Question 1: (Section 6.4, #2)

$g \circ h = g(h(x)) = g(3x+2) = (3x+2)^2$.

$h \circ g = h(g(x)) = h(x^3) = 3x^3 + 2$.

$g \circ h \neq h \circ g$, thus composition is not commutative.

Question 2:

(a) The image of f is all the odd integers. The image of $f(X)$ where $X = \{ x \in \mathbb{Z} \mid x \text{ is even} \}$ is every other odd integer.

(b) The image of f is every positive real number. The image of $f(X)$ where $X = (0, 1]$ is $(\infty, 1]$.

(c) The image of f is $[-1, 1]$. The image of $f(X)$ where $X = [0, \pi)$ is also $(0, 1]$.

Question 3:

(a) *Proof.* Suppose A and B are sets with subsets $U, V \subseteq A$, and $f : A \to B$ is a function. Let b be an aribtrary element of $f(U \cap V)$. This means that there exists an $x \in U \cap V$ such that $f(x) = b$. Therefore, $\exists x \in U$ such that $f(x) = b$, and $\exists x \in V$ such that $f(x) = b$. Therefore, $b \in f(U) \wedge b \in f(V) \to b \in f(U) \cap f(V)$. So, $f(U \cap V) \subseteq f(U) \cap f(V)$. □

(b) *Proof.* Suppose the same conditions as in part (a), but now suppose also that f is injective. Let b be an aribtrary element of $f(U) \cap f(V)$. Then, $b \in f(U)$ and $b \in f(V)$. So, $\exists x \in U$ such that $f(x) = b$ and $\exists y \in V$ such that $f(y) = b$. By the injectivity of f, $x = y$. WLOG, consider x. $x \in U$, and now, $x \in V$. So, $b \in f(U \cap V)$. Thus, $f(U) \cap f(V) \subseteq f(U \cap V)$. □

(c) $f(x) = x^2$, $f : \mathbb{R} \to \mathbb{R}$.

Question 4:

(a) $f^{-1}(Y) = [0, \infty)$.

(b) $f^{-1}(Y) = \{ [n\pi, (n+1)\pi] \mid n \in \{ \dots, -4, -2, 0, 2, 4, \dots \} \}$.

(c) $f^{-1}(Y) = \{ 1, 10, 12, 14, 16, 18 \}$.

Question 5:

(a) e^x, when taken from $\mathbb{R} \to \mathbb{R}^+$ is injective and surjective.

(b) $\sin(x)$ is neither injective nor surjective.

(c) The function in part (c) of Question 4 is not injective, but it is surjective.

Question 6:

(a) *Proof.* Suppose $f : A \to B$ is a function taking A to B, and that $X \subseteq A$. Let x be an aribtary element of X. Consider the image of X, $f(X) = \{ b \in B \mid b = f(a), a \in X \}$. $x \in X$, so $f(x) \in f(X)$. Then, $f^{-1}(f(X)) = \{ a \in A \mid f(a) \in f(X) \}$. $x \in A$, and $f(x) \in f(X)$, so $x \in f^{-1}(f(X))$. Thus, $X \subseteq f^{-1}(f(X))$. □

(b) *Proof.* Suppose $f : A \to B$ is a function taking A to B, $X \subseteq A$, and that f is injective. We prove that $X = f^{-1}f(X)$ by proving that each set is a subset of the other.

($\subseteq$) *Proof.* The result from part (a) is true whether or not f is injective. $\square$

($\supseteq$) Let a be an aribtrary element of $f^{-1}(f(X))$. So, $a \in A$ and $f(a) \in f(X)$. $f(a) \in f(X) = \{\, f(a') \mid a' \in X \,\}$. By the injectivity of f, $a = a'$, so it must be the case that $a \in X$. Thus, $f^{-1}(f(X)) \subseteq X$.

Thus, we are done. $\square$

(c) $f(x) = x^2$, $f : \mathbb{R} \to \mathbb{R}$.

Question 7: *Proof.* We prove that f is surjective if and only if $f^{-1}(Y) \neq \varnothing$ for all nonempty $Y \subseteq B$ by proving both directions.

($\to$) *Proof.* Suppose $f : A \to B$ is a function that takes A to B, and suppose that $f^{-1}(Y) \neq \varnothing$ and $Y \neq \varnothing \subseteq B$. Consider the family of Singleton sets whose union is B, $\mathcal{F} = \{\, \{\, b \,\} \mid b \in B \,\}$. Clearly, every $\{\, b \,\} \neq \varnothing \subseteq B$, so $\exists a \in A$ such that $f(a) = b$. So, f is surjective. $\square$

($\leftarrow$)

$\square$

Question 8: (a) $V(x^2 - 2) = \{\, \sqrt{2}, -\sqrt{2} \,\}$. $V(x^3 - 1) = \{\, 1 \,\}$. $V(x^2 + 1) = \varnothing$. $V(0) = \mathbb{R}$.

(b) V is not injective. We show this by a counterexample: consider the polynomials $P \coloneqq x + 1 = 0$ and $Q \coloneqq x^3 + 1 = 0$. We can see that $V(P) = \{\, -1 \,\}$ and $V(Q) = \{\, -1 \,\}$. Thus, we found have $P, Q \in \mathbb{Z}[x]$ such that $P \neq Q$ but $V(P) = V(Q)$

V is also not surjective. We show this by a counterexample: consider $\left\{\, \pi^{\frac{1}{n}} \;\middle|\; n \in \mathbb{Z} \,\right\} \in \mathcal{P}(\mathbb{R})$. Informally, $\pi^{\frac{1}{n}}$, $n \in \mathbb{Z}$ cannot be a solution to any $P \in \mathbb{Z}[x]$ because π is not the ratio of any two integers. Thus, it would be impossible to obtain any radical of π as a solution.

Question 9: (Should be question 10)

Proof. We prove that f is a bijection by showing it has an inverse. Consider the function $g : \mathbb{R} \setminus \{\, 1 \,\} \to \mathbb{R} \setminus \{\, 1 \,\}$ given by $g(x) = \frac{x+1}{x-1}$. We show that g is both a left and right inverse of f.

($g \circ f$) We have that $g \circ f = g(f(x)), x \in \mathbb{R} \setminus \{\, 1 \,\}$. So,

$$g(f(x)) = \frac{\frac{x+1}{x-1} + 1}{\frac{x+1}{x-1} - 1} = \frac{\frac{2x}{x-1}}{\frac{2}{x-1}} = x.$$

Therefore, $g \circ f = I_A$.

$(f \circ g)$ Similarly, we get that

$$f(g(x)) = \frac{\frac{x+1}{x-1}+1}{\frac{x+1}{x-1}-1} = \frac{\frac{2x}{x-1}}{\frac{2}{x-1}} = x.$$

Therefore, $f \circ g = I_B$.

By showing that g is a left and right inverse of f, we have shown that f is invertible. Therefore, f is a bijection. NOTE: I think this is right, but clean it up $\square$