: map $t \to X[t] = X(t, t_0, X^0)$

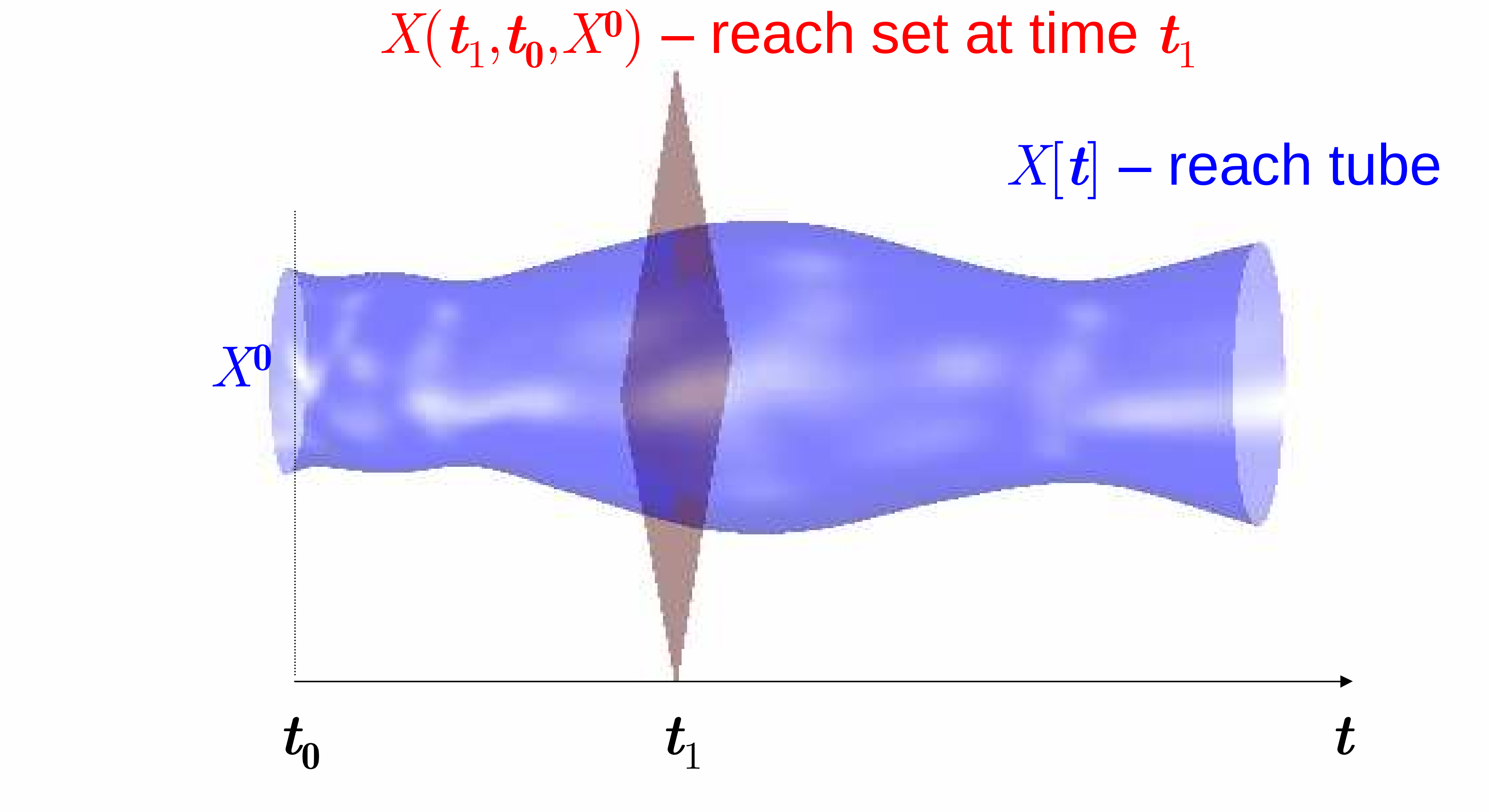
- Reach set at some time within $[t_1, t_2]$:

$$\mathcal{X}(t_2, t_1, X^0) = \bigcup_{t_1\leq\tau\leq t_2} X(\tau, t_0, X^0)$$

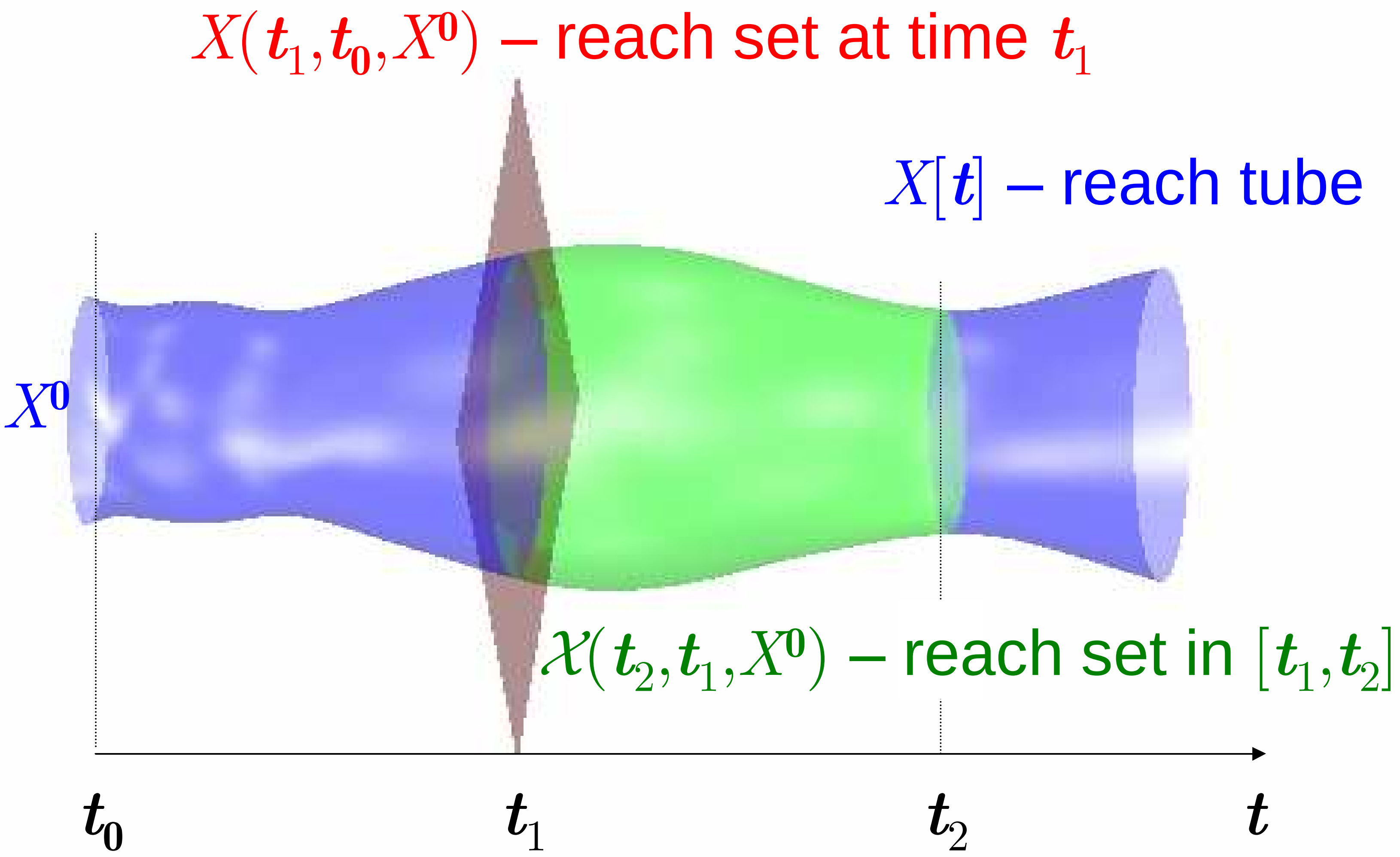
Reachability (illustrations)



Reachability (illustrations)



Reachability (illustrations)



Reachability (properties)

- The reach sets are the same for open-loop and closed-loop controls

- Reach set $X(t, t_0, X^0)$ satisfies the semigroup property:

$$X(t, t_0, X^0) = X(t, \tau, X(\tau, t_0, X^0))$$

Also true for the reach tube $X[t]$

Backward Reach Set

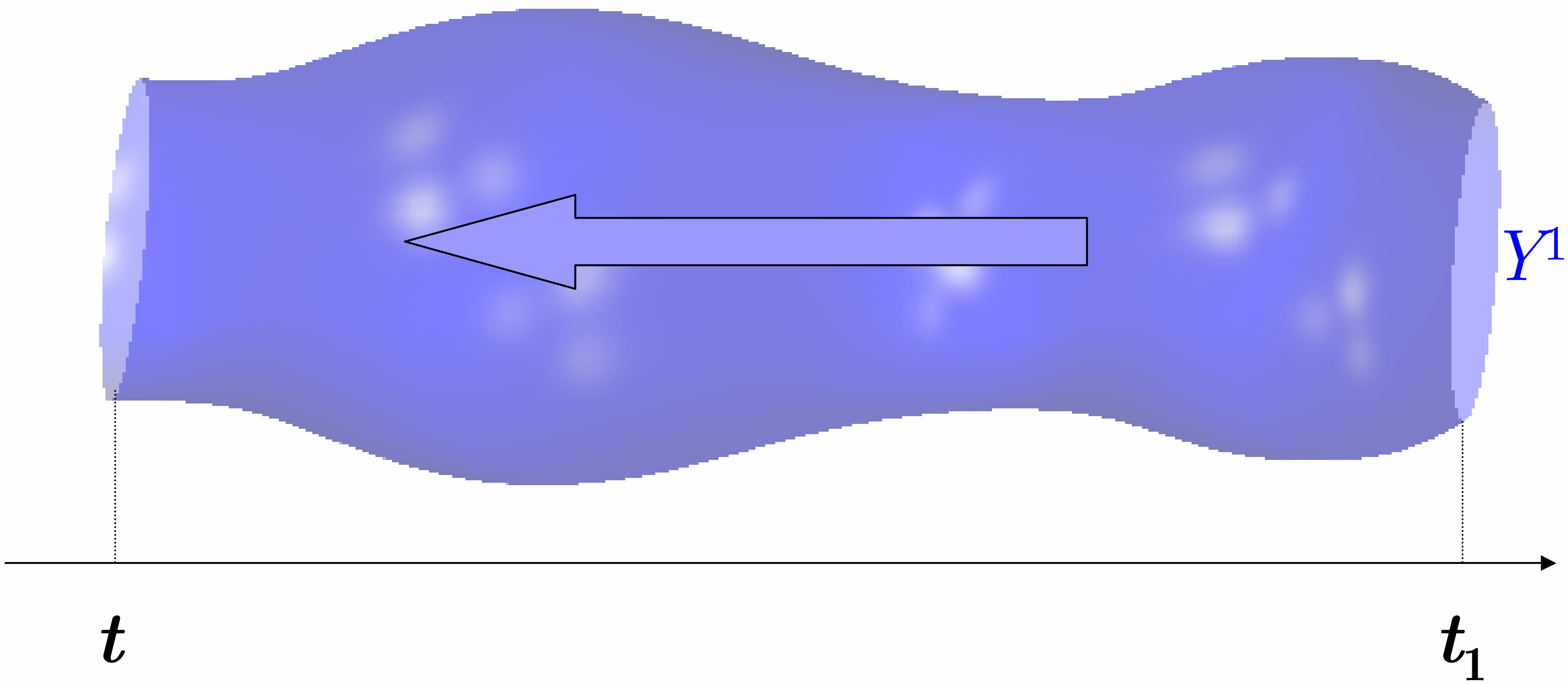
Given:

- Target set Y^1
- Terminating time t_1

Backward reach set $Y(t, t_1, Y^1)$ at time t – set of all states y for each of which there exists control $u(\tau),\ t_0 \leq \tau < t$, such that $y(t) = y$ and $y(t_1) \in Y^1$

Backward Reachability (illustration)

$Y(t, t_1, Y^1)$ – backward reach set at time t



Linear Systems

- Continuous-time:

$$\dot{\boldsymbol{x}}(\boldsymbol{t}) = A(\boldsymbol{t})\boldsymbol{x}(\boldsymbol{t}) + B(\boldsymbol{t})\boldsymbol{u}(\boldsymbol{t})$$

- Discrete-time:

$$\boldsymbol{x}(\boldsymbol{t}+1) = A(\boldsymbol{t})\boldsymbol{x}(\boldsymbol{t}) + B(\boldsymbol{t})\boldsymbol{u}(\boldsymbol{t})$$

$$\boldsymbol{x}(\boldsymbol{t}_{\mathbf{0}}) \in X^{\mathbf{0}},\ \boldsymbol{u}(\boldsymbol{t}) \in \mathcal{P}(\boldsymbol{t})$$

Outline

- Problem setting and basic definitions
- Overview of existing methods and tools
- Ellipsoidal approach
- Systems with disturbances
- Hybrid systems
- Summary and outlook

Algorithmic Methods

Polytopes (*MPT*)	linear systems exact reach set	ETHZ
Zonotopes (*MATISSE*)	linear systems external apprx.	UPenn/ Verimag
Hyperrectangles (*d/dt*)	linear systems external apprx.	Verimag
Oriented Rectangles (*CheckMate*)	autonomous systems external apprx.	CMU

Analytic Methods

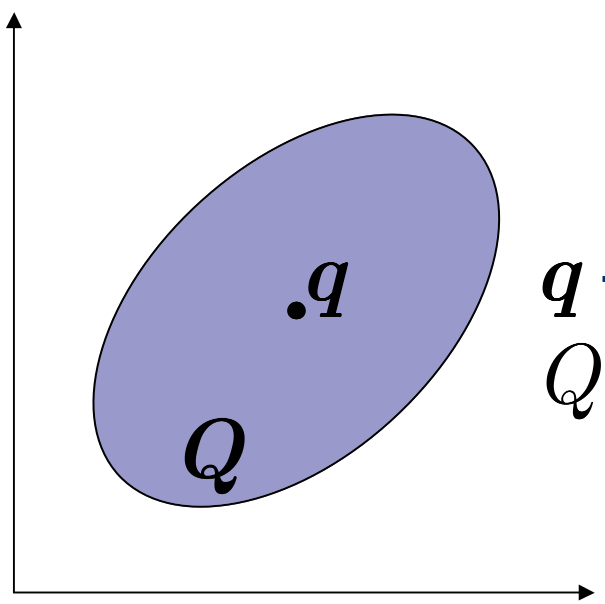
Quantifier Elimination (*Requiem*)	linear nilpotent systems exact reach set	UPenn
Parallelotopes	linear systems external/internal apprx.	IMM
Level Sets (*Level Set Toolbox*)	any systems exact reach set	UBC
Barrier Certificates	polynomial systems no reach set computation	Caltech

Outline

- Problem setting and basic definitions
- Overview of existing methods and tools
- Ellipsoidal approach
- Systems with disturbances
- Hybrid systems
- Summary and outlook

Ellipsoid

$$\mathcal{E}(\boldsymbol{q}, Q) = \{\boldsymbol{x} \in \mathbb{R}^{n} \mid \langle (\boldsymbol{x} - \boldsymbol{q}),\ Q^{-1}(\boldsymbol{x} - \boldsymbol{q}) \rangle \leq 1\}$$

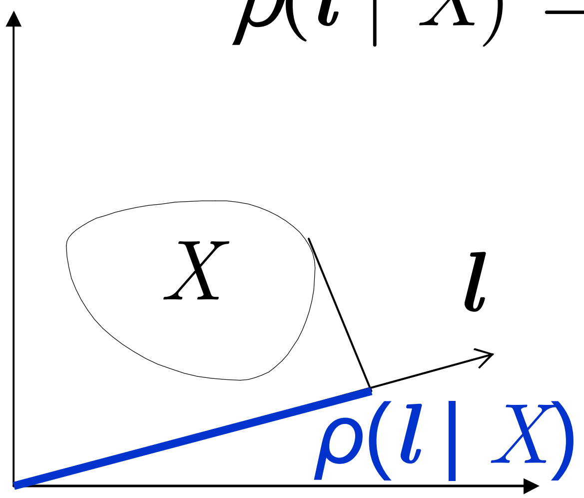


$\boldsymbol{q}$ – center of ellipsoid

Q – shape matrix ($Q = Q^{T} > 0$)

Support Function

$$\rho(\boldsymbol{l} \mid X) = \mathbf{sup} \ \{\langle \boldsymbol{l},\ \boldsymbol{x}\rangle \mid \boldsymbol{x} \in X\}$$



Support function of ellipsoid:

$$\rho(\boldsymbol{l} \mid \mathcal{E}(\boldsymbol{q}, Q)) = \langle \boldsymbol{l},\ \boldsymbol{q}\rangle + \langle \boldsymbol{l},\ Q\boldsymbol{l}\rangle^{1/2}$$

Linear Systems

- Continuous-time:

$$\dot{\boldsymbol{x}}(\boldsymbol{t}) = A(\boldsymbol{t})\boldsymbol{x}(\boldsymbol{t}) + B(\boldsymbol{t})\boldsymbol{u}(\boldsymbol{t})$$

- Discrete-time:

$$\boldsymbol{x}(\boldsymbol{t}+1) = A(\boldsymbol{t})\boldsymbol{x}(\boldsymbol{t}) + B(\boldsymbol{t})\boldsymbol{u}(\boldsymbol{t})$$

$$\boldsymbol{x}(\boldsymbol{t}_{\mathbf{0}}) \in \mathcal{E}(\boldsymbol{x}_{\mathbf{0}}, X_{\mathbf{0}}),\ \boldsymbol{u}(\boldsymbol{t}) \in \mathcal{E}(\boldsymbol{p}(\boldsymbol{t}), P(\boldsymbol{t}))$$

Linear Systems

- Continuous-time:

$$\dot{\boldsymbol{x}}(\boldsymbol{t}) = A(\boldsymbol{t})\boldsymbol{x}(\boldsymbol{t}) + B(\boldsymbol{t})\boldsymbol{u}(\boldsymbol{t})$$

- Discrete-time:

$$\boldsymbol{x}(\boldsymbol{t}+1) = A(\boldsymbol{t})\boldsymbol{x}(\boldsymbol{t}) + B(\boldsymbol{t})\boldsymbol{u}(\boldsymbol{t})$$

$$\boldsymbol{x}(\boldsymbol{t}_{\mathbf{0}}) \in \boxed{\mathcal{E}(\boldsymbol{x}_{\mathbf{0}}, X_{\mathbf{0}})},\ \boldsymbol{u}(\boldsymbol{t}) \in \boxed{\mathcal{E}(\boldsymbol{p}(\boldsymbol{t}), P(\boldsymbol{t}))}$$

Reach Set of Linear System

Symmetric convex compact set in $\mathbb{R}^n$
evolving in time

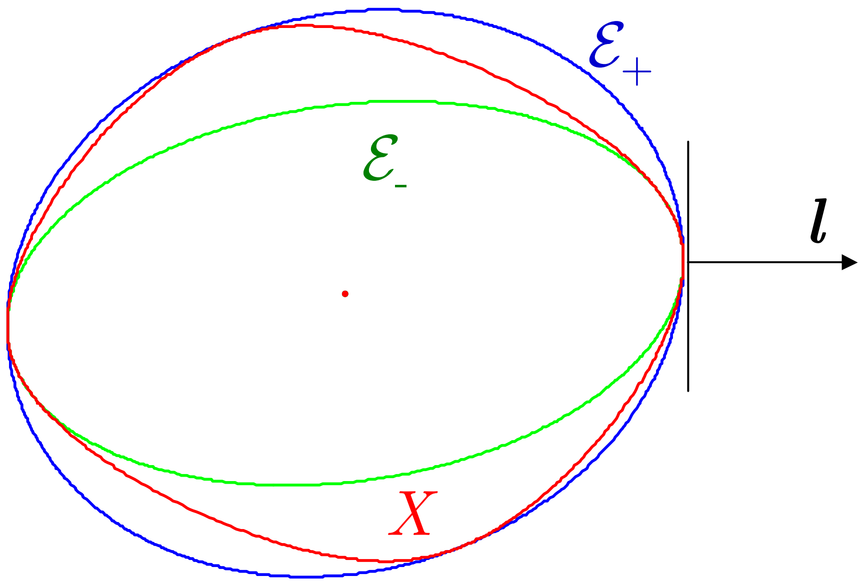
Tight Approximations

- External ellipsoidal approximation $\mathcal{E}_+$ of symmetric convex set X is tight if
  - $X \subseteq \mathcal{E}_+$
  - There exists $\boldsymbol{l}$ such that $\rho(\pm\boldsymbol{l} \mid \mathcal{E}_+) = \rho(\pm\boldsymbol{l} \mid X)$
- Internal ellipsoidal approximation $\mathcal{E}_-$ of symmetric convex set X is tight if
  - $\mathcal{E}_- \subseteq X$
  - There exists $\boldsymbol{l}$ such that $\rho(\pm\boldsymbol{l} \mid \mathcal{E}_-) = \rho(\pm\boldsymbol{l} \mid X)$

Reach Set Approximation

For any $\boldsymbol{l}$ there exist $\mathcal{E}_+$ and $\mathcal{E}_-$:

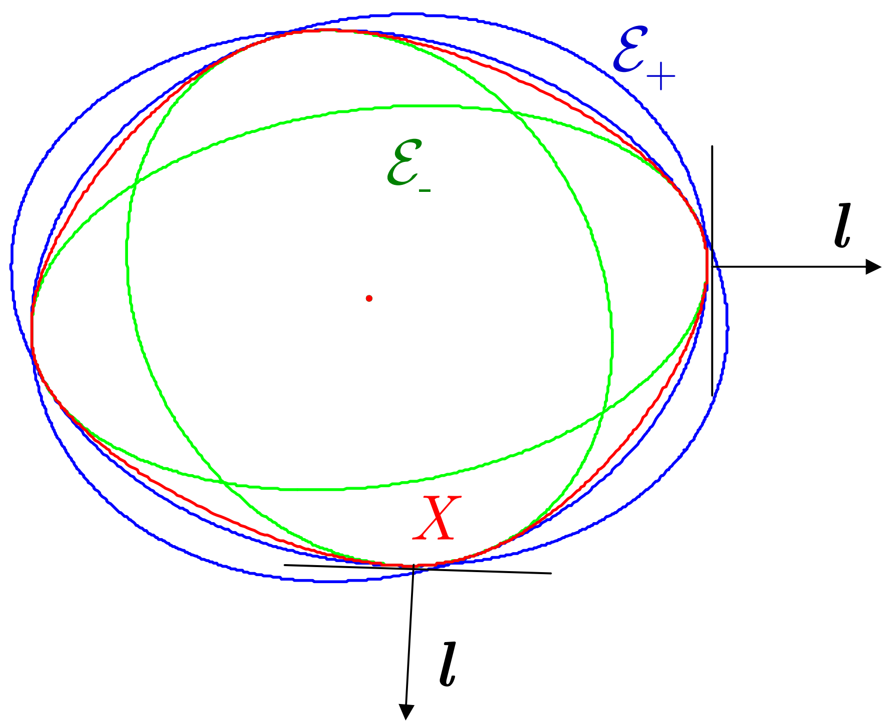
- $\mathcal{E}_- \subseteq X \subseteq \mathcal{E}_+$
- $\rho(\pm \boldsymbol{l} \mid \mathcal{E}_-) = \rho(\pm \boldsymbol{l} \mid X) = \rho(\pm \boldsymbol{l} \mid \mathcal{E}_+)$



Reach Set Approximation

For any $\boldsymbol{l}$ there exist $\mathcal{E}_+$ and $\mathcal{E}_-$:

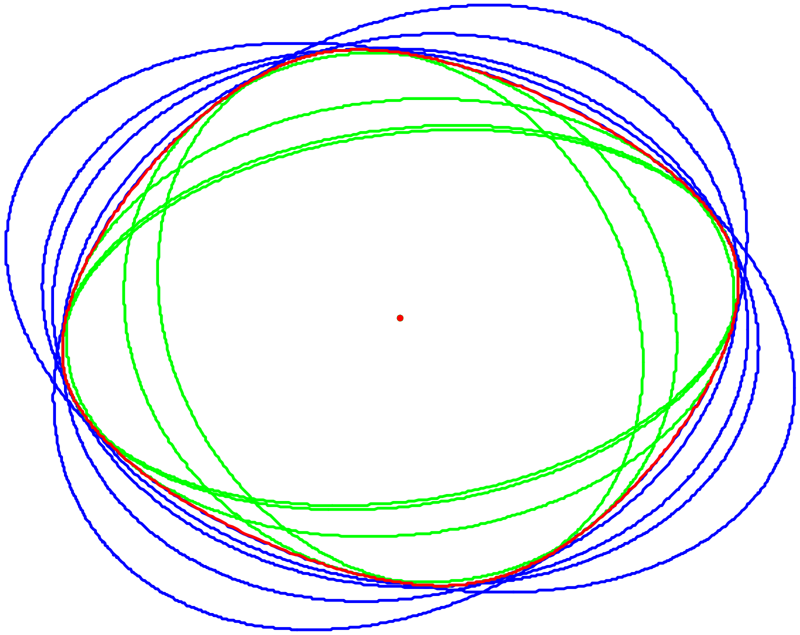
- $\mathcal{E}_- \subseteq X \subseteq \mathcal{E}_+$
- $\rho(\pm\boldsymbol{l} \mid \mathcal{E}_-) = \rho(\pm\boldsymbol{l} \mid X) = \rho(\pm\boldsymbol{l} \mid \mathcal{E}_+)$



Reach Set Approximation

For any $\boldsymbol{l}$ there exist $\mathcal{E}_+$ and $\mathcal{E}_-$:

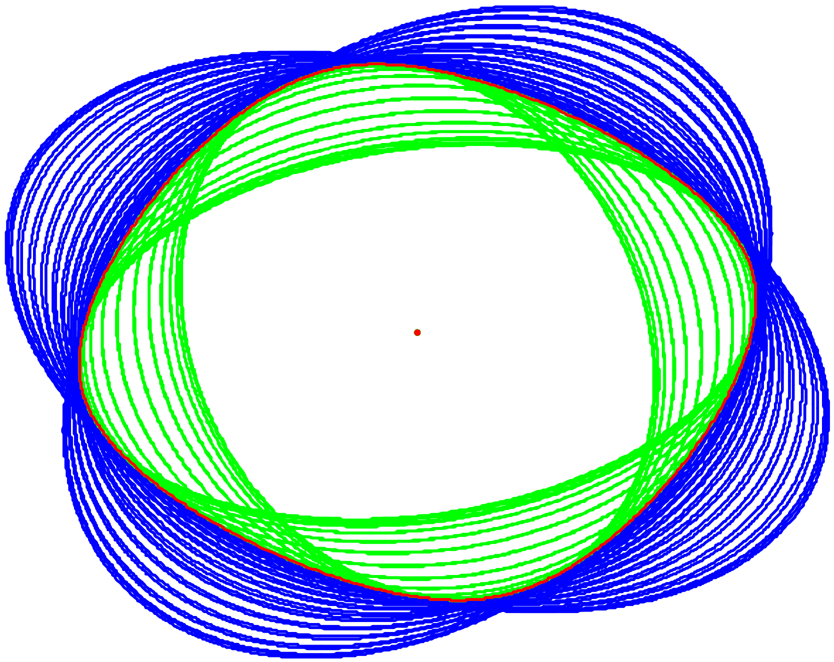
- $\mathcal{E}_- \subseteq X \subseteq \mathcal{E}_+$
- $\rho(\pm \boldsymbol{l} \mid \mathcal{E}_-) = \rho(\pm \boldsymbol{l} \mid X) = \rho(\pm \boldsymbol{l} \mid \mathcal{E}_+)$



Reach Set Approximation

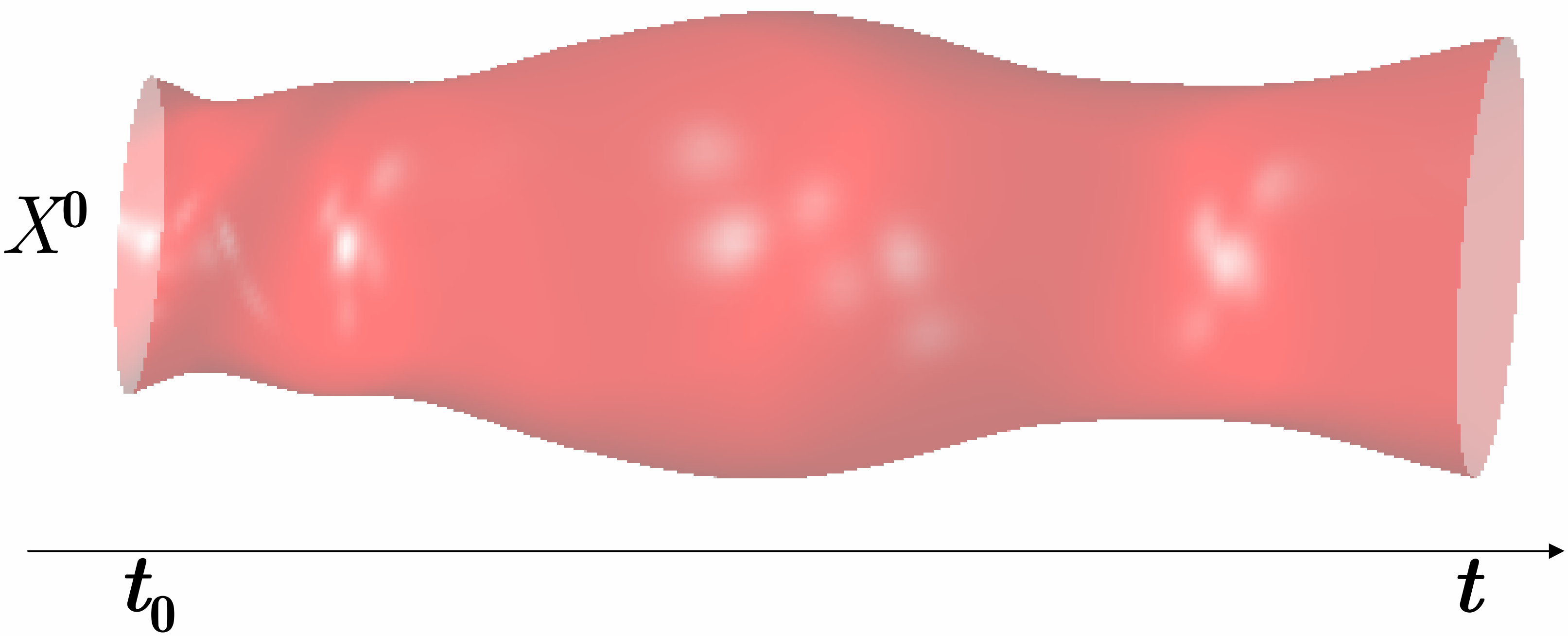
For any $\boldsymbol{l}$ there exist $\mathcal{E}_+$ and $\mathcal{E}_-$:

- $\mathcal{E}_- \subseteq X \subseteq \mathcal{E}_+$
- $\rho(\pm \boldsymbol{l} \mid \mathcal{E}_-) = \rho(\pm \boldsymbol{l} \mid X) = \rho(\pm \boldsymbol{l} \mid \mathcal{E}_+)$



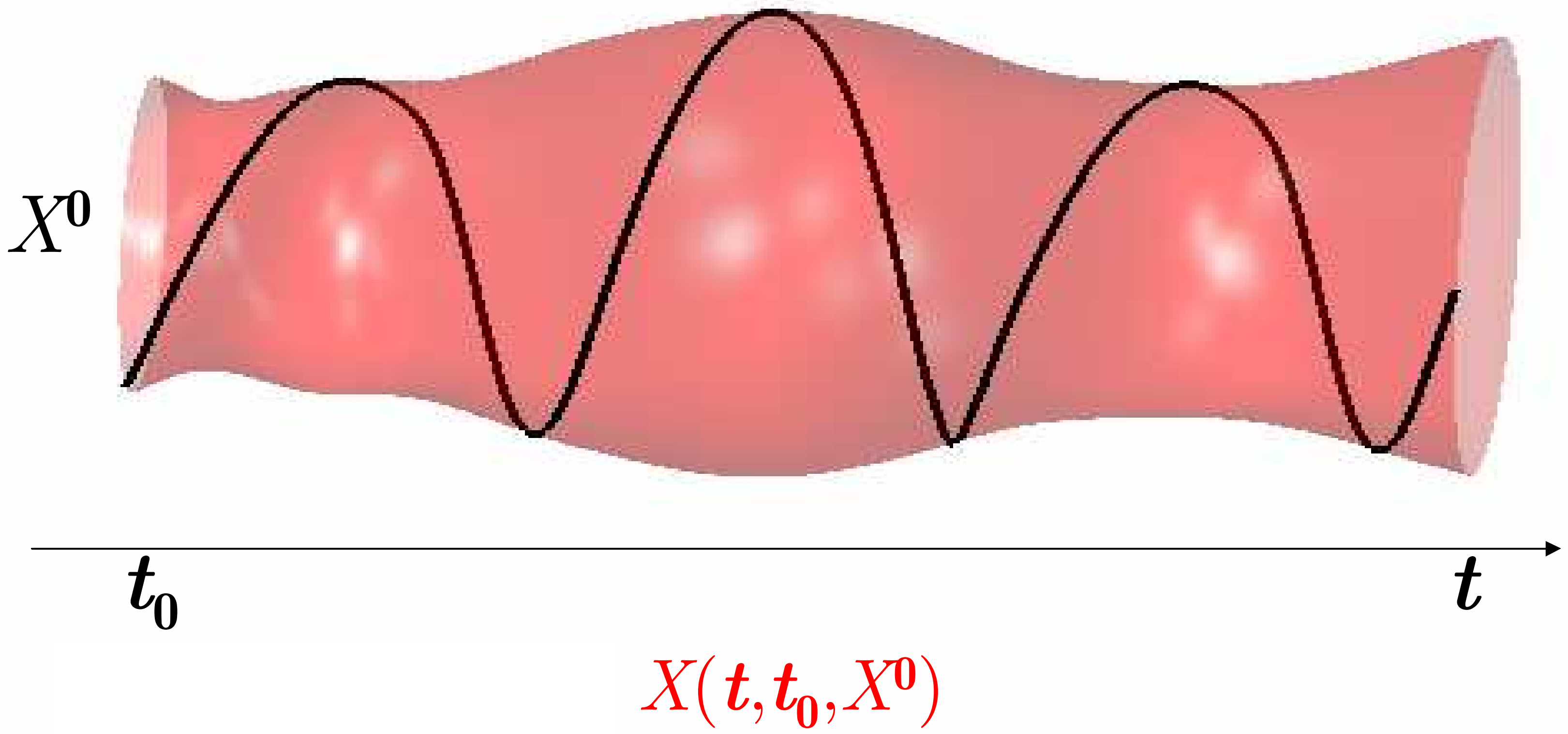
$$\bigcup_{\boldsymbol{l}} \mathcal{E}_- = X = \bigcap_{\boldsymbol{l}} \mathcal{E}_+$$

Good Curves (concept)

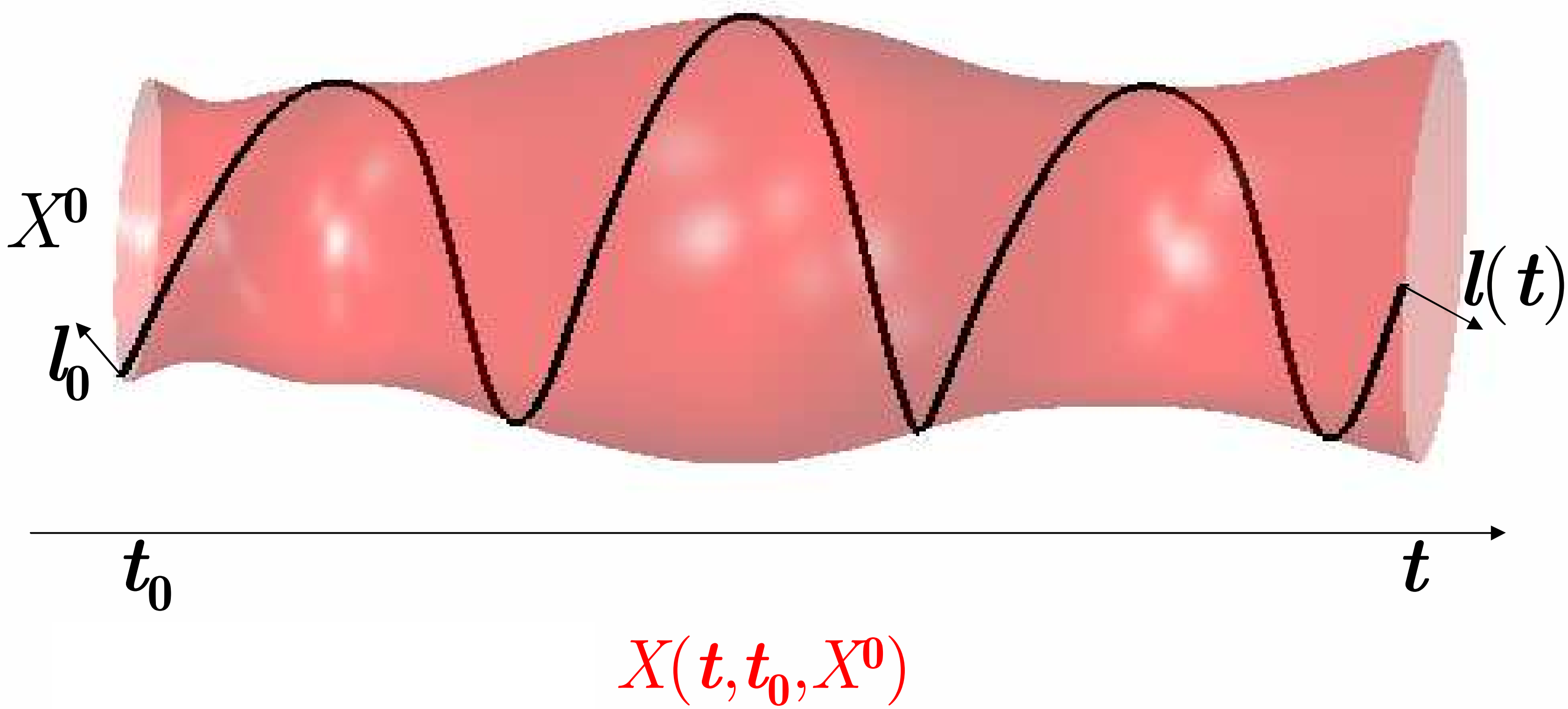


$X(t, t_0, X^0)$

Good Curves (concept)



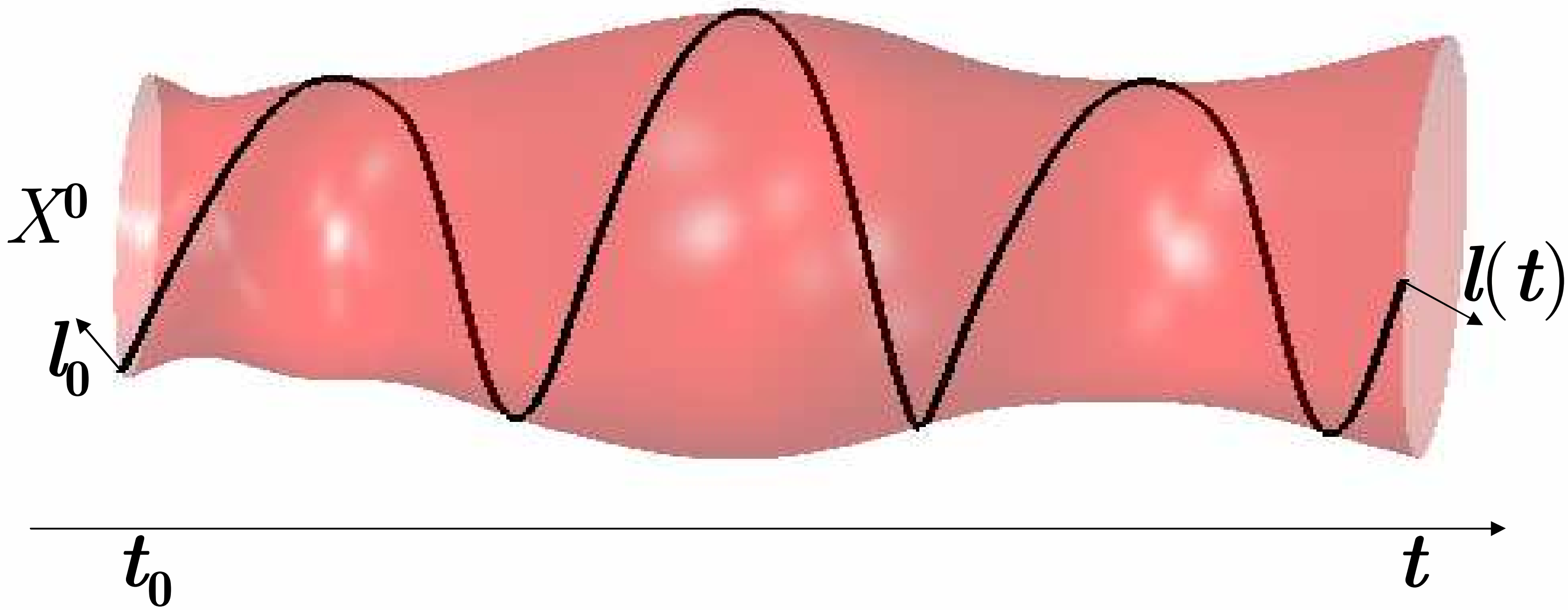
Good Curves (concept)



Good Curves (concept)

$$\dot{l}(t) = -A^{\mathrm{T}}(t)\,l(t),\ l(t_0) = l_0$$

<u>good curve</u>



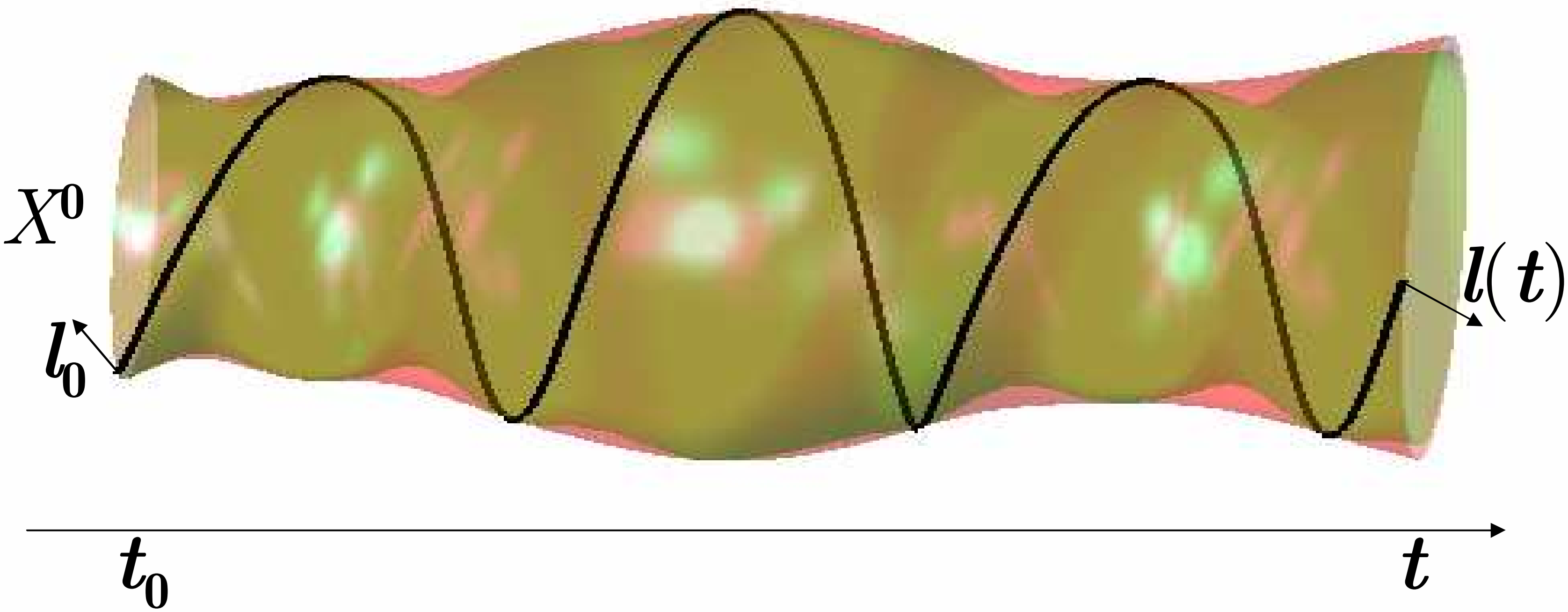
$X(t, t_0, X^0)$

GC

Good Curves (concept)

$$\dot{l}(t) = -A^{\mathrm{T}}(t)l(t),\ l(t_0) = l_0$$

good curve

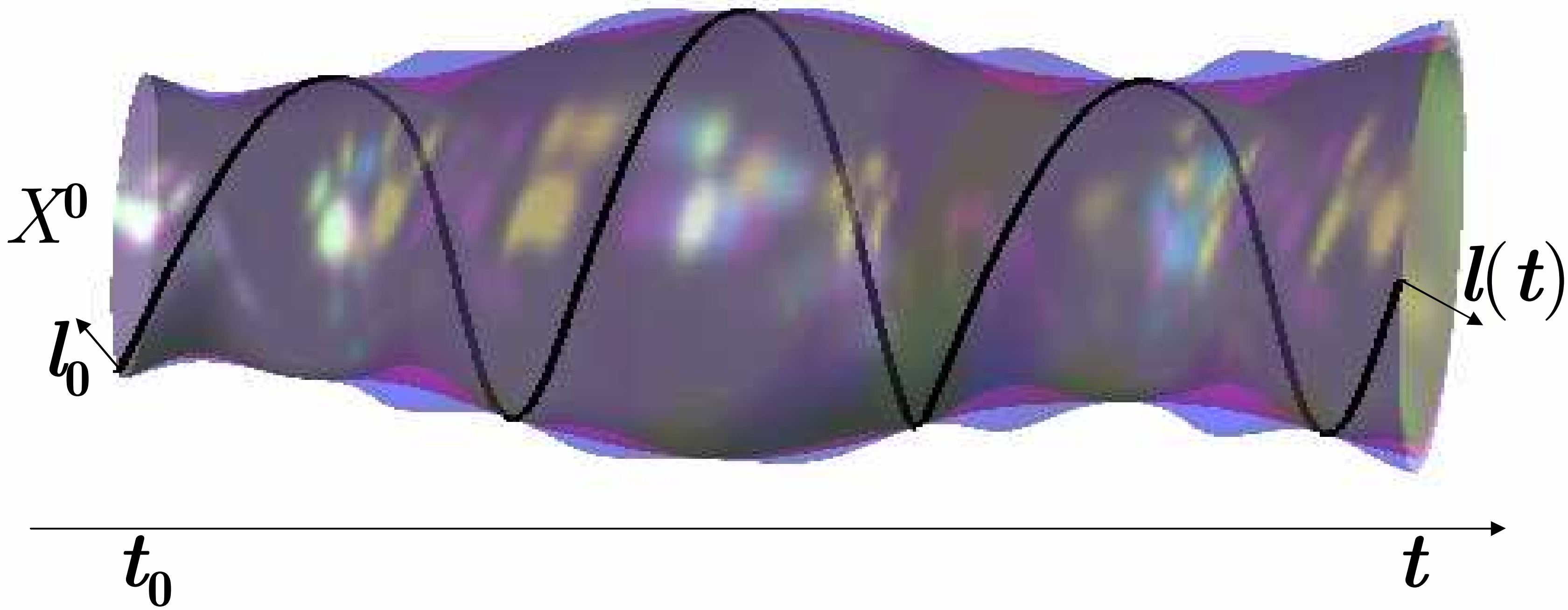


$$\mathcal{E}(x_c(t), X_l^-(t)) \subseteq X(t, t_0, X^0)$$

Good Curves (concept)

$$\dot{\boldsymbol{l}}(\boldsymbol{t}) = -A^{\mathrm{T}}(\boldsymbol{t})\boldsymbol{l}(\boldsymbol{t}),\ \boldsymbol{l}(\boldsymbol{t}_0) = \boldsymbol{l}_0$$

<u>good curve</u>

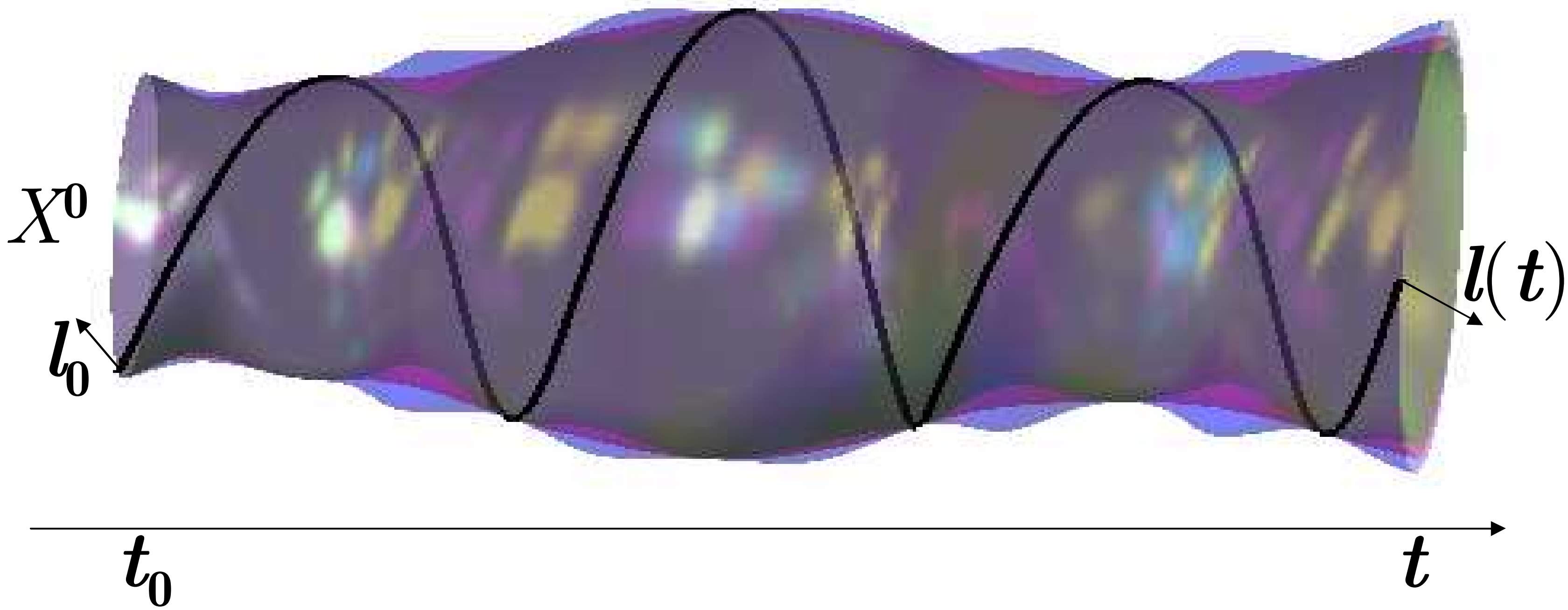


$$\mathcal{E}(\boldsymbol{x}_c(\boldsymbol{t}), X_l^-(\boldsymbol{t})) \subseteq X(\boldsymbol{t}, \boldsymbol{t}_0, X^0) \subseteq \mathcal{E}(\boldsymbol{x}_c(\boldsymbol{t}), X_l^+(\boldsymbol{t}))$$

Good Curves (concept)

$$\dot{l}(t) = -A^{\mathrm{T}}(t)\,l(t),\ l(t_0) = l_0$$

good curve



$$\rho(l(t)\,|\,\mathcal{E}(x_c(t), X_l^-(t))) = \rho(l(t)\,|\,X(t, t_0, X^0)) = \rho(l(t)\,|\,\mathcal{E}(x_c(t), X_l^+(t)))$$

Good Curves (summary)

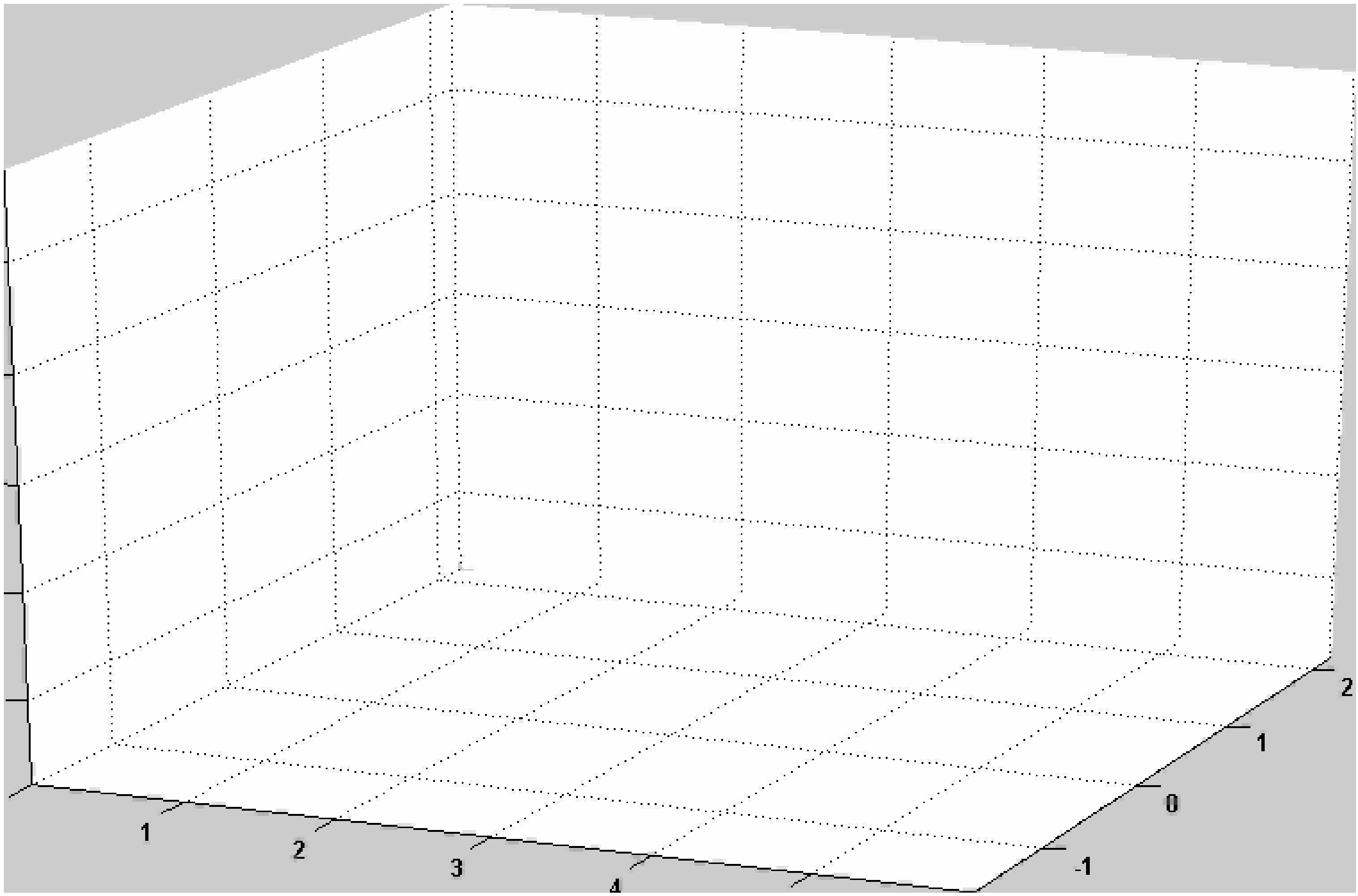
If $\boldsymbol{l}(\boldsymbol{t})$ satisfies $\dot{\boldsymbol{l}}(\boldsymbol{t}) = -A^{\mathrm{T}}(\boldsymbol{t})\boldsymbol{l}(\boldsymbol{t}),\ \boldsymbol{l}(\boldsymbol{t}_0) = \boldsymbol{l}_0$, then

- $\mathcal{E}(\boldsymbol{x}_c(\boldsymbol{t}), X_l^-(\boldsymbol{t})) \subseteq X(\boldsymbol{t}, \boldsymbol{t}_0, X^0) \subseteq \mathcal{E}(\boldsymbol{x}_c(\boldsymbol{t}), X_l^+(\boldsymbol{t}))$
- $\rho(\boldsymbol{l}(\boldsymbol{t}) | \mathcal{E}(\boldsymbol{x}_c(\boldsymbol{t}), X_l^-(\boldsymbol{t}))) = \rho(\boldsymbol{l}(\boldsymbol{t}) | X(\boldsymbol{t}, \boldsymbol{t}_0, X^0)) = \rho(\boldsymbol{l}(\boldsymbol{t}) | \mathcal{E}(\boldsymbol{x}_c(\boldsymbol{t}), X_l^+(\boldsymbol{t})))$

where $\dot{\boldsymbol{x}}_c(\boldsymbol{t}) = A(\boldsymbol{t})\boldsymbol{x}_c(\boldsymbol{t}) + B(\boldsymbol{t})\boldsymbol{p}(\boldsymbol{t}),\ \boldsymbol{x}_c(\boldsymbol{t}_0) = \boldsymbol{x}_0$,
and the shape matrices $X_l^+(\boldsymbol{t})$, $X_l^-(\boldsymbol{t})$ are governed by <u>single ODEs</u>

- *<u>On ellipsoidal techniques for reachability analysis</u>*
  by A.B.Kurzhanski, P.Varaiya (2000)

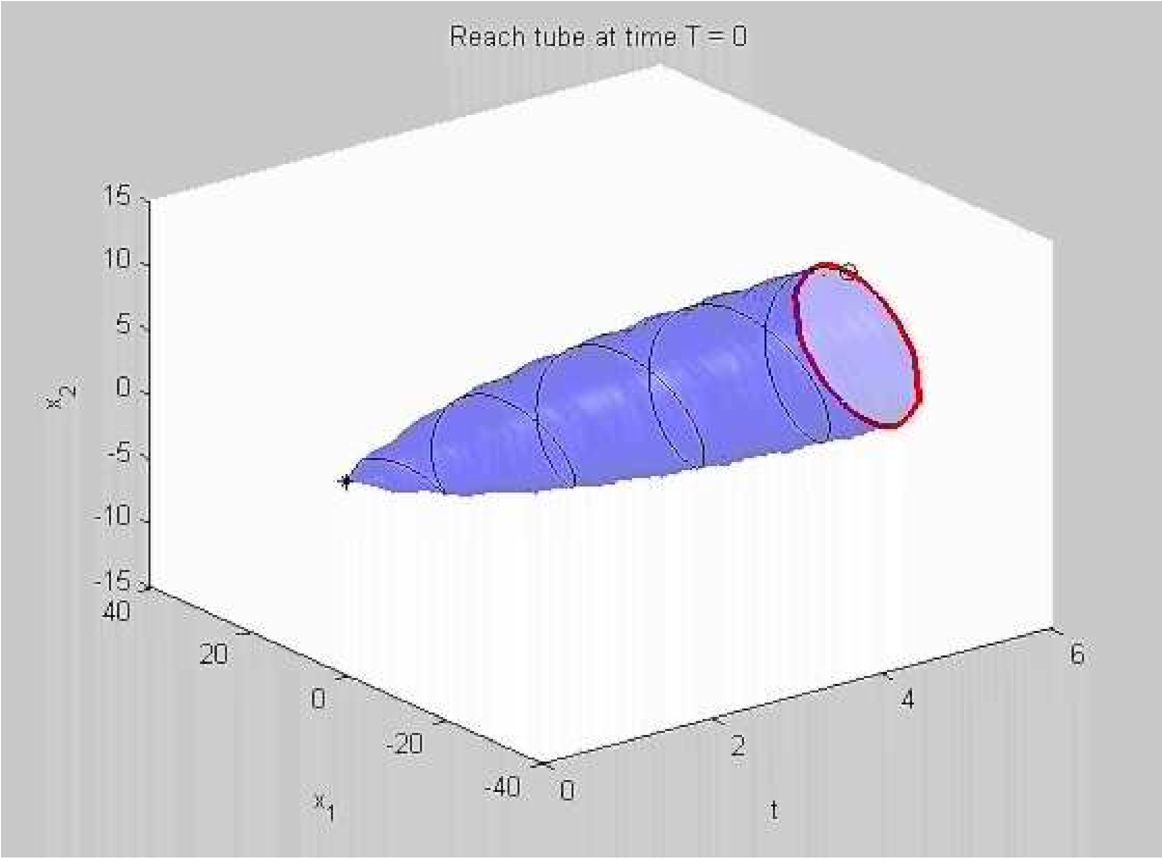
Good Curves (movie)



Steering the system to a given target point at given time

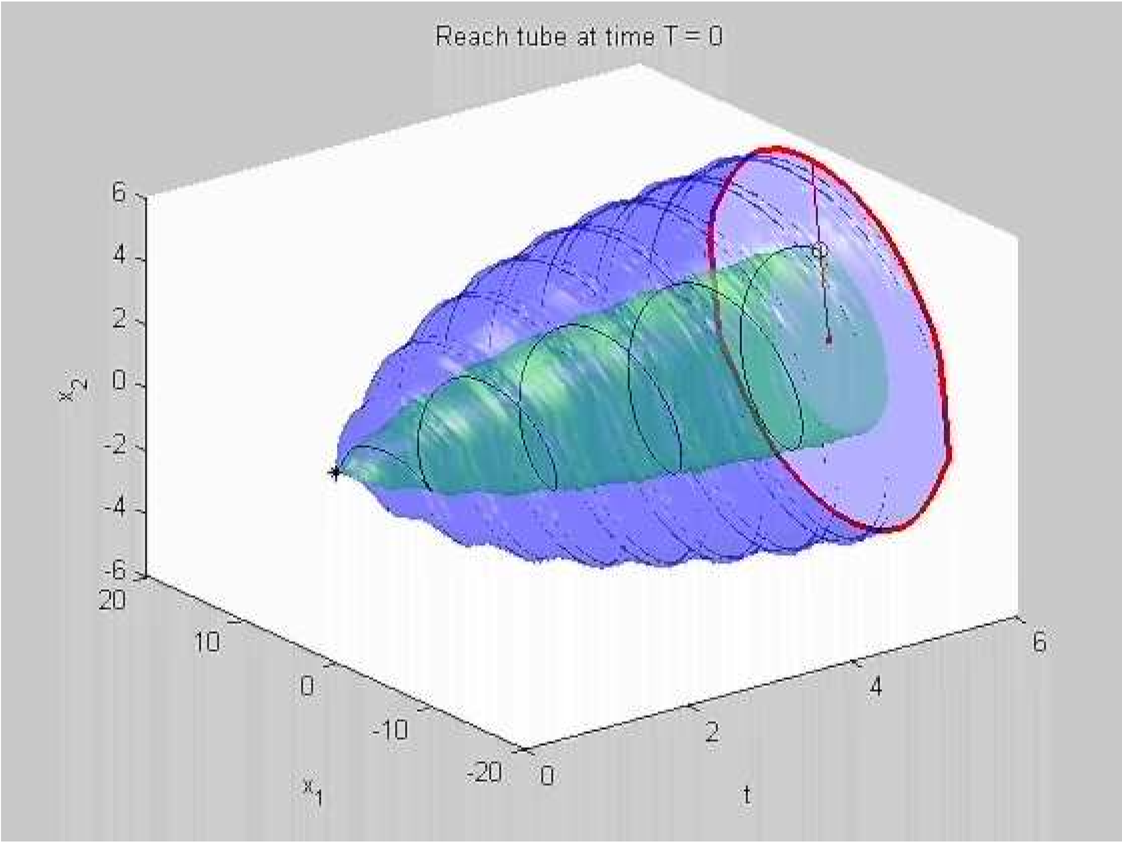
Good Curves (control)

$$u_l(t) = p(t) + \frac{P(t)B^T(t)\Phi(t_0,t)l_0}{\langle l_0, \Phi(t_0,t)B(t)P(t)B^T(t)\Phi(t_0,t)l_0\rangle^{1/2}}$$



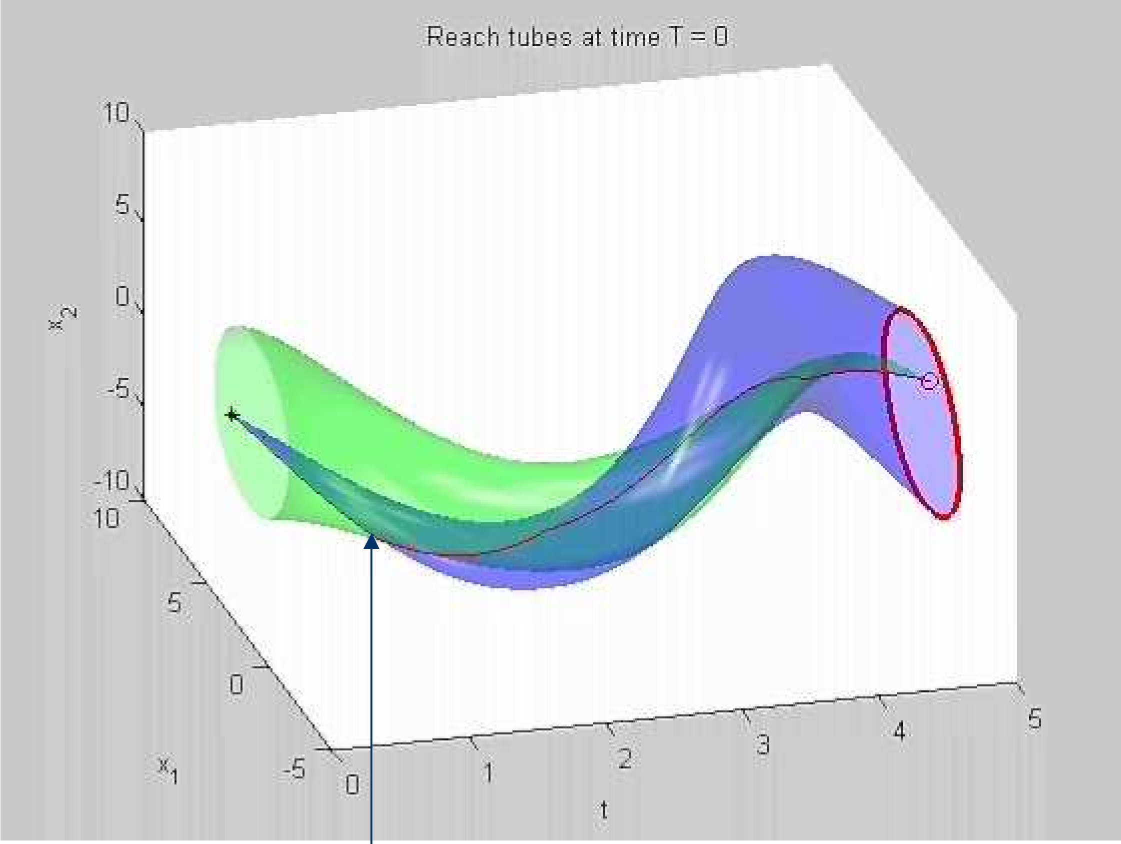
Reaching Internal Point

Scale the set of controls: $\mathcal{E}(\boldsymbol{p}(\boldsymbol{t}), \boldsymbol{\mu}^2 P(\boldsymbol{t})),\ |\boldsymbol{\mu}| \leq 1$



Reaching Internal Point

Colliding forward and backward reach tubes



switch good curves

*E*llipsoidal *T*oolbox©

www.eecs.berkeley.edu/~akurzhan/ellipsoids

Ellipsoidal Toolbox

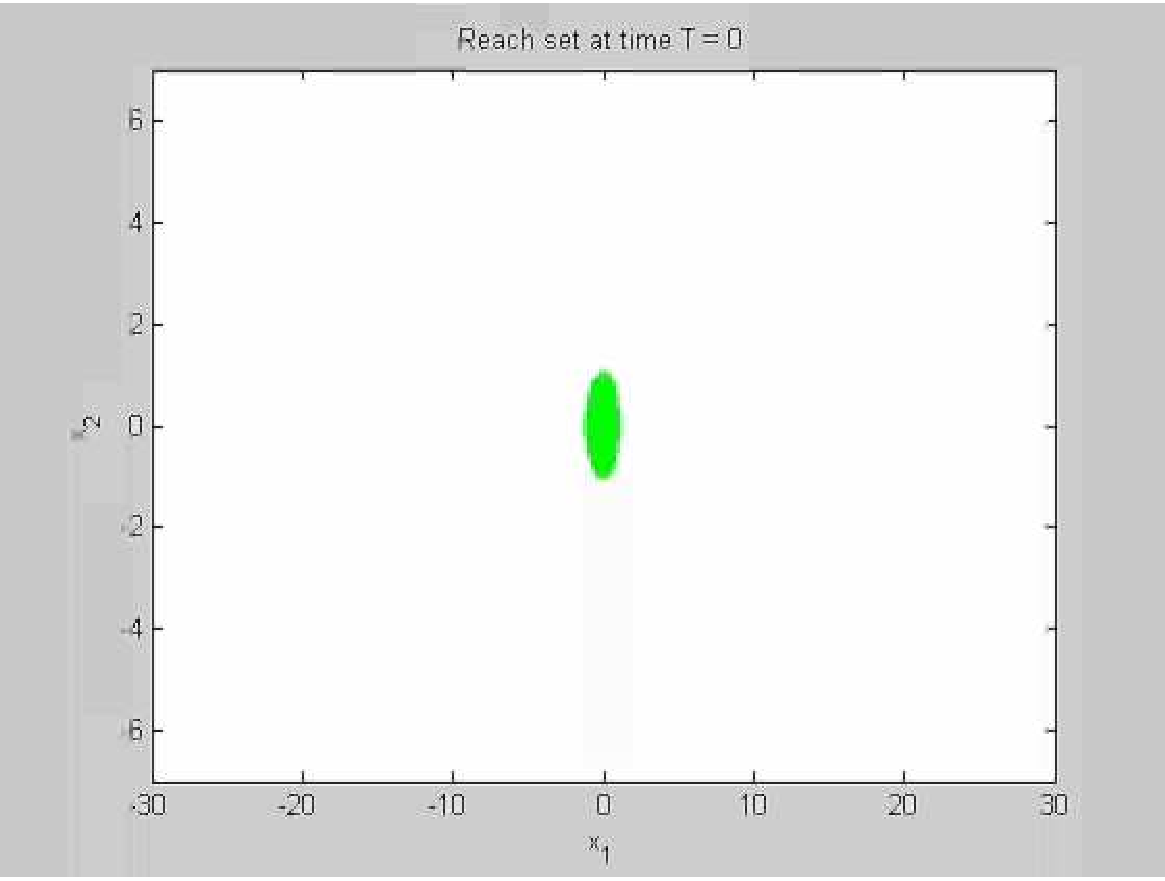
- Ellipsoidal calculus
  - Geometric sums and differences
  - Intersections with ellipsoids, hyperplanes, polyhedra
- Reachability analysis
  - Continuous- and discrete-time linear systems
  - Forward and backward reach sets
- Visualization (2D and 3D)
  - Plotting of ellipsoids, hyperplanes, reach sets
  - Projections

MATLAB Types

$\boldsymbol{ET}$ implements classes:

- `ellipsoid`
- `hyperplane`
- `linsys`
- `reach`

Approximation Refinement

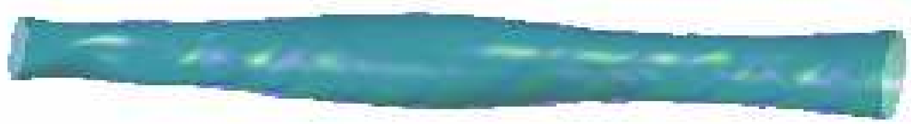


$\boldsymbol{ET}$ function: **`refine`**

Semigroup Property

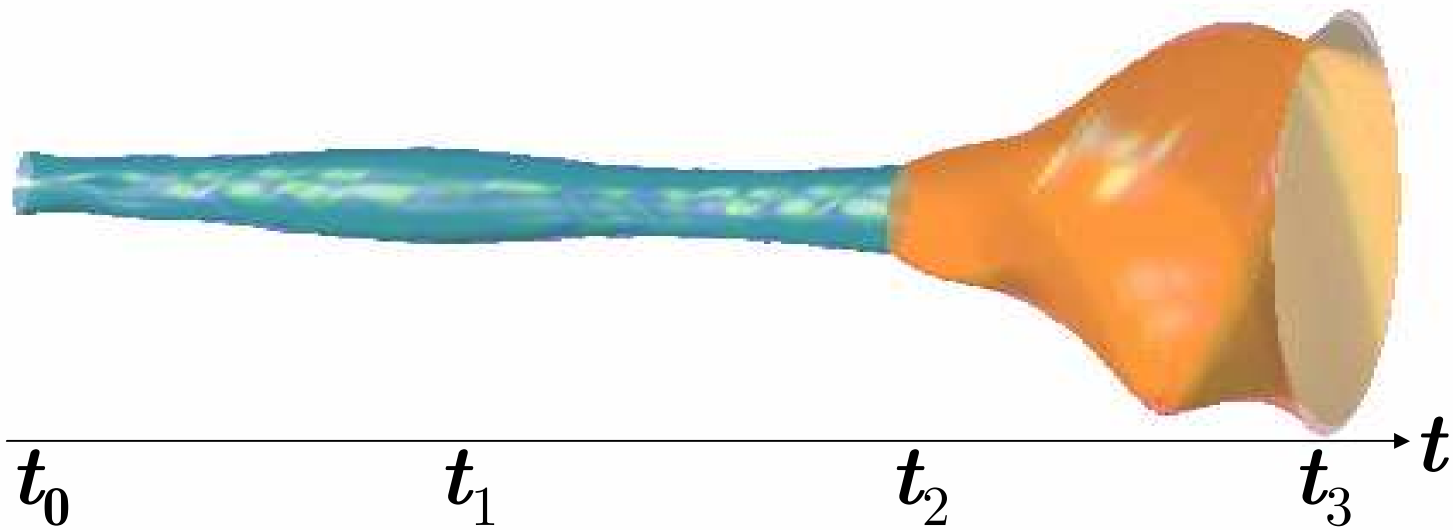
ET function: **`evolve`**

Semigroup Property



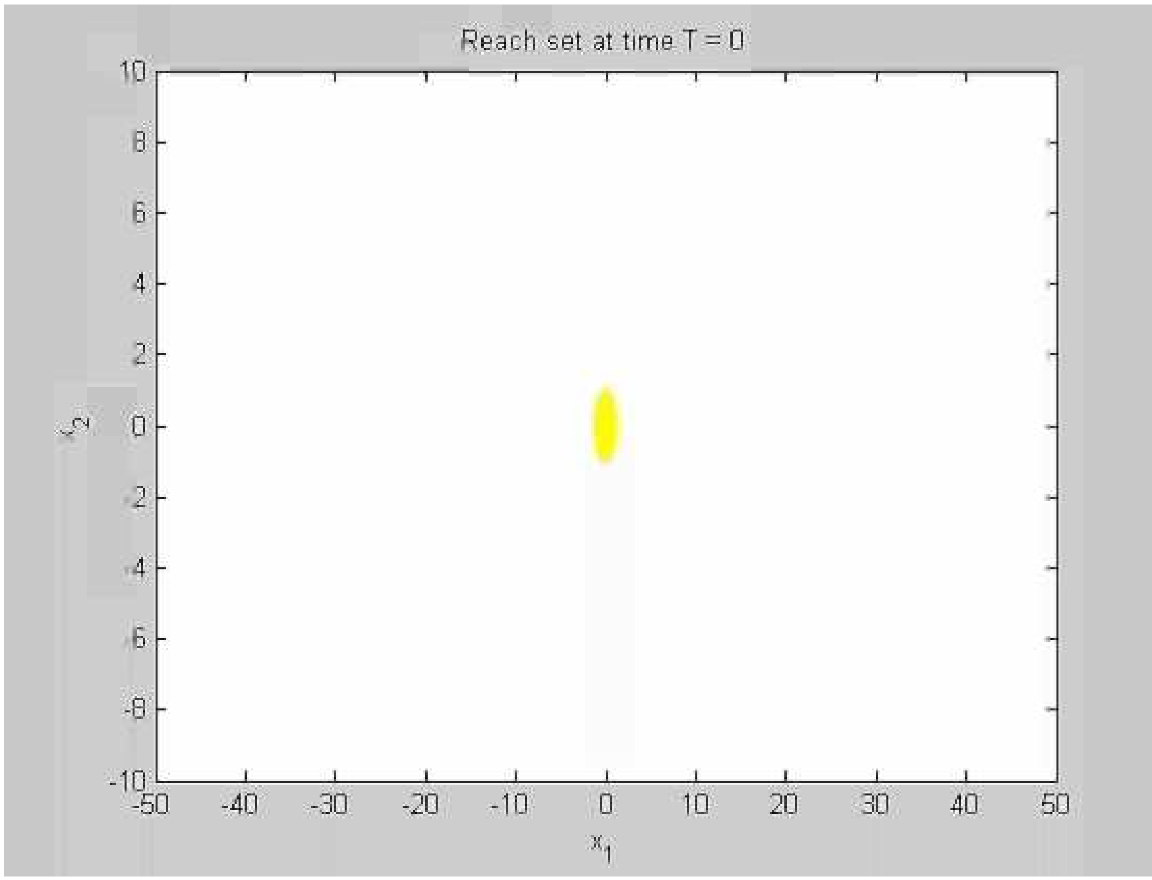
ET function: **`evolve`**

Semigroup Property



ET function: **`evolve`**

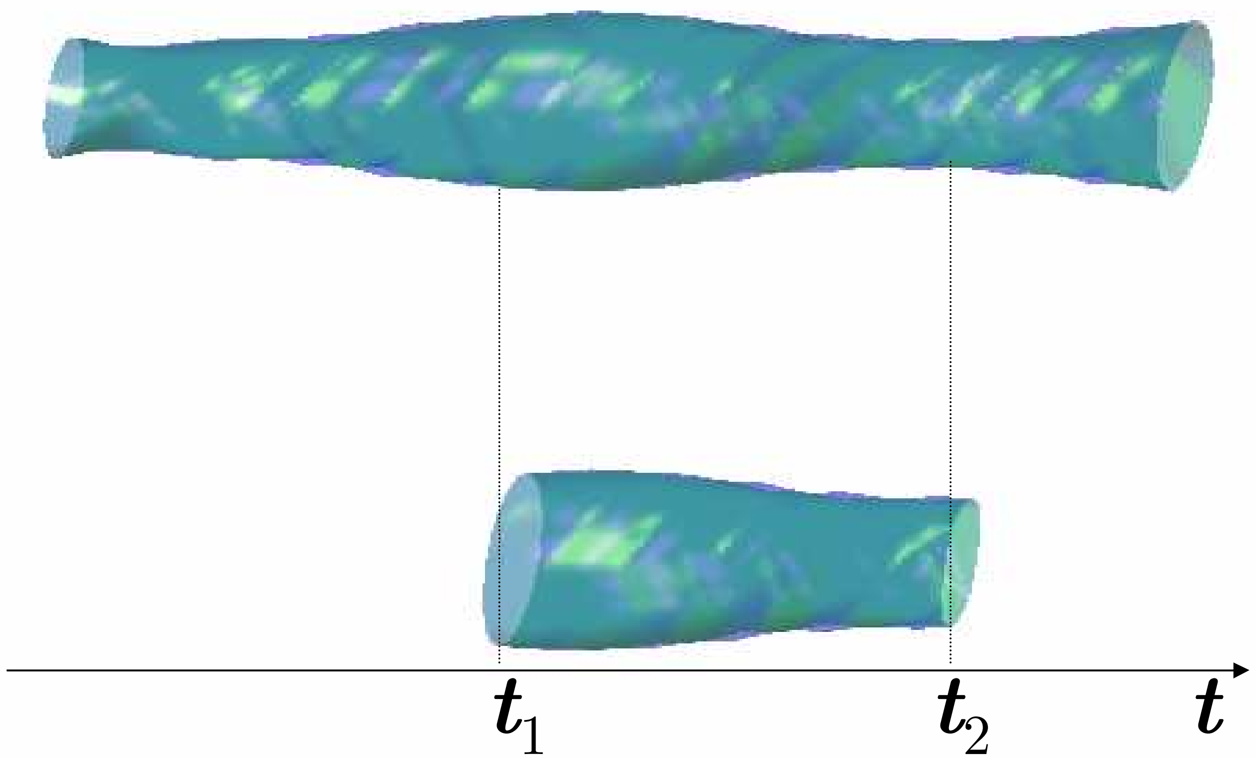
Switched System



ET function: **`evolve`**

3D

Cutting the Reach Tube



ET function: **`cut`**

Verification

- Check if reach set external (internal) approximation intersects with given object: **`ellipsoid`**, **`hyperplane`**, **`polytope`**

$\boldsymbol{ET}$ function: **`intersect`**

Discrete-Time Systems

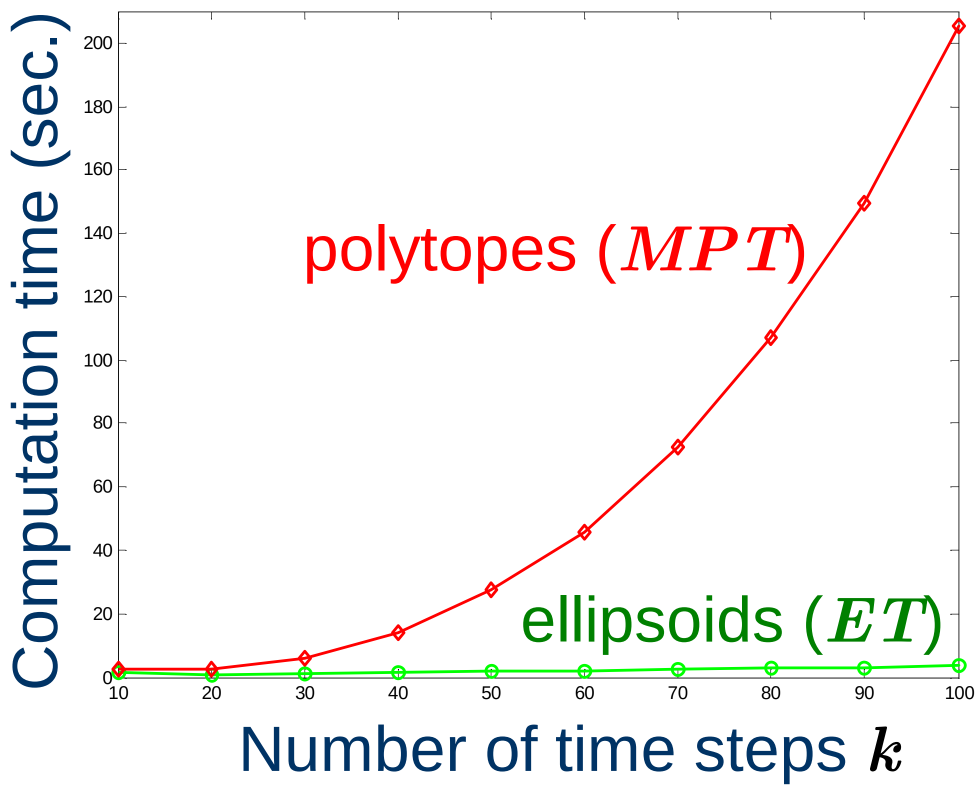
$$\boldsymbol{x}[\boldsymbol{k}+1] = A[\boldsymbol{k}]\boldsymbol{x}[\boldsymbol{k}] + B[\boldsymbol{k}]\boldsymbol{u}[\boldsymbol{k}]$$

$$\boldsymbol{x}[\boldsymbol{k_0}] \in \mathcal{E}(\boldsymbol{x_0}, X_{\boldsymbol{0}}),\ \boldsymbol{u}[\boldsymbol{k}] \in \mathcal{E}(\boldsymbol{p}[\boldsymbol{k}], P[\boldsymbol{k}])$$

Same ellipsoidal theory applies with some adjustments

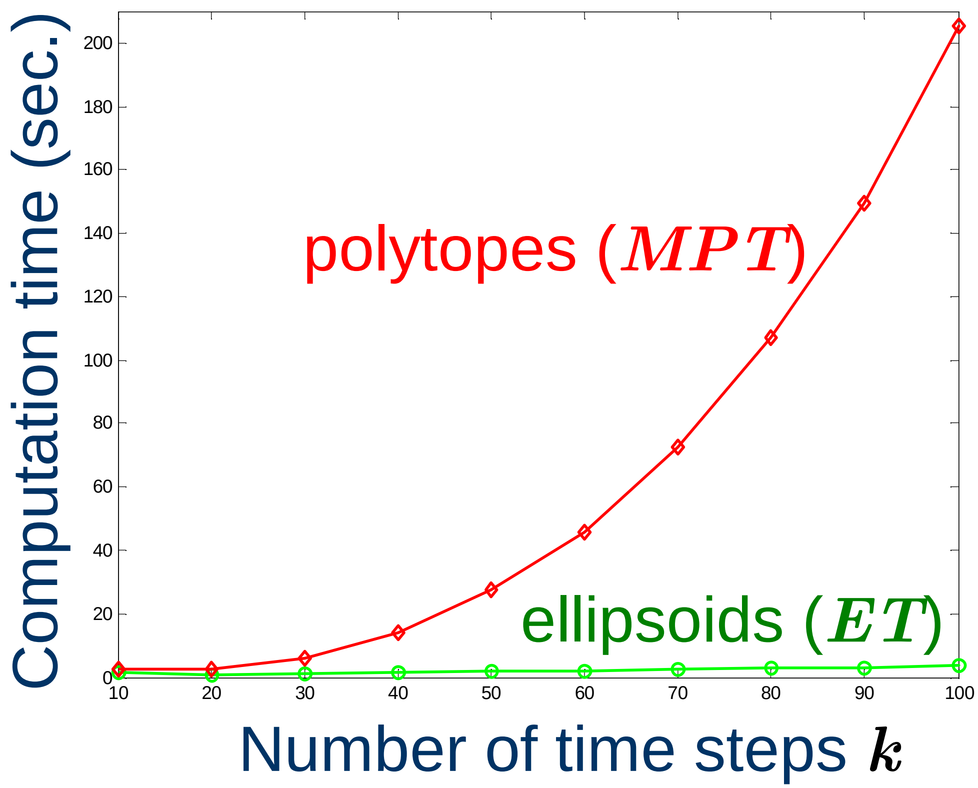
Ellipsoids vs Polytopes

$$x[k+1] = \begin{bmatrix} \cos(1) & -\sin(1) \\ \sin(1) & \cos(1) \end{bmatrix} x[k] + u[k], \; x[0] \in X^0, \; u[k] \in \mathcal{P}$$



Ellipsoids vs Polytopes

$$x[k+1] = \begin{bmatrix} \cos(1) & -\sin(1) \\ \sin(1) & \cos(1) \end{bmatrix} x[k] + u[k],\ x[0] \in X^0,\ u[k] \in \mathcal{P}$$

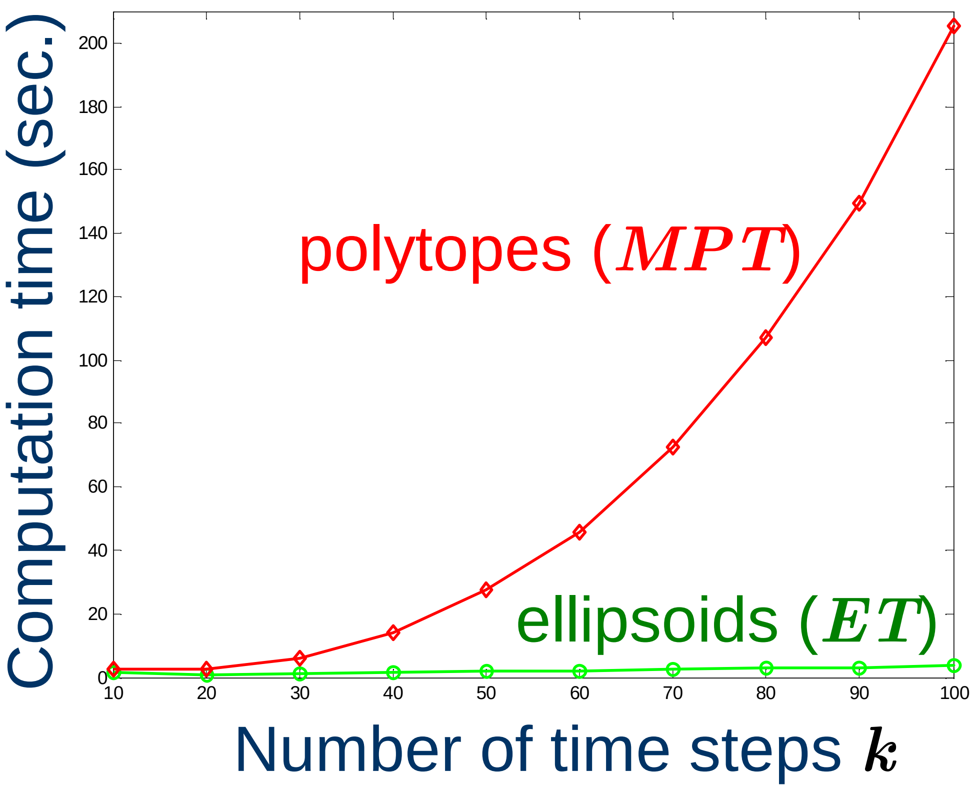


Complexity:

$$\boldsymbol{k}(L(8\mathrm{n}^3 + 4\mathrm{n}^2 + 2\mathrm{n}) + 2\mathrm{n}^2)$$

Ellipsoids vs Polytopes

$$x[k+1] = \begin{bmatrix} \cos(1) & -\sin(1) \\ \sin(1) & \cos(1) \end{bmatrix} x[k] + u[k],\ x[0] \in X^0,\ u[k] \in \mathcal{P}$$



<u>Complexity:</u>

$$\boldsymbol{k}(L(8n^3 + 4n^2 + 2n) + 2n^2)$$

number of directions $\boldsymbol{l_0}$

state space dimension

Outline

- Problem setting and basic definitions
- Overview of existing methods and tools
- Ellipsoidal approach
- Systems with disturbances
- Hybrid systems
- Summary and outlook

Linear System with Disturbance

- System equation:

$$\dot{\boldsymbol{x}}(\boldsymbol{t}) = A(\boldsymbol{t})\boldsymbol{x}(\boldsymbol{t}) + B(\boldsymbol{t})\boldsymbol{u}(\boldsymbol{t},\boldsymbol{x}(\boldsymbol{t})) + G(\boldsymbol{t})\boldsymbol{v}(\boldsymbol{t})$$

- Initial state: $\boldsymbol{x}(\boldsymbol{t}_{\mathbf{0}}) \in X^{\mathbf{0}}=\mathcal{E}(\boldsymbol{x}_{\mathbf{0}},X_{\mathbf{0}})$
- Control
  - Open-loop: $\boldsymbol{u}(\boldsymbol{t}) \in \mathcal{P}(\boldsymbol{t})=\mathcal{E}(\boldsymbol{p}(\boldsymbol{t}),P(\boldsymbol{t}))$
  - Closed-loop: $\boldsymbol{u}(\boldsymbol{t},\boldsymbol{x}(\boldsymbol{t})) \in \mathcal{P}(\boldsymbol{t})=\mathcal{E}(\boldsymbol{p}(\boldsymbol{t}),P(\boldsymbol{t}))$

- Disturbance: $\boldsymbol{v}(\boldsymbol{t}) \in \mathcal{Q}(\boldsymbol{t})=\mathcal{E}(\boldsymbol{q}(\boldsymbol{t}),Q(\boldsymbol{t}))$

Linear System with Disturbance

- System equation:

$$\dot{\boldsymbol{x}}(\boldsymbol{t}) = A(\boldsymbol{t})\boldsymbol{x}(\boldsymbol{t}) + B(\boldsymbol{t})\boldsymbol{u}(\boldsymbol{t},\boldsymbol{x}(\boldsymbol{t})) \boxed{+ G(\boldsymbol{t})\boldsymbol{v}(\boldsymbol{t})}$$

- Initial state: $\boldsymbol{x}(\boldsymbol{t}_{\mathbf{0}}) \in X^{\mathbf{0}} = \mathcal{E}(\boldsymbol{x}_{\mathbf{0}}, X_{\mathbf{0}})$
- Control
  - Open-loop: $\boldsymbol{u}(\boldsymbol{t}) \in \mathcal{P}(\boldsymbol{t}) = \mathcal{E}(\boldsymbol{p}(\boldsymbol{t}), P(\boldsymbol{t}))$
  - Closed-loop: $\boldsymbol{u}(\boldsymbol{t},\boldsymbol{x}(\boldsymbol{t})) \in \mathcal{P}(\boldsymbol{t}) = \mathcal{E}(\boldsymbol{p}(\boldsymbol{t}), P(\boldsymbol{t}))$

- Disturbance: $\boldsymbol{v}(\boldsymbol{t}) \in \mathcal{Q}(\boldsymbol{t}) = \mathcal{E}(\boldsymbol{q}(\boldsymbol{t}), Q(\boldsymbol{t}))$

Linear System with Disturbance

- System equation:

$$\dot{\boldsymbol{x}}(\boldsymbol{t}) = A(\boldsymbol{t})\boldsymbol{x}(\boldsymbol{t}) + B(\boldsymbol{t})\boldsymbol{u}(\boldsymbol{t},\boldsymbol{x}(\boldsymbol{t})) \boxed{+\ G(\boldsymbol{t})\boldsymbol{v}(\boldsymbol{t})}$$

- Initial state: $\boldsymbol{x}(\boldsymbol{t}_{\mathbf{0}}) \in X^{\mathbf{0}} = \mathcal{E}(\boldsymbol{x}_{\mathbf{0}}, X_{\mathbf{0}})$
- Control
  - Open-loop: $\boldsymbol{u}(\boldsymbol{t}) \in \mathcal{P}(\boldsymbol{t}) = \mathcal{E}(\boldsymbol{p}(\boldsymbol{t}), P(\boldsymbol{t}))$
  - Closed-loop: $\boldsymbol{u}(\boldsymbol{t},\boldsymbol{x}(\boldsymbol{t})) \in \mathcal{P}(\boldsymbol{t}) = \mathcal{E}(\boldsymbol{p}(\boldsymbol{t}), P(\boldsymbol{t}))$

- Disturbance: $\boldsymbol{v}(\boldsymbol{t}) \in \mathcal{Q}(\boldsymbol{t}) = \mathcal{E}(\boldsymbol{q}(\boldsymbol{t}), Q(\boldsymbol{t}))$

Reach Sets

Open-loop reach set (OLRS) and closed-loop reach set (CLRS) of a system with disturbance are different

OLRS of MAXMIN Type

- Given initial set X^0, time $t > t_0$, $X^-(t,t_0,X^0)$ is the set of all x, such that for any $v(\tau) \in \mathcal{Q}(\tau)$ there exists $x^0 \in X^0$ and $u(\tau) \in \mathcal{P}(\tau)$, $t_0 \le \tau < t$, which steer the system from $x(t_0) = x^0$ to $x(t) = x$

- $X^-(t,t_0,X^0)$ is subzero level set of

$$V^-(t,x) = \max_v \min_u \{\mathbf{dist}(x(t_0), X^0) \mid x(t) = x\}$$

OLRS of MINMAX Type

- Given initial set $X^{\mathbf{0}}$, time $\boldsymbol{t} > \boldsymbol{t}_{\mathbf{0}}$, $X^{+}(\boldsymbol{t},\boldsymbol{t}_{\mathbf{0}},X^{\mathbf{0}})$ is the set of all $\boldsymbol{x}$, for which there exists $\boldsymbol{u}(\boldsymbol{\tau}) \in \mathcal{P}(\boldsymbol{\tau})$, that for all $\boldsymbol{v}(\boldsymbol{\tau}) \in \mathcal{Q}(\boldsymbol{\tau})$ assigns $\boldsymbol{x}^{\mathbf{0}} \in X^{\mathbf{0}}$ such that trajectory $\boldsymbol{x}(\boldsymbol{\tau})$, $\boldsymbol{t}_{\mathbf{0}} \leq \boldsymbol{\tau} < \boldsymbol{t}$, leads from $\boldsymbol{x}(\boldsymbol{t}_{\mathbf{0}}) = \boldsymbol{x}^{\mathbf{0}}$ to $\boldsymbol{x}(\boldsymbol{t}) = \boldsymbol{x}$

- $X^{+}(\boldsymbol{t},\boldsymbol{t}_{\mathbf{0}},X^{\mathbf{0}})$ is subzero level set of

$$V^{+}(\boldsymbol{t},\boldsymbol{x}) = \mathbf{min}_{u}\mathbf{max}_{v}\{\mathbf{dist}(\boldsymbol{x}(\boldsymbol{t}_{\mathbf{0}}), X^{\mathbf{0}}) \mid \boldsymbol{x}(\boldsymbol{t}) = \boldsymbol{x}\}$$

OLRS Properties

- MAXMIN reach set:

$$X^{-}(t, t_0, X^0) =$$
$$\left(\Phi(t, t_0)X^0 \oplus \int_{t_0}^{t} \Phi(t, \tau)B(\tau)\mathcal{P}(\tau)d\tau\right) \dot{-} \int_{t_0}^{t} \Phi(t, \tau)(-G(\tau))\mathcal{Q}(\tau)d\tau$$

- MINMAX reach set:

$$X^{+}(t, t_0, X^0) =$$
$$\left(\Phi(t, t_0)X^0 \dot{-} \int_{t_0}^{t} \Phi(t, \tau)(-G(\tau))\mathcal{Q}(\tau)d\tau\right) \oplus \int_{t_0}^{t} \Phi(t, \tau)B(\tau)\mathcal{P}(\tau)d\tau$$

- $X^{+}(\boldsymbol{t}, \boldsymbol{t_0}, X^{\mathbf{0}}) \subseteq X^{-}(\boldsymbol{t}, \boldsymbol{t_0}, X^{\mathbf{0}})$

OLRS Properties

- MAXMIN reach set:

$$X^-(t,t_0,X^0) =$$
$$\left(\Phi(t,t_0)X^0 \oplus \int_{t_0}^{t} \Phi(t,\tau)B(\tau)\mathcal{P}(\tau)d\tau\right) \dot{-} \int_{t_0}^{t} \Phi(t,\tau)(-G(\tau))\mathcal{Q}(\tau)d\tau$$

geometric difference

- MINMAX reach set:

$$X^+(t,t_0,X^0) =$$
$$\left(\Phi(t,t_0)X^0 \dot{-} \int_{t_0}^{t} \Phi(t,\tau)(-G(\tau))\mathcal{Q}(\tau)d\tau\right) \oplus \int_{t_0}^{t} \Phi(t,\tau)B(\tau)\mathcal{P}(\tau)d\tau$$

- $X^+(t,t_0,X^0) \subseteq X^-(t,t_0,X^0)$

Sequential MAXMIN

- Correction at t_1: $[t_0,\ t] = [t_0,\ t_1] \cup [t_1,\ t]$

$$X^-_1(t,t_0,X^0) = X^-(t,t_1,X^-(t_1,t_0,X^0))$$

$$X^-_1(t,t_0,X^0) \subseteq X^-(t,t_0,X^0)$$

- k corrections: $t_0 \le t_1 \le \dots \le t_k \le t$

$$X^-_k(t,t_0,X^0) = X^-(t,t_k,X^-_{k-1}(t_1,t_0,X^0))$$

$$X^-_k(t,t_0,X^0) \subseteq \dots \subseteq X^-_1(t,t_0,X^0) \subseteq X^-(t,t_0,X^0)$$

Sequential MINMAX

- Correction at $\boldsymbol{t}_1$: $[\boldsymbol{t}_\mathbf{0},\ \boldsymbol{t}] = [\boldsymbol{t}_\mathbf{0},\ \boldsymbol{t}_1] \cup [\boldsymbol{t}_1,\ \boldsymbol{t}]$

$$X^+{}_1(\boldsymbol{t}, \boldsymbol{t}_\mathbf{0}, X^\mathbf{0}) = X^+(\boldsymbol{t}, \boldsymbol{t}_1, X^+(\boldsymbol{t}_1, \boldsymbol{t}_\mathbf{0}, X^\mathbf{0}))$$

$$X^+(\boldsymbol{t}, \boldsymbol{t}_\mathbf{0}, X^\mathbf{0}) \subseteq X^+{}_1(\boldsymbol{t}, \boldsymbol{t}_\mathbf{0}, X^\mathbf{0})$$

- $\boldsymbol{k}$ corrections: $\boldsymbol{t}_\mathbf{0} \le \boldsymbol{t}_1 \le \ldots \le \boldsymbol{t}_{\boldsymbol{k}} \le \boldsymbol{t}$

$$X^+{}_{\boldsymbol{k}}(\boldsymbol{t}, \boldsymbol{t}_\mathbf{0}, X^\mathbf{0}) = X^+(\boldsymbol{t}, \boldsymbol{t}_{\boldsymbol{k}}, X^+{}_{\boldsymbol{k}\text{-}1}(\boldsymbol{t}_1, \boldsymbol{t}_\mathbf{0}, X^\mathbf{0}))$$

$$X^+(\boldsymbol{t}, \boldsymbol{t}_\mathbf{0}, X^\mathbf{0}) \subseteq X^+{}_1(\boldsymbol{t}, \boldsymbol{t}_\mathbf{0}, X^\mathbf{0}) \subseteq \ldots \subseteq X^+{}_{\boldsymbol{k}}(\boldsymbol{t}, \boldsymbol{t}_\mathbf{0}, X^\mathbf{0})$$

Piecewise Open-Loop

$$X^{+}(\boldsymbol{t},\boldsymbol{t}_{\mathbf{0}},X^{\mathbf{0}}) \subseteq X^{+}{}_{\boldsymbol{k}}(\boldsymbol{t},\boldsymbol{t}_{\mathbf{0}},X^{\mathbf{0}}) \subseteq X^{-}{}_{\boldsymbol{k}}(\boldsymbol{t},\boldsymbol{t}_{\mathbf{0}},X^{\mathbf{0}}) \subseteq X^{-}(\boldsymbol{t},\boldsymbol{t}_{\mathbf{0}},X^{\mathbf{0}})$$

Piecewise Open-Loop

$$\boxed{X^{+}(t,t_0,X^0) \subseteq X^{+}_{k}(t,t_0,X^0)} \subseteq \boxed{X^{-}_{k}(t,t_0,X^0) \subseteq X^{-}(t,t_0,X^0)}$$

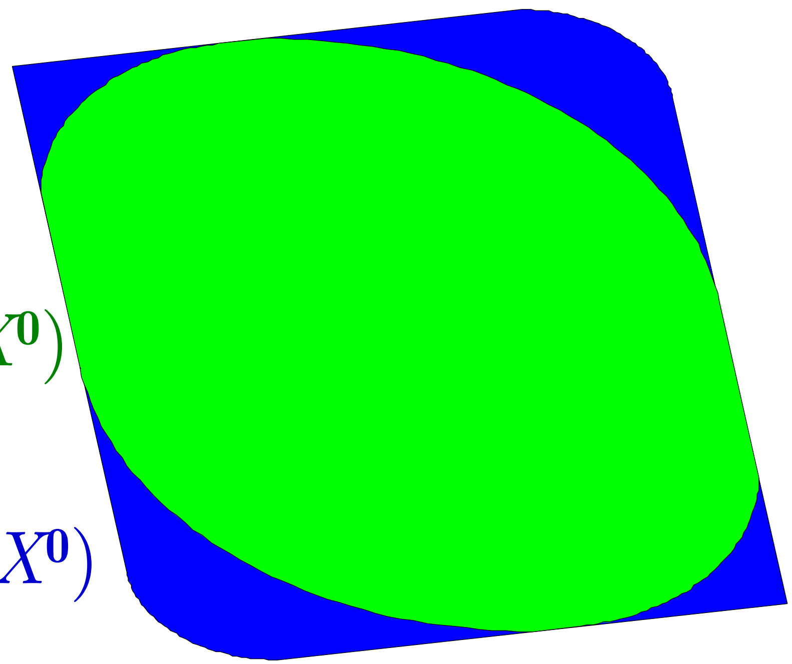
MINMAX MAXMIN

Piecewise Open-Loop

$$\boxed{X^{+}(t,t_0,X^0) \subseteq X^{+}{}_{k}(t,t_0,X^0)} \subseteq \boxed{X^{-}{}_{k}(t,t_0,X^0) \subseteq X^{-}(t,t_0,X^0)}$$

MINMAX MAXMIN

$X^{+}{}_{k}(t,t_0,X^0)$



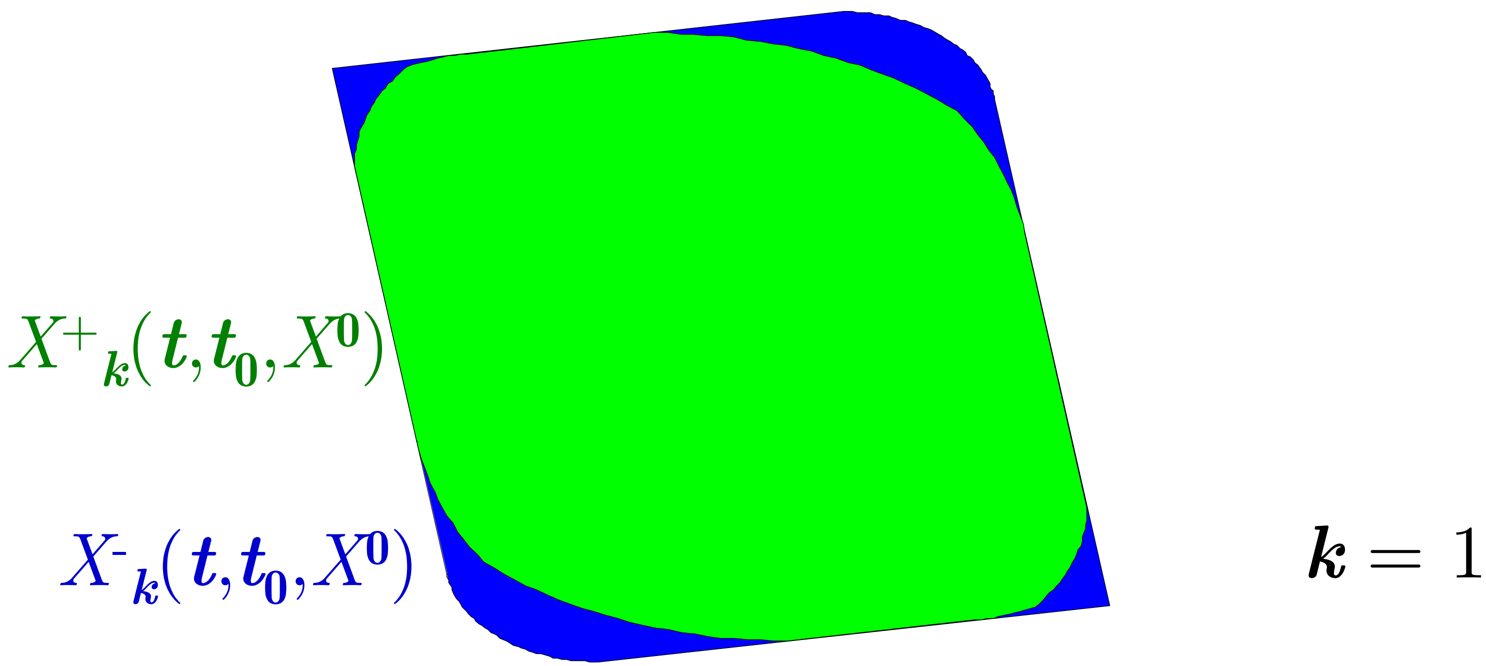
$X^{-}{}_{k}(t,t_0,X^0)$

$k = 0$

Piecewise Open-Loop

$$\boxed{X^{+}(\boldsymbol{t},\boldsymbol{t_0},X^{\mathbf{0}}) \subseteq X^{+}{}_{\boldsymbol{k}}(\boldsymbol{t},\boldsymbol{t_0},X^{\mathbf{0}})} \subseteq \boxed{X^{-}{}_{\boldsymbol{k}}(\boldsymbol{t},\boldsymbol{t_0},X^{\mathbf{0}}) \subseteq X^{-}(\boldsymbol{t},\boldsymbol{t_0},X^{\mathbf{0}})}$$

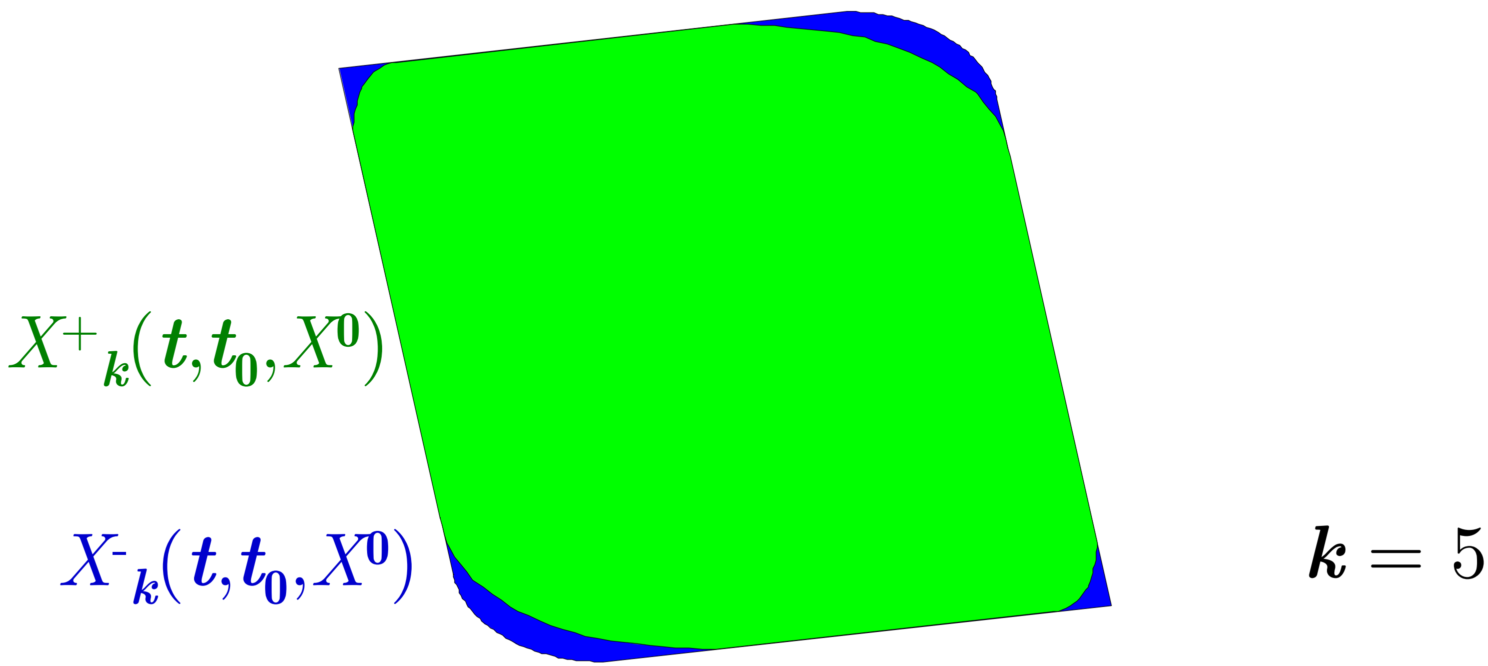
MINMAX MAXMIN



Piecewise Open-Loop

$$\boxed{X^{+}(t,t_0,X^0) \subseteq X^{+}{}_k(t,t_0,X^0)} \subseteq \boxed{X^{-}{}_k(t,t_0,X^0) \subseteq X^{-}(t,t_0,X^0)}$$

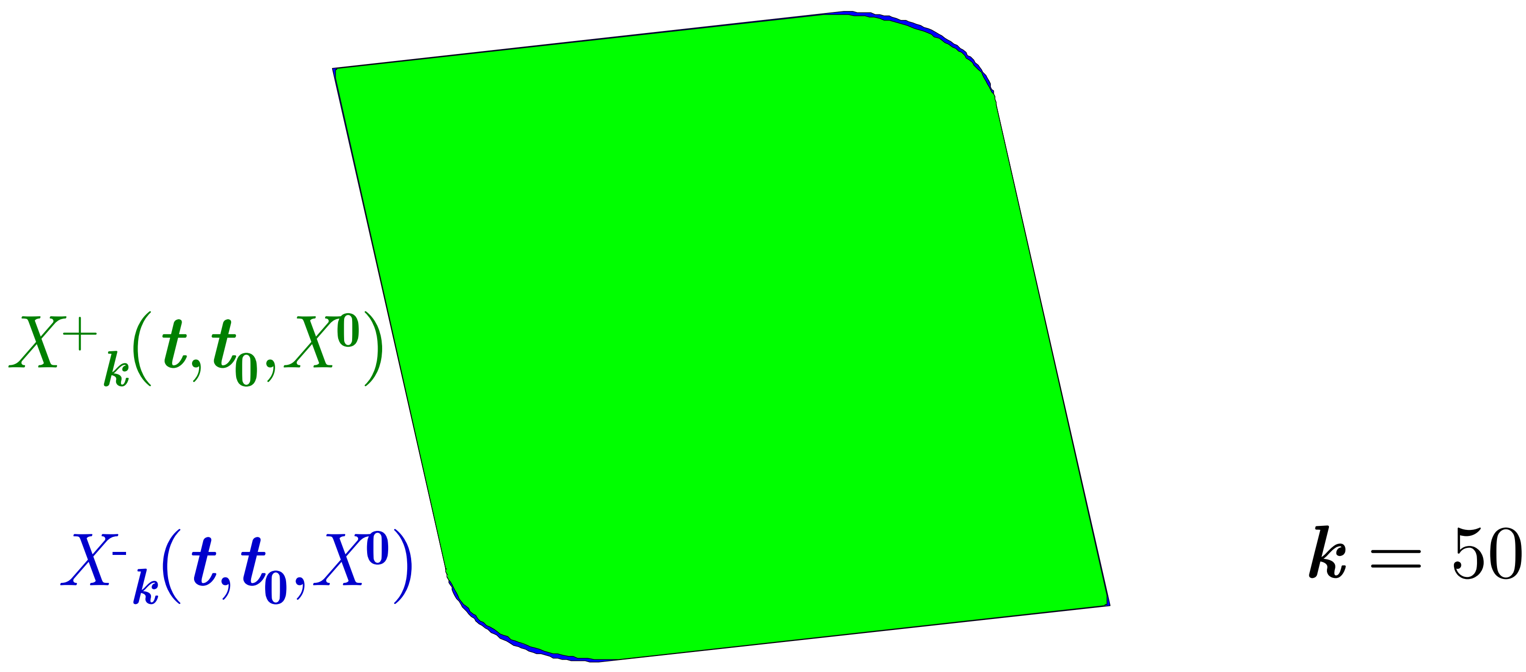
MINMAX MAXMIN



Piecewise Open-Loop

$$\boxed{X^{+}(t,t_0,X^0) \subseteq X^{+}{}_{k}(t,t_0,X^0)} \subseteq \boxed{X^{-}{}_{k}(t,t_0,X^0) \subseteq X^{-}(t,t_0,X^0)}$$

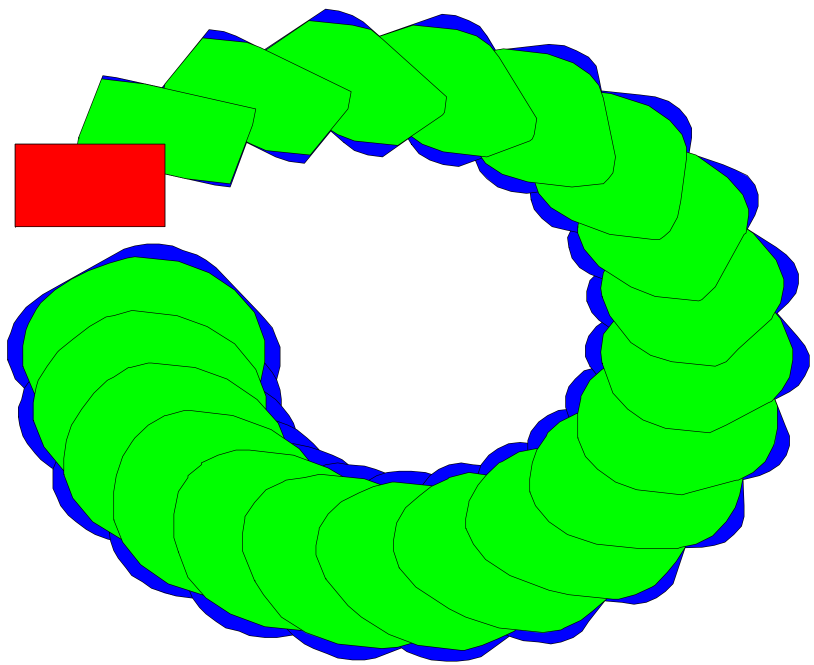
MINMAX MAXMIN



Piecewise Open-Loop

$$\boxed{X^{+}(t,t_0,X^0) \subseteq X^{+}{}_{k}(t,t_0,X^0)} \subseteq \boxed{X^{-}{}_{k}(t,t_0,X^0) \subseteq X^{-}(t,t_0,X^0)}$$

MINMAX MAXMIN

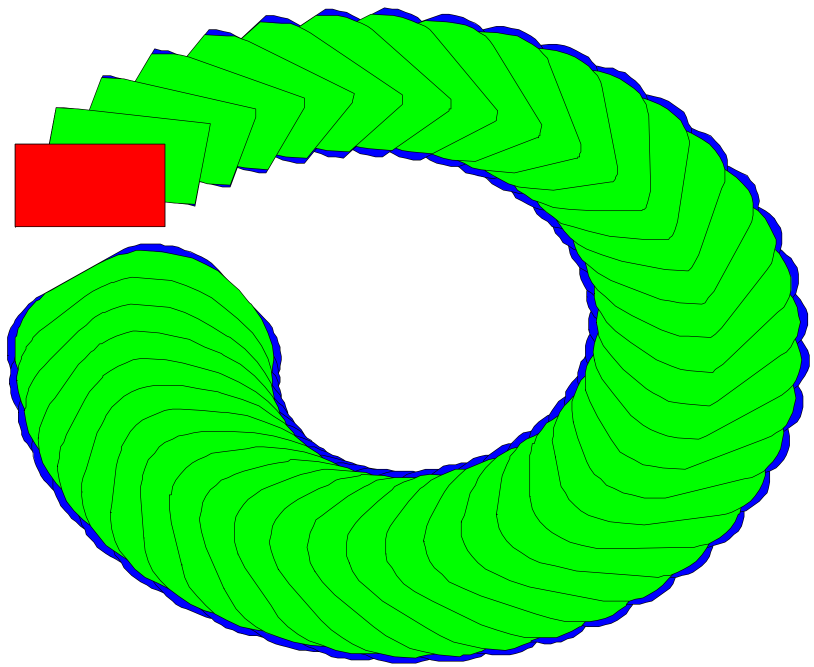


Computed in *MPT*

Piecewise Open-Loop

$$\boxed{X^{+}(t,t_0,X^0) \subseteq X^{+}_{k}(t,t_0,X^0)} \subseteq \boxed{X^{-}_{k}(t,t_0,X^0) \subseteq X^{-}(t,t_0,X^0)}$$

MINMAX MAXMIN

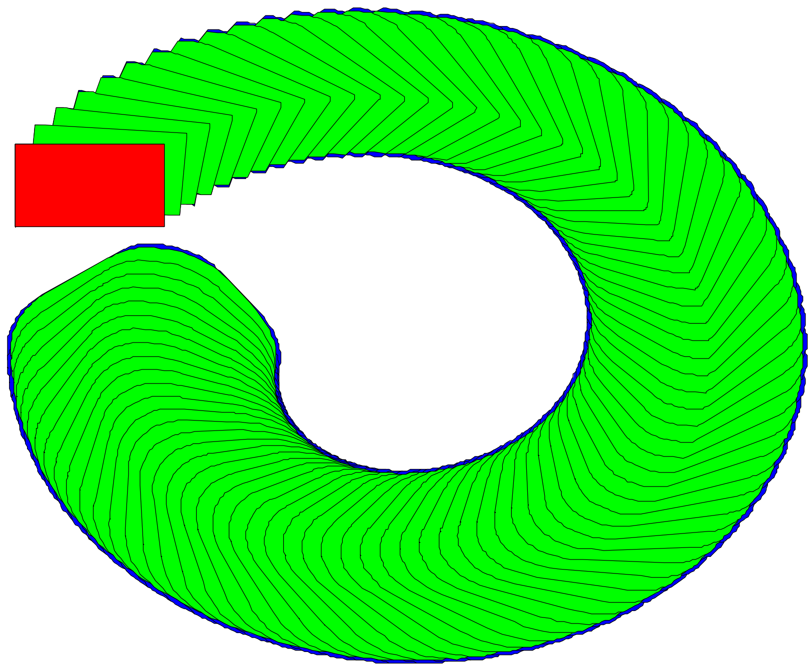


Computed in *MPT*

Piecewise Open-Loop

$$\boxed{X^{+}(t,t_0,X^0) \subseteq X^{+}{}_{k}(t,t_0,X^0)} \subseteq \boxed{X^{-}{}_{k}(t,t_0,X^0) \subseteq X^{-}(t,t_0,X^0)}$$

MINMAX MAXMIN



Computed in *MPT*

Piecewise Open-Loop

$$\boxed{X^{+}(\boldsymbol{t},\boldsymbol{t_0},X^{\mathbf{0}}) \subseteq X^{+}{}_{\boldsymbol{k}}(\boldsymbol{t},\boldsymbol{t_0},X^{\mathbf{0}})} \subseteq \boxed{X^{-}{}_{\boldsymbol{k}}(\boldsymbol{t},\boldsymbol{t_0},X^{\mathbf{0}}) \subseteq X^{-}(\boldsymbol{t},\boldsymbol{t_0},X^{\mathbf{0}})}$$

MINMAX MAXMIN

$$\boldsymbol{k} \rightarrow \infty$$

Piecewise Open-Loop

$$\boxed{X^+(t,t_0,X^0) \subseteq X^+_k(t,t_0,X^0)} \subseteq \boxed{X^-_k(t,t_0,X^0) \subseteq X^-(t,t_0,X^0)}$$

MINMAX MAXMIN

$$k\to\infty$$

$$X^+_\infty(t,t_0,X^0) = X^-_\infty(t,t_0,X^0) = X(t,t_0,X^0)$$

- *On reachability under uncertainty*
  by A.B.Kurzhanskiy, P.Varaiya

Piecewise Open-Loop

$$\boxed{X^{+}(t,t_0,X^0) \subseteq X^{+}_{k}(t,t_0,X^0)} \subseteq \boxed{X^{-}_{k}(t,t_0,X^0) \subseteq X^{-}(t,t_0,X^0)}$$

MINMAX MAXMIN

$$k \rightarrow \infty$$

$$X^{+}_{\infty}(t,t_0,X^0) = X^{-}_{\infty}(t,t_0,X^0) = \boxed{X(t,t_0,X^0)}$$

CLRS

- *On reachability under uncertainty*

  by A.B.Kurzhanskiy, P.Varaiya

Closed-Loop Reach Set (CLRS)

Given initial set X^0, time $t > t_0$,

$X(t, t_0, X^0)$ is the set of all x, for each of which there exist $x^0 \in X^0$ and $u(\tau, x(\tau)) \in \mathcal{P}(\tau)$ that for every $v(\tau) \in \mathcal{Q}(\tau)$ assigns trajectory $x(\tau)$:

$$\dot{x}(\tau) \in A(\tau)x(\tau) + B(\tau)u(\tau, x(\tau)) + G(\tau)v(\tau)$$

where $t_0 \leq \tau < t$, such that $x(t_0) = x_0$ and $x(t) = x$

CLRS Computation

- Tight ellipsoidal approximations for $X(\boldsymbol{t},\boldsymbol{t_0},X^{\mathbf{0}})$:

$$X(\boldsymbol{t},\boldsymbol{t_0},X^{\mathbf{0}}) = \bigcap \mathcal{E}(\boldsymbol{x}_{\mathrm{c}}(\boldsymbol{t}),X_{\boldsymbol{l}}^{+}(\boldsymbol{t})) = \bigcup \mathcal{E}(\boldsymbol{x}_{\mathrm{c}}(\boldsymbol{t}),X_{\boldsymbol{l}}^{-}(\mathrm{t}))$$

where $\boldsymbol{x}_{\mathrm{c}}(\boldsymbol{t})$ satisfies

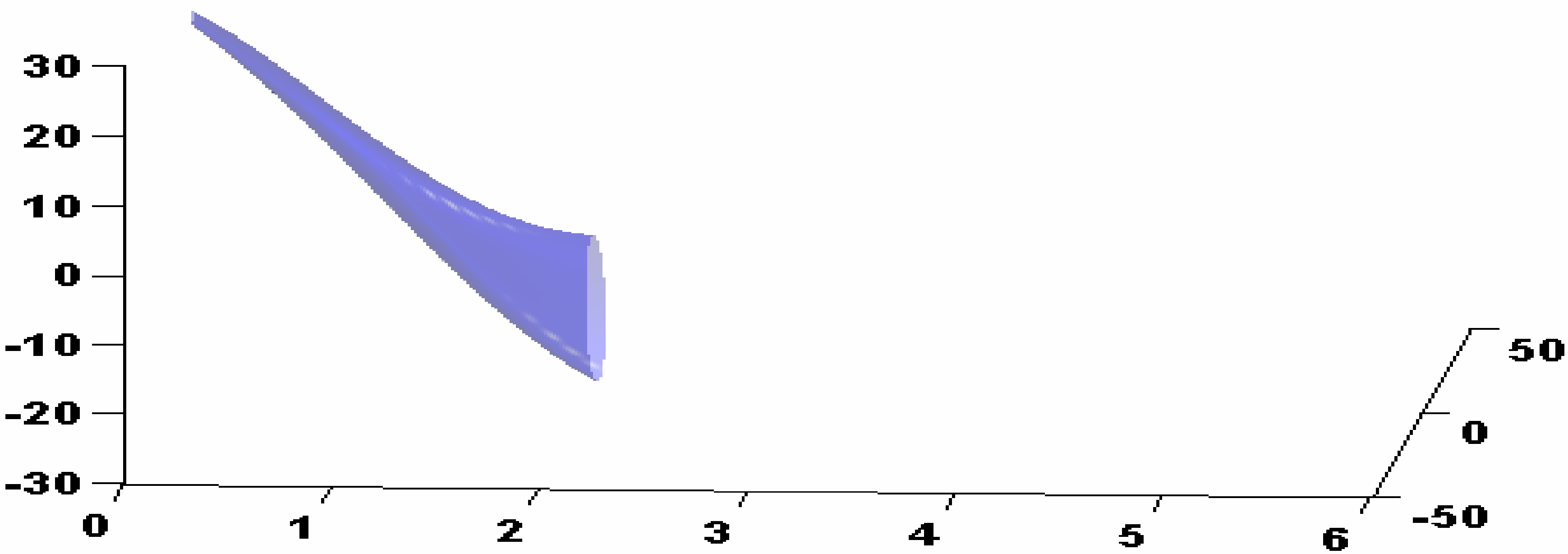
$$\dot{\boldsymbol{x}}_{\mathrm{c}}(\boldsymbol{t}) = A(\boldsymbol{t})\boldsymbol{x}(\boldsymbol{t}) + B(\boldsymbol{t})\boldsymbol{p}(\boldsymbol{t}) + G(\boldsymbol{t})\boldsymbol{q}(\boldsymbol{t})$$

and $X_{\boldsymbol{l}}^{+}(\boldsymbol{t})$, $X_{\boldsymbol{l}}^{-}(\boldsymbol{t})$ are obtained from ODEs

- Implemented in $\boldsymbol{ET}$

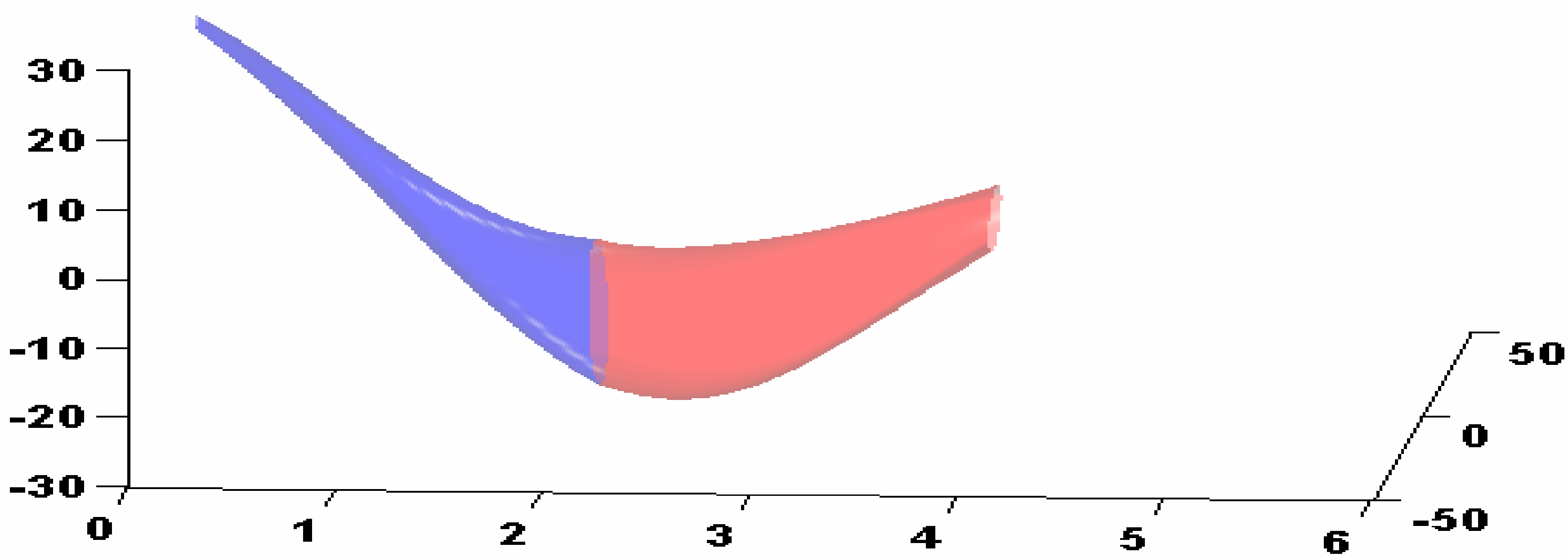
Example

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -0.2 & 1 \\ -1 & -0.2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v(t)$$



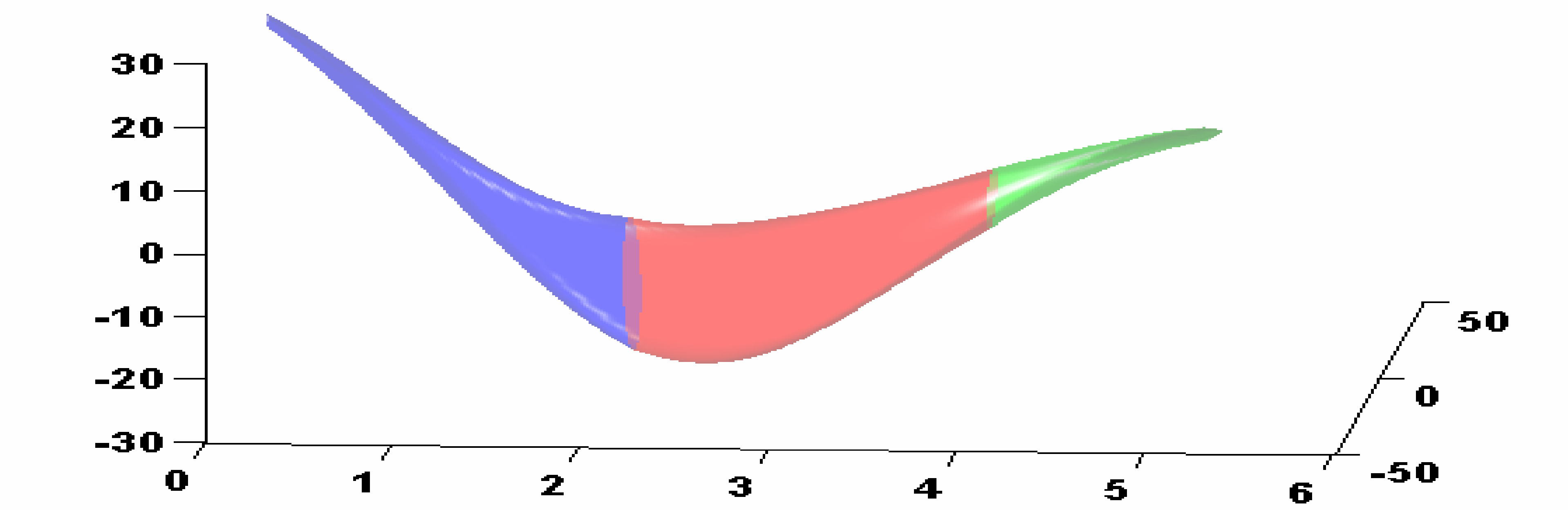
Example

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -0.2 & 1 \\ -1 & -0.2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v(t)$$



Example

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -0.2 & 1 \\ -1 & -0.2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v(t)$$



Steering the System to a Target

SPRING-MASS SYSTEM

- *Reachability approaches and ellipsoidal techniques for closed-loop control of oscillating systems under uncertainty*

  by A.N.Daryin, A.B.Kurzhanski, I.V.Vostrikov

Outline

- Problem setting and basic definitions
- Overview of existing methods and tools
- Ellipsoidal approach
- Systems with disturbances
- Hybrid systems
- Summary and outlook

Hybrid Setting

- Discrete states (modes)
- Continuous dynamics – affine
- Enabling zones (guards) – hyperplanes, ellipsoids, polyhedra
- Resets – affine

- No Zeno

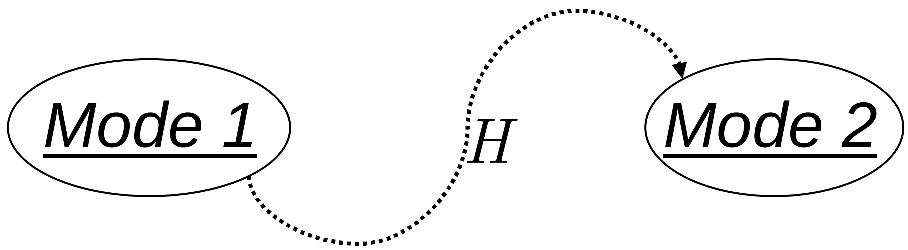
Hybrid System Example

- *Mode 1*:

$$\dot{\boldsymbol{x}}(\boldsymbol{t}) = A_1\boldsymbol{x}(\boldsymbol{t})+B_1\boldsymbol{u}(\boldsymbol{t}),\ \boldsymbol{u}(\boldsymbol{t})\in\mathcal{P}_1(\boldsymbol{t})$$

- *Mode 2*:

$$\dot{\boldsymbol{x}}(\boldsymbol{t}) = A_2\boldsymbol{x}(\boldsymbol{t})+B_2\boldsymbol{u}(\boldsymbol{t})+G_2\boldsymbol{v}(\boldsymbol{t}),\ \boldsymbol{u}(\boldsymbol{t})\in\mathcal{P}_2(\boldsymbol{t}),\ \boldsymbol{v}(\boldsymbol{t})\in\ \mathcal{Q}_2(\mathrm{t})$$

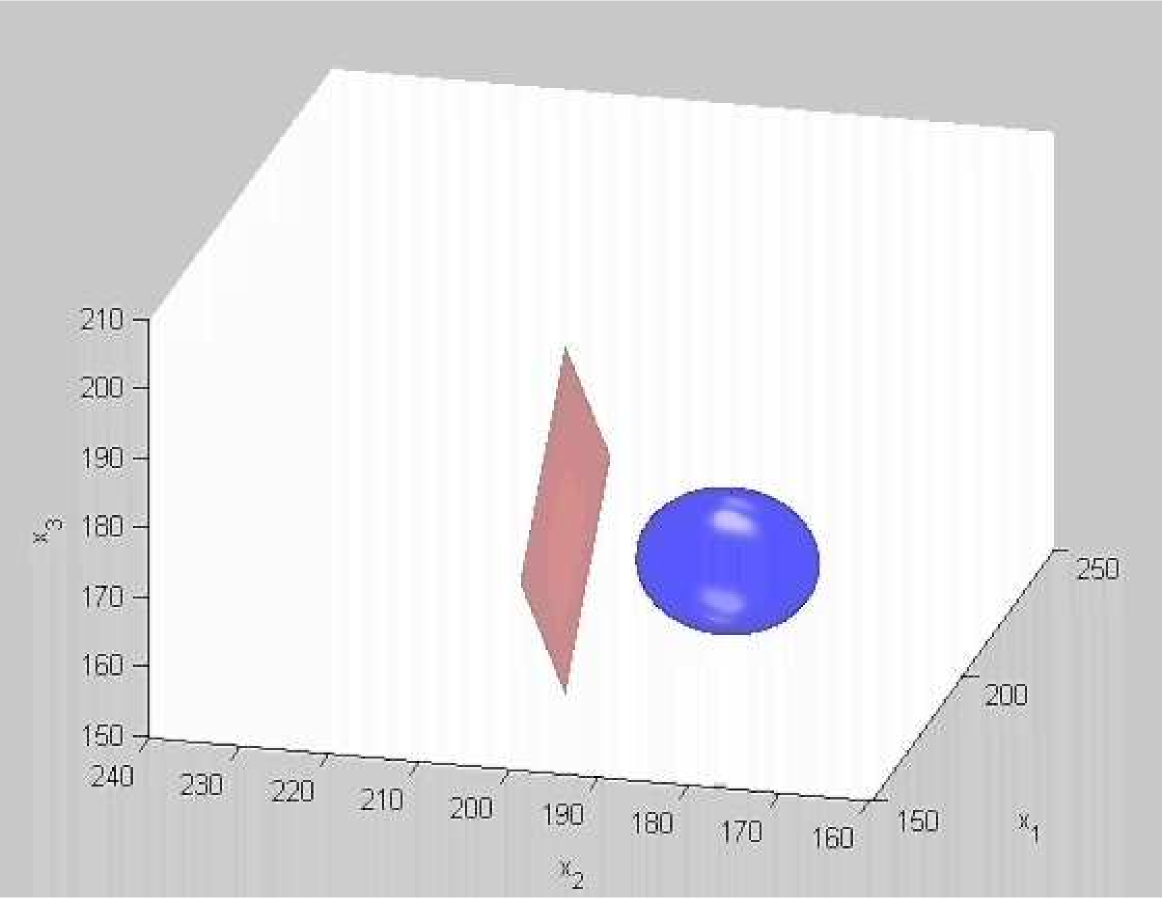


- Guard: hyperplane H
- Reset: identity

Hybrid Reach Set Computation

- Initial conditions: ***Mode 1***, $\boldsymbol{t_0}$, $X^{\mathbf{0}}$
- Compute reach set for ***Mode 1***: $X_1(\boldsymbol{t},\boldsymbol{t_0},X^{\mathbf{0}})$
- Detect when $X_1(\boldsymbol{\tau},\boldsymbol{t_0},X^{\mathbf{0}}) \cap H \neq \emptyset$, $\boldsymbol{t_0} \leq \boldsymbol{\tau} \leq \boldsymbol{t}$

Guard Detection

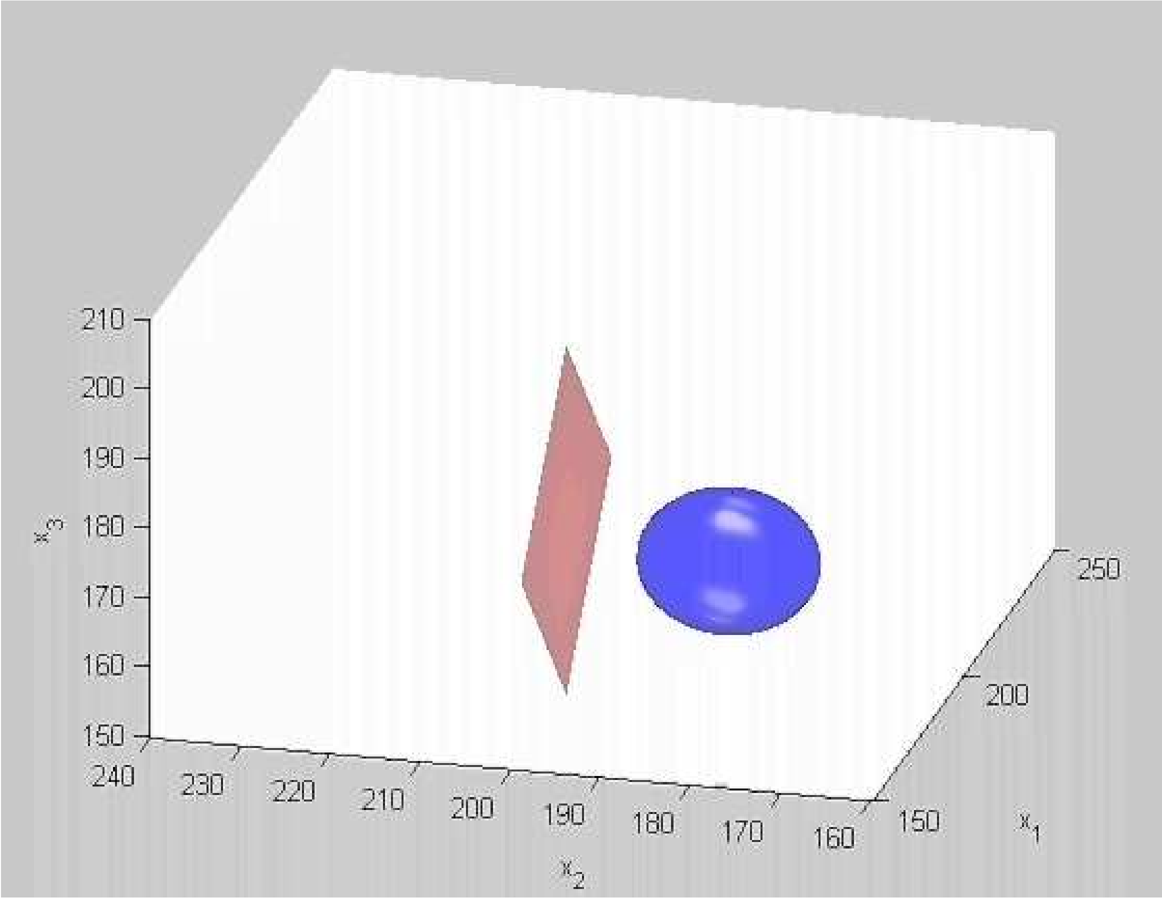


Hybrid Reach Set Computation

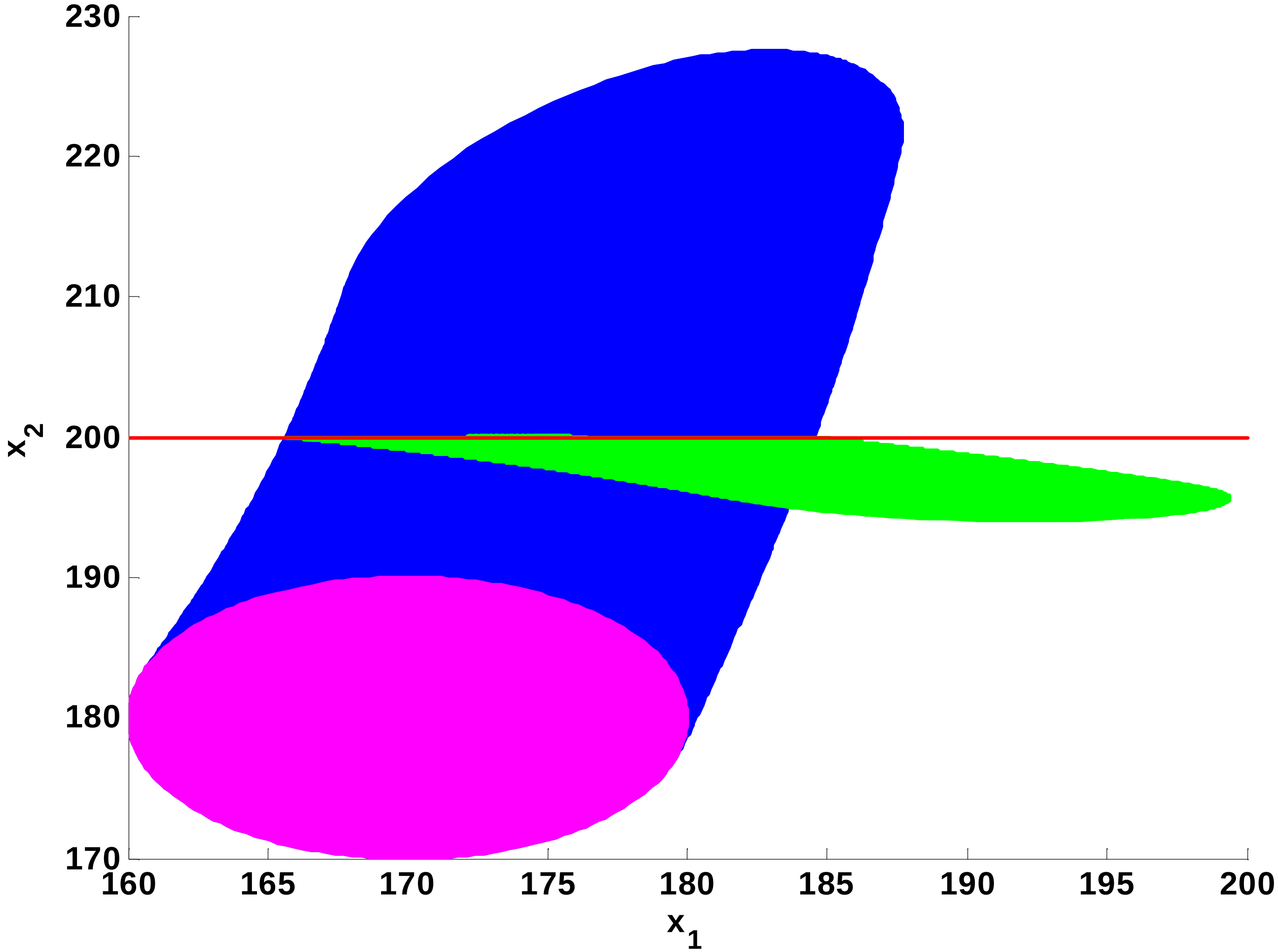
- Initial conditions: ***Mode 1***, $\boldsymbol{t_0},\ X^{\mathbf{0}}$
- Compute reach set for ***Mode 1***: $X_1(\boldsymbol{t},\boldsymbol{t_0},X^{\mathbf{0}})$
- Detect when $X_1(\boldsymbol{\tau},\boldsymbol{t_0},X^{\mathbf{0}}) \cap H \neq \emptyset,\ \boldsymbol{t_0} \leq \boldsymbol{\tau} \leq \boldsymbol{t}$
- For each such $\boldsymbol{\tau}$, compute reach set for ***Mode 2***: $X_2(\boldsymbol{t},\boldsymbol{\tau},(X_1(\boldsymbol{\tau},\boldsymbol{t_0},X^{\mathbf{0}}) \cap H))$
- Reach set of the whole system:

$$X_1(\boldsymbol{t},\boldsymbol{t_0},X^{\mathbf{0}}) \bigcup_{\boldsymbol{\tau}} X_2(\boldsymbol{t},\boldsymbol{\tau},(X_1(\boldsymbol{\tau},\boldsymbol{t_0},X^{\mathbf{0}}) \cap H))$$

Hybrid Reach Set

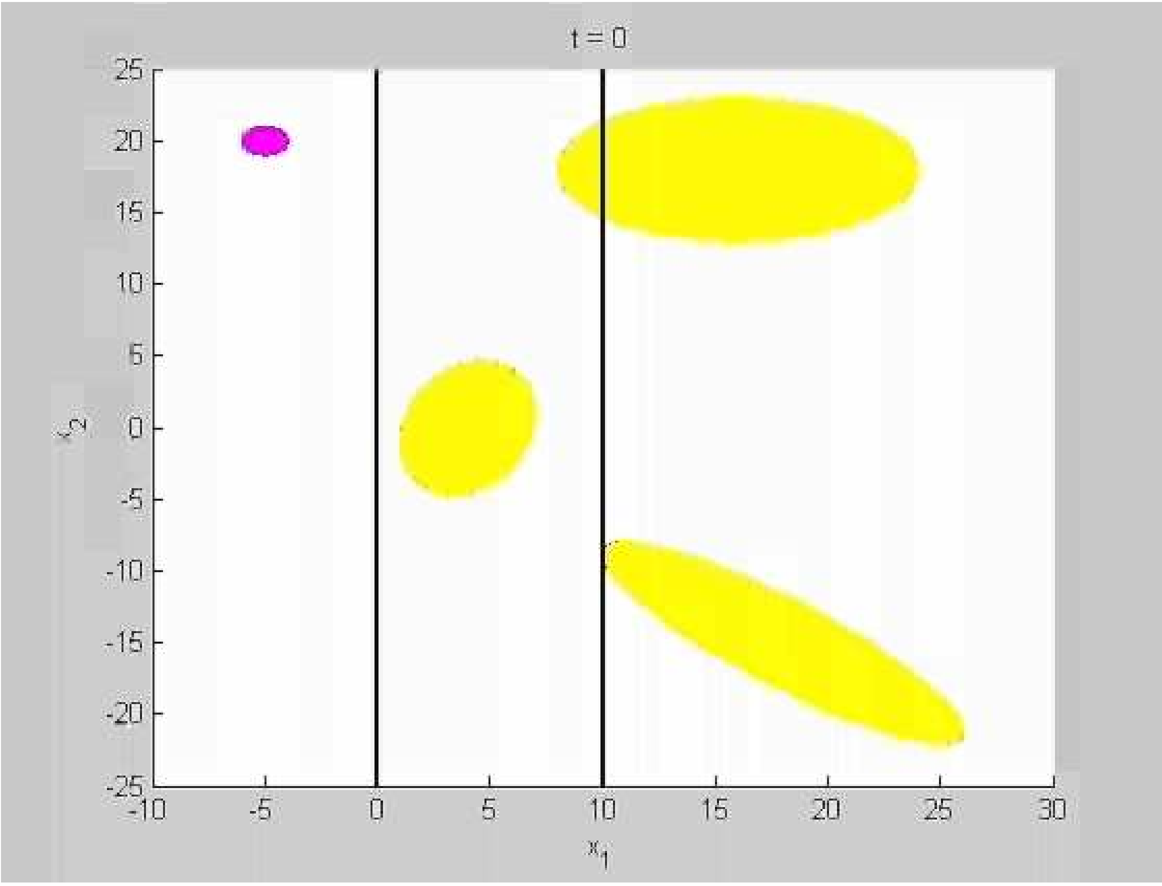


Reach Set Trace Projection



2D

Hybrid Reach Set (concept)



Outline

- Problem setting and basic definitions
- Overview of existing methods and tools
- Ellipsoidal approach
- Systems with disturbances
- Hybrid systems
- Summary and outlook

Road Ahead

- State estimation
- Discrete-time systems with disturbance
- Obstacle problems
- Stochastic systems

Discrete-Time Systems

$$\boldsymbol{x}[\boldsymbol{k}+1] = A[\boldsymbol{k}]\boldsymbol{x}[\boldsymbol{k}] + B[\boldsymbol{k}]\boldsymbol{u}[\boldsymbol{k}]$$

$$\boldsymbol{x}[\boldsymbol{k}_0] \in \mathcal{E}(\boldsymbol{x}_0, X_0),\ \boldsymbol{u}[\boldsymbol{k}] \in \mathcal{E}(\boldsymbol{p}[\boldsymbol{k}], P[\boldsymbol{k}])$$

- Same ellipsoidal theory applies with some adjustments
  - Controllability

    CT: $\int_{t_0}^{t} \Phi(t,\tau)(BPB^T)(\tau)\Phi^T(t,\tau)d\tau > 0$ for all $\boldsymbol{t} > \boldsymbol{t}_0$

    DT: $\sum_{i=k_0}^{k-1} \Phi(k,i+1)(BPB^T)[i]\Phi^T(k,i+1) > 0$ for some $\boldsymbol{k} > \boldsymbol{k}_0$

  - Singular $A[\boldsymbol{k}]$

Regularization

- $A_{\delta}[\boldsymbol{k}] = A[\mathrm{k}] + \boldsymbol{\delta} U V^{\mathrm{T}}$
  (U, V come from SVD of $A[k]$)

Regularization

- $A_{\boldsymbol{\delta}}[\boldsymbol{k}] = A[\mathrm{k}] + \boldsymbol{\delta} UV^{\mathrm{T}}$

  (U, V come from SVD of $A[k]$)
- $(BPB^{\mathrm{T}})_{\boldsymbol{\alpha}}[\boldsymbol{k}] = (BPB^{\mathrm{T}})[\boldsymbol{k}] + \boldsymbol{\alpha} I$

Regularization

- $A_{\boldsymbol{\delta}}[\boldsymbol{k}] = A[\mathrm{k}] + \boldsymbol{\delta} UV^{\mathrm{T}}$

  (U, V come from SVD of $A[k]$)
- $(BPB^{\mathrm{T}})_{\boldsymbol{\alpha}}[\boldsymbol{k}] = (BPB^{\mathrm{T}})[\boldsymbol{k}] + \boldsymbol{\alpha} I$
- Compute reach set $X_{\boldsymbol{\alpha},\boldsymbol{\delta}}(\boldsymbol{k}, \boldsymbol{k_0}, X^{\mathbf{0}})$ for $A_{\boldsymbol{\delta}}[\boldsymbol{k}]$ and $(BPB^{\mathrm{T}})_{\boldsymbol{\alpha}}[\boldsymbol{k}]$

Regularization

- $A_{\boldsymbol{\delta}}[\boldsymbol{k}] = A[\mathrm{k}] + \boldsymbol{\delta} UV^{\mathrm{T}}$

  (U, V come from SVD of $A[k]$)
- $(BPB^{\mathrm{T}})_{\boldsymbol{\alpha}}[\boldsymbol{k}] = (BPB^{\mathrm{T}})[\boldsymbol{k}] + \boldsymbol{\alpha} I$
- Compute reach set $X_{\boldsymbol{\alpha},\boldsymbol{\delta}}(\boldsymbol{k}, \boldsymbol{k}_{\mathbf{0}}, X^{\mathbf{0}})$ for $A_{\boldsymbol{\delta}}[\boldsymbol{k}]$ and $(BPB^{\mathrm{T}})_{\boldsymbol{\alpha}}[\boldsymbol{k}]$
- For any $\boldsymbol{\varepsilon} > 0$ and given $\boldsymbol{k}$ there exist $\boldsymbol{\alpha}$ and $\boldsymbol{\delta}$:

$$0 < \boldsymbol{\rho}(\boldsymbol{l} \,|\, X_{\boldsymbol{\alpha},\boldsymbol{\delta}}(\boldsymbol{k}, \boldsymbol{k}_{\mathbf{0}}, X^{\mathbf{0}})) - \boldsymbol{\rho}(\boldsymbol{l} \,|\, X(\boldsymbol{k}, \boldsymbol{k}_{\mathbf{0}}, X^{\mathbf{0}})) < \boldsymbol{\varepsilon}$$

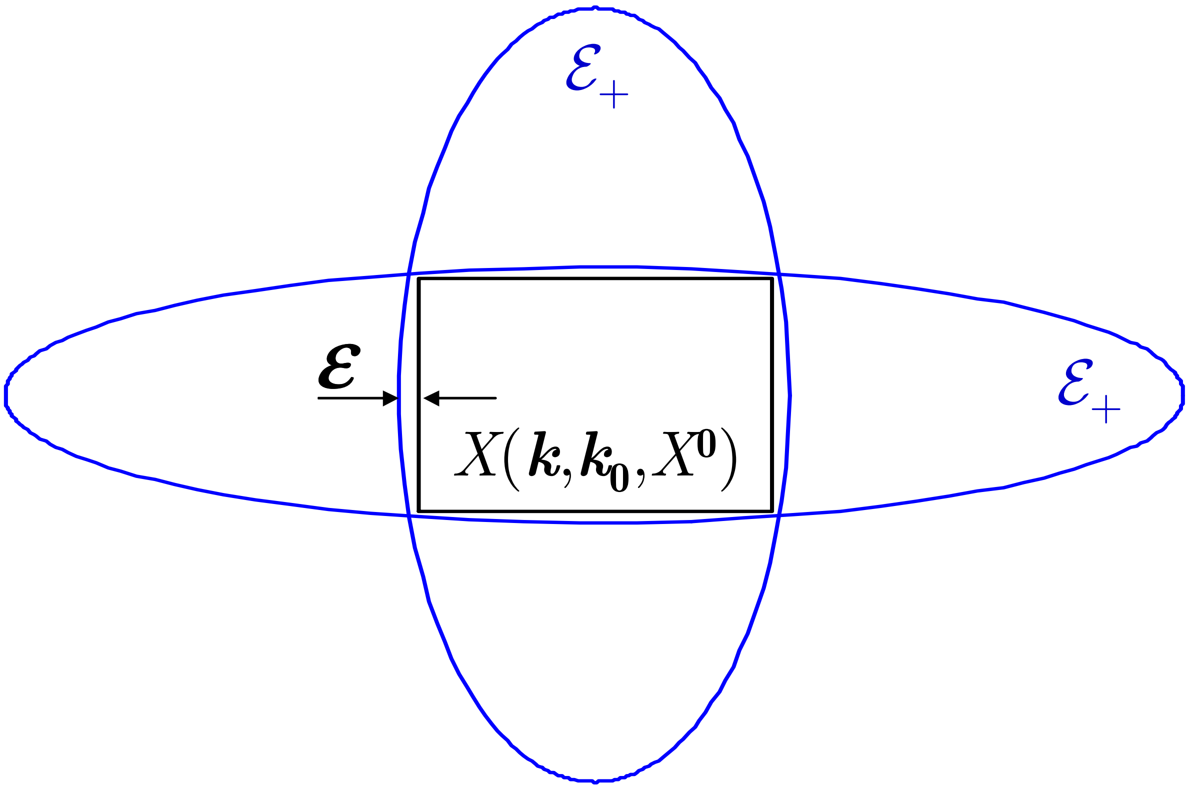
for all $\boldsymbol{l}$, $\langle \boldsymbol{l}, \boldsymbol{l} \rangle = 1$

Regularization (illustration)

$$\boxed{X(\boldsymbol{k}, \boldsymbol{k_0}, X^{\mathbf{0}})}$$

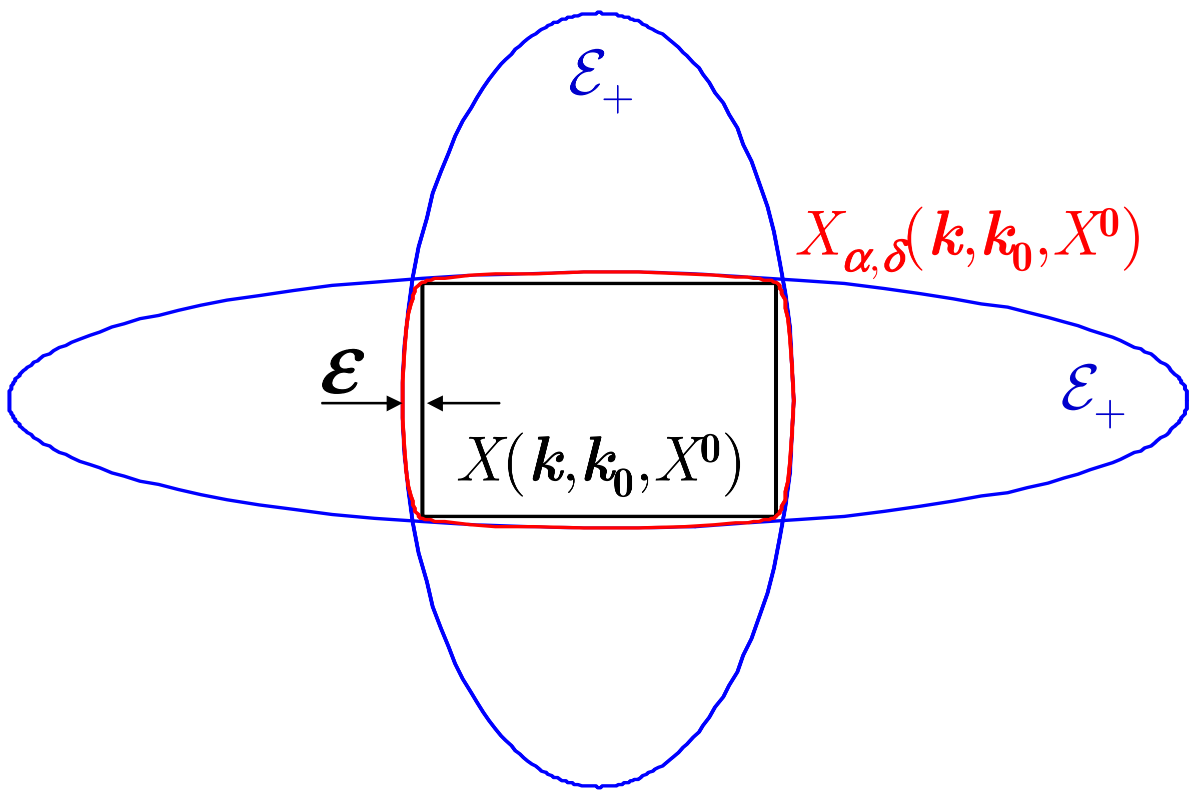
- *Ellipsoidal techniques for reachability analysis of discrete-time linear systems* by A.A.Kurzhanskiy, P.Varaiya (2005)

Regularization (illustration)



- *Ellipsoidal techniques for reachability analysis of discrete-time linear systems* by A.A.Kurzhanskiy, P.Varaiya (2005)

Regularization (illustration)



- *Ellipsoidal techniques for reachability analysis of discrete-time linear systems* by A.A.Kurzhanskiy, P.Varaiya (2005)

ICM

Value Functions

- Correction at t_1: $[t_0,\ t] = [t_0,\ t_1] \cup [t_1,\ t]$

$$V^-_1(t,x) = \mathbf{max}_v \mathbf{min}_u \{\, V^-(t_1, x(t_1)) \mid x(t) = x \}$$

$$V^+_1(t,x) = \mathbf{min}_u \mathbf{max}_v \{\, V^+(t_1, x(t_1)) \mid x(t) = x \}$$

- k corrections: $t_0 \le t_1 \le \ldots \le t_k \le t$

$$V^-_k(t,x) = \mathbf{max}_v \mathbf{min}_u \{\, V^-_{k\text{-}1}(t_k, x(t_k)) \mid x(t) = x \}$$

$$V^+_k(t,x) = \mathbf{min}_u \mathbf{max}_v \{\, V^+_{k\text{-}1}(t_k, x(t_k)) \mid x(t) = x \}$$

CLRS as Level Set of HJBI Solution

- Value function:

$$V(\boldsymbol{t},\boldsymbol{x}) = V^{-}_{\infty}(\boldsymbol{t},\boldsymbol{x}) = V^{+}_{\infty}(\boldsymbol{t},\boldsymbol{x}) = \mathbf{dist}(\boldsymbol{x}, X(\boldsymbol{t},\boldsymbol{t_0},X^{\mathbf{0}}))$$

- Hamilton-Jacobi-Bellman-Isaacs equation:

$$V_{\boldsymbol{t}} + \mathbf{max}_{\boldsymbol{u}}\mathbf{min}_{\boldsymbol{v}}\langle D_{\boldsymbol{x}}V,\ A(\boldsymbol{t})\boldsymbol{x} + B(\boldsymbol{t})\boldsymbol{u} + G(\boldsymbol{t})\boldsymbol{v}\rangle = 0$$

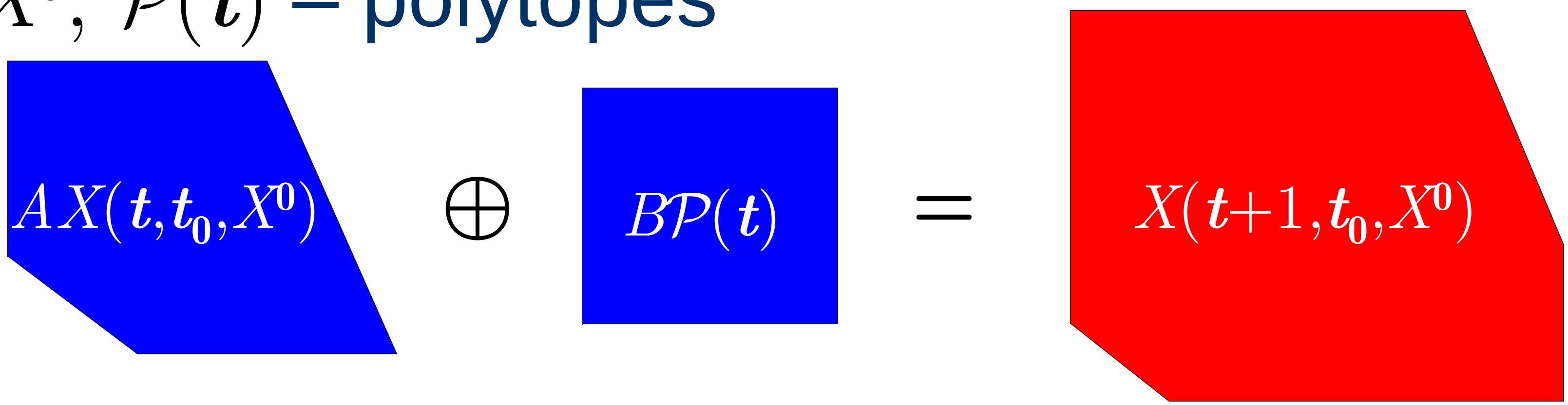
with initial condition

$$V(\boldsymbol{t_0},\boldsymbol{x}) = \mathbf{dist}(\boldsymbol{x}, X^{\mathbf{0}})$$

- Reach set: $X(\boldsymbol{t},\boldsymbol{t_0},X^{\mathbf{0}}) = \{\boldsymbol{x} \mid V(\boldsymbol{t},\boldsymbol{x}) \leq 0\}$

Polytopes (*MPT*)

- $X^{\mathbf{0}}$, $\mathcal{P}(\boldsymbol{t})$ – polytopes



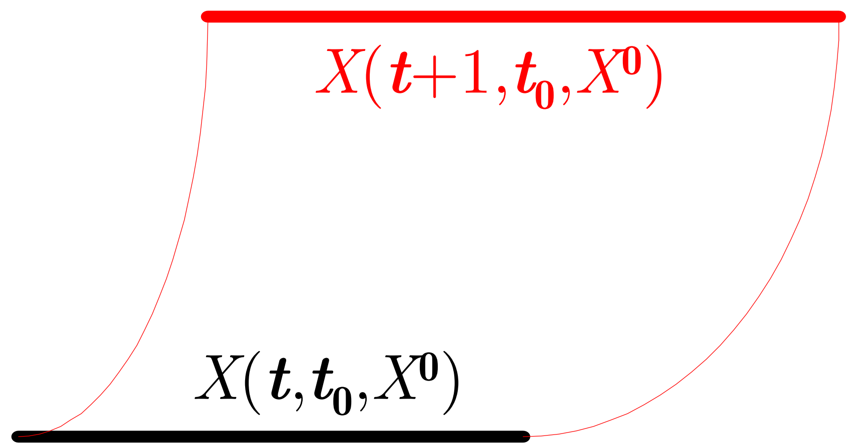
- Convex hull computation complexity:

$$(\text{number\_of\_vertices})^{\mathbf{n}}$$

- *Multi-Parametric Toolbox (MPT)*
  by M.Kvasnica, P.Grieder, M.Baotić, M.Morari (2004)
- `control.ee.ethz.ch/~mpt`

BACK

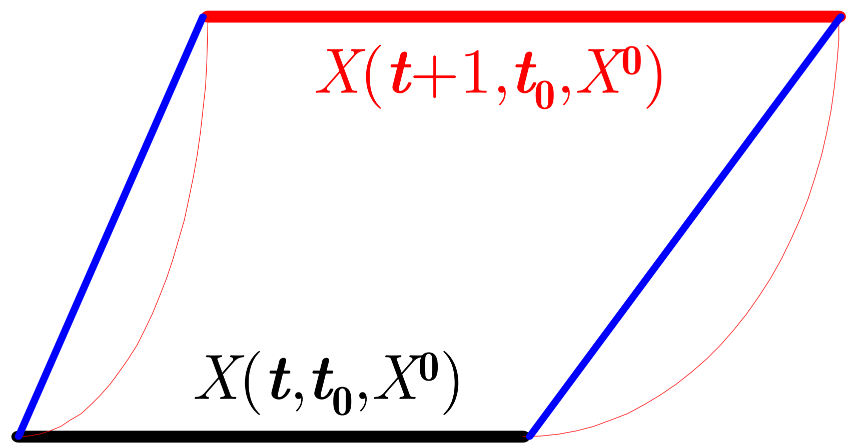
Hyperrectangles (*d/dt*)



- *Approximate reachability analysis of piecewise linear dynamical systems* by E.Asarin, O.Bournez, T.Dang, O.Maler
- `www-verimag.imag.fr/~tdang/ddt.html`

BACK

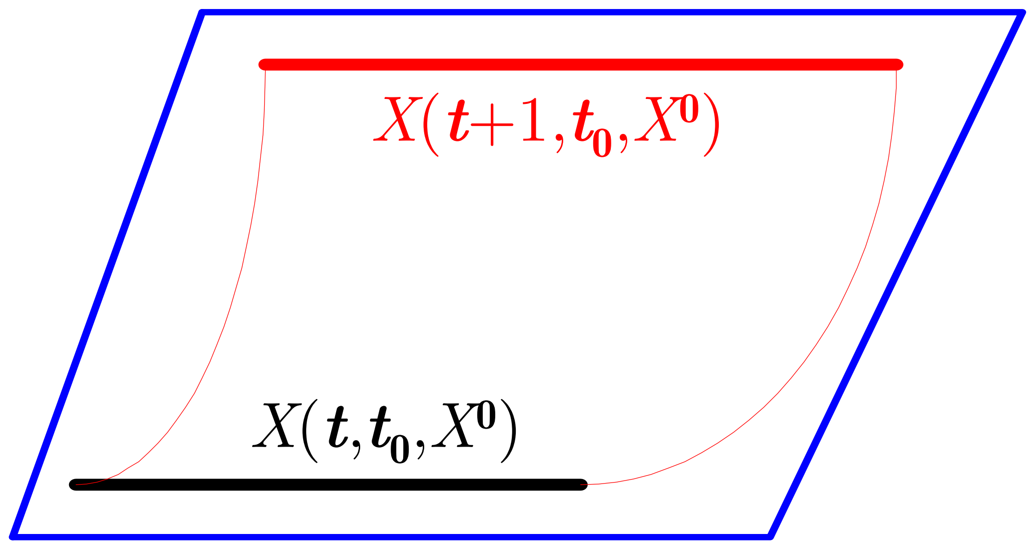
Hyperrectangles (*d/dt*)



- *Approximate reachability analysis of piecewise linear dynamical systems* by E.Asarin, O.Bournez, T.Dang, O.Maler
- `www-verimag.imag.fr/~tdang/ddt.html`

BACK

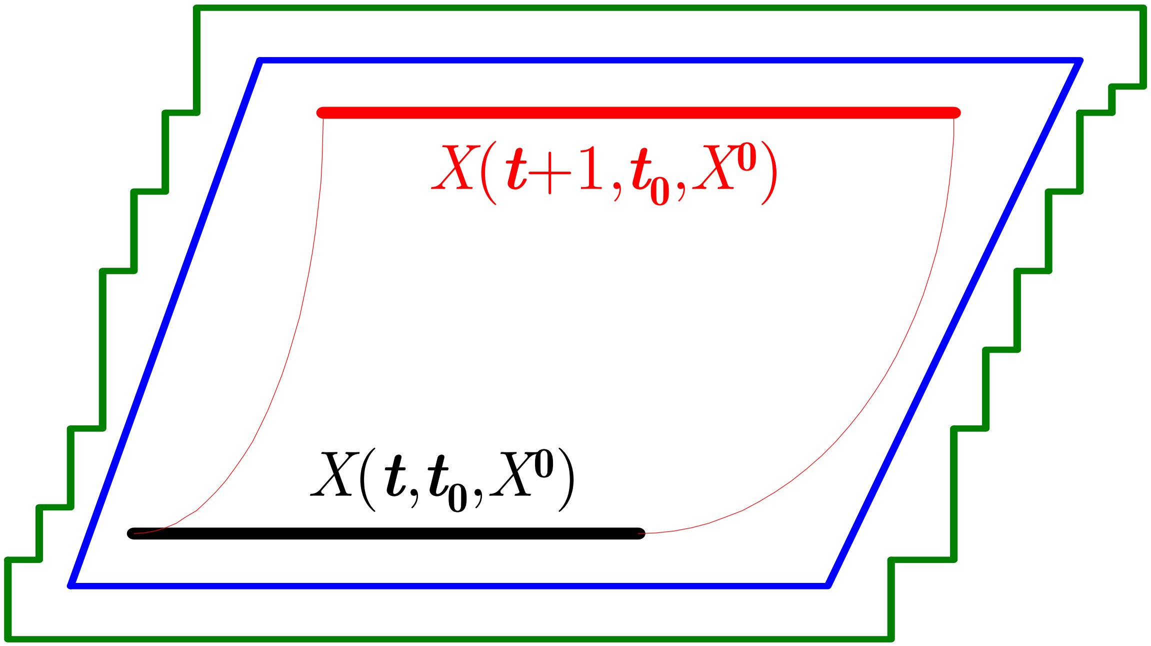
Hyperrectangles (*d/dt*)



- *Approximate reachability analysis of piecewise linear dynamical systems* by E.Asarin, O.Bournez, T.Dang, O.Maler
- `www-verimag.imag.fr/~tdang/ddt.html`

BACK

Hyperrectangles (*d/dt*)



- *Approximate reachability analysis of piecewise linear dynamical systems* by E.Asarin, O.Bournez, T.Dang, O.Maler
- `www-verimag.imag.fr/~tdang/ddt.html`

BACK

Zonotopes (*MATISSE*)

- Zonotope is symmetric polytope:

$$\mathcal{Z}(\boldsymbol{z},Z) = \{\boldsymbol{x} \in \mathbb{R}^{\mathrm{n}} \mid \boldsymbol{x} = \boldsymbol{z} + \sum \boldsymbol{\alpha_i} Z^{\boldsymbol{i}},\ \ |\boldsymbol{\alpha_i}| \leq 1\}$$

- Compact representation:

$$\mathcal{Z}(\boldsymbol{z}_1,Z_1) \oplus \mathcal{Z}(\boldsymbol{z}_2,Z_2) = \mathcal{Z}(\boldsymbol{z}_1+\boldsymbol{z}_2,\ [Z_1\ \ Z_2])$$

- *Reachability of uncertain linear systems using zonotopes* by A.Girard (2005)
- `www.seas.upenn.edu/~agirard/Software/MATISSE`

BACK

Oriented Rectangles (*CheckMate*)

- Computes trajectories of the vertices of the initial polytope
- Constructs oriented rectangular hull of the reach set at every time step
- Orientation is determined by the SVD of sample covariance matrix of the reachable states

- *Efficient representation and computation of reachable sets for hybrid systems* by O.Stursberg, B.Krogh (2003)
- `www.ece.cmu.edu/~webk/checkmate`

BACK

Quantifier Elimination (*Requiem*)

- Symbolic computation of the exact reach set by removing quantifiers $\forall$, $\exists$ from the initial expression of the reach set and substituting it with formula that contains only

$$+,\ \times,\ =,\ <,\ \wedge,\ \vee$$

- *Symbolic reachability computation for families of linear vector fields* by G.Lafferriere, G.Pappas, S.Yovine (2001)
- `www.seas.upenn.edu/~hybrid/requiem`

BACK

Quantifier Elimination (*Requiem*)

- Example:

$$\boldsymbol{x}(\boldsymbol{t}+1) = A\boldsymbol{x}(\boldsymbol{t}) + B\boldsymbol{u}(\boldsymbol{t})$$

with $$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- Initial conditions: $x(0) = \{x \mid \|x\|_\infty \leq 1\}$
- Control bounds: $-1 \leq u(t) \leq 1, \;\; t \geq 0$

BACK

Quantifier Elimination (*Requiem*)

- Reach set (quantified formula):

$$\{x \mid \exists x_0,\ \exists t \geq 0,\ \exists u(i),\ 0 \leq i \leq t :\ x = A^t x_0 + \sum_{i=0}^{t-1} A^{t-i-1} B u(i)\}$$

- Reach set (quantifier-free expression):

$$-1 \leq [1\ \ 0]x \leq 1\ \ \wedge\ \ -1 \leq [0\ \ 1]x \leq 1$$

- *Symbolic reachability computation for families of linear vector fields* by G.Lafferriere, G.Pappas, S.Yovine (2001)
- `www.seas.upenn.edu/~hybrid/requiem`

BACK

Parallelotopes

- Parallelotope:

$$\mathcal{P}(\boldsymbol{p}, P) = \{\boldsymbol{x} \in \mathbb{R}^{\mathrm{n}} \mid \boldsymbol{x} = \boldsymbol{p} + \sum \boldsymbol{\alpha_i} P^{\boldsymbol{i}},\ |\boldsymbol{\alpha_i}| \leq 1\}$$

- Reach set:

$$X(t, t_0, X^0) = \bigcup_l \mathcal{P}(p(t), P_l^-(t)) = \bigcap_l \mathcal{P}(p(t), P_l^+(t))$$

  shape matrices $P_{\boldsymbol{l}}^-$ and $P_{\boldsymbol{l}}^+$ are solutions to ODEs that depend on parameter $\boldsymbol{l}$

- *Control synthesis via parallelotopes: optimization and parallel computations* by E.Kostousova (2001)

BACK

Level Sets (*Level Set Toolbox*)

- Value function:

$$V(t,x) = \min_u\{\mathbf{dist}(x(t_0),X^0) \mid x(t)=x\}$$

- Reach set: $X(t,t_0,X^0)=\{x \mid V(t,x)\leq 0\}$
- Hamilton-Jacobi-Bellman (HJB) equation:

$$V_t(t,x) + \max_u\langle D_x V(t,x), f(t,x,u)\rangle = 0$$

with initial condition

$$V(t_0,x) = \mathbf{dist}(x,X^0)$$

- *Level set methods for computation in hybrid systems* by I.Mitchell, C.Tomlin (2000)
- `www.cs.ubc.ca/~mitchell/ToolboxLS`

BACK

Barrier Certificates

- No reach set computation
- Find function $C(\boldsymbol{x})$:
  - $C(\boldsymbol{x}) > 0$ in the unsafe set
  - $C(\boldsymbol{x}) \leq 0$ in $X^{\mathbf{0}}$
  - $\langle D_{\boldsymbol{x}} C(\boldsymbol{x}), \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}) \rangle \leq 0$, where $C(\boldsymbol{x}) = 0$
- If such $C(\boldsymbol{x})$ exists, system $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u})$ is safe

- *Safety verification of hybrid systems using barrier certificates* by S.Prajna, S.Jadbabaie (2004)

BACK

ODEs for Shape Matrices

The equation for the shape matrix of external ellipsoid is

$$\begin{aligned}
\dot{X}_l^+(t) &= A(t)X_l^+(t) + X_l^+(t)A^T(t) && (3.3)\\
&+ \pi_l(t)X_l^+(t) + \frac{1}{\pi_l(t)}B(t)P(t)B^T(t) && (3.4)\\
&- X_l^{+1/2}(t)S_l(t)(G(t)Q(t)G^T(t))^{1/2} - (G(t)Q(t)G^T(t))^{1/2}S_l^T(t)X_l^{+1/2}(t), && (3.5)\\
& X_l^+(t_0) = X_0, && (3.6)
\end{aligned}$$

where

$$\pi_l(t) = \frac{\langle l, \Phi(t_0,t)B(t)P(t)B^T(t)\Phi^T(t_0,t)l\rangle^{1/2}}{\langle l, \Phi(t_0,t)X_l^+(t)\Phi^T(t_0,t)l\rangle^{1/2}},$$

and matrix $S_l(t)$ is orthogonal ($S_l(t)S_l^T(t) = I$) and determined from equation

$$S_l(t)(G(t)Q(t)G^T(t))^{1/2}\Phi^T(t_0,t)l = \frac{\langle l, \Phi(t_0,t)G(t)Q(t)G^T(t)\Phi^T(t_0,t)l\rangle^{1/2}}{\langle l, \Phi(t_0,t)X_l^+\Phi^T(t_0,t)l\rangle^{1/2}}X_l^{+1/2}\Phi^T(t_0,t)l.$$

In the presence of disturbance, the reach set may be empty, and then matrix $X_l^+(t)$ will be sign indefinite. For systems without disturbance, the part (3.5) naturally vanishes from the equation (3.3–3.6).

The equation for the shape matrix of internal ellipsoid is

$$\begin{aligned}
\dot{X}_l^-(t) &= A(t)X_l^-(t) + X_l^-(t)A^T(t) && (3.7)\\
&+ X_l^{-1/2}(t)T_l(t)(B(t)P(t)B^T(t))^{1/2} + (B(t)P(t)B^T(t))^{1/2}T_l^T(t)X_l^{-1/2}(t) && (3.8)\\
&- \eta_l(t)X_l^-(t) - \frac{1}{\eta_l(t)}G(t)Q(t)G^T(t), && (3.9)\\
& X_l^-(t_0) = X_0, && (3.10)
\end{aligned}$$

where

$$\eta_l(t) = \frac{\langle l, \Phi(t_0,t)G(t)Q(t)G^T(t)\Phi^T(t_0,t)l\rangle^{1/2}}{\langle l, \Phi(t_0,t)X_l^+(t)\Phi^T(t_0,t)l\rangle^{1/2}},$$

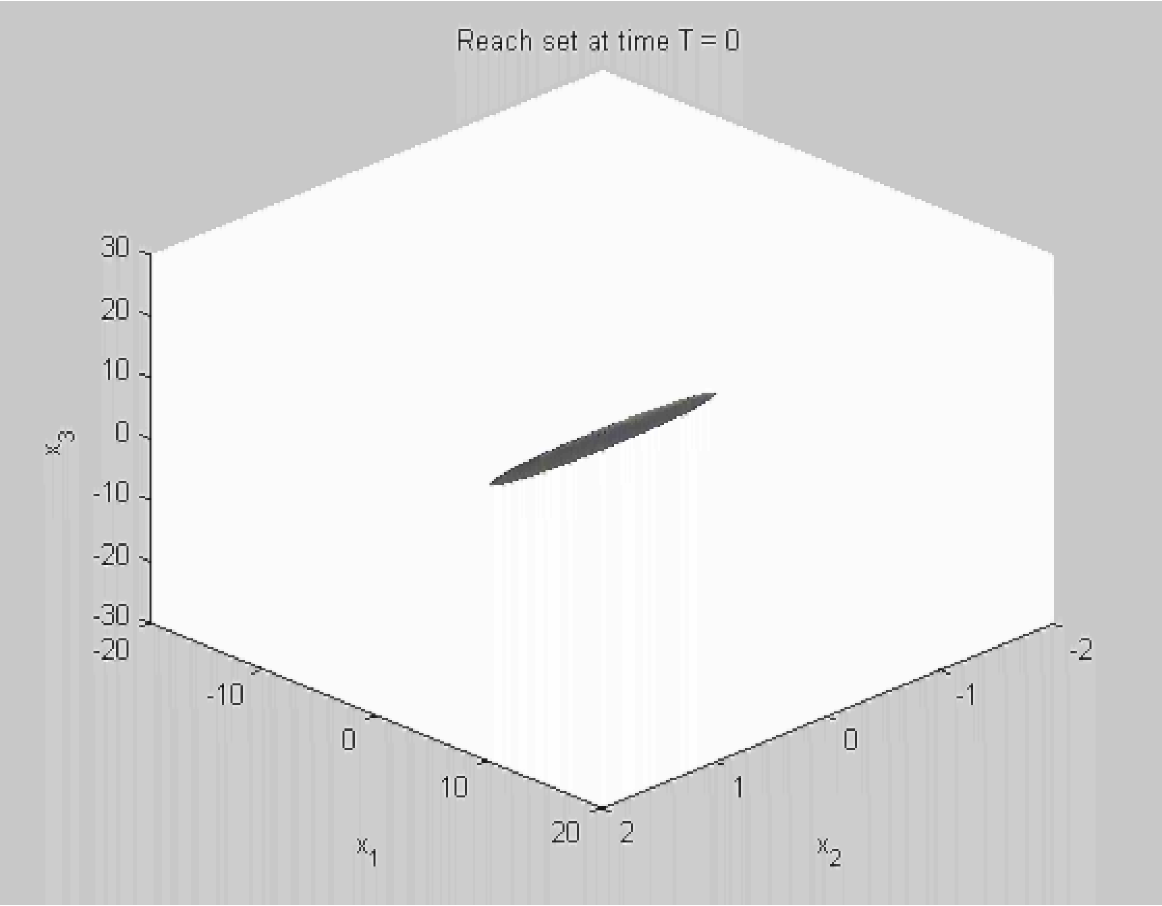
and matrix $T_l(t)$ is orthogonal and determined from equation

$$T_l(t)(B(t)P(t)B^T(t))^{1/2}\Phi^T(t_0,t)l = \frac{\langle l, \Phi(t_0,t)B(t)P(t)B^T(t)\Phi^T(t_0,t)l\rangle^{1/2}}{\langle l, \Phi(t_0,t)X_l^-\Phi^T(t_0,t)l\rangle^{1/2}}X_l^{-1/2}\Phi^T(t_0,t)l.$$

BACK

BACK

Switched System in 3D



ET function: **`evolve`**

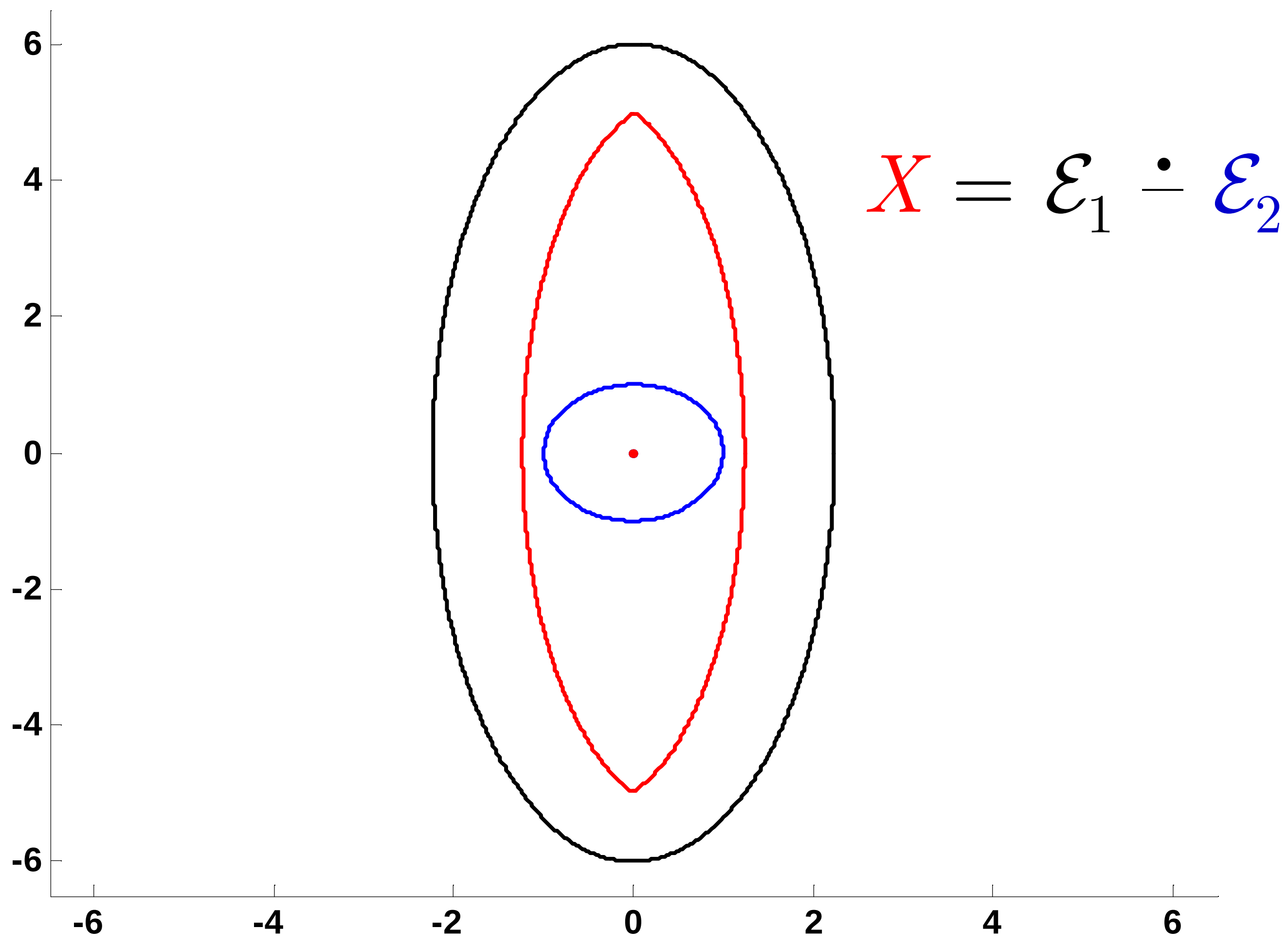
BACK

Geometric (Minkowski) Difference

$$\mathcal{P} \stackrel{\bullet}{-} \mathcal{Q} = \{\boldsymbol{x} \mid \boldsymbol{x} + \mathcal{Q} \subseteq \mathcal{P}\}$$

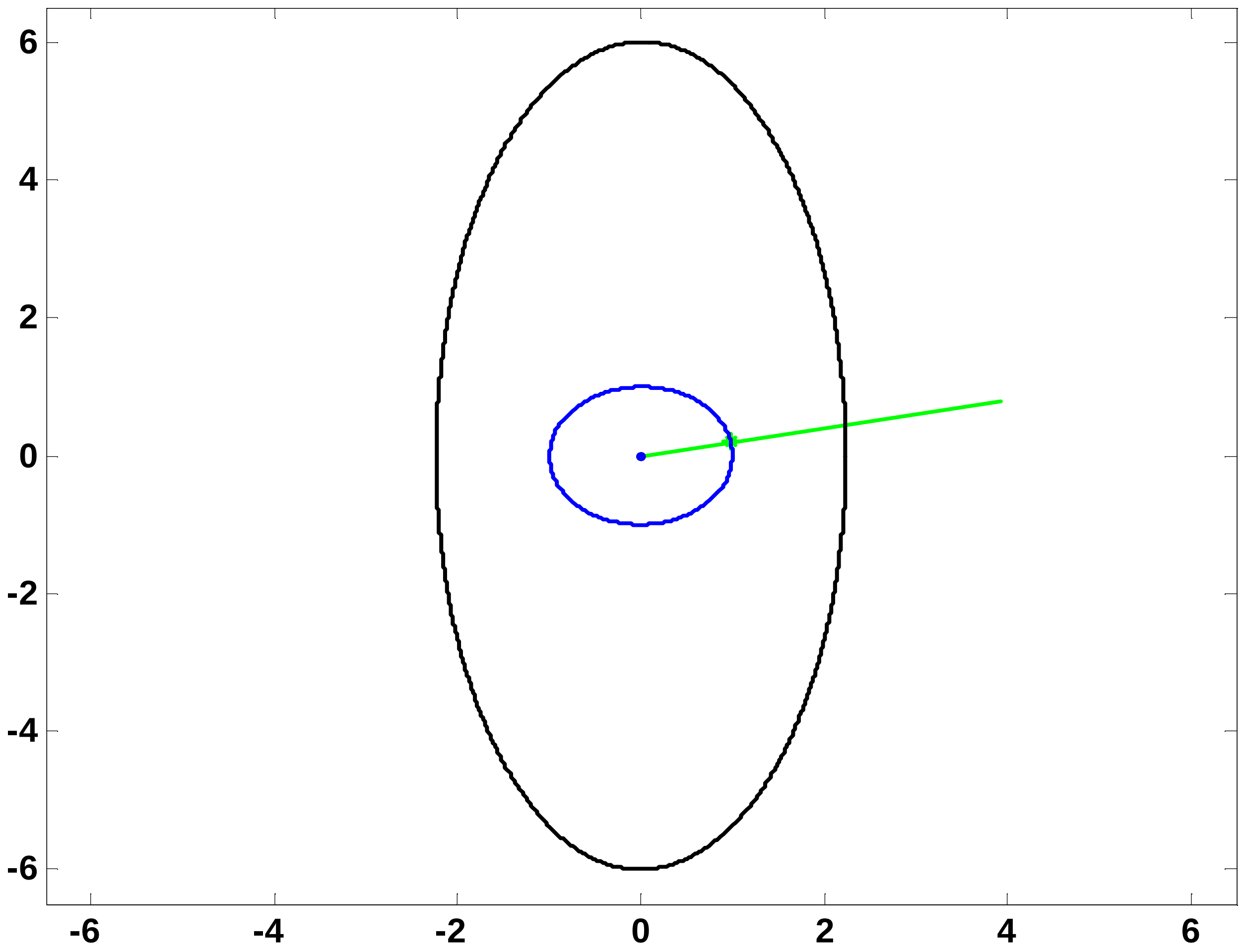
BACK

Geometric Difference (example)

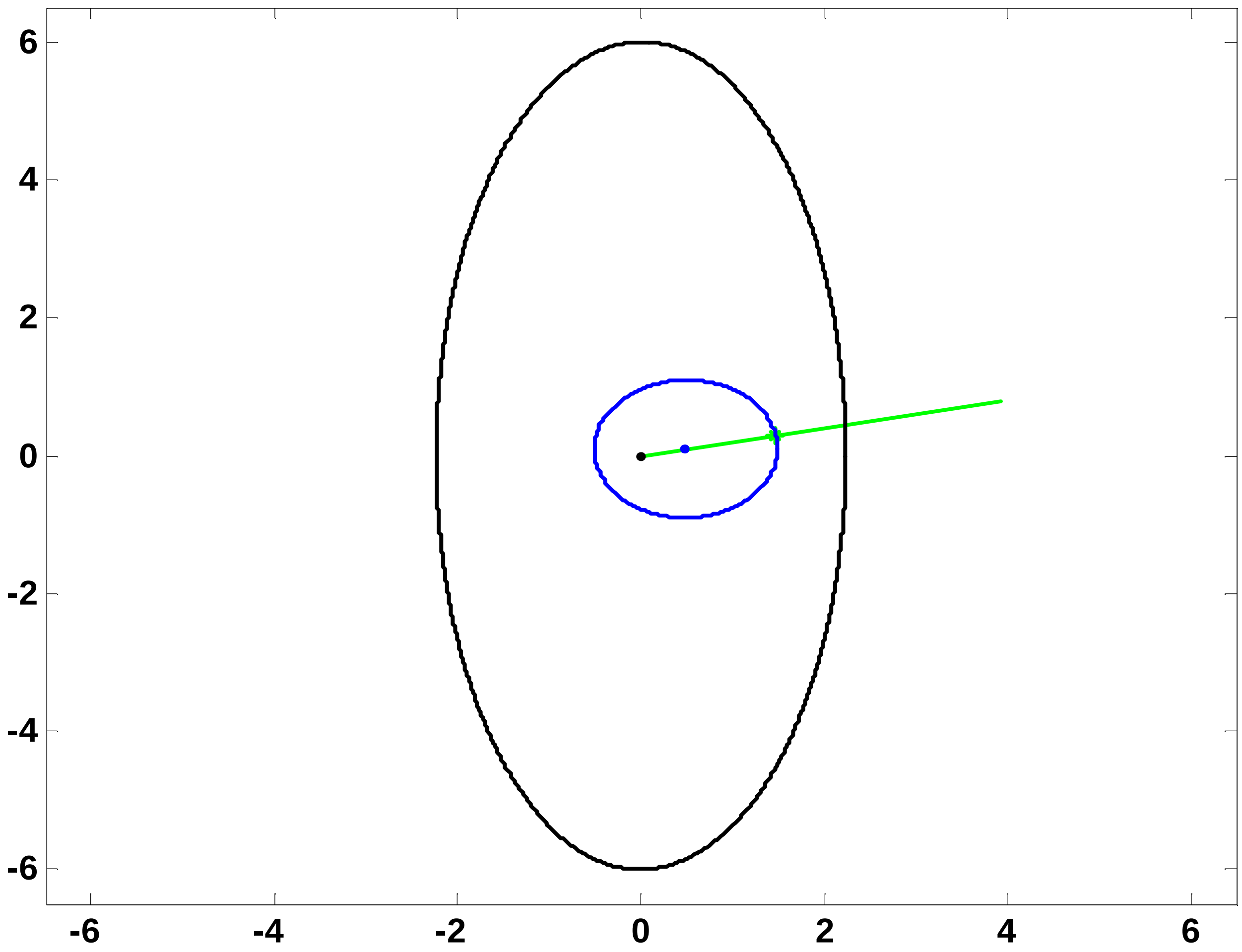


BACK

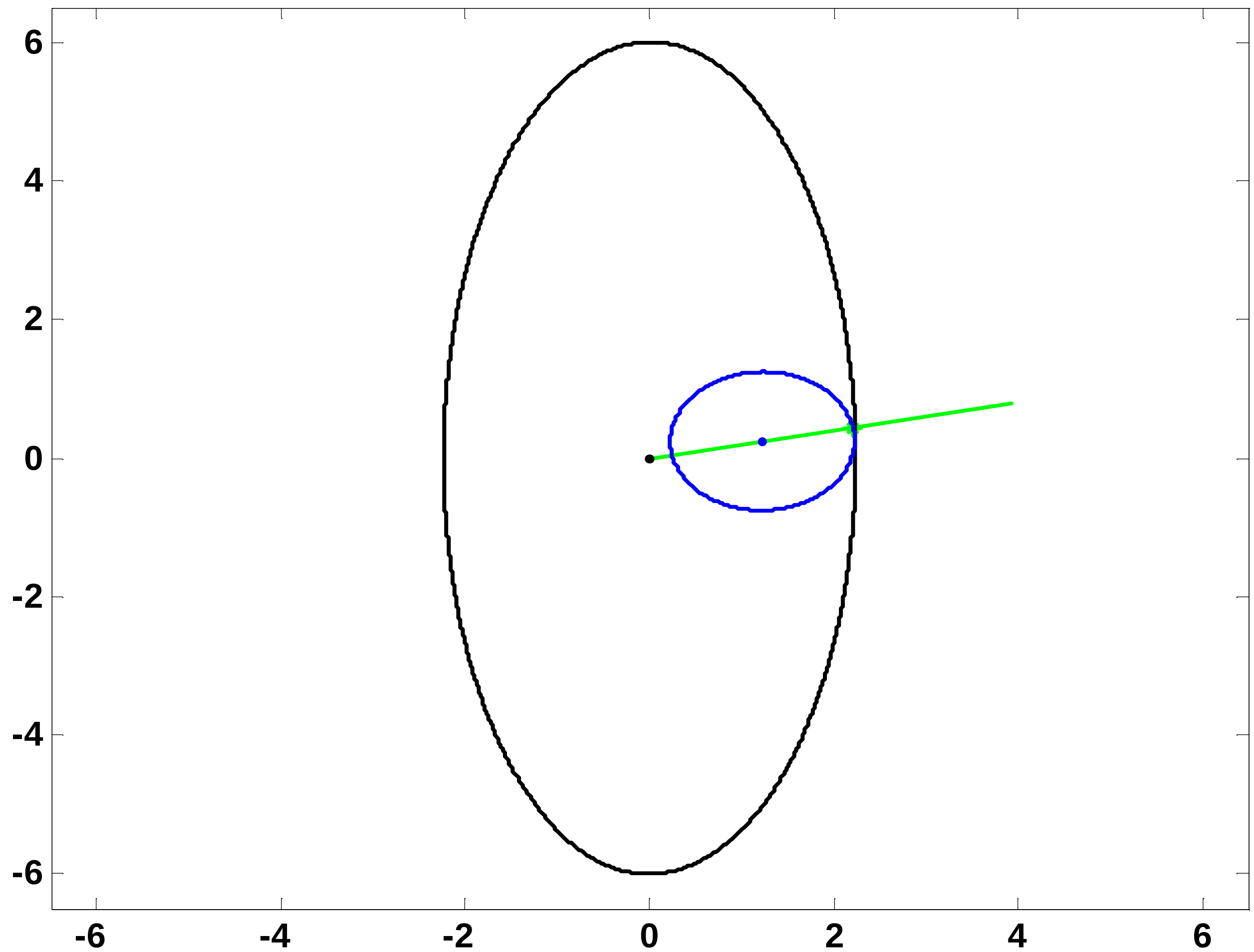
Good Direction



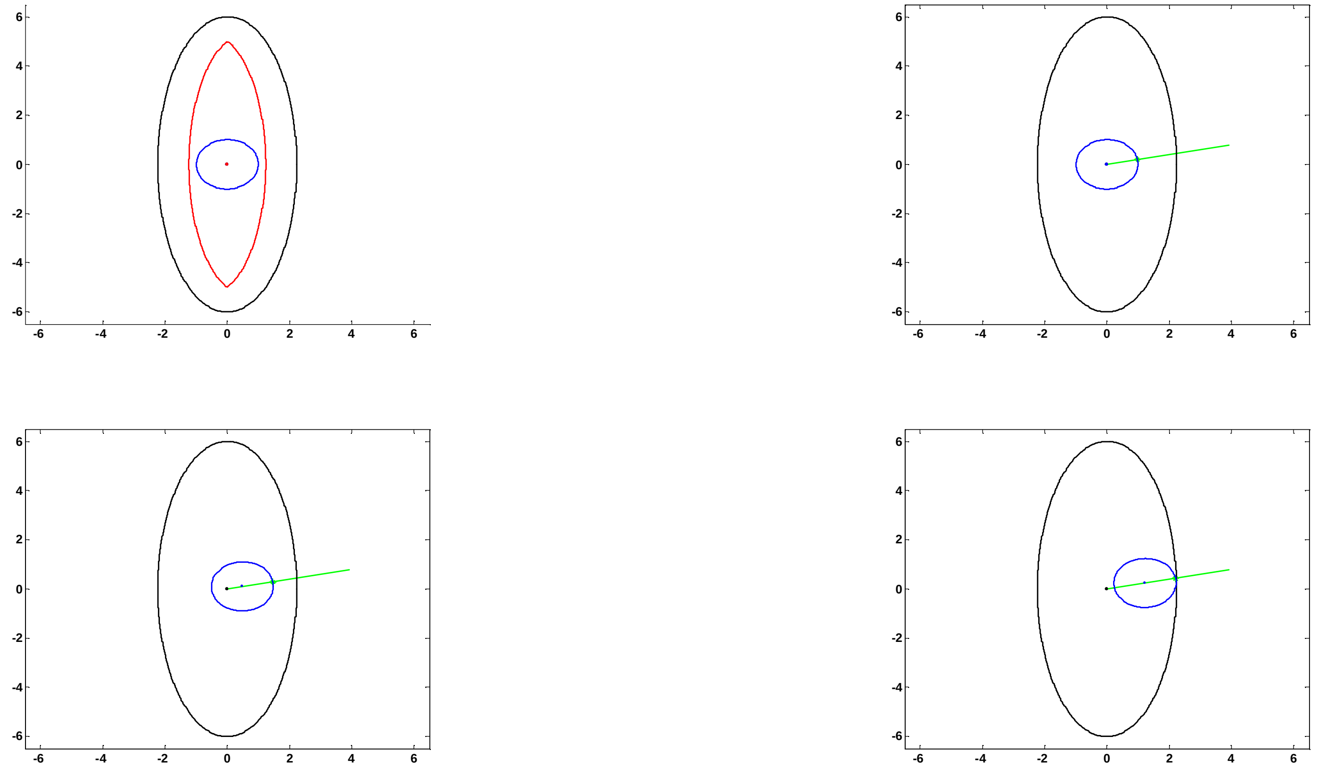
Good Direction



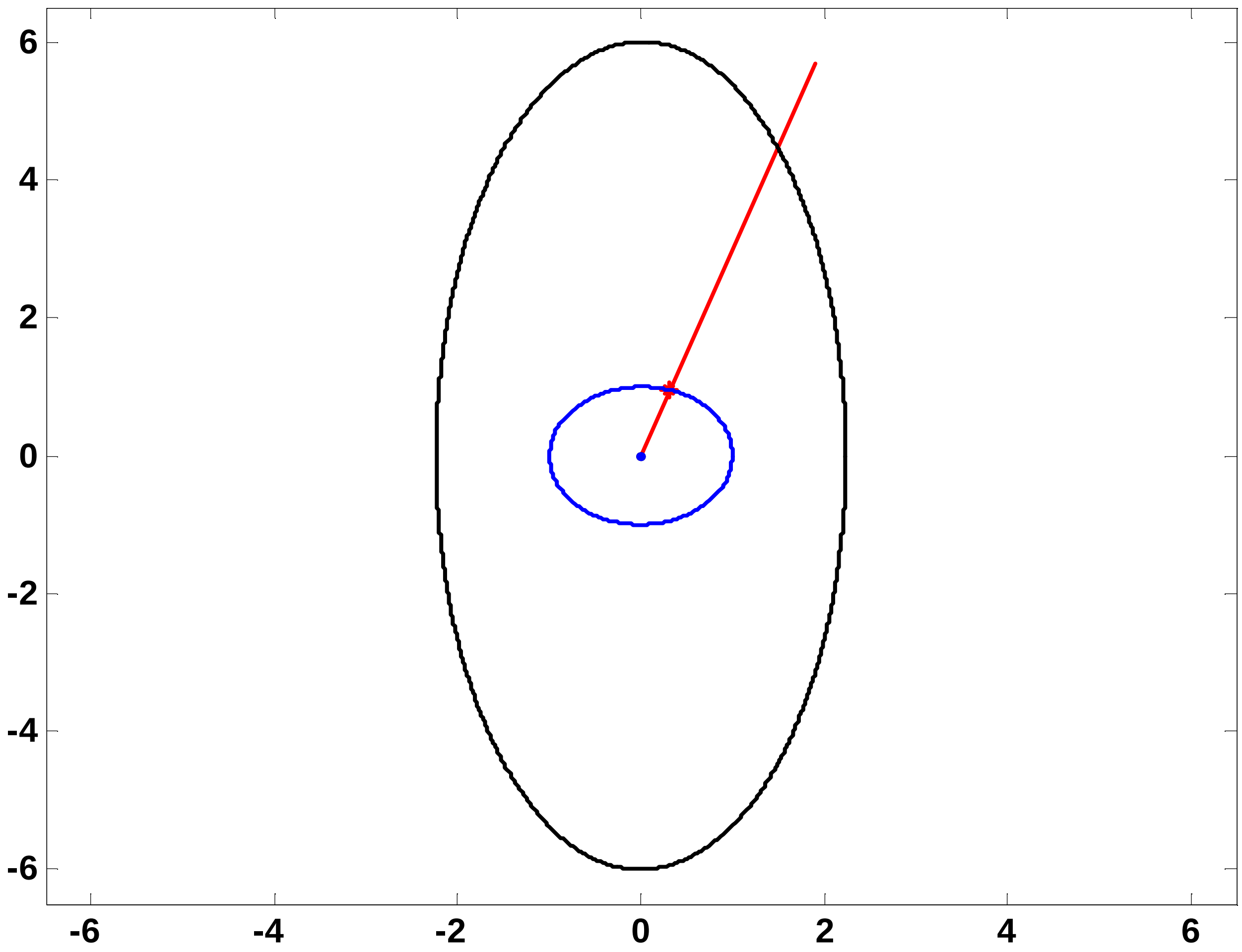
Good Direction



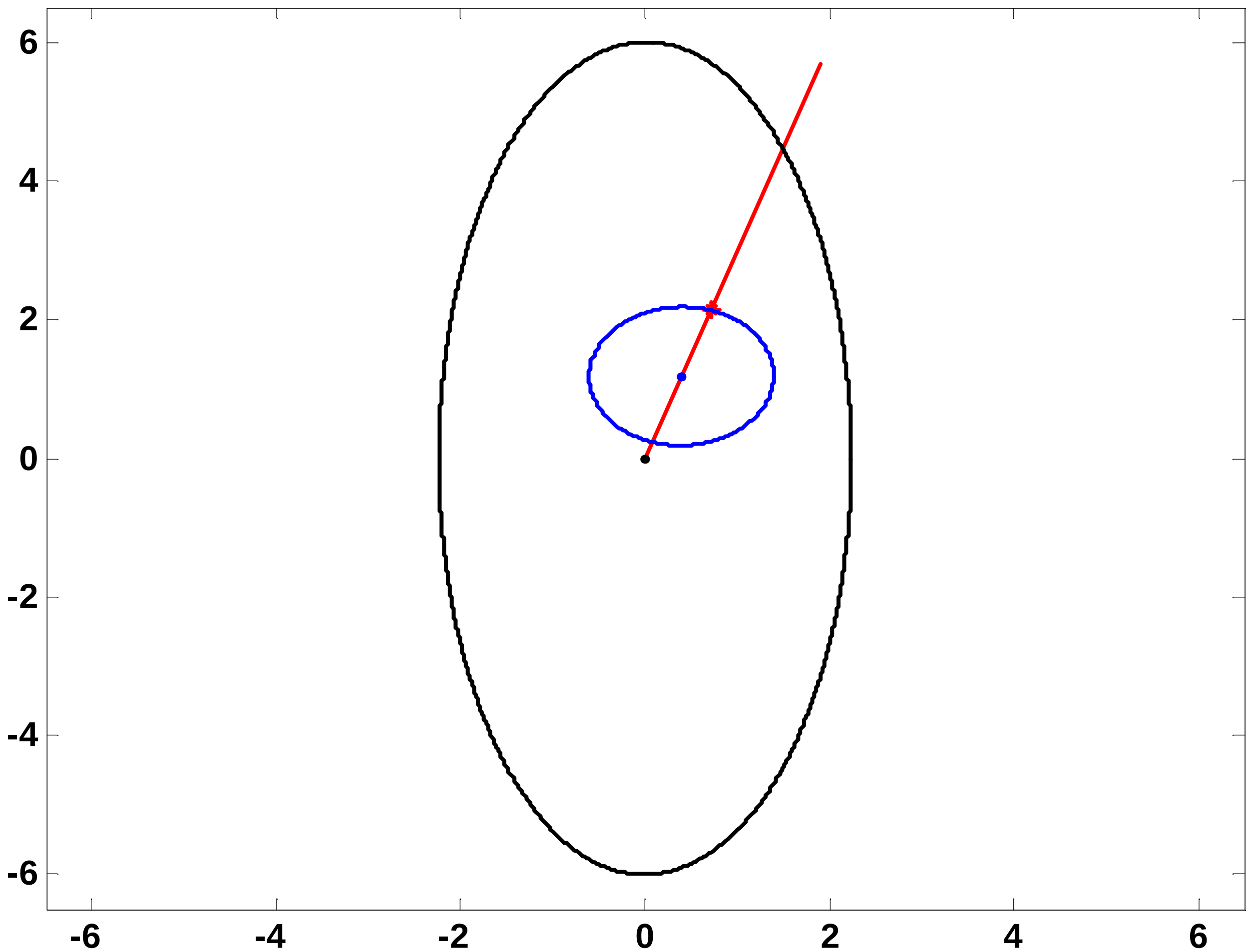
Good Direction



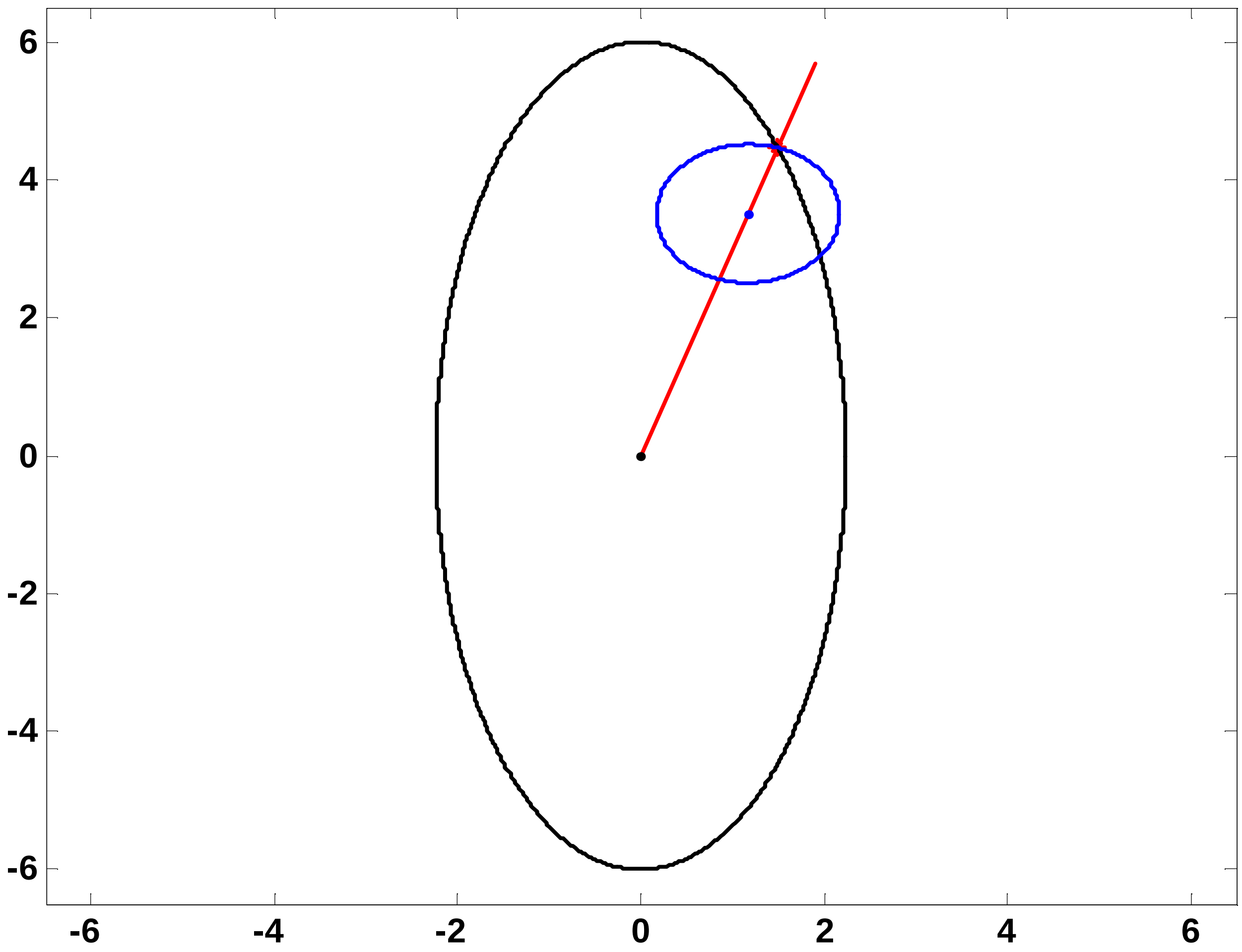
Bad Direction



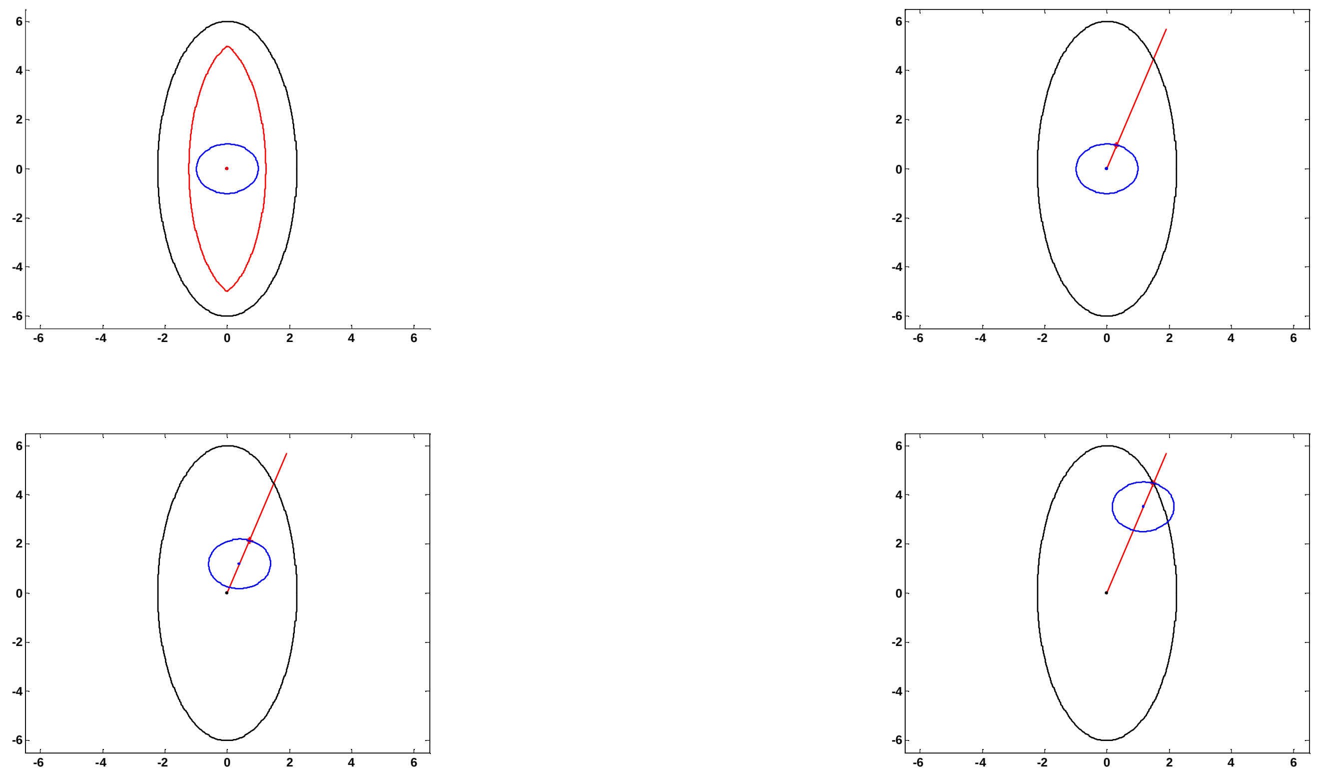
Bad Direction



Bad Direction



Bad Direction



Inverting Ill-Conditioned Matrix

- Square matrix A is nonsingular but ill-conditioned
- Compute inverse of A by usual method (e.g. Gauss-Jordan) and denote it A^{+}
- Matrix $\mathrm{A}^{+}\mathrm{A}$ is well-conditioned

- $\mathrm{A}^{-1} = (\mathrm{A}^{+}\mathrm{A})^{-1}\mathrm{A}^{+}$

BACK

Viscosity Solution

- Hamilton-Jacobi equation

$$V_t(\boldsymbol{t},\boldsymbol{x}) + H(D_{\boldsymbol{x}}V(\boldsymbol{t},\boldsymbol{x}),\boldsymbol{x}) = 0$$

with initial condition: $V(0,\boldsymbol{x}) = \boldsymbol{g}$

BACK
BACK

Viscosity Solution

- Hamilton-Jacobi equation

$$V_t(\boldsymbol{t},\boldsymbol{x}) + H(D_x V(\boldsymbol{t},\boldsymbol{x}),\boldsymbol{x}) = 0$$

with initial condition: $V(0,\boldsymbol{x}) = \boldsymbol{g}$

- Substituted by

$$V^{\varepsilon}{}_t(\boldsymbol{t},\boldsymbol{x}) + H(D_x V^{\varepsilon}(\boldsymbol{t},\boldsymbol{x}),\boldsymbol{x}) - \underline{\boldsymbol{\varepsilon}\Delta V^{\varepsilon}(\boldsymbol{t},\boldsymbol{x})} = 0$$

$$V^{\varepsilon}(0,\boldsymbol{x}) = \boldsymbol{g}$$

- $V^{\varepsilon} \to V$ as $\boldsymbol{\varepsilon} \to 0$

BACK

BACK

Formal Definition

Bounded, uniformly continuous function V is viscosity solution of Hamilton-Jacobi equation if

- $V(0,\boldsymbol{x}) = \boldsymbol{g}$
- For each smooth $\boldsymbol{u}$, if $(V - \boldsymbol{u})$ has local maximum at $(\boldsymbol{t}_0,\boldsymbol{x}_0)$, then

$$\boldsymbol{u}_t(\boldsymbol{t}_0,\boldsymbol{x}_0) + H(D_x\boldsymbol{u}(\boldsymbol{t}_0,\boldsymbol{x}_0),\boldsymbol{x}_0) \leq 0$$

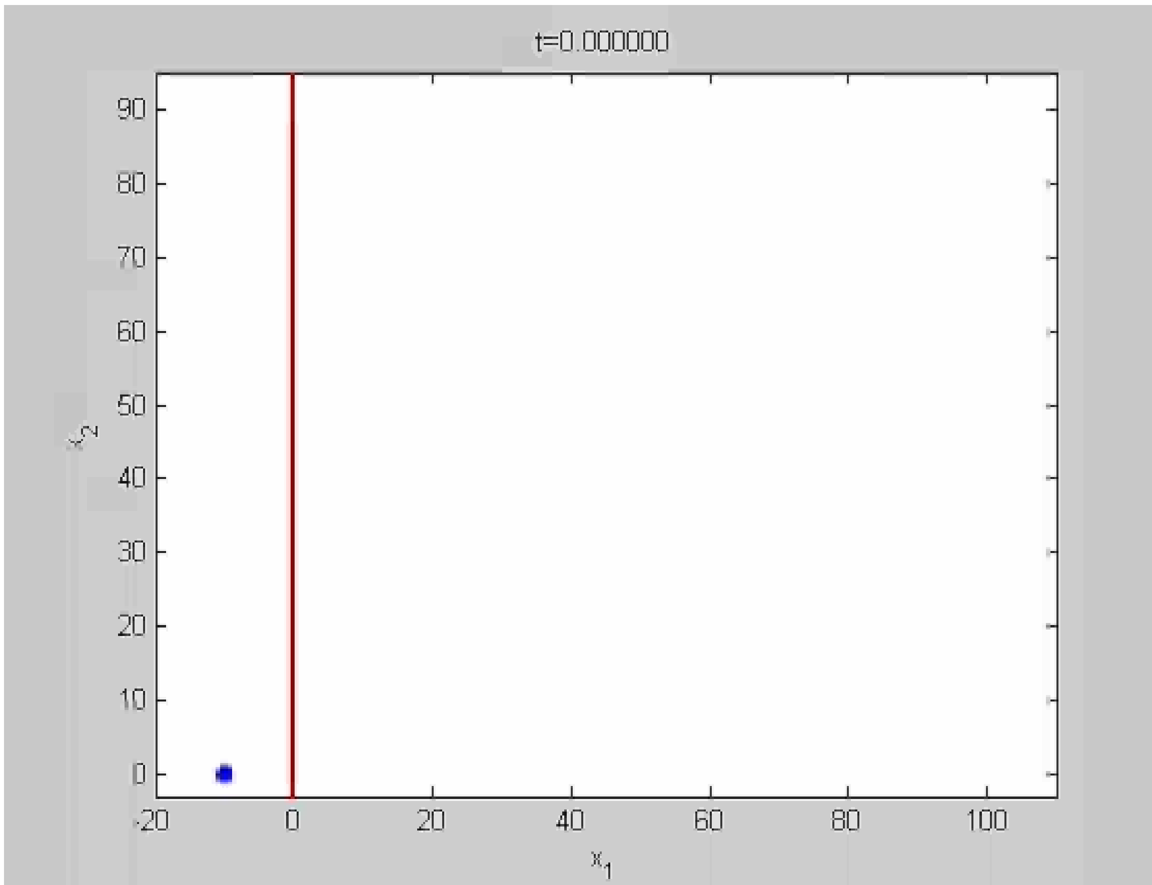
- For each smooth $\boldsymbol{u}$, if $(V - \boldsymbol{u})$ has local minimum at $(\boldsymbol{t}_0,\boldsymbol{x}_0)$, then

$$\boldsymbol{u}_t(\boldsymbol{t}_0,\boldsymbol{x}_0) + H(D_x\boldsymbol{u}(\boldsymbol{t}_0,\boldsymbol{x}_0),\boldsymbol{x}_0) \geq 0$$

BACK

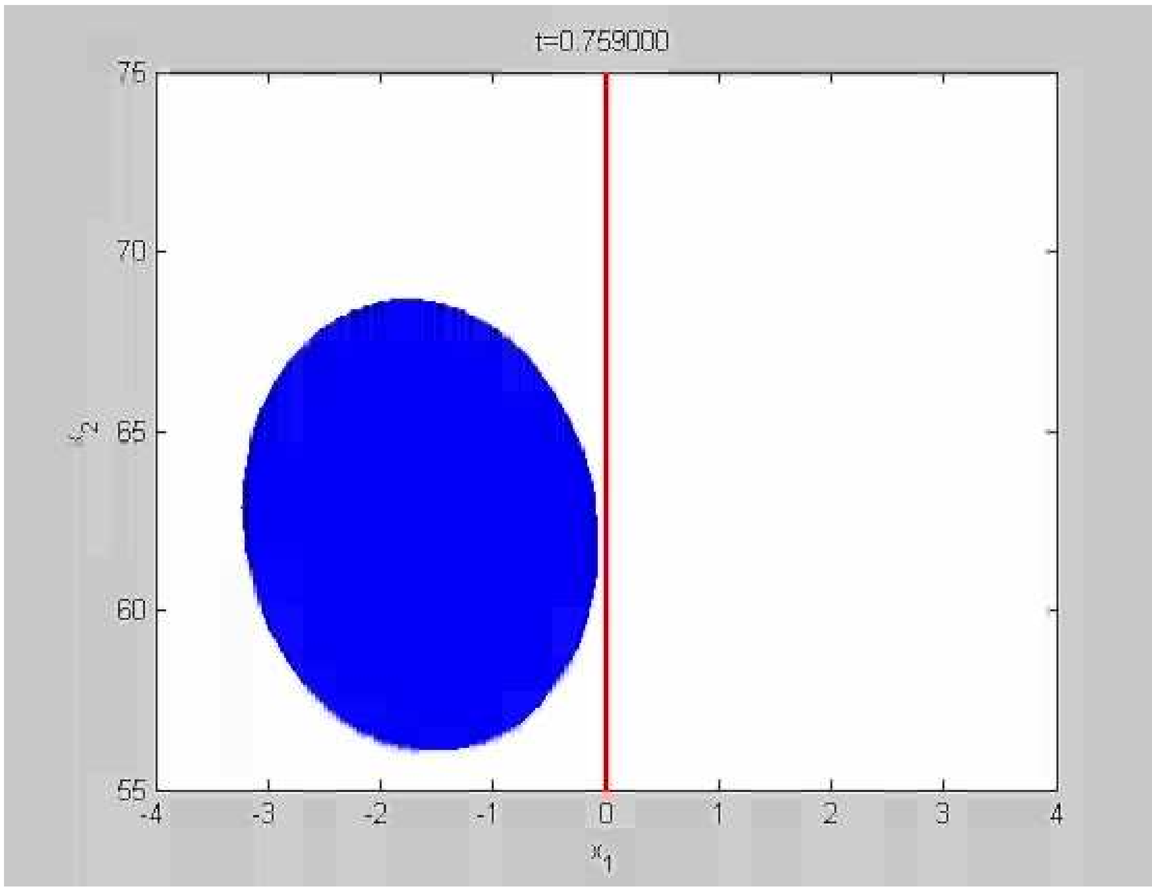
BACK

Hybrid Reach Set



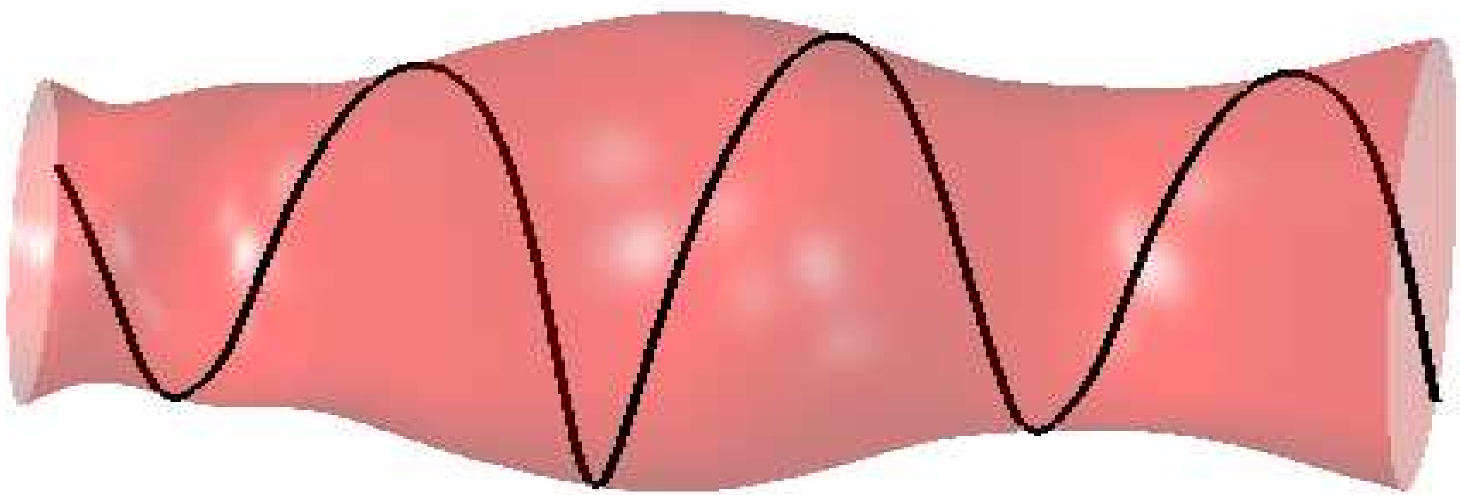
BACK

Guard Crossing



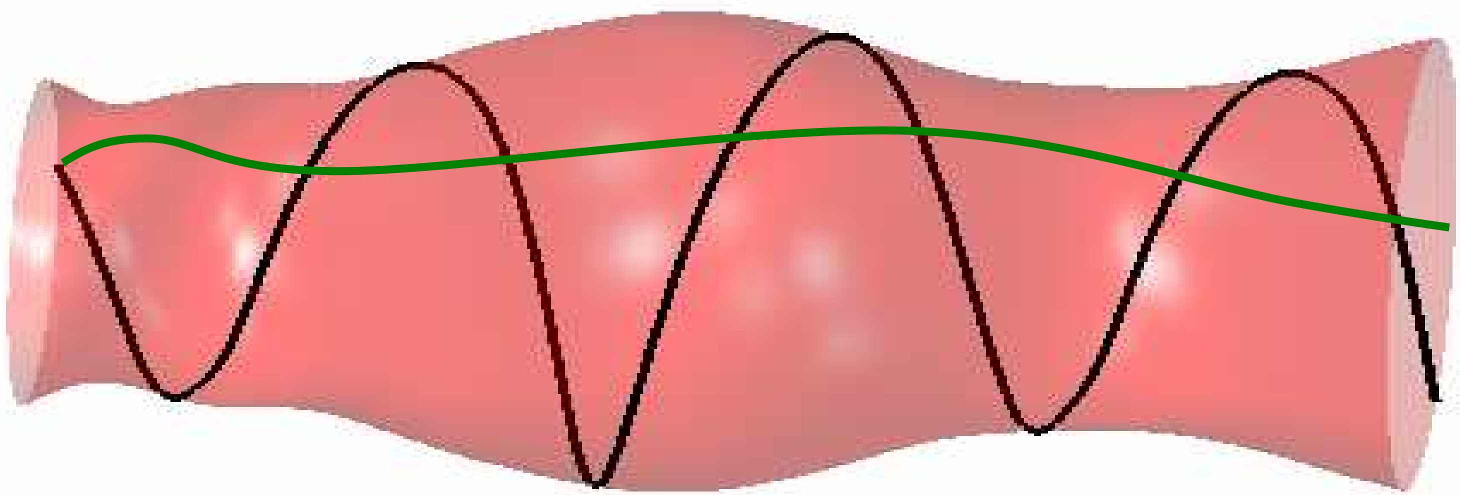
BACK

Good Curve



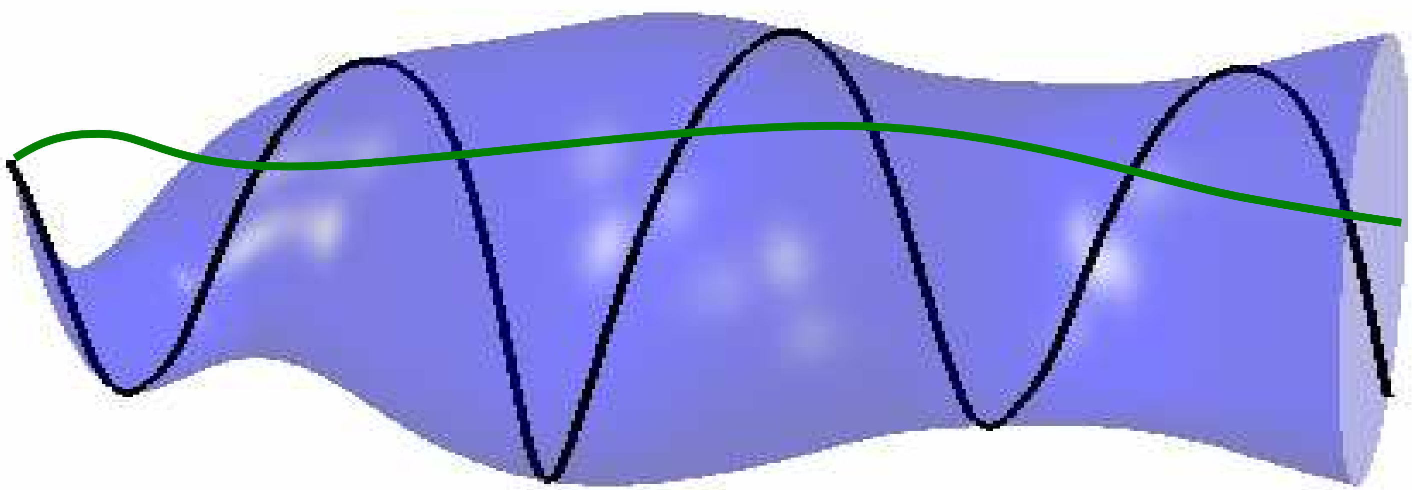
BACK

Good Curve



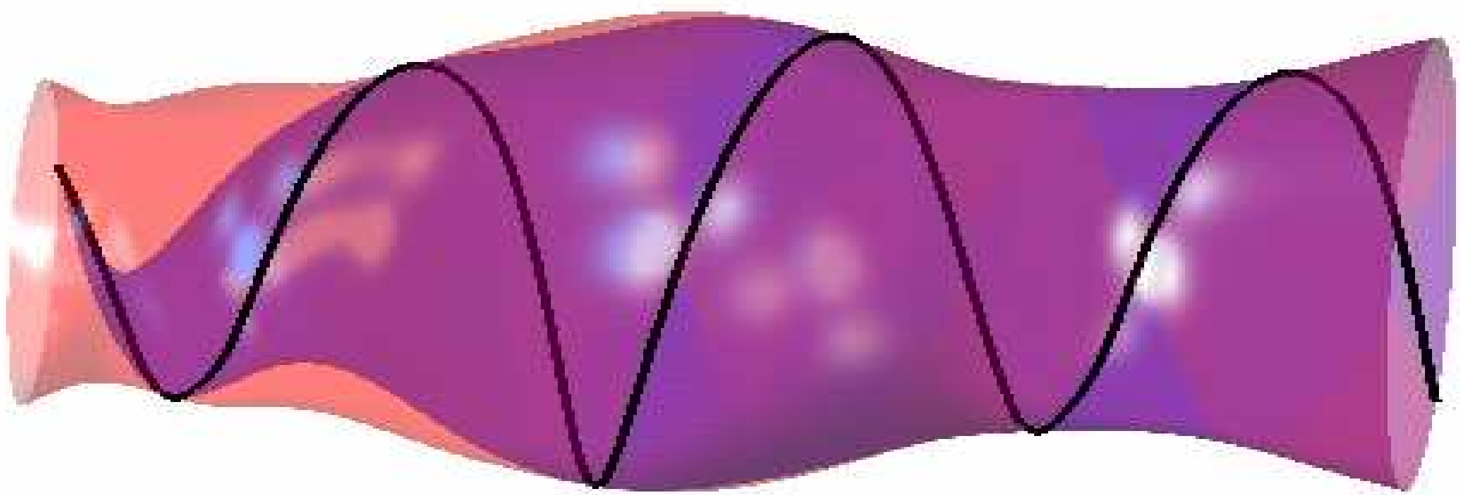
BACK

Good Curve



BACK

Good Curve



BACK