This integral appears in problem 62 on page 360 of *A Treatise on the Integral Calculus, Volume II*, by Joseph Edwards.

$$
\begin{aligned}
\int_0^4 \frac{\ln x}{\sqrt{4x-x^2}}\,dx &= \int_0^4 \frac{\ln x}{\sqrt{4-4+4x-x^2}}\,dx \\
&= \int_0^4 \frac{\ln x}{\sqrt{4-(x^2-4x+4)}}\,dx \\
&= \int_0^4 \frac{\ln x}{\sqrt{4-(x-2)^2}}\,dx \\
&= \int_{-\pi/2}^{\pi/2} \frac{2\ln(2+2\sin\theta)}{\sqrt{4-4\sin^2\theta}}\cos\theta\,d\theta && x-2=2\sin\theta \\
&= \int_{-\pi/2}^{\pi/2} \ln(2(1+\sin\theta))\,d\theta && \text{Simplification} \\
&= \int_{-\pi/2}^{\pi/2} (\ln 2+\ln(1+\sin\theta))\,d\theta \\
&= \int_{-\pi/2}^{\pi/2} \ln 2\,d\theta + \int_{-\pi/2}^{\pi/2} \ln(1+\sin\theta)\,d\theta \\
&= \theta\ln 2\Big|_{-\pi/2}^{\pi/2} + \int_{-\pi/2}^{\pi/2} \ln(1+\sin\theta)\,d\theta \\
&= \pi\ln 2 + \int_{-\pi/2}^{\pi/2} \ln(1+\sin\theta)\,d\theta
\end{aligned}
$$

$$
\begin{aligned}
I_1 &= \int_{-\pi/2}^{\pi/2} \ln(1+\sin\theta)\,d\theta \\
&= \int_{\pi/2}^{-\pi/2} -\ln(1-\sin\phi)\,d\phi && \theta=-\phi \\
&= \int_{-\pi/2}^{\pi/2} \ln(1-\sin\phi)\,d\phi \\
&= \int_{-\pi/2}^{\pi/2} \ln(1-\sin\theta)\,d\theta && \phi=\theta
\end{aligned}
$$

$$
\begin{aligned}
2I_1 &= \int_{-\pi/2}^{\pi/2} \ln(1+\sin\theta)\,d\theta + \int_{-\pi/2}^{\pi/2} \ln(1-\sin\theta)\,d\theta \\
&= \int_{-\pi/2}^{\pi/2} (\ln(1+\sin\theta)+\ln(1-\sin\theta))\,d\theta \\
&= \int_{-\pi/2}^{\pi/2} \ln\{(1+\sin\theta)(1-\sin\theta)\}\,d\theta \\
&= \int_{-\pi/2}^{\pi/2} \ln(1-\sin^2\theta)\,d\theta \\
&= \int_{-\pi/2}^{\pi/2} \ln\cos^2\theta\,d\theta
\end{aligned}
$$

$$
\begin{aligned}
&= \int_{-\pi/2}^{\pi/2} 2\ln\cos\theta\,d\theta \\
&= 2\int_{-\pi/2}^{\pi/2} \ln\cos\theta\,d\theta
\end{aligned}
$$

$$
I_1 = \int_{-\pi/2}^{\pi/2} \ln\cos\theta\,d\theta
$$

$$
\begin{aligned}
I_2 &= \int_0^{\pi/2} \ln\cos\theta\,d\theta \\
&= \int_{\pi/2}^{0} -\ln\cos\left(\frac{\pi}{2}-\psi\right)d\psi && \theta = \frac{\pi}{2}-\psi \\
&= \int_0^{\pi/2} \ln\cos\left(\frac{\pi}{2}-\psi\right)d\psi \\
&= \int_0^{\pi/2} \ln\sin\psi\,d\psi \\
&= \int_0^{\pi/2} \ln\left(2\sin\frac{\psi}{2}\cos\frac{\psi}{2}\right)\,d\psi \\
&= \int_0^{\pi/2} \left(\ln 2 + \ln\sin\frac{\psi}{2} + \ln\cos\frac{\psi}{2}\right)\,d\psi \\
&= \int_0^{\pi/2} \ln 2\,d\psi + \int_0^{\pi/2} \ln\sin\frac{\psi}{2}\,d\psi + \int_0^{\pi/2} \ln\cos\frac{\psi}{2}\,d\psi \\
&= \psi\ln 2\Big|_0^{\pi/2} + \int_{\pi/2}^{\pi/4} -2\ln\sin\left(\frac{\pi}{2}-\theta\right)\,d\theta + \int_0^{\pi/2} \ln\cos\frac{\psi}{2}\,d\psi && \frac{\psi}{2} = \frac{\pi}{2}-\theta \\
&= \frac{\pi}{2}\ln 2 + 2\int_{\pi/4}^{\pi/2} \ln\cos\theta\,d\theta + \int_0^{\pi/2} \ln\cos\frac{\psi}{2}\,d\psi \\
&= \frac{\pi}{2}\ln 2 + 2\int_{\pi/4}^{\pi/2} \ln\cos\theta\,d\theta + \int_0^{\pi/4} 2\ln\cos\theta\,d\theta && \frac{\psi}{2} = \theta \\
&= \frac{\pi}{2}\ln 2 + 2\int_{\pi/4}^{\pi/2} \ln\cos\theta\,d\theta + 2\int_0^{\pi/4} \ln\cos\theta\,d\theta \\
&= \frac{\pi}{2}\ln 2 + 2\left(\int_0^{\pi/4} \ln\cos\theta\,d\theta + \int_{\pi/4}^{\pi/2} \ln\cos\theta\,d\theta\right) \\
&= \frac{\pi}{2}\ln 2 + 2\int_0^{\pi/2} \ln\cos\theta\,d\theta \\
&= \frac{\pi}{2}\ln 2 + 2I_2
\end{aligned}
$$

$$
I_2 = -\frac{\pi}{2}\ln 2
$$

$$
I_3 = \int_{-\pi/2}^{0} \ln\cos\theta\,d\theta
$$

$$
\begin{aligned}
&= \int_{\pi/2}^{0} -\ln\cos\psi \, d\psi && \theta = -\psi \\
&= \int_{0}^{\pi/2} \ln\cos\psi \, d\psi \\
&= \int_{0}^{\pi/2} \ln\cos\theta \, d\theta && \psi = \theta \\
&= I_2 \\
&= -\frac{\pi}{2}\ln 2
\end{aligned}
$$

$$
\begin{aligned}
\int_0^4 \frac{\ln x}{\sqrt{4x - x^2}} \, dx &= \pi\ln 2 + \int_{-\pi/2}^{\pi/2} \ln(1+\sin\theta)\, d\theta \\
&= \pi\ln 2 + I_1 \\
&= \pi\ln 2 + \int_{-\pi/2}^{\pi/2} \ln\cos\theta \, d\theta \\
&= \pi\ln 2 + \int_{-\pi/2}^{0} \ln\cos\theta \, d\theta + \int_{0}^{\pi/2} \ln\cos\theta \, d\theta \\
&= \pi\ln 2 + I_3 + I_2 \\
&= \pi\ln 2 - \frac{\pi}{2}\ln 2 - \frac{\pi}{2}\ln 2 \\
&= 0
\end{aligned}
$$