Rohan

Logic week 2

1\. a) i. Suppose, for $\divideontimes$, that $V_M(\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi), w) = 0$, where M is an MPL model $\langle W, R, I \rangle$ and $w \in W$.

(*)

Then $V(\Box(\phi\to\psi), w) = 1$ and $V(\Box\phi \to \Box\psi, w) = 0$

∴ $V(\Box\phi, w) = 1$ and $V(\Box\psi, w) = 0$

So ϕ is true in every world accessible from w, and ψ is false in some " " " " .
But this means there is some world v ~~where~~ s.t. Rwv and $V(\phi, v) = 1$ and $V(\psi, v) = 0$. So $V(\phi \to \psi)$ is not true in all worlds accessible from w, but this is $\divideontimes$ with (*).

ii. ~~Suppose~~ Let M be an MPL model $\langle W, R, I \rangle$. and Suppose, for $\divideontimes$, that $V_M(\Box\phi \to \Diamond\phi, w) = 0$ where $w \in W$

∴ $V(\Box\phi, w) = 1$ and $V(\Diamond\phi, w) = 0$

∴ $V(\Box\phi, w) = 1$ and $V(\Box\sim\phi, w) = 1$

So ϕ is true in every world ~~[illegible]~~ accessible from w, and $\sim\phi$ is also true in every accessible world. But ~~then there is~~ Since R is serial, there exists some v s.t. Rwv holds. And then $V(\phi, v) = 1$, and $V(\sim\phi, v) = 1$

$\divideontimes$

iii. Let M be an MPL model $\langle W, R, I \rangle$
Suppose, for $\divideontimes$, that $V_M(\Box\phi \to \phi, w) = 0$ where $w \in W$

∴ $V(\Box\phi, w) = 1$ and $V(\phi, w) = 0$.

But as R is reflexive, Rww holds.
So in every world accessible from w, ϕ is true
~~so~~ $V(\phi, w) = 1$ and $V(\phi, w) = 0$ $\divideontimes$

iv. Let M be an MPL model $\langle W, R, I \rangle$.
Suppose, for $\divideontimes$, that $V_M(\phi \to \Box\Diamond\phi, w) = 0$ where $w \in W$

(*)

$\therefore V_M(\phi, w) = 1$ and $V(\Box\Diamond\phi, w) = 0$

So there is some world accessible from w in which $\Diamond\phi$ is false. Let v be a world s.t. Rwv and $V(\Diamond\phi, v) = 0$ $\therefore V(\Box\sim\phi, v) = 1$

(R is reflexive so w accesses at least one other world).

But as R is symmetric, Rvw holds also. So w is accessible from v, and in all worlds accessible from v, $\sim\phi$ is true. # with (*).

v. ~~Let~~ Let M be an ~~and~~ MPL model $\langle W, R, I \rangle$.
Suppose, for #, that $V_M(\Box\phi \to \Box\Box\phi, w) = 0$ where $w \in W$.
$\therefore V(\Box\phi, w) = 1$ and $V(\Box\Box\phi, w) = 0$.
So in every world accessible from w, ϕ is true (i)
and there is some world v accessible from w in which $\Box\phi$ is false. (ii)
$\therefore$ there's a world u accessible from a world v accessible from w in which ϕ is false (from ii) (iii)
But R is transitive. So ~~A~~ Rwv and Rvu means Rwu.
~~So~~ But then $V(\phi, u) = 0$ is # with (i)

(*) Do I need to appeal to reflexivity here, to avoid some case where (i) is vacuously true? or all OK?

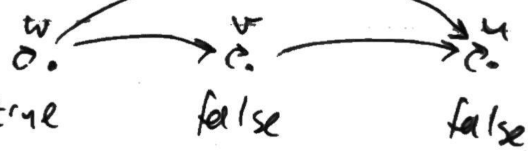
This proof is all good! Nicely done.

vi. Let M be an MPL model $\langle W, R, I \rangle$. Suppose, for #, that $V_M(\Diamond\phi \to \Box\Diamond\phi, w) = 0$, where $w \in W$.
$\therefore V(\Diamond\phi, w) = 1$ and $V(\Box\Diamond\phi, w) = 0$.
- So there is some world u accessible from w where ϕ is true(*) and some ~~all~~ worlds v accessible from w where $\Diamond\phi$ is false (+)
- Since R is ~~reflexive~~ ~~transitive~~ symmetric, Ruw and Rvw hold. Since it's transitive, [illegible] Rvw and Rwu jointly means Rvu holds.
- If $V(\Diamond\phi, v) = 0$ then ~~the~~ ϕ is false in all worlds accessible from v, using (+). But given Rvu, this is # with (*).

b) ii Consider the model $M = \langle W, R, I \rangle$
where $W = \{ w \}$, $R = \emptyset$ and any I s.t. $I(\phi, w) = 0$
Then the antecedent is vacuously true but the consequent false.

iii $M = \langle W, R, I \rangle$ where
$W = \{ w, v \}$, $R = \{ \langle w,v \rangle, \langle v,w \rangle \}$ and
any I s.t. $I(\phi, w) = 0$ and $I(\phi, v) = 1$.
Then in w, $\Box\phi$ is true but ϕ is false.

iv. $M = \langle W, R, I \rangle$ where
$W = \{ w, u, v \}$, $R = \{ \langle w,w \rangle, \langle u,u \rangle, \langle v,v \rangle, \langle w,v \rangle, \langle v,u \rangle, \langle w,u \rangle \}$
and any I s.t. $I(\phi, w) = 1$ and $I(\phi, v) = 0$ and $I(\phi, u) = 0$.
Then in w, ϕ is true but it can access v where $\Diamond\phi$ is false, so $\Box\Diamond\phi$ is false in w.

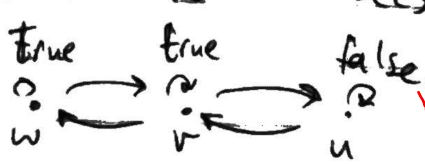


v. $M = \langle W, R, I \rangle$ where
$W = \{ w, u, v \}$, $R = \{ \langle w,w \rangle, \langle u,u \rangle, \langle v,v \rangle, \langle w,v \rangle, \langle v,w \rangle, \langle v,u \rangle, \langle u,v \rangle \}$
and any I s.t. $I(\phi, w) = 1$ and $I(\phi, v) = 1$ and $I(\phi, u) = 0$.

(*) [is this needed?]

Then in all worlds accessible from w, ϕ is true, but w sees v and v sees u, (where ϕ is false).

Well, I appreciated it being there :)



vi. $M = \langle W, R, I \rangle$ where (#S4)

$W = \{ w, v \}$, $R = \{ \langle w,w \rangle, \langle v,v \rangle, \langle w,v \rangle \}$
and any I s.t. $I(\phi, w) = 1$ and $I(\phi, v) = 0$.

Then ϕ is true in some world accessible from w, but w sees a world which cannot access any world in which ϕ is true

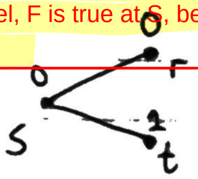
($\nvDash_B$) $M = \langle W, I, R \rangle$
where $W = \{w, v, u\}$ and R is the reflexive closure of $\{\langle w,u\rangle, \langle w,v\rangle, \langle u,v\rangle\}$
and I is s.t. $I(\phi, w) = 1$, $I(\phi, v) = 0$, $I(\phi, u) = 0$.

2a) i. Not valid. Consider
$M = \langle T, I, \le \rangle$ where $\le$ is the reflexive closure of $\{\langle s,t\rangle, \langle s,t\rangle\}$
$T = \{t, s, r\}$ and

and I is s.t. $I(\phi, t) = 1$, $I(\phi, s) = 0$, $I(\phi, r) = 0$

(*) How do I indicate that "now" is t? or unimportant even if using $<$ rather than $\le$

Notice that in your model, F is true at S, because there is at least one future point (in this case T) where is true.But



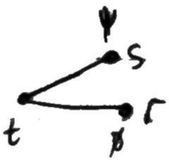
Then at time t, ϕ is true but it was not always the case that ϕ will always be true.

In Tense Logic models you don't really have to indicate "now", in the same way in Modal Logic models you never really had to indicate "this world is the actual world" -- the point is that the logical claims should hold *across* points in time/possible worlds!

ii. Valid. Suppose for ~~*~~, that $V_M(PP\phi \to P\phi, t) = 0$ where $M = \langle T, I, \le \rangle$ and $t \in T$.
$\therefore V(PP\phi, t) = 1^{(i)}$ and $V(P\phi, t) = 0^{(ii)}$
So all times which see t, are seen by a time where ϕ is true. (Since $\le$ is reflexive, t is seen by at least one time) and by transitivity of $\le$ t is seen by a time where ϕ is true. But this is * with (ii)

If transitivity is assumed, you're not wrong. But in PTL, transitivity is not assumed by default.

iii. Not valid. Consider $M = \langle T, I, \le \rangle$ where $T = \{t, s, r\}$ and $\le$ is the reflexive closure of $\{\langle t,s\rangle, \langle t,r\rangle\}$



and I is s.t. $I(\phi, t) = I(\psi, t) = I(\phi, s) = I(\psi, r) = 0$
and $I(\psi, s) = I(\phi, r) = 1$.

Then t sees a time where ψ is true and where ϕ is true but the branching makes the consequent false.

b) i. We can impose a non-branching strong connectedness restriction s.t. for all t, t', either $t \leq t'$ or $t' \leq t$. Then, suppose for # that $V_M(\phi \to HF\phi, t) = 0$ for $t \in T$. $\therefore V(\phi, t) = 1$ and $V(HF\phi, t) = 0$.

So there ~~are~~ is some ~~no~~ times weakly earlier than t which ~~A~~ doesn't see a time where ϕ is ~~false~~ true. ~~Both~~ ~~Consider one time weakly earlier than t, s. (by reflexivity some s exists).~~ ~~Then~~ But this is a #, since in a linear timeline all times before t can see t, and it's true at t.

Hah! Nice catch. Well done.

[hmm, actually my answer to (a) was probably wrong].

iii. Impose a strong connectedness restriction. Then ~~suppose for~~ ~~that~~ ~~V_M~~ if ϕ will at some future point be true, and ψ will at some future point be true, given that the timeline is linear, ~~at~~ it must be true that:

- at some point in future, ϕ will be true and ψ will sometime later be true or
- at some point in future, ψ will be true and ϕ will be later, or
- ~~they~~ ~~at~~ some future point, both will be true.

Since both occur in the same future timeline.

5 a) Base case: any atomic formula has 0 parentheses and 0 occurrences of $\to$, so the condition F holds.

Inductive case: ~~Assume~~ ~~about~~ Let ϕ be an arbitrary formula, s.t. $cp(\phi) = n$. Assume for all ψ s.t. $cp(\psi) < n$, that F holds. ~~Then~~ ϕ can be written as either $\sim\psi$ or $(\psi_1 \to \psi_2)$

(a) If ϕ is $\sim\psi$, then ϕ has the same number of material cond.s and parenth. as ψ, so F holds for ϕ.

Good stuff.

(b) If ϕ is $(\psi_1 \to \psi_2)$ then $P(\phi) = P(\psi_1) + P(\psi_2) + 2$ and $I(\phi) = I(\psi_1) + I(\psi_2) + 1$ where P, I is the number of parenth., implications. But $P(\psi_1) = 2I(\psi_1)$; $P(\psi_2) = 2I(\psi_2)$ so $2I(\phi) = P(\phi)$ and F holds.

So F holds for all PL sentences by strong induction.

b) Let ~~F~~ $F(\phi)$ be the property of having $V_{I^+}(\phi) = 1$
Base case: F holds for all atomic formulae, since I^+ is s.t. $I^+(\alpha) = 1$ for all sentence letters α.
Inductive case ~~assume that~~ let ϕ be an arbitrary formula*.
Assume that for all formulae ψ with lower complexity* than ϕ, F holds. Then ϕ can be written $\psi_1 \to \psi_2$.
Since $V_{I^+}(\psi_1) = 1$ and $V_{I^+}(\psi_2) = 1$, $V_{I^+}(\phi) = 1$.
So by strong induction, F holds for ~~any~~ all sentences without any occurrences of $\neg$.

* with no occurrences of negation.

c) Let $F(\phi)$ be the property of having there exist some interpretation I, I' s.t. $V_I(\phi) = 1$, $V_{I'}(\phi) = 0$.
Base case: For some atomic formula α, let I be s.t. $I(\alpha) = 1$ and I' be s.t. $I'(\alpha) = 0$.
Inductive case: Let ϕ be an arbitrary formula*. Assume that for all formulae ψ with lower complexity*, F holds. ϕ can be written as either ~~ϕ~~ $\neg\psi$ or $\psi_1 \to \psi_2$.
(a) if ϕ is $\neg\psi$, then switch I and I' for ψ to get interpretations on which ϕ is false and true respectively. So F holds.
(b) if ϕ is $\psi_1 \to \psi_2$ then there exists I, I', J, J' s.t. $I(\psi_1) = 1$, $I'(\psi_1) = 0$, $J(\psi_2) = 1$, $J'(\psi_2) = 0$.
Since ~~the~~ ψ_1 and ψ_2 cannot share sentence letters, we can ~~use~~ combine I and J to make ϕ true, and I and J' to make ϕ false, by taking the values each interpretation assigns to the sentence letters in its ~~sub~~ subformula ψ_1, ψ_2.
So F holds.
So by strong induction, F holds for all ϕ ~~without~~ with only one occurrence of each sentence letter.
But then such a formula cannot be a tautology, as a tautology must be true under all interpretations.

* with at most one occurrence of each sentence letter.

Nicely done!