Week 7

1(a) i. Base case: ϕ is atomic. Then it is of the form either:

(i) $\psi_1 = \psi_2$ where ψ_1, ψ_2 are terms.

~~And~~ If g and h agree on the free variables in ϕ, then $[\psi_1]_{M,g} = [\psi_1]_{M,h}$ and $[\psi_2]_{M_g} = [\psi_2]_{M,h}$

So $V_{M,g}(\psi_1 = \psi_2, w) = 1$ iff $V_{M,h}(\psi_1 = \psi_2, w)$

(ii) $\Pi^n \psi_1, \ldots, \psi_n$ where each ψ_k is a term

If g and h agree on the free variables in ϕ, then $[\psi_k]_{M,g} = [\psi_k]_{M,h}$ for each k, and as ~~and~~ ~~thus~~ $V_g(\Pi^n \psi_1, \ldots, \psi_n, w) = 1$ iff $\langle [\psi_1]_g, \ldots [\psi_n]_g, w \rangle \in I(\Pi^n)$, $V_g(\phi, w) = V_h(\phi, w)$.

Inductive hypothesis: assume for all wffs less complex than ϕ, the property holds,

ϕ is of one of the following forms:

(i) $\neg\psi$ as in week 5

(ii) $\psi_1 \to \psi_2$ "

(iii) $\forall\alpha\psi$. ~~in which case~~ For every $d \in D_w$ g^α_d and h^α_d agree on the free variables in ψ, since the only variable possibly free in ψ but not ϕ is α and we set that to d by variant assignment. By hypothesis, then,

$V_{g^\alpha_d}(\psi, w) = V_{h^\alpha_d}(\psi, w)$ and so $V_g(\forall\alpha\psi) = 1$ iff $V_h(\forall\alpha\psi) = 1$.

(iv) $\Box\psi$. Since $V_g(\psi, w) = V_h(\psi, w)$ for each $w \in W$ by hypothesis, certainly it holds for each w' s.t. Rww'.

So ~~as~~ as $V_g(\Box\psi, w^n) = 1$ iff for every w' s.t. Rww', $V_g(\psi, w') = 1$, it follows that $V_g(\Box\psi, w) = V_h(\Box\psi, w)$.

As this holds for any arbitrary $w \in W$, we have proven the desired result by strong induction.

ii. Base case: ϕ is atomic. Then it's of the form either

(i) $\psi_1 = \psi_2$, where ψ_1, ψ_2 are terms.

So $\phi(\beta/\alpha) = \psi_1(\beta/\alpha) = \psi_2(\beta/\alpha)$ and since $[\beta]_{M,g} = [\alpha]_{M,g}$

$V_g(\phi, w) = V_g(\phi(\beta/\alpha), w)$.

(ii) an n-place predicate as week 5.

Inductive cases: assume the property holds for all wffs simpler than ϕ. ϕ is one of:

(i) $\neg\psi$

(ii) $\psi_1 \to \psi_2$ } as in week 5

(iii) $\forall\alpha\,\psi$ using all $d \in D_w$ now

(iv) $\Box\psi$. So $\phi(\beta/\alpha) = \Box(\psi(\beta/\alpha))$ and as by hypothesis $V_g(\psi, w') = V_g(\psi(\beta/\alpha), w')$ for each world s.t. Rww', $V_g(\Box\psi, w) = V_g(\Box(\psi(\beta/\alpha)), w)$

Nice. We've done lots of induction proofs so I'm not too worried that you haven't spelled out all the details here, but it might be worth hashing them out for practice once or twice when revising!

b) i. Invalid. Consider ~~the~~ a model M s.t.

$D_w = \{a\}$ $D_v = \{a, b\}$

$[\beta]_{M,g} = b$

$\langle a, w\rangle \in I(P)$, $\langle a, v\rangle \in I(P)$, $\langle b, v\rangle \notin I(P)$

~~Then [illegible]~~

Then at world w the antecedent is true but $V_{M,g}(P\beta, w) = 0$.

[Semantically, it seems strange to assign b to β given it's not in the domain of w, but still ~~[illegible]~~ evaluate it there).

is the M or g important here? and is the syntax ~~[illegible]~~ correct, or should it say β/α somewhere?

Striiiictly speaking it doesn't matter too much, and I personally don't like Sider's notation much anyway, but I would recommend sticking to the way the lecture notes/textbook does it just for the sake of continuity

ii. Valid.

Suppose for ~~*~~ that at some world w, $V_{M,g}(\forall\alpha\phi, w) = 1$ (i) and $V_{M,g}(\exists\gamma(\gamma = \beta), w) = 1$ (ii) and $V(\phi(\beta/\alpha), w) = 0$ (iii)

From (ii), there is some $d \in D_w$ s.t. $[\beta]_{M,g} = d$.

From (i), for every $d \in D_w$, $V_{g^\alpha_d}(\phi, w) = 1$

In particular, this holds for $g^\alpha_{[\beta]_{M,g}}$.

By a)ii, then, $V(\phi(\beta/\alpha), w) = 1$. But this is ※ with (iii). ∎

iii. Valid.

Suppose for # that in some world w, in model M under assignment g, $V_{M,g}(\underbrace{\forall\alpha(\phi \to \psi)}_{(i)}, w) = 1$ and $V_{M,g}(\underbrace{\phi \to \forall\alpha\psi}_{(ii)}, w) = 0$.

(I) From (i), for every $d \in D_w$, $V_{Mg^\alpha_d}(\phi, w) = 0$ or $V_{Mg^\alpha_d}(\psi, w) = 1$.

(II) From (ii), $V_{M,g}(\phi, w) = 1$ and for some $d' \in D_w$, $V_{Mg^\alpha_{d'}}(\psi, w) = 0$

~~But this implies there is some g', namely g~~

Since α doesn't occur freely in ϕ, $V_{M,g^\alpha_d}(\phi, w) = V_{M,g}(\phi, w)$.

But this implies, with (I), that $V_{Mg^\alpha_{d'}}(\psi, w) = 1$, which is # with (II). ∎

2a) Suppose, for #, that there exists a ~~interpretation~~ frame $\langle W, R, D, Q \rangle$ and ~~interpretation~~ assignment g s.t. at some $w \in W$ $V_g(\underbrace{\Box\forall\alpha\phi, w}_{(i)}) = 1$

(⇐) and $V_g(\underbrace{\forall\alpha\Box\phi, w}_{(ii)}) = 0$.

We want to assume increasing frame + assume for reductio that CBF is invalid

From (ii), there is some $d \in D_w$ and some world w' s.t. Rww' and $V_{g^\alpha_d}(\phi, w') = 0$. (II)

From increasing domains, $D_w \subseteq D_{w'}$, so $d \in D_{w'}$.

From (i), in all accessible worlds v, for each $d' \in D_v$, $V_{g^\alpha_{d'}}(\phi, v) = 1$. But d is one such d' and w' one such v, which is # with (II).

(⇒) Assume F is not ~~dec~~ increasing. Then consider the following counterexample to CBF:

A model M such that Ruw and $D_u = \{a, b\}$ and $D_w = \{a\}$, and $\langle a, w\rangle \in I(P)$, $\langle a, u\rangle \in I(P)$, $\langle b, u\rangle \in I(P)$

then at u, $V_g(\Box\forall x Px, u) = 1$

but $V_g(\forall x \Box Px, u) = 0$ since $V_{g^x_b}(Px, w) = 0$.

So by contrapositive this direction holds. ∎

Good

~~Assume that the frame is locally constant.~~

b) ~~Suppose ... that ... does not hold~~ ~~$D_u = D_w \Rightarrow$~~

($\Rightarrow$) ~~$D_u \subseteq D_w$ So from a), CBF will hold.~~

~~Suppose, for ..., that B does not hold. Then~~

From (a), we know that CBF is valid only if F is increasing. The (B) axiom is valid iff the symmetry relation R is symmetric. So, Ruw implies Rwu, which ~~...~~ imply $D_u \subseteq D_w$ and $D_w \subseteq D_u$ respectively.

So, Ruw implies $D_u = D_w$.

Very good — you just apply the result of part (a) twice, because of symmetry.

3 a) i. ~~...~~

$V^2_g(D@\phi, w, w) = 1$ iff for all worlds $w' \in W$,

$V^2_g(@\phi, w, w') = 1$ iff ~~for all~~ $V^2_g(\phi, w, w) = 1$

So $V^2_g(D@\phi, w, w) = V^2_g(\phi, w, w)$ and the wff is 2D-valid. (in all worlds $w \in W$ in all models M under any g)

ii. $V^2_g(D@\phi, v, w) = 1$ iff for all worlds $w' \in W$,

$V^2_g(@\phi, v, w') = 1$ iff $V^2_g(\phi, v, v) = 1$

but then $V^2_g(D@\phi, v, w)$ doesn't necessarily $= V^2_g(\phi, v, w)$ in all $v, w \in W$ in all models M under any g,

e.g. consider some model s.t. $\langle a, w\rangle \in I(P)$ and $\langle a, v\rangle \notin I(P)$,

and letting $\phi := Pa$.

So not generally 2D-valid.

b) i. To be ~~that~~ 2D-valid, we need $V_g^2(\Box(\phi \leftrightarrow \Box@\phi), w, w)$ to be true in all ~~worlds~~ ~~[illegible]~~ models, assignments, and worlds.

$$V_g^2(\Box(\phi \leftrightarrow \Box@\phi), w, w) = 1 \text{ iff}$$

$$V(\phi \leftrightarrow \Box@\phi, w, w') = 1 \text{ for all } w' \in W \quad (\text{I})$$

$$\text{iff} \quad V(\phi, w, w') = \underbrace{V(\Box@\phi, w, w')}_{= 1 \text{ iff}}$$

$$V(@\phi, w, w'') = 1 \text{ for all } w'' \in W$$

$$\text{iff} \quad V(\phi, w, w) = 1$$

So ~~[illegible]~~ the wff is valid ~~[illegible]~~ iff $V(\phi, w, w') = V(\phi, w, w)$ for all $w' \in W$, for all models and assignments.

But consider some model M s.t. $\langle a, w \rangle \in I(P)$ and $\langle a, w' \rangle \notin I(P)$, with $\phi := Pa$. ~~[illegible]~~ Then $V(\Box(\phi \leftrightarrow \Box@\phi), w, w)$ is false, since

$$V(\phi, w, w') = 0 \quad \neq \quad V(\phi, w, w) = 1$$

ii. Not generally 2D-valid, since you can consider (w, w) as in i.

Yup, whenever its not 2D-valid its also not generally 2D-valid

c) i. $V(\Box X(\phi \leftrightarrow \Box@\phi), w, w) = 1$ iff for all $w' \in W$

$$V(X(\phi \leftrightarrow \Box@\phi), w, w') = 1, \text{ iff}$$

we already showed this was 2D-valid in (a)

$$V(\phi, w', w') = V(\Box@\phi, w', w') \quad (\text{I})$$

$$V(\Box@\phi, w', w') = 1 \text{ iff for every } w'' \in W \; V(@\phi, w', w'') = 1$$

$$\text{iff} \quad V(\phi, w', w')$$

So since from (a), formula (I) holds at all worlds in all models under all assignments, this formula is 2_0-valid.

ii. $V(\mathcal{D}X(\phi \leftrightarrow \Box @ \phi), v, w) = 1$ iff for all $w' \in W$

$V(X(\phi \leftrightarrow \mathcal{D} @ \phi), v, w') = 1$ iff $V(\phi \leftrightarrow \Box @ \phi, w', w') = 1$

So again use the result in (a). Note how the X means that we evaluate the inner biconditional at each (w', w'), removing the counterexample from (a). So it's generally 2_0-v.

You should try the formalisation exercises in Question 4, they're fun!