Week 3

1a). $p \to (p \to p)$ $\square$ PL1

b)

1. $p \to ((p \to p) \to p)$ PL1
2. $(p \to ((p \to p) \to p)) \to ((p \to (p \to p)) \to (p \to p))$ PL2
3. $(p \to (p \to p)) \to (p \to p)$ 1, 2, MP
4. $p \to (p \to p)$ PL1
5. $p \to p$ $\square$ 3, 4, MP

c)

1. $\sim p \to \sim p$ by using the proof above but with everywhere $\sim p$
2. $(\sim p \to \sim p) \to ((\sim p \to p) \to p)$ PL3
3. $(\sim p \to p) \to p$ $\square$ 1, 2, MP

2a)

1. ϕ Ass.
2. $\phi \to \psi$ Ass.
3. ψ 1, 2, MP

So $\phi, \phi \to \psi \vdash_{PL} \psi$

$\therefore \phi \vdash (\phi \to \psi) \to \psi$ DT

$\vdash \phi \to ((\phi \to \psi) \to \psi)$ $\square$ DT

b)

1. $\phi \to (\psi \to \phi)$ PL1
2. ϕ Ass.
3. $\psi \to \phi$ $\square$ 1, 2, MP

or

1. ϕ Ass.
2. ψ Ass
3. So $\phi, \psi \vdash \phi$

$\phi \vdash \psi \to \phi$ DT

c)

1. ϕ Ass.
2. $\sim \phi$ Ass.
3. $(\sim \psi \to \sim \phi) \to ((\sim \psi \to \phi) \to \psi)$ PL3
4. $\sim \phi \to (\sim \psi \to \sim \phi)$ PL1
5. $\sim \psi \to \sim \phi$ 2, 4, MP
6. $(\sim \psi \to \phi) \to \psi$ 3, 5, MP
7. $\phi \to (\sim \psi \to \phi)$ PL1
8. $\sim \psi \to \phi$ 1, 7, MP
9. ψ 6, 8, MP

So $\phi, \sim \phi \vdash \psi$

$\sim \phi \vdash \phi \to \psi$ DT

$\vdash \sim \phi \to (\phi \to \psi)$ DT $\square$

Good stuff

d) 1. $\sim\sim\phi$ Ass.

2. $\sim\sim\phi \to (\sim\phi \to \sim\sim\phi)$ PL7

3. $(\sim\phi \to \sim\sim\phi) \to ((\sim\phi \to \sim\phi) \to \phi)$

4. ~~$(\ldots)$~~ $\sim\phi \to \sim\sim\phi$ 1, 2, MP

5. $(\sim\phi \to \sim\phi) \to \phi$ 3, 4, MP

6. ϕ 5, MP with 1b) solution

So $\sim\sim\phi \vdash \phi$

$\therefore \vdash \sim\sim\phi \to \phi$ $\Box$ DT

e) 1. $(\sim\sim\sim\phi \to \sim\phi) \to ((\sim\sim\sim\phi \to \phi) \to \sim\sim\phi)$ PL3

2. $\sim\sim\sim\phi \to \sim\phi$ from 2(d)

3. $(\sim\sim\sim\phi \to \phi) \to \sim\sim\phi$ 1, 2, MP so $\sim\sim\sim\phi \to \phi \vdash \sim\sim\phi$ DT

4. $\phi \to (\sim\sim\sim\phi \to \phi)$ PL7 so $\phi \vdash \sim\sim\sim\phi \to \phi$ DT

~~5. $\sim\sim\phi$~~ ~~Ass.~~ so $\phi \to \sim\sim\phi$ by 3, 4, cut

~~6.~~ $\therefore$ $\phi \vdash \sim\sim\phi$ $\Box$ DT

4a) 1. $\Box P \to P$ T

2. $\Box\Box P \to \Box P$ T

3. $(\Box P \to P) \to (\Box\Box P \to (\Box P \to P))$ PL7

4. $\Box\Box P \to (\Box P \to P)$ 1, 3, MP

5. $(\Box\Box P \to (\Box P \to P)) \to ((\Box\Box P \to \Box P) \to (\Box\Box P \to P))$ PL2

6. $(\Box\Box P \to \Box P) \to (\Box\Box P \to P)$ ~~MP~~ 4, 5, MP

7. $\Box\Box P \to P$ $\Box$ 1, 6, MP

Or just use the PL tautology "syllogism"

b) 1. $\Box\sim P \to \sim P$ T

2. $(\psi \to \sim\phi) \to (\phi \to \sim\psi)$ PL i.e. contrapositive.

3. $P \to \sim\Box\sim P$ 1, 2, MP

You don't want to write the schema (but rather the relevant instance of it) in a proof

c) Goal: show $\vdash_{S4} \sim\Box\sim\sim\Box\sim P \to \sim\Box\sim P$

1. $\Box\sim P \to \Box\Box\sim P$ S4

2. $\Box\sim P \to \sim\sim\Box\sim P$ PL

3. $\Box\Box\sim P \to \Box\sim\sim\Box\sim P$ 2, K, NEC, MP

Yup, or just say "Becker Box"

4. $\Box\sim P \to \Box\sim\sim\Box\sim P$ 1, 3, ~~MP~~ PL

5. $\Box\sim\sim\Box\sim P \to \Box\sim\sim\Box\sim\sim\Box\sim P$ (repeat 1-4)

6. $\Box{\sim}P \to \Box{\sim}{\sim}\Box{\sim}{\sim}\Box{\sim}P$ 4,5, PL

7. ${\sim}\Box{\sim}{\sim}\Box{\sim}{\sim}\Box{\sim}P \to {\sim}\Box{\sim}P$ ■ 6, PL

Good

d) Goal: show $\vdash_{S4} \Box P \to \Box{\sim}\Box{\sim}\Box P$

1. $\Box{\sim}\Box P \to {\sim}\Box P$ T
2. $\Box P \to {\sim}\Box{\sim}\Box P$ 1, PL

Contraposition (btw it helps an exam marker a lot to mention the PL tautology you're appealing to!)

3. $\Box\Box P \to \Box{\sim}\Box{\sim}\Box P$ 2, K, NEC, MP
4. $\Box P \to \Box\Box P$ S4
5. $\Box P \to \Box{\sim}\Box{\sim}\Box P$ ■ 3, 4, PL

e)

1. $(\Box P \to \Box Q) \vee (\Box Q \to \Box P)$ PL
2. $(\Box P \to \Box Q) \to \Box(\Box P \to \Box Q)$ S4
3. $(\Box Q \to \Box P) \to \Box(\Box Q \to \Box P)$
4. $\Box(\Box Q \to \Box P) \vee \Box(\Box P \to \Box Q)$ ■ 2,3, PL

not totally sure how to justify this but I think it's true.

~~is this S4.2 or S4.3 is strictly weaker than S5.~~

Yeah it's a tough one, isn't it! Hope you caught the solution we covered in class -- drop me an email if you missed it

5a) i:

~~1. $\Box(\phi \wedge \psi) \to \Box\phi$ PL, Becker~~
~~2. $\Box(\phi \wedge \psi) \to \Box\psi$ "~~
~~3. $\Box(\phi \wedge \psi) \to \Box\phi \wedge \Box\psi$ 1,2, PL~~
~~4. ${\sim}(\Box\phi \wedge \Box\psi) \to {\sim}\Box(\phi \wedge \psi)$ 3, PL~~

1. $\Box((\phi \wedge \psi) \to \phi)$ PL, NEC
2. $\Box((\phi \wedge \psi) \to \psi)$ "
3. $\Diamond(\phi \wedge \psi) \to \Diamond\phi$ 1, K$\Diamond$
4. $\Diamond(\phi \wedge \psi) \to \Diamond\psi$ 2, K$\Diamond$
5. $\Diamond(\phi \wedge \psi) \to \Diamond\phi \wedge \Diamond\psi$ ■ 3, 4, PL

Yup, that'll do!

Nice

ii:

~~1. $\Box((\phi \to \psi) \to ({\sim}\psi \to {\sim}\phi))$ NEC, PL~~
~~2. $\Diamond(\phi \to \psi) \to \Diamond({\sim}\psi \to {\sim}\phi)$ K$\Diamond$, MP, 1~~
~~3. ${\sim}{\sim}\Box{\sim}({\sim}\psi \to {\sim}\phi) \to {\sim}{\sim}\Box{\sim}(\phi \to \psi)$ 2, PL~~

1. $\Box(\sim\psi \to \sim\phi) \to (\Diamond\sim\psi \to \Diamond\sim\phi)$ $K\Diamond$ ✓

2. $\Box(\phi \to \psi) \to (\sim\Diamond\sim\phi \to \sim\Diamond\sim\psi)$ PL

3. $\Box(\phi \to \psi) \to (\sim\sim\Box\sim\sim\phi \to \sim\sim\Box\sim\sim\psi)$ $\Diamond$df

4. $\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$. PL ∎

Are we sure about that?
It looks like you're using
the distribution of Box over
--> here, together with
the definition of Box.
But you can't help yourself to K in
this question!

b) ~~(⇒) If $\vdash_K \gamma$ then $\vdash_{K\Diamond} \gamma$~~

Goal: Show that $\vdash_K \gamma$ iff $\vdash_{K\Diamond} \gamma$.

(⇒) If some wff is provable in K, then take any parts of that proof using the (K) axiom, insert a subproof as in ii, then continue with the proofs in K$\Diamond$.

(⇐) If some wff is provable in K$\Diamond$, then take any parts of that proof using (K$\Diamond$) or ($\Diamond$df), inserting above a subproof of the axiom in K. Then continue with the proof in K. ∎

You have roughly the right idea! In an exam I might be a bit more explicit and write out things like "assume there exists a proof of Phi from [system], then every line of this proof is either an axiom or derived from prior lines via MP or NEC,.... you know how it goes.