Question 1: (Section 2.4: #1a, #1b)

(a) There exists a rational number x, such that $x^2 - 3x - 7 = 0$. This statement is false, as the solutions to this polynomial are $\frac{3}{2} \pm \frac{\sqrt{37}}{2}$, and the square root of a non-square integer is irrational.

(b) There exists a real number x, such that $x^2 + 1 = 0$. This is false because the solutions to this polynomial are $\pm i$, which are imaginary.

Question 2: (Section 2.4: #2d, #2f)

(d) Consider $m = 4$. Then, $\frac{m}{3} = \frac{4}{3} \notin \mathbb{Z}$. The negation of the statement in English is 'there exists an integer m such that m divided by 3 is not an integer.'

(f) Consider $x = \frac{\pi}{2}$. Then, $\tan^2 x + 1$ is undefined, and thus there can be no equality. The negation of the statement in English is that 'there exists a real number x such that the one more than the tangent squared of x is not equal to the secant squared of x.'

Question 3: (Section 2.4: #10a, #10b)

(a) $[\forall x, y \in \mathbb{R}][(x < y) \to (f(x) < f(y))]$.

(b) $[\exists x, y \in \mathbb{R}][(x \geq y) \wedge f(x) < f(y)]$.

Question 4: (Section 3.2: #5)

Proof. We proceed by the contrapositive, which is the statement

If a and b are odd, then ab is odd.

Suppose that a and b are odd integers. So, $a = 2k+1, k \in \mathbb{Z}$ and $b = 2l+1, l \in \mathbb{Z}$. Then $ab = (2k+1)(2l+1) = 4lk + 2k + 2l + 1$. Factoring, we get $2(2lk + k + l) + 1$, which is odd. $\square$

Question 5: (Section 3.2: #10)

Proof. Suppose n is an arbitrary integer.

($\to$) Suppose n is even. Then, $n = 2k, k \in \mathbb{Z}$. So, $n^2 = 4k^2$. Then,

$$n^2 \equiv 0 \pmod{4}.$$

Thus, $4 \mid n^2$.

($\leftarrow$) We proceed by the contrapositive, which is the statement

If n is odd, then $4 \nmid n^2$.

Suppose n is odd. Then, $n = 2k+1, k \in \mathbb{Z}$, and so $n^2 = 4k^2 + 4k + 1$. Factoring, we get $n^2 = 4(k^2 + k) + 1$. Thus,

$$n^2 \equiv 1 \pmod 4,$$

so $4 \nmid n^2$.

□

Question 6: (Section 3.3: #4)

Proof. We proceed via a proof by contradiction. Suppose that $\exists r \in \mathbb{Q}$ such that $r^3 = 2$. Then, r can be expressed as $\frac{a}{b}$ such that $(a, b \in \mathbb{Z}), \gcd(a, b) = 1$. Substituting in, we have $\frac{a^3}{b^3} = 2$. Rearranging, we get that

$$a^3 = 2b^3$$

By observation, we see that a^3 is even.

Lemma 1. *If n^3 is even, then n is even.*

Proof. We proceed via a proof by the contrapositive, which is the statement

If n is odd, then n^3 is odd.

Suppose n is odd. Then, $n = 2k + 1, k \in \mathbb{Z}$. Expanding, we get $n^3 = 8k^3 + 12k^2 + 6k + 1$. Thus, n^3 is odd. □

By Lemma 1, because a^3 is even, a is also even. So, it can be written as $a = 2k, k \in \mathbb{Z}$. Plugging $2k$ in for a, we find that $8k^3 = 2b^3$. So,

$$4k^3 = b^3.$$

By observation, b^3 must also be even, and by Lemma 1, b must be even as well. So, a and b both have 2 as a factor; however, we stated previously that $\gcd(a, b) = 1$. Therefore, the assumption that r is a rational number leads to a contradiction, thus r must be irrational. □

Question 7: (a) *Proof.* From the definition of the biconditional, we have $P \Leftrightarrow Q \equiv (P \to Q) \wedge (Q \to P)$. Rewriting the implications in their logical form, we get

$$(P \leftarrow Q) \wedge (Q \to P) \equiv (\neg P \vee Q) \wedge (\neg Q \vee P).$$

Using the distributive properties of conjunctions and disjunctions, we find that

$$(\neg P \vee Q) \wedge (\neg Q \vee P) \equiv (\neg Q \wedge \neg P) \vee (\neg Q \wedge Q) \vee (\neg P \wedge P) \vee (P \wedge Q).$$

Eliminating the contradictions $\neg P \wedge P$ and $\neg Q \wedge Q$, we obtain $(\neg Q \wedge \neg P) \vee (P \wedge Q)$. Thus, $P \Leftrightarrow Q \equiv (P \wedge Q) \vee (\neg P \wedge \neg Q)$.

□

(b) *Proof.* Rewriting the LHS in terms of logical operators, we get

$$(P \to Q) \vee (P \to Q) \equiv (\neg P \vee Q) \vee (\neg P \vee R).$$

Using the associative properties of disjunctions, we can rewrite the expression into the form

$$(\neg P \vee Q) \vee (\neg P \vee R) \equiv (\neg P \vee \neg P) \vee (Q \vee R).$$

By idempotency, this is equivalent to $\neg P \vee (Q \vee R)$. This is the definition of an implication, so $\neg P \vee (Q \vee R) \equiv P \to (Q \vee R)$. Thus, $(P \to Q) \vee (P \to R) \equiv P \to (Q \vee R)$. □

Question 8: *Proof.* We proceed by contradiction. Suppose there exists $x_0, y_0 \in \mathbb{Z}$ such that $x_0^2 = 4y_0 + 3$. We consider two cases.

Case 1: x_0 is even. Then, $x_0 = 2k, k \in \mathbb{Z}$. Plugging into the original expression, we get $4k^2 = 4y + 3$. Taking the expression modulo 4, we get $0 \equiv 3 \pmod 4$, which is a contradiction.

Case 2: x_0 is not even. So, x_0 is odd, and can be expressed as $x_0 = 2k + 1, k \in \mathbb{Z}$. Plugging into the original expression, we get $4k^2 + 4k + 1 = 4(k^2 + k) + 1 = 4y + 3$. Taking the expression modulo 4, we get $1 \equiv 3 \pmod 4$, which is a contradiction.

In both cases, we have contradictions, thus there cannot exist integer solutions $x, y \in \mathbb{Z}$ such that $x^2 = 4y + 3$. □

Question 9: *Proof.* Suppose $x, y \in \mathbb{R}$, and $0 < x < y$. Because x and y are both non-zero, $\frac{1}{xy}$ exists. We multiply the expression $0 < x < y$ by $\frac{1}{xy}$, which preserves the order of the inequality by the multiplicativity axiom. This gives us

$$(0 < x < y)(\tfrac{1}{xy}) = 0 < \tfrac{1}{y} < \tfrac{1}{x}.$$

□

Question 10: *Proof.* Suppose $\epsilon \in \mathbb{R}$ and $\epsilon > 0$. Then, we know that $\frac{1}{\epsilon} \in \mathbb{R}$ exists since $\epsilon \neq 0$. Because $\frac{1}{\epsilon} \in \mathbb{R}$, by the Archimedean Principle, $\exists n \in \mathbb{Z}$ such that $n > \frac{1}{\epsilon}$. Because $\epsilon > 0$, we can say $n \in \mathbb{Z}^+$. By the transitivity axiom, because $\frac{1}{\epsilon} > 0$ and $n > \frac{1}{\epsilon}$, we have the inequality

$$0 < \frac{1}{\epsilon} < n. \tag{1}$$

We showed in Problem 9 that if the inequality $0 < x < y$ holds for some $x, y \in \mathbb{R}$, then $0 < \frac{1}{y} < \frac{1}{x}$. Applying that result to Equation (1), we know that $0 < \frac{1}{n} < \epsilon$. So, $\frac{1}{n} < \epsilon$. Thus, we are done. □