

This integral appears in problem 62 on page 360 of *A Treatise on the Integral Calculus, Volume II*, by Joseph Edwards.

$$\begin{aligned}
\int_0^4 \frac{\ln x}{\sqrt{4x-x^2}} dx &= \int_0^4 \frac{\ln x}{\sqrt{4-4+4x-x^2}} dx \\
&= \int_0^4 \frac{\ln x}{\sqrt{4-(x^2-4x+4)}} dx \\
&= \int_0^4 \frac{\ln x}{\sqrt{4-(x-2)^2}} dx \\
&= \int_{-\pi/2}^{\pi/2} \frac{2 \ln(2+2\sin\theta)}{\sqrt{4-4\sin^2\theta}} \cos\theta d\theta && x-2 = 2\sin\theta \\
&= \int_{-\pi/2}^{\pi/2} \ln(2(1+\sin\theta)) d\theta && \text{Simplification} \\
&= \int_{-\pi/2}^{\pi/2} (\ln 2 + \ln(1+\sin\theta)) d\theta \\
&= \int_{-\pi/2}^{\pi/2} \ln 2 d\theta + \int_{-\pi/2}^{\pi/2} \ln(1+\sin\theta) d\theta \\
&= \theta \ln 2 \Big|_{-\pi/2}^{\pi/2} + \int_{-\pi/2}^{\pi/2} \ln(1+\sin\theta) d\theta \\
&= \pi \ln 2 + \int_{-\pi/2}^{\pi/2} \ln(1+\sin\theta) d\theta
\end{aligned}$$

$$\begin{aligned}
I_1 &= \int_{-\pi/2}^{\pi/2} \ln(1+\sin\theta) d\theta \\
&= \int_{\pi/2}^{-\pi/2} -\ln(1-\sin\phi) d\phi && \theta = -\phi \\
&= \int_{-\pi/2}^{\pi/2} \ln(1-\sin\phi) d\phi \\
&= \int_{-\pi/2}^{\pi/2} \ln(1-\sin\theta) d\theta && \phi = \theta
\end{aligned}$$

$$\begin{aligned}
2I_1 &= \int_{-\pi/2}^{\pi/2} \ln(1+\sin\theta) d\theta + \int_{-\pi/2}^{\pi/2} \ln(1-\sin\theta) d\theta \\
&= \int_{-\pi/2}^{\pi/2} (\ln(1+\sin\theta) + \ln(1-\sin\theta)) d\theta \\
&= \int_{-\pi/2}^{\pi/2} \ln\{(1+\sin\theta)(1-\sin\theta)\} d\theta \\
&= \int_{-\pi/2}^{\pi/2} \ln(1-\sin^2\theta) d\theta \\
&= \int_{-\pi/2}^{\pi/2} \ln\cos^2\theta d\theta
\end{aligned}$$

$$\begin{aligned}
&= \int_{-\pi/2}^{\pi/2} 2 \ln \cos \theta \, d\theta \\
&= 2 \int_{-\pi/2}^{\pi/2} \ln \cos \theta \, d\theta
\end{aligned}$$

$$I_1 = \int_{-\pi/2}^{\pi/2} \ln \cos \theta \, d\theta$$

$$\begin{aligned}
I_2 &= \int_0^{\pi/2} \ln \cos \theta \, d\theta \\
&= \int_{\pi/2}^0 -\ln \cos \left(\frac{\pi}{2} - \psi \right) d\psi & \theta = \frac{\pi}{2} - \psi \\
&= \int_0^{\pi/2} \ln \cos \left(\frac{\pi}{2} - \psi \right) d\psi \\
&= \int_0^{\pi/2} \ln \sin \psi \, d\psi \\
&= \int_0^{\pi/2} \ln \left(2 \sin \frac{\psi}{2} \cos \frac{\psi}{2} \right) d\psi \\
&= \int_0^{\pi/2} \left(\ln 2 + \ln \sin \frac{\psi}{2} + \ln \cos \frac{\psi}{2} \right) d\psi \\
&= \int_0^{\pi/2} \ln 2 \, d\psi + \int_0^{\pi/2} \ln \sin \frac{\psi}{2} \, d\psi + \int_0^{\pi/2} \ln \cos \frac{\psi}{2} \, d\psi \\
&= \psi \ln 2 \Big|_0^{\pi/2} + \int_{\pi/2}^{\pi/4} -2 \ln \sin \left(\frac{\pi}{2} - \theta \right) d\theta + \int_0^{\pi/2} \ln \cos \frac{\psi}{2} \, d\psi & \frac{\psi}{2} = \frac{\pi}{2} - \theta \\
&= \frac{\pi}{2} \ln 2 + 2 \int_{\pi/4}^{\pi/2} \ln \cos \theta \, d\theta + \int_0^{\pi/2} \ln \cos \frac{\psi}{2} \, d\psi \\
&= \frac{\pi}{2} \ln 2 + 2 \int_{\pi/4}^{\pi/2} \ln \cos \theta \, d\theta + \int_0^{\pi/4} 2 \ln \cos \theta \, d\theta & \frac{\psi}{2} = \theta \\
&= \frac{\pi}{2} \ln 2 + 2 \int_{\pi/4}^{\pi/2} \ln \cos \theta \, d\theta + 2 \int_0^{\pi/4} \ln \cos \theta \, d\theta \\
&= \frac{\pi}{2} \ln 2 + 2 \left(\int_0^{\pi/4} \ln \cos \theta \, d\theta + \int_{\pi/4}^{\pi/2} \ln \cos \theta \, d\theta \right) \\
&= \frac{\pi}{2} \ln 2 + 2 \int_0^{\pi/2} \ln \cos \theta \, d\theta \\
&= \frac{\pi}{2} \ln 2 + 2I_2
\end{aligned}$$

$$I_2 = -\frac{\pi}{2} \ln 2$$

$$I_3 = \int_{-\pi/2}^0 \ln \cos \theta \, d\theta$$

$$\begin{aligned}
&= \int_{\pi/2}^0 -\ln \cos \psi \, d\psi & \theta = -\psi \\
&= \int_0^{\pi/2} \ln \cos \psi \, d\psi \\
&= \int_0^{\pi/2} \ln \cos \theta \, d\theta & \psi = \theta \\
&= I_2 \\
&= -\frac{\pi}{2} \ln 2
\end{aligned}$$

$$\begin{aligned}
\int_0^4 \frac{\ln x}{\sqrt{4x-x^2}} dx &= \pi \ln 2 + \int_{-\pi/2}^{\pi/2} \ln(1 + \sin \theta) \, d\theta \\
&= \pi \ln 2 + I_1 \\
&= \pi \ln 2 + \int_{-\pi/2}^{\pi/2} \ln \cos \theta \, d\theta \\
&= \pi \ln 2 + \int_{-\pi/2}^0 \ln \cos \theta \, d\theta + \int_0^{\pi/2} \ln \cos \theta \, d\theta \\
&= \pi \ln 2 + I_3 + I_2 \\
&= \pi \ln 2 - \frac{\pi}{2} \ln 2 - \frac{\pi}{2} \ln 2 \\
&= 0
\end{aligned}$$

References

- [1] <https://math.stackexchange.com/questions/1066006/evaluate-int-04-frac-ln-x-sqrt4x-x2-mathrm-dx/1078278>