Problem set 1

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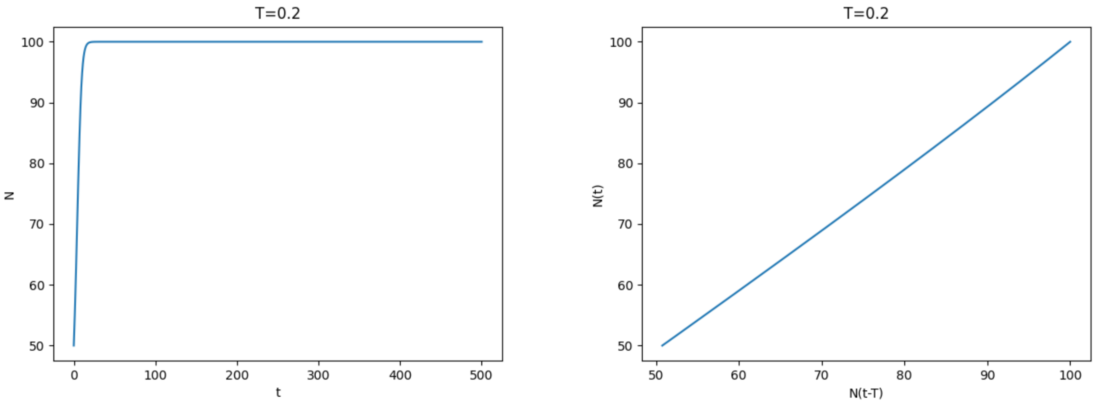
February 2023

Problem 1

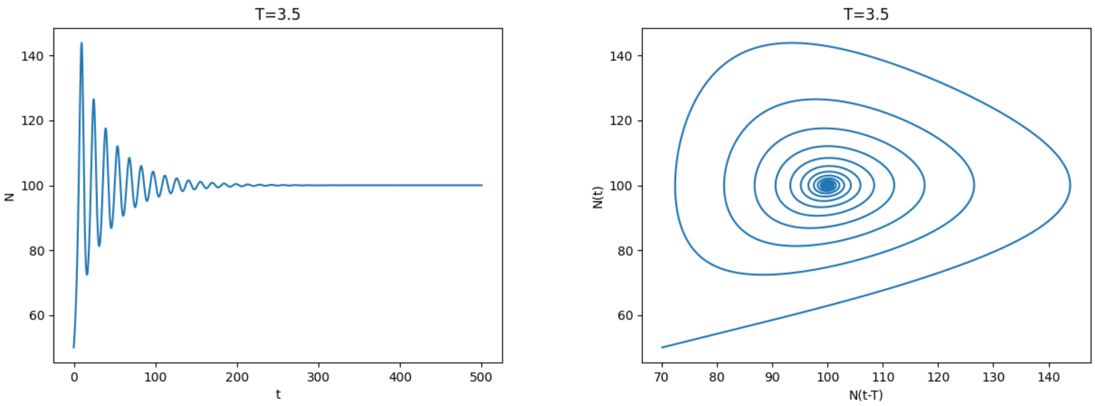
a

Figure 1 shows the solution to the dynamics for the parameter value $T = 0.2$. For this value of T, the system does not exhibit any oscillations.



Figur 1: An example graph showing the dynamics for a value of T that does not lead to oscillations. The left plot shows the solution $N(t)$ plotted against the time t. The right plot shows the solution $N(t)$ plotted against the delayed solution $N(t-T)$.

Figure 2 shows the solution to the dynamics for the parameter value $T = 3.5$. For this value of T, the system exhibits damped oscillations.

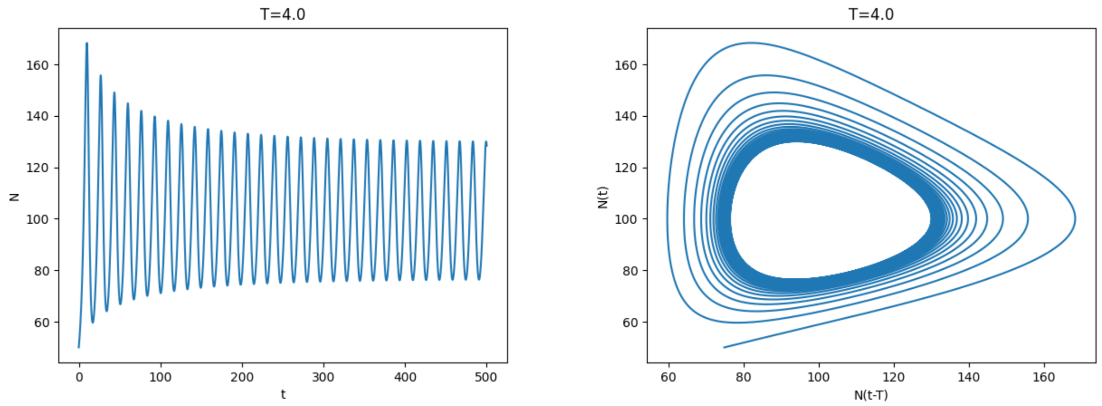


Figur 2: An example graph showing the dynamics for a value of T that leads to damped oscillations. The left plot shows the solution $N(t)$ plotted against the time t. The right plot shows the solution $N(t)$ plotted against the delayed solution $N(t-T)$.

Figure 3 shows the solution to the dynamics for the parameter value $T = 4$. For this value of T, the system exhibits stable oscillations after an initial transient period.

b

At $T = 1.1$, the system makes a small overshoot over its equilibrium. Thus this is where the system starts to exhibit damped oscillations.



Figur 3: An example graph showing the dynamics for a value of T that leads to stable oscillations. The left plot shows solution $N(t)$ plotted against the time t. The right plot shows the solution $N(t)$ plotted against the delayed solution $N(t-T)$.

c

A Hopf bifurcation occurs when a dynamical system, upon the variation of a parameter, changes its behaviour from a stable spiral around a fixed point into a limit cycle, and thereafter to an unstable spiral around the same fixed point. When searching for this behaviour while varying the time delay T, we find that $T_H = 4$. For $T = 3.9$, the system has no limit cycle but for T_H it appears after a number of iterations.

d

We start with the dynamics

$$\dot{N} = rN(t)\left(1 - \frac{N(t-T)}{K}\right)\left(\frac{N(t)}{A} - 1\right). \tag{1}$$

Let $\tau = \frac{t}{t_0}$ and $u(\tau) = \frac{N(t)}{N0}$. In dimensionless units, equation (1) can be rewritten as

$$\frac{du}{d\tau} = \frac{1}{N_0}\frac{dN}{dt} = \frac{t_0}{N_0}\frac{dN}{dt} = \ldots = t_0 r u(\tau) \cdot \left(1 - \frac{N_0}{K}u(\tau - D)\right)\left(\frac{N_0 u(\tau)}{A} - 1\right),$$

where $D := \frac{T}{t_0}$. By choosing $t_0 = \frac{1}{r}$ and $N_0 = K$ we obtain

$$\frac{du}{d\tau} = u(\tau)\left(1 - u(\tau - D)\right)\left(\frac{K}{A}u(\tau) - 1\right). \tag{2}$$

From equation (2) we see that the fixed points of the system are

$$u_1^* = 0, \quad u_2^* = 1, \quad u_3^* = \frac{A}{K},$$

where the fixed point $u_2^* = 1$ in the dimensionless system corresponds to the steady-state $N^* = K$ in the original system. Now, linearise and let $u(\tau) = u^* + \eta(\tau)$ and insert the linearisation into equation (2).

$$\frac{d}{d\tau}\left(u^* + \eta(\tau)\right) = \left(u^* + \eta(\tau)\right)\left(1 - u^* - \eta(\tau - D)\right)\left(\frac{K}{A}\left(u^* + \eta(\tau)\right) - 1\right)$$

Neglect non-linear terms, use that $u^* = 1$, (the fixed point we wanted to linearise around) and that $A = 20$ and $K = 100$ to get

$$\frac{d}{d\tau}\eta(\tau) = \eta(\tau - D)\left(1 - \frac{K}{A}\right) + \mathcal{O}(\eta^2) \approx \eta(\tau - D)\left(1 - \frac{K}{A}\right) = -4\eta(\tau - D). \tag{3}$$

Make the ansatz $\eta(\tau) = Ae^{\lambda\tau}$ and assume that $\lambda = \lambda' + i\lambda''$. Equation (3) now becomes

$$\begin{aligned}
&\lambda A e^{\lambda\tau} = -4Ae^{\lambda(\tau - D)} \\
&\lambda = -4e^{-\lambda D} \\
&\lambda' + i\lambda'' = -4e^{-D(\lambda' + i\lambda'')} = -4e^{-\lambda' D} \cdot (\cos(\lambda'' D) - i\sin(\lambda'' D)) \\
&\implies \begin{cases} \lambda' = -4e^{-\lambda' D}\cos(\lambda'' D), \\ \lambda'' = 4e^{-\lambda' D}\sin(\lambda'' D). \end{cases}
\end{aligned}$$

We are searching for the critical value of the delay parameter D that leads to a Hopf bifurcation. We call the critical value D_c. A Hopf bifurcation occurs as the real part of λ passes zero and therefore we solve

$$\begin{cases} 0 = -4\cos(\lambda'' D_c), \\ \lambda'' = 4\sin(\lambda'' D_c). \end{cases}$$

We get

$$\lambda'' D_c = \pm\frac{\pi}{2},$$

and

$$\lambda'' = 4\sin(\lambda'' D_c) \iff \frac{\lambda}{4} = \sin(\lambda'' D_c). \tag{4}$$

Let $\frac{\lambda''}{4} = x$ and $C = 4D_c$. Equation (4) now becomes

$$x = \sin(Cx),$$

which has the solution $x = \pm 1$. Changing back to our original variables we get that $\lambda'' = \pm 4$ and

$$\lambda'' D_c = \pm 4 D_c = \pm\frac{\pi}{2}.$$

We only keep the positive solutions and find that $D_c = \frac{\pi}{8}$. Since $D = \frac{T}{t_0} = rT = 0.1 \cdot T$, we get

$$T_H = \frac{D_c}{0.1} = \frac{\pi}{0.8} \approx 3.92699.$$

Code

```
import numpy as np
import matplotlib.pyplot as plt

A = 20
K = 100
r = 0.1
```

```
N_0 = 50
T = 1.1
T_arr = np.linspace(0.1, 5, num=50)
dt = 0.01
totalTime = 500

t = []
Ndot = []

nTimesteps = int(totalTime / dt)
N = np.empty(nTimesteps)

for i in range(nTimesteps):
    N[i] = N_0
    t.append(i * dt)

Tind = int(T/dt)

for i in range(nTimesteps-1):
    NdotTemp = r * N[i] * (1 - N[i-Tind] / K) * (N[i] / A - 1)
    Ndot.append(NdotTemp)
    N[i+1] = (NdotTemp*dt+N[i])


fig = plt.figure()
plt.plot(t, N)
plt.xlabel("t")
plt.ylabel("N")
plt.title("T=%.1f" % T)
plt.savefig('graphics/saved_img/oscillations_T'+str(T)+'.png')


NCut = N[:len(N)-Tind]
NTCut = N[Tind:]

fig = plt.figure()
plt.plot(NTCut, NCut)
plt.xlabel("N(t-T)")
plt.ylabel("N(t)")
plt.title("T=%.1f" % T)
plt.savefig('graphics/saved_img/cycle_T'+str(T)+'.png')


fig, (ax1, ax2) = plt.subplots(1, 2, figsize=(8, 3))
fig.suptitle("T=%.1f" % T)

ax1.plot(t, N)
ax1.set_xlabel('t')
ax1.set_ylabel('N')
ax2.set_xlabel("N(t-T)")
ax2.set_ylabel("N(t)")
ax2.plot(NTCut, NCut)
\end{verbatim}
```