Folhan

Week 4

2a) Result to show: if $\Gamma \vdash_S \delta$ and $\Sigma, \delta \vdash_S \phi$ then $\Gamma, \Sigma \vdash_S \phi$

- By assumption, there is a proof A of δ from Γ, and a proof B of ϕ from Σ, δ. Let C be the result of concatenating A and B, in that order.
- Let m, n be the lengths of the proofs A, B respectively
- The first m lines of C will all be acceptable in axiomatic proof, since they are from A and thus must either be an S-axiom, a member of Γ, or the application of an S-rule to an earlier lines.
- The following n lines are from B, and are either
  - (i) an S-axiom
  - (ii) a member of Σ
  - (iii) δ
  - (iv) an application of an S-rule to lines from m+1 onwards

  In cases (i), (ii) and (iv), the line is certainly acceptable in the axiomatic proof of ϕ from Γ, Σ. In case (iii), note that δ is already in our proof at line m, and so has already been established as acceptable.
- So, every line in C is acceptable as an axiomatic proof of ϕ from Γ, Σ. ■

Cut follows by a simple modification of the above argument. ~~inserting~~ concatenating all the proofs A_i for $\delta_1, \dots \delta_N$ together before our proof B.

b) Result to show: if $\Gamma, \phi \vdash_{PL} \psi$ then $\Gamma \vdash_{PL} \phi \to \psi$

- By assumption, there is a proof A of ψ from Γ, ϕ.
- Each line in A is either a PL-axiom, a member of $\Gamma \cup \{\phi\}$, or an application of MP to earlier lines.

~~If a line α_i is a PL axiom,~~

- We can use strong induction to show that, for each line α_i in A, the property F_i: $\Gamma \vdash_{PL} \phi \to \alpha_i$ holds, and reach our result.

(i) If α_i is an axiom, then we can show F holds as follows:

1. α_i axiom
2. $\alpha_i \to (\phi \to \alpha_i)$ PL1
3. $\phi \to \alpha_i$ 1,2, MP

(ii) If α_i is a member of Γ, then

1. α_i premiss
2. $\alpha_i \to (\phi \to \alpha_i)$ PL1
3. $\phi \to \alpha_i$ 1,2, MP

again shows F holds

(iii) If α_i is ϕ, then as $\phi \to \phi$ is PL-valid, F holds.

You don't really need the Cut rule in the proof of the Deduction Theorem! Your proof here goes through completely.

(iv) IH: Assume that for all $j < i$, the property F_j holds.

- Then there are, in particular, two earlier lines (with the property F) of the form χ and $\chi \to \alpha_i$, in order for α_i to be obtained by MP.
- So 1. $\Gamma \vdash_{PL} \phi \to \chi$ and 2. $\Gamma \vdash_{PL} \phi \to (\chi \to \alpha_i)$
- ~~By Cut~~, we can show further that
  3. $(\phi \to (\chi \to \alpha_i)) \to ((\phi \to \chi) \to (\phi \to \alpha_i))$ PL2
  4. $(\phi \to \chi) \to (\phi \to \alpha_i)$ 2, 3, MP
  5. $\phi \to \alpha_i$ 1, 4, MP

(*) Sider talks about also requiring Cut - why?

and thus $\Gamma \vdash_{PL} \phi \to \alpha_i$, so F holds for α_i.

So in all cases F holds. ■

3a) Result to show: $\dfrac{\phi \to \psi}{O\phi \to O\psi}$ ~~preserves~~ has the preserving property F of a K-derivation is

Base case: O is an empty string. Then the second line is just $\phi \to \psi$ which is assumed to be ~~derivable~~ a theorem, so it is F.

The base case is actually where O is one instance of box or diamond!

IH: assume for ~~the~~ some string O, F holds.

Prefixing a further modal operator comes in two cases (i) $\Box O\phi \to \Box O\psi$ (ii) $\Diamond O\phi \to \Diamond O\psi$.

But you did give me the necessary proof that Box and Diamond can be appended in a proof below

(i) Consider the axiomatic proof

This is correct

1. $O\phi \to O\psi$ theorem by hypothesis
2. $\Box(\phi \to O\psi) \to (\Box O\phi \to \Box O\psi)$ K
3. $\Box(O\phi \to O\psi)$ 1, Nec
4. $\Box O\phi \to \Box O\psi$ 2, 3, MP

showing that F holds ~~with~~ in this case, since we derived ~~it~~ the new off in an ~~[illegible]~~ axiomatic manner.

~~(ii) Again, consider~~

~~Note that~~ we can show

~~1. $\Box O\phi \to \Box O\psi$~~ as above

~~2. $\Diamond O\psi \to \Diamond O\phi$ PL (contrapositive)~~

and ~~thus exchanging~~ ϕ for ψ in the proof for (i), ~~and vice versa,~~

(ii) If $\phi \to \psi$ is derivable, by PL so is $\neg\psi \to \neg\phi$.

Applying IH to this, we get

1. $O\neg\psi \to O\neg\phi$ by hypothesis
2. $\Box O\neg\psi \to \Box O\neg\phi$ 1, Nec, K, MP as in (i)
3. $\Diamond O\phi \to \Diamond O\psi$ 2, PL (contrapositive)

and so again F holds

So by induction Becker is K-admissible

And then you need to generalise, by saying that we proved this for arbitrary formulae possibly already containing strings of modal operators, so we can always add one additional box or one additional diamond

b) Suppose the derivation of $\vdash_K \chi(a)$ is a proof with n steps.

BC: If $n = 0$ then $\chi(a)$ is an axiom. Then uniformly substituting ϕ for α will be an axiom licensed by the same schema, so the property F of being K-derivable holds.

IH Assume for all ~~[illegible]~~ theorems provable in m steps s.t. $m < n$, F holds.

The formula $\chi(\alpha)$ must be reached through the application of either (i) MP or (ii) Nec, given $n > 0$.

(i) $\chi(\alpha)$ may be reached from $\psi(\alpha) \to \chi(a)$ and $\psi(\alpha)$. By IH, these are both derivable after substitution, and thus so is $\chi(\phi)$

(ii) $\chi(\alpha)$ may be reached from $\psi(\alpha)$ if $\chi := \Box\psi(a)$. Again by IH $\psi(\alpha)$ is derivable after subs and so

In an exam, to play it safe, it might be worth giving the formal axiomatic derivation in each of the two cases (MP and NEC)

too will be $\chi(a)$.
So if holds in both cases and by induction the rule is K-admissible.

c) ~~Since $(\phi_1 \to \ldots (\phi_n \to \psi))$ is an MPL tautology, we can obtain a PL tautology~~
Since $\phi_1, \ldots, \phi_n$ are all K-derivable, we can concatenate the proofs for each and have the start of a proof A, every line in which is either an axiom or theorem of K.

As $(\phi_1 \to \ldots (\phi_n \to \psi))$ is an MPL tautology, there exists some PL tautology from which it is obtainable by uniform substitutions of MPL wffs for sentence letters.

You didn't try 4; drop me an email if you have any questions about 4 or regarding maximal consistency!

By completeness there exists some PL proof of that tautology. And as K has all the PL axioms plus MP, the tautology is provable in K. Using Subst 1_P repeatedly, we can further see that the MPL-tautology is provable, B.

Yup!

Finally, we can use MP repeatedly with the earlier lines in A obtaining $\phi_1, \ldots, \phi_n$ and our final line in B, to obtain ψ as K-derivable.

d) We can show this by induction on complexity of $\chi(a)$, supposing we have proven $\phi_1 \leftrightarrow \phi_2$ already.
BC: Suppose $\chi(a)$ is just the sentence-letter a. Then trivially* $\chi(\phi_1) \leftrightarrow \chi(\phi_2)$ being provable whenever $\phi_1 \leftrightarrow \phi_2$ is provable holds, since $\chi(\phi_1)$, $\chi(\phi_2)$ simply are ϕ_1, ϕ_2.

* the property F of

There are technically two subcases for the base case: Chi(alpha) could be equal to or distinct from alpha --- not terribly interesting, but you do need to consider both

IH: assume for all formulae χ' less complex than $\chi(a)$, $F_{\chi'(a)}$ holds.
Our formula $\chi(a)$ is either (i) the negation of a simpler formula or (ii) an implication between two simpler formulae, or (iii) necessitation

(i) $\chi(a) := \sim\chi'(a)$.

By IH, as $\chi'(a)$ is simpler than $\chi(a)$, we have $\chi'(\phi_1) \leftrightarrow \chi'(\phi_2)$.

By PL, we know that it follows that

$$\sim\chi'(\phi_1) \leftrightarrow \sim\chi'(\phi_2)$$

So F holds.

(ii) $\chi(a) := \chi_1'(a) \to \chi_2'(a)$

By IH, $\chi_1'(\phi_1) \leftrightarrow \chi_1'(\phi_2)$ and $\chi_2'(\phi_1) \leftrightarrow \chi_2'(\phi_2)$

($\Rightarrow$) If $\chi_1'(\phi_1) \to \chi_2'(\phi_1)$ is true, then on the assumption of $\chi_1'(\phi_1)$, $\chi_2'(\phi_1)$ holds. But this means also that it holds on the assumption of $\chi_1'(\phi_2)$ and, further, that on that assumption $\chi_2'(\phi_2)$ holds

We can repeat the analagous argument in the other direction, to show F holds, i.e.

$$(\chi_1'(\phi_1) \to \chi_2'(\phi_1)) \leftrightarrow (\chi_1'(\phi_2) \to (\chi_2'(\phi_2))$$

(iii) $\chi(a) := \Box\chi'(a)$

By IH, $\chi'(\phi_1) \leftrightarrow \chi'(\phi_2)$, ~~and with then~~

i.e. $\chi'(\phi_1) \to \chi'(\phi_2)$ and $\chi'(\phi_2) \to \chi'(\phi_1)$.

Using Becker, $\Box\chi'(\phi_1) \to \Box\chi'(\phi_2)$ and the converse,

so $\Box\chi'(\phi_1) \leftrightarrow \Box\chi'(\phi_2)$ ~~iso~~

and F holds.

So in all cases F holds, so Subst is admissible by strong induction.

For question 1: In order to argue that some axiomatic system is valid, you need to show

1. that all of its axioms are valid
2. that its rules of inference are valid

In the case of the axiom system S5, the axioms are PL1 to PL3, and then the modal axioms K, T and S5. We have demonstrated the semantic S5-validity of all of these axioms in the Week 2 problem sheet, and so what we have to do is prove that the rules of inference in S5 *preserve* validity. And, of course, the rules of inference that we need to consider are MP and NEC.