

# Homework 3

## Exercises

1. Problem #2 from Section 6.4 of *Mathematical Reasoning: Writing and Proof* [1].
  2. For the following, compute the image of  $f$  and then the image  $f(X)$  of the given subset  $X$  (no proof necessary):
    - (a)  $f : \mathbb{Z} \rightarrow \mathbb{Z}$ ,  $f(n) = 2n + 1$ ,  $X = \{n \in \mathbb{Z} : n \text{ is even}\}$ .
    - (b)  $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ ,  $f(x) = \frac{1}{x}$ ,  $X = (0, 1]$ .
    - (c)  $f : \mathbb{R} \rightarrow \mathbb{R}$ ,  $f(x) = \cos(x)$ ,  $X = [0, \pi]$ .
  3. Let  $A$  and  $B$  be sets with subsets  $U, V \subseteq A$ . Let  $f : A \rightarrow B$  be a function.
    - (a) Prove that  $f(U \cap V) \subseteq f(U) \cap f(V)$ .
    - (b) Now suppose  $f$  is injective. Prove that  $f(U \cap V) = f(U) \cap f(V)$ . Note: You already did half the work in 3a!
    - (c) Give an example of a function  $f : A \rightarrow B$  and  $U, V \subseteq A$  for which  $f(U \cap V) \neq f(U) \cap f(V)$ .
  4. In the following, compute the inverse image  $f^{-1}(Y)$  for the given subset  $Y$  (no proof necessary):
    - (a)  $f : \mathbb{R} \rightarrow \mathbb{R}^+$ ,  $f(x) = e^x$ ,  $Y = [1, \infty)$ .
    - (b)  $f : \mathbb{R} \rightarrow \mathbb{R}$ ,  $f(x) = \sin(x)$ ,  $Y = [0, 1]$ .
    - (c)  $f : \mathbb{Z} \rightarrow \mathbb{Z}$ ,  $f(n) = \begin{cases} n/2 & \text{if } n \text{ is even} \\ 2n + 4 & \text{if } n \text{ is odd} \end{cases}$ ,  $Y = \{5, 6, 7, 8, 9\}$ .
  5. Which of the functions in the previous problem are injective? What about surjective?
  6. Let  $f : A \rightarrow B$  be a function and take  $X \subseteq A$ .
    - (a) Prove that  $X \subseteq f^{-1}(f(X))$ .
    - (b) Now suppose  $f$  is injective. Prove that  $X = f^{-1}(f(X))$ .
    - (c) Give an example of a function  $f : A \rightarrow B$  and  $X \subseteq A$  for which  $X \neq f^{-1}(f(X))$ .
  7. Let  $f : A \rightarrow B$  be a function. Prove that  $f$  is surjective if and only if  $f^{-1}(Y) \neq \emptyset$  for all nonempty  $Y \subseteq B$ . (Bonus)
  8. Here we write  $\mathbb{Z}[x]$  for the set of *polynomials in the indeterminant  $x$  with integer coefficients*. Remember that a polynomial  $p(x)$  is a formal expression in  $x$ ; there is always an  $n \in \mathbb{Z}^+$  such that  $p(x) = a_n x^n + \dots + a_1 x + a_0$ , with each  $a_k \in \mathbb{Z}$ . Thus we have  $8x^5 - 4x + 1 \in \mathbb{Z}[x]$  and  $0, 1 \in \mathbb{Z}[x]$ , but things like
 
$$\frac{1}{x^6 + 1}, \quad \frac{1}{5}x, \quad 7x^4 + 3x^{-2}, \quad \text{and} \quad 1 + x + x^2 + x^3 + \dots \quad (\text{an infinite series})$$
 are *not* elements of  $\mathbb{Z}[x]$ . Two polynomials  $p(x), q(x) \in \mathbb{Z}[x]$  are equal if and only if all their coefficients match.
- Consider the function  $V : \mathbb{Z}[x] \rightarrow \mathcal{P}(\mathbb{R})$  given by
- $$V(p(x)) = \{r \in \mathbb{R} : p(r) = 0\}.$$
- (a) Compute  $V(x^2 - 2)$ ,  $V(x^3 - 1)$ ,  $V(x^2 + 1)$ , and  $V(0)$ .
  - (b) Is  $V$  injective or surjective? Explain.
9. Let  $f : A \rightarrow B$  and  $g : B \rightarrow C$ .
    - (a) Prove that if  $f$  is surjective and  $g$  is *not* injective, then  $g \circ f$  is also not injective.
    - (b) Is the claim necessarily true if  $f$  is not surjective? Prove or give a counterexample.
  10. Consider the function  $f : \mathbb{R} \setminus \{1\} \rightarrow \mathbb{R} \setminus \{1\}$  given by  $f(x) = \frac{x+1}{x-1}$ . Prove that  $f$  is a bijection.

*Hint: You can show this directly, but the shrewd among you might instead verify a condition that is equivalent to being bijective.*

## References

- [1] Ted Sundstrom. *Mathematical Reasoning: Writing and Proof*. Grand Valley State University Libraries, 3rd edition, 2020.