Homework 3

Exercises

1. Problem #2 from Section 6.4 of *Mathematical Reasoning: Writing and Proof* [1].
2. For the following, compute the image of f and then the image $f(X)$ of the given subset X (no proof necessary):
   (a) $f : \mathbb{Z} \to \mathbb{Z}$, $f(n) = 2n + 1$, $X = \{n \in \mathbb{Z} : n \text{ is even}\}$.
   (b) $f : \mathbb{R}^+ \to \mathbb{R}^+$, $f(x) = \frac{1}{x}$, $X = (0, 1]$.
   (c) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \cos(x)$, $X = [0, \pi)$.
3. Let A and B be sets with subsets $U, V \subseteq A$. Let $f : A \to B$ be a function.
   (a) Prove that $f(U \cap V) \subseteq f(U) \cap f(V)$.
   (b) Now suppose f is injective. Prove that $f(U \cap V) = f(U) \cap f(V)$. *Note: You already did half the work in 3a!*
   (c) Give an example of a function $f : A \to B$ and $U, V \subseteq A$ for which $f(U \cap V) \neq f(U) \cap f(V)$.
4. In the following, compute the inverse image $f^{-1}(Y)$ for the given subset Y (no proof necessary):
   (a) $f : \mathbb{R} \to \mathbb{R}^+$, $f(x) = e^x$, $Y = [1, \infty)$.
   (b) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \sin(x)$, $Y = [0, 1]$.
   (c) $f : \mathbb{Z} \to \mathbb{Z}$, $f(n) = \begin{cases} n/2 & \text{if } n \text{ is even} \\ 2n + 4 & \text{if } n \text{ is odd} \end{cases}$, $Y = \{5, 6, 7, 8, 9\}$.
5. Which of the functions in the previous problem are injective? What about surjective?
6. Let $f : A \to B$ be a function and take $X \subseteq A$.
   (a) Prove that $X \subseteq f^{-1}(f(X))$.
   (b) Now suppose f is injective. Prove that $X = f^{-1}(f(X))$.
   (c) Give an example of a function $f : A \to B$ and $X \subseteq A$ for which $X \neq f^{-1}(f(X))$.
7. Let $f : A \to B$ be a function. Prove that f is surjective if and only if $f^{-1}(Y) \neq \varnothing$ for all nonempty $Y \subseteq B$. (*Bonus*)
8. Here we write $\mathbb{Z}[x]$ for the set of *polynomials in the indeterminant x with integer coefficients*. Remember that a polynomial $p(x)$ is a formal expression in x; there is always an $n \in \mathbb{Z}^+$ such that $p(x) = a_n x^n + \cdots + a_1 x + a_0$, with each $a_k \in \mathbb{Z}$. Thus we have $8x^5 - 4x + 1 \in \mathbb{Z}[x]$ and $0, 1 \in \mathbb{Z}[x]$, but things like
   $$\frac{1}{x^6+1}, \quad \tfrac{1}{5}x, \quad 7x^4 + 3x^{-2}, \quad \text{and} \quad 1 + x + x^2 + x^3 + \cdots \text{(an infinite series)}$$
   are *not* elements of $\mathbb{Z}[x]$. Two polynomials $p(x), q(x) \in \mathbb{Z}[x]$ are equal if and only if all their coefficients match.

   Consider the function $V : \mathbb{Z}[x] \to \mathcal{P}(\mathbb{R})$ given by
   $$V(p(x)) = \{r \in \mathbb{R} : p(r) = 0\}.$$
   (a) Compute $V(x^2 - 2)$, $V(x^3 - 1)$, $V(x^2 + 1)$, and $V(0)$.
   (b) Is V injective or surjective? Explain.
9. Let $f : A \to B$ and $g : B \to C$.
   (a) Prove that if f is surjective and g is *not* injective, then $g \circ f$ is also not injective.
   (b) Is the claim necessarily true if f is not surjective? Prove or give a counterexample.
10. Consider the function $f : \mathbb{R} \setminus \{1\} \to \mathbb{R} \setminus \{1\}$ given by $f(x) = \dfrac{x+1}{x-1}$. Prove that f is a bijection.

    Hint: You can show this directly, but the shrewd among you might instead verify a condition that is equivalent to being bijective.

References

[1] Ted Sundstrom. *Mathematical Reasoning: Writing and Proof*. Grand Valley State University Libraries, 3rd edition, 2020.