Ellipsoidal Techniques for Reachability Under State Constraints*

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Abstract

The paper presents a scheme to calculate approximations of reach sets and tubes for linear control systems with time-varying coefficients, bounds on the controls, and *constraints on the state*. The scheme provides tight external approximations by ellipsoidal-valued tubes. These tubes touch the reach tubes from the outside at *each point* of their boundary so that the surface of the reach tube is totally covered by curves that belong to the approximating tubes. The result is an exact parametric representation of reach tubes through families of external ellipsoidal tubes.

The parameters that characterize the approximating elliposids are solutions of ordinary differential equations with coefficients given partly in explicit analytical form and partly through the solution of a recursive optimization problem. The scheme combines the calculation of external approximations of infinite sums and intersections of ellipsoids, and suggests an approach to calculate reach sets of hybrid systems.

1 Introduction

Recent activities in automation of real-time processes have renewed interest in the calculation of reach sets $\mathcal{X}(\tau, t_0, x^0)$ of linear controlled systems of the form

$$\dot{x} = A(t)x + B(t)u, \quad \tau \geq t_0. \tag{1}$$

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Typically, there are additional constraints. The most difficult calculations are for state constraints, and for hybrid or switched systems in which the state and dynamics are reset [22, 6]. State constraints are considered in this paper. Hybrid systems are considered in a separate paper [16].

Among methods for reach set calculation are those based on ellipsoidal techniques. The first publications in this area, e.g. [1, 2], determined a *single* (minimum volume) ellipsoid that externally approximates the reach set at a *single* point in time.

In a series of papers [11, 12, 14, 15] the authors developed a parametrized family of ellipsoidal approximations, both external and internal, which are

• *tight*—no other ellipsoid in the family can be 'squeezed' between the reach set and the approximating ellipsoid;
• *exact*—the intersection of the external ellipsoids and the union of the internal ellipsoids equals the reach set; and
• *recursive*—a single differential equation provides parameters that specify ellipsoids which approximate the reach set for *all* times.

The approach in these papers is first to convert the problem of calculating the reach set into an equivalent *optimization* problem; then to show that the corresponding value function satisfies a forward Hamilton-Jacobi-Bellman (HJB) equation; and finally to use convex analysis to approximate the level sets of the value function . The same approach is followed here.

The situation in which (1) contains a disturbance term

$$\dot{x} = A(t)x + B(t)u + C(t)v, \quad \tau \geq t_0,$$

and reachability must be guaranteed independent of the disturbance v, requires consideration of *feedback* control. That situation is treated in [13].

This paper deals with reach tubes for control systems (1), with bounds on the control—$u(t) \in \mathcal{P}(t)$, and state constraints—$x(t) \in \mathcal{Y}(t)$. As in the case without state constraints [15], the approximating tubes are generated by ellipsoidal-valued maps that satisfy a semi-group property and thus form a generalized dynamical system. The ellipsoidal maps are chosen in a certain class, which can be characterized through ordinary differential equations. But, unlike in [15], these differential equations now contain parameters that are the on-line solutions of a recursive dual optimization problem. The optimizers of these dual problems are the Lagrange multipliers of the state constraints, and may contain generalized Dirac delta functions. However, simpler equations are obtained if we drop the requirement of tightness (exactness).

Section 2 defines the reachability problem for general nonlinear systems; section 3 formulates the associated optimization problem, and shows that its value function satisfies a HJB equation. Section 4 uses convex analysis to characterize the reach sets for linear systems with state constraints in terms of the maximum principle. The state constraints introduce a Lagrange multiplier in the adjoint equation. Additional properties of the multiplier are derived in section 5. Section 6 derives a different version of the maximum prinicple that permits a recursive calculation of the reach sets. Section 7 developes the scheme for ellipsoidal approximnations. Section 8 exhibits additional properties of the multipliers. Section 9 works out the example of a double integrator. A brief conclusion is presented in section 10.

2 The reachability problem with state constraints

The controlled system is described by the differential equation,

$$\dot{x} = f(t, x, u), \quad t_0 \leq t \leq \tau, \tag{2}$$

in which the state $x \in \mathbb{R}^n$ and the control $u \in \mathbb{R}^m$ are restricted by

$$\text{(a)} \quad u(t) \in \mathcal{P}(t) \quad \text{(b)} \quad x(t) \in \mathcal{Y}(t), \tag{3}$$

for all $t \geq t_0$. Here $\mathcal{P}(t), \mathcal{Y}(t)$ are compact set-valued functions in $\mathbb{R}^m, \mathbb{R}^n$ respectively, continuous in the Hausdorff metric. The initial state is restricted by $x(t_0) \in \mathcal{X}^0$, a compact subset of $\mathbb{R}^n$. The function $f(t, x, u)$ is assumed to ensure uniqueness and uniform extension of solutions to any finite interval of time for any $x(t_0) = x^0,\ u(t) \in \mathcal{P}(t),\ t \geq t_0$.

Definition 2.1 *Given set-valued* **positions** $\{t_0, \mathcal{X}^0\}$, $\mathcal{X}^0 \cap \mathcal{Y}(t_0) \neq \emptyset$, *the* **reach set** *(or "attainability domain")* $\mathcal{X}(\tau, t_0, \mathcal{X}^0)$ **at time** $\tau > t_0$ *is the set*

$$\mathcal{X}[\tau] = \mathcal{X}(\tau, t_0, \mathcal{X}^0)$$

of all states $x[\tau] = x(\tau, t_0, x^0),\ x^0 = x(t_0) \in \mathcal{X}(t_0) = \mathcal{X}^0$, *that can be reached at time* τ *by system (2), from some* $x^0 \in \mathcal{X}^0$, *using all possible controls* u *that ensure constraints (3). The set-valued function* $\tau \mapsto \mathcal{X}[\tau] = \mathcal{X}(\tau, t_0, \mathcal{X}^0)$ *is the* **reach tube** *from* $\{t_0, \mathcal{X}^0\}$.

3 A related optimization problem and the value function

The basic problem is simply stated.

Problem 3.1. *Calculate the reach sets* $\mathcal{X}(\tau, t_0, \mathcal{X}^0), \tau \geq t_0$.

A fairly general approach is to relate reach sets to an optimization problem [14] . This could be done by calculating certain *value functions* which may be selected in several ways. Namely, consider first the value function

$$V(\tau, x) = \min_u \{d^2(x(t_0), \mathcal{X}^0) + \int_{t_0}^{\tau} d^2(x(t), \mathcal{Y}(t))dt\}, \tag{4}$$

under the restriction $x(\tau) = x$. Here the minimum is over all measurable functions $u(t) \in \mathcal{P}(t)$, $x(t), t \geq t_0$, is the corresponding trajectory and

$$d^2(x, \mathcal{X}) = \min\{(x - z, x - z) | z \in \mathcal{X}\},$$

is the square of the distance function $d(x, \mathcal{X})$.

Lemma 3.1 *The following relation is true*

$$\mathcal{X}(\tau, t_0, \mathcal{X}^0) = \{x : V(\tau, x) \leq 0\}.$$

This follows from the definition of the reach set $\mathcal{X}(\tau, t_0, \mathcal{X}^0)$, which is thus a *level set* for $V(\tau, x)$.

To state the important semigroup property of the value function we extend definition (4) to more general boundary conditions, namely,

$$V(\tau, x \mid t_0, V(t_0, \cdot)) = \min_u\{V(t_0, \cdot) + \int_{t_0}^{\tau} d^2(x(t), \mathcal{Y}(t))dt\},$$

under the restriction $x(\tau) = x$. The function $V(t_0, \cdot)$ is any boundary condition. In (4) the boundary condition is $V(t_0, x) = d^2(x(t_0), \mathcal{X}^0)$.

Theorem 3.1 *The value function $V(\tau, x)$ satisfies* **the principle of optimality**, *which has the semigroup form*

$$V(\tau, x \mid t_0, V(t_0, \cdot)) = V(\tau, x | t, V(t, \cdot \mid t_0, V(t_0, \cdot))), \;\; t_0 \leq t \leq \tau. \tag{5}$$

These properties are established through conventional arguments [4] which also yield similar properties for reach sets. The solution of the reachability problem can be cast in the form of a solution of the "forward" HJB equation that follows from (5). To develop this approach we further assume that the functions $V(t, x)$ and $\varphi(t, x)$ are *differentiable*.

A standard procedure [4] yields

$$V_t(t, x) + \max_u\{(V_x(t, x), f(t, x, u)) - d^2(x, \mathcal{Y}(t))\} = 0, \;\; t_0 \leq t \leq \tau. \tag{6}$$

with boundary condition $V(t_0, x) = d^2(x, \mathcal{X}^0)$. Here V_t, V_x stand for the partial derivatives of $V(t, x)$. Note that the term $d^2(x, \mathcal{Y}(t)) \neq 0$ only outside the state constraint $\mathcal{Y}(t)$.

An alternative scheme relies on the value function

$$V(\tau, x) = \min_u\{\varphi_0(t_0, x(t_0))\}. \tag{7}$$

The minimum in (7) is over all $u(t) \in \mathcal{P}(t)$, under restrictions $x(\tau) = x$ and $\varphi(t, x) \leq 1,\ t \in [t_0, \tau]$. Here

$$\varphi_0(t_0, x) = (x - x^0, X^0(x - x^0))^{1/2},\ X^0 = X^{0\prime} > 0,$$
$$\varphi(t, x) = (x - y(t), Y(t)(x - y(t)))^{1/2},\ Y(t) = Y'(t) > 0.$$

Thus, in this scheme, the initial condition and the state constraints are ellipsoidal, which respectively take the form

$$\varphi_0(t_0, x) \leq 0 \text{ and } \varphi(t, x(t)) \leq 1,\ \ t \in [t_0, \tau].$$

The reach set at time t is

$$\mathcal{X}(\tau, t_0, \mathcal{X}^0) = \{x \mid V(\tau, x) \leq 1\},$$

which is thus a level set of $V(\tau, x)$ and the previously stated *semigroup property* of the value function (the principle of optimality) is again true.

The solution of the reachability problem can again be cast in the form of a solution of the "forward" HJB equation—now somewhat different from (6). Again assume the functions $V(t, x)$ and $\varphi(t, x)$ to be *differentiable*.

Denote

$$\mathcal{H}(t, x, V, u) = V_t(t, x) + (V_x(t, x), f(t, x, u)). \tag{8}$$

Lemma 3.2 *The formal HJB equation is*

$$\begin{aligned} \max_{u \in \mathcal{P}(t)} \mathcal{H}(t, x, V, u) &= 0,\ \textit{if}\ \varphi(t, x) < 1, & (9)\\ \textit{and}\quad \max_{u \in \mathcal{P}_s(t)} \mathcal{H}(t, x, V, u) &= 0,\ \textit{if}\ \varphi(t, x) = 1, & (10)\\ \textit{with}\ \ \mathcal{P}_s(t) &= \mathcal{P}(t) \cap \{u \mid \mathcal{H}(t, x, \varphi, u) \leq 0\}, & (11) \end{aligned}$$

and with the boundary condition

$$V(t_0, x) = \varphi_0(t_0, x). \tag{12}$$

We sketch a proof of Lemma 3.2.

With $\varphi(t, x) < 1$ the principle of optimality (5), with $t_0 = \tau - \sigma$, gives

$$0 = \min_u \{V(\tau - \sigma, x(\tau - \sigma)) - V(\tau, x)\}, \tag{13}$$

or

$$\max_{u \in \mathcal{P}(t)} \{\mathcal{H}(t, x, V, u)\} = \mathcal{H}(t, x, V, u^0) = 0. \tag{14}$$

With $\varphi(t, x) = 1$ we apply the same principle but only through those controls that do not allow the trajectory to move outside the state constraint. These are

$$\mathcal{P}_s = \{u \in \mathcal{P}(t) \cap \{u \mid \mathcal{H}(t, x, \varphi, u) \leq 0\}\}.$$

Let $u^0(t, x)$ be the optimal control for the trajectory $x^0(t)$ that starts at $x(\tau) = x$ and minimizes $\varphi_0(t_0, x(t_0))$ under constraints $\varphi(t) \leq 1,\ t \in [t_0, \tau]$.

Note that with $\varphi(t, x(t)) = 1$ we have two cases: either $\mathcal{H}(t, x(t), \varphi, u^0) = 0$, which means the related optimal trajectory runs along the state constraint, or

$$d\varphi(t, x(t))/dt|_{u=u^0} = \mathcal{H}(t, x(t), \varphi, u^0) < 0,$$

so the optimal trajectory departs from the state constraint, and for $\sigma > 0$ we have $\varphi(t + \sigma, x(t + \sigma)) < 1$. Relations $\mathcal{H}(t, x(t), \varphi, u^0) \equiv 0$ and (14) allow one to find the control u^0 along the state constraint.

If all the operations in (6) (9), (10) result in smooth functions, then these equations may have a classical solution [4]. Otherwise (9), (10) is a symbolic generalized HJB equation, which has to be described in terms of subdifferentials, Dini derivatives or their equivalents.

However, the typical situation is that V *is not differentiable.* The treatment of (6), (9), (10) then involves the notion of generalized "viscosity" solution for this equation or its equivalents [4, 28]. One approach is to use the method of characteristics as developed for this type of equation [3, 28]. But it is a fairly complicated procedure, especially in the nonsmooth case for which the method requires additional refinement. Another route is to look for the solution through level set methods [27].

However, for the specific "linear-convex" problems of this paper the function $V(t, x)$ is indeed differentiable. Moreover, in this case an effective ellipsoidal technique may be applied, which allows one to bypass calculation of solutions to the HJB equation.

4 Linear-convex systems

Consider the linear system

$$\dot{x} = A(t)x + B(t)u, \quad t_0 \leq t \leq t_1, \tag{15}$$

in which $A(t), B(t)$ are continuous, and the system is completely controllable (see [17]). The control set $\mathcal{P}(t)$ is a nondegenerate ellipsoid, $\mathcal{P}(t) = \mathcal{E}(p(t), P(t))$, and

$$\mathcal{E}(p(t), P(t)) = \{u \mid (u - p(t), P^{-1}(t)(u - p(t)) \leq 1\}, \tag{16}$$

with $p(t) \in \mathbb{R}^p$ (the center of the ellipsoid) and symmetric positive definite matrix function $P(t) \in \mathbb{R}^{p \times p}$ (the shape matrix of the ellipsoid) continuous in t. The *support function* of the ellipsoid is

$$\rho(l \mid \mathcal{E}(p(t), P(t))) = \max\{(l, x) \mid x \in \mathcal{E}(p(t), P(t))\} = (l, p(t)) + (l, P(t)l)^{1/2}.$$

The *state constraint* is

$$x(t) \in \mathcal{Y}(t), \quad t \in [t_0, t_1], \tag{17}$$

in which $\mathcal{Y}(t)$ is an ellipsoidal-valued function: $\mathcal{Y}(t) = \mathcal{E}(y(t), Y(t)),\ \ Y(t) = Y'(t) > 0, Y \in \mathbb{R}^{n\times n}$ and with $y(t), Y(t)$ absolutely continuous. Lastly $\mathcal{X}^0 = \mathcal{E}(x^0, X^0)$ is also an ellipsoid.

In [11] and [14] it is also assumed that the constraints on the controls and initial values are ellipsoidal which make the explanations more transparent. However these methods are applicable to box-valued constraints as well [14], [7].

Lemma 4.1 *For the linear system (15), with convex-valued restrictions (3), the reach set $\mathcal{X}(\tau, t_0, \mathcal{X}^0)$ at time τ is convex and compact.*

Problem 3.1 of calculating the reach set is now formulated as follows.

Problem 4.1 *Calculate the support functions $\rho(l \mid \mathcal{X}(\tau, t_0, \mathcal{X}^0)),\ l \in \mathbb{R}^n$.*

This is equivalent to solving the following optimal control problem with state constraints:

$$\begin{array}{rcl} \rho(l \mid \mathcal{X}(\tau, t_0, \mathcal{X}^0)) = & \max & (l, x(\tau)) \\ & \text{subject to} & x(t_0) \in \mathcal{X}^0,\ u(t) \in \mathcal{P}(t),\ x(t) \in \mathcal{Y}(t),\ t_0 \le t \le \tau. \end{array}$$

This optimal control problem may be approached through the maximum principle [21, 10, 29]. To handle the state constraints $\mathcal{X}[\tau]$ one usually imposes the following *constraint qualification.*

Assumption 4.1 *There exist a control $u(t) \in \mathcal{P}(t),\ t \in [t_0, \tau],\ x^0 \in \mathcal{X}^0$, and $\varepsilon > 0$, such that the trajectory $x[t] = x(t, t_0, x^0) = x(t, t_0, x^0 \mid u(\cdot))$ generated by $u(t)$ produces a tube*

$$x(t, t_0, x^0) + \epsilon \mathcal{B}_m(0) \subset \mathcal{Y}(t),\ t \in [t_0, \tau].$$

Here $\mathcal{B}_m(0) = \{z : (z, z) \le 1,\ \ z \in \mathbb{R}^m\}$.

The techniques of convex analysis yield the following assertion [10, 8].

Theorem 4.1 *Under Assumption 4.1 the support function $\rho(l \mid \mathcal{X}[\tau])$ is given by*

$$\rho(l \mid \mathcal{X}[\tau]) = \min\{\Psi(\tau, t_0, l, \Lambda(\cdot)) \mid \Lambda(\cdot) \in \mathbf{V}_n[t_0, \tau]\}, \tag{18}$$

in which

$$\Psi(\tau, t_0, l, \Lambda(\cdot)) = s(t_0, \tau, l \mid \Lambda(\cdot))x^0 + (s(t_0, \tau, l \mid \Lambda(\cdot))X^0 s'(t_0, \tau, l \mid \Lambda(\cdot)))^{1/2}$$

$$+ \int_{t_0}^{\tau} \Big(s(t, \tau, l \mid \Lambda(\cdot))B(t)p(t) + (s(t, \tau, l \mid \Lambda(\cdot))B(t)P(t)B'(t)s'(t, \tau, l \mid \Lambda(\cdot)))^{1/2}\Big)dt$$

$$+ \int_{t_0}^{\tau} (d\Lambda(t)y(t) + (d\Lambda(t)Y(t)d\Lambda(t))^{1/2}.$$

Above, $s(t, \tau, l \mid \Lambda(\cdot)),\ t \leq \tau$, is the row-vector solution to the adjoint equation

$$ds = -sA(t)dt + d\Lambda(t), \quad s(\tau) = l, \tag{19}$$

and $\Lambda(\cdot) \in \mathbf{V}_n[t_0, \tau]$, the space of n-dimensional functions of bounded variation on $[t_0, \tau]$. We also used the notation

$$\int_{t_0}^{\tau} (d\Lambda(\cdot)Y(t)d\Lambda(\cdot))^{1/2} = \max\Big(\int_{t_0}^{\tau} d\Lambda(t)h(t) \mid h(t) \in \mathcal{E}(0, Y(t)), t \in [t_0, \tau]\Big).$$

This is the maximum of a Stieltjes integral over continuous functions $h(t) \in \mathcal{E}(0, Y(t))$.

The minimum over $\Lambda(\cdot)$ in Theorem 4.1 is reached because of Assumption 4.1. Let $\Lambda^{(0)}(\cdot) = \arg\min\{\Psi(\tau, t_0, l, \Lambda(\cdot)) \ |\Lambda(\cdot) \in \mathbf{V}_n[t_0, \tau]\}$ be the minimizer of (18) and let $s^{(0)}[t] = s(t, \tau, l \mid \Lambda^{(0)}(\cdot))$ be the solution to (19) with $\Lambda(\cdot) = \Lambda^{(0)}(\cdot)$.

Theorem 4.2 The "standard" maximum principle under state constraints*[10, 8]. For Problem 4.1 under Assumption 4.1, the optimal control $u^{(0)}(t)$, initial condition $x^{(0)}$ and trajectory $x^{(0)}[t] = x(t, t_0, x^{(0)}|u^{(0)}(\cdot)),\ z^0(t) = x^0(t)$, must satisfy the "maximum principle"*

$$s^{(0)}[t]u^{(0)}(t) = \max\{s^{(0)}[t]u \mid u \in \mathcal{E}(p(t), P(t))\}, \tag{20}$$

and the "maximum conditions"

$$\int_{t_0}^{\tau} (d\Lambda^{(0)}(t)y(t) + (d\Lambda^{(0)}(t)Y(t)d\Lambda^{(0)'}(t))^{1/2}) = \int_{t_0}^{\tau} d\Lambda^{(0)}(t)z^{(0)}[t] \tag{21}$$

$$= \max\{\int_{t_0}^{\tau} (d\Lambda^{(0)}(t)z(t)) \mid z(t) \in \mathcal{E}(y(t), Y(t)))\},$$

$$s^{(0)}[t_0]x^{(0)}[t_0] = s^{(0)}[t_0]x^0 + (s^{(0)}[t_0], X^0 s^{(0)}[t_0])^{1/2} = \max\{s^{(0)}[t_0]x \mid x \in \mathcal{E}(x^0, X^0)\}. \tag{22}$$

As indicated in [5], the minimum over $\Lambda(\cdot) \in \mathbf{V}_n[t_0, \tau]$ in (18) may be replaced by the minimum over the pair $\{M(t), \lambda(t)\}$, with $d\Lambda(t) = l'M(t)d\lambda(t)$, in which the $n \times n$ matrix $M(t)$ is continuous, and $\lambda(\cdot) \in \mathbf{V}_1[t_0, \tau]$ is a scalar function of bounded variation. Moreover, $M(t)$ may be chosen within a compact set $\mathcal{C}_0$ of continuous functions. We summarize this result as follows.

Lemma 4.2 *The multiplier $\Lambda^{(0)}(t)$ may be represented as*

$$\Lambda^{(0)}(t) = l'M^{(0)}(t)\lambda^{(0)}(t), \tag{23}$$

with $M^{(0)}(\cdot) \in \mathcal{C}_0,\ \lambda^{(0)}(\cdot) \in \mathbf{V}_1[t_0, \tau]$.

Denote by $S(t, \tau \mid M(\cdot), \lambda(\cdot))$ the solution of the matrix equation

$$dS = -SA(t)dt + M(t)d\lambda(t), \quad S(\tau) = I. \tag{24}$$

This is a symbolic expression for the linear differential equation whose solution for fixed τ is

$$S(t, \tau) = \exp\left(\int_t^\tau A(s)ds\right) - \int_t^\tau \left(\exp \int_t^s A(\xi)d\xi\right) M(s)d\lambda(s),$$

in which the second integral is a Stiltjes integral.

As in [9], the single-valued functional $\Psi(\tau, t_0, l, \Lambda(\cdot))$ may be substituted by the set-valued integral

$$R(\tau, t_0, M(\cdot), \lambda(\cdot), \mathcal{X}^0) = S(t_0, \tau \mid M(\cdot), \lambda(\cdot))\mathcal{E}(x^0, X^0) +$$

$$+ \int_{t_0}^\tau S(t, \tau \mid M(\cdot), \lambda(\cdot))B(t)\mathcal{E}(p(t), P(t))dt + \int_{t_0}^\tau d\lambda(t)M(t)\mathcal{E}(y(t), Y(t))dt, \tag{25}$$

which yields the next result [9].

Lemma 4.3 *The following equalities hold*

$$\mathcal{X}[\tau] = \cap\{R(\tau, t_0, M(\cdot), \lambda(\cdot), \mathcal{X}^0) \mid M(\cdot), \lambda(\cdot))\} = \cap\{R(\tau, t_0, M(\cdot), \lambda(\cdot), \mathcal{X}^0) \mid M(\cdot), \lambda \equiv t\},$$

for $M(\cdot) \in \mathcal{C}_0, \lambda \in \mathbf{V}_1[t_0, \tau]$.

Note that $l'M(t)d\lambda(t) = d\Lambda(\cdot)$ is the Lagrange multiplier responsible for the state constraint, with $M(t)$ continuous in t, and $\lambda(t)$, with bounded variation, responsible for the jumps of the multiplier $\Lambda(\cdot)$. Such properties of the multipliers are due to the linearity of the system and the type of the constraints on u, x, taken in this paper (see [8], [5]). In general, the multipliers responsible for the state constraints may be represented through a measure of general type and therefore may contain singular components. This is not the case, however, in the problems of this paper.

Assumption 4.2 *The multipliers* $\Lambda^{(0)}(t), \lambda^{(0)}(t)$ *do not have singular components.*

This assumption holds if the support function $\rho(q \mid \mathcal{Y}(t))$ is absolutely continuous in t, uniformly in $q, (q, q) \leq 1$, and for each l the optimal trajectory for Problem 4.1 reaches the boundary $\partial\mathcal{Y}(t)$ on a finite number of intervals.

Lemma 4.3 gives the reach set at time τ as an intersection of sets $R(\tau, t_0, M(\cdot), \lambda(\cdot)), \mathcal{X}^0)$ parametrized by functions $M(\cdot) \in \mathcal{C}_0$. Theorem 4.1 or Lemma 4.3 may be used to calculate the reach set $\mathcal{X}[\tau]$ *at each time* τ.

In order to use Lemma 3.2 we must interpret the solutions of this section in terms of value functions. Let us find the value function that allows us to obtain the reach set $\mathcal{X}(\tau, t_0, X^0)$

for linear systems under a state constraint given by continuous, convex, compact-valued function $\mathcal{Y}(t)$.

Let
$$V(\tau, x) = \min_u \{d(x(t_0), X^0) \mid x(\tau) = x, x(t) \in \mathcal{Y}(t),\ t \in [t_0, \tau]\}.$$
Then, using the techniques of convex analysis along the lines of [9], [11], [8] we come to

$$V(\tau, x) =$$
$$\max\left\{ \int_{t_0}^{\tau} (s(t, t_0, l), x) - \int_{t_0}^{\tau} \rho\Big(s(\xi, t_0, l) \mid B(\xi)\mathcal{P}(\xi)\Big) d\xi - \int_{t_0}^{\tau} \rho(\lambda(s) \mid \mathcal{Y}(s)) ds \right.$$
$$\left. \mid (l, l) = 1, \lambda(\cdot) \in \mathbf{C}[t_0, \tau] \right\}.$$

Here $\mathbf{C}[t_0, \tau]$ is the space of n-dimensional continuous vector functions and $s(t, t_0, l)$ is the solution of (19) with $d\Lambda = \lambda(t)dt$. This value function will eventually lead to a reach set formula similar to those used above in this section and to relations for reach sets of Definition 2.1.

To proceed further we need additional information about the structure of solutions to Problem 4.1.

5 Additional facts

One difficulty in solving the control problem under state constraints (Problem 4.1) is to determine the set of times $\{t \mid x(t) \in \partial\mathcal{Y}(t)\}$. This is a union of closed intervals during which the optimal trajectory is on the boundary of the state constraint.

Here are some helpful facts. For Problem 4.1 under Assumptions 4.1, 4.2, properties 5.1, 5.2, 5.3 are shown in [10, 8, 5].

Property 5.1 *For any $l \in \mathbb{R}^n$, the minimizer of (18) $\Lambda^{(0)}(t) = const$ and the corresponding $\lambda^{(0)}(t) = const$ during time intervals for which $x(t) \in int\mathcal{Y}(t)$, the interior of $\mathcal{Y}(t)$.*

Since $l'M^{(0)}(t)d\lambda^{(0)}(t) = d\Lambda^{(0)}(t)$ we may track whether the trajectory is on the boundary of $\mathcal{Y}(t)$ by the multiplier $\lambda^{(0)}(t)$. Thus, we need not be interested in values of $M^{(0)}(t)$ when the trajectory is not on the boundary $\partial\mathcal{Y}(t)$.

Property 5.2 *Suppose $l \in \mathbb{R}^n$ is given and $x^0(t)$ is the solution of Problem 4.1. For the function $\Lambda^{(0)}(t)$ to have a jump at t^*, under our smoothness conditions on the state constraint, t^* must be a time of arrival or departure from the boundary of the tube*

$\mathcal{E}(y(t), Y(t))$, $t_0 \le t \le \tau$, *and the trajectory* $x^{(0)}(t)$ *must be* tangent *to the tube* $\mathcal{E}(y(t), Y(t))$ *at* t^* *(see [8]). Thus* $x^{(0)}(t)$ *is differentiable at* t^* *and*

$$(\chi, \dot{x}^{(0)}(t^* - 0)) = ((\chi, \dot{x}^{(0)}(t^* + 0)) = 0. \tag{26}$$

Here χ *is the support vector to the state constraint* $\mathcal{Y}(t^*)$ *at* $x^{(0)}(t^*)$*. In general, if the jump is* $\Lambda(t^* + 0) - \Lambda(t^* - 0) = \chi$*, then the necessary condition for such a jump is*

$$(\chi, \dot{x}^{(0)}(t^* + 0) - \dot{x}^{(0)}(t^* - 0)) \ge 0. \tag{27}$$

For example, if $x^{(0)}(t^* - \sigma), \sigma > 0$, lies inside the interior of the constraint set and $x(t^* + \sigma)$ lies on the boundary, then

$$(\chi, \dot{x}^{(0)}(t^* + 0) - \dot{x}^{(0)}(t^* - 0)) \le 0,$$

and (27) will be fulfilled only if the last relation gives zero.

We also need the following assumption.

Assumption 5.1 *For Problem 4.1, for given* l*, there exists no control* $u^*(s)$ *that satisfies the maximum principle*

$$l'S(s, \tau \mid M^{(0)}(\cdot), \lambda^{(0)}(\cdot))B(s)u^*(s) = \max\{l'S(s, \tau \mid M^{(0)}(\cdot), \lambda^{(0)}(\cdot))B(s)u \mid u \in \mathcal{E}(p(s), P(s))\}$$

for $\{s \mid l'S(s, \tau \mid M^{(0)}(\cdot), \lambda^{(0)}(\cdot))B(s) \ne 0\}$ *and at the same time ensures for these values of* s *that the corresponding trajectory* $x(s) \in \partial\mathcal{Y}(s)$*.*

Property 5.3 . *Assumption 5.1 does hold.*

This means that the control $u^{(0)}(s)$, determined by the maximum principle, with $h^{(0)}(\tau, s) = l'S(s, \tau \mid M^{(0)}(\cdot), \lambda^{(0)}(\cdot)))B(s) \ne 0$, cannot also keep the corresponding trajectory $x^{(0)}(s)$ along the boundary $\partial\mathcal{Y}(s)$. In other words, in this case *the maximum principle is degenerate along the state constraint*, i.e., $h^{(0)}(\tau, s) \equiv 0$, and does not help to find the control when the trajectory lies on the boundary $\partial\mathcal{Y}(s)$.

Lemma 5.1 *Under Assumption 5.1 applied to Problem 4.1 the function*

$$h^{(0)}(\tau, s) = l'S(s, \tau \mid M^{(0)}(\cdot), \lambda^{(0)})(\cdot)B(s) \equiv 0,$$

whenever $x^{(0)}(s) \in \partial\mathcal{Y}(s)$*.*

We now look for a "recursive" version of the Maximum Principle.

6 The maximum principle in recursive form

In Theorem 4.1 we indicated the solution to Problem 4.1 for any *fixed* time $\tau > t_0$. However, our objective is to recursively calculate the whole tube $\mathcal{X}[\tau], \tau \geq t_0$. This means that while solving Problem 4.1 for increasing values of τ, we want a procedure that does not require one to solve the whole problem "afresh" for each new value of τ, but allows the use of the solutions for previous values.

However, given $M(\cdot), \lambda(\cdot)$, one may observe that in general $R(\tau, t_0, M(\cdot), \lambda(\cdot), \mathcal{X}^0) \neq R(\tau, t, M(\cdot), \lambda(\cdot), R(t, t_0, M(\cdot), \lambda(\cdot), \mathcal{X}^0))$, that is, R given by (25) does not satisfy the semi-group property.

Therefore Theorem 4.2 and Lemmas 4.2, 4.3 have to be modified to meet the recursion requirements.

Under Assumptions 4.1, 4.2 let us first restrict Λ in (23) to satisfy the relation $d\Lambda(t) = l'M(t)d\lambda(t)$ where $\lambda(t)$ is *absolutely continuous*. This means $\Lambda(t) = M(t)l(t)$, and $l(t) = d\lambda(t)/dt \equiv 0$ for $t \in T_Y = \{t \mid x_l^{(0)}(t) \in int\mathcal{Y}(t)\}$, when $x_l^{(0)}(t)$ is the optimal trajectory for Problem 4.1, for the given l.

Remark 6.1 *We may always take $l(t)$ as*

$$l(t) \equiv 1,\ t \notin T_Y;\ \ l(t) \equiv 0,\ t \in T_Y.$$

Denoting $L(t) = S^{-1'}(t, \tau | M(\cdot), l(\cdot))M(t)l(t)$, we may replace equation

$$dS/dt = -SA(t) + M(t)l(t), \quad S(\tau, \tau \mid M(\cdot), l(\cdot)) = I,$$

whose solution is $S(t, \tau \mid M(\cdot), l(\cdot))$ (similar to (24)) by

$$dS_L/dt = -S_L(A(t) - L(t)), \quad S_L(\tau, \tau) = I,$$

whose solution is $S_L(t, \tau)$. Then $S(t, \tau \mid M(\cdot), l(\cdot)) \equiv S_L(t, \tau),\ t \in [t_0, \tau]$.

In this case $R(\tau, t_0, M(\cdot), l(\cdot), \mathcal{X}^0)$ transforms into

$$\mathcal{R}(\tau, t_0, L(\cdot)) = S_L(t_0, \tau)\mathcal{E}(x^0, X^0) + \int_{t_0}^{\tau} S_L(t, \tau)(\mathcal{E}(B(t)p(t), B(t)P(t)B'(t)) +$$

$$+ L(t)\mathcal{E}(y(t), Y(t)))dt = \mathcal{X}(\tau, t_0, L(\cdot), \mathcal{X}^0) = \mathcal{X}_L[\tau],$$

and $\mathcal{X}_L[\tau]$ turns out to be the solution to the differential inclusion

$$\dot{x}(t) \in (A(t) - L(t))x + L(t)\mathcal{E}(y(t), Y(t))) + \mathcal{E}(B(t)p(t), B(t)P(t)B'(t)), \tag{28}$$

$$t \geq t_0, \quad x^0 \in \mathcal{E}(x^0, X^0). \tag{29}$$

Here also the compact set $\mathcal{C}_0$ of functions $M(t)$ transforms into a compact set $\mathcal{C}_{00}$ of functions $L(t)$.

We thus arrive at the following important property proved in [9].

Lemma 6.1 *The reach set* $\mathcal{X}[\tau]$ *is the intersection*

$$\mathcal{X}[\tau] = \cap\{\mathcal{X}_L(\tau, t_0, \mathcal{E}(x^0, X^0)) \mid L(\cdot)\} \tag{30}$$

of the 'cuts' or 'sections' $\mathcal{X}_L(\tau, t_0, \mathcal{E}(x^0, X^0))$ *of the reach tubes (solution tubes)* $\mathcal{X}_L(\cdot) = \{\mathcal{X}_L[t] : t \geq t_0\}$ *of the differential inclusion (28), (29). The intersection is over all* $L(\cdot) \in \mathcal{C}_{00}$*, a compact set of continuous matrix functions* $L(t), t \in [t_0, \tau]$*.*

A calculation using convex analysis, similar to [9], yields the next formula.

Theorem 6.1 *The support function*

$$\rho(l \mid \mathcal{X}(t, t_0, \mathcal{E}(x^0, X^0)) = \inf\{\rho(l \mid \mathcal{X}_L(\tau, t_0, \mathcal{E}(x^0, X^0)) \mid L(\cdot)\}, \tag{31}$$

with

$$\rho(l \mid \mathcal{X}_L(\tau, t_0, \mathcal{E}(x^0, X^0))) = (l, x_L^\star(t)) + (l, G_L(t, t_0)X^0 G_L'(t, t_0)l)^{1/2} + \tag{32}$$

$$\int_{t_0}^t (l, G_L(t, s)B(s)P(s)B'(s)G_L'(t, s)l)^{1/2} ds + \int_{t_0}^t (l, G_L(t, s)L(s)Y(s)L'(s)G_L'(t, s)l)^{1/2} ds.$$

Here

$$x_L^\star(t) = G_L(t, t_0)x^0 + \int_{t_0}^t G_L(t, s)B(s)p(s)ds,$$

and $G_L(t, s) = S_L(s, t)$ is the transition matrix for the homogeneous system

$$\dot{x} = (A(t) - L(t))x \ ; \ \ G_L(t, t) = I, \ \ L(\cdot) \in \mathcal{C}_{00}.$$

The significance of the last result is that in Problem 4.1 the support function of the intersection (30) is equal to the pointwise infimum (31) of the support functions rather than to their infimal convolution as specified by general theory [24].

It is not unimportant to specify when the infimum in (31) is attained, that is, it is actually a minimum. Indeed it may happen that for a given l the minimum over $L(\cdot)$ is in the class $\mathcal{C}_{00}$. But to ensure the minimum is always reached, we have to broaden the class of functions $L(\cdot)$.

To illustrate how to continue the procedure we will assume the following.

Assumption 6.1 *For each* $l \in \mathbb{R}^n$ *the optimal trajectory* $x_l^0(t)$ *of Problem 4.1 visits the boundary* $\partial\mathcal{Y}(t)$ *only during* **one time interval** $[t_1, t_2], \ t_0 \leq t_1, \ t_2 \leq \tau$*.*

(The case of finite or countable variety of such intervals is treated in a similar way.)

Then, instead of the product $M(t)l(t)$, we must deal with multipliers of the form $M_*(t) = M(t)l(t) + M_1\delta(t-t_1) + M_2\delta(t-t_2)$, where M_1, M_2 are $n \times n$ matrices.

In order to match the formulas for $R(\tau, t_0, M_*(\cdot), l(\cdot))$ and its transformed version $\mathcal{R}(\tau, t_0, L_*(\cdot))$, we must introduce a new multiplier $L_*(t) = L(t) + L_1\delta(t-t_1) + L_2\delta(t-t_2)$ under transformation $L_*(t) = S^{-1}(t, \tau \mid M_*(\cdot))M_*(t)$.

Following the schemes of [5], [9], it is possible to rewrite the preceding assertions.

Lemma 6.2 *The support function*

$$\rho(l \mid \mathcal{X}(t, t_0, \mathcal{E}(x^0, X^0)) = \min\{\rho(l \mid \mathcal{X}_{L_*}(t, t_0, \mathcal{E}(x^0, X^0)) \mid L_*(\cdot)\}, \tag{33}$$

with

$$\rho(l \mid \mathcal{X}_{L_*}(\tau, t_0, \mathcal{E}(x^0, X^0)) = \Phi(l, L_*(\cdot), t, t_0), \tag{34}$$

and

$$\Phi(l, L_*(\cdot), t, t_0) = (l, x_*(t)) + (l, G_*(t, t_0)X^0G_*'(t, t_0)l)^{1/2}+ \tag{35}$$

$$\int_{t_0}^{t}(l, G_*(t,s)B(s)P(s)B'(s)G_*'(t,s)l)^{1/2}ds + \int_{t_0}^{t}(l, G_*(t,s)L(s)Y(s)L'(s)G_*'(t,s)l)^{1/2}ds+$$

$$\int_{t_0}^{t}(l, G_*(t,s)L_1Y(s)L_1'G_*'(t,s)l)^{1/2}d\chi(s,t_1) + \int_{t_0}^{t}(l, G_*(t,s)L_2Y(s)L_2'G_*'(t,s)l)^{1/2}d\chi(s,t_2).$$

Here $G_*(t,s)$ is the transition matrix for the homogeneous system

$$dx(t) = (A(t) - L(t))xdt - L_1xd\chi(s,t_1) - L_2xd\chi(s,t_2),$$

namely,

$$G_*(t,s) = \exp\Big(\int_s^t (A(\xi) - L(\xi))dt - \int_s^t (L_1d\chi(s,t_1) + L_2d\chi(s,t_2))\Big),$$

so that

$$x(t) = G_*(t, t_0)x^0;$$

and $\chi(s,t')$ is the step function,

$$\chi(s,t') \equiv 0, s < t';\ \chi(s,t') \equiv 1, s \geq t',\ d\chi(s,t')/ds = \delta(s-t').$$

The vector $x_*(t)$ in (35) may be described by

$$\begin{gathered} dx_*(t) = ((A(t) - L(t))x_* + B(t)p(t) + L(t)y(t))dt, \\ -L_1(x_* - y(t))d\chi(s,t_1) - L_2(x_* - y(t))d\chi(s,t_2)), x_*(0) = x^0. \end{gathered} \tag{36}$$

Remark 6.2 *Note that in the exponent $G_*(t,s)$ we have the difference of a Riemann integral and a Riemann-Stiltjes integral. On the other hand, the last two integrals in (35) should formally be interpreted as a Lebesgue-Stieltjes integrals, [19, 25]. This does not cause additional difficulty since the multipliers L_1, L_2 are among the optimizers in (33).*

A result similar to (30) is all the more true for the sets $\mathcal{X}_{L_*}(\tau, t_0, \mathcal{E}(x^0, X^0))$.

Corollary 6.1 *The following intersection holds:*

$$\mathcal{X}[\tau] = \mathcal{X}(\tau, t_0, \mathcal{E}(x^0, X^0)) = \cap\{\mathcal{X}_{L_*}(\tau, t_0, \mathcal{E}(x^0, X^0) \mid L_*(\cdot)\}.$$

The difference between Lemmas 6.1 and 6.2 is that in the former it is not guaranteed that the boundary $\partial\mathcal{X}[\tau]$ is touched at each point by one of the intersecting sets $\mathcal{X}_L(\tau, t_0, \mathcal{E}(x^0, X^0))$, whereas in the latter the boundary $\partial\mathcal{X}[\tau]$ is indeed touched at each point by one of the sets $\mathcal{X}_{L_*}(\tau, t_0, \mathcal{E}(x^0, X^0))$. This is because the minimum in (33) is attained.

Under the Assumptions 4.1, 4.2, 5.1, 6.1 the reasoning above leads to the next result.

Lemma 6.3 *The reach set $\mathcal{X}[t] = \mathcal{X}(t, t_0, \mathcal{E}(x^0, X^0))$ is a convex compact set in $\mathbb{R}^n$ which evolves continuously in t.*

The boundary of the reach set $\mathcal{X}[t]$ has an important characterization. Consider a point x^* on the boundary $\partial\mathcal{X}[\tau]$ of the reach set $\mathcal{X}[\tau] = \mathcal{X}(\tau, t_0, \mathcal{E}(x^0, X^0))$.[1]

Then there exists a *support vector* z^* such that

$$(z^*, x^*) = \rho(z^* \mid \mathcal{X}[\tau]).$$

Let $L_*^0(\cdot)$ be the minimizer for the problem (see (33))

$$\rho(l \mid \mathcal{X}[\tau]) = \min_{L_*}\{\Phi(l^*, L_*(\cdot), \tau, t_0) \mid L_*(\cdot)\} = \Phi(z^*, L_*^0(\cdot), \tau, t_0). \tag{37}$$

Then the control $u = u^*(t)$, the initial state $x(t_0) = x^* \in \mathcal{E}(x^0, X^0)$, and the corresponding trajectory $x^*(t)$ along which system (15) is transferred from state $x^*(t_0) = x^*$ to $x(\tau) = x^*$ are specified by the following "modified maximum principle".

Theorem 6.2 *(The modified maximum principle under state constraints) For Problem 4.1 suppose state x^* is such that*

$$(l^*, x^*) = \rho(l^* \mid \mathcal{X}[\tau]).$$

[1]The boundary $\partial\mathcal{X}[\tau]$ of $\mathcal{X}[\tau]$ may here be defined as the set $\partial\mathcal{X}[\tau] = \mathcal{X}[\tau] \setminus int\mathcal{X}[\tau]$. Under the controllability assumption, $\mathcal{X}[\tau]$ has a non-empty interior, $int\mathcal{X}[\tau] \neq \emptyset$, $\tau > t_0$.

Then the control $u^*(t)$, which transfers the state of the system (15) along the trajectory from $x^*(t_0) = x^{*0}$ under constraints $u(t) \in \mathcal{E}(p(t), P(t))$, $x(t) \in \mathcal{E}(y(t), Y(t))$ while ensuring

$$(l^*, x^*) = \max\{(l^*, x) \mid x \in \mathcal{X}[\tau]\},$$

satisfy the following pointwise "maximum principle" for the control ($s \in [t_0, \tau]$),

$$(z^{*\prime} G^0_*(\tau, s)B(s), u^*(s) = \max_u\{(l^{*\prime} G^0_*(\tau, s)B(s), u) \mid u \in \mathcal{E}(p(s), P(s))\}, \tag{38}$$

$$= (z^*, G_*({}^0\tau, s)B(s)p(s)) + (z^*, G^0_*(\tau, s)B(s)P(s)B'(s)G^0_*(\tau, s)z^*)^{1/2}, s \in [t_0, \tau],$$

and the "maximum conditions" for the system trajectory (pointwise),

$$(z^{*'} G^0_*(\tau, s)L^0_*, x^*(s)) = \max_p\{(z^{*'} G^0_*(\tau, s)L^0_*, p) \mid p \in \mathcal{E}(y(s), Y(s))\} \tag{39}$$

$$= (z^{*'} G^0_*(\tau, s)L^0_* y(s) + (l^*, G^0_*(\tau, s)L^0_*(s)Y(s)L^0_* G^0_*(\tau, s)l^*)^{1/2},$$

and the initial state,

$$(z^*, G^0_*(\tau, t_0)x^{*0}) = \max\{(z^*, x) \mid x \in G_*(\tau, t_0)\} = \rho(z^* \mid \mathcal{E}(x^0, X^0)) = \tag{40}$$

$$(z^*, G^0_*(\tau, t_0)x^0) + (z^*, G^0_*(\tau, t_0)X^0 G^{0\prime}_*(\tau, t_0)z^*)^{1/2}.$$

Here $G^0_*(\tau, s)$ stands for the matrix function $G_*(\tau, s)$ taken for $L^0_*(t)$—the minimizer of problem (33).

The function $h(\tau, s)) = l^{*\prime} G^0_*(\tau, s)B(s)$ is taken right-continuous.[2]

Remark 6.3 *Suppose we want to solve Problem 4.1, seeking for $\rho(l^*(t) \mid \mathcal{X}[t])$ along a curve $l^*(t), t > t_0$. Then, taking $l^*(t) = l'(G^0_*)^{-1}(t, s), l = l_*$, one may observe that the* integrands *in functional $\Phi(l^*(t), L^0_*(\cdot), t, t_0))$ of (33)-(35) will be independent of t. This property ensures the existence of a* recursive *computational procedure as indicated in the next section (see also [14]). The modified maximum principle of this section thus allows a solution in* recursive form*. This is not the case for the standard maximum principle.*

We now pass to the construction of ellipsoidal approximations for the reach sets.

[2] In the general case, under Assumption 4.2, the optimal trajectory may visit the boundary during a countable set of closed intervals, and the function $L^0_*(\cdot)$ allows not more than a countable set of discontinuities of the first order.

7 External ellipsoids

Despite the linearity of the system, the calculation of reach sets directly from the relations above is rather difficult. Among effective methods for such calculations are those that rely on ellipsoidal techniques, as given in [11, 14]. Indeed, although the initial set $\mathcal{E}(x^0, X^0))$ and the control set $\mathcal{E}(q(t), Q(t))$ are ellipsoids, the reach set $\mathcal{X}[t] = \mathcal{X}(t, t_0, \mathcal{E}(x^0, X^0))$ will of course *not* generally be an ellipsoid.

As shown in [11, 14], in the absence of state constraints the reach set $\mathcal{X}[t]$ may be approximated both externally and internally by ellipsoids $\mathcal{E}_-$ and $\mathcal{E}_+$, with $\mathcal{E}_- \subseteq \mathcal{X}[t] \subseteq \mathcal{E}_+$. Here we deal only with external approximations, but taken under state constraints.

An approximation $\mathcal{E}(x_+, X_+)$ is said to be *tight* if there exists a vector $z \in \mathbb{R}^n$ such that $\rho(z \mid \mathcal{E}(x_+, X_+)) = \rho(z \mid \mathcal{X}[t])$ (the ellipsoid $\mathcal{E}(x_+, X_+)$ *touches* $\mathcal{X}[t]$ along direction z). We shall look for external approximations that are tight, on one hand, and are also recursively computable, on the other.

In order to apply ellipsoidal techniques to state state-constrained problems, recall Lemma 6.1, Corollary 6.1 which indicate how the reach set $\mathcal{X}[t] = \mathcal{X}(t, t_0, \mathcal{X}^0)$ may be presented as an intersection of reach sets $\mathcal{X}_L$ or $\mathcal{X}_{L_*}$ for the system (28) or (36) *without state constraints.* We therefore first study how to approximate sets $\mathcal{X}_L$ by ellipsoids.

Problem 7.1 *Given a vector function $l^*(t)$, continuously differentiable in t, find external ellipsoids $\mathcal{E}_+[t] \supset \mathcal{X}_L[t]$ such that* **for all** $t \geq t_0$, *the equalities*

$$\rho(l^*(t) \mid \mathcal{X}_L[t]) = \rho(l^* \mid \mathcal{E}_+[t]) = (l^*(t), x^*(t))$$

hold, so that the supporting hyperplane for $\mathcal{X}_L[t]$ generated by $l^*(t)$, namely, the plane $(x - x^*(t), l^*(t)) = 0$ which touches $\mathcal{X}_L[t]$ at point $x^*(t)$, is also a supporting hyperplane for $\mathcal{E}_+[t]$ and touches it at the same point. Apart from Assumptions 4.1, 4.2 let us assume in this section that the functions $L(\cdot), L_*(\cdot)$ of the sequel do not have any delta function components. (The case when such components are present is treated in the next section).

The solution to Problem 7.1 is given within the following statement.

Theorem 7.1 *With $l^*(t)$ given, the solution to Problem 7.1 is an ellipsoid $\mathcal{E}_+[t] = \mathcal{E}(x^*(t), X_+^*[t])$, in which*

$$X_+^*[t] = \left(\int_{t_0}^{t} (p_u(t,s) + p_Y(t,s))ds + p_0(t,s)\right) \times \tag{41}$$

$$\Bigg(\int_{t_0}^{t} (p_u(t,s))^{-1} G_*^0(t,s) B(s) P(s) B'(s) {G'_*}^0(t,s) ds$$

$$+ \int_{t_0}^{t} (p_Y(t,s))^{-1} G_*^0(t,s) L_{*t}^0(s) Y(s) {L_{*t}^0}'(s) {G'_*}^0(t,s) ds + p_0^{-1}(t) G_*^0(t,t_0) X^0 {G'_*}^0(t,t_0)\Bigg),$$

and

$$p_u(t,s) = (l(t), G_*^0(t,s)P(s){G'_*}^0(t,s)l(t))^{1/2}, \tag{42}$$

$$p_Y(t,s) = (l(t), G_*^0(t,s)L_{*t}^0(s)Y(s)H'L_{*t}^0G'_*(t,s)l(t))^{1/2},$$

$$p_0(t) = (l(t_0), G_*^0(t,t_0)X^0{G_*^0}'(t,t_0)l(t_0))^{1/2}.$$

This result follows from [11], [14]. Since the calculations have to be made for all t, the parametrizing functions $p_u(t,s), p_Y(t,s), s \in [t_0,t], p_0(t)$ must also formally depend on t.

In other words, relations (41),(42) need to be calculated "afresh" for each t. It may be more convenient for computational purposes to have them given in recursive form. As indicated in [14], in the absence of state constraints this could be done by selecting function $l^*(t)$ in an appropriate way. For the case of state constraints we follow Remark 6.3, arriving at the next assumption.

Assumption 7.1 *The function $l^*(t)$ is of the form, ${l^*}'(t) = l'_*G_*(t_0,t)$, with $l_* \in \mathbb{R}^n$ given.*

Under this Assumption the function $l^*(t)$ is the solution to the equation

$$\dot{l}^* = -(A'(t) - {L_*}'(t))l^*,$$

Note that under our assumptions *the minimum over $L_*(\cdot)$ in* (33) *is attained for any $l = z_* \in \mathbb{R}^n$*, the minimizers being denoted as $L_*^0(t) = \{L^0(\cdot) \in \mathcal{C}_{00}\}$.

But prior to moving ahead we have to investigate the following. Suppose element $L_{*t}^0(s)$ is the minimizer of function $\Phi(l, L_{*t}, t, t_0)$. The question is: if we minimize function $\Phi(l, L_{*(t+\sigma)}(s), t+\sigma, t_0), \sigma > 0$, over a larger interval $[t_0, t+\sigma]$ than before, will the minimizer $L_{*(t+\sigma)}^0(s)$, for the latter problem, taken within $s \in [t_0,t]$, be the same as the minimizer $L_{*t}^0(s), s \in [t_0,t]$ for $\Phi(z^*(\cdot), L_{*t}^0(\cdot), t, t_0)$?

The answer to this question is given by the next lemma.

Lemma 7.1 *Taking $l = l^*(t)$ according to Assumption 7.1, suppose $L_{*t}^0(s), s \in [t_0,t]$and $L_{*(t+\sigma)}^0(s), s \in [t_0, t+\sigma], \sigma > 0$ are the minimizers of functionals $\Phi(l, L_*(\cdot), t, t_0)$ and $\Phi(l, L_*(\cdot), t+\sigma, t_0)$ respectively. Then*

$$L_{*t}^0(s) \equiv L_{*(t+\sigma)}^0(s), \quad s \in [t_0,t].$$

The proof is achieved by contradiction.

Following Assumption 7.1, we proceed to deal with the given $L_*^0(\cdot)$, which depends on $l = l_*$ but does not depend on t.

Then $p_u(t,s), p_Y(t,s)), p_0(t)$ of (42) transform into

$$p_u(t,s) = (l_*, G^0_*(t_0,s)P(s){G'_*}^0(t_0,s)l_*)^{1/2} = p_u(s); \tag{43}$$

$$p_Y(t,s) = (l_*, G^0_*(t_0,s)L^0_*(s)Y(s)L^{0'}_*(s){G'_*}^0(t_0,s)l_*)^{1/2} = p_Y(s),$$

$$p_0(t) = (l_*, X^0 l_*)^{1/2} = p_0,$$

matrix $X^*_+[t]$ transforms into

$$X^*_+[t] = \left(\int_{t_0}^t (p_u(s) + p_Y(s))ds + p_0\right)\times \tag{44}$$

$$\left(\int_{t_0}^t (p_u(s))^{-1} G^0_*(t_0,s)B(s)P(s)B'(s){G'_*}^0(t_0),s)ds\right.$$

$$\left.+\int_{t_0}^t (p_Y(s))^{-1} G^0_*(t_0,s)L^0_*(s)Y(s)L^{0'}_*(s){G'_*}^0(t_0,s)ds + p_0^{-1}X^0\right),$$

and the function $\Phi(l^*(t), L^0_{*t}(\cdot), t, t_0)$ transforms into

$$\Phi(l_*, L^0_{*t}(\cdot), t, t_0) = \tag{45}$$

$$(l_*, x^*(t)) + (l_*, X^0 l_*)^{1/2} + \int_{t_0}^t (l_*, G^0_*(t_0,s)B(s)P(s)B'(s){G'_*}^0(t_0,s)l_*)^{1/2}ds +$$

$$\int_{t_0}^t (l_*, G^0_*(t_0,s)L^0_{*t}(s)Y(s){L^0}'_{*t}(s)G^{0'}_*(t_0,s)l_*)^{1/2}ds,$$

$$x^*(t) = x^0 + \int_{t_0}^t G^0_*(t_0,s)(B(s)p(s) + L^0_{*t}(s)y(s))ds.$$

We may now differentiate $X^*_+[t], x^*[t]$. According to our earlier assumptions the necessary condition (26) for the jump in $L^0_*(t)$ is not fulfilled and therefore $L^0_*(t) \equiv L^0(t)$.

Denoting

$$\pi_u(t) = p_u(t)(\int_{t_0}^t (p_u(s) + p_Y(s))ds + p_0)^{-1},$$

$$\pi_Y(t) = p_Y(t)(\int_{t_0}^t (p_u(s) + p_Y(s))ds + p_0)^{-1}, \tag{46}$$

and differentiating $X^*_+[t], x^*[t]$, we arrive at

$$\dot{X}^*(t) = (\pi_u(t) + \pi_Y(t))X^* + (\pi_u(t))^{-1} G^0_*(t_0,t)B(t)P(t)B'(t)G^{0'}_*(t_0,t) \tag{47}$$

$$+(\pi_Y(t))^{-1} G^0_*(t_0,t)L^0(t)Y(t)L^0(t)G^{0'}_*(t_0,t),$$

$$\dot{x}^*(t) = (A(t) - L^0(t))x^* + B(t)p(t) + L^0_*(t)y(t),\ x^*(t_0) = x^0, \tag{48}$$

$$X^*[t_0] = X^0,\ \ x^* = x^0. \tag{49}$$

Returning to the function $l^*(t) = G^0_*(t_0, t)l_*$, we have $X[t] = G^0_*(t, t_0)X^*[t]G^0_*(t, t_0)$ and

$$\dot{X}[t] = (A(t) - L^0(t))X[t] + X[t](A(t) - L^0(t))' + G^0_*(t, t_0)\dot{X}^*[t]G^0_*(t, t_0), \tag{50}$$

so that

$$\dot{X}[t] = (A(t) - L^0(t))X[t] + X[t](A(t) - L^0(t)) \tag{51}$$
$$+(\pi_u(t) + \pi_Y(t))X[t] + (\pi_u(t))^{-1}B(t)P(t)B'(t) + (\pi_Y(t))^{-1}L^0(t)Y(t)L^{0'}(t).$$

Summarizing the results we come to the following.

Theorem 7.2 *Under Assumptions 4.1, 4.2, if the multipliers $L^0_*(t) \equiv L^0(t)$ responsible for the state constraint have no jumps ($d\Lambda(t) = \lambda(t)dt$), the external ellipsoids $\mathcal{E}_+(x^*(t), X[t]) \supseteq \mathcal{X}[t]$ have the form (48),(51) with $X[t_0] = X^0$, $x^* = x^0$.*

Moreover, with $l^*(t) = G^{0'}_*(t_0, t)l_*$, $l_* \in \mathbb{R}^n$, the ellipsoids $\mathcal{E}_+(x^*(t), X[t])$ touch the set $X[t]$ along the curve $x^*(t)$ which satisfies the condition

$$(l^*(t), x^*(t)) = \rho(l^*(t) \mid X[t]),$$

so that

$$(l^*(t), x^*(t)) = \rho(l^*(t) \mid X[t]) = \rho(l^*(t) \mid \mathcal{E}_+(l^*(t), X[t])). \tag{52}$$

We now pass to the more general case, in which the multipliers may have delta function components.

8 Generalized multipliers

In view of Assumptions 4.1, 4.2, 5.1, in this section we assume the following.

Assumption 8.1 *(i) The trajectory $x^*(t)$, $t \in [t_0, \tau]$, which maximizes $(l^*(\tau), x^*(\tau))$, with $l^*(t) = l_* {G'}^0_*(t_0, t)$, $l_* \in \mathbb{R}^n$, satisfies the inclusion $x^*(t) \in \partial\mathcal{Y}(t)$ within a finite number of (closed) intervals $e_i = [\tau_i, \tau_{i+1}]$, $i = 1, \ldots, k$; $t_0 \leq \tau_1$; $\tau_{k+1} \leq t_1$.*

(ii) The minimizer of $\Phi(l^*(t), L_*(\cdot), \tau)$ has the form

$$L^0_*(t) = L^0(t) + \Sigma^k_{i=1}(L^0_{i+1}\delta(t - \tau_{i+1}) - L^0_i\delta(t - \tau_i)),$$

for all $l^*(\tau)$ with $L^0(t)$ absolutely continuous and $L^0(t) \equiv 0$ whenever $x^*(t) \in int\mathcal{Y}(t)$.

Here $L^0_*(\cdot)$ is the generalized derivative of function $\Lambda^0(\cdot)$ – the generalized Lagrange multiplier – which is assumed piecewise absolutely right-continuous, with jumps at points τ_i and possible jump at t_0 and/or τ if it happens that $t_0 = \tau_1$ and/or $\tau = \tau_k$. Function $\Lambda^0 \equiv const$ whenever $x^*(t) \in int\mathcal{Y}(t)$.

Following the reasoning of the previous section, we may derive equations similar to (48) and (47) or (51). The necessary prerequisites for such a move are similar to Lemma 7.1 and Theorems 7.1, 7.2.

Thus, we come to

$$dx^*(t) = ((A(t) - L^0_*(t))x^* + (B(t)p(t) + L^0_*(t)y(t))dt - \sum_{i=1}^{k} L^0_i(x_* - y(t))d\chi_i(t,\tau_i), \quad (53)$$

$$dX[t] = ((A(t) - L^0_*(t))X[t] + X[t](A'(t) - L^{0'}_*(t))dt \quad (54)$$

$$+(\pi_u(t) + \pi_Y(t) + \sum_{i=1}^{k} \pi_i)X[t]dt + (\pi_u(t))^{-1}B(t)P(t)B'(t)dt$$

$$+(\pi_Y(t))^{-1}L^0(t)Y(t)L^{0'}(t)dt + \sum_{i=1}^{k} \pi_i^{-1}L^0_iY(t)L^{0'}_i d\chi(t,\tau_i).$$

$$X[t_0] = X^0, \;\; x^*(t_0) = x^0.$$

The $\pi_i > 0$ are additional parametrizing coefficients similar to $\pi_Y(t)$.

The last equation may be interpreted as before (see Remark 6.2).

Theorem 8.1 *Under Assumption 8.1, with $z(t) = G'^{\,0}_*(t_0,t)l_*,\; l_* \in \mathbb{R}^n$, the external ellipsoids $\mathcal{E}_+(x^*(t), X[t]) \supseteq \mathcal{X}[t]$ have the form (48),(51) with $X[t_0] = X^0,\;\; x^* = x^0$. One may select the parametrizing coefficients $\pi_u, \pi_{Y(t)}, p_i$ so that the ellipsoids $\mathcal{E}_+(x^*(t), X[t])$ touch set $X[t]$ along the curve $x^*(t)$ which satisfies condition (52).*

Remark 8.1 *The results of this paper remain true if the state constraint is applied to the system output $w(t) = Wx(t),\; w \in \mathbb{R}^k,\; k < n$, rather than to the whole state vector x. The state constraint $\mathcal{Y}$ is then given by an elliptical cylinder in $\mathbb{R}^n$. The proofs may be achieved either by directly following the scheme of this paper or by bounding the elliptical cylinder by a sphere of radius r with subsequent limit transition $r \to \infty$.*

9 Example

The system is the double integrator

$$\dot{x}_1 = x_2, \;\; \dot{x}_2 = u, \;\; x(0) = 0, \quad (55)$$

under constraints

$$|u| \le \mu, \quad |x_2| \le \nu, \quad (56)$$

The state constraint may be treated either directly or as a limit as $\epsilon \to 0$ of the ellipsoid $\varepsilon^2 x_1^2 + x_2^2 \le \nu^2$.

The treatment of this example in [8], p.119, indicated the following:

1. the boundary of the reach set $\mathcal{X}[t]$ at each time t consists of three types of points:
   (a) points reached without visiting the state constraint,
   (b) points reached after visiting the state constraint for some time,
   (c) points on the state constraint;
2. each optimal trajectory $x^{(0)}(t)$, for given $l \in \mathbb{R}^2$, visits the state constraint not more than once;
3. the necessary conditions for a jump of the multiplier responsible for the state constraint are not fulfilled—there are no jumps;
4. Assumption 4.1 is fulfilled.

Case (a) is treated according to [14], p.202, while case (c) is trivial. Therefore we concentrate on case (b). The exact parametric equations for case (b) from [14] are

$$x_1 = -\mu(t_1 - \sigma)^2 + c, \;\; x_2 = \mu\sigma + d, \; \tau_1 \leq \sigma \leq \tau.$$

Here $\tau_1 = t_0 + \nu/\mu$ is the first instant when the trajectory reaches the constraint, while σ is a parameter that indicates the instant τ_2 when the trajectory leaves the constraint. Also $c = \nu(\tau - t_0) - \nu^2/2\mu; d = \nu - \mu\tau$.

A typical trajectory for case (b) reaches the constraint at time τ_1, then runs along the constraint and leaves it at time τ_2. Later it runs towards the boundary $\partial\mathcal{X}[\tau]$, while staying inside the constraint.

For the interval $[t_0, \tau]$ the adjoint system is

$$\dot{s} = -sA(t) + H\lambda(t), \;\; H = (0, 1)$$

and Assumption 4.1 implies

$$\lambda(t) \equiv -l_1, \; t \in [\tau_1, \tau_2]; \;\; \lambda(t) \equiv 0, \; t \notin [\tau_1, \tau_2].$$

and $s_1(t) \equiv -l_1$. Here $l_1 = \mu/(\mu^2 + \nu^2)^{1/2}, \; l_2 = \nu/(\mu^2 + \nu^2)^{1/2}$.

To have a recursive formula, consider $l_1 = z_{*1} = \mu/(\mu^2 + \nu^2)^{1/2}, \; l_2 = z_{*2} = \nu/(\mu^2 + \nu^2)^{1/2}$, taking the curve $z^*(t)$ according to Remark 6.3. Then the transformed adjoint equation of Section 6 will have the form

$$\dot{S}_L = -S_L(A(t) - L(t)), \; L(t) = H'K(t), \; K(t) = (0, k(t)), \; H = (1, 0),$$

with $k(t) \equiv 0, \; t < \tau_1, \; t > \tau_2$. Direct calculation alows one to find $k(t), t \in [\tau_1, \tau_2]$, from the equality $l'M(t) = H\lambda(t), \;\; L(t) = S^{-1}(t)M(t)$.

Recalling the ellipsoidal equations we have,

$$\dot{X} = (A(t) - L(t))X + X(A(t) - L(t))' + \tag{57}$$
$$(\pi_Q(t) + \pi_Y(t))X + \pi^{-1}(t)BQ(t)B' + \pi_Y^{-1}(t)L(t)Y(t)L(t),$$

The algorithm for this example involves three steps:

1. For the given starting direction z_* find time $\tau_1 = t_0 + \nu/\mu$ of first exit (encounter with the constraint);

2. Construct the ellipsoidal approximations for the system without constraint, taking $L(t) \equiv 0$. Denote these as $\mathcal{E}(0, X_l^+(\tau))$;

3. For each $\tau_2 = \sigma \in [\tau_1, \tau]$ construct the ellipsoidal approximation following equations (57).

The part of the boundary related to case (b) is described by the intersection $\cap\{\mathcal{E}(0, X(\tau)) \mid L(\cdot)\}$, where $L(\cdot)$ actually is reduced to $K(\cdot)$, and the latter to the parameter $\sigma = \tau_2$. Denote these as $\mathcal{E}(0, X_\sigma^+(\tau))$.

The total boundary is the intersection of the approximations for each group (a), (b) and (c), namely,

$$\mathcal{X}[0] = \mathcal{X}(\tau, 0, 0) = \bigcap_l \mathcal{E}(0, X_l^+(\tau)) \bigcap_\sigma \mathcal{E}(0, X_\sigma^+(\tau)) \bigcap \mathcal{E}(0, Y).$$

Figure 1 illustrates the reach tube for this example; figure 2 shows the same tube together with the reach tube without state constraint.

10 Conclusion

In this paper we presented an ellipsoidal technique to calculate reach sets for linear systems with constraints on the control and state. The suggested scheme introduces parametrized varieties of tight ellipsoidal-valued tubes that approximate the exact reach tube from above, touching the tube along specially selected "good" curves that cover the entire exact tube. This leads to recursive relations that simplify calculations compared to other approaches. The proofs rely a special "recursive" version of the maximum principle under state constraints.

References

[1] BOYD S., EL GHAOUI L., FERON E., BALAKRISHNAN V., *Linear Matrix Inequalities in System and Control Theory*, SIAM, Studies in Applied Mathematics, 1994.

[2] CHERNOUSKO F.L., *State Estimation for Dynamic Systems*, CRC Press, 1994.

[3] COURANT R., HILBERT D., *Methods of Mathematical Physics*, Vols. 1,2, NY, Wiley, 1953, 1962.

[4] FLEMING W.H., SONER H.M., *Controlled Markov Processes and Viscosity Solutions*, Springer-Verlag, 1993.

[5] GUSEV M.I., KURZHANSKI A. B., Optimization of controlled systems with bounds on the controls andd the phase coordinates. *Differential Equations–"Differentsial'nye uravneniya"*, Vol 7, No 9, 1591–1602, 1971.

[6] HENZINGER T.A., KOPKE P.W., PURI A. and VARAIYA P., What's Decidable about Hybrid Automata? *Proc. 27-th STOC*, 373–382, 1995.

[7] KIRILIN M.N., KURZHANSKI A.B. Ellipsoidal Methods Under Nonellipsoidal Constraints. NOLCOS-2001 IFAC Symp., Plenum Press, 2001.

[8] KURZHANSKI A.B. *Control and Observation Under Uncertainty Conditions* Nauka, Moscow, 1977. (In Russian)

[9] KURZHANSKI A.B., FILIPPOVA T.F., On the Theory of Trajectory Tubes—a Mathematical Formalism for Uncertin Dynamics, Viability and Control. *Advances in Nonlinear Dynamics and Control. A Report from Russia*, 122–188, Birkhauser, Boston, 1993.

[10] KURZHANSKI A.B., OSIPOV Yu.S., On Optimal Control under Restricted Coordinates. *Prikl. Mat. i Mehan.* Vol 33, No 4, 1969. (Translated from as "Applied Mathematics and Mechanics").

[11] KURZHANSKI A.B., VALYI I., *Ellipsoidal Calculus for Dynamics and Control* Birkhauser, Boston, ser. SCFA, 1997.

[12] KURZHANSKI A.B., VARAIYA P., Ellipsoidal techniques for reachability analysis: internal approximation. *Systems and Control Letters*, Vol 41, 201–211, 2000.

[13] KURZHANSKI A.B., VARAIYA P., On reachability under uncertainty. *SIAM J. Control and Optimization*, Vol 41, No 1, 181–216, 2002.

[14] KURZHANSKI A.B., VARAIYA P., On ellipsoidal techniques for reachability analysis. *Optimization Methods and Software*, Vol 17, 177–207, 2002.

[15] KURZHANSKI A.B., VARAIYA P., Dynamic optimization for reachability problems. *J. Opt. Th. and Appl.*, Vol 108, No 2, 227–251, 2002.

[16] KURZHANSKI A.B., VARAIYA P., Ellipsoidal techniques in reachability analysis for hybrid systems, *preprint*, August 2003.

[17] LEE E.B., MARCUS L., *Foundations of Optimal Control Theory*, Wiley, NY, 1967.

[18] LEITMANN G., Optimality and reachability via feedback controls. In :*Dynamic Systems and Mycrophysics*, Blaquiere A., Leitmann G. eds., 1982.

[19] McSHANE E.J. *Integration*, Princeton Univ.Press, 1947.

[20] PAPPAS G.L., SASTRY S., Straightening out differential inclusions. *System and Control Letters*, Vol 35 ,No 2, 79–85, Sept. 1998.

[21] PONTRYAGIN L.S., BOLTYANSKI V.G., GAMKRELIDZE R.V., MISCHENKO E.F., *Mathematical Theory of Optimal Processes*, Wiley, NY, 1962.

[22] PURI A., BORKAR V. and VARAIYA P., ϵ-Approximations of Differential Inclusions, in: R.Alur, T.A.Henzinger, and E.D.Sonntag eds., *Hybrid Systems*, 109–123, LNCS 1201, Springer, 1996.

[23] PURI A., VARAIYA P., Decidability of hybrid systems with rectangular inclusions. In D.Dill ed., *Proc. CAV-94*, 95–104, LNCS 1066, Springer, 1996.

[24] ROCKAFELLAR, R. T., *Convex Analysis*, second ed., Princeton University Press, 1999.

[25] SAKS S., *The theory of the integral*, Dover, 1964.

[26] SCHWARTZ L., *La théorie des distributions*, Dunod, Paris.

[27] SETHIAN J.A. *Level-set Methods and Fast-marching Methods*. Cambridge Univ. Press, 1999.

[28] SUBBOTIN A.I. *Generalized solutions of first-order PDE's*, Birkhauser, Boston, 1995.

[29] VARAIYA P. Reach Set Computation Using Optimal Control, in *Proc.of KIT Workshop on Verification of Hybrid Systems*, Verimag, Grenoble, 1998.

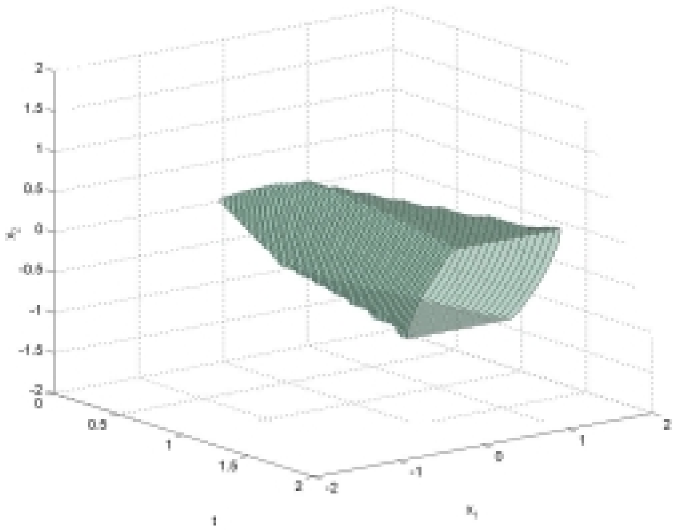

Figure 1: Reach tube with state constraints

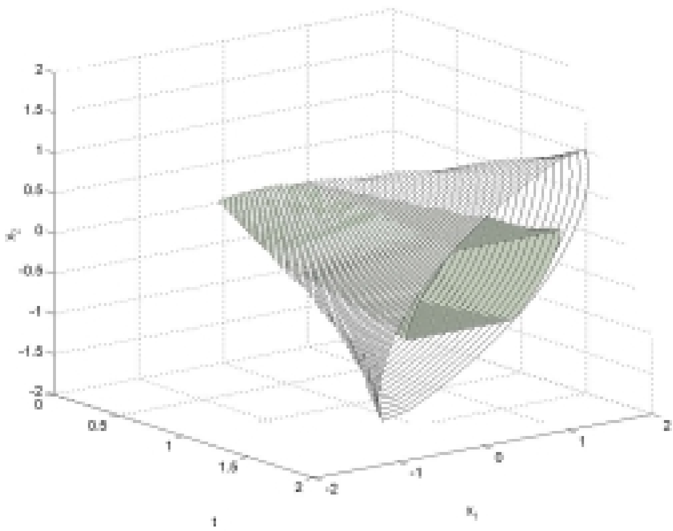

Figure 2: Reach tubes with and without state constraints