Homework 1

Exercises

1-6. Problems from *Mathematical Reasoning: Writing and Proof* [1]:

- Section 2.4, # 1a–b,[1] 2d–f, 10a–b
- Section 3.2, # 5, 10
- Section 3.3, # 4[2]

7. Show the following logical identities without using a truth table:

   (a) $P \leftrightarrow Q \equiv (P \wedge Q) \vee (\neg P \wedge \neg Q)$.

   (b) $(P \to Q) \vee (P \to R) \equiv P \to (Q \vee R)$.

   Hint: Remember that, by definition, $P \leftrightarrow Q$ *is given by* $(P \to Q) \wedge (Q \to P)$. *Use the predicate calculus!*

8. Argue by contradiction that there are no integer solutions to the equation $x^2 = 4y + 3$.

   In the final two problems, you are asked to prove statements that you could have imagined proving in Calculus—they rely on your intuition of the real numbers and the basic arithmetic operations, both of which will be developed formally later in the course. For now, do your best to develop cogent arguments using things we know to be true. For handling inequalities, you should use the following axioms:

   - **Trichotomy.** For any a and b, one (and only one) of the following statements must hold: $a < b$, $a = b$, or $a > b$.
   - **Transitivity.** If $a < b$ and $b < c$, then $a < c$. *(So it makes sense to write something like* $a < b < c$*)*
   - **Additivity.** If $a < b$ and c is any real number, then $a + c < b + c$.
   - **Multiplicativity.** If $a < b$ and $0 < c$, then $a \cdot c < b \cdot c$.

   You are free to use the usual algebraic and arithmetic principles that you are used to. For example:

   $$\forall a \in \mathbb{R}, a + 0 = a \qquad \text{and } \forall b \in \mathbb{R}, b \cdot 1 = b.$$

   Here is an example of these ideas in action.

   Proposition: *If* $x > 0$, *then* $\frac{1}{x} > 0$.

   Proof. Let $x > 0$ and consider $\frac{1}{x}$, which must exist since $x \neq 0$. By the trichotomy axiom, there are three options:

   Firstly, if $\frac{1}{x} = 0$, then $x \cdot \frac{1}{x} = 1$ by definition of multiplicative inverse, yet at the same time $x \cdot \frac{1}{x} = 0$ because $\frac{1}{x} = 0$. Hence $0 = 1$, which is nonsense. This contradiction means we cannot have $\frac{1}{x} = 0$.

   Secondly, assume $\frac{1}{x} < 0$. Since $x > 0$, we can multiply on both sides of the inequality to see that $1 = x \cdot \frac{1}{x} < x \cdot 0 = 0$. But this means $1 < 0$, which is also nonsense. Thus we cannot have $\frac{1}{x} < 0$.

   The only remaining possibility is $\frac{1}{x} > 0$, which must hold by the trichotomy axiom. We are done! □

9. Suppose $x, y \in \mathbb{R}$ and $0 < x < y$. Prove that $0 < \frac{1}{y} < \frac{1}{x}$.
   Hint: Can you multiply through the inequality by a certain positive quantity to get the desired result?

10. Use the Archimedean Principle (Theorem 1.1 in the notes) to prove: For every $\varepsilon > 0$, there is an $n \in \mathbb{Z}^+$ such that $\frac{1}{n} < \varepsilon$.
    Hint: Use mathematical specificity to prove the result by invoking the inequality Axioms, the Proposition I've just proved, what you showed in Problem 9, and the Archimedean Principle. Informally, the Archimedean Principle says that there are no infinitely large real numbers; by proving this statement, you are showing that there are no infinitely small numbers.

[1] You may use the quadratic equation for (a) and the fact that the square root of a non-square integer is irrational—we prove that later in the course.
[2] Hint: It might be useful to prove a Lemma: For any $n \in \mathbb{Z}$, n is even if and only if n^3 is even. If you use this Lemma, prove it!

References

[1] Ted Sundstrom. *Mathematical Reasoning: Writing and Proof*. Grand Valley State University Libraries, 3rd edition, 2020.