Tute sheet 5 - games + institutions.

college 1. a) A strategy is strictly dominated if it leads to worse outcomes for an agent than some other specific strategy does, for all possible actions taken by their opponents. ~~[illegible]~~

✓

Underline the best-response payoffs

	A	B
A	3,3	1,4
B	4,1	2,2

	A	B
A	5,5	0,4
B	4,0	3,3

	A	B
A	1,-1	-1,1
B	-1,1	1,-1

A is strictly dominated by B

Neither is strictly dominant.

Neither is strictly dominant.

b) A Nash equilibrium is ~~when~~ is a strat. profile no player in a game could improve their outcome by changing ~~action~~ strategy, given the other players' strategies. i.e. each agent plays the best response to the opponents' actually-chosen strategies.

✓

~~Equilibria circled above.~~

Pure Nash equilibria circled above.

[Solving mixed equilibria: "both pure strategies must themself be a best response"?]

Not sure what you mean.

ii) There is also a mixed-strategy equilibrium. Let column play A with probability p. Then expected payoffs for row are $EU(A) = 5p + 0$, ~~$EU(B) = 3(1-p)$~~. These must be "best responses" equal, if it's the best response to mix in this way. ~~opponent player optimal strat.~~

$EU(B) = 4p + 3(1-p)$.

So both play A with $p = \cancel{\frac{3}{8}}$, $P(B) = \cancel{\frac{5}{8}}$ (due to symmetry).

iii) There is only a mixed-strategy equilibrium. Both players choose A with probability $\frac{1}{2}$, and $P(B) = \frac{1}{2}$ also. This leads their expected payoffs to be 0, the maximum for this 0-sum game.

c) If player 1 goes first, the subgame perfect equilibrium is ~~{A, A}~~ with outcome 5, 5.

- Say ① chooses B. Then ②'s best response is B. ① will get 3 payoff
- Say instead ① chooses A. Then ②'s best reply is A. ① gets 5 payoff.

So, the optimal opening from ① is A, which leads to A from ② also.

Define SPE. Solve by "backwards induction" ← Say this.

Then write out the full SPE strat. profile

[illegible]

dept 5a)

	C	N
C	$2b-1, 2b-1$	$b-1, b^*$
N	$b^*, b-1$	$0^*, 0^*$

As $0.5 < b < 1$,
- $2b-1 > 0$
- $b-1 < 0$
- $2b-1 < b$

Nash equilibrium is $\{N, N\}$

We know the outcome is not Pareto-efficient: $\{C, C\}$ is a Pareto-improvement, since both players move from payoff 0 to $2b-1$ and $2b-1 > 0$. So, it's possible to make someone (in fact everyone) better off without making anyone worse off $\Rightarrow$ not Pareto-efficient.

b) If $b < 0.5$ then $2b < 1$, $2b-1 < 0$ i.e. the Pareto-optimal outcome is that the public good is not supplied as the social benefit of it is less than the cost of supplying it. So the Nash equilibrium $\{N,N\}$ is not inefficient; N is the strictly dominant strategy (as before).

c) If $b > 1$ then $b-1 > 0$ i.e. the private benefit of the good is greater than the cost of supplying it, so each individual will rationally contribute. So, the Nash equilibrium is $\{C, C\}$; ~~[illegible]~~ C is the strictly dominant strategy. There is no Pareto-inefficiency.

d) Consider ~~first~~ the cases where the outcome is Pareto efficient:

I) As in b), perhaps it is ~~optimal~~ efficient for the good to not be supplied. For this to be the case, the total cost > total benefit, i.e.

$k \times 1 > k \times (bk)$ since the aggregate contribution $= k$

$b < \frac{1}{k}$

II) As in c), perhaps each agent's best strategy is simply to ~~supply~~ contribute, as the private benefit is greater than private cost. i.e.

$b > 1$

So, if $\frac{1}{k} < b < 1$, then the Nash equilibrium <u>is not</u> Pareto-efficient.

e)

	C	N
C	$4b-2,\ 4b-2$	$2b-1,\ 2b-1$
N	$2b-1,\ 2b-1$	$0,\ 0$

As noted, $2b-1>0$ and so $4b-2>2b-1$

(if 1 contributes, $\sum b_i = 2b$, cost $=1$
if 2 contribute, $\sum_i b_i = 4b$, cost $=2$)

The externality has now been internalised into each agent's payoff. The Nash equilibrium is thus $\{C, C\}$, which is Pareto-optimal.

f) One way to ensure Pareto-efficient provision would be to internalise the externality - that is, provide a subsidy to those who contribute so that the private benefit to them of supplying a marginal unit is equal to the ~~pat~~ social benefit. This is like how in e) the agents have altruistic preferences and are incentivised to maximise social welfare as a result - and hence the Nash equilibrium is Pareto-efficient.

6a) The utility function is increasing in x_1, holding constant the inequality-aversion part $-\frac{\alpha}{2}\left[\sum_{j=1}^{2}(x_j-\bar{x})^2\right]$. This part shows inequality-aversion because the value of the sum is large when x_1 and x_2 are further apart, and it contributes negatively to u_1 (it's subtracted). Note that without the parameter ~~α~~, the inequality-aversion here is simply the variance of the distribution. Greater variance $\leftrightarrow$ greater inequality. If $\alpha=0$ then the utility function is purely egoistic.

b) $u_1 = x_1 - \frac{\alpha}{2}\left((x_1-\tfrac{1}{2})^2 + (x_2-\tfrac{1}{2})^2\right)$ as $\bar{x} = \frac{1}{2}$

$= x_1 - \frac{\alpha}{4}\left(2x_1^2 + 2x_2^2 - 2x_1 - 2x_2 + 1\right)$

$= x_1 - \frac{\alpha}{4}\left(2x_1^2 + 2x_2^2 - 1\right)$ since $x_1 + x_2 = 1 \therefore -2(x_1+x_2) = -2$

$= x_1 - \frac{\alpha}{4}\left(2x_1^2 + 2(x_1^2 - 2x_1 + 1) - 1\right)$ as $(x_2)^2 = (1-x_1)^2$

$= x_1 - \frac{\alpha}{4}\left(4x_1^2 - 4x + 1\right) = x_1 - \frac{\alpha}{4}(2x_1-1)^2 = x_1 - \frac{\alpha}{4}(x_1-(1-x_1))^2$

$= x_1 - \frac{\alpha}{4}(x_1 - x_2)^2$

c) Player 1 wishes to maximise their own utility ~~subject~~ by varying x_1. Taking the form of u_1 in x_1 only, we have ~~un~~constrained optimisation:

optimisation ⊛ with inequality constraints.

$$\max_{x_1}\left\{u_1 = x_1 - \frac{\alpha}{4}(2x-1)^2\right\} \text{ s.t. } 0 \le x_1 \le 1$$

Good

Is it "constrained"? You can't use a Lagrangian though?

You can, but we'll see how to formally deal with inequality constraints via Lagrangian next year

The first-order condition is that $\frac{du_1}{dx_1} = 0$ but $x_1 \le 1$ so $x_1 = \min\left\{1, \frac{\alpha+1}{2\alpha}\right\}$

i.e. $1 - \alpha(2x_1 - 1) = 0$

so $x_1 = \frac{1}{2}\left(\frac{1}{\alpha}+1\right) = \frac{\alpha+1}{2\alpha}$

$x_2 = \max\left\{0, \frac{1}{2} - \frac{1}{2\alpha}\right\}$

and $x_2 = 1 - x_1 = 1 - \frac{\alpha+1}{2\alpha} = \frac{1}{2} - \frac{1}{2\alpha}$

d) $\frac{dx_2}{d\alpha} = \frac{1}{2\alpha^2}$. Since for all $\alpha \in \mathbb{R}$, $\alpha^2 > 0$, $\frac{dx_2}{d\alpha} > 0$ for all α, so the offer to x_2 is increasing in α.

e) Intuitively, consider an agent with arbitrarily high inequality-aversion. They would want to minimise the variance of $\{x_1, x_2\}$, which is 0 when $x_1 = x_2 = \frac{1}{2}$. Offering any stake greater than this would mean that $\left[\sum_{j=1}^{2}(x_j - \bar{x})^2\right] > 0$, i.e. they would be reducing their utility via increasing inequality, in addition to reducing x_1, which would also contributes negatively to utility. Mathematically, consider the limit as $\alpha \to \infty$ of $x_1 = \frac{\alpha+1}{2\alpha}$: this is $\frac{1}{2}$, so the utility-maximising offer for an infinitely loss-averse agent in this model is $1 - \frac{1}{2} = \frac{1}{2}$.

f) i. $x_2 = \frac{1}{4}$ in this case and we know $x_2 = \frac{1}{2} - \frac{1}{2\alpha}$

so $\frac{1}{2\alpha} = \frac{1}{4}$, $\alpha = 2$

ii. $x_2 = 0$ in this case; $\frac{1}{2} = \frac{1}{2\alpha}$ so $\alpha = 1$. Not quite.

This is somewhat surprising: one might expect that only for $\alpha = 0$ (i.e. pure egoism) will an offer of 0 be made. But the direct effect on utility via a ~~smaller~~ bigger x_1 outweighs the negative inequality effect from $0 \le \alpha \le 1$, leaving x_2 at 0. The naive expressions for x_1, x_2 above seem to suggest offers/retained stakes outside of $0 < x < 1$ if $\alpha < 1$, so I refined them.

Ah! Yes

should it ⊛ be min{-} max{-}?