(a) List the main differences between first- and second-order logic.

The fundamental difference between first- and second-order logic is that the latter introduces the ability to quantify over properties, as well as over objects in the domain of discourse. As a result, second-order logic has strictly greater expressive power than first-order logic – it can capture the concepts like mathematical induction and Leibniz's law. The quantification over properties also leads to a few other differences: second-order logic lacks a complete axiomatic proof system, and does not satisfy compactness (meaning that it can express the idea of finiteness, unlike first-order logic).

(b) Briefly expound and critically assess what you take to be the best argument for

i. taking second-order logic to be logic (500 words)

The strongest argument for taking second-order logic to be logic is that it builds naturally on first-order logic and addresses some defects that the weaker system has in its ability to express what are intuitively *logical* statements. As Boolos notes, there are no fundamental changes needed to the theoretical apparatus of first-order logic in order to reach second-order logic. We add in two additional clauses each for the definition of a well-formed formula and a PC-valuation, as well as lightly extending the notion of variable & variant assignment from ordinary variables to also cover predicate variables, but the remainder of first-order predicate logic remains intact, with the definition of semantic consequence and logical validity the same as before (and the structure of a PC-model entirely unchanged). This semantic and syntactic continuity with first-order logic means that it would be strange to describe second-order logic as a different kind of system altogether.

Not only does second-order logic follow smoothly from first-order logic, it also helps to iron out some apparent shortcomings from the first system. The compactness of first-order logic means that there is no way to reach the result that the set of statements {"There are finitely many things", "There is at least one thing", "There are at least two things", … } is inconsistent, even while we can (with the addition of identity) show the inconsistency in a set like {"There are at least three things", "There are no more than two things"}. Similarly, second-order logic is able to show the truth of the claim "There is some predicate whose extension contains all objects in the domain of discourse", formalised as $\exists X\, \forall x\, Xx$, which first-order logic does not recognise.

Certainly it is not the case that a stronger system is better and "more logical" – as Quine suggests, the more sophisticated the system, the further it tends towards set theory and mathematics, as opposed to logic proper. One might object that the concepts of induction and finiteness are straying out past the territory allotted for logic, but as Boolos convincingly argues, concepts like the ancestral relation and mathematical induction are fundamentally logical in nature – they are universal, topic-neutral truths about the formal structure of reasoning, not anything subject-specific.

To conclude, drawing the boundary between logic and not-logic at the transition from first-order logic to second-order logic seems overly conservative: the latter is semantically very similar to the former, with a few small additions to the underlying machinery. These additions help second-order logic to better achieve the goal of identifying tautologies and contradictions which seem distinctly logical (rather than mathematical), and as a result, it makes sense to treat it as another form of logic.

ii. taking second-order logic not to be logic (500 words)

The strongest argument against taking second-order logic to be logic is that it has substantial metatheoretical departures from first-order logic, demonstrating that it is qualitatively different and crosses the line over into set theory. One of the primary issues is that second-order logic extends quantification to predicate variables, thereby introducing commitments that go well beyond the neutral domain of objects found in first-order logic. In first-order logic, quantifiers range solely over individuals in the domain, ensuring that the logical apparatus remains agnostic about the nature of the entities involved. In contrast, by quantifying over predicates or properties, second-order logic implicitly assumes that these predicates have extensions – in effect, are sets or classes. This implicit commitment to set-theoretic notions suggests that second-order logic is not merely a conservative extension of first-order logic but one that imports the ontological baggage of set theory.

Moreover, the metatheoretical properties of second-order logic further distance it from the traditional conception of logic. Unlike first-order logic, which is characterized by a complete axiomatic proof system (as guaranteed by Gödel's completeness theorem), second-order logic lacks such a complete system. No recursively enumerable set of axioms can capture all its validities. One could reasonably argue that this incompleteness is not merely a technical shortcoming but a profound indication that second-order logic does not fulfil the role of a pure logical system. Logic is typically expected to provide a mechanical, effective method for determining truth or validity—qualities that second-order logic does not fully exhibit due to its inherent complexity.

As Boolos points out, though, the ontological commitments needed for second-order logic are in fact quite mild. Second-order quantifiers do not force us to adopt an extravagant set-theoretic ontology; rather, they merely require that we acknowledge that every non-empty domain possesses a power set, or equivalently, a collection of all properties defined over that domain. For example, the logical validity of the sentence $\exists X\, \forall x\, Xx$ should be understood as asserting that, in any given domain, there is a predicate whose extension simply coincides with the domain itself. Similarly, the existence of an empty predicate reflects only the minimal assumption that the domain has a structure rich enough to support the notion of an empty collection. The additional ontological commitments of second-order logic are hardly more burdensome than the assumption that the domain of discourse is non-empty, and they do not commit us to the existence of problematic entities like a universal set. These mild commitments can be seen as a natural and modest extension of first-order semantics, rather than an onerous importation of set theory.

It would be good to be able to list some specific things that SOL adds/imports in to FOL which causes Quine and those of like mind to suspect that SOL veers into something other than logic. Beyond listing those things, it would be good to have a rough idea of *why* those things might fail to be pure logic. Spend some time thinking about what is and is not a principle of pure logic. Take some simple set theory axioms like "there exists an empty set" and "two sets are identical if they contain the same things" – are these logical, or are they topic-specific? (If they fail to be logical, then it must be the case that you can think of some nontrivial domain of reasoning in which they might fail to apply). Then consider Modus Ponens – logical, or topic-specific? Then consider all of mathematics: what *is* mathematics about? Is it about some specific domain, or is it a purely neutral science of reasoning patterns which can be shipped off to physics or biology or economics or other such topic-specific sciences? (Look up "logicism", "deductivism" and "game formalism" for philosophical views that all of mathematics is topic-neutral in some important sense)