Online Supplement: Appendices

Appendix A: RLT Relaxation of Problem (8)

Suppose we have a symmetric matrix variable $V \in \mathbb{S}^{m\times m}$ such that $V = ww^\top$. The following can be applied to replace the convex term $||A^\top w||_2$:

$$||A^\top w||_2 = \sqrt{w^\top AA^\top w} = \sqrt{\mathrm{tr}\,(w^\top AA^\top w)} = \sqrt{\mathrm{tr}\,(A^\top ww^\top A)}.$$

Therefore, we use the following concave reformulation of $||A^\top w||_2$:

$$||A^\top w||_2 = \sqrt{\mathrm{tr}\,(A^\top VA)}$$

This concave reformulation is of course based on the assumption $V = ww^\top$. Moreover, we use the main idea of RLT and multiply each of the original constraints $\alpha_i^\top w - \beta_i \le 0,\ \alpha_j^\top w - \beta_j \le 0$ to obtain:

$$\begin{aligned}
&(\alpha_i^\top w - \beta_i)(\alpha_j^\top w - \beta_j) \ge 0 \\
\iff &\alpha_i^\top ww^\top \alpha_j - (\beta_i\alpha_j + \beta_j\alpha_i)^\top w + \beta_i\beta_j \ge 0 \qquad (29)\\
\iff &\alpha_i^\top V\alpha_j - (\beta_i\alpha_j + \beta_j\alpha_i)^\top w + \beta_i\beta_j \ge 0. \qquad (30)
\end{aligned}$$

Although $V = ww^\top$ is assumed, it is a non-convex constraint, so we relax it as:

$$V \succeq ww^\top \iff \begin{pmatrix} V & w \\ w^\top & 1 \end{pmatrix} \succeq 0. \qquad (31)$$

Thus, problem (8) is relaxed by the following convex optimization problem:

$$\begin{array}{lll}
\sup\limits_{V\in\mathbb{S}^{m\times m},\ w\in\mathbb{R}^m} & \rho\sqrt{\mathrm{tr}(A^\top VA)} + a^\top A^\top w + b^\top w - f^*(w) & \\
\mathrm{s.t.} & \alpha_i^\top w - \beta_i \le 0, & i = 1,\dots,d \\
& \alpha_i^\top V\alpha_j - (\beta_i\alpha_j + \beta_j\alpha_i)^\top w + \beta_i\beta_j \ge 0, & i \le j = 1,\dots,d \\
& \begin{pmatrix} V & w \\ w^\top & 1 \end{pmatrix} \succeq 0, &
\end{array} \qquad (32)$$

which concludes the proof.

Appendix B: Proof of Theorem 3

We showed that problem (20) can be represented as ARO problem (22). As shown by Roos et al. (2018), we can lift the nonlinear term $f^*(w)$ to the uncertainty set by introducing an auxiliary uncertain parameter w_0. Hence, the set of constraints of the ARO problem is equivalent to:

$$\forall \begin{pmatrix} w_0 \\ w \end{pmatrix} \in W,\ \exists \lambda \in \mathbb{R}^q : \begin{cases} d^\top\lambda + b^\top w + w_0 \le \tau & (33a) \\ D^\top\lambda \ge A^\top w & (33b) \\ \lambda \ge \mathbf{0}, & (33c) \end{cases}$$

where we define the new uncertainty set as

$$W = \left\{ \begin{pmatrix} w_0 \\ w \end{pmatrix} \in \mathbb{R}^{m+1} :\ w_0 + f^*(w) \le 0 \right\}. \qquad (34)$$

A safe approximation of the constraint set is obtained by using a linear decision rule for the adjustable variable:

$$\lambda = u + Vw + rw_0,$$

where $u \in \mathbb{R}^q, V \in \mathbb{R}^{q\times m}$ and $r \in \mathbb{R}^q$. Substituting this LDR in (33a) leads to

$$\begin{aligned}
& d^\top \lambda + b^\top w + w_0 \le \tau && \forall \left(w_0 \ w^\top\right)^\top \in W \\
\iff & d^\top (u + Vw + rw_0) + b^\top w + w_0 \le \tau && \forall \left(w_0 \ w^\top\right)^\top \in W \\
\iff & d^\top u + \begin{pmatrix} w_0 \\ w \end{pmatrix}^\top \begin{pmatrix} 1 + d^\top r \\ V^\top d + b \end{pmatrix} \le \tau && \forall \left(w_0 \ w^\top\right)^\top \in W \\
\iff & d^\top u + \delta^* \left(\left(\begin{matrix} 1 + d^T r \\ V^T d + b \end{matrix} \right) \middle| W \right) \le \tau. &&
\end{aligned} \tag{35}$$

To be able to find the tractable robust counterpart of (35), we derive the support function of the new uncertainty set W, which is

$$\begin{aligned}
\delta^* \left(\begin{pmatrix} z_0 \\ z \end{pmatrix} \middle| W \right) &= \sup_{(w_0, w)^\top \in W} \{ z_0 w_0 + z^\top w \} \\
&= \begin{cases} \sup\limits_{w \in \mathbb{R}^m} \{ z^\top w - z_0 f^*(w) \} & \text{if } z_0 > 0 \\ \sup\limits_{w \in \operatorname{dom} f^*} \{ z^\top w \} & \text{if } z_0 = 0 \\ +\infty & \text{otherwise} \end{cases} \\
&= \begin{cases} z_0 f\left(\frac{z}{z_0}\right) & \text{if } z_0 \ge 0 \\ +\infty & \text{otherwise.} \end{cases}
\end{aligned} \tag{36}$$

Above, for the case of $z_0 = 0$, we use the property $\sup\limits_{w \in \operatorname{dom} f^*} \{z^\top w\} = \delta^*(z|\operatorname{dom} f^*) = \lim\limits_{z_0 \downarrow 0} z_0 f\left(\frac{z}{z_0}\right)$, and the result follows since for $z_0 = 0$ we have the understanding of $\lim\limits_{z_0 \downarrow 0} z_0 f\left(\frac{z}{z_0}\right)$ for the perspective (Rockafellar 1970). By substituting (36) into (35) we obtain:

$$\begin{aligned}
& d^\top u + \delta^* \left(\left(\begin{matrix} 1 + d^T r \\ V^T d + b \end{matrix} \right) \middle| W \right) \le \tau \\
\iff & \begin{cases} d^\top u + (1 + d^\top r) f\left(\dfrac{V^\top d + b}{1 + d^\top r} \right) \le \tau \\ 1 + d^\top r \ge 0. \end{cases}
\end{aligned}$$

Hence, by using LDRs (33a) becomes exactly (24a).

Following the same steps for (33b) yields us to (24b):

$$\begin{aligned}
& D^\top \lambda \ge A^\top w && \forall \left(w_0 \ w^\top\right)^\top \in W \\
\iff & D_i^\top \lambda \ge A_i^\top w && \forall \left(w_0 \ w^\top\right)^\top \in W, \ i = 1, \dots, n \\
\iff & \begin{cases} -D_i^\top u + (-D_i^\top r) f\left(\dfrac{A_i - V^\top D_i}{-D_i^\top r} \right) \le 0 \\ -D_i^\top r \ge 0 \end{cases} && i = 1, \dots, n.
\end{aligned}$$

Similarly (33c) becomes (24c):

$$\begin{aligned}
& \lambda \ge 0 \\
\iff & -u_i - V_{(i)} w - r_i w_0 \le 0 && \forall \left(w_0 \ w^\top\right)^\top \in W, \ i = 1, \dots, q \\
\iff & \begin{cases} -u_i + (-r_i) f\left(\frac{-V_{(i)}^\top}{-r_i} \right) \\ -r_i \ge 0 \end{cases} && i = 1, \dots, q.
\end{aligned}$$

As we use an LDR for the adjustable variable, the optimal objective value of (24) is an upper bound to (20).

Appendix C: Upper and Lower Bound Approximation of Problem (20)

We summarize the process of finding upper and lower bounds on the global optimum objective value of problem (20).

Algorithm 1: Obtaining upper and lower bounds for problem (20)

---

input : f, A, b, U

output: Upper bound value $\hat{\tau}$, lower bound solution x^* with value $f(Ax^*+b)$

1. Obtain the upper bound solution by solving (24), i.e., $(\hat{u}, \hat{V}, \hat{r}, \hat{\tau}) \in$

$$\underset{u\in\mathbb{R}^q, V\in\mathbb{R}^{q\times m}, r\in\mathbb{R}^q, \tau\in\mathbb{R}}{\arg\inf} \tau \text{ s.t.} \begin{cases} d^\top u + \left(1+d^\top r\right) f\left(\dfrac{V^\top d + b}{1+d^\top r}\right) \le \tau & \\ 1 + d^\top r \ge 0 & \\ -D_i^\top u + \left(-D_i^\top r\right) f\left(\dfrac{A_i - V^\top D_i}{-D_i^\top r}\right) \le 0 & i=1,\ldots,n \\ -D_i^\top r \ge 0 & \\ -u_i + (-r_i) f\left(\dfrac{V_{(i)}^\top}{r_i}\right) \le 0 & i=1,\ldots,q \\ -r_i \ge 0, & \end{cases}$$

or alternatively via the adversarial approach. $\hat{\tau}$ is an upper bound value.

2. Generate a finite set of (potential) worst-case ARO scenarios by plugging the optimal LDR back in the safe approximation of the original ARO, and by collecting worst-case scenario of each constraint, i.e., $\overline{W} = \overline{W}^1 \cup [\bigcup_{i=1}^n \overline{W}_i^2] \cup [\bigcup_{i=1}^q \overline{W}_i^3]$ with:

$$\begin{aligned} &\overline{W}^1 \in \underset{w\in\operatorname{dom} f^*}{\arg\sup} \left\{-(1+d^\top r) f^*(w) + (d^\top \hat{V} + b^\top) w + d^\top \hat{u} - \hat{\tau}\right\}, && \\ &\overline{W}_i^2 \in \underset{w\in\operatorname{dom} f^*}{\arg\sup} \left\{(D_i^\top r) f^*(w) + (A_i^\top - D_i^\top \hat{V}) w - D_i^\top \hat{u}\right\}, && i=1,\ldots,n \\ &\overline{W}_i^3 \in \underset{w\in\operatorname{dom} f^*}{\arg\sup} \left\{-\hat{u}_i - \hat{V}_i w + \hat{r}_i f^*(w)\right\}, && i=1,\ldots,q. \end{aligned}$$

3. For all $w \in \overline{W}$, solve the linear optimization problem:

$$x \in \underset{x\in U}{\arg\sup}\{(A^\top w)^\top x\},$$

and return x that achieves the highest $f(Ax+b)$ as the lower bound solution of problem (20).

Appendix D: Complete Derivation of Specific Problems in Section 3.2

D.1. Quadratic Optimization

Here we consider problem (20) when the objective function is a convex quadratic function. For the problem of maximizing a convex quadratic function over a polyhedron, we can find an upper bound by solving a second-order cone optimization problem, and we can find a lower bound by solving a linear optimization problem.

Consider the convex quadratic function $g : \mathbb{R}^n \mapsto \mathbb{R}$ defined by:

$$g(x) = x^\top Qx + \ell^\top x,$$

where $\ell \in \mathbb{R}^n$ and Q is a symmetric positive semi-definite (psd) matrix. Maximizing this function over a polyhedral set can be written as the robust optimization problem:

$$\begin{aligned} &\inf \ \ \tau \\ &\text{s.t. } x^\top Qx + \ell^\top x \le \tau, \quad \forall x \in U, \end{aligned} \tag{38}$$

where $U = \{x \in \mathbb{R}^n_+ : \ Dx \le d\}$ for $D \in \mathbb{R}^{q\times n}, \ d \in \mathbb{R}^q$. We use the conic representation of the constraints of problem (38):

$$\left\| \begin{pmatrix} (1+\ell^\top x - \tau)/2 \\ Lx \end{pmatrix} \right\|_2 - (1 - \ell^\top x + \tau)/2 \le 0,$$

where L is the psd decomposition $Q = L^\top L$. Therefore, the constraint of problem (38) can be written as a robust conic constraint. Define $f : \mathbb{R}^{m+1} \times \mathbb{R} \mapsto \mathbb{R}$ by:

$$f\begin{pmatrix} z \\ \tilde{z} \end{pmatrix} = \|z\|_2 + \tilde{z}, \tag{39}$$

with $z \in \mathbb{R}^{m+1}$ and $\tilde{z} \in \mathbb{R}$. It can be verified that f is positively homogeneous and that the conjugate of this function for $w \in \mathbb{R}^{m+1}, \ \tilde{w} \in \mathbb{R}$ is:

$$f^*\begin{pmatrix} w \\ \tilde{w} \end{pmatrix} = \begin{cases} 0 & \text{if } \tilde{w} = 1 \text{ and } ||w||_2 \le 1 \\ +\infty & \text{otherwise.} \end{cases}$$

Defining

$$A = \begin{bmatrix} L \\ \ell^\top/2 \\ \ell^\top/2 \end{bmatrix}, \quad b = \begin{bmatrix} \mathbf{0} \\ (1-\tau)/2 \\ (-1-\tau)/2 \end{bmatrix}, \tag{40}$$

it follows that:

$$f(Ax+b) = f\left(\begin{pmatrix} Lx \\ (1+\ell^\top x - \tau)/2 \\ (\ell^\top x - 1 - \tau)/2 \end{pmatrix}\right) = \left\| \begin{pmatrix} Lx \\ (1+\ell^\top x - \tau)/2 \end{pmatrix} \right\|_2 - (1 - \ell^\top x + \tau)/2.$$

Hence, the constraint of problem (38) is equivalent to $f(Ax+b) \le 0$. Therefore, problem (38) can be rewritten as:

$$\begin{aligned} &\inf \ \ \tau \\ &\text{s.t. } f\left(\begin{pmatrix} Lx \\ (1+\ell^\top x - \tau)/2 \\ (\ell^\top x - 1 - \tau)/2 \end{pmatrix}\right) \le 0, \ \forall x \in U. \end{aligned} \tag{41}$$

An upper bound of this problem can now be obtained by applying Theorem 3 and exploiting the positive homogeneity of f (see Appendix E). The upper bound is the optimal value of the problem:

$$\begin{aligned}
\inf\ & \tau \\
\text{s.t. } & d^\top u + \bar{v}^\top d - (1+\tau)/2 + \left\| \begin{pmatrix} \tilde{V}^\top d \\ \hat{v}^\top d + (1-\tau)/2 \end{pmatrix} \right\|_2 \leq 0 \\
& -D_i^\top u + \frac{\ell_i}{2} - \bar{v}^\top D_i + \left\| \begin{pmatrix} L_i - \tilde{V}^\top D_i \\ \ell_i/2 - \hat{v}^\top D_i \end{pmatrix} \right\|_2 \leq 0, \qquad i = 1,\dots,n \\
& -u_i - \bar{v}_i + \left\| \begin{pmatrix} -\tilde{V}_{(i)}^\top \\ -\hat{v}_i \end{pmatrix} \right\|_2 \leq 0, \qquad i = 1,\dots,q,
\end{aligned} \tag{42}$$

in which the variables are $\tau \in \mathbb{R}, u \in \mathbb{R}^q, \bar{v} \in \mathbb{R}^q, \hat{v} \in \mathbb{R}^q, \tilde{V} \in \mathbb{R}^{q\times m}$.

In order to compute a lower bound, we use the optimal solution $(\tau, u, \bar{v}, \hat{v}, \tilde{V})$ to problem (42) by obtaining a collection of worst case scenarios $\overline{W}$ from (26), (27), and (28). These problems can be solved analytically as explained in Appendix F. This yields the scenarios:

$$\begin{aligned}
\overline{W}^1 &= \begin{bmatrix} h\left(\begin{pmatrix} \tilde{V}^\top d \\ d^\top \hat{v} + (1-\tau)/2 \end{pmatrix}\right) \\ 1 \end{bmatrix} \\
\overline{W}_i^2 &= \begin{bmatrix} h\left(\begin{pmatrix} L_i - \tilde{V}^\top D_i \\ \ell_i/2 - D_i^\top \hat{v} \end{pmatrix}\right) \\ 1 \end{bmatrix} \qquad i = 1,\dots,n \\
\overline{W}_i^3 &= \begin{bmatrix} h\left(\begin{pmatrix} -\tilde{V}_{(i)}^\top \\ -\hat{v}_i \end{pmatrix}\right) \\ 1 \end{bmatrix} \qquad i = 1,\dots,q,
\end{aligned} \tag{43}$$

where $h(a) = a/||a||_2$ normalizes its input. Using these worst-case scenarios, the candidate solutions $\bar{x}^{(j)}$ are obtained by solving (25), and we can substitute them in the main objective function as $f(A\bar{x}^{(j)} + b)$ to find the best lower bound.

D.2. Geometric Optimization

Geometric Optimization (GO) is a class of optimization problems originally introduced by Duffin (1967). A practical tutorial can be found in the work of Boyd et al. (2007). Even though it can have many representations, we focus on the GO variant where the objective is maximizing the convex log-sum-exp objective. The log-sum-exp function $f : \mathbb{R}^m \mapsto \mathbb{R}$ is defined as

$$f(z) = \log\left(\sum_{i=1}^m \exp(z_i)\right), \tag{44}$$

and we are interested in solving problems of the following type

$$\begin{aligned}
\max_x\ & f(Ax+b) = \log\left(\textstyle\sum_{i=1}^m \exp(A_{(i)}x + b_i)\right) \\
\text{s.t. }\ & x \in U,
\end{aligned} \tag{45}$$

where $U = \{x \in \mathbb{R}^n_+ \ :\ Dx \leq d\}$. This problem may appear in robust geometric optimization problems. If one applies the adversarial approach for such problems, in the step of adding worst-case uncertainty realization to the discrete uncertainty set, one will need to maximize the convex geometric function.

The conjugate $f^* : \mathbb{R}^m \mapsto \mathbb{R}$ of the log-sum-exp function (44) is

$$f^*(w) = \begin{cases} \sum_{i=1}^m w_i \log(w_i) & \text{if } w \in \mathbb{R}^m_+ \text{ and } \sum_{i=1}^m w_i = 1 \\ \infty & \text{otherwise.} \end{cases} \tag{46}$$

We observe that $f^*(w)$ is the negative-entropy function of w on its domain, which is a standard m-dimensional simplex. It is well known that the negative entropy is a strictly convex function. Next we show that the upper bound and lower bound approximation problems (of problem (45)) are exponential cone representable. This allows one to use the power of today's conic programming solvers, e.g., Mosek's exponential cone optimization solver. We start by introducing the exponential cone, which is the following convex subset of $\mathbb{R}^3$:

$$\mathcal{K}_{\exp} = \{(x_1, x_2, x_3) : x_1 \geq x_2 \exp(x_3/x_2), x_2 > 0\} \cup \{(x_1, 0, x_3) : x_1 \geq 0, x_3 < 0\}.$$

So, the exponential cone is the closure of the set of points which satisfy $x_1 \geq x_2 \exp(x_3/x_2),\ x_1, x_2 > 0$. In the next corollary we show that the upper bound problem (24) is exponential cone representable.

Corollary 4 (Upper Bound Approximation). Upper bound problem (24) is exponential cone representable with the following problem with variables $r \in \mathbb{R}^q, u \in \mathbb{R}^q, V \in \mathbb{R}^{q\times m}, \tau \in \mathbb{R}, z^{(1)} \in \mathbb{R}^m, z^{(2)} \in \mathbb{R}^{n\times m}, z^{(3)} \in \mathbb{R}^{q\times m}$:

$$\inf \tau \quad \text{s.t.} \begin{cases} 1 + d^\mathsf{T} r \geq \sum_{j=1}^m z_j^{(1)} & & (47a) \\ \left(z_j^{(1)}, 1 + d^\mathsf{T} r, \left(V_j^\top d + b_j - \tau + d^\mathsf{T} u\right)\right) \in \mathcal{K}_{\exp}, \quad j = 1, \ldots, m & & \\ -D_i^\mathsf{T} r \geq \sum_{j=1}^m z_{ij}^{(2)} & i = 1, \ldots, n & (47b) \\ \left(z_{ij}^{(2)}, -D_i^\mathsf{T} r, (A_{i,(j)} - V_j^\mathsf{T} D_i - D_i^T u)\right) \in \mathcal{K}_{\exp}, \quad j = 1, \ldots, m & & \\ -r_i \geq \sum_{j=1}^m z_{ij}^{(3)} & i = 1, \ldots, q & (47c) \\ \left(z_{ij}^{(3)}, -r_i, (-V_{j,(i)} - u_i)\right) \in \mathcal{K}_{\exp}, \quad j = 1, \ldots, m. & & \end{cases}$$

■

Proof Roos et al. (2018) show that if a function is conically representable, so is its perspective in the same cone. Log-sum-exp is an exponential cone representable function (MOSEK ApS 2019a), and we show how to represent a convex inequality system of its perspective with exponential cones. Consider the following set of constraints:

$$\begin{cases} t \geq x_0 \log(\exp(x_1/x_0) + \ldots + \exp(x_m/x_0)) \\ x_0 > 0. \end{cases} \tag{48}$$

By using the proof in (Roos et al. 2018), we can write the following equivalent constraint set:

$$\begin{cases} x_0 \geq \sum_{j=1}^m z_j \\ (z_j, x_0, (x_j - t)) \in \mathcal{K}_{\exp}, \quad j = 1, \ldots, m. \end{cases} \tag{49}$$

Since constraints of type (48) appear in the upper bound approximation problem (24), we can use the equivalent representation (49) in each of the constraints to obtain problem (47). □

The lower bound can also be obtained by solving exponential cone programs. The worst-case scenario collection is obtained by solving (26), (27), (28). Here (r, u, V, r, τ) are all parameters taken from the solution of the upper bound problem. This set of problems can be formulated as exponential conic problems, which is shown in the next corollary.

Corollary 5 (Lower Bound Scenarios). The problems (26), (27), and (28) can be written as the following exponential cone problems:

$$\begin{array}{llr} (26): \underset{w,t}{\arg\sup} & \left\{(1+d^\top r)(\sum_{j=1}^m t_j) + (d^\top V + b^\top)w + d^\top u - \tau\right\} & \\ \text{s.t.} & (1, w_j, t_j) \in \mathcal{K}_{\exp}, \quad j = 1,\dots,m & \\ & \sum_{j=1}^m w_j = 1, & \\ (27): \underset{w,t}{\arg\sup} & \left\{(-D_i^\top r)(\sum_{j=1}^m t_j) + (A_i^\top - D_i^\top V)w - D_i^\top u\right\} & \\ \text{s.t.} & (1, w_j, t_j) \in \mathcal{K}_{\exp}, \quad j = 1,\dots,m & i = 1,\dots,n \\ & \sum_{j=1}^m w_j = 1, & \\ (28): \underset{w,t}{\arg\sup} & \left\{(-r_i)(\sum_{j=1}^m t_j) + (-V_{(i)})w - u_i\right\} & \\ \text{s.t.} & (1, w_j, t_j) \in \mathcal{K}_{\exp}, \quad j = 1,\dots,m & i = 1,\dots,q \\ & \sum_{j=1}^m w_j = 1. & \end{array}$$

Proof Serrano (2015) shows that the negative entropy function is exponential conically representable, and the following problems are equivalent (here $w \in \mathbb{R}^m$):

$$\begin{array}{ll} \sup_w & \{c_0(-\sum_{i=1}^m w_i \log(w_i))\} \\ \text{s.t.} & w_i \geq 0, \qquad i = 1,\dots,m. \end{array} = \begin{array}{ll} \sup_{w,t} & \{c_0 \sum_{i=1}^m t_i\} \\ \text{s.t.} & (1, w_i, t_i) \in \mathcal{K}_{\exp}, \quad i = 1,\dots,m. \end{array} \tag{50}$$

The result now follows by substitution of the conjugate (46) in (26), (27), and (28), respectively and then applying equivalence (50). □

D.3. Sum-of-Max-Linear-Terms Optimization

Formally, the sum-of-max-terms function $f : \mathbb{R}^m \mapsto \mathbb{R}$ is written as

$$f(z) = \sum_{k=1}^K \max_{j \in \mathcal{I}_k} \{z_j\}, \tag{51}$$

where the set $\mathcal{I}_k \subseteq \{1,\dots,m\}$ for each $k \in \{1,\dots,K\}$. Moreover, we can assume $\mathcal{I}_k \cap \mathcal{I}_\ell = \emptyset$ for any $k \neq \ell$ and $\cup_{k=1}^K = \{1,\dots,m\}$ without loss of generality, since otherwise we can add components to z to make this statement hold. The sum-of-max-linear-terms function we cover at this section is represented as

$$f(Ax+b) = \sum_{k=1}^K \max_{j \in \mathcal{I}_k} \{A_{(j)}x + b_j\},$$

which is a convex and positively homogeneous function. The main convex maximization problem we are interested in is maximizing $f(Ax+b)$ over $U = \{x \in \mathbb{R}^n_+ : \ Dx \leq d\}$, formally:

$$\begin{array}{ll} \max\limits_x & \sum_{k=1}^K \max\limits_{j \in \mathcal{I}_k} \{A_{(j)}x + b_j\} \\ \text{s.t.} & x \in U. \end{array} \tag{52}$$

This problem naturally arises when one applies the adversarial approach to robust optimization problems with uncertain sum-of-max-linear-terms constraints.

The conjugate of sum-of-max-linear-terms (51) is given by Roos et al. (2018) as:

$$f^*(w) = \begin{cases} 0 & \text{if } w_i \geq 0 \ \forall i = 1,\dots,m, \ \sum_{j \in \mathcal{I}_k} w_j = 1 \\ \infty & \text{otherwise.} \end{cases}$$

The formulation of the upper bound approximation for maximizing sum-of-max-linear-terms function over a polyhedron can be greatly simplified. This is due to the fact that sum-of-max-linear-terms function is a positively homogeneous function as well as the trick of introducing auxiliary variables which give us a linear optimization problem in return.

Since the function is a positively homogeneous function, we can write the upper bound approximation problem (24) as:

$$\begin{array}{lll} \inf\limits_{u\in\mathbb{R}^q, V\in\mathbb{R}^{q\times m}, \tau\in\mathbb{R}} & \tau & \\ \text{s.t.} & d^\top u + \sum_{k=1}^K \max\limits_{j\in\mathcal{I}_k}\{V_j^\top d + b_j\} & \leq \tau \\ & -D_i^\top u + \sum_{k=1}^K \max\limits_{j\in\mathcal{I}_k}\{A_{i,(j)} - V_j^\top D_i\} & \leq 0 \quad i=1,\ldots,n \\ & -u_i - \sum_{k=1}^K \max\limits_{j\in\mathcal{I}_k}\{V_{j,(i)}\} & \leq 0 \quad i=1,\ldots,q. \end{array} \tag{53}$$

Problem (53) can be reformulated as a linear optimization problem by using auxiliary variables. The worst-case scenarios for computing the lower bound are obtained from problems (26), (27), and (28), which simplify to:

$$(26): \operatorname*{arg\,sup}_{w\in \operatorname{dom} f^*} \left\{d^\top(u+Vw) + b^\top w - \tau\right\}, \tag{54}$$

$$(27): \operatorname*{arg\,sup}_{w\in \operatorname{dom} f^*} \left\{A_i^\top w - D_i^\top(u+Vw)\right\}, \qquad i=1,\ldots,n \tag{55}$$

$$(28): \operatorname*{arg\,sup}_{w\in \operatorname{dom} f^*} \left\{-u_i - V_{(i)}w\right\}, \qquad i=1,\ldots,q, \tag{56}$$

where we do not have the conjugate terms since f is a homogeneous function so its conjugate takes value 0. Therefore, the worst-case scenarios of each constraint are given by the following problems:

- For (54):

$$\begin{array}{lll} \sup_w & d^\top(u+Vw) + b^\top w - \tau & \\ \text{s.t.} & w_j \geq 0, & j=1,\ldots,m \\ & \sum_{j\in\mathcal{I}_k} w_j = 1, & k=1,\ldots,K. \end{array} \tag{57}$$

  Recalling the only variable here is w, this is a linear optimization problem. Moreover, since we have $\mathcal{I}_k \cap \mathcal{I}_{k'} = \emptyset$ for $k\neq k'$, we can separate this problem to K independent optimization problems, where each problem k is:

$$\begin{array}{ll} c_k = \sup\limits_{w\geq 0} & d^\top(u+V_{\{j\}}y) + b_{\{j\}}^\top y \\ \text{s.t.} & \sum_{i=1}^{|\mathcal{I}_k|} y_i = 1. \end{array} \tag{58}$$

  Here $y\in\mathbb{R}^{|\mathcal{I}_k|}$ is the w components corresponding to the k-th term in the sum-of-max-linear-terms function definition. Similarly, $V_{\{j\}},\ b_{\{j\}} \in \mathbb{R}^{|\mathcal{I}_k|}$ are the components of V, b corresponding to the k-th term. Notice that problem (58) is a linear optimization problem over a simplex. The optimal value will have $y_i = 1$ for some i and $y_{i'} = 0$ for all $i' \neq i$. Therefore, the solution is

$$c_k = \max_{i=1,\ldots,|\mathcal{I}_k|}\{d^\top(u+V_{\{j\},i})\} + b_{\{j\},i},$$

  where $V_{\{j\},i}, b_{\{j\},i}$ represent the i-th columns of $V_{\{j\}}$ and $b_{\{j\}}$, respectively. Hence, the optimal value of (57) is given by $-\tau + \sum_{k=1}^K c_k$. The $\arg\max$ value can be retrieved easily by detecting which y_i variables took value 1; there will be exactly K ones in the result and the rest will be zeros.

- For (55), for all $i=1,\ldots,n$:

$$\begin{array}{lll} \sup_w & A_i^\top w - D_i^\top(u+Vw) & \\ \text{s.t.} & w_j \geq 0, & j=1,\ldots,m \\ & \sum_{j\in\mathcal{I}_k} w_j = 1, & k=1,\ldots,K. \end{array}$$

  Similarly, this problem can be separated to K independent linear optimization problems over simplices. The optimal solution can be found analytically.

- For (56), for all $i = 1, \ldots, q$:

$$\begin{array}{lll} \sup_w & -u_i - V_{(i)} w & \\ \text{s.t.} & w_j \geq 0, & j = 1, \ldots, m \\ & \sum_{j \in \mathcal{I}_k} w_j = 1, & k = 1, \ldots, K. \end{array}$$

  This problem can be solved analytically once again, concluding that all of the worst-case scenario finding procedure can be solved analytically.

Appendix E: Upper Bound Approximation of Quadratic Maximization via SOCO

We follow Theorem 3 to apply the upper bound approximation for problem (41). Because f is a positively homogeneous function, the upper bound problem (24) reduces to the following problem for the variables $u \in \mathbb{R}^q, V \in \mathbb{R}^{q \times (m+2)}, \tau \in \mathbb{R}$:

$$\begin{array}{lll} \inf & \tau & \\ \text{s.t.} & d^\top u + f\left(V^\top d + b\right) \leq 0 & \\ & -D_i^\top u + f\left(A_i - V^\top D_i\right) \leq 0, & i = 1, \ldots, n \\ & -u_i + f\left(-V_{(i)}^\top\right) \leq 0, & i = 1, \ldots, q. \end{array} \tag{59}$$

Since V has $m+2$ columns, we represent it as:

$$V = \begin{bmatrix} \tilde{V} & \hat{v} & \bar{v} \end{bmatrix} \quad \text{where} \quad \tilde{V} \in \mathbb{R}^{q \times m}, \hat{v} \in \mathbb{R}^q, \bar{v} \in \mathbb{R}^q.$$

Constraints of problem (59) can be simplified to respectively:

$$\begin{cases} \quad d^\top u + f\left(V^\top d + b\right) \leq 0 \\ = d^\top u + f\left(\begin{pmatrix} \tilde{V}^\top d \\ \hat{v}^\top d + 1/2 - \tau/2 \\ \bar{v}^\top d - 1/2 - \tau/2 \end{pmatrix}\right) \leq 0 \\ = d^\top u + \bar{v}^\top d - \dfrac{1}{2} - \dfrac{\tau}{2} + \left\| \begin{pmatrix} \tilde{V}^\top d \\ \hat{v}^\top d + \dfrac{1}{2} - \dfrac{\tau}{2} \end{pmatrix} \right\|_2 \leq 0 \end{cases}$$

$$\begin{cases} \quad -D_i^\top u + f\left(A_i - V^\top D_i\right) \leq 0 \\ = -D_i^\top u + f\left(\begin{pmatrix} L_i - \tilde{V}^\top D_i \\ \ell_i/2 - \hat{v}^\top D_i \\ \ell_i/2 - \bar{v}^\top D_i \end{pmatrix}\right) \leq 0 \\ = -D_i^\top u + \dfrac{\ell_i}{2} - \bar{v}^\top D_i + \left\| \begin{pmatrix} L_i - \tilde{V}^\top D_i \\ \ell_i/2 - \hat{v}^\top D_i \end{pmatrix} \right\|_2 \leq 0 \end{cases}$$

$$\begin{cases} \quad -u_i + f\left(-V_{(i)}^\top\right) \leq 0 \\ = -u_i + f\left(\begin{pmatrix} -\tilde{V}_{(i)}^\top \\ -\hat{v}_i \\ -\bar{v}_i \end{pmatrix}\right) \leq 0 \\ = -u_i - \bar{v}_i + \left\| \begin{pmatrix} -\tilde{V}_{(i)}^\top \\ -\hat{v}_i \end{pmatrix} \right\|_2 \leq 0. \end{cases}$$

Thus the upper bound approximation problem can be represented as the following second-order conic program:

$$\begin{aligned}
\inf\ & \tau \\
\text{s.t.}\ & d^\top u + \bar{v}^\top d - (1+\tau)/2 + \left\| \begin{pmatrix} \tilde{V}^\top d \\ \hat{v}^\top d + (1-\tau)/2 \end{pmatrix} \right\|_2 \le 0 \\
& -D_i^\top u + \frac{\ell_i}{2} - \bar{v}^\top D_i + \left\| \begin{pmatrix} L_i - \tilde{V}^\top D_i \\ \ell_i/2 - \hat{v}^\top D_i \end{pmatrix} \right\|_2 \le 0, \qquad i = 1,\dots,n \\
& -u_i - \bar{v}_i + \left\| \begin{pmatrix} -\tilde{V}_{(i)}^\top \\ -\hat{v}_i \end{pmatrix} \right\|_2 \le 0, \qquad i = 1,\dots,q.
\end{aligned} \tag{60}$$

Appendix F: Lower Bound Scenarios for Quadratic Maximization

In the light of problems (26), (27), (28), the worst-case scenarios are collected by:

$$(26):\ \underset{(w,\tilde{w})\in\operatorname{dom} f^*}{\arg\sup} \left\{ -(1+d^\top r) f^* \begin{pmatrix} w \\ \tilde{w} \end{pmatrix} + (d^\top V + b^\top) \begin{pmatrix} w \\ \tilde{w} \end{pmatrix} + d^\top u \right\}, \tag{61}$$

$$(27):\ \underset{(w,\tilde{w})\in\operatorname{dom} f^*}{\arg\sup} \left\{ (D_i^\top r) f^* \begin{pmatrix} w \\ \tilde{w} \end{pmatrix} + (A_i^\top - D_i^\top V) \begin{pmatrix} w \\ \tilde{w} \end{pmatrix} - D_i^\top u \right\}, \qquad i = 1,\dots,n \tag{62}$$

$$(28):\ \underset{(w,\tilde{w})\in\operatorname{dom} f^*}{\arg\sup} \left\{ -u_i - V_{(i)} \begin{pmatrix} w \\ \tilde{w} \end{pmatrix} + r_i f^* \begin{pmatrix} w \\ \tilde{w} \end{pmatrix} \right\}, \qquad i = 1,\dots,q. \tag{63}$$

We already showed the convex conjugate of f takes value 0 in its domain. Recalling

$$A = \begin{bmatrix} L \\ \ell^\top/2 \\ \ell^\top/2 \end{bmatrix}, \quad b = \begin{bmatrix} \mathbf{0} \\ (1-\tau)/2 \\ (-1-\tau)/2 \end{bmatrix}, V = \begin{bmatrix} \tilde{V} & \hat{v} & \bar{v} \end{bmatrix},$$

we can rewrite problem (61) as:

$$\begin{aligned}
\sup_{w\in\mathbb{R}^{m+1},\tilde{w}\in\mathbb{R}}\ & \left(d^\top \begin{bmatrix} \tilde{V} & \hat{v} & \bar{v} \end{bmatrix} + \begin{bmatrix} \mathbf{0} & (1-\tau)/2 & (-1-\tau)/2 \end{bmatrix} \right) \begin{pmatrix} w \\ \tilde{w} \end{pmatrix} + d^\top u \\
\text{s.t.}\ & \tilde{w} = 1 \\
& \|w\|_2 \le 1.
\end{aligned}$$

By using $\tilde{w} = 1$, we can eliminate $\tilde{w}$ from the problem. Moreover, w only appears in a linear term, so we can change $\|w\|_2 \le 1$ constraint to be $\|w\|_2 = 1$ instead, i.e., w is a unit vector. So the problem becomes finding the value of:

$$\sup_{w:\|w\|_2=1} \left\{ \begin{bmatrix} \tilde{V}^\top d \\ \hat{v}^\top d + (1-\tau)/2 \end{bmatrix}^\top w \right\} + d^\top u + d^\top \bar{v} - (1+\tau)/2.$$

Hence we need to maximize a linear function over the unit ball, which can be solved analytically. This yields the objective value:

$$\left\| \begin{pmatrix} \bar{V}^\top d \\ d^\top \hat{v} + (1-\tau)/2 \end{pmatrix} \right\|_2 + d^\top u + d^\top \bar{v} - (1+\tau)/2, \tag{64}$$

and the maximizer is:

$$\overline{W}_1 = \begin{bmatrix} h\left(\begin{pmatrix} \bar{V}^\top d \\ d^\top \hat{v} + (1-\tau)/2 \end{pmatrix} \right) \\ 1 \end{bmatrix},$$

where $h(a) = a/\|a\|$ normalizes its input. Notice that the last element 1 comes since $\tilde{w} = 1$ is in the domain of convex conjugate. The worst-case of constraints (62) and (63) are obtained via similar calculations.

Appendix G: Data of Problems in Numerical Experiments

Here we explain the construction of the test data. The exact problem data, solutions, and the codes are available for download on GitHub at `https://github.com/selvi-aras/convex-max`.

G.1. Experiments of Section 4.1

Problem 1 The problem data is:

$$A=\begin{bmatrix}1&1&1&1&1\\1&0&0&0&1\\1&0&1&0&0\\0&0&1&1&1\\0&1&0&0&0\end{bmatrix},\quad b=\begin{bmatrix}-1\\0\\1\\1\\1\end{bmatrix},\quad a=\begin{bmatrix}0\\2\\1\\1\\0\end{bmatrix},\quad \rho=3.$$

For the next problems we generate larger problems by uniform random sampling (denoted simply as $\sim$). Remember that $A\in\mathbb{R}^{m\times n},\ b\in\mathbb{R}^m,\ a\in\mathbb{R}^n$. We let A_{ij} denote the elements of A.

Problem 2: $m=15,\ n=20,\ A_{ij}\sim\{0,1\},\ b_i\sim[-5,5],\ a_j\sim[0,3],\rho=8$

Problem 3: $m=120,\ n=100,\ A_{ij}\sim\{0,1\},\ b_i\sim[-5,5],\ a_j\sim[0,4],\rho=14$

Problem 4: $m=40,\ n=20,\ A_{ij}\sim\{-4,-3,\dots,3,4\},\ b_i\sim[-5,5],\ a_j\sim[0,4],\rho=10$

Problem 5: $m=100,\ n=50,\ A_{ij}\sim[-5,5],\ b_i\sim[-2,2],\ a_j\sim[-4,4],\rho=12$

Problem 6: $m=100,\ n=100,\ A_{ij}\sim[-4,4],\ b_i\sim[-3,3],\ a_j\sim[-4,4],\rho=15$

Problem 7: $m=30,\ n=200,\ A_{ij}\sim[-4,2],\ b_i\sim[-1,1],\ a_j\sim[-3,3],\rho=16$

Problem 8: $m=80,\ n=400,\ A_{ij}\sim[-2,1],\ b_i\sim[-\frac{1}{2},\frac{1}{2}],\ a_j\sim[-1,1],\rho=12$

Problem 9: $m=20,\ n=50,\ A_{ij}\sim[0,8],\ b_i\sim[-1,1],\ a_j\sim[0,4],\rho=14$

Problem 10: $m=100,\ n=10{,}000,\ A_{ij}\sim[-\frac{1}{2},\frac{1}{2}],\ b_i\sim[-\frac{1}{4},\frac{1}{4}],\ a_j\sim[-\frac{1}{2},\frac{1}{2}],\rho=15$

Problem 11: $m=1{,}000,\ n=1{,}000,\ A_{ij}\sim[-\frac{1}{2},\frac{1}{2}],\ b_i\sim[-\frac{1}{4},\frac{1}{4}],\ a_j\sim[-\frac{1}{2},\frac{1}{2}],\rho=18$

Problem 12: $m=700,\ n=2{,}000,\ A_{ij}\sim[-\frac{1}{2},\frac{1}{2}],\ b_i\sim[-\frac{1}{4},\frac{1}{4}],\ a_j\sim[-\frac{1}{2},\frac{1}{2}],\rho=24.$

G.2. Experiments of Section 4.2

Problem 1: $A=\begin{bmatrix}-1&-1&1&-1&1\\1&-1&0&0&0\\0&-1&0&0&-1\\0&0&-1&-1&-1\\0&0&0&0&-1\end{bmatrix},\ b=\mathbf{0}_{5\times1},a=\mathbf{0}_{5\times1},\rho=3$

Problem 2: $m=5,\ n=20,A_{ij}\sim[-1,1],\ b_i\sim[-2,2],\ a_j\sim[0,1],\rho=5$

Problem 3: $m=20,\ n=50,A_{ij}\sim[-10,10],\ b_i\sim[-3,3],\ a_j\sim[-2,2],\rho=6$

Problem 4: $m=20,\ n=180,A_{ij}\sim[-1,0.5],\ b_i=0,\ a_j\sim[0,1],\rho=1$

Problem 5: $m=300,\ n=30,A_{ij}\sim[-1,1],\ b_i=0,\ a_j\sim[0,1],\rho=2$

G.3. Experiments of Section 4.3

Problem 1 (Enkhbat et al. 2006) In this example, we solve:

$$\begin{aligned}\max_{x\in\mathbb{R}^{20}_+}\ & \tfrac{1}{2}\textstyle\sum_{i=1}^{20}(x_i-2)^2\\ \text{s.t.}\ \ & Dx\le d,\end{aligned}$$

where

$$D^\top = \begin{bmatrix} -3 & 7 & 0 & -5 & 1 & 1 & 0 & 2 & -1 & 1 \\ 7 & 0 & -5 & 1 & 1 & 0 & 2 & -1 & -1 & 1 \\ 0 & -5 & 1 & 1 & 0 & 2 & -1 & -1 & -9 & 1 \\ -5 & 1 & 1 & 0 & 2 & -1 & -1 & -9 & 3 & 1 \\ 1 & 1 & 0 & 2 & -1 & -1 & -9 & 3 & 5 & 1 \\ 1 & 0 & 2 & -1 & -1 & -9 & 3 & 5 & 0 & 1 \\ 0 & 2 & -1 & -1 & -9 & 3 & 5 & 0 & 0 & 1 \\ 2 & -1 & -1 & -9 & 3 & 5 & 0 & 0 & 1 & 1 \\ -1 & -1 & -9 & 3 & 5 & 0 & 0 & 1 & 7 & 1 \\ -1 & -9 & 3 & 5 & 0 & 0 & 1 & 7 & -7 & 1 \\ -9 & 3 & 5 & 0 & 0 & 1 & 7 & -7 & -4 & 1 \\ 3 & 5 & 0 & 0 & 1 & 7 & -7 & -4 & -6 & 1 \\ 5 & 0 & 0 & 1 & 7 & -7 & -4 & -6 & -3 & 1 \\ 0 & 0 & 1 & 7 & -7 & -4 & -6 & -3 & 7 & 1 \\ 0 & 1 & 7 & -7 & -4 & -6 & -3 & 7 & 0 & 1 \\ 1 & 7 & -7 & -4 & -6 & -3 & 7 & 0 & -5 & 1 \\ 7 & -7 & -4 & -6 & -3 & 7 & 0 & -5 & 1 & 1 \\ -7 & -4 & -6 & -3 & 7 & 0 & -5 & 1 & 1 & 1 \\ -4 & -6 & -3 & 7 & 0 & -5 & 1 & 1 & 0 & 1 \\ -6 & -3 & 7 & 0 & -5 & 1 & 1 & 0 & 2 & 1 \end{bmatrix}, d = \begin{bmatrix} -5 \\ 2 \\ -1 \\ -3 \\ 5 \\ 4 \\ -1 \\ 0 \\ 9 \\ 40. \end{bmatrix} \tag{65}$$

Problem 2 Same as problem 1, but the objective function is $\frac{1}{2}\sum_{i=1}^{20}(x_i+5)^2$.

Problems 3-7 use the following problem:

$$\begin{aligned} \max_{x \in \mathbb{R}^n_+} \quad & x^\top L^\top L x \\ \text{s.t.} \quad & Dx \le d \\ & x \le x_u, \end{aligned}$$

where $L \in \mathbb{R}^{m \times n}$ is a matrix generated randomly where all the entries are sampled uniformly from $0-1$. Moreover, let D be a similar matrix with all $0-1$ random coefficients (except the last problem uses $0-2$ random coefficients) and d to have integer entries uniformly distributed in range $[d_l, d_u]$. Denote q to be the number of total constraints.

Problem 3 $x_u = 5, d_l = 20, d_u = 60, n = 10, q = 15$

Problem 4 $x_u = 3, d_l = 30, d_u = 60, n = 50, q = 62$

Problem 5 $x_u = 2, d_l = 80, d_u = 120, n = 100, q = 130$

Problem 6 $x_u = 2, d_l = 160, d_u = 240, n = 200, q = 240$

Problem 7 $x_u = 1, d_l = 150, d_u = 300, n = 240, q = 280.$

G.4. Experiments of Section 4.4

Problem 1 considers the following problem:

$$\begin{aligned} \max_{x \in \mathbb{R}^n_+} \quad & \log\left(\textstyle\sum_{i=1}^m \exp(A_{(i)}x)\right) \\ \text{s.t.} \quad & -\frac{i}{n} \le x_i \le \frac{n}{i}, \quad i = 1, \ldots, n, \end{aligned}$$

where $A_{ij} \sim [-3, 3]$. In the numerical experiments n will vary.

Problems 2-4 consider the following problem:

$$\begin{aligned} \max_{x \in \mathbb{R}^n_+} \quad & \log\left(\textstyle\sum_{i=1}^m \exp(A_{(i)}x + b_i)\right) \\ \text{s.t.} \quad & Dx \le d. \end{aligned}$$

Problem 2 $n=q=m=20, A_{ij}\sim[-3,3],\ b_i\sim[-2,2],\ D_{i,j}\sim[0,1],\ d_i\sim[10,30]$

Problem 3 $n=q=m=50, A_{ij}\sim[-3,3],\ b_i\sim[-1,1],\ D_{i,j}\sim[0,1],\ d_i\sim[20,60]$

Problem 4 $n=q=m=100, A_{ij}\sim[-3,3],\ b_i\sim[-1,1],\ D_{i,j}\sim[0,1],\ d_i\sim[25,75]$

Problems 5-6 consider:

$$\begin{array}{ll}\max\limits_{x\in\mathbb{R}^n_+} & \log\left(\sum_{i=1}^m \exp(A_{(i)}x+b_i)\right)\\ \text{s.t.} & x_i\le c,\quad i=1,\dots,n\\ & x_i+x_j\le u_{ij},\quad i,j=1,\dots,n,\ i\ne j,\end{array}$$

Problem 5 $n=m=10,\ A_{ij}\sim[-3,3],\ b_i\sim[-1,1],\ u_{ij}\sim[5,15], c=8$

Problem 6 $n=m=30,\ A_{ij}\sim[-3,3],\ b_i\sim[-1,1],\ u_{ij}\sim[4,12], c=6.$

G.5. Experiments of Section 4.5

For the easiness of bookkeeping, we generate problem with every max-term having the same number of elements, i.e., $|\mathcal{I}_k|=|\mathcal{I}_{k'}|\ \forall k,k'\in\{1,\dots,K\}$.

Problems 1-6 are defined by:

$$\begin{array}{ll}\max\limits_{x\in\mathbb{R}^n_+} & \sum_{k=1}^K \max\limits_{j\in\mathcal{I}_k}\{A_{(j)}x\}\\ \text{s.t.} & -\dfrac{i}{n}\le x_i\le\dfrac{n}{i},\quad i=1,\dots,n,\end{array}$$

where:

Problem 1 $n=5,\ A_{ij}\sim[-5,5],\ |\mathcal{I}_k|=5,\ K=1$

Problem 2 $n=5,\ A_{ij}\sim[-5,5],\ |\mathcal{I}_k|=5,\ K=10$

Problem 3 $n=20,\ A_{ij}\sim[-5,5],\ |\mathcal{I}_k|=10,\ K=10$

Problem 4 $n=30,\ A_{ij}\sim[-5,5],\ |\mathcal{I}_k|=20,\ K=20$

Problem 5 $n=100,\ A_{ij}\sim[-5,5],\ |\mathcal{I}_k|=40,\ K=30$

Problem 6 $n=200,\ A_{ij}\sim[-4,4],\ |\mathcal{I}_k|=50,\ K=50.$

Problems 7-10 are defined by:

$$\begin{array}{ll}\max\limits_{x\in\mathbb{R}^n_+} & \sum_{k=1}^K \max\limits_{j\in\mathcal{I}_k}\{A_{(j)}x+b_j\}\\ \text{s.t.} & Dx\le d,\end{array}$$

Problem 7 $n=10,\ A_{ij}\sim[-5,5],\ b_j\sim[-10,10],\ D_{ij}\sim[0,1],\ d_i\sim[5,15],\ |\mathcal{I}_k|=5,\ K=2$

Problem 8 $n=10,\ A_{ij}\sim[-5,5],\ b_j\sim[-10,10],\ D_{ij}\sim[0,1],\ d_i\sim[5,15],\ |\mathcal{I}_k|=50,\ K=50$

Problem 9 $n=30,\ A_{ij}\sim[-5,5],\ b_j\sim[-10,10],\ D_{ij}\sim[0,1],\ d_i\sim[5,15],\ |\mathcal{I}_k|=50,\ K=50$

Problem 10 $n=50,\ A_{ij}\sim[-5,5],\ b_j\sim[-10,10],\ D_{ij}\sim[0,1],\ d_i\sim[5,15],\ |\mathcal{I}_k|=60, K=60$.

Problems 10-13 consider the same problem as above, but D and d are as given in (65) with $n=20$. The objective function varies as:

Problem 11 $A_{ij}\sim[-5,10],\ b_j\sim[-10,10],\ |\mathcal{I}_k|=10,\ K=10$

Problem 12 $A_{ij}\sim[-5,10],\ b_j\sim[-10,10],\ |\mathcal{I}_k|=50,\ K=10$

Problem 13 $A_{ij}\sim[-5,10],\ b_j\sim[-10,10],\ |\mathcal{I}_k|=100,\ K=50$.