

Question 1: (Section 6.4, #2)

$$g \circ h = g(h(x)) = g(3x + 2) = (3x + 2)^2.$$

$$h \circ g = h(g(x)) = h(x^3) = 3x^3 + 2.$$

$g \circ h \neq h \circ g$, thus composition is not commutative.

Question 2: (a) The image of f is all the odd integers. The image of $f(X)$ where $X = \{x \in \mathbb{Z} \mid x \text{ is even}\}$ is every other odd integer.

(b) The image of f is every positive real number. The image of $f(X)$ where $X = (0, 1]$ is $(\infty, 1]$.

(c) The image of f is $[-1, 1]$. The image of $f(X)$ where $X = [0, \pi)$ is also $(0, 1]$.

Question 3: (a) *Proof.* Suppose A and B are sets with subsets $U, V \subseteq A$, and $f : A \rightarrow B$ is a function. Let b be an arbitrary element of $f(U \cap V)$. This means that there exists an $x \in U \cap V$ such that $f(x) = b$. Therefore, $\exists x \in U$ such that $f(x) = b$, and $\exists x \in V$ such that $f(x) = b$. Therefore, $b \in f(U) \wedge b \in f(V) \rightarrow b \in f(U) \cap f(V)$. So, $f(U \cap V) \subseteq f(U) \cap f(V)$. \square

(b) *Proof.* Suppose the same conditions as in part (a), but now suppose also that f is injective. Let b be an arbitrary element of $f(U) \cap f(V)$. Then, $b \in f(U)$ and $b \in f(V)$. So, $\exists x \in U$ such that $f(x) = b$ and $\exists y \in V$ such that $f(y) = b$. By the injectivity of f , $x = y$. WLOG, consider x . $x \in U$, and now, $x \in V$. So, $b \in f(U \cap V)$. Thus, $f(U) \cap f(V) \subseteq f(U \cap V)$. \square

(c) $f(x) = x^2$, $f : \mathbb{R} \rightarrow \mathbb{R}$.

Question 4: (a) $f^{-1}(Y) = [0, \infty)$.

(b) $f^{-1}(Y) = \{[n\pi, (n+1)\pi] \mid n \in \{\dots, -4, -2, 0, 2, 4, \dots\}\}$.

(c) $f^{-1}(Y) = \{1, 10, 12, 14, 16, 18\}$.

Question 5: (a) e^x , when taken from $\mathbb{R} \rightarrow \mathbb{R}^+$ is injective and surjective.

(b) $\sin(x)$ is neither injective nor surjective.

(c) The function in part (c) of Question 4 is not injective, but it is surjective.

Question 6: (a) *Proof.* Suppose $f : A \rightarrow B$ is a function taking A to B , and that $X \subseteq A$. Let x be an arbitrary element of X . Consider the image of X , $f(X) = \{b \in B \mid b = f(a), a \in X\}$. $x \in X$, so $f(x) \in f(X)$. Then, $f^{-1}(f(X)) = \{a \in A \mid f(a) \in f(X)\}$. $x \in A$, and $f(x) \in f(X)$, so $x \in f^{-1}(f(X))$. Thus, $X \subseteq f^{-1}(f(X))$. \square

- (b) *Proof.* Suppose $f : A \rightarrow B$ is a function taking A to B , $X \subseteq A$, and that f is injective. We prove that $X = f^{-1}f(X)$ by proving that each set is a subset of the other.

(\subseteq) *Proof.* The result from part (a) is true whether or not f is injective. \square

(\supseteq) Let a be an arbitrary element of $f^{-1}(f(X))$. So, $a \in A$ and $f(a) \in f(X)$. $f(a) \in f(X) = \{ f(a') \mid a' \in X \}$. By the injectivity of f , $a = a'$, so it must be the case that $a \in X$. Thus, $f^{-1}(f(X)) \subseteq X$.

Thus, we are done. \square

- (c) $f(x) = x^2$, $f : \mathbb{R} \rightarrow \mathbb{R}$.

Question 7: *Proof.* We prove that f is surjective if and only if $f^{-1}(Y) \neq \emptyset$ for all nonempty $Y \subseteq B$ by proving both directions.

(\rightarrow) *Proof.* Suppose $f : A \rightarrow B$ is a function that takes A to B , and suppose that $f^{-1}(Y) \neq \emptyset$ and $Y \neq \emptyset \subseteq B$. Consider the family of Singleton sets whose union is B , $\mathcal{F} = \{ \{ b \} \mid b \in B \}$. Clearly, every $\{ b \} \neq \emptyset \subseteq B$, so $\exists a \in A$ such that $f(a) = b$. So, f is surjective. \square

(\leftarrow) Suppose $f : A \rightarrow B$ is a function that takes A to B , and f is surjective. Let $Y \neq \emptyset \subseteq B$. Clearly, every Y contains at least one element $b \in B$, and appealing to the surjectivity of f , there must be an $a \in A$ such that $f(a) = b$. Thus, $f^{-1}(Y) \neq \emptyset$. \square

Question 8: (a) $V(x^2 - 2) = \{ \sqrt{2}, -\sqrt{2} \}$. $V(x^3 - 1) = \{ 1 \}$. $V(x^2 + 1) = \emptyset$. $V(0) = \mathbb{R}$.

(b) V is not injective. We show this by a counterexample: consider the polynomials $P := x + 1 = 0$ and $Q := x^3 + 1 = 0$. We can see that $V(P) = \{ -1 \}$ and $V(Q) = \{ -1 \}$. Thus, we found have $P, Q \in \mathbb{Z}[x]$ such that $P \neq Q$ but $V(P) = V(Q)$

V is also not surjective. We show this by a counterexample: consider $\{ \pi \} \in \mathcal{P}(\mathbb{R})$. Informally, π cannot be a solution to any $P \in \mathbb{Z}[x]$ because π is not the ratio of any two integers. Thus, it would be impossible to obtain any radical of π as a solution, since P has only integer coefficients.

Question 9: (a) *Proof.* Suppose $f : A \rightarrow B$ is a surjective function, and $g : B \rightarrow C$ is not injective. Then, consider an element $c \in C$ such that $\exists b, b' \in B$ such that $f(b) = f(b') = c$. Because f is surjective, $\forall k \in B$ there exists an $a \in A$ such that $f(a) = k$. So, there must be an $f(a) = f(b) = f(b') = c$, so $g \circ f$ is not injective. \square

(b) It is not necessarily true.

Proof. We prove this by a counterexample: consider $f : \mathbb{R} \rightarrow \mathbb{R}$ given by $f(x) = 2^x$, and $g : \mathbb{R} \rightarrow \mathbb{R}$ given by $g(x) = x^2$. f is not surjective, because the negative reals are never accessed. g is not injective because $g(1) = g(-1) = 1$. However, $g \circ f = (2^x)^2 = 4^x$, which is injective. \square

Question 10: *Proof.* We prove that f is a bijection by showing it has an inverse. Consider the function $g : \mathbb{R} \setminus \{1\} \rightarrow \mathbb{R} \setminus \{1\}$ given by $g(x) = \frac{x+1}{x-1}$, which is only defined when $x \neq 1$. We show that g is both a left and right inverse of f .

$(g \circ f)$ We have that $g \circ f = g(f(x))$, $x \in \mathbb{R} \setminus \{1\}$. So,

$$g(f(x)) = \frac{\frac{x+1}{x-1} + 1}{\frac{x+1}{x-1} - 1} = \frac{\frac{2x}{x-1}}{\frac{2}{x-1}} = x.$$

Therefore, $g \circ f = I_A$.

$(f \circ g)$ Similarly, we get that

$$f(g(x)) = \frac{\frac{x+1}{x-1} + 1}{\frac{x+1}{x-1} - 1} = \frac{\frac{2x}{x-1}}{\frac{2}{x-1}} = x.$$

Therefore, $f \circ g = I_B$.

By showing that g is a left and right inverse of f , we have shown that f is invertible. Therefore, f is a bijection. \square