This integral appears in problem 62 on page 360 of *A Treatise on the Integral Calculus, Volume II*, by Joseph Edwards.

$$\begin{aligned}
\int_0^4 \frac{\ln x}{\sqrt{4x - x^2}}\,dx &= \int_0^4 \frac{\ln x}{\sqrt{4 - 4 + 4x - x^2}}\,dx \\
&= \int_0^4 \frac{\ln x}{\sqrt{4 - (x^2 - 4x + 4)}}\,dx \\
&= \int_0^4 \frac{\ln x}{\sqrt{4 - (x - 2)^2}}\,dx \\
&= \int_{-\pi/2}^{\pi/2} \frac{2\ln(2 + 2\sin\theta)}{\sqrt{4 - 4\sin^2\theta}}\cos\theta\,d\theta && x - 2 = 2\sin\theta \\
&= \int_{-\pi/2}^{\pi/2} \ln(2(1 + \sin\theta))\,d\theta && \text{Simplification} \\
&= \int_{-\pi/2}^{\pi/2} \ln 2 + \ln(1 + \sin\theta)\,d\theta \\
&= \theta\ln 2\Big|_{-\pi/2}^{\pi/2} + \int_{-\pi/2}^{\pi/2} \ln(1 + \sin\theta)\,d\theta \\
&= \pi\ln 2 + \int_{-\pi/2}^{\pi/2} \ln(1 + \sin\theta)\,d\theta
\end{aligned}$$

$$\begin{aligned}
I_1 &= \int_{-\pi/2}^{\pi/2} \ln(1 + \sin\theta)\,d\theta \\
&= \int_{\pi/2}^{-\pi/2} -\ln(1 - \sin\phi)\,d\phi && \theta = -\phi \\
&= \int_{-\pi/2}^{\pi/2} \ln(1 - \sin\phi)\,d\phi \\
&= \int_{-\pi/2}^{\pi/2} \ln(1 - \sin\theta)\,d\theta && \phi = \theta
\end{aligned}$$

$$
\begin{aligned}
2I_1 &= \int_{-\pi/2}^{\pi/2} \ln\left(1+\sin\theta\right) d\theta + \int_{-\pi/2}^{\pi/2} \ln\left(1-\sin\theta\right) d\theta \\
&= \int_{-\pi/2}^{\pi/2} \ln\left(1+\sin\theta\right) + \ln\left(1-\sin\theta\right) d\theta \\
&= \int_{-\pi/2}^{\pi/2} \ln\left\{(1+\sin\theta)(1-\sin\theta)\right\} d\theta \\
&= \int_{-\pi/2}^{\pi/2} \ln\left(1-\sin^2\theta\right) d\theta \\
&= \int_{-\pi/2}^{\pi/2} \ln\cos^2\theta \, d\theta \\
&= \int_{-\pi/2}^{\pi/2} 2\ln\cos\theta \, d\theta \\
&= 2\int_{-\pi/2}^{\pi/2} \ln\cos\theta \, d\theta
\end{aligned}
$$