Risk

Define in terms of preferences

Yes, this is a result when you have an EU representation.

3. $u(y) = \ln y$; $\omega = 10$

a) An agent i is risk averse iff u_i is concave.

We can show concavity of u by the second-order derivative:

$$\frac{du}{dy} = \frac{1}{y} \; ; \; \frac{d^2u}{dy^2} = -\frac{1}{y^2}$$

and since $-\frac{1}{y^2} \leq 0$ for all $y > 0$,

the utility function is concave.

So Charlie is risk averse. This is equivalent to the statement that for every lottery $\mathbb{L}$, $EU(\mathbb{L}) \leq u(EV(\mathbb{L}))$.

b) The Absolute Arrow-Pratt Measure $:= -\frac{u''(y)}{u'(y)}$ which in this case is $-\frac{(-y^{-2})}{y^{-1}} = \frac{1}{y}$.

The Relative Arrow-Pratt Measure $:= -\frac{u''(y)}{u'(y)} \cdot y$ which in this case is $\frac{1}{y} \cdot y = 1$.

[relationship with Kelly criterion?]

So, Charlie exhibits ~~constant~~ ~~relative~~ constant relative risk aversion. This is an empirically common finding - individuals' risk aversion tends to decrease proportionally with wealth, as in Charlie's case.

Kelly essentially assumes log utility, if I'm not mistaken.

c) We can describe this investment as the following lottery $\mathbb{L}$:

$$\mathbb{L} = [\tfrac{1}{2}, \tfrac{1}{2}; 10+c, 10-\tfrac{1}{2}c]$$

i.e. if the project succeeds he ends up with $c + 10$ and otherwise he ends up with $10 - \frac{1}{2}c$.

$$EU(\mathbb{L}) := \sum_i p_i \cdot u(x_i) = \frac{1}{2} \cdot \ln(10+c) + \frac{1}{2}\ln(10-\tfrac{1}{2}c)$$

Charlie's problem is to $\max_c EU(\mathbb{L}_c) = \frac{1}{2}\ln(10+c) + \frac{1}{2}\ln(10-\frac{1}{2}c)$

for which the FOC is $\frac{\partial EU(\mathbb{L}_c)}{\partial c} = 0$

i.e. $\frac{1}{2} \cdot \frac{1}{10+c} - \frac{1}{4} \cdot \frac{1}{10-\frac{1}{2}c} = 0$

$\therefore \frac{10+c}{2} = 10 - \frac{1}{2}c \Rightarrow \underline{c = 5}$

d) If $\omega = 20$ instead, the FOC will be

$$\frac{1}{2} \cdot \frac{1}{20+c} - \frac{1}{4} \cdot \frac{1}{20-\frac{1}{2}c} = 0$$

which yields $c = 10$.

~~Therefore~~ So, he invests twice as much as before, with twice as much wealth to start with — or to put it another way, the fraction of his wealth used to invest in the risky asset is constant at $\frac{1}{2}$. This is as expected since his utility function exhibits CRRA as discussed in (b): risk tolerance increases proportionally with wealth, and so too does the optimal investment level.

Yes

4. Loss $L > 0$ w.p. π; insurance available at cost pq for q units.

a) We assume that Perdita has well-behaved preferences, meaning that utility is monotonic increasing in money, i.e. $u'(y) > 0$ for all $y \geq 0$. As Perdita is risk averse, her utility function is concave, and therefore $u''(y) < 0$ for all $y > 0$.

So, since the marginal utility of money is decreasing, as given by concavity, we know that $u'(w-L) > u'(w)$, since $w - L < w$.

b) When $p = \pi$, the cost of insuring one unit is simply the probability of that unit being lost. So, in expectation, the value of one unit of insurance is exactly equal to its cost, i.e. the expected value of all the lotteries generated by choices of $q \in [0, L]$ is the same, and a risk-neutral agent would be indifferent between them all. This is actuarially fair because the insurer generates zero expected profits.

c) If the loss is not realised, she ends up with wealth $w - pq$.
If it is realised, she ends up with $w - pq - L + q = w + (1-p)q - L$
So her expected utility is $(1-\pi) \cdot u(w-pq) + \pi \cdot u(w + (1-p) \cdot q - L)$

Her utility-maximisation problem is $\max_q EU(\mathbb{L}_q)$
for which the FOC is $\frac{\partial EU(\mathbb{L}_q)}{\partial q} = 0$,
i.e. at the optimal q^*,

$$-p\cdot(1-\pi)\cdot u'(w - pq^*) + (1-p)\cdot\pi\cdot u'(w+(1-p)\cdot q^* - L) = 0$$

as required.

Great

d). ~~From~~ Rearranging, the FOC states that

$$\frac{u'(w+(1-p)q^* - L)}{u'(w-p\cdot q^*)} = \frac{p(1-\pi)}{(1-p)\cdot\pi}$$

but by inspection the RHS $= 1$ when $p=\pi$. And from strict concavity, we know that if $u'(\phi) = u'(\psi)$ then $\phi = \psi$, i.e. $w+(1-p)q^* - L = w - pq^*$, so $q^* = L$ by simplifying.

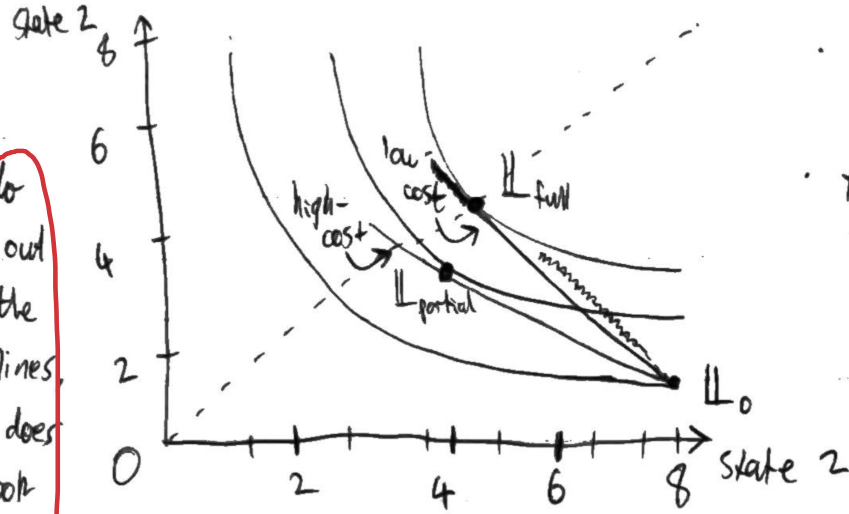
e). If $p > \pi$ then the RHS $= \frac{p-k}{\pi-k}$ where $k = p\pi$ so RHS > 1. assuming $p < 1$ (See below for special case where $p = 1$)
This means that $u'(w+(1-p)q^* - L) > u'(w-pq^*)$
and from (a) $w+(1-p)q^* - L < w - pq^*$
which implies that $q^* < L$, i.e. she only buys partial insurance. Intuitively, because the price is higher than actuarially fair, each unit of insurance bought reduces the expected value of the resulting lottery. At some quantity q^* the utility benefits of risk reduction are outweighed by the EV losses, and no more insurance is demanded.

Note that in the case where $p = 1$, the RHS is undefined. Here, the insurance is so expensive that she is no better off from having bought it even in the worlds where the loss is suffered (as the insurance costs as much as the loss reimbursement provides) so given $\pi > 0$ she's strictly better off buying no insurance, no matter her risk aversion.

· More mathematically / technically for these ICs:
  • Can we say anything about their slope at the 45° line? Yes: −1
  • ~~Would any agent who's risk neutral have ICs with slope ≠ −1 in this example?~~
  How do you go from knowing the probabilities in lottery to constructing ICs? (and utility func.)

7. $\mathbb{L}_0^A = [\frac{1}{2}, \frac{1}{2}; 8, 2]$

a)



· Arthur has downward-sloping ICs by monotonicity
· His risk aversion means they are convex: he prefers a sure thing to a lottery with the same expected value.

• how do you work out slope for the insurance lines, and what does price = 1 look like?

flat

Hope this was answered in class.

Depending on the cost of the insurance, he may buy either partial or full insurance, as it would allow him to reach a higher indifference curve as shown above. Intuitively, he may be willing to sell some of his risky future income for a sure thing now, at a price which is above his certainty equivalent for $\mathbb{L}_0$, making him better off.

b) Norma has linear indifference curves of slope −1. Since her $EU(\mathbb{L}) = EV(\mathbb{L})$ for all lotteries (by definition of risk neutrality), and the two events are equiprobable, she's indifferent between all lotteries where the payoff in state 1 + payoff in state 2 is some constant, i.e. $x + y = k$.

$\mathbb{L}_0^N = [\frac{1}{2}, \frac{1}{2}; 3, 7]$ and $\mathbb{L}_1^N = [\frac{1}{2}, \frac{1}{2}; 6, 4]$

(What is a better way to word this?)

Norma is actuarially fair: she is willing (or at least indifferent) to insuring Arthur at a price equal to the probability of loss. So Arthur will seek full insurance coverage from her – that is, sign a contract that makes his payoff independent of the state which ends up occurring. Norma will weakly prefer any contract with at least as much expected value as her endowed lottery (5). So Arthur will give her 3 if s_1 occurs, and she will give him 3 if s_2 occurs, for $\mathbb{L}_1^A = [1; 5]$

c) Arthur's endowed lottery has expected utility

$$EU(\mathbb{L}_0^A) = \sum_i p_i \cdot u(x_i) = \tfrac{1}{2}\ln 8 + \tfrac{1}{2}\ln 2 = 2\ln 2.$$

and he will only accept contracts at least as good in expectation. Norma simply wants to maximise $x_1 + x_2$ (this is identical to maximising her expected utility).

(Lagrangian?)

We're interested in the ~~greatest~~ x and y which maximises

$$\tfrac{1}{2}(3+x) + \tfrac{1}{2}(7-y) \quad \text{s.t.} \quad \tfrac{1}{2}\ln(8-x) + \tfrac{1}{2}\ln(2+y) \geq 2\ln 2$$

and $0 \leq x \leq 8,\ 0 \leq y \leq 7$

$$\iff \ln((8-x)(2+y)) \geq \ln 16$$

$$\iff (8-x)(2+y) \geq 16$$

At the optimum, the constraint will bind, since Norma's EU is increasing in x and decreasing in y, and vice versa for Arthur. So rearranging, $y = \frac{16}{8-x} - 2$.

OK good, although more convoluted than necessary. Solve diagrammatically using your Edgeworth box.

Now we can solve unconstrained

$$\max_x EU(x) = \tfrac{1}{2}(3+x) + \tfrac{1}{2}\left(7 - \left(\tfrac{16}{8-x} - 2\right)\right) = 6 + \tfrac{1}{2}x - \tfrac{8}{8-x}$$

which the FOC is $\frac{\partial EU(x)}{\partial x} = 0$

$$\Rightarrow \tfrac{1}{2} - \frac{8}{(8-x)^2} = 0$$

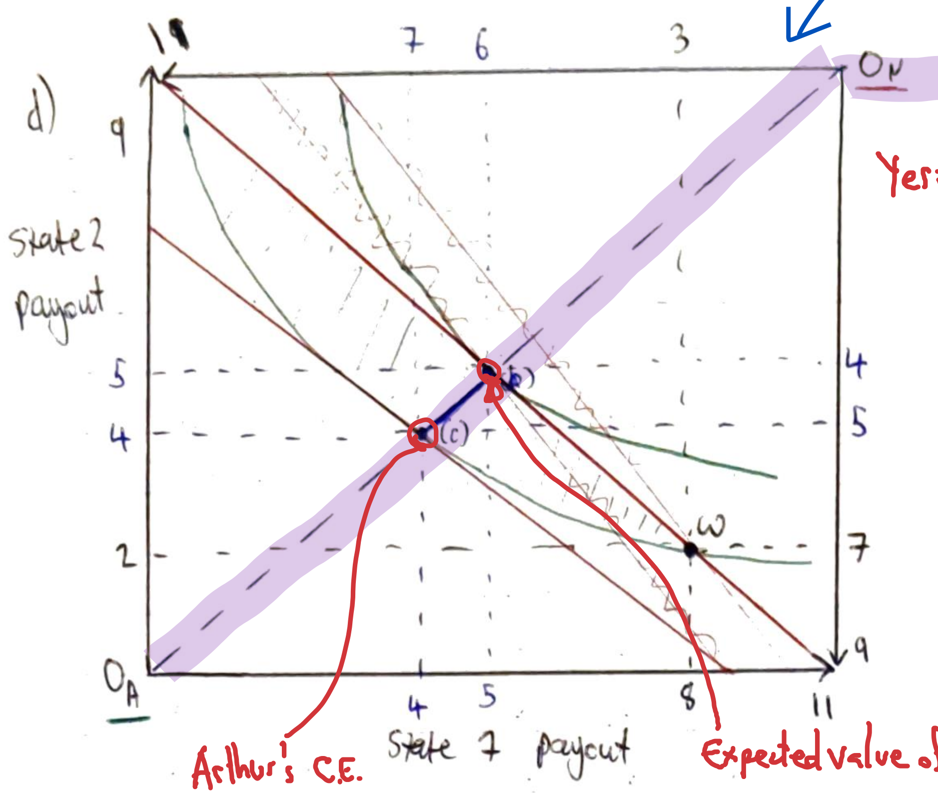
$$(8-x)^2 = 16, \quad 8-x = \pm 4, \quad x = 4 \text{ or } 12 \text{ but } 0 \leq x \leq 8 \text{ so } x = 4$$

We can check the SOC to ensure this is a maximum,

$$\left.\frac{\partial^2 EU(x)}{\partial x^2}\right|_4 = \left. -16(8-x)^{-3}\right|_4 < 0 \text{ so yes, a maximum.}$$

So the best contract for Norma has $x=4$, $y=2$ yielding lotteries $\mathbb{L}_2^N = [\tfrac{1}{2}, \tfrac{1}{2}; 7, 5]$ and $\mathbb{L}_2^A = [\tfrac{1}{2}, \tfrac{1}{2}; 4, 4]$ and $EU_L = 6$, $EU_A = 2\ln 2$.

7. d)



The 45* line shouldn't hit O_N, it's meant go through (9,9). So the x-axis ought to be a bit wider - imagine it finishes where the purple line (for Pareto set) does.

Yes: tangencies are along Arthur's certainty line.

are the tangencies right? or should it be different

Should the green ICs look different around the extremes?

They look just like Cobb-Douglas ICs.

(I think this is probably wrong...)

All contracts leading to ~~our~~ lotteries in the striped lens (////) are Pareto improvements on the initial situation. The solid portion of the blue line — between the extremes for the solutions to (b) and (c) are the efficient risk-sharing contracts.

Oh, oops, that's the contract curve. Proper Pareto set highlighted in purple (hard to correctly do as my Edgeworth box is a bit wrong)

9 a) $u(y) = \ln y$. $\mathbb{L} = [\frac{1}{2}, \frac{1}{2}; 40, 10]$.

The certainty equivalent $CE(\mathbb{L})$ of a lottery $\mathbb{L}$ is the amount of money as a sure thing that has the same utility as the lottery's EV.

i.e. $CE(\mathbb{L}) := u^{-1}(EU(\mathbb{L}))$

And $EU(\mathbb{L}) := \sum_i p_i \cdot u(x_i) = \frac{1}{2}\ln 40 + \frac{1}{2}\ln 10 = 3.00$

so $u^{-1}(EU(\mathbb{L})) = e^{3.00} = \underline{20}$

The risk premium $RP(\mathbb{L})$ ~~is~~ is the amount of money someone is willing to give up in expectation to avoid taking on a lottery $\mathbb{L}$'s risk

i.e. $RP(\mathbb{L}) := EV(\mathbb{L}) - CE(\mathbb{L})$

here, $EV(\mathbb{L}) = \frac{1}{2}\cdot 40 + \frac{1}{2}\cdot 10 = 25$, so $RP(\mathbb{L}) = 5$. The fact the risk premium is +ve is a consequence of Bill's risk aversion.

He shouldn't invest in the project as the certainty equivalent of it, 20, is less than the value of his reservation option of not investing (22). Put differently, investing is expected-utility negative.

b) The new lottery $\mathbb{L}' = [\frac{1}{2}, \frac{1}{2}; 31, 16]$ has EU of $\frac{1}{2}\ln 31 + \frac{1}{2}\ln 16 = 3.103$ and so CE of $e^{3.103} = 22.27722$.

So yes, he should accept the offer to share the risk, for ~~similar~~ ~~by the~~ converse ~~reasoned~~ reasons to above.

Sharing the risk in this way does reduce ~~preserves~~ the expected value of ~~net gain~~ ~~from~~ ~~of~~ the investment for each party, but moves probability mass away from the extremes towards the centre, ~~i.e.~~ ~~~~ ~~a mean~~ ~~preserving spread of~~ in a way that makes it more attractive for risk-averse agents. (and EV is still ~~higher than~~ +ve)

c) Yes, they should. The chance of both failing is $\frac{1}{4}$, both succeeding $\frac{1}{4}$, and different outcomes between the two $\frac{1}{2}$. So this generates the following lottery for each of them:

$\mathbb{L}^* = [\frac{1}{4}, \frac{1}{2}, \frac{1}{4}; 10, 25, 40]$. The associated expected utility $EU(\mathbb{L}^*) = 3.107$ and CE of 22.36, so it is even better than the risk sharing, which was already better (i.e. EU-positive) than not making any investment at all and just keeping the 22.

Note that the EV of $\mathbb{L}^*$ is identical to that of $\mathbb{L}$, just with probability mass moved from extreme outcomes to the centre. So, $\mathbb{L}$ is a mean-preserving spread of $\mathbb{L}^*$ and by second-order stochastic dominance, all risk-averse agents would prefer $\mathbb{L}^*$ to $\mathbb{L}$.

d) • Regarding the initial lottery (i.e. full investment on his own as in (a)) no change - still reject it.
~~mathematically, $\ln kx = \ln x + \ln k$~~
- If we calculate the CE of the scaled lottery we find that it is 200, compared to initial wealth of 220. The CE scales in perfect proportion to the payoffs here because his log preferences display constant relative risk aversion, i.e. the tolerance for risk is proportional to wealth, but the absolute amount of risk (as given by the RP) here has scaled by exactly the same factor, so decision shouldn't change.

• Similarly, he will still accept the risk pooling offer in (b), and ~~(since all risk averse agents, as discussed)~~ the risk sharing in (c).

$$RRA := ARA(x)\cdot x = -\frac{u''(x)}{u'(x)}\cdot x = 1.$$