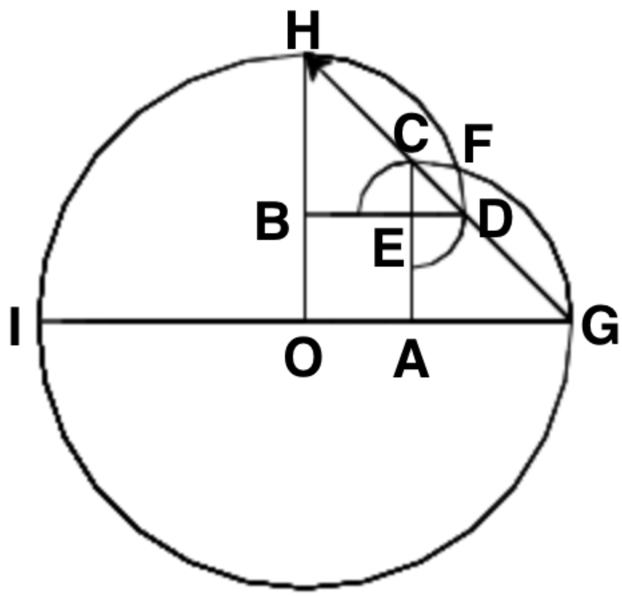
Solution to the Challenge in Exercise 2 Given in the Article "How to Solve Equations That Are Stubborn as a Goat" in Quanta Magazine

In this document, I provide an outline of the solution to the challenge in exercise 2 in the following article: `https://www.quantamagazine.org/solve-math-equations-that-are-stubborn-as-a-goat-20210506/` (archived at: `https://web.archive.org/web/20230615014008/https://www.quantamagazine.org/solve-math-equations-that-are-stubborn-as-a-goat-20210506/`). You can see the challenge by clicking on the link titled "Click for Answer 2".

The following diagram shows the area that the goat can graze:



Note that the goat is tied at O.

To find the area that the goat can graze in, you would have to add the area of the region GIH (which is three quarters of a circle of radius 10 units), the area of the region $BDFH$ (which is a quarter of a circle of radius 6 units) and the area of the region $AGFC$ (which is a quarter of a circle of radius 6 units). Then you have to subtract the area of the region $EDFC$.

The main challenge here is to calculate the area of the region $EDFC$.

First, note that arcs CF and DF are of equal length. To see this, join A and B, B and F, and A and F. Then, in $\triangle ABF$, $AF = BF$ because $AF = AC$ (since they are the radii of the same circle) and $BF = BD$ (again, since they are the radii of the same circle) and $AC = BD = 6$ units. Therefore, $\angle ABF = \angle BAF$. But since $\angle ABE = \angle BAE = 45°$, therefore $\angle DBF = \angle EBF = \angle ABF - \angle ABE = \angle BAF - \angle BAE = \angle EAF = \angle CAF$. Now, since the radii and the angles of the arcs CF and DF are equal, so the two arcs are of equal length.

Now, to find the area of the region $EDFC$, we'll use calculus. Here's the approach we'll use. We'll draw a line FJ perpendicular to OG. This line will intersect OG at J and BD at K. Similarly, we'll draw a line FL perpendicular to OH. This line will intersect OH at L and AC at M. Now, we'll use integration to calculate the areas of the regions FKD and FMC. The sum of these two areas and the area of the rectangle $EKFM$ would give us the area of the region $EDFC$.

To start with, we'll have a coordinate system with the origin at O, OG as the X-axis and OH as the Y-axis. Also, draw DP perpendicular to OG. Here, P is the point of intersection between DP and OG. Similarly, draw CQ perpendicular to OH, where Q is the point of intersection of CQ with OH.

Now, let's find the coordinates of the point F. Note that the point F is the intersection of the arcs of the circles with equations $(x-4)^2+y^2 = 6^2$ and $x^2+(y-4)^2 = 6^2$. If you expand the two equations, you'll get $x^2 - 8x + 16 + y^2 = 36$ and $x^2 + y^2 - 8y + 16 = 36$. Now, if you subtract the first equation from the second, you'll get $8x - 8y = 0$ or $x = y$. Substituting this in either equation, you'll get $2x^2 - 8x + 16 = 36$. Simplifying, you'll get $x^2 - 4x - 10 = 0$. This can be solved to get $x = y = \frac{4\pm\sqrt{16+40}}{2} = 2 \pm \sqrt{14}$. Since we're only interested in values in the first quadrant, we'll take $x = y = 2 + \sqrt{14}$. So, the coordinates of F are $(2 + \sqrt{14}, 2 + \sqrt{14})$.

$$\begin{aligned}
\text{Now, area of } FJPD &= \int_{2+\sqrt{14}}^{6} 4 + \sqrt{36-x^2}\,dx \\
&= \int_{2+\sqrt{14}}^{6} 4\,dx + \int_{2+\sqrt{14}}^{6} \sqrt{36-x^2}\,dx \\
&= 4x\Big|_{2+\sqrt{14}}^{6} + \frac{6^2}{2}\left(\frac{x\sqrt{6^2-x^2}}{6^2} - \cos^{-1}\frac{x}{6}\right)\Big|_{2+\sqrt{14}}^{6} \\
&= 4(6-2-\sqrt{14}) + 18\left(0 - \frac{(2+\sqrt{14})\sqrt{6^2-(2+\sqrt{14})^2}}{36} + \cos^{-1}\frac{2+\sqrt{14}}{6}\right) \\
&\approx 1.335
\end{aligned}$$

Note that the integral $\int\sqrt{36-x^2}\,dx$ that appears above is of the form $\int\sqrt{a^2-x^2}\,dx$ which can be easily computed by using the substitution $x = a\cos\theta$. Also, $\sqrt{6^2-(2+\sqrt{14})^2}$ can be simplified to $\sqrt{14}-2$.

Now, you can get the area of the region FKD by subtracting the area of the rectangle $KJPD$ (which is equal to $4(6-(2+\sqrt{14})) \approx 1.033$ square units) from the area of $FJPD$.

$$\begin{aligned}
\text{Similarly, area of } CAJF &= \int_{4}^{2+\sqrt{14}} \sqrt{36-(x-4)^2}\,dx \\
&= \frac{6^2}{2}\left(\frac{(x-4)\sqrt{6^2-(x-4)^2}}{6^2} - \cos^{-1}\frac{x-4}{6}\right)\Big|_{4}^{2+\sqrt{14}} \\
&\approx 10.301
\end{aligned}$$

The above integral can be computed by transforming it to an integral of the from $\int\sqrt{a^2-x^2}\,dx$ by using the substitution $X = x - 4$.

Now, the area of the region FMC can be computed by subtracting the area of the rectangle $MAJF$ (which is equal to $(2+\sqrt{14})(2+\sqrt{14}-4) = 14-4 = 10$ square units) from the area of $CAJF$.