Week 5

1a) i. Base case: ϕ is atomic, i.e. an n-place predicate $\Pi\alpha_1, \ldots, \alpha_n$, where each α is a term.
Since $V_g(\phi) = 1$ iff $\langle [\alpha_1]_g, \ldots, [\alpha_n]_g \rangle \in I(\Pi)$, and g and h agree on variable assignments, $V_h(\phi) = 1$ iff $V_g(\phi) = 1$, as $[\alpha_k]_h = [\alpha_k]_g$ for all k.
So $V_g(\phi) = V_h(\phi)$

Inductive hypothesis: Let ϕ have complexity w. Assume for all wff ψ with complexity $v < w$, the property F_ψ – if g and h agree on variable assignments in ψ, $V_g(\psi) = V_h(\psi)$ – holds.

ϕ can be written as one of (i) $\neg\psi$, (ii) $\psi_1 \to \psi_2$, or (iii) $\forall\alpha\psi$

exactly the same variables are free in ψ as in ϕ

(i) If ϕ is $\neg\psi$ then $V_g(\phi) = 1 - V_g(\psi)$; $V_h(\phi) = 1 - V_h(\psi)$ and by hypothesis $V_g(\psi) = V_h(\psi)$, so F_ϕ holds, ~~given~~ ~~greater~~ ~~appearing~~.

(ii) If ϕ is $\psi_1 \to \psi_2$, since we have $V_g(\psi_1) = V_h(\psi_1)$ and $V_g(\psi_2) = V_h(\psi_2)$, $V_g(\phi) = 1$ iff $V_g(\psi_1) = 0$ or $V_g(\psi_2) = 1 \Leftrightarrow V_h(\psi_1) = 0$ or $V_g(\psi_2) = 1$ iff $V_h(\phi) = 1$, so F holds.

(iii) If ϕ is $\forall\alpha\psi$ then α is not free in ϕ, but all other free variables in ψ remain free in ϕ.
And by hypothesis $V_g(\psi) = V_h(\psi)$, given g and h agreeing. Substituting in any $u \in D$ for α, it will hold that $V_{g^u_\alpha}(\psi) = V_{h^u_\alpha}(\psi)$, since g and h agree on all the free variables in ψ other than α.
So $V_g(\forall\alpha\phi) = V_h(\forall\alpha\phi)$, and F holds.
The result is proven with strong induction. ■

ii. Assume that $[\alpha]_g = [\beta]_g$.
Base case: ϕ is atomic, i.e. $\Pi\alpha_1, \ldots, \alpha_n$.
Then $\phi(\beta/\alpha)$ replaces every free occurrence of α with β,
for each $\alpha_k \to \beta_k$.
But since the extension of β in g is the same as α's,

$[\alpha_k]_g = [\beta_k]_g$ for all k. So $V_g(\phi) = V_g(\phi(\beta/\alpha))$

Inductive hypothesis: Let ϕ have complexity w. Assume for all ψ with complexity $v < w$, $V_g(\psi) = V_g(\psi(\beta/\alpha))$. ϕ is one of (i) $\neg\psi$, (ii) $\psi_1 \to \psi_2$, (iii) $\forall x\psi$.

(i) $\phi(\beta/\alpha) = \neg\psi(\beta/\alpha)$, and $V_g(\phi) = 1 - V_g(\psi)$, which by hypothesis $= 1 - V_g(\psi(\beta/\alpha))$. So $V_g(\phi) = V_g(\beta/\alpha)$ as required.

(ii) $\phi(\beta/\alpha) = \psi_1(\beta/\alpha) \to \psi_2(\beta/\alpha)$, and $V_g(\phi) = 1$ iff $V_g(\psi_1) = 0$ or $V_g(\psi_2) = 1$. But this is equivalent to $V_g(\psi_1(\beta/\alpha)) = 0$ or $V_g(\psi_2(\beta/\alpha)) = 1$ by hypothesis, so $V_g(\phi) = V_g(\phi(\beta/\alpha))$ as required.

(iii) $\phi(\beta/\alpha) = \forall x\,\psi(\beta/\alpha)$. For every g' differing from g at most in x, $V_{g'}(\psi) = V_{g'}(\psi(\beta/\alpha))$ (by hypothesis and since β is free for α, i.e. $x \neq \alpha$) So $V_g(\forall x\psi) = V_g(\forall x\,\psi(\beta/\alpha))$ as required.

So the result holds by strong induction. ∎

b) i. ~~$V(\forall\alpha\phi) = 1$ means that for all assignments $u \in D$, $V_{g^\alpha_u}(\phi) = 1$.~~
~~Take one such assignment g, where $V_g(\phi) = 1$.~~
~~In particular, $V_{g^\alpha_{[\beta]_g}}(\phi) = 1$.~~

~~Take any PC-model M and assignment g s.t.~~
Suppose, for reductio, that there exists some M, g, such that $V_{M,g}(\forall\alpha\phi \to \phi(\beta/\alpha)) = 0$.

Then $\underset{(i)}{V_{M,g}(\forall\alpha\phi) = 1}$ and $\underset{(ii)}{V_{M,g}(\phi(\beta/\alpha)) = 0}$.

~~Beat~~ From (i), for all $u \in D$, $V_{M, g^{\alpha}_{u}}(\phi) = 1$.
In particular, $V_{M, g^{\alpha}_{[\beta]_g}}(\phi) = 1$. Call this g^*

By a)ii., and since here we have ~~set~~ things such that $[\alpha]_{g^*} = [\beta]_{g^*}$; $V_{M, g^*}(\phi(\beta/\alpha)) = V_{M, g^{\alpha}_{[\beta]_g}}(\phi) = 1$
with (ii), since g and g^* agree on all free variables in $\phi(\beta/\alpha)$ (they differ at most in α but α doesn't appear.)

ii. Let M, g be an arbitrary PC-model, assignment.

Assume that $V_{M,g}(\forall \alpha(\phi \to \psi)) = 1$ and $V_{M,g}(\phi) = 1$.

For every $u \in D$, $V_{M, g^{\alpha}_{u}}(\phi \to \psi) = 1$. By a)ii, since g and g^{α}_{u} agree on the free variables in ϕ (as α doesn't occur freely in ϕ), $V_{M, g^{\alpha}_{u}}(\phi) = V_{M,g}(\phi) = 1$ by hyp, for every $u \in D$.

This means, for every $u \in D$, $V_{M, g^{\alpha}_{u}}(\psi) = 1$, and thus $V_{M,g}(\forall\alpha\psi) = 1$

~~So~~ ~~V~~ And since M and g were arbitrary ~~the~~ ~~sentence~~ schema $\forall \alpha(\phi \to \psi) \to (\phi \to \forall \alpha \psi)$ is PC-valid.

iii. If $\models_{PC} \phi$, then $V_{M,g}(\phi) = 1$ for all PC-models M and assignments g. (i)

Pick some arbitrary M, g. ~~and select another g' s.t. for all variables other, $g(\alpha) \neq g'(\alpha)$~~.
Consider g^{α}_{u}, for every $u \in D$. ϕ is true under each of these assignments, from (i). So then $V_{M,g}(\forall \alpha \phi) = 1$.

As M, g were arbitrary, this is PC-valid.

2a) ① $\exists x\,(Kx \wedge \sim Bx \wedge \forall y\,(Ky \to x=y))$ narrow

② $\sim\exists x\,(Kx \wedge Bx \wedge \forall y\,(Ky \to x=y))$ wide scope

b) $\sim B(\iota x.\, Kx)$ is equivalent to ②.

- Suppose there is ~~some~~ one king of France. Then this formulation will be true just in case he is not bald, as with ①.
- Alternatively, there are multiple kings of France or no king at all. Then the ι-term is undefined and $B(\cdot)$ is false, making the sentence true, just as in ② how $\exists x(\cdot)$ is false and the sentence true.

c) $\lambda x \sim Bx(\iota x.\, Kx)$ is equivalent to ①.

- Suppose there is one king of France. Then this will be true just in case he is not bald, since the term $\iota x.\, Kx$ is defined and we only need to evaluate the complex predicate $\lambda x \sim Bx$ on that term; This is the same as in ①.
- Alternatively, there are multiple kings or no king. Then the descriptive term is undefined. This means the complex predicate evaluates as false, as with the existence claim in ①.

So, $\models_M \lambda x \sim Bx\,(\iota x.\, Kx)$ iff $\models_M \exists x\,(Kx \wedge \sim Bx \wedge \forall y\,(Ky \to x=y))$

d) Both symbolisations are semantically identical and have the same meaning. The difference is syntactic / grammatical — our formulation in (c) is more concise and arguably ~~[illegible]~~ reflects the English construction better: we have "the king of France" represented as a single unit to which the property of baldness may apply, rather than mixing up syntactically the definite description and the baldness predicate.

Nicely done here.