Question 1: (a) $[18] = \{ 4n + 2 \mid n \in \mathbb{Z} \}$.

(b) $[31] = \{ 4n + 2 \mid n \in \mathbb{Z} \}$.

(c) There are 4 equivalence classes.

Question 2: *Proof.* We prove that $\equiv_n$ on $\mathbb{Z}$ is an equivalence relation by showing that it is reflexive, transitive, and symmetric. Suppose $n \in \mathbb{Z}^+$.

Reflexive Let a be an arbitrary element in $\mathbb{Z}$. Clearly, $a - a = 0 = kn, k \in \mathbb{Z}$. So, $\forall a \in \mathbb{Z}, a \equiv_n a$.

Symmetric Let a, b be arbitrary elements of $\mathbb{Z}$. Suppose $a \equiv_n b$. By definition, $a - b = kn, k \in \mathbb{Z}$. Multiplying the equation by -1, we get

$$b - a = -kn.$$

$-k$ is still an integer, and thus by definition, $b \equiv_n a$.

Transitive Let a, b, c be aribtrary elements of $\mathbb{Z}$. Suppose $a \equiv_n b$ and $b \equiv_n c$. By definition, we have $a - b = kn, k \in \mathbb{Z}$ and $b - c = gn, g \in \mathbb{Z}$. Then, $b = a - nk$. Subbing in, we get that

$$a - kn - c = gn$$
$$a - c = (k + g)n$$

By definition, we have $a \equiv_n c$.

□

Question 3: *Proof.* Suppose $a, b, c, d \in \mathbb{Z}$, and $a \equiv_n c$ and $b \equiv_n d$. Then,

$$a - c = nk, k \in \mathbb{Z}$$
$$b - d = nl, l \in \mathbb{Z}.$$

Adding the equations, we get

$$a + b - c - d = n(k + l).$$

Simplifying, we get

$$(a + b) - (c + d) = n(k + l).$$

Thus, by definition $a + b \equiv_n c + d$. □

Question 4: (a) *Proof.* The theorem given in the proposition is equivalent to claiming that 'If ab is even, then a is even or b is even'. We have proven this theorem to be true, so it is the case that if $ab \equiv_2 0$, then $a \equiv_2 0$ or $b \equiv_2 0$. □

(b) The answer is no. $10 \times 2 = 20 \equiv_4 0$, but $10 \not\equiv_4 0$ and $2 \not\equiv_4 0$.

(c) The answer to the question will be yes when n is a prime.

Proof. Suppose n is a prime, and $a, b \in \mathbb{Z}$. Also suppose that $ab \equiv_n 0$. This is equivalent to saying that $n \mid ab$. Because n is prime, $\gcd(a, n)$ is either 1 or n. Consider two cases:

($\gcd(a, n) = n$). Then, $n \mid a$.

($\gcd(a, b) = 1$). luke said to appeal to the fundamental theorem of arithemtic

$\square$