1 Standards:

2 Find the conjugate of a complex number; use conjugates to find moduli and quotients of complex numbers.

3 Represent addition, subtraction, multiplication, and conjugation of complex numbers geometrically on the complex plane; use properties of this representation for computation.

6 Analyze possible zeros for a polynomial function over the complex numbers by applying the Fundamental Theorem of Algebra, using a graph of the function, or factoring with algebraic identities.

2 Discourse

With the introduction of the imaginary unit, we are able to form a new set of numbers. This is analogous to when negative numbers were introduced to you. When you first learned subtraction, you were limited to subtracting the smaller number from the larger number, e.g. $4 - 3$ was possible but $3 - 4$ was not. Up until the introduction of the imaginary unit $\sqrt{x}$ where $x < 0$ was unsolvable because you did not have the tools needed to solve it and we said this problem had *no real solution*. We use that terminology because there are in fact solutions but they are *complex solutions*.

2.1 Sets of numbers

2.1.1 Natural

These are the numbers you first learned to count with $1, 2, 3, 4, \ldots$ and the symbolic representation of this set is $\mathbb{N}$.

2.1.2 Whole

Many texts do not include this set or symbol, but for completeness sake I will include it here. The texts that do include this define it as $\mathbb{N}$ with 0 included. Therefore it is $0, 1, 2, 3, \ldots$ with symbol $\mathbb{W}$.

2.1.3 Integers

The integers are the set of natural numbers along with their opposites (negatives) and 0. So, $\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$ and they are symbolically denotes as $\mathbb{Z}$. It is important to note that $\mathbb{Z}$ contains all of $\mathbb{N}$.

2.1.4 Rational

The rational numbers are all numbers that can be represented as a **ratio** of integers. That means any number that can be written as a fraction is a rational numbers. The symbol for the rational numbers is $\mathbb{Q}$ and some examples are $\frac{2}{3}$, $\frac{22}{9}$, and 7. How is seven considered a rational number? We can represent it as a fraction such as $\frac{7}{1}$ or $\frac{28}{4}$. Notice this means that $\mathbb{Q}$ contains $\mathbb{Z}$.

2.1.5 Irrational

The irrational numbers a literally the *not rational* numbers. These are numbers that cannot be written as a fraction. These are numbers that are non-repeating, non-terminating decimals. Examples include π and $\sqrt{2}$. There are infinitely many others but they are difficult to represent. The irrational numbers do not have a symbol of their own, but are often notated as $\mathbb{R}\backslash\mathbb{Q}$

2.1.6 Real

The real numbers contain all of the sets previously listed. This is the set of numbers you are used to working with. It is denoted as $\mathbb{R}$ and can be thought of as combining the rational and irrational numbers.

2.1.7 Complex

These are the numbers that allow us to solve problems involving the square roots of negative numbers. These are also used to model alternating current and other electronic symbols. The symbol for the complex numbers are $\mathbb{C}$ and they contain all of the real numbers.

These numbers are two dimensional, having both a real part and an imaginary part. Thus to create these numbers we need to know what an imaginary number is. The imaginary unit, i, is defined as

Definition 1. $i = \sqrt{-1}$.

Complex numbers take the form of $z = a + bi$ where $a, b \in \mathbb{R}$. The *real part*, $\text{Re } z$, of the complex number is a and b is considered the *imaginary part*, $\text{Im } z$. For purely real numbers, $\text{Im } z = 0$, and for purely imaginary numbers $\text{Re } z = 0$.

2.1.8 Examples

State the real and imaginary part of each complex number.

1. $3 + 4i$
2. $-3 - i$
3. $17 - 8i$
4. $11i$
5. $5i$
6. $-2i$
7. 8
8. -3
9. 7

2.1.9 Quaternions and Octonions

The quaternions and octonions have no bearing on this course, but they are introduced to illustrate the plethora of sets of numbers. As indicated by their names the quaternions and octonions are four and eight dimensional numbers respectively.

3 Operations with Complex Numbers

We can operate with complex numbers in much the same way as we can with real numbers.

3.1 Addition and Subtraction

Given two complex numbers, $z_1 = a + bi$ and $z_2 = c + di$. Then to add the numbers we have

Definition 2. $z_1 + z_2 = a + c + (b + d)i$

Similarly, for subtraction

Definition 3. $z_1 + z_2 = a - c + (b - d)i$

3.1.1 Examples

1. $(3 + 4i) - (2 + 5i)$
2. $(12 - 5i) + (24 - 3i)$
3. $(-5 + 16i) + (7 - 18i)$
4. $(9 - i) - (1 - 2i)$
5. $(-2 + 7i) - (4 + 8i)$
6. $(7) - (19 - 12i)$
7. $(3i) - (2i)$
8. $(2i) + (7)$
9. $(4i) + (3 - 4i)$

3.2 Multiplication

Multiplication of complex numbers is similar to multiplying binomials together using the distributive property and a mnemonic device such as FOIL can be helpful.

Definition 4. *Given complex numbers $z_1 = a + bi$ and $z_2 = c + di$ then $z_1 \cdot z_2 = (ac - bd) + (ad + cb)i$*

This can be somewhat intimidating and less than clear; therefore, during the video of this lesson this definition will rarely if ever be used.

3.2.1 Examples

1. $(2 + 5i) \cdot (1 - i)$
2. $(4 - 3i) \cdot (-4 + 2i)$
3. $(-6 + 4i) \cdot (-9 - 12i)$
4. $(-3 - 7i) \cdot (4 + 6i)$
5. $2i \cdot (3 + 5i)$
6. $12 \cdot (-3 - 9i)$
7. $5 \cdot 5i$
8. $(11 - 6i) \cdot 2$
9. $(-4 + 12i) \cdot -7i$

3.3 Division

Division of complex numbers is accomplished through multiplying by the conjugate.

Definition 5. *Given a complex number $z = a + bi$ its conjugate, denoted $\overline{z}$, is $\overline{z} = a - bi$.*

Using this conjugate, we are able to perform division in the following manner.

Definition 6. *Given complex numbers $z_1 = a + bi$ and $z_2 = c + di$ then $\frac{z_1}{z_2} = \frac{a+bi}{c+di} = \frac{a+bi}{c+di} \cdot \frac{c-di}{c-di} = \frac{(ac+bd)-(ad-bc)i}{c^2+d^2}$*

3.3.1 Examples

1. $\frac{3-2i}{5-i}$
2. $\frac{18-12i}{6}$
3. $\frac{-5-4i}{7+4i}$
4. $\frac{-4+6i}{2i}$
5. $\frac{7-19i}{2+3i}$
6. $\frac{6}{2-3i}$
7. $\frac{12i}{11-2}$
8. $\frac{5-2i}{6+3i}$
9. $\frac{-15-5i}{5}$

4 Exercises

Perform the indicated operation.

1. $(8-2i)+(-12-3i)$
2. $(2-7i)\cdot(-1+3i)$
3. $4i-(4-2i)$
4. $2\cdot(8-6i)$
5. $(5-2i)+(3i)$
6. $(2-7i)-(12+7i)$
7. $(4+9i)\cdot(4-9i)$
8. $\frac{3+4i}{2-7i}$
9. $(2-i)-(2-i)$
10. $(3+12i)+(5-7i)$
11. $\frac{2-3i}{2-3i}$
12. $(8-7i)\cdot 6$
13. $(2+3i)-(8-3i)$
14. $\frac{8-i}{8-i}$
15. $(6-i)\cdot(3+i)$
16. $(3+2i)-2i$
17. $\frac{2+i}{1+i}$
18. $\frac{7i}{2+3i}$
19. $1\cdot(7-i)$
20. $(2+4i)-(9+2i)$
21. $(12-9i)+(-12+9i)$

5 Answer Document

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